Pacific Journal of Mathematics

BOUNDEDNESS OF THE RIESZ PROJECTION ON SPACES WITH WEIGHTS

Stephen D. Abbott and Irina Marinov

Given a bounded, non-negative operator W and a projection P on a Hilbert space, we find necessary and sufficient conditions for the existence of a non-trivial, non-negative operator V such that P is bounded from $L^2(W)$ to $L^2(V)$. This leads to a vector-valued version of a theorem of Koosis and Treil' concerning the boundedness of the Riesz projection in spaces with weights.

1. Introduction.

Let $\partial \mathbf{D}$ be the unit circle in the complex plane, define the function χ on $\partial \mathbf{D}$ by $\chi(e^{i\theta}) = e^{i\theta}$, and set $\mathcal{P} = \{p : p = \sum_{k=-N}^{N} c_k \chi^k\}$. Let σ be normalized Lebesgue measure on $\partial \mathbf{D}$. The *Riesz projection* P_+ is defined on $\mathcal{P}$ by the formula $P_+(\sum_{k=-N}^{N} c_k \chi^k) = \sum_{k=0}^{N} c_k \chi^k$.

In [**4**], Paul Koosis proved:

Theorem 1 (Koosis). *Given a non-negative function $w \in L^1$, there exists a non-negative, non-trivial function $v \in L^1$ such that*

$$\int_{\partial \mathbf{D}} |P_+ f|^2 v \, d\sigma \le \int_{\partial \mathbf{D}} |f|^2 w \, d\sigma \qquad \forall f \in \mathcal{P}$$

if and only if $\frac{1}{w} \in L^1$.

The $w^{-1} \in L^1$ requirement may look familiar to readers acquainted with the theorems of prediction theory, and indeed, in his proof of Theorem 1, Koosis observes that the necessity of the $w^{-1} \in L^1$ condition is a consequence of:

Theorem 2 (Kolmogorov's infimum). *Given $w \ge 0$ in L^1,*

$$\inf \left\{ \int_{\partial \mathbf{D}} |1 - p|^2 \omega d\sigma : p \in \mathcal{P}, \int_{\partial \mathbf{D}} p d\sigma = 0 \right\} = \left[\int_{\partial \mathbf{D}} \frac{1}{\omega} \, d\sigma \right]^{-1},$$

where the infimum is understood to be zero if $w^{-1} \notin L^1$.

Koosis' proof that $w^{-1} \in L^1$ is sufficient in Theorem 1 is short and elegant, but it uses techniques from analytic function theory that tie it to the scalar-valued setting. A version of Theorem 1 for vector-valued functions

258 STEPHEN D. ABBOTT AND IRINA MARINOV

and operator-valued weights was proved in a very different way by S.R. Treil'
in [**6**]. Treil' takes an interesting geometric approach, and it is this view-
point that prompted us to study more deeply the nature of the relationship
between the weights w, v, and the projection P_+.

Starting with an extremely general formulation of the Koosis result in
Section 2, we prove a version of Theorem 1 for a projection P and a non-
negative, bounded operator W on an arbitrary Hilbert space $\mathcal{L}$. The re-
sulting theorem (Theorem 4) has some interesting implications when we
specialize to L^2 of the unit circle. We cannot, however, use it to recover the
Koosis result (for bounded weight functions) since the positive operator V
that appears in the theorem need not be a multiplication operator. This
issue is addressed in Section 3, where we introduce a bilateral shift U on $\mathcal{L}$
and require that our weights W and V commute with U. The main result
of this section (Theorem 7) is a strengthening of Treil's vector-valued result
referenced above.

This research owes a great debt to Treil' in that the proof of Theorem 7
uses the same line of attack discovered by him, albeit with two notable dif-
ferences. One substantial simplification comes from the use of Theorem 4
below which is essentially a corollary to the main result in [**1**]. A second,
more significant, improvement is achieved by replacing Treil's geometric con-
struction with an algebraic argument that enables us to drop the hypothesis
of invertibility assumed in Treil's work. (See Corollary 8.) The result is a
stronger theorem with, what is in our opinion, a more elegant proof.

2. Koosis' Theorem for an Arbitrary Projection.

Let $\mathcal{L}$ be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$, and let $\mathcal{B}(\mathcal{L})$ be the
algebra of bounded linear operators on $\mathcal{L}$. Given a projection $P \in \mathcal{B}(\mathcal{L})$
onto a subspace $\mathcal{C} \subseteq \mathcal{L}$ and a non-negative operator $W \in \mathcal{B}(\mathcal{L})$, we ask
when there exists a non-trivial, non-negative operator $V \in \mathcal{B}(\mathcal{C})$ satisfying

$$\langle VPf, Pf \rangle \leq \langle Wf, f \rangle \quad \forall f \in \mathcal{L}.$$

It may seem surprising that one could say anything interesting at all without
the addition of some more hypotheses, but we get hope from the fact that
Kolmogorov's infimum has a useful analogue in this very general setting.
The result appears in [**1**], and is stated here as:

Theorem 3. *Let $W \in \mathcal{B}(\mathcal{L})$ be non-negative, and let $P \in \mathcal{B}(\mathcal{L})$ be the
projection onto a subspace $\mathcal{C} \subseteq \mathcal{L}$. If $k \in \mathcal{C}$, then*

$$(1) \qquad \inf \{ \langle W(k+f), k+f \rangle : Pf = 0 \} = \lim_{\epsilon \to 0^+} \langle [PW_\epsilon^{-1}|_{\mathcal{C}}]^{-1}k, k \rangle,$$

where $W_\epsilon = W + \epsilon I$, and I is the identity operator on $\mathcal{L}$.

The two inverses in Equation (1) refer to different spaces. For each $\epsilon > 0$,
the operator W_ϵ is invertible in $\mathcal{B}(\mathcal{L})$. Letting $A_\epsilon = PW_\epsilon^{-1}|_{\mathcal{C}} \in \mathcal{B}(\mathcal{C})$, we

have that A_ϵ is bounded below and thus is invertible in $\mathcal{B}(\mathcal{C})$. The limit in Equation (1) is monotone decreasing with decreasing ϵ, and a polarization argument ensures that $\lim_{\epsilon \to 0^+} \langle [PW_\epsilon^{-1}|_\mathcal{C}]^{-1} f, g \rangle$ exists for all $f, g \in \mathcal{C}$. Thus it makes sense to define $V \in \mathcal{B}(\mathcal{C})$ to be the weak limit of $[PW_\epsilon^{-1}|_\mathcal{C}]^{-1}$ as ϵ tends to zero from the right.

Combining these observations with Treil's geometric insight into Koosis' theorem gives us:

Theorem 4. *Let $W \in \mathcal{B}(\mathcal{L})$ be non-negative, and let $P \in \mathcal{B}(\mathcal{L})$ be a projection onto $\mathcal{C} \subseteq \mathcal{L}$. Then $V = \text{wk-}\lim_{\epsilon \to 0^+} [PW_\epsilon^{-1}|_\mathcal{C}]^{-1}$ satisfies*

$$(2) \qquad \langle VPf, Pf \rangle \leq \langle Wf, f \rangle \qquad \forall f \in \mathcal{L},$$

and is maximal in the sense that $V \geq B$ for any B that also satisfies (2).

Proof. For $f \in \mathcal{L}$, write $f = k + g$ where $k \in \mathcal{C}$ and $g \in \mathcal{C}^\perp$. By Theorem 3,

$$\langle VPf, Pf \rangle = \langle Vk, k \rangle = \inf_{Pg'=0} \langle W(k + g'), k + g' \rangle \leq \langle Wf, f \rangle.$$

If B satisfies (2), then for any $g' \in \mathcal{C}^\perp$ it must be that $\langle Bk, k \rangle \leq \langle W(k + g'), k + g' \rangle$. Thus

$$\langle Bk, k \rangle \leq \inf_{Pg'=0} \langle W(k + g'), k + g' \rangle = \langle Vk, k \rangle.$$

$\square$

Corollary 5. *Given W and P in $\mathcal{B}(\mathcal{L})$ as in Theorem 4, there exists a non-negative, non-trivial $V \in \mathcal{B}(\mathcal{C})$ satisfying (2) if and only if $\lim_{\epsilon \to 0^+} \langle [PW_\epsilon^{-1}|_\mathcal{C}]^{-1} k, k \rangle > 0$ for some $k \in \mathcal{C}$.*

Corollary 5 is just a slightly weaker reformulation of Theorem 4 that more accurately parallels the statement of Koosis' result (Theorem 1). The next proposition gives a condition sufficient for proving the existence of a non-trivial weight V. Although it is no longer necessary, this condition is somewhat easier to verify than the one given in Corollary 5.

Corollary 6. *Given W and P in $\mathcal{B}(\mathcal{L})$ as in Theorem 4, there exists a non-trivial, non-negative operator $V \in \mathcal{B}(\mathcal{C})$ satisfying (2) provided $\lim_{\epsilon \to 0^+} \langle W_\epsilon^{-1} k, k \rangle < \infty$ for some non-trivial $k \in \mathcal{C}$.*

Proof. Let P_k be the projection onto the one dimensional subspace spanned by the vector k. A straightforward calculation shows that the operator $P_k W_\epsilon^{-1}|_{P_k \mathcal{L}}$ is just multiplication by the constant $\langle W_\epsilon^{-1} \frac{k}{\|k\|}, \frac{k}{\|k\|} \rangle$. Since $k \in \text{ran} P$,

$$\inf_{Pf=0} \langle W(k + f), k + f \rangle \geq \inf_{P_k f=0} \langle W(k + f), k + f \rangle.$$

Now using Theorem 3 we can write

$$\lim_{\epsilon \to 0^+} \langle [PW_\epsilon^{-1}|_{\mathcal{C}}]^{-1}k, k \rangle \geq \lim_{\epsilon \to 0^+} \langle [P_k W_\epsilon^{-1}|_{P_k \mathcal{L}}]^{-1}k, k \rangle = \lim_{\epsilon \to 0^+} \frac{\|k\|^4}{\langle W_\epsilon^{-1}k, k \rangle},$$

and the result follows from Corollary 5. $\square$

3. Laurent Operators.

The generality of Theorem 4 has a strong appeal; however, the original Koosis result deals with multiplication operators, and this quality is ignored in Theorem 4. Consider this example on L^2 of the unit circle.

Let $w \geq 0$ be a bounded function satisfying (i) $\log w \in L^1$ and (ii) $\frac{1}{w} \notin L^1$, and define W to be multiplication by w on $\mathcal{L} = L^2$. The Hardy space $H^2 = \{f \in L^2 : \hat{f}(n) = 0, \forall n < 0\}$ is a closed subspace of L^2 and the orthogonal projection P_H onto H^2 agrees with the Riesz projection P_+ on polynomials. Now condition (i) implies that there exists an $h \in H^2$ such that $|h|^2 = w$ a.e. on the unit circle, which means that $\lim_{\epsilon \to 0^+} \langle W_\epsilon^{-1}h, h \rangle = 1$. By Corollary 6, then, there exists a non-trivial, non-negative operator $V \in \mathcal{B}(H^2)$ satisfying $\langle V P_H f, P_H f \rangle \leq \int_{\partial \mathbf{D}} |f|^2 w \, d\sigma$ for all $f \in L^2$. However, using Theorem 1, we see that condition (ii) above implies that there is no way to extend V to be multiplication by some non-negative function v on L^2.

This example illustrates that to fully recover Koosis' theorem from the abstract setting, we must introduce a bilateral shift $U \in \mathcal{B}(\mathcal{L})$ and consider operators that commute with U.

Definition. A unitary operator $U \in \mathcal{B}(\mathcal{L})$ is a *bilateral shift* if there exists a projection $P_0 \in \mathcal{B}(\mathcal{L})$ satisfying

  (i) $P_0 U^j P_0 = \delta_{j,0} P_0, \quad \forall j \in \mathbf{Z}$; and,

  (ii) as $n \to \infty$, $\sum\limits_{j=-n}^{n} U^j P_0 U^{*j}$ converges strongly to the identity on $\mathcal{L}$.

Letting $\mathcal{P}_0 = P_0 \mathcal{L}$, we can write $\mathcal{L} = \sum\limits_{j=-\infty}^{\infty} \oplus U^j \mathcal{P}_0$. Theorem 7 will deal specifically with the projection

$$P_\mathcal{H} = \sum_{j=0}^{\infty} U^j P_0 U^{*j}$$

onto the half-space $\mathcal{H} = \sum\limits_{j=0}^{\infty} \oplus U^j \mathcal{P}_0$.

Definition. An operator $A \in \mathcal{B}(\mathcal{L})$ is *Laurent* (*with respect to U*) if $AU = UA$.

In the case of the unit circle, if $U \in \mathcal{B}(L^2)$ is given by $Uf = \chi f$, then $A \in \mathcal{B}(L^2)$ is Laurent if and only if $Af = \phi f$ for some $\phi \in L^\infty$. An analogous fact holds in the vector-valued case ([**5**, p. 110]).

We are now ready to prove:

Theorem 7. *Let $W \in \mathcal{B}(\mathcal{L})$ be non-negative and Laurent. Then there exists a non-trivial, non-negative Laurent operator $V \in \mathcal{B}(\mathcal{L})$ satisfying*

$$(3) \qquad \langle V P_{\mathcal{H}} f, P_{\mathcal{H}} f \rangle \leq \langle W f, f \rangle \quad \forall f \in \mathcal{L}$$

if and only if $V_0 = $ wk-$\lim_{\epsilon \to 0}[P_0 W_\epsilon^{-1}|_{\mathcal{P}_0}]^{-1}$ is non-trivial.

Moreover, if V_0 is non-trivial, then V can be constructed to satisfy

$$(4) \qquad \langle V c, c \rangle \geq \frac{1}{4} \langle V_0 c, c \rangle \quad \forall c \in \mathcal{P}_0.$$

Proof. Assume V exists. Then for any $f \in \mathcal{L}$,

$$
\begin{aligned}
\langle V P_0 f, P_0 f \rangle^{\frac{1}{2}} &= \| V^{\frac{1}{2}} P_0 f \| = \| V^{\frac{1}{2}} (P_{\mathcal{H}} - U P_{\mathcal{H}} U^*) f \| \\
&\leq \| V^{\frac{1}{2}} P_{\mathcal{H}} f \| + \| V^{\frac{1}{2}} U P_{\mathcal{H}} U^* f \| \\
&\leq \| V^{\frac{1}{2}} P_{\mathcal{H}} f \| + \| V^{\frac{1}{2}} P_{\mathcal{H}} U^* f \| \\
&\leq \langle W f, f \rangle^{\frac{1}{2}} + \langle W U^* f, U^* f \rangle^{\frac{1}{2}} \\
&= 2 \langle W f, f \rangle^{\frac{1}{2}}.
\end{aligned}
$$

Thus, $\frac{1}{4} \langle V P_0 f, P_0 f \rangle \leq \langle W f, f \rangle$ for all $f \in \mathcal{L}$, and so by Theorem 4, $\frac{1}{4} P_0 V|_{\mathcal{P}_0} \leq V_0$. Since V is non-trivial and Laurent, its kernel cannot contain $\mathcal{P}_0$ and it follows that V_0 is non-trivial as well.

Conversely, assume $V_0 = $ wk-$\lim_{\epsilon \to 0^+}[P_0 W_\epsilon^{-1}|_{\mathcal{P}_0}]^{-1}$ is non-trivial. For $n \geq 1$, define $P_n = \sum_{j=0}^{n} U^j P_0 U^{*j}$ to be the projection onto the subspace $\mathcal{P}_n = P_n \mathcal{L}$, and let $V_n = $ wk-$\lim_{\epsilon \to 0^+}[P_n W_\epsilon^{-1}|_{\mathcal{P}_n}]^{-1}$. By Theorem 4,

$$\langle V_n P_n f, P_n f \rangle \leq \langle W f, f \rangle \quad \forall f \in \mathcal{L},$$

and the sequence V_n is monotone in the sense that if $0 \leq m < n$ and $p_m \in \mathcal{P}_m$ then

$$
\begin{aligned}
\langle V_m p_m, p_m \rangle &= \inf_{P_m f = 0} \langle W(p_m + f), p_m + f \rangle \\
&\leq \inf_{P_n f = 0} \langle W(p_m + f), p_m + f \rangle = \langle V_n p_m, p_m \rangle.
\end{aligned}
$$

Roughly speaking, we intend to define V via the limit of the monotone sequence V_n. The dilemma is that the argument will require each successive operator to be a dilation of the previous one (i.e., $P_n V_{n+1}|_{\mathcal{P}_n} = V_n$) which is not true of the sequence V_n. Thus, we first need to move to a new sequence A_n satisfying $0 \leq A_n \leq V_n$ which does have this property.

To this end set $A_0 = V_0$, and define A_{n+1} inductively as follows. Write $\mathcal{P}_{n+1} = \mathcal{P}_n \oplus U^{n+1}\mathcal{P}_0$, and denote $V_{n+1} \in \mathcal{B}(\mathcal{P}_{n+1})$ by the 2×2 matrix

$$V_{n+1} = \begin{pmatrix} B & D \\ D^* & C \end{pmatrix}$$

where B, C, and D are acting on the appropriate spaces. Now $V_{n+1} \geq 0$ is equivalent to $B \geq 0$, $C \geq 0$ and the existence of a contraction $W : \mathrm{ran}\overline{C} \to \overline{\mathrm{ran}B}$ satisfying $D = B^{\frac{1}{2}}WC^{\frac{1}{2}}$ [**3**, p. 547]. Letting $X = WC^{\frac{1}{2}}$ leads to the LU-factorization

$$(5) \qquad V_{n+1} = \begin{pmatrix} B & D \\ D^* & C \end{pmatrix} = \begin{pmatrix} B^{\frac{1}{2}} & 0 \\ X^* & Y^{\frac{1}{2}} \end{pmatrix} \begin{pmatrix} B^{\frac{1}{2}} & X \\ 0 & Y^{\frac{1}{2}} \end{pmatrix}$$

where $Y = C - X^*X = C^{\frac{1}{2}}(I - W^*W)C^{\frac{1}{2}} \geq 0$. By the induction hypothesis, $0 \leq A_n \leq B$ which implies that $A_n = B^{\frac{1}{2}}ZB^{\frac{1}{2}}$ for a positive contraction Z [**2**, p. 413]. Now it is straightforward to verify that

$$A_{n+1} = \begin{pmatrix} B^{\frac{1}{2}} & 0 \\ X^* & Y^{\frac{1}{2}} \end{pmatrix} \begin{pmatrix} Z & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} B^{\frac{1}{2}} & X \\ 0 & Y^{\frac{1}{2}} \end{pmatrix}$$

satisfies $0 \leq A_{n+1} \leq V_{n+1}$, and $P_n A_{n+1}|_{\mathcal{P}_n} = A_n$.

By construction, $\langle A_n P_n f, P_n f \rangle \leq \langle V_n P_n f, P_n f \rangle \leq \langle Wf, f \rangle$ for all $n \geq 0$ and $f \in \mathcal{L}$, and the sesquilinear form $a(p, q) = \lim_{n \to \infty} \langle A_n p, q \rangle$ is well defined for $p, q \in \mathcal{P}_+ = \bigcup_{n=0}^{\infty} \mathcal{P}_n$. The operators A_n are uniformly bounded on the diagonal by $\|W\|$, so a is as well, and hence there exists an operator $A \in \mathcal{B}(\mathcal{H})$ such that $\langle Ap, q \rangle = a(p, q)$ for all $p, q \in \mathcal{P}_+$. The operator A satisfies $\langle AP_n f, P_n f \rangle \leq \langle Wf, f \rangle$ for all $f \in \mathcal{L}$ from which we can conclude that $\langle AP_\mathcal{H} f, P_\mathcal{H} f \rangle \leq \langle Wf, f \rangle$.

We now use A to construct a Laurent operator $V \in \mathcal{B}(\mathcal{L})$ with the required properties. For $k \geq 0$, let F_k be the operator on $\mathcal{L}$ defined by $F_k = \frac{1}{k+1} \sum_{n=0}^{k} U^{*n}AP_\mathcal{H}U^n$. For $n \geq 1$ and $f \in \mathcal{L}$,

$$\begin{aligned} \langle AU^n P_\mathcal{H} f, U^n P_\mathcal{H} f \rangle^{\frac{1}{2}} &= \|A^{\frac{1}{2}}U^n P_\mathcal{H} f\| = \|A^{\frac{1}{2}}(P_\mathcal{H} - P_{n-1})U^n f\| \\ &\leq \|A^{\frac{1}{2}}P_\mathcal{H}U^n f\| + \|A_{n-1}^{\frac{1}{2}}P_{n-1}U^n f\| \\ &\leq 2\langle Wf, f \rangle^{\frac{1}{2}}. \end{aligned}$$

This implies $\langle F_k P_\mathcal{H} f, P_\mathcal{H} f \rangle \leq \langle 4Wf, f \rangle$ for all $k \geq 0$. Letting V be a weak limit point of the set $\{\frac{1}{4}F_k : k \geq 0\}$, it follows that V satisfies (3) and is Laurent as desired.

It remains to show that V satisfies (4), which will follow if we can demonstrate that $\langle AU^n c, U^n c \rangle \geq \langle V_0 c, c \rangle$ for all $n \geq 0$ and $c \in \mathcal{P}_0$. If $n = 0$,

$A_0 = V_0$ and the result is clear. For a fixed $n \geq 0$, the inductive construction of A_{n+1} yields

$$\begin{aligned}
\langle AU^{n+1}c, U^{n+1}c \rangle &= \langle A_{n+1}U^{n+1}c, U^{n+1}c \rangle \\
&= \langle X^*ZXU^{n+1}c, U^{n+1}c \rangle + \langle YU^{n+1}c, U^{n+1}c \rangle \\
&\geq \langle YU^{n+1}c, U^{n+1}c \rangle.
\end{aligned}$$

Thus it is sufficient to prove $\langle YU^{n+1}c, U^{n+1}c \rangle \geq \langle V_0 c, c \rangle$ for all $c \in \mathcal{P}_0$.

Let $z \in \mathcal{P}_n$, so that $z + U^{n+1}c \in \mathcal{P}_n \oplus U^{n+1}\mathcal{P}_0 = \mathcal{P}_{n+1}$. Using the LU-factorization for V_{n+1} given in (5), we have

$$\begin{aligned}
&\langle V_{n+1}(z + U^{n+1}c), z + U^{n+1}c \rangle \\
&= \langle Bz, z \rangle + 2Re\langle B^{\frac{1}{2}}z, XU^{n+1}c \rangle + \langle CU^{n+1}c, U^{n+1}c \rangle.
\end{aligned}$$

Recall that the operator V_{n+1} was generated via Theorem 4. This, together with the assumption that W is Laurent allows us to write

$$\begin{aligned}
&\langle V_{n+1}(z + U^{n+1}c), z + U^{n+1}c \rangle \\
&= \inf \left\{ \langle W(z + U^{n+1}c + f), z + U^{n+1}c + f \rangle : P_{n+1}f = 0 \right\} \\
&\geq \inf \left\{ \langle W(U^{n+1}c + f), U^{n+1}c + f \rangle : U^{n+1}P_0U^{*(n+1)}f = 0 \right\} \\
&= \inf \left\{ \langle W(c + f), c + f \rangle : P_0 f = 0 \right\} \\
&= \langle V_0 c, c, \rangle.
\end{aligned}$$

Combining these observations we have

$$(6) \quad \langle Bz, z \rangle + 2Re\langle B^{\frac{1}{2}}z, XU^{n+1}c \rangle + \langle CU^{n+1}c, U^{n+1}c \rangle - \langle V_0 c, c \rangle \geq 0$$

for all $z \in \mathcal{P}_n$ and $c \in \mathcal{P}_0$. Let r be an arbitrary real number. Since $\text{ran}X \subseteq \overline{\text{ran}B} = \overline{\text{ran}B^{\frac{1}{2}}}$, there exists a sequence z_m in $\mathcal{P}_n$ such that $B^{\frac{1}{2}}z_m \to rXU^{n+1}c$. Substituting into (6) and taking limits we get

$$r^2\|XU^{n+1}c\|^2 + 2r\|XU^{n+1}c\|^2 + \left(\langle CU^{n+1}c, U^{n+1}c \rangle - \langle V_0 c, c \rangle \right) \geq 0.$$

Evidently, this quadratic equation in r has at most one real root which means that its discriminant is not positive. Translating this into a statement about the coefficients yields $\|XU^{n+1}c\|^2 \leq \langle CU^{n+1}c, U^{n+1}c \rangle - \langle V_0 c, c \rangle$, which is equivalent to

$$\langle V_0 c, c \rangle \leq \langle CU^{n+1}c, U^{n+1}c \rangle - \langle X^*XU^{n+1}c, U^{n+1}c \rangle = \langle YU^{n+1}c, U^{n+1}c \rangle.$$

Altogether then, if $c \in \mathcal{P}_0$, we have

$$\langle V_0 c, c \rangle \leq \langle YU^{n+1}c, U^{n+1}c \rangle \leq \langle AU^{n+1}c, U^{n+1}c \rangle,$$

and the lower estimate in (4) follows. $\qquad\qquad\qquad\qquad\qquad\qquad\square$

In the algebraic language of this paper, Treil's result in [6] essentially takes the form of:

Corollary 8 (Treil'). *Let $W \in \mathcal{B}(\mathcal{L})$ be non-negative and Laurent. Then there exists a non-trivial, non-negative Laurent operator $V \in \mathcal{B}(\mathcal{L})$ with $P_0 V|_{\mathcal{P}_0}$ invertible in $\mathcal{B}(\mathcal{P}_0)$ and satisfying*

$$\langle V P_{\mathcal{H}} f, P_{\mathcal{H}} f \rangle \leq \langle W f, f \rangle \quad \forall f \in \mathcal{L}$$

if and only if $V_0 = \text{wk-}\lim_{\epsilon \to 0} [P_0 W_\epsilon^{-1}|_{\mathcal{P}_0}]^{-1}$ is invertible.

Proof. If V exists, then as before, we can show that $V_0 \geq \frac{1}{4} P_0 V|_{\mathcal{P}_0}$. It follows that V_0 is bounded below and consequently invertible. Conversely, the construction in Theorem 7 yields an operator V satisfying (4). Thus, if V_0 is invertible then $P_0 V|_{\mathcal{P}_0}$ is invertible as well. □

References

[1] S. Abbott, *A unified approach to some prediction problems*, Proceedings of the American Mathematical Society, **123** (1995), 425-431.

[2] R.G. Douglas, *On majorization, factorization and range inclusion of operators on Hilbert Space*, Proceedings of the American Mathematical Society, **17** (1966), 413-416.

[3] C. Foias and A. Frazho, *The Commutant Lifting Approach to Interpolation Problems*, Operator Theory: Advances and Applications, **44**, Birkhauser, Basel (1990).

[4] P. Koosis, *Moyennes quadratiques pondérées de fonctions périodiques et de leurs conjuguées harmoniques*, C.R. Academie Science Paris, **A 291** (1980), 255-257.

[5] M. Rosenblum and J. Rovnyak, *Hardy Classes and Operator Theory*, Oxford University Press, New York (1985).

[6] S.R. Treil', *Geometric Methods in Spectral Theory of Vector-Valued Functions: Some Recent Results*, Operator Theory: Advances and Applications, **42**, Birkhauser, Basel, (1989), 209-280.

MIDDLEBURY COLLEGE
MIDDLEBURY, VT 05753
E-mail address: abbott@middlebury.edu

DEPT. OF ATMOSPHERIC AND OCEANIC SCIENCES
PRINCETON UNIVERSITY
PRINCETON, NJ 08544
E-mail address: imarinov@princeton.edu

APPLICATION OF RESTRICTION OF FOURIER TRANSFORMS TO AN EXAMPLE FROM REPRESENTATION THEORY

L. Barchini and Mark R. Sepanski

This paper uses restriction of Fourier transforms to construct explicit realizations of certain irreducible unitary representations of $SU(n,n)$. The realizations begin with generalizations of the classical Szegö map. Boundary values of these Szegö maps naturally lead to certain restrictions of Fourier transforms. The image of these restrictions provide concrete constructions of unitary representations as L^2 spaces on certain orbits. The $SU(n,n)$ invariance of the L^2 spaces and inner products follows immediately from the restriction maps and the natural pairing between certain degenerate principal series.

1. Introduction.

Calculating explicit and natural constructions of unitary representations–especially singular ones–has been a very fruitful field of study in representation theory. Consider, for instance, the Metaplectic representation and its many applications. One of the reasons explicit realizations are so useful is because detailed knowledge of a representation frequently comes through use of a good realization. This paper studies and constructs a number of explicit realizations for certain unitary representations of $SU(n,n)$. The central technique employs certain restrictions of Fourier transforms ([**15**]) that arise naturally in the study of the representation theory of $SU(n,n)$. Our approach is different than the one usually adopted in such studies of this kind (e.g., [**8**], [**12**], [**13**], [**14**]) where "extensions" of Fourier transforms are mainly used.

In our approach, the representations naturally arise from an examination of various Szegö maps and their boundary values which immediately lead to certain restrictions of Fourier transforms. There are several advantages to this line of study. The first is that the invariance of our spaces and inner products are very natural from this point of view. The second is that precise knowledge of the K-types is not needed. The third is that the techniques employed are independent of a multiplicity one assumption on K-types.

To be more specific about the results of this study, write $G = SU(n,n)$, $K = S(U(n) \times U(n))$, and G/P for the closed G-orbit in the boundary of G/K. The unbounded realization of G/K may be identified with $\mathcal{D}^+ = H + iH^+$ where H and H^+ are the set of $n \times n$ Hermitian matrices and $n \times n$ positive definite Hermitian matrices, respectively. More generally, the *semi-definite* G-orbits of $G_{\mathbb{C}}/P_{\mathbb{C}}$ may be described on an open dense set as $H + i\mathcal{O}_p$ where

$$\mathcal{O}_p = \{X \in H \mid \text{signature of } X \text{ is } (p,0)\}.$$

$\mathcal{O}_p$ is an orbit under the action of the Levi part of P and comes equipped with a uniquely defined equivariant measure, $d\mu_p$. Then the main application of our study of the restriction of the Fourier transform shows that

$$L^2(\mathcal{O}_p, d\mu_p)$$

is an irreducible unitary representation of $SU(n,n)$ (Theorem 10.4). Though this statement is already known ([**12**]), we believe the techniques in our new approach yield a more complete understanding of this representation. We also expect the same techniques to be applicable to a wider family of representations–at least including the representations associated to certain orbits in real semisimple Jordan algebras ([**13**]).

In more detail, we begin with certain pairs of degenerate principal series on G/P. For certain parameters, depending on each choice of $\mathcal{O}_p$, the appropriate principal series may be realized in the noncompact picture as $L^2(H, \det(I + X^2)^{\pm(n-p)} dX)$ and is denoted by $L^2(H)^{\pm}$, respectively. Using techniques similar to [**11**] and [**1**], we write down a *Szegö map*, $S : L^2(H)^+ \to C^\infty(\mathcal{D}^+)$. It turns out that S acts on a function $f \in L^2(H)^+$ by the particularly easy formula

$$(1.1) \qquad Sf(\eta) = \int_H \det(X - \eta)^{-p} f(X) \, dX$$

for each $\eta \in \mathcal{D}^+$ (Theorem 5.3). Writing B for the *boundary value map* taking $\mathcal{D}^+$ to H, it is possible to form a commutative diagram defining an intertwining map, $A : L^2(H)^+ \to L^2(H)^-$, of the form

$$(1.2) \qquad \begin{array}{ccc} & A & \\ L^2(H)^+ & \longrightarrow & L^2(H)^-. \\ S \downarrow & \nearrow & \\ & B & \\ C^\infty(\mathcal{D}^+) & & \end{array}$$

For functions $\phi \in \mathcal{S}(H)$, the Schwartz functions on H, it is possible to see that the action of A may be rewritten as

$$(1.3) \qquad A\phi(X) = i^{np} \int_{\mathcal{O}_p} e^{i\,\mathrm{tr}(X\xi)} \, \check{\phi}(\xi) \, d\mu_p(\xi)$$

where $\check{\phi}$ is the inverse Fourier transform of ϕ on H (Theorem 7.2). Equation (1.3) suggests a second splitting of the singular integral defining A. Namely, consider the two maps $F_R : L^2(H)^+ \to L^2(\mathcal{O}_p, d\mu_p)$ and $F_E : L^2(\mathcal{O}_p, d\mu_p) \to L^2(H)^-$ given by

$$F_R\phi = \check{\phi}|_{\mathcal{O}_p}$$

when $\phi \in \mathcal{S}(H)$, where $\check{\phi}|_{\mathcal{O}_p}$ denotes restriction of $\check{\phi}$ to $\mathcal{O}_p$, and

$$F_E\psi(X) = i^{np} \int_{\mathcal{O}_p} e^{i\,\mathrm{tr}(\xi X)}\, \psi(\xi)\, d\mu_p(\xi)$$

for $\psi \in \mathrm{Im}(F_R)$. The first is a restriction of the Fourier transform and the second is the more usual "extension" of the Fourier transform. These maps are proved to be continuous (Theorem 9.1) and yield the commutative diagram

$$
(1.4) \qquad
\begin{array}{ccc}
 & A & \\
L^2(H)^+ & \longrightarrow & L^2(H)^- \\
 & \searrow \quad \uparrow & \qquad F_E. \\
F_R & & \\
 & L^2(\mathcal{O}_p,\, d\mu_p) &
\end{array}
$$

This diagram is used to make $L^2(\mathcal{O}_p, d\mu_p)$ into a representation of G by requiring all maps to be G-maps (Theorem 10.2).

The point of working with $L^2(\mathcal{O}_p, d\mu_p)$ is that it comes equipped with its own inner product denoted by

$$\langle f_1, f_2 \rangle_{\mathcal{O}_p} = \int_{\mathcal{O}_p} f_1(\xi)\overline{f_2(\xi)}\, d\mu_p(\xi)$$

for $f, g \in L^2(\mathcal{O}_p, d\mu_p)$. In fact, it is proved that this structure makes $L^2(\mathcal{O}_p, d\mu_p)$ into a irreducible unitary representation (Theorem 10.4) of G. The key to seeing the invariance of the inner product is to relate it to an invariant form on $L^2(H)^+$. Indeed, consider the natural G-invariant pairing of $L^2(H)^+$ and $L^2(H)^-$ given by

$$\langle f, g \rangle_A = \int_H f(X)\overline{Ag(X)}\, dX$$

for $f, g \in L^2(H)^+$. The central identity fitting everything together is

$$\langle f, g \rangle_A = \langle F_R f, F_R g \rangle_{\mathcal{O}_p}$$

(Theorem 10.3). Thus the G-invariance of the L^2 inner product follows directly and immediately.

Finally, denoting the kernel of R_F as $\mathcal{K}$ (which is the same as the kernel of $\langle \cdot, \cdot \rangle_A$), R_F therefore induces a bijective intertwining isometry between the completion of $L^2(H)^+/\mathcal{K}$ and $L^2(\mathcal{O}_p, d\mu_p)$.

2. Preliminaries.

Let $G = SU(n, n)$. Breaking up the $2n \times 2n$ matrices into four $n \times n$ blocks, write:

$$J = \begin{pmatrix} 0 & -iI_n \\ iI_n & 0 \end{pmatrix}.$$

Unless noted otherwise, we will use the following realization throughout the paper:

$$G = \{g \in SL(2n, \mathbb{C}) \mid g^* J g = J\}.$$

It is useful to gather a few simple facts about G for later use.

Lemma 2.1. *Write the $2n \times 2n$ matrix $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ as four $n \times n$ blocks.*

1) *G consists of the matrices $g \in SL(2n, \mathbb{C})$ satisfying*

$$(2.1) \qquad A^*C = C^*A, \quad D^*B = B^*D, \quad A^*D - C^*B = I.$$

2) *G consists of the matrices $g \in SL(2n, \mathbb{C})$ satisfying*

$$(2.2) \qquad AB^* = BA^*, \quad CD^* = DC^*, \quad AD^* - BC^* = I.$$

3) *For $g \in G$,*

$$(2.3) \qquad g^{-1} = \begin{pmatrix} D^* & -B^* \\ -C^* & A^* \end{pmatrix}.$$

4) *$K = \{g \in G \mid A = D, B = -C\}$ is a maximally compact subgroup of G. K may also be described as*

$$K = \left\{ \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \ \Big| \ (A + iB, A - iB) \in S(U(n) \times U(n)) \right\}.$$

5) *$P = \{g \in G \mid C = 0\}$ and $\overline{P} = \{g \in G \mid B = 0\}$ are parabolic subgroups of G. P can also be described as*

$$(2.4) \quad P = \left\{ \begin{pmatrix} A & B \\ 0 & A^{*-1} \end{pmatrix} \ \Big| \ A \in GL(n, \mathbb{C}), \ \det(A) \in \mathbb{R}^\times, \right.$$

$$\left. B \in \mathfrak{gl}(n, \mathbb{C}), \ A^{-1}B = (A^{-1}B)^* \right\}.$$

6) *P admits a Langlands decomposition $P = LN$ with $L = MA$ where*

$$M = \left\{ \begin{pmatrix} A & 0 \\ 0 & A^{*-1} \end{pmatrix} \; \middle| \; A \in GL(n, \mathbb{C}), \; \det(A) = \pm 1 \right\},$$

$$A = \left\{ \begin{pmatrix} aI & 0 \\ 0 & a^{-1}I \end{pmatrix} \; \middle| \; a \in \mathbb{R}^{>0} \right\},$$

$$N = \left\{ \begin{pmatrix} I & X \\ 0 & I \end{pmatrix} \; \middle| \; X^* = X \right\}.$$

Likewise, $\overline{P} = L\overline{N}$ where

$$\overline{N} = \left\{ \begin{pmatrix} I & 0 \\ X & I \end{pmatrix} \; \middle| \; X^* = X \right\}.$$

The the representations of G to be studied will be induced from the following characters.

Definition 2.1. Let $p \in \mathbb{Z}$, $0 \leq p \leq n$.

1) The character $\chi_p : K \to S^1$ acts by

$$\chi_p \begin{pmatrix} A & B \\ -B & A \end{pmatrix} = \det(A + iB)^p.$$

2) The character $\delta_p : M \to \{\pm 1\}$ acts by

$$\delta_p \begin{pmatrix} A & 0 \\ 0 & A^{*-1} \end{pmatrix} = \det(A)^p.$$

3) Write $\mathfrak{a}_0 = \text{Lie}(A)$. Let $\epsilon : \mathfrak{a}_0 \to \mathbb{R}$ by

$$\epsilon \begin{pmatrix} aI & 0 \\ 0 & -aI \end{pmatrix} = a$$

and $\nu_p : \mathfrak{a}_0 \to \mathbb{R}$ by

$$\nu_p = n(n - p)\epsilon$$

(the differential of the character $\det(A)^{n-p}$).

4) Write the Cartan decomposition for $\text{Lie}(G)$ as $\text{Lie}(G) = \text{Lie}(K) + \mathfrak{p}$, write $\mathfrak{a}_\mathfrak{p}$ for the maximal Abelian subalgebra of $\mathfrak{p}$ consisting of diagonal matrices, and $\rho : \mathfrak{a}_\mathfrak{p} \to \mathbb{R}$ for the half sum of restricted weights. An easy calculation shows $\rho|_{\mathfrak{a}_0} = n^2 \epsilon$.

Explicitly, we will study the degenerate principal series induced from the characters $\delta_p \otimes \pm \nu_p$ of the maximal parabolic P.

Definition 2.2. Let

$$I_p^\pm = \text{Ind}_P^G(\delta_p \pm \nu_p)$$

(smooth, normalized induction).

In other words,

(2.5) $\quad I_p^\pm = \{f : G \to \mathbb{C}, \text{ smooth} \mid f(xman) = \delta_p^{-1}(m) \, \mathrm{e}^{-(\pm\nu_p+\rho)\,\log(a)} f(x)\}$

with a G action of

$$gf(x) = f(g^{-1}x).$$

Beginning in Section 5, we will also make use of the noncompact picture of these induced representations ([**9**], §7.1). To that purpose, decompose G as $KMAN$ and write the A part of g as $\mathrm{e}^{H(g)}$. Then the associated Hilbert space for $I_p^\pm$ is

(2.6) $\qquad\qquad\qquad L^2(\overline{N}, \mathrm{e}^{\pm 2\,\mathrm{Re}\,\nu_p H(\overline{n})} \, d\overline{n})$

where $d\overline{n}$ is Haar measure.

Definition 2.3. Let $X, Y \in \mathfrak{gl}(n, \mathbb{C})$.

1) Given a fixed presentation $Z = X + iY$, define

$$\overline{Z} = X - iY,$$
$$Z^* = X^* + iY^*$$

so that $\overline{Z}^*$ is the normal transpose complex conjugation.

2) Given $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathfrak{gl}(2n, \mathbb{C})$, let

$$\eta_g = (C + iD)(A + iB)^{-1}$$

which is well defined for almost all g and let

$$\alpha_g = A + iB.$$

3) Write $H = H(n)$ for the set of $n \times n$ Hermitian matrices,

$$H = \{X \in \mathfrak{gl}(n, \mathbb{C}) \mid X^* = X\},$$

and $H^\pm$ for the positive, respectively negative, definite ones,

$$H^\pm = \{X \in H \mid \pm X > 0\}.$$

4) Write

$$\mathcal{D} = \{X + iY \mid X, Y \in H\},$$
$$\mathcal{D}^+ = \{X + iY \mid X \in H, \ Y \in H^+\}$$
$$\mathcal{D}^- = \{X + iY \mid X \in H, \ Y \in H^-\}.$$

It is easy to check the following.

Lemma 2.2. *The mapping*

$$g \to \eta_g$$

implements an isomorphism between G/K and $\mathcal{D}^+$. In particular, $\eta_g^* = \eta_g$ and α_g is invertible.

3. The Szegö Map to Sections on G/K.

Definition 3.1. Write $C^\infty(G/K, \chi_p)$ for the smooth sections on G/K of the line bundle induced by χ_p. We will view this as

$$\{f : G \to \mathbb{C}, \text{ smooth } \mid f(gk) = \chi_p^{-1}(k)f(g) \, \forall g \in G, \, k \in K\}.$$

The central tool used to analyze the representations in this paper is the following Szegö map.

Definition 3.2. Define the *Szegö map*,

$$S : I_p^+ \to C^\infty(G/K, \chi_p),$$

to be the G-intertwining operator mapping $f \to Sf$ given by

$$Sf(g) = \frac{1}{\text{Vol}(M \bigcap K)} \int_K f(gk)\chi_p(k)\,dk.$$

In the following, we show that the map S is a kernel operator. This will permit us to switch to the noncompact picture for I_p^+ and identify $C^\infty(G/K, \chi_p)$ as the set of smooth functions on the tube domain $\mathcal{D}^+$.

The first step is to rewrite S as an integral over $\overline{N}$. As usual, given $g \in G$, decompose g according to

$$G = K \, \exp(\mathfrak{m}_0 \cap \mathfrak{p}) \, A \, N$$

where $\mathfrak{m}_0 = \text{Lie}(M)$. Therefore write $g = K(g) \, M(g) \, A(g) \, N(g)$, $A(g) = e^{H(g)}$, and $L(g) = M(g) \, A(g)$. For future reference, observe $\delta_p|_{\exp(\mathfrak{m}_0)} = 1$ since δ_p is trivial on the connected component of M.

Theorem 3.1. *For $f \in I_p^+$,*

$$Sf(g) = \int_{\overline{N}} e^{(\nu_p - \rho)H(g^{-1}\overline{n})} \, \chi_p(K(g^{-1}\overline{n}))f(\overline{n})\,d\overline{n}.$$

Proof. This is a standard change of variables. For instance, see [**11**].

Lemma 3.2. *Let* $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G$, $\overline{n} = \begin{pmatrix} I & 0 \\ X & I \end{pmatrix} \in \overline{N}$, *and* $l = \begin{pmatrix} A_1 & 0 \\ 0 & A_1^{*-1} \end{pmatrix} \in L$. *Let* $\eta = \eta_g$ *and* $\alpha = \alpha_g$.

1) *Then*

$$e^{2n\epsilon H(g^{-1}\overline{n}l)} = \det(A_1^* A_1) \det(\alpha^*\alpha) \det(X - \eta) \det(X - \overline{\eta}).$$

2) *Both* $\det(\alpha^*\overline{\alpha})$ *and* $\det(X - \eta)\det(X - \overline{\eta})$ *are in* $\mathbb{R}^{\geq 0}$.
3) *As a special case,*

$$e^{n\epsilon H(\overline{n})} = \det(I + X^2)^{\frac{1}{2}}.$$

4) *Finally,*

$$\chi_1(K(g^{-1}\overline{n}l)) = i^{-n}\det(A_1)\det(\overline{\alpha}^*)\det(X-\overline{\eta})\,\mathrm{e}^{-n\epsilon H(g^{-1}\overline{n}l)}.$$

Proof.

Set $x = g^{-1}\overline{n}l$ and write $x = K(x)M(x)A(x)N(x)$. Let $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, $A(x) = \begin{pmatrix} R_1 & 0 \\ 0 & R_2 \end{pmatrix}$, and $L(x) = M(x)A(x) = \begin{pmatrix} L_1 & 0 \\ 0 & L_2 \end{pmatrix}$. Then

$$\mathrm{e}^{2n\epsilon H(x)} = \det(R_1)^2 = |\det(L_1)|^2.$$

On the other hand, if we write $x = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}$ and calculate x^*x and compare the $K(x)L(x)N(x)$ expansion to straightforward multiplication, the upper left hand corner yields the equality

$$L_1^* L_1 = x_1^* x_1 + x_3^* x_3.$$

But now Equation (2.3) allows us to calculate $x = g^{-1}\overline{n}l$ and so compute that $x_1 = (D^* - B^*X)A_1$ and $x_3 = (-C^* + A^*X)A_1$. A simple calculation using Equation (2.1) and the fact that G/K is fixed by $*$ verifies that

$$x_1^* x_1 + x_3^* x_3 = A_1^*(X - \overline{\eta})\overline{\alpha}\alpha^*(X - \eta)A_1.$$

Taking determinants finishes the first part.

The second claim follows by observing both terms are complex numbers times their conjugates. The third is a special case of (1) since (2) allows square roots. For the fourth claim, set $x = g^{-1}\overline{n}l$. Using Equation (2.3) to calculate g^{-1} and then expressing the result in the form $K(x)L(x)N(x)$, it is easy to see that if we write $K(x) = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$, then $(a+ib)L_1 = x_1 - ix_3$ by looking at the upper left and lower left entries in the equality $x = K(x)L(x)N(x)$. A simple calculation then shows that

$$(a+ib)L_1 = -i\overline{\alpha}^*(X - \overline{\eta})A_1.$$

Taking determinants, noting that $\det(L) = |\det(L)|$, and using Equation (2.3) finishes the job. $\qquad\square$

We are now in a position to rewrite Theorem 3.1.

Definition 3.3. Let $f \in I_p^+$ and $X \in H$. Define $\overline{f}$ a function on H by the restriction to $\overline{N}$:

$$\overline{f}(X) = f\left(\begin{pmatrix} I & 0 \\ X & I \end{pmatrix}\right).$$

Write dX for Haar measure on H.

Theorem 3.3. *Let $f \in I_p^+$, $g \in G$, $X \in H$, $\eta = \eta_g$, and $\alpha = \alpha_g$.   Then $Sf \in C^\infty(G/K, \chi_p)$ can be calculated by*

$$Sf(g) = i^{-np} \det(\alpha)^{-p} \int_H \det(X - \eta)^{-p} \overline{f}(X) \, dX.$$

Proof. From Theorem 3.1, Lemma 3.2 (with $l = I$), and Definitions 2.1 and 2.3, it is easy to check that

$$Sf(g) = i^{-np} \det(\overline{\alpha}\alpha^*)^{-p} \det(\overline{\alpha}^*)^p \int_H \det(X - \eta)^{-p} \overline{f}(X) \, dX.$$

But Equation (2.2) can be used to check that $\alpha\overline{\alpha}^* = (AA^* + BB^*) = \overline{\alpha}\alpha^*$ which finishes the proof.   □

4. The Szegö Map to Functions on $\mathcal{D}^+$.

Taking our cue from Lemma 2.2 and Theorem 3.3, it is reasonable to rewrite the Szegö map, S, and $C^\infty(G/K, \chi_p)$ in terms of the tube domain $\mathcal{D}^+$.

Definition 4.1. Let $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G_\mathbb{C}$.   Identify $H \cong \overline{N}$ and $\mathcal{D} \cong \overline{N}_\mathbb{C}$ by the map

$$Z \rightarrow \begin{pmatrix} I & 0 \\ Z & I \end{pmatrix}.$$

This implements an embedding $\mathcal{D} \rightarrow G_\mathbb{C}/P_\mathbb{C}$ whose image is open and dense. For almost all $Z \in \mathcal{D}$, left multiplication by g in $G_\mathbb{C}/P_\mathbb{C}$ may be pulled back to $\mathcal{D}$ by the linear fractional transformation action defined as

$$gZ = (DZ + C)(BZ + A)^{-1}.$$

Note that H and $\mathcal{D}^\pm$ are G orbits under this action and that η_g from Lemma 2.2 is simply g acting on iI.

If $\sigma \in C^\infty(G/K, \chi_p)$ does not vanish, there is an isomorphism

(4.1) $$C^\infty(G/K, \chi_p) \cong C^\infty(G/K)$$

established by mapping $f \in C^\infty(G/K)$ to $\sigma f \in C^\infty(G/K, \chi_p)$. If we let

(4.2) $$\sigma(x, g) = \frac{\sigma(g^{-1}x)}{\sigma(x)}$$

for $g \in G$ and $x \in G/K$ and define a G action on $C^\infty(G/K)$ by

(4.3) $$gf(x) = \sigma(x, g) \, f(g^{-1}x),$$

then the map $f \rightarrow f\sigma$ is a G map as well. Below we choose a section σ and use it to push the Szegö map down to functions on G/K. Finally, identify

$G/K \cong \mathcal{D}^+$ as in Lemma 2.2 by identifying $x \in G/K$ with $\eta = \eta_x \in \mathcal{D}^+$. Thus

$$(4.4) \qquad C^\infty(G/K) \cong C^\infty(\mathcal{D}^+).$$

Definition 4.2. Write $\eta = \eta_g$. Let $\sigma \in C^\infty(G/K, \chi_p)$ be defined by

$$\sigma(g) = e^{(\nu_p - \rho)H(g^{-1})} \chi_p(K(g^{-1})) \det(-\eta)^p.$$

To make sure the above definition is valid, we check that

$$\sigma(gk) = \chi_p(k)^{-1}\sigma(g)$$

for $g \in G$ and $k \in K$. But this follows immediately by observing that $H(k^{-1}g^{-1}) = H(g^{-1})$, $K(k^{-1}g^{-1}) = k^{-1}K(g^{-1})$ and that $\eta_{gk} = \eta_g$.

Lemma 4.1. *For $g \in G$, write $\alpha = \alpha_g$. Then*

$$\sigma(g) = i^{-np} \det(\alpha)^{-p}.$$

Proof. This calculation follows from Lemma 3.2 with $\overline{n} = l = I$. $\qquad\square$

Since $C^\infty(G/K, \chi_p) \cong C^\infty(\mathcal{D}^+)$, we may view $C^\infty(\mathcal{D}^+)$ as a G space and view S as the G-map taking $I_p^+ \to C^\infty(\mathcal{D}^+)$. We continue to denote the resulting map as S as the range will remove ambiguity. We now apply Equations (4.1) and (4.4) and Lemma 4.1 to rewrite Theorem 3.3 in terms of $C^\infty(\mathcal{D}^+)$.

Theorem 4.2. *The G intertwining map $S : I_p^+ \to C^\infty(\mathcal{D}^+)$ acts by*

$$Sf(\eta) = \int_H \det(X - \eta)^{-p}\overline{f}(X)\,dX.$$

It is useful to write the G action on $C^\infty(\mathcal{D}^+)$ explicitly.

Lemma 4.3. *Identifying $G/K \cong \mathcal{D}^+$ and writing $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G$,*

$$\sigma(\eta, g) = \det(D^* - B^*\eta)^{-p}.$$

Proof. It is enough to compute $\sigma(\cdot, g)$ on $\overline{N}L$. Writing

$$x = \begin{pmatrix} I & 0 \\ X & I \end{pmatrix} \begin{pmatrix} L & 0 \\ 0 & L^{*-1} \end{pmatrix},$$

it is easy to compute

$$\alpha(g^{-1}x)\alpha(x)^{-1} = D^* - B^*\eta_x.$$

Applying Lemma 4.1 to Equation (4.2) finishes the proof. $\qquad\square$

Thus using Lemma 3.2 and Equation (4.3), we can write the G action on $C^\infty(\mathcal{D}^+)$.

Theorem 4.4. *For* $f \in C^\infty(\mathcal{D}^+)$, $\eta \in \mathcal{D}^+$, *and* $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G$,

$$gf(\eta) = \det(D^* - B^*\eta)^{-p} f(g^{-1}\eta).$$

5. $I_p^\pm$ as Functions on H.

This section looks at an explicit form of the closure of $I_p^\pm$.

Lemma 5.1. *Restriction to* $\overline{N} \cong H$ *establishes an isomorphism of G spaces between the closure of*

$$I_p^\pm$$

and

$$L^2(H, \det(I + X^2)^{\pm(n-p)}\, dX)$$

where dX is Haar measure on H.

Proof. Apply Lemma 3.2 (with $g = l = I$) to Equation (2.6) and make the identification of H with $\overline{N}$ as in Definition 4.1. $\qquad\square$

In this section we explicitly compute the action of G on $L^2(H, \det(I + X^2)^{\pm(n-p)}\, dX)$ and extend the Szegö map accordingly.

Definition 5.1. Write

$$L^2(H)^\pm = L^2(H, \det(I + X^2)^{\pm(n-p)}\, dX)$$

and

$$L^2(H) = L^2(H, dX).$$

Lemma 5.2. $L^2(H)^+ \subseteq L^2(H) \subseteq L^2(H)^-$.

Proof. Since $0 \le p \le n$ and $\det(I + X^2) \ge 1$,

$$0 < \det(I + X^2)^{-(n-p)} \le 1 \le \det(I + X^2)^{+(n-p)}.$$

Thus the Lemma follows immediately from the definition of $L^2(H)^+$ in Definition 5.1. $\qquad\square$

For functions $f \in I_p^\pm$, this Lemma can also be proved directly. Since two formulas arising from this approach will be needed later, we sketch the idea. For instance, by starting with $f \in I_p^+$, applying Equation (2.5) to the $KMAN$ decomposition, and using Lemma 3.2 (with $g = l = I$) it is easy to show

$$(5.1) \qquad\qquad |\overline{f}(X)|^2 \le C \det(I + X^2)^{p-2n}$$

where C is a constant bounding $|f|^2$ on K. This is enough to finish the first inclusion since it is known ([**7**], §2.1, p. 38) that

$$(5.2) \qquad \int_H \det(I + X^2)^m \, dX < \infty$$

whenever $m < -n + \frac{1}{2}$. Though not needed immediately, we will also have recourse to make use of a formula for the Jacobian of the change of variables on H given by $X \to gX$. It is

$$(5.3) \qquad \det(BX + A)^{-2n}.$$

It is now apparent that Theorem 4.2 may be completed to the following (remember everything can be written as an integral over K).

Theorem 5.3. *The G intertwining map $S : L^2(H)^+ \to C^\infty(\mathcal{D}^+)$ acts by*

$$Sf(\eta) = \int_H \det(X - \eta)^{-p} f(X) \, dX.$$

We finish this section by writing the G action on $L^2(H)^\pm$ explicitly.

Lemma 5.4. *If $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G$ and $X \in H$, then $\det(A + BX)$, $\det(D - XB) \in \mathbb{R}$.*

Proof. Recall $gX \in H$ by Definition 4.1. Thus

$$g' = \begin{pmatrix} I & 0 \\ -gX & I \end{pmatrix} g \begin{pmatrix} I & 0 \\ X & I \end{pmatrix} \in G.$$

In fact it is easy to calculate that $g' = \begin{pmatrix} A + BX & * \\ 0 & -(gX)B + D \end{pmatrix}$ and so $g' \in P$. Equation (2.4) finishes the theorem. As an additional point we see that

$$(A + BX)^{-1} = D^* - B^*(gX)$$

whenever it is invertible. To prove the second assertion, apply the first to g^{-1} and then apply $(\cdot)^*$. $\qquad\square$

Theorem 5.5. *Let $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G$. If $f_\pm \in L^2(H)^\pm$, then*

$$gf_+(X) = \det(D - XB)^{-2n+p} f_+(g^{-1}X)$$

and

$$gf_-(X) = \det(D - XB)^{-p} f_-(g^{-1}X).$$

Proof. If we write $g^{-1} \begin{pmatrix} I & 0 \\ X & I \end{pmatrix}$ in the form

$$\begin{pmatrix} I & 0 \\ Y & I \end{pmatrix} \begin{pmatrix} L & 0 \\ 0 & L^{*-1} \end{pmatrix} \begin{pmatrix} I & Y' \\ 0 & I \end{pmatrix},$$

we can solve

$$Y = g^{-1}X,$$

$$L = D^* - B^*X,$$

and $Y' = (-C^* + A^*X)^{-1}(I - L^{*-1})$. For $f \in I_p^+$, this says that $gf(X) = $ sgn$(\det(D^* - B^*X))^{-p}|\det(D^* - B^*X)|^{-2n+p} f(g^{-1}X)$. Equation (2.4) applied to g^{-1} shows that $\det(D^* - B^*X) \in \mathbb{R}$. We can therefore change the $-p$ in the previous formula to $-2n+p$ since the parity does not change modulo 2. Hence we get that $gf(X) = \det(D^* - B^*X)^{-2n+p} f(g^{-1}X)$ which is equal to $\det(D - XB)^{-2n+p} f(g^{-1}X)$. The work for I_p^- is done similarly by replacing the $-2n + p$ by $-p$. $\qquad\square$

6. The Orbits $\mathcal{O}_p$.

Let

$$l = \begin{pmatrix} A & 0 \\ 0 & A^{*-1} \end{pmatrix} \in L.$$

When convenient, we make use of the identification

(6.1) $$L \cong \{A \in Gl(n, \mathbb{C}) \,|\, \det(A) \in \mathbb{R}^\times\}$$

implemented by $l \to A$ above. Definition 4.1 calculates the action of L on $\overline{N}$ which pulls down to an action of L on H as

(6.2) $$l \cdot X = A^{-1*}XA^{-1}$$

for $X \in H$. Hence the L-orbits on H are parameterized by signature. The study of these orbits will be of fundamental importance.

Definition 6.1. Let $\mathcal{O}_p$ be the L-orbit in H consisting of Hermitian matrices of signature $p, 0$.

For general reasons, there exist L-equivariant measures on $\mathcal{O}_p$ ([**13**]), but they are also easy to write explicitly. We briefly outline their construction. In this paragraph only, write $\mathcal{O}_p(n)$ for the matrices in $H(n)$ of signature $p, 0$. Then there is a smooth emedding $\mathbb{C}^{p(n-p)} \times \mathcal{O}_p(p) \to \mathcal{O}_p(n)$ with dense open image given by

$$(Z, X) \to \begin{pmatrix} I_p & 0 \\ Z^* & I_{n-p} \end{pmatrix} \begin{pmatrix} X & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} I_p & Z \\ 0 & I_{n-p} \end{pmatrix}.$$

Using this embedding, the desired L-equivariant measure on $\mathcal{O}_p$ (up to a scalar multiple) pulls back to

$$(6.3) \qquad \det(X)^{n-p} \, dZ_p dX_p$$

where dZ_p is Haar measure on $\mathbb{C}^{p(n-p)}$ and dX_p is Haar measure on $H(p)$ restricted to the open orbit $\mathcal{O}_p(p)$. We omit the details.

Definition 6.2. Write τ for the character on L acting by $\tau(l) = \det(A)$.

1) Let

$$d\mu_p$$

be the unique (up to scalar multiplication) L-equivariant measure on $\mathcal{O}_p$ that transforms by the character τ^{2p} (so the change of variables $l \cdot X \to X$ multiplies the measure by $\det(A)^{2p}$). The measure $d\mu_p$ will be normalized below.

2) Write

$$L^2(\mathcal{O}_p) = L^2(\mathcal{O}_p, d\mu_p).$$

Normalize $d\mu_p$ so that $\int_{\mathcal{O}_p} e^{-\operatorname{tr}(Y)} \, d\mu_p(Y) = 1$. This allows us to verify the following well known identity.

Lemma 6.1. Let $Z \in \mathcal{D}^+$. Then

$$i^{np} \det(Z)^{-p} = \int_{\mathcal{O}_p} e^{i \operatorname{tr}(Z\xi)} \, d\mu_p(\xi).$$

Proof. We first show

$$\det(Y)^{-p} = \int_{\mathcal{O}_p} e^{\operatorname{tr}(Y\xi)} \, d\mu_{p,0}(\xi)$$

for any $Y \in H^+$. Using the L action from (6.2), write $Y = A^{-1*}A^{-1}$ with $\det(A) \in \mathbb{R}^\times$. Then, making use of the L-equivariance and normalization, we calculate:

$$\int_{\mathcal{O}_p} e^{-\operatorname{tr}(Y\xi)} \, d\mu_p(\xi) = \int_{\mathcal{O}_p} e^{-\operatorname{tr}(A^{-1*}A^{-1}\xi)} \, d\mu_p(\xi)$$

$$= \int_{\mathcal{O}_p} e^{-\operatorname{tr}(A^{-1}\xi A^{-1*})} \, d\mu_p(\xi)$$

$$= \det(A^*)^{2p} \int_{\mathcal{O}_p} e^{-\operatorname{tr}(\xi)} \, d\mu_p(\xi)$$

$$= \det(Y)^{-p}.$$

To finish the Lemma, write $Z = X + iY$ with $X \in H$ and $Y \in H^+$. We see $-iZ = Y - iX$. The statement of the Lemma then follows by analytic continuation. $\qquad\square$

To apply this Lemma in the setting of our Szegö map, we need the Fourier transform.

Definition 6.3. Write $\mathcal{S}(H)$ for the set of all *Schwartz* functions on H. As this space is not G invariant, write $\mathcal{S}(H)^+$ for the smallest G invariant space containing $\mathcal{S}(H)$:

$$\mathcal{S}(H)^+ = \operatorname{span}\{g\phi \,|\, g \in G,\, \phi \in \mathcal{S}(H) \subseteq L^2(H)^+\}.$$

For $\phi \in \mathcal{S}(H)$, define its Fourier transform, $\widehat{\phi} \in \mathcal{S}(H)$, by

$$\widehat{\phi}(\xi) = \int_H \mathrm{e}^{i\,\mathrm{tr}(\xi X)}\,\phi(X)\,dX$$

and, up to a scalar multiple, the inverse Fourier transform, $\check{\phi} \in \mathcal{S}(H)$, by

$$\check{\phi}(\xi) = \int_H \mathrm{e}^{-i\,\mathrm{tr}(\xi X)}\,\phi(X)\,dX.$$

Since we will eventually be looking at boundary values of the map S, the following Lemma will be needed.

Lemma 6.2. Let $\phi \in \mathcal{S}(H)$, $Z \in \mathcal{D}^+$, and $Y \in H$. Then

$$\lim_{Z \to Y} \int_H \det(X + Z)^{-p}\phi(X)\,dX = i^{-np}\int_{\mathcal{O}_p} \mathrm{e}^{i\,\mathrm{tr}(Y\xi)}\,\widehat{\phi}(\xi)\,d\xi.$$

Proof. We begin by using Lemma 6.1 and compute

$$\int_H \det(X + Z)^{-p}\phi(X)\,dX = i^{-np}\int_H \int_{\mathcal{O}_p} \mathrm{e}^{i\,\mathrm{tr}(X+Z)\xi}\,\phi(X)\,d\mu_p(\xi)dX.$$

To apply Fubini's theorem, we need to check the L^1 condition. Write $Z = X' + iY$ with $X' \in H$ and $Y \in H^+$ and use Lemma 6.1:

$$\int_H \int_{\mathcal{O}_p} |\,\mathrm{e}^{i\,\mathrm{tr}(X+Z)\xi}\,\phi(X)|\,d\mu_p(\xi)dX = \int_H \int_{\mathcal{O}_p} \mathrm{e}^{-\,\mathrm{tr}(Y\xi)}\,|\phi(X)|\,d\mu_p(\xi)dX$$

$$= \det(Y)^{-p}\int_{\mathcal{O}_p} |\phi(X)|\,dX < \infty.$$

Hence

$$\int_H \int_{\mathcal{O}_p} \mathrm{e}^{i\,\mathrm{tr}(X+Z)\xi}\,\phi(X)\,d\mu_p(\xi)dX = \int_{\mathcal{O}_p} \mathrm{e}^{i\,\mathrm{tr}(Z\xi)}\int_H \mathrm{e}^{i\,\mathrm{tr}(X\xi)}\,\phi(X)\,dX d\mu_p(\xi)$$

$$= \int_{\mathcal{O}_p} \mathrm{e}^{i\,\mathrm{tr}(Z\xi)}\,\widehat{\phi}(\xi)\,d\mu_p(\xi).$$

Since $\widehat{\phi}$ is still Schwartz and the measure (see Equation (6.3)) is only of polynomial growth, the above integrand is an L^1 function. Hence, when we take the limit as $Z \to Y$, we may move the limit past the integral to finish the Lemma. $\square$

7. Boundary Values of the Szegö Map.

Definition 7.1. For $f \in C^\infty(\mathcal{D}^+)$ and $X \in H$, define

$$Bf(X) = \lim_{Z \to X} f(Z)$$

where $Z \in \mathcal{D}^+$.

In general, Bf may not be well defined. However, we see below that it is at least well behaved on $\mathrm{Im}(S)$.

Theorem 7.1. Let $\phi \in \mathcal{S}(H)$ and $\eta \in \mathcal{D}^+$. Then

$$S\phi(\eta) = i^{np} \int_{\mathcal{O}_p} e^{i \, \mathrm{tr} \, \eta \xi} \, \check{\phi}(\xi) \, d\xi.$$

Moreover, B is well defined on $S(\mathcal{S}(H))$ and $BS\phi$ is alternately written as the smooth function

$$BS\phi(Y) = i^{np} \int_{\mathcal{O}_p} e^{i \, \mathrm{tr}(Y\xi)} \, \check{\phi}(\xi) \, d\xi.$$

Proof. Lemma 6.2 computes that

$$\lim_{Z \to Y} \int_H \det(X + Z)^{-p} \phi(X) \, dX$$

$$= i^{-np} \lim_{Z \to Y} \int_{\mathcal{O}_p} e^{i \, \mathrm{tr} \, ZW} \, \widehat{\phi}(W) \, d\mu_p(W)$$

$$= i^{-np} \int_{\mathcal{O}_p} e^{i \, \mathrm{tr}(YW)} \, \widehat{\phi}(W) \, d\mu_p(W).$$

Multiplying both sides by $(-1)^{np}$ and making the change of variables $X \to -X$ finishes the identity. Regarding smoothness, recall that $\widehat{\phi}$ is still Schwartz and the measure (see Equation (6.3)) is only of polynomial growth so that the integrand is an L^1 function. $\square$

Definition 7.2. If $f \in \mathcal{S}(H)^+$, define Af by

$$Af = BSf.$$

So far this map is well defined on $\mathcal{S}(H)$ by Theorem 7.1.

Theorem 7.2. Let $f \in \mathcal{S}(H)^+$ and $\psi = Sf$. Then (1) Af and $B\psi$ are well defined almost everywhere; (2) $Af = B\psi \in L^2(H)^-$; and (3) A and B are G-maps on $\mathcal{S}(H)^+$ and $S(\mathcal{S}(H)^+)$, respectively. Finally, for $\phi \in \mathcal{S}(H)$, $\eta \in \mathcal{D}$, and $Y \in H$,

$$S\phi(\eta) = i^{np} \int_{\mathcal{O}_p} e^{i \, \mathrm{tr} \, \eta \xi} \, \check{\phi}(\xi) \, d\xi$$

and

$$A\phi(Y) = i^{np} \int_{\mathcal{O}_p} e^{i \, \mathrm{tr}(Y\xi)} \, \check{\phi}(\xi) \, d\xi.$$

Proof. Begin with any function $\psi \in C^\infty(\mathcal{D}^+)$ for which $B\psi$ is well defined. First we check B commutes with the group actions given in Theorems 5.5 and 4.4. Write $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G$. Then using Theorems 5.5 and 4.4 for the group action and making use of Lemma 5.4 in the second set of equations below, we calculate:

$$(7.1) \qquad gB\psi(X) = \det(D - XB)^{-p}B\psi(g^{-1}X)$$
$$= \det(D - XB)^{-p} \lim_{Z \to g^{-1}X} \psi(Z).$$

On the other hand, we have

$$(7.2) \qquad Bg\psi\,(X) = \lim_{\zeta \to X} g\psi(\zeta)$$
$$= \lim_{\zeta \to X} \det (D^* - B^*\zeta)^{-p} \; \psi(g^{-1}\zeta)$$
$$= \overline{\det(D - XB)}^{-p} \lim_{\zeta \to X} \psi(g^{-1}\zeta)$$
$$= \det(D - XB)^{-p} \lim_{Z \to g^{-1}X} \psi(Z)$$

so that $gB\psi = Bg\psi$ which proves part (3) since S is a G-map (Theorem 5.3). Coupled with Theorem 7.1, Equations (7.2) show $Bg\psi$ is well defined almost everywhere. Since S is a G-map, this finishes part (1). That the range of B restricted to $\mathcal{S}(H)^+$ is contained in $L^2(H)^-$ follows from the G action. The argument is completely analogous to the one around Equation (5.1) (since each $f \in \mathcal{S}(H)^+$ comes from the smooth principal series) except that the final bound will be

$$(7.3) \qquad |Bf(X)|^2 \le C \det(I + X^2)^{-p}.$$

The final equations come from Theorem 7.1.   □

Note that though Bf is well defined almost everywhere for $\psi \in S(\mathcal{S}(H)^+)$, it need not be given by the formula in Theorem 7.1 for $\psi \notin S(\mathcal{S}(H))$. A similar cautionary remark applies to A on $\mathcal{S}(H)$ versus $\mathcal{S}(H)^+$.

Theorem 7.2 establishes the following commutative diagram of G maps:

$$(7.4) \qquad \begin{array}{ccc} & A & \\ \mathcal{S}(H)^+ & \longrightarrow & L^2(H)^-. \\ S \downarrow & \nearrow & \\ & B & \\ S(\mathcal{S}(H)^+) & & \end{array}$$

8. Functions on $\mathcal{O}_p$ and an Inner Product.

Definition 8.1. For $\phi \in \mathcal{S}(H)$, define $F_R\phi \in L^2(\mathcal{O}_p, d\mu_p)$ (F for Fourier transform and R for *restriction*) by

$$F_R\phi(\xi_p) = \check{\phi}(\xi_p).$$

For $\psi \in \mathrm{Im}(F_R)$, define $F_E\psi \in L^2(H)^-$ (E for *extension*) by

$$F_E\psi\,(X) = i^{np} \int_{\mathcal{O}_p} \mathrm{e}^{i\,\mathrm{tr}(\xi X)}\,\psi(\xi)\,d\mu_p(\xi).$$

First note that $\widehat{\phi}$ is still Schwartz and the measure $d\mu_p$ (see Equation (6.3)) is only of polynomial growth so that $F_E\psi$ is well defined. Second, note that Theorem 7.2 immediately implies that on $\mathcal{S}(H)$,

$$(8.1) \qquad\qquad F_E \circ F_R = A|_{\mathcal{S}(H)} = B \circ S|_{\mathcal{S}(H)}$$

where $A|_{\mathcal{S}(H)}$ denotes the map A restricted to $\mathcal{S}(H)$. In other words, there is a commutative diagram of maps (compare to diagram 7.4)

$$
(8.2) \qquad
\begin{array}{ccc}
& A & \\
\mathcal{S}(H) & \longrightarrow & L^2(H)^- \\
& \searrow & \uparrow \qquad F_E. \\
& F_R & \\
& L^2(\mathcal{O}_p) &
\end{array}
$$

Also note that $L^2(\mathcal{O}_p)$ comes equipped with its own inner product denoted by

$$\langle \cdot, \cdot \rangle_{\mathcal{O}_p}.$$

This pairing can be related to A as follows.

Definition 8.2. If $\phi_1, \phi_2 \in \mathcal{S}(H)^+$, let

$$\langle \phi_1, \phi_2 \rangle_A = i^{np} \int_H \phi_1(X)\overline{A\phi_2(X)}\,dX.$$

Equations (5.1), (7.3), and (5.2) can be used to show that $\langle \cdot, \cdot \rangle_A$ is well defined for functions coming from the principal series I_p^+ which includes $\mathcal{S}(H)^+$ (see the proof of Lemma 10.1 or a more general result under Definition 10.2 below).

Theorem 8.1. *If $\phi_1, \phi_2 \in \mathcal{S}(H)$, then*

$$\langle \phi_1, \phi_2 \rangle_A = \langle F_R\phi_1, F_R\phi_2 \rangle_{\mathcal{O}_p}.$$

Moreover, the form $\langle \cdot, \cdot \rangle_A$ is G-invariant on $\mathcal{S}(H)^+$.

Proof. We make use of Theorem 7.2 to calculate:

$$\langle\phi_1,\phi_2\rangle_A = i^{np}\int_H \phi_1(X)\overline{A\phi_2(X)}\,dX$$

$$= \int_H\int_{\mathcal{O}_p}\phi_1(X)\,e^{-i\,\mathrm{tr}(X\xi)}\,\overline{\check{\phi}(\xi)}\,d\xi dX$$

$$= \int_{\mathcal{O}_p}\int_H\phi_1(X)\,e^{-i\,\mathrm{tr}(X\xi)}\,\overline{\check{\phi}(\xi)}\,d\xi dX$$

$$= \int_{\mathcal{O}_p}\check{\phi}_1(\xi)\overline{\check{\phi}_2(\xi)}\,d\mu_p(\xi).$$

To check G-invariance, write $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G$. In the following equations, make use of the actions given in Theorem 5.5, Equation (5.3) for the change of variables, Lemma 5.4 for conjugation issues, and Equation (2.1) and Definition 4.1 to check that $(D^* - B^*\,gX) = (BX + A)^{-1}$:

$$\langle g\phi_1, g\phi_2\rangle = i^{np}\int_H g\phi_1(X)\overline{Ag\phi_2(X)}\,dX$$

$$= i^{np}\int_H \det(D - XB)^{-2n+p}\phi_1(g^{-1}X)\overline{gA\phi_2(X)}\,dX$$

$$= i^{np}\int_H \det(D - XB)^{-2n}\phi_1(g^{-1}X)\overline{A\phi_2(g^{-1}X)}\,dX$$

$$= i^{np}\int_H \det(D^* - B^*X)^{-2n}\phi_1(g^{-1}X)\overline{A\phi_2(g^{-1}X)}\,dX$$

$$= i^{np}\int_H \det(D^* - B^*\,gX)^{-2n}\det(BX + A)^{-2n}\phi_1(X)\overline{A\phi_2(X)}\,dX$$

$$= i^{np}\int_H \phi_1(X)\overline{A\phi_2(X)}\,dX$$

$$= \langle\phi_1,\phi_2\rangle.$$

$\square$

The equality of the two pairings in the above Theorem will be extended to a larger domain as soon as the map F_R is extended.

9. Continuity of A, F_R, and F_E.

As it stands, most operators are only defined on dense sets such as $\mathcal{S}(H) \subseteq L^2(H)^+$. To complete the picture, we need to prove the operators are continuous.

Theorem 9.1. *The maps*

$$A : \mathcal{S}(H) \subseteq L^2(H)^+ \to L^2(H)^-,$$

$$F_R : \mathcal{S}(H) \subseteq L^2(H)^+ \to L^2(\mathcal{O}_p),$$

and

$$F_E : \mathrm{Im}(F_R|_{\mathcal{S}(H)}) \subseteq L^2(\mathcal{O}_p) \to L^2(H)^-$$

are continuous maps. The notation $\mathrm{Im}(F_R|_{\mathcal{S}(H)})$ *denotes the image of* F_R *restricted to* $\mathcal{S}(H)$.

This section is devoted to the proof of Theorem 9.1. The first step is the following Lemma.

Lemma 9.2. *If the operator*

$$F_E : \mathrm{Im}(F_R|_{\mathcal{S}(H)}) \subset L^2(\mathcal{O}_p) \to L^2(H)^-$$

is continuous, then

$$A : \mathcal{S}(H) \subseteq L^2(H)^+ \to L^2(H)^-$$

and

$$F_R : \mathcal{S}(H) \subseteq L^2(H)^+ \to L^2(\mathcal{O}_p)$$

are bounded operators as well.

Proof. Suppose the hypothesis of this Lemma is in effect and let $f \in \mathcal{S}(H)$. Consider the map A first. Then for some constant C,

$$||Af||^2_{L^2(H)^-} = ||F_E F_R f||^2_{L^2(H)^-} \le C||F_R f||^2_{L^2(\mathcal{O}_p)}.$$

On the other hand, $||F_R f||^2_{L^2(\mathcal{O}_p)} = \langle F_R f, F_R f \rangle_{\mathcal{O}_p} = \langle f, f \rangle_A$ by Theorem 8.1. But Definition 8.2 says

$$\langle f, f \rangle_A = |\langle f, f \rangle_A| = \left| \int_H f(X)\overline{Af(X)} \, dX \right|$$

$$\le \int_H |[f(X) \det(I + X^2)^{\frac{n-p}{2}}]\,[\overline{Af(X)} \det(I + X^2)^{\frac{-n+p}{2}}]| \, dX$$

$$\le ||f||_{L^2(H)^+} ||Af||_{L^2(H)^-}.$$

Putting these equations together gives

$$||Af||^2_{L^2(H)^-} \le C||f||_{L^2(H)^+} ||Af||_{L^2(H)^-}.$$

Division finishes the proof for the continuity of A.

Now consider F_R. Using the relations above, $||F_R f||^2_{L^2(\mathcal{O}_p)} = \langle f, f \rangle_A \le ||f||_{L^2(H)^+} ||Af||_{L^2(H)^-} \le C||f||^2_{L^2(H)^+}$. Division again finishes the proof for continuity of F_R. $\qquad\square$

Thus we devote the rest of the section to proving that $F_E : \mathrm{Im}(F_R|_{\mathcal{S}(H)}) \subset L^2(\mathcal{O}_p) \to L^2(H)^-$ is a bounded map. This amounts to showing that

$$(9.1) \qquad \int_H |Af(X)|^2 \det(I + X^2)^{-n+p} \, dX \le C \left\| \check{f} \right\|_{L^2(\mathcal{O}_p)}$$

for some constant, C, and all $f \in \mathcal{S}(H)$. In the special case of $p = n$, this statement is trivial to verify using the Plancherel theorem as $\mathcal{O}_n$ is open in H, $d\mu_n = dX|_{\mathcal{O}_n}$, and $\det(I + X^2)^{\pm(n-p)} = 1$. In the case of $p < n$, much more work is required.

Let S_p be the stabilizer of

$$E_p = \begin{pmatrix} I_p & \\ & 0_{n-p} \end{pmatrix}$$

in L so that $\mathcal{O}_p \cong L/S_p$ and let $\widetilde{S}_{n-p}$ be the stabilizer of

$$\widetilde{E}_{n-p} = \begin{pmatrix} 0_p & \\ & I_{n-p} \end{pmatrix}$$

so that $\mathcal{O}_{n-p} \cong L/\widetilde{S}_{n-p}$.

Lemma 9.3. *Given a smooth function of compact support, f, on $\mathcal{O}_p \times \mathcal{O}_{n-p}$, pull and push it to a function f^* on $\mathcal{O}_n$ using the double fibration:*

$$
\begin{array}{ccc}
 & L & \\
\swarrow & & \searrow \\
\mathcal{O}_p \times \quad \mathcal{O}_{n-p} & & \mathcal{O}_n
\end{array}
$$

by

$$f^*(l \cdot E_n) = \int_{S_n/S_p \cap \widetilde{S}_{n-p}} f((ls) \cdot E_p, \ (ls) \cdot \widetilde{E}_{n-p}) \, ds$$

for $l \in L$ where ds is an S_n-invariant measure. The function f^* satisfies

$$\int_{\mathcal{O}_n} f^*(\xi_n) \, d\mu_n(\xi_n) = \int_{\mathcal{O}_p \times \mathcal{O}_{n-p}} f(\xi_p, \xi_{n-p}) \, d\mu_p(\xi_p) d\mu_{n-p}(\xi_{n-p}).$$

In particular, if f_p and f_{n-p} are functions on $\mathcal{O}_p$ and $\mathcal{O}_{n-p}$, respectively, then $f_p \times f_{n-p}$ is a function on $\mathcal{O}_p \times \mathcal{O}_{n-p}$. Define

$$f_p * f_{n-p}$$

to be the function on $\mathcal{O}_n$ given by $(f_p \times f_{n-p})^*$.

Proof. Easy calculations show that $S_n/S_p \cap \widetilde{S}_{n-p} \cong U(n)/U(p) \times U(n-p)$ so that ds exists since have a quotient of reductive groups. Then it is easy to see that it suffices to prove the injection

$$L/S_p \cap \widetilde{S}_{n-p} \to \mathcal{O}_p \times \mathcal{O}_{n-p}$$

induced by the diagonal action, $l \to (l \cdot E_p, \ l \cdot \widetilde{E}_{n-p})$, has a dense open image. Using the transitivity of the L-action on $\mathcal{O}_p$, it suffices to show that S_p can be used to conjugate almost all elements in $\mathcal{O}_{n-p}$ to $\widetilde{E}_{n-p}$. But for this, it suffices to show that almost all $A \in Gl(n, \mathbb{C})$, with $\det(A) \in \mathbb{R}^\times$, can be written as the product of elements from S_p and $\widetilde{S}_{n-p}$. But this is an easy calculation we omit. $\square$

By previous remarks, the following Lemma will finish the proof of Theorem 9.1.

Theorem 9.4. *For some constant, C, and all $f \in \mathcal{S}(H)$,*

$$\int_H |Af(X)|^2 \det(I + X^2)^{-n+p}\, dX \leq C\, \big\|\check{f}\big\|_{L^2(\mathcal{O}_p)}.$$

Proof. By Theorem 7.2, we know $\int_H |Af(X)|^2 \det(I + X^2)^{-n+p}\, dX$ is equal to

$$\int_H \left| \int_{\mathcal{O}_p} \mathrm{e}^{i\,\mathrm{tr}(X\xi)}\, \check{f}(\xi)\, d\mu_p(\xi) \right|^2 \det(I + X^2)^{-n+p}\, dX.$$

On the other hand, Lemma 6.1 shows that

$$|\det(iI + X)|^{-n+p} = \left| \int_{\mathcal{O}_{n-p}} \mathrm{e}^{i\,\mathrm{tr}[(iI+X)\xi]}\, d\mu_{n-p}(\xi) \right|.$$

Since $\det(I + X^2) = \det(iI + X)\det(-iI + X)$ and $|\det(iI + X)| = |\det(-iI + X)|$, we therefore know

$$\det(I + X^2)^{-n+p} = \left| \int_{\mathcal{O}_{n-p}} \mathrm{e}^{i\,\mathrm{tr}[(iI+X)\xi]}\, d\mu_{n-p}(\xi) \right|^2.$$

Thus $\int_H |Af(X)|^2 \det(I + X^2)^{-n+p}\, dX$ is equal to

$$\int_H \left| \int_{\mathcal{O}_p} \mathrm{e}^{i\,\mathrm{tr}(X\xi_{p,0})}\, \check{f}(\xi_p)\, d\mu_p(\xi_p) \int_{\mathcal{O}_{n-p}} \mathrm{e}^{i\,\mathrm{tr}[(iI+X)\xi_{n-p}]}\, d\mu_{n-p}(\xi_{n-p}) \right|^2 dX.$$

But Lemma 9.3 allows this to be rewritten as

$$\int_H \left| \int_{\mathcal{O}_n} \left(\mathrm{e}^{i\,\mathrm{tr}(X\cdot)}\, \check{f}(\cdot) * \mathrm{e}^{i\,\mathrm{tr}[(iI+X)\cdot]} \right)(\xi_n)\, d\mu_n(\xi_n) \right|^2 dX.$$

However, it is easy to check that the definition of $*$ in Lemma 9.3 implies

$$\left(\mathrm{e}^{i\,\mathrm{tr}(X\cdot)}\, \check{f}(\cdot) * \mathrm{e}^{i\,\mathrm{tr}[(iI+X)\cdot]} \right)(\xi_n) = \mathrm{e}^{i\,\mathrm{tr}(X\xi_n)} \left(\check{f}(\cdot) * \mathrm{e}^{-\,\mathrm{tr}\,\cdot} \right)(\xi_n).$$

Thus $\int_H |Af(X)|^2 \det(I + X^2)^{-n+p}\, dX$ is equal to

$$\int_H \left| \int_{\mathcal{O}_n} \mathrm{e}^{i\,\mathrm{tr}(X\xi_n)} \left(\check{f}(\cdot) * \mathrm{e}^{-\,\mathrm{tr}\,\cdot} \right)(\xi_n)\, d\mu_n(\xi_n) \right|^2 dX$$

$$= \int_H \left| \int_H \mathrm{e}^{i\,\mathrm{tr}(XY)}\, \chi_{\mathcal{O}_n}(Y) \left(\check{f}(\cdot) * \mathrm{e}^{-\,\mathrm{tr}\,\cdot} \right)(Y)\, dY \right|^2 dX$$

where $\chi_{\mathcal{O}_n}$ is the characteristic function for the open set $\mathcal{O}_n$ inside H. The next step uses the Plancherel Theorem on the above integral to rewrite it as

$$\int_H \left| \chi_{\mathcal{O}_n}(X) \left(\check{f}(\cdot) * \mathrm{e}^{-\,\mathrm{tr}\,\cdot} \right)(X) \right|^2 dX = \int_{\mathcal{O}_n} \left| \left(\check{f}(\cdot) * \mathrm{e}^{-\,\mathrm{tr}\,\cdot} \right)(X) \right|^2 dX.$$

To justify this step, we verify that $\check{f}(\cdot) * \mathrm{e}^{-\,\mathrm{tr}\,\cdot} \in L^1(\mathcal{O}_n) \cap L^2(\mathcal{O}_n)$. By the definition of $*$, we have

$$\left(\check{f}(\cdot) * \mathrm{e}^{-\,\mathrm{tr}\,\cdot}\right)(l \cdot E_n) = \int_{S_n/S_p \cap \widetilde{S}_{n-p}} \check{f}((ls) \cdot E_p)\, \mathrm{e}^{-\,\mathrm{tr}\left[(ls)\cdot \widetilde{E}_{n-p}\right]}\, ds.$$

So

$$\int_{\mathcal{O}_n} \left|\left(\check{f}(\cdot) * \mathrm{e}^{-\,\mathrm{tr}\,\cdot}\right)(X)\right|\, dX$$

$$= \int_{L/S_n} \left|\int_{S_n/S_p \cap \widetilde{S}_{n-p}} \check{f}((ls) \cdot E_p)\, \mathrm{e}^{-\,\mathrm{tr}\left[(ls)\cdot \widetilde{E}_{n-p}\right]}\, ds\right|\, dl$$

$$\leq \int_{L/S_n} \int_{S_n/S_p \cap \widetilde{S}_{n-p}} \left|\check{f}((ls) \cdot E_p)\, \mathrm{e}^{-\,\mathrm{tr}\left[(ls)\cdot \widetilde{E}_{n-p}\right]}\right|\, ds\, dl$$

$$= \int_{\mathcal{O}_p \times \mathcal{O}_{n-p}} \left|\check{f}(\xi_p)\, \mathrm{e}^{-\,\mathrm{tr}\,\xi_{n-p}}\right|\, d\mu_p(\xi_p) d\mu_{n-p}(\xi_{n-p}) < \infty$$

and so the L^1 condition follows. For the L^2 condition, the key observation is that $S_n/S_p \cap \widetilde{S}_{n-p}$ is compact. In fact, recall $S_n/S_p \cap \widetilde{S}_{n-p}$ is isomorphic to $U(n)/U(p) \times U(n-p)$. Thus Hölder's inequality can be made use of below to check

$$\int_{\mathcal{O}_n} \left|\left(\check{f}(\cdot) * \mathrm{e}^{-\,\mathrm{tr}\,\cdot}\right)(X)\right|^2\, dX$$

$$= \int_{L/S_n} \left|\int_{S_n/S_p \cap \widetilde{S}_{n-p}} \check{f}((ls) \cdot E_p)\, \mathrm{e}^{-\,\mathrm{tr}\left[(ls)\cdot \widetilde{E}_{n-p}\right]}\, ds\right|^2\, dl$$

$$\leq \int_{L/S_n} k \int_{S_n/S_p \cap \widetilde{S}_{n-p}} \left|\check{f}((ls) \cdot E_p)\, \mathrm{e}^{-\,\mathrm{tr}\left[(ls)\cdot \widetilde{E}_{n-p}\right]}\right|^2\, ds$$

$$= k \int_{\mathcal{O}_p \times \mathcal{O}_{n-p}} \left|\check{f}(\xi_p)\, \mathrm{e}^{-\,\mathrm{tr}\,\xi_{n-p}}\right|^2\, d\mu_p(\xi_p) d\mu_{n-p}(\xi_{n-p}) < \infty$$

where $k = \mathrm{Vol}(S_n/S_p \cap \widetilde{S}_{n-p})$. Thus the use of the Plancherel Theorem is valid and we may write

$$\int_H |Af(X)|^2 \det(I + X^2)^{-n+p}\, dX = \int_{\mathcal{O}_n} \left|\left(\check{f}(\cdot) * \mathrm{e}^{-\,\mathrm{tr}\,\cdot}\right)(X)\right|^2\, dX.$$

However, the above calculation that checks the L^2 condition now implies

$$\int_H |Af(X)|^2 \det(I + X^2)^{-n+p}\, dX \leq C \int_{\mathcal{O}_p} \left|\check{f}(\xi_p)\right|^2\, d\mu_p(\xi_p) = C \left\|\check{f}\right\|_{L^2(\mathcal{O}_p)}$$

as desired where $C = k \int_{\mathcal{O}_{n-p}} \mathrm{e}^{-2\,\mathrm{tr}\,\xi_{n-p}}\, d\mu_{n-p}(\xi_{n-p}) < \infty$. $\qquad\square$

10. The Main Theorem.

Theorem 9.1 allows the completion of the maps A, F_R, and F_E.

Definition 10.1. Let
$$A : L^2(H)^+ \to L^2(H)^-$$
be the continuous extension of $A : \mathcal{S}(H) \to L^2(H)^-$ where we view $\mathcal{S}(H) \subseteq L^2(H)^+$,
$$F_R : L^2(H)^+ \to L^2(\mathcal{O}_p)$$
be the continuous extension of $F_R : \mathcal{S}(H) \to L^2(\mathcal{O}_p)$, and
$$F_E : \mathrm{Im}(F_R) \to L^2(H)^-$$
be the continuous extension of $F_E : \mathrm{Im}(F_R|_{\mathcal{S}(H)}) \to L^2(H)^+$ where $\mathrm{Im}(F_R)$ denotes the image of F_R on $L^2(H)^+$.

Two notes are in order. The first is that Definition 7.2 already gives a definition of $A = BS$ on *all* of $\mathcal{S}(H)^+$ which Theorem 7.2 shows is well defined. However, it is apriori possible (though not true) that Definition 10.1 defines A differently on $\mathcal{S}(H)^+ \backslash \mathcal{S}(H)$. This ambiguity is removed in Lemma 10.1 below.

The second note is that the closure of $\mathrm{Im}(F_R)$ is in fact all of $L^2(\mathcal{O}_p)$. This is shown in Theorem 10.4 below.

Lemma 10.1. *When restricted to $A : \mathcal{S}(H)^+ \to L^2(H)^+$, both Definitions 7.2 and 10.1 coincide.*

Proof. We break the proof of this Lemma up into steps. In this proof only, write $\overline{A}$ for the operator defined in Definition 10.1 by extending continuously from $\mathcal{S}(H)$ to $\mathcal{S}(H)^+$ with respect to the $L^2(H)^+$ and $L^2(\mathcal{O}_p)$ norms, respectively. Likewise for this proof only, write A for the operator defined in Definition 7.2 as $B \circ S$ on $\mathcal{S}(H)^+$ (Theorem 7.2 shows it is well defined).

(1).   The first step is found in [**14**], Lemma 1.   Since the proof is straightforward and identical to the one in [**14**], we simply state the result. Namely, $f \in C^\infty(H)$ has a (unique) *smooth* extension via the open dense embedding $H \cong \overline{N} \hookrightarrow G/P$ to a function in the smooth principal series $I_p^\pm$ if and only if the function
$$X \to (gf)(X),$$
initially defined for $X \in H$ with $\det(D - XB) \neq 0$ by Theorem 5.5, extends to a smooth function on H for each $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G$.

(2).   Suppose $X_0 \in H$ and $\det(A + BX_0) = 0$. Choose X_t, $t \in \mathbb{R}$, a smooth path in H with $\det(D - X(t)B)$ not identically zero as a function of t. Let $\| \cdot \|$ be a norm on H. Then $\lim_{t \to 0} \| gX_t \| = \infty$. This follows by

the definition of the linear fractional action in Definition 4.1 combined with expressing the inverse of a matrix in terms of cofactors and its determinant.

(3). Steps (1) and (2) may now be combined with Theorem 5.5 (see also Definition 4.1 and Equation (2.3)) to show that each $\phi \in \mathcal{S}(H)$ has a smooth extension to I_p^+. In particular, the smooth extension of $X \to (g\phi)(X)$ is the map sending X to

$$\det(D^* - B^*X)^{-2n+p}\phi((-C^* + A^*X)/(D^* - B^*X)^{-1})$$

if $\det(D^* - B^*X) \neq 0$ and sending X to 0 if $\det(D^* - B^*X) = 0$. By G invariance of I_p^+, we also conclude that each function in $\mathcal{S}(H)^+$ has a smooth extension to I_p^+. Moreover using (1), it is easy to check that if $\psi \in I_p^+$ is the extension of an element in $\mathcal{S}(H)^+$, then ψ is identically zero on all points of G/P in the compliment of $\overline{P}$.

(4). Each $\psi \in I_p^+$ is bounded and when restricted to H satisfies the growth condition $|\psi(X)| \leq C\|X\|^{-2n+p}$ for some constant C. This follows easily from Equation (5.1) and unitary diagonalization of X.

(5). Fix $\psi \in \mathcal{S}(H)^+$. Also denote by ψ its smooth extension to G/P. For $r > 0$ choose cut-off functions $\phi_r \in C_0^\infty(H)$ with range in $[0, 1]$ so that ϕ_r is identically 1 on the ball of radius r about the origin and identically 0 outside the ball of radius $r + 1$. Then the fact that $\psi \in L^2(H)^+$ and points (3) and (4) show that as $r \to \infty$ that $\phi_r\psi \to \psi$ in the L^2-norm and that $\phi_r\psi \to \psi$ uniformly as functions on either H or G/P.

(6). $S : I_p^+ \to C^\infty(G/K)$ is continuous in the smooth topology of uniform convergence on compact sets. This follows since S is an integral of smooth functions over a compact set (Definition 3.2).

(7). Suppose that $f, f_i \in \mathcal{S}(H)^+$ so that $f_i \to f$ uniformly. Then $Af_i \to Af$ pointwise. To see this, use the definition of A in the first step below, uniform convergence in the second, and point (6) in the third to calculate

$$\lim_{i\to\infty} Af_i(X) = \lim_{i\to\infty} \lim_{\eta\in\mathcal{D}^+, \, \eta\to X} Sf_i(X)$$
$$= \lim_{\eta\in\mathcal{D}^+, \, \eta\to X} \lim_{i\to\infty} Sf_i(X)$$
$$= \lim_{\eta\in\mathcal{D}^+, \, \eta\to X} Sf(X)$$
$$= Af(X).$$

(8). We now prove the Lemma. Let $\psi \in \mathcal{S}(H)^+$ and pick ϕ_r as in point (5). Note that $\phi_r \in \mathcal{S}(H)$. Thus, by definition,

$$\overline{A}\psi = \lim_{r\to\infty} A\phi_r\psi$$

in the L^2-sense. We may therefore choose a subsequence so that $\overline{A}\psi = \lim_{r\to\infty} A\phi_r\psi$ pointwise almost everywhere. But points (5) and (7) imply

that $\lim_{r \to \infty} A\phi_r \psi = A\psi$ pointwise everywhere. In particular, we see $\overline{A}\psi = A\psi$ almost everywhere so that $\overline{A} = A$ in the L^2-sense on $\mathcal{S}(H)^+$. $\square$

We now show that A remains a G map.

Theorem 10.2. *The map*

$$A : L^2(H)^+ \to L^2(H)^-$$

is a G map and $A = F_R \circ F_E$.

Proof. It follows from Lemma 10.1 and Theorem 7.2 that A is a G-map on $\mathcal{S}(H)^+$. Theorem 7.2 also shows that $A = F_R \circ F_E$ on $\mathcal{S}(H)$. The denseness of $\mathcal{S}(H)$ and a continuity argument suffice to finish the proof of this Theorem. $\square$

We are able to complete Definition 8.2 and Theorem 8.1 as follows.

Definition 10.2. If $f_1, f_2 \in L^2(H)^+$, let

$$\langle f_1, f_2 \rangle_A = i^{np} \int_H f_1(X) \overline{Af_2(X)} dX.$$

By Hölder's inequality (multiply by the det and its inverse), the form $\langle f_1, f_2 \rangle_A$ is well defined and bounded by the product of the norm of $f_1 \in L^2(H)^+$ and the norm of $Af_2 \in L^2(H)^-$. Theorem 8.1 and continuity imply the following.

Theorem 10.3. *If $f_1, f_2 \in L^2(H)^+$, then*

$$\langle f_1, f_2 \rangle_A = \langle F_R f_1, F_R f_2 \rangle_{\mathcal{O}_p}.$$

Moreover, the form $\langle \cdot, \cdot \rangle_A$ is G-invariant.

This suggests that we try to make $L^2(\mathcal{O}_p)$ into a G-space in such a way that F_R is a G-map. In turn, this Theorem 10.3 ought to induce a G-invariant structure on a quotient of the principal series. First observe that F_E is injective (for instance, by the Stone-Weierstrass theorem and the fact that the characters $e^{i \operatorname{tr}(\xi \cdot)}$ separate points). This implies that

(10.1) $$\ker(A) = \ker(F_R)$$

and in particular that $\ker(F_R)$ is G-invariant. Thus there is a G-action on

$$L^2(H)^+ / \ker(F_R)$$

for which $\langle \cdot, \cdot \rangle_A$ descends to a G-invariant, positive (by Theorem 10.3), Hermitian two-form. Moreover, the following is well defined.

Definition 10.3. For $g \in G$, $f \in L^2(H)^+$, and $h = F_R f \in L^2(\mathcal{O}_p)$, define

$$g(h) = F_R(gf)$$

or equivalently

$$gh = F_E^{-1}(gF_E h).$$

This equivalence is trivial to check using Theorem 10.2. This definition makes $\operatorname{Im}(F_R) \subseteq L^2(\mathcal{O}_p)$ into a representation of G so that both F_R and F_E are now G-maps. In general, we have to take closures to complete the picture.

Definition 10.4. Let $\mathcal{K} = \{f \in L^2(H)^+ \mid \langle f, f \rangle_A = 0\} = \ker(F_R) = \ker(A) = \{f \in L^2(H)^+ \mid \langle f, g \rangle_A = 0, \ \forall g \in L^2(H)^+\}$. Let

$$\overline{L^2(H)^+/\mathcal{K}}$$

be the completion of $L^2(H)^+/\mathcal{K}$ with respect to $\langle \cdot, \cdot \rangle_A$.

We use continuity to extend the G-action to all of $\overline{L^2(H)^+/\mathcal{K}}$, continuity to extend the map

$$F_R : \overline{L^2(H)^+/\mathcal{K}} \to L^2(\mathcal{O}_p),$$

and the fact that $A = F_E \circ F_R$ to extend the map

$$A : \overline{L^2(H)^+/\mathcal{K}} \to L^2(H)^-.$$

Then we have the following.

Theorem 10.4. $\operatorname{Im}(F_R) = L^2(\mathcal{O}_p)$. *Moreover, $L^2(\mathcal{O}_p)$ is an irreducible unitary representation of G that, in fact, remains irreducible under restriction to $\overline{P}$.*

Proof. Unitarity follows immediately from Theorem 10.3. The rest of the argument is entirely classical. Let $I = \operatorname{Im}(F_R)$. Recall that F_R is an isometry mapping the closure of $L^2(H)^+/\mathcal{K}$ into $L^2(\mathcal{O}_p)$. I is therefore a closed, non-trivial subspace of $L^2(\mathcal{O}_p)$ since, in particular, $\check{\phi}_{|\mathcal{O}_p} \in I$ for each $\phi \in \mathcal{S}(H)$. It is easy to check (Definitions 10.3 and 8.1, Theorem 5.5, and Equation (5.3)) that

$$\begin{pmatrix} I & 0 \\ X & I \end{pmatrix} f(\xi) = e^{-i\operatorname{tr}(X\xi)} f(\xi),$$

$$\begin{pmatrix} A & 0 \\ 0 & A^{*-1} \end{pmatrix} f(\xi) = \det(A)^{-p} f(A^{-1}\xi A^{*-1})$$

for each $f \in I$. We can use the same formulas above to extend the action of $\overline{P}$ on I to an action on all of $L^2(\mathcal{O}_p)$. It is a fact that this action is irreducible. We sketch the idea (see [**16**] §1.2, Theorem 2.1 or [**13**] §3.1). It suffices to show that any bounded intertwining operator, T, is a constant. However, it can be shown that commuting with $\overline{N}$ implies that T is multiplication by a bounded function. The transitivity of the L action then implies that T is a constant. Thus $L^2(\mathcal{O}_p)$ is irreducible under $\overline{P}$. Since I is $\overline{P}$-invariant, this implies that $I = L^2(\mathcal{O}_p)$ which finishes the proof. □

This Theorem induces a unitary structure on a quotient of the principal series.

Corollary 10.5. $\langle\cdot,\cdot\rangle_A$ *induces a* G-*invariant unitary structure on the quotient space* $\overline{L^2(H)^+/\mathcal{K}}$ *that is unitarily isomorphic to* $L^2(\mathcal{O}_p)$ *by* F_R. *In particular,*

$$\langle f_1, f_2\rangle_A = \langle F_R f_1, F_R f_2\rangle_{\mathcal{O}_p}.$$

for all $f_1, f_2 \in L^2(H)^+$.

Proof. This is contained in Theorem 10.3, 10.4, and Equation (10.1). □

This completes diagram 8.2 to the following diagram of unitary G-maps where F_R is an isomorphism and A (viewed as a map on the quotient space) and F_E are injective:

(10.2)
$$
\begin{array}{ccc}
 & A & \\
\overline{L^2(H)^+/\mathcal{K}} & \longrightarrow & L^2(H)^- \\
 & \searrow \quad \uparrow & F_E. \\
 & F_R & \\
 & L^2(\mathcal{O}_p) &
\end{array}
$$

This Diagram then fits into a larger diagram that incorporates Diagram 7.4. Namely, define $F_{\widetilde{E}} : L^2(\mathcal{O}_p) \to C^\infty(\mathcal{D}^+)$ by

$$F_{\widetilde{E}} f(\eta) = i^{np} \int_{\mathcal{O}_p} e^{i\,\mathrm{tr}\,\eta\xi}\, f(\xi)\, d\xi$$

which is well defined by Hölder's inequality. It is immediate that we obtain a commuting diagram of G-maps,

(10.3)
$$
\begin{array}{ccccc}
 & & A & & \\
 & L^2(H)^+/\mathcal{K} & \longrightarrow & L^2(H)^- & \\
S & \downarrow & \quad B \quad F_R & \uparrow & F_E. \\
 & \mathrm{Im}(S) & \nearrow \quad \searrow & L^2(\mathcal{O}_p) & \\
 & & \longleftarrow & & \\
 & & F_{\widetilde{E}} & &
\end{array}
$$

Also note that the injectivity of $F_{\widetilde{E}}$ is enough to strengthen Corollary 10.5 so that B is injective and

(10.4) $\qquad \mathcal{K} = \ker(A) = \ker(S) = \ker(F_R) = \ker(\langle\cdot,\cdot\rangle_A).$

References

[1] L. Barchini, *Szegö kernels associated with Zuckerman modules*, J. Funct. Anal., **131** (1995), 145-182.

[2] L. Barchini, S. Gindikin and H.-W. Wong, *The geometry of the flag manifold and holomorphic extension of Szegö kernels for $U(p,q)$*, Pacific J. Math., to appear.

[3] L. Barchini, A. Knapp and R. Zierau, *Intertwining operators into Dolbeault cohomology representations*, J. Funct. Anal., **107** (1992), 302-341.

[4] J. Faraut and A. Korányi, *Analysis on Symmetric Cones*, Clarendon Press, Oxford, 1994.

[5] Harish-Chandra, *Representations of a semisimple Lie group on a Banach space* I, Trans. Amer. Math. Soc., **70** (1953), 185-243.

[6] S. Helgason, *Groups and Geometric Analysis: Integral Geometry, Invariant Differential Operators, and Spherical Functions*, Academic Press, 1984.

[7] L.K. Hua, *Harmonic Analysis of Functions of Several Complex Variables in the Classical Domains*, American Mathematical Society, Providence, 1963.

[8] M. Kashiwara and M. Vergne, *Functions on the Shilov boundary of the generalized half plane*, Springer-Verlag Lecture Notes in Math., **728** (1979), 136-176.

[9] A. Knapp, *Representation Theory of Semisimple Groups: An Overview Based on Examples*, Princeton University Press, Princeton, 1986.

[10] A. Knapp and K. Okamoto, *Limits of holomorphic discrete series*, J. Funct. Anal., **9** (1972), 375-409.

[11] A. Knapp and N. Wallach, *Szegö kernels associated with discrete series*, Invent. Math., **34** (1976), 163-200.

[12] H. Rossi and M. Vergne, *Analytic continuation of the holomorphic discrete series of a semi-simple Lie group*, Acta Math., **136** (1976), 1-59.

[13] S. Sahi, *Explicit Hilbert spaces for certain unipotent representations*, Invent. Math., **110** (1992), 409-418.

[14] S. Sahi and E. Stein, *Analysis in matrix space and Speh's representations*, Invent. Math., **101** (1990), 379-393.

[15] E. Stein and T. Murphy, *Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals*, (Princeton Mathematical, No 43), Princeton University Press, Princeton, 1993.

[16] M. Taylor, *Noncommutative Harmonic Analysis*, Mathematical Surveys and Monographs No. 22, American Mathematical Society, Providence, RI 1946.

DEPARTMENT OF MATHEMATICS
401 MATH SCIENCE
OKLAHOMA STATE UNIVERSITY
STILLWATER, OK 74078-1058

ON KRONECKER PRODUCTS OF SPIN CHARACTERS OF THE DOUBLE COVERS OF THE SYMMETRIC GROUPS

Christine Bessenrodt and Alexander S. Kleshchev

In this article, restrictions on the constituents of Kronecker products of spin characters of the double covers of the symmetric groups are derived. This is then used to classify homogeneous and irreducible products of spin characters; as an application of this, certain homogeneous 2-modular tensor products for the symmetric groups are described.

1. Introduction.

In recent years, a number of results on Kronecker products of complex S_n-characters have been obtained. In particular, the rectangular hull for the constituents in such products was found, and this was used for the classification of products with few homogeneous components; see [1] for this classification result and references to related work.

Here, we provide similar results for products of spin characters for the double covers $\widetilde{S}_n$ of the symmetric groups. The rectangular hull for spin products is determined in Theorem 3.2; this result serves as a crucial tool for the classification of homogeneous spin products in Theorem 4.2. (A module is called *homogeneous* if all of its composition factors are isomorphic to each other.) Finally, Theorem 4.2 is applied to prove a recent conjecture of Gow and Kleshchev describing certain homogeneous 2-modular tensor products for the symmetric groups (see Theorem 5.1).

2. Preliminaries.

We denote by $P(n)$ the set of partitions of n. For a partition $\lambda \in P(n)$, $l(\lambda)$ denotes its length, i.e., the number of (non-zero) parts of λ. The set of partitions of n into odd parts only is denoted by $O(n)$, and the set of partitions of n into distinct parts is denoted by $D(n)$. We write $D^+(n)$ (resp. $D^-(n)$) for the sets of partitions λ in $D(n)$ with $n - l(\lambda)$ even (resp. odd); the partition λ is then also called even (resp. odd).

We write S_n for the symmetric group on n letters, and $\widetilde{S}_n$ for one of its double covers; so $\widetilde{S}_n$ is a non-split extension of S_n by a central subgroup $\langle z \rangle$ of order 2. It is well-known that the representation theory of these

double covers is 'the same' for all representation theoretical purposes. The spin characters of $\widetilde{S}_n$ are those that do not have z in their kernel. For an introduction to the properties of spin characters (resp. for some results we will need in the sequel) we refer to [6], [10], [11], [13]. Below we collect some of the necessary notation and some results from [13] that are crucial in later sections.

For $\lambda \in P(n)$, we write $[\lambda]$ for the corresponding irreducible character of S_n; this is identified with the corresponding character of $\widetilde{S}_n$. The associate classes of spin characters of $\widetilde{S}_n$ are labelled canonically by the partitions in $D(n)$. For each $\lambda \in D^+(n)$ there is a self-associate spin character $\langle\lambda\rangle = \mathrm{sgn}\,\langle\lambda\rangle$, and to each $\lambda \in D^-(n)$ there is a pair of associate spin characters $\langle\lambda\rangle, \langle\lambda\rangle' = \mathrm{sgn}\,\langle\lambda\rangle$. We write

$$\widehat{\langle\lambda\rangle} = \begin{cases} \langle\lambda\rangle & \text{if } \lambda \in D^+(n) \\ \langle\lambda\rangle + \langle\lambda\rangle' & \text{if } \lambda \in D^-(n) \end{cases}$$

$$\varepsilon_\lambda = \begin{cases} 1 & \text{if } \lambda \in D^+(n) \\ \sqrt{2} & \text{if } \lambda \in D^-(n) \end{cases} .$$

In [13], Stembridge introduces a projective analogue of the outer tensor product, called the reduced Clifford product, and proves a shifted analogue of the LR rule which we will need in the sequel. To state this, we first have to define some further combinatorial notions.

Let A' be the ordered alphabet $\{1' < 1 < 2' < 2 < ...\}$. The letters $1', 2', \ldots$ are said to be *marked*, the others are *unmarked*. The notation $|a|$ refers to the unmarked version of a letter a in A'. To a partition $\lambda \in D(n)$ we associate a shifted diagram

$$Y'(\lambda) = \{(i, j) \in \mathbb{N}^2 \mid 1 \leq i \leq l(\lambda), i \leq j \leq \lambda_i + i - 1\}.$$

A shifted tableau T of shape λ is a map $T : Y'(\lambda) \to A'$ such that $T(i, j) \leq T(i + 1, j)$, $T(i, j) \leq T(i, j + 1)$ for all i, j and the following additional property holds. Every $k \in \{1, 2, \ldots\}$ appears at most once in each column of T, and every $k' \in \{1', 2', \ldots\}$ appears at most once in each row of T. For $k \in \{1, 2, \ldots\}$, let c_k be the number of boxes (i, j) in $Y'(\lambda)$ such that $|T(i, j)| = k$. Then we say that the tableau T has content $(c_1, c_2, \ldots)$. Analogously, we define skew shifted diagrams and skew shifted tableaux of skew shape λ/μ if μ is a partition with $Y'(\mu) \subseteq Y'(\lambda)$. For a (possibly skew) shifted tableau S we define its associated word $w(S) = w_1 w_2 \cdots$ by reading the rows of S from left to right and from bottom to top. By erasing the marks of w, we obtain the word $|w|$.

Given a word $w = w_1 w_2 \ldots$, we define

$m_i(j) = $ multiplicity of i among $w_{n-j+1}, \ldots, w_n$ (for $0 \leq j \leq n$),

$m_i(n + j) = m_i(n) + $ multiplicity of i' among $w_1, \ldots, w_j$ (for $0 < j \leq n$).

This function m_i corresponds to reading the rows of the tableau first from right to left and from top to bottom, counting the letter i on the way, and then reading from bottom to top and left to right, counting the letter i' on this way.

The word w satisfies the lattice property if, whenever $m_i(j) = m_{i-1}(j)$, then

$$\begin{aligned} w_{n-j} &\neq i, i', & \text{if } 0 \leq j < n, \\ w_{j-n+1} &\neq i-1, i', & \text{if } n \leq j < 2n. \end{aligned}$$

For two partitions μ and ν we denote by $\mu \cup \nu$ the partition which has as its parts all the parts of μ and ν together.

Theorem 2.1 ([**13**], 8.1 and 8.3]). *Let $\mu \in D(k)$, $\nu \in D(n-k)$, $\lambda \in D(n)$, and form the reduced Clifford product $\langle \mu \rangle \times_c \langle \nu \rangle$. Then we have*

$$((\langle \mu \rangle \times_c \langle \nu \rangle) \uparrow^{\widetilde{S}_n}, \langle \lambda \rangle) = \frac{1}{\varepsilon_\lambda \varepsilon_{\mu \cup \nu}} 2^{(l(\mu)+l(\nu)-l(\lambda))/2} f_{\mu\nu}^\lambda,$$

unless λ is odd and $\lambda = \mu \cup \nu$. In that latter case, the multiplicity of $\langle \lambda \rangle$ is 0 or 1, according to the choice of associates.

The coefficient $f_{\mu\nu}^\lambda$ is the number of shifted tableaux S of shape λ/μ and content ν such that the tableau word $w = w(S)$ satisfies the lattice property and the leftmost i of $|w|$ is unmarked in w for $1 \leq i \leq l(\nu)$.

We will also use the following result from [**13**] on inner tensor products with the basic spin character $\langle n \rangle$:

Theorem 2.2 ([**13**], 9.3]). *Let $\lambda \in D(n)$, μ a partition of n. We have*

$$(\langle n \rangle [\mu], \langle \lambda \rangle) = \frac{1}{\varepsilon_\lambda \varepsilon_{(n)}} 2^{(l(\lambda)-1)/2} g_{\lambda\mu},$$

unless $\lambda = (n)$, n is even, and μ is a hook partition. In that case, the multiplicity of $\langle \lambda \rangle$ is 0 or 1 according to choice of associates.

The coefficient $g_{\lambda\mu}$ is the number of "shifted tableaux" S of unshifted shape μ and content λ such that the tableau word $w = w(S)$ satisfies the lattice property and the leftmost i of $|w|$ is unmarked in w for $1 \leq i \leq l(\lambda)$.

3. On bounds for the constituents of spin products.

First we prove a spin version of a result of Dvir [**4**] (resp. Clausen and Meier [**3**]), describing the rectangular hull of the constituents in the Kronecker product of two spin characters.

Definition 3.1. Let $\mu, \nu \in D(n)$. Define the coefficients $d_{\mu\nu}^\lambda$, $\lambda \in P(n)$, by

$$\langle \mu \rangle \cdot \langle \nu \rangle = \sum_{\lambda \vdash n} d_{\mu\nu}^\lambda [\lambda].$$

Then define the *width w* and the *height h* of the product by

$$w(\langle\mu\rangle \cdot \langle\nu\rangle) \quad = \quad \max\{\lambda_1 \mid \exists \lambda = (\lambda_1, \dots) : d_{\mu\nu}^\lambda \neq 0\}$$

$$h(\langle\mu\rangle \cdot \langle\nu\rangle) \quad = \quad \max\{l \mid \exists \lambda = (\lambda_1, \dots, \lambda_l > 0) : d_{\mu\nu}^\lambda \neq 0\}.$$

The rectangular partition (w^h) is then called the *rectangular hull* $R(\langle\mu\rangle \cdot \langle\nu\rangle)$ of the product $\langle\mu\rangle \cdot \langle\nu\rangle$.

Remark. Since $\langle\mu\rangle' = \text{sgn} \cdot \langle\mu\rangle$, we can easily obtain all products of spin characters if we know the coefficients $d_{\mu\nu}^\lambda$ above.

Theorem 3.2. *Let $\mu, \nu \in D(n)$.*
  Then the rectangular hull of the product $\langle\mu\rangle \cdot \langle\nu\rangle$ is given by

$$R(\langle\mu\rangle \cdot \langle\nu\rangle) = (|\mu \cap \nu|^{|\mu\cap\nu|}),$$

except in the case when $\mu = \nu \in D^-(n)$.
  In the case $\mu = \nu \in D^-(n)$, but μ not a staircase partition $(k, k-1, \dots, 2, 1)$, we have

$$R(\langle\mu\rangle \cdot \langle\mu\rangle) = \begin{cases} ((n-1)^n) & \text{if } n - l(\mu) \equiv 1 \bmod 4 \\ (n^{n-1}) & \text{if } n - l(\mu) \equiv 3 \bmod 4 \end{cases}.$$

More precisely,

$$|\{j \mid \mu_j > \mu_{j+1} + 1\}| = \begin{cases} (\langle\mu\rangle \cdot \langle\mu\rangle, [n-1,1]) & \text{if } n - l(\mu) \equiv 1 \bmod 4 \\ (\langle\mu\rangle \cdot \langle\mu\rangle, [2, 1^{n-2}]) & \text{if } n - l(\mu) \equiv 3 \bmod 4 \end{cases}.$$

Finally, if $n = \binom{k+1}{2}$ is a triangular number with $k \equiv 2$ or $3 \bmod 4$, and $\mu = \nu = (k, k-1, \dots, 2, 1) \in D^-(n)$, then we have

$$R(\langle\mu\rangle \cdot \langle\mu\rangle) = \begin{cases} ((n-2)^n) & \text{if } k \equiv 2 \text{ or } 7 \bmod 8 \\ (n^{n-2}) & \text{if } k \equiv 3 \text{ or } 6 \bmod 8 \end{cases}.$$

More precisely,

$$1 = \begin{cases} (\langle\mu\rangle \cdot \langle\mu\rangle, [n-2, 1^2]) & \text{if } k \equiv 2 \text{ or } 7 \bmod 8 \\ (\langle\mu\rangle \cdot \langle\mu\rangle, [3, 1^{n-3}]) & \text{if } k \equiv 3 \text{ or } 6 \bmod 8 \end{cases}.$$

Remark. The number $|\{j \mid \mu_j > \mu_{j+1} + 1\}|$ is almost the number of boxes A that can be removed such that the resulting partition $\mu \setminus A$ is in $D(n-1)$; the only exception is that a final part 1 is not counted. This is the reason for the further exception in the case of staircase partitions.

Proof. Let $\lambda = (\lambda_1, \lambda_2, \dots)$ be a partition of n with $(\langle\mu\rangle \cdot \langle\nu\rangle, [\lambda]) \neq 0$. Put $k = \lambda_1$. Then $([k], [\lambda] \downarrow_{\widetilde{S}_k}) \neq 0$, and hence

$$0 \neq (\langle\mu\rangle \downarrow_{\widetilde{S}_k} \cdot \langle\nu\rangle \downarrow_{\widetilde{S}_k}, [k]) = \left(\langle\nu\rangle \downarrow_{\widetilde{S}_k}, \overline{\langle\mu\rangle} \downarrow_{\widetilde{S}_k}\right)$$

$$= \begin{cases} (\langle\nu\rangle \downarrow_{\widetilde{S}_k}, \langle\mu\rangle' \downarrow_{\widetilde{S}_k}), & \text{if } n - l(\mu) \equiv 1 \bmod 4 \\ (\langle\nu\rangle \downarrow_{\widetilde{S}_k}, \langle\mu\rangle \downarrow_{\widetilde{S}_k}), & \text{else.} \end{cases}$$

Indeed, the last equation follows since clearly $\overline{\langle \rho \rangle} = \langle \rho \rangle$ for each $\rho \in D^+(n)$, and for $\rho \in D^-(n)$ the only possibly non-real values can occur at classes of cycle type ρ, where $\langle \rho \rangle$ takes value $i^{(n-l(\rho)+1)/2}\sqrt{\frac{\prod_j \rho_j}{2}}$ (see [6], [10]). Hence $\overline{\langle \rho \rangle} = \langle \rho \rangle$ if $n - l(\rho) \equiv 3 \bmod 4$, and $\overline{\langle \rho \rangle} = \langle \rho \rangle'$ if $n - l(\rho) \equiv 1 \bmod 4$. In any case, there exists $\alpha \in D(k)$ such that $\alpha \subseteq \mu \cap \nu$, so $k \le |\mu \cap \nu|$.

Multiplying with the sign character, the same argument also gives the inequality $l(\lambda) \le |\mu \cap \nu|$.

Put $m := |\mu \cap \nu|$. The character $\langle \mu \cap \nu \rangle$ or its associate is a common constituent of $\langle \mu \rangle \downarrow_{\widetilde{S}_m}$ and $\langle \nu \rangle \downarrow_{\widetilde{S}_m}$ (and their associates), unless $\mu = \nu \in D^-(n)$, when $(\langle \mu \rangle, \langle \mu \rangle') = 0$. Hence, if we are not in the exceptional case, then by the argument above

$$0 \ne (\langle \mu \rangle \downarrow_{\widetilde{S}_m} \cdot \langle \nu \rangle \downarrow_{\widetilde{S}_m}, [m]).$$

Hence there must be a constituent $[\lambda]$ in the product with first part $\lambda_1 \ge m$, and by what we have already proved, we have in fact equality $\lambda_1 = m$. This argument is independent of the choice of associates so we also obtain $h(\langle \mu \rangle \cdot \langle \nu \rangle) = m$ unless we are in the exceptional case.

We now have to deal with the case that $\mu = \nu \in D^-(n)$. If $n - l(\mu) \equiv 1 \bmod 4$, then $\overline{\langle \mu \rangle} = \langle \mu \rangle'$, as we have already noted above, so $(\langle \mu \rangle \cdot \langle \mu \rangle, [n]) = 0$. Hence

$$
\begin{aligned}
(\langle \mu \rangle \cdot \langle \mu \rangle, [n-1,1]) &= (\langle \mu \rangle \cdot \langle \mu \rangle, [n-1]\uparrow^{\widetilde{S}_n}) \\
&= (\langle \mu \rangle \downarrow_{\widetilde{S}_{n-1}} \cdot \langle \mu \rangle \downarrow_{\widetilde{S}_{n-1}}, [n-1]) \\
&= (\langle \mu \rangle \downarrow_{\widetilde{S}_{n-1}}, \langle \mu \rangle' \downarrow_{\widetilde{S}_{n-1}}) \\
&= |\{ j \mid \mu_j > \mu_{j+1} + 1 \}|,
\end{aligned}
$$

where the last equality follows from the spin branching rule. In particular, if μ is not the staircase $(k, k-1, \dots, 1)$, then $w(\langle \mu \rangle \cdot \langle \mu \rangle) = n - 1$. Since

$$(\langle \mu \rangle \cdot \langle \mu \rangle, [1^n]) = (\langle \mu \rangle, \langle \mu \rangle) = 1,$$

the assertion on the height follows immediately.

If $n - l(\mu) \equiv 3 \bmod 4$, then $\overline{\langle \mu \rangle} = \langle \mu \rangle$. By a similar reasoning as above we obtain the assertion in this case except for the height statement in the case of a staircase partition.

Finally, we have to deal with the case where $\mu = (k, k-1, \dots, 1) \in D^-(n)$. Here,

$$
n - l(\mu) \equiv \begin{cases} 1 \bmod 4 & \text{if } k \equiv 2 \text{ or } 7 \bmod 8 \\ 3 \bmod 4 & \text{if } k \equiv 3 \text{ or } 6 \bmod 8 \end{cases}.
$$

We consider the case where $k \equiv 2$ or $7 \bmod 8$; the other case is dual. As $n - l(\mu) \equiv 1 \bmod 4$, we already know from the previous arguments that

$w(\langle\mu\rangle \cdot \langle\mu\rangle) \leq n-2$. Hence

$$
\begin{aligned}
(\langle\mu\rangle \cdot \langle\mu\rangle, [n-2,2]) & \\
&= (\langle\mu\rangle \downarrow_{\widetilde{S}_{n-2,2}}, \langle\mu\rangle' \downarrow_{\widetilde{S}_{n-2,2}}) \\
&= (\langle k, k-1, \ldots, 3, 1\rangle \times_c \langle 2\rangle, \langle k, k-1, \ldots, 3, 1\rangle \times_c \langle 2\rangle') \\
&= 0,
\end{aligned}
$$

where the restriction to $\widetilde{S}_{n-2,2}$ follows from Theorem 2.1. Thus, using the spin branching rule we obtain

$$
\begin{aligned}
(\langle\mu\rangle \cdot \langle\mu\rangle, [n-2,1^2]) &= (\langle\mu\rangle \downarrow_{\widetilde{S}_{n-2,1^2}}, \langle\mu\rangle' \downarrow_{\widetilde{S}_{n-2,1^2}}) \\
&= (\langle k, k-1, \ldots, 3, 1\rangle, \langle k, k-1, \ldots, 3, 1\rangle) \\
&= 1.
\end{aligned}
$$

Hence $R(\langle\mu\rangle \cdot \langle\mu\rangle) = ((n-2)^n)$ in this case, as claimed. $\qquad\square$

For making the previous result slightly more precise, we need spin versions of some results in [1]:

Lemma 3.3. *Let $a_1, a_2, b_1, b_2 \in \mathbb{N}_0$ with $a_1 > a_2$, $b_1 > b_2$, $a_1+a_2 \geq b_1+b_2$. Then*

$$
b_1 + b_2 = \min(a_1 + a_2, b_1 + b_2) \leq 2(\min(a_1, b_1) + \min(a_2, b_2)) - 1,
$$

and equality holds if and only if (a_1, a_2, b_1, b_2) is of the form $(a_1, 0, b_1, b_1-1)$ or $(a_1, a_2, 1, 0)$ or $(a_1, a_1-1, 2a_1-1, 0)$.

Proof. First we consider the case when $b_1 \leq a_1$. Then the right hand side is

$$
2(b_1 + \min(a_2, b_2)) - 1 \geq 2b_1 - 1 \geq b_1 + b_2,
$$

and equality holds if and only if $\min(a_2, b_2) = 0$ and $b_1 = b_2 + 1$. These are the cases where (a_1, a_2, b_1, b_2) is of the form $(a_1, 0, b_1, b_1-1)$ or $(a_1, a_2, 1, 0)$.

In the case when $a_1 < b_1$, we have $b_2 < a_2 < a_1 < b_1$. Hence the right hand side is

$$
2(a_1 + b_2) - 1 = (2a_1 - 1) + 2b_2 \geq a_1 + a_2 + 2b_2 \geq b_1 + 3b_2 \geq b_1 + b_2,
$$

and equality holds if and only if $a_2 = a_1-1$, $b_1+b_2 = a_1+a_2$ and $b_2 = 0$. This is exactly the third situation described in the statement of the lemma. $\qquad\square$

We denote by $b_{ij}(\mu)$ the (i,j)-bar length of μ; in particular, $b_{11}(\mu) = \mu_1 + \mu_2$ for $\mu = (\mu_1, \mu_2, \ldots)$. The partition μ has an ℓ-bar if $b_{ij}(\mu) = \ell$ for some (i,j). For details on the combinatorics of bars we refer to [12].

Lemma 3.4. *Let $\mu, \nu \in D(n)$. Then*

$$
\min(b_{11}(\mu), b_{11}(\nu)) \leq 2|\mu \cap \nu| - 1,
$$

and equality holds if and only if one of μ, ν is (n) and the other one has first two parts $k, k-1$.

Proof. Let $\mu = (\mu_1, \mu_2, \dots)$, $\nu = (\nu_1, \nu_2, \dots)$. Then we have $b_{11}(\mu) = \mu_1 + \mu_2$ and $b_{11}(\nu) = \nu_1 + \nu_2$. Furthermore, $|\mu \cap \nu| = \sum_j \min(\mu_j, \nu_j) \geq \min(\mu_1, \nu_1) + \min(\mu_2, \nu_2)$. By the previous Lemma, applied to $(a_1, a_2, b_1, b_2) = (\mu_1, \mu_2, \nu_1, \nu_2)$, we immediately obtain the assertion. $\square$

We denote by $h_{ij}(\lambda)$ the (i,j)-hook length of $\lambda = (\lambda_1, \dots, \lambda_l)$; in particular, $h_{11}(\lambda) = \lambda_1 + l - 1$.

Theorem 3.5. *Let $n \geq 4$, μ, $\nu \in D(n)$, and assume we are not in the case that one of μ, ν is (n) and the other one has first two parts $k, k-1$. Let λ be a partition such that $[\lambda]$ is a constituent of $\langle \mu \rangle \cdot \langle \nu \rangle$. Then*

$$h_{11}(\lambda) < 2|\mu \cap \nu| - 1.$$

Proof. Set $l = 2|\mu \cap \nu| - 1$. Take $\widetilde{\pi} \in \widetilde{S}_n$ of type $(l, 1^{n-l}) \in O(n)$, π the corresponding element in S_n. By Morris' recursion formula [**11**] we have:

$$\langle \mu \rangle \cdot \langle \nu \rangle(\widetilde{\pi}) = 0,$$

since either μ or ν does not have an l-bar, by Lemma 3.4. By Theorem 3.2, for any constituent $[\lambda]$ of $\langle \mu \rangle \cdot \langle \nu \rangle$ we have $\lambda_1 \leq |\mu \cap \nu|$ and $l(\lambda) \leq |\mu \cap \nu|$, so $h_{11}(\lambda) \leq 2|\mu \cap \nu| - 1$. Thus λ has an l-hook if and only if $\lambda_1 = |\mu \cap \nu| = l(\lambda)$, which is then the hook $H_{11} = H_{11}(\lambda)$ of leg length $|\mu \cap \nu| - 1$. So assume now that $[\lambda]$ is a constituent which has an l-hook; then by the Murnaghan-Nakayama formula [**8**]:

$$[\lambda](\widetilde{\pi}) = [\lambda](\pi) = (-1)^{|\mu \cap \nu| - 1}[\lambda \setminus H_{11}](1) \neq 0.$$

So all such constituents $[\lambda]$ contribute a summand of the same sign to $\langle \mu \rangle \cdot \langle \nu \rangle(\widetilde{\pi})$. Since this latter value is zero, there can be no such constituent, and hence $h_{11}(\lambda) < 2|\mu \cap \nu| - 1$ for all constituents of $\langle \mu \rangle \cdot \langle \nu \rangle$. $\square$

Corollary 3.6. *Let $n \geq 4$, μ, $\nu \in D(n)$, and assume we are not in the situation that one of μ, ν is (n) and the other one has first two parts $k, k-1$. Then $\langle \mu \rangle \cdot \langle \nu \rangle$ has at least two different constituents.*

Proof. Clearly, if $\mu \in D^-(n)$, then $\langle \mu \rangle \cdot \langle \mu \rangle$ contains one of $[n]$ or $[1^n]$ and at least one further constituent different from these. So we may assume now that we are not in the exceptional case of Theorem 3.2. Then the assertion follows from the inequality in Theorem 3.5 as the $(1,1)$-hook of the rectangle $(|\mu \cap \nu|^{|\mu \cap \nu|})$ has length $2|\mu \cap \nu| - 1$. $\square$

4. Homogeneous spin character products.

We start by proving the following combinatorial lemma.

Lemma 4.1. *Let $k \in \mathbb{N}$, $n = k(k+1)/2$, and let $\lambda = (k, k-1, \dots, 2, 1)$ be the staircase partition of height k. Let $H(k)$ be the product of the hook lengths in λ, and let $B(k)$ be the product of the bar lengths in λ. Then*

$$2^{n-k}H(k) = B(k).$$

Proof. We prove the claim by induction on k. For $k = 1$ the assertion is clear. Now assume that $k \geq 2$ and that the assertion holds for $k - 1$, i.e.,

$$2^{(n-k)-(k-1)} H(k-1) = B(k-1) .$$

When the staircase diagram of $(k-1, k-2, \ldots, 2, 1)$ is extended by the first row k, it is clear how the products of hook (resp. bar) lengths change:

$$H(k) = H(k-1) \cdot \prod_{i=1}^{k} (2i - 1) , \ \ B(k) = B(k-1) \cdot \prod_{j=0}^{k-1} (k + j) .$$

As

$$\prod_{j=0}^{k-1} (k+j) = \frac{(2k-1)!}{(k-1)!} = \frac{(1 \cdot 3 \cdots (2k-1))(2 \cdot 4 \cdots 2(k-1))}{(k-1)!}$$

$$= 1 \cdot 3 \cdots (2k-1) \cdot 2^{k-1}$$

the result now follows immediately. $\qquad \square$

Theorem 4.2. *Let $n \geq 4$, μ, $\nu \in D(n)$. Then $\langle \mu \rangle \cdot \langle \nu \rangle$ is homogeneous if and only if n is a triangular number, say $n = \binom{k+1}{2}$, one of μ, ν is (n) and the other one is $(k, k-1, \ldots, 2, 1)$. In this case, we have*

$$\langle n \rangle \cdot \langle k, k-1, \ldots, 2, 1 \rangle = 2^{a(k)} [k, k-1, \ldots, 2, 1]$$

where

$$a(k) = \begin{cases} \frac{k-2}{2} & \text{if } k \text{ is even} \\ \frac{k-1}{2} & \text{if } k \equiv 1 \bmod 4 \\ \frac{k-3}{2} & \text{if } k \equiv 3 \bmod 4 \end{cases} .$$

In particular, $\langle \mu \rangle \cdot \langle \nu \rangle$ is irreducible if and only if $n = 6$ and the product is $\langle 6 \rangle \cdot \langle 3, 2, 1 \rangle = [3, 2, 1]$.

Proof. By Corollary 3.6 we only have to deal with the case $\mu = (n)$ and $\nu = (k, k-1, \ldots)$ for some k, where we have to show that the product is homogeneous if and only if ν is a staircase partition $(k, k-1, k-2, \ldots, 2, 1)$.

First, we assume that the product $\langle n \rangle \cdot \langle \nu \rangle$, with $\nu = (k, k-1, \ldots)$, is homogeneous. In this case, we use the information given by Stembridge on products with the basic spin representation $\langle n \rangle$, i.e., Theorem 2.2. By this result, clearly $[\nu]$ is a constituent of $\langle n \rangle \cdot \langle \nu \rangle$. On the other hand, by Theorem 3.2 we know that there is a constituent $[\lambda]$ in the product with $l(\lambda) = k = |(n) \cap \nu|$. Since the product is assumed to be homogeneous, we must have $\nu = \lambda$, but as $\nu_1 = k$ and $\nu \in D(n)$, we can only have $l(\nu) = k$ if $\nu = (k, k-1, \ldots, 2, 1)$.

Finally, we have to prove the assertion on the products of the special form $\langle n \rangle \cdot \langle k, k-1, \ldots, 2, 1 \rangle$. As we have remarked above, by Theorem 2.2 we

know that $[\nu]$ appears in the product $\langle n \rangle \cdot \langle \nu \rangle$, and for $\nu = (k, k-1, \ldots, 2, 1)$ the multiplicity is

$$\frac{1}{\varepsilon_\nu \varepsilon_{(n)}} 2^{(l(\nu)-1)/2} g_{\nu\nu} = \frac{1}{\varepsilon_\nu \varepsilon_{(n)}} 2^{(k-1)/2}.$$

One easily checks that this equals $2^{a(k)}$, with $a(k)$ as given in the statement of the theorem.

To show that no other constituent occurs, we just check dimensions on both sides. Let $B(k)$ be the product of the bar lengths in $\nu = (k, k - 1, \ldots, 2, 1)$, and let $H(k)$ be the product of the hook lengths in ν. Then by the hook formula (resp. the bar formula) for the character degrees we have to show that

$$2^{\left[\frac{n-1}{2}\right]} \cdot 2^{\left[\frac{n-k}{2}\right]} \frac{n!}{B(k)} = 2^{a(k)} \frac{n!}{H(k)}$$

with $a(k)$ as in the statement of the theorem. Considering the different cases depending on $k \bmod 4$, this is easily seen to be equivalent to

$$2^{n-k} H(k) = B(k).$$

Hence the assertion follows from Lemma 4.1.   □

Remark 4.3. A completely different proof of Theorem 4.2 is based on the observation that the product of two non-associate spin characters gives an ordinary character of S_n which vanishes on all 2-elements, and is thus the character of a projective module at characteristic 2. Hence, if the product is homogeneous, this character is a multiple of an irreducible projective character of S_n; these irreducible characters are exactly the ones labelled by 2-cores, i.e., staircase partitions. So the right hand side in Theorem 4.2 comes as no surprise.

5. Application to 2-modular tensor products.

In this section we want to apply the result about spin products for a proof of a recent conjecture by Gow and Kleshchev [5] describing certain homogeneous 2-modular tensor products for the symmetric groups. The argument for this proof was developed jointly with R. Gow.

Let F be a field of characteristic 2. We denote the irreducible FS_n-module labelled by the partition $\lambda \in D(n)$ by D^λ. We define the spin module S to be the irreducible FS_n-module labelled by $(m+1, m-1)$ if $n = 2m$, and by $(m, m-1)$ if $n = 2m - 1$.

Theorem 5.1. Let $n = \binom{k+1}{2}$ be a triangular number, and set $\mu = (k, k-1, \ldots, 2, 1)$ and $\lambda = (2k-1, 2k-5, 2k-9, \ldots) \in D(n)$. Then

$$S \otimes D^\lambda \simeq 2^{[(k-1)/4]} D^\mu.$$

Proof. By Theorem 4.2 we have

$$\langle n \rangle \cdot \langle k, k-1, \dots, 2, 1 \rangle = 2^{a(k)}[k, k-1, \dots, 2, 1]$$

with $a(k)$ described explicitly in Theorem 4.2. We know that the Brauer character ϕ of the reduction modulo 2 of the basic spin character $\langle n \rangle$ is irreducible, and it is the Brauer character for the spin module S. Furthermore, $\mu = (k, k-1, \dots, 2, 1)$ is a 2-core, so the corresponding Brauer character χ of $[\mu]$ mod 2 is also irreducible. Let ψ be the Brauer character of the reduction mod 2 of $\langle \mu \rangle$.

By the formula above we have

$$\phi \cdot \psi = 2^{a(k)}\chi.$$

Suppose that ψ contains two different irreducible Brauer characters α and β. Then $\phi\alpha = s\chi$ and $\phi\beta = t\chi$ for certain positive integers s and t. But the basic spin character $\langle n \rangle$ and hence also ϕ is non-zero on every 2-regular element and therefore $\alpha(g) = s\chi(g)/\phi(g)$ and $\beta(g) = t\chi(g)/\phi(g)$ holds for all 2-regular elements $g \in \widetilde{S}_n$. This implies that α and β are linearly dependent, contradicting the assumption that they are two different irreducible Brauer characters.

Hence ψ is a multiple of an irreducible Brauer character. For obtaining the precise decomposition of ψ we use some results on the 2-decomposition matrix for $\widetilde{S}_n$ in [2]. First we have to introduce some notation. For a partition $\alpha \in D(n)$ let $\mathrm{dbl}^2(\alpha)$ be the 2-regular partition obtained as follows. First "double" α by breaking each part into two halves, i.e., an odd part $2t-1$ is replaced by $t, t-1$, and an even part $2t$ is replaced by t, t. Let β be the resulting partition. Then we regularize β into the 2-regular partition $\beta^R =: \mathrm{dbl}^2(\alpha)$ by pushing up nodes along the diagonal ladders of the 2-residue diagram (we refer the reader to [8], p. 282 for more details on this process). Now by [2], Theorem (5.2) we know that the final 2-decomposition number for each spin character $\langle \alpha \rangle$ occurs in the column labelled by $\mathrm{dbl}^2(\alpha)$, and this entry is precisely $2^{[m(\alpha)/2]}$, where $m(\alpha)$ is the number of even parts of α.

We apply this to $\mu = (k, k-1, \dots, 2, 1)$. It is easy to check that $\mathrm{dbl}^2(\mu) = \lambda$. Keeping in mind that the Brauer character ψ corresponding to μ is a multiple of an irreducible Brauer character, we thus obtain

$$\psi = 2^{[m(\mu)/2]}\theta$$

where θ is the irreducible Brauer character labelled by λ.

As $m(\mu) = [k/2]$ we have thus shown that

$$2^{[[k/2]/2]}\phi \cdot \theta = 2^{a(k)}\chi.$$

Using the explicit description of $a(k)$, we then obtain

$$\phi \cdot \theta = 2^{[(k-1)/4]}\chi \ .$$

This is equivalent to the assertion on the tensor product.   □

References

[1] C. Bessenrodt and A. Kleshchev, *On Kronecker products of complex representations of the symmetric groups*, Pacific J. Math., **190** (1999), 201-223.

[2] C. Bessenrodt and J. Olsson, *The 2-blocks of the covering groups of the symmetric groups*, Advances in Math., **129** (1997), 261-300.

[3] M. Clausen and H. Meier, *Extreme irreduzible Konstituenten in Tensordarstellungen symmetrischer Gruppen*, Bayreuther Math. Schriften, **45** (1993), 1-17.

[4] Y. Dvir, *On the Kronecker product of S_n characters*, J. Algebra, **154** (1993), 125-140.

[5] R. Gow and A. Kleshchev, *Connections between the representations of the symmetric group and the symplectic group in characteristic 2*, J. Algebra, **221** (1999), 60-89.

[6] P. Hoffman and J.F. Humphreys, *Projective Representations of the Symmetric Groups*, Oxford, 1992.

[7] G. James, *The Representation Theory of the Symmetric Groups*, Springer Lecture Notes Math., **682** (1978).

[8] G. James and A. Kerber, *The Representation Theory of the Symmetric Group*, Addison-Wesley, London, 1981.

[9] I.G. Macdonald, *Symmetric Functions and Hall Polynomials*, 2nd edition, Oxford Univ. Press, Oxford, 1995.

[10] A.O. Morris, *The spin representation of the symmetric group*, Proc. London Math. Soc. (3), **12** (1962), 55-76.

[11] A.O. Morris, *The spin representation of the symmetric group*, Canad. J. Math., **17** (1965), 543-549.

[12] J.B. Olsson, *Combinatorics and Representations of Finite Groups*, Vorlesungen aus dem FB Mathematik der Univ. Essen, Heft 20, 1993.

[13] J. Stembridge, *Shifted tableaux and the projective representations of symmetric groups*, Adv. Math., **74** (1989), 87-134.

Fakultät für Mathematik
Otto-von-Guericke-Universität Magdeburg
D-39016 Magdeburg
Germany
E-mail address: bessen@mathematik.uni-magdeburg.de

Department of Mathematics
University of Oregon
Eugene, OR 97403

E-mail address: klesh@math.uoregon.edu

INNER QUASIDIAGONALITY AND STRONG NF ALGEBRAS

Bruce Blackadar and Eberhard Kirchberg

Continuing the study of generalized inductive limits of finite-dimensional C^*-algebras, we define a refined notion of quasidiagonality for C^*-algebras, called inner quasidiagonality, and show that a separable C^*-algebra is a strong NF algebra if and only if it is nuclear and inner quasidiagonal. Many natural classes of NF algebras are strong NF, including all simple NF algebras, all residually finite-dimensional nuclear C^*-algebras, and all approximately subhomogeneous C^*-algebras. Examples are given of NF algebras which are not strong NF.

1. Introduction.

This paper is a sequel to Blackadar & Kirchberg [**BKb**], to which we will frequently refer. In Blackadar & Kirchberg, we studied a generalized inductive limit construction for C^*-algebras and gave various characterizations of C^*-algebras which can be written as generalized inductive limits of finite-dimensional C^*-algebras. We recall the definitions for the convenience of the reader:

Definition 1.1. A separable C^*-algebra A is an *MF algebra* if it can be written as $\varinjlim(A_n, \phi_{m,n})$ for a generalized inductive system with the A_n finite-dimensional. If the connecting maps $\phi_{m,n}$ can be chosen to be completely positive contractions, then A is an *NF algebra*, and A is a *strong NF algebra* if the $\phi_{m,n}$ can be chosen to be complete order embeddings.

There is a close relation between these notions and quasidiagonality and nuclearity: a (separable) C^*-algebra A is an MF algebra if and only if it has an essential quasidiagonal extension by the compact operators **K** [**BKb**, 3.2.2], and A is an NF algebra if and only if it is nuclear and quasidiagonal [**BKb**, 5.2.2]. A number of other characterizations of MF, NF, and strong NF algebras are given in [**BKb**].

One major problem left unresolved in [**BKb**] is whether every NF algebra is a strong NF algebra. The purpose of this paper is to answer this question. We will characterize strong NF algebras in terms of a sharpened version of quasidiagonality we call *inner quasidiagonality*. The exact definition of inner quasidiagonality will be given in Section 2; roughly (and possibly not quite

correctly), a C^*-algebra is inner quasidiagonal if it has a separating family of quasidiagonal irreducible representations.

As a consequence, we show that "most", but not all, NF algebras are strong NF. In particular, we show that the following C^*-algebras are strong NF:

> All (separable) strongly quasidiagonal nuclear C^*-algebras.
> All (separable) residually finite-dimensional nuclear C^*-algebras.
> All (separable) approximately subhomogeneous C^*-algebras.
> All prime antiminimal NF algebras.
> All simple NF algebras.

We actually show that a prime strong NF algebra has a strong NF system $(A_n, \phi_{m,n})$ in which each A_n is a single matrix algebra.

On the other hand, there are NF algebras which are not strong NF: if A is a (separable) prime nuclear C^*-algebra containing an ideal isomorphic to $\mathbf{K}$, then A is strong NF if and only if its (unique) faithful irreducible representation is quasidiagonal. Thus the examples of [**Bn**] and [**BnD**] are not strong NF. We also show that if A is a separable nuclear C^*-algebra which is not strong NF, then CA and SA (which are NF by [**BKb**, 5.3.3]) are not strong NF, answering [**BKb**, 6.2.3(c)].

2. Inner quasidiagonality.

We begin by noting the following characterization of quasidiagonality from [**Vo2**, Theorem 1]:

Proposition 2.1. *A C^*-algebra A is quasidiagonal if and only if, for every $x_1, \ldots, x_m \in A$ and $\varepsilon > 0$, there is a representation π of A on a Hilbert space $\mathbf{H}$ and a finite-rank projection $p \in \mathbf{B}(\mathbf{H})$ with $\|p\pi(x_j)p\| > \|x_j\| - \varepsilon$ and $\|[p, \pi(x_j)]\| < \varepsilon$ for all j.*

Definition 2.2. A C^*-algebra A is *inner quasidiagonal* if, for every $x_1, \ldots, x_m \in A$ and $\varepsilon > 0$, there is a representation π of A on a Hilbert space $\mathbf{H}$ and a finite-rank projection $p \in \pi(A)'' \subseteq \mathbf{B}(\mathbf{H})$ with $\|p\pi(x_j)p\| > \|x_j\| - \varepsilon$ and $\|[p, \pi(x_j)]\| < \varepsilon$ for all j.

It obviously suffices in this definition to assume that the x_j have norm 1.

The term "inner quasidiagonal" should really be "weakly inner quasidiagonal," but we have rejected this terminology on pedantic grounds.

An inner quasidiagonal C^*-algebra is obviously quasidiagonal. The converse is false (2.7).

Proposition 2.3. *In the definition of inner quasidiagonality (2.2), the representation π may be taken to be a direct sum of a finite number of mutually inequivalent irreducible representations.*

Proof. Suppose A is inner quasidiagonal, and let $x_1, \dots, x_m \in A$ and $\varepsilon > 0$. Choose π and p as in 2.2. Let z be the central support projection for p in $\pi(A)''$; then $\pi(A)''z$ and hence also $\pi(A)'z$ are type I von Neumann algebras with finite-dimensional centers. If q is an abelian projection in $\pi(A)'$ with central support z, then $\pi|_{q\mathbf{H}}$ and pq are the desired representation and projection. $\qquad\square$

This will be generalized later (3.7).

The following is an immediate consequence of the definition.

Proposition 2.4. *If A has a separating family of quasidiagonal irreducible representations, then A is inner quasidiagonal. In particular, every residually finite-dimensional C^*-algebra is inner quasidiagonal.*

We do not know whether the converse of 2.4 is true. (See note added in proof.) But an important special case of the converse is true, even in stronger form:

Proposition 2.5. *If A is separable and prime, then A is inner quasidiagonal if and only if some (hence every) faithful irreducible representation of A is quasidiagonal.*

Proof. A C^*-algebra with a quasidiagonal faithful irreducible representation is obviously inner quasidiagonal. For the converse, consider the cases A antiliminal (NGCR) and A not antiliminal separately. If A is antiliminal, separable, prime, and inner quasidiagonal, then A is quasidiagonal, so by Voiculescu's Weyl-von Neumann Theorem [**Vo1**] every faithful representation not hitting the compacts (in particular, every faithful irreducible representation) of A is quasidiagonal.

Now suppose A is separable, prime, inner quasidiagonal, and not antiliminal. Then A has an essential ideal isomorphic to $\mathbf{K}$, and has a unique faithful irreducible representation π_0 on a Hilbert space $\mathbf{H}_0$. Let $\{x_i\}$ be a dense sequence in A, and let $\{e_{ij}\}$ be a set of matrix units in $\mathbf{K} \subseteq A$. For each n let π_n and p_n be as in 2.3 for the set $\{x_1, \dots, x_n, e_{11}, \dots, e_{nn}\}$ and for $\varepsilon = 1/n$. Then π_0 must be one of the irreducible subrepresentations of π_n for each n since $\|\pi_n(e_{11})\| > \|e_{11}\| - 1/n \geq 0$. Let q_n be the component of p_n in π_0. Then, for any j, $\lim_{n\to\infty} \|q_n \pi_0(e_{jj})q_n\| = 1$ and $\lim_{n\to\infty}[q_n, \pi_0(x_j)] = 0$, so $q_n \to 1_{\mathbf{H}_0}$ strongly and π_0 is quasidiagonal. $\qquad\square$

2.5 will be generalized in 3.18.

Corollary 2.6. *Every separable antiliminal quasidiagonal prime C^*-algebra is inner quasidiagonal. Every separable simple quasidiagonal C^*-algebra is inner quasidiagonal.*

Example 2.7. (a) The examples of [**Bn**] and [**BnD**] are quasidiagonal, but not inner quasidiagonal by 2.5.

We recall for the reader that the example of [**Bn**] is an essential extension of the continuous functions on the real projective plane $\mathbf{R}P^2$ by the compacts:

$$0 \to \mathbf{K} \to A \to C(\mathbf{R}P^2) \to 0.$$

The examples of [**BnD**] are slight variations of this, and have the additional feature that the extension has real rank zero.

(b) A similar example is the C^*-algebra generated by the direct sum of the unilateral shift s and its adjoint. The C^*-algebra is not prime; we have the extension

$$0 \to \mathbf{K} \oplus \mathbf{K} \to C^*(s \oplus s^*) \to C(\mathbf{T}) \to 0.$$

$C^*(s \oplus s)$ is quasidiagonal, but has only two irreducible representations nonzero on the $\mathbf{K} \oplus \mathbf{K}$, neither of which is quasidiagonal; hence it is easily seen not to be inner quasidiagonal.

Proposition 2.8. *Let A be a separable C^*-algebra. The following are equivalent:*

(i) *Every quotient of A is inner quasidiagonal.*
(ii) *Every primitive quotient of A is inner quasidiagonal.*
(iii) *Every irreducible representation of A is quasidiagonal.*
(iv) *A is strongly quasidiagonal, i.e., every representation of A is quasidiagonal.*

Proof. (iv) $\Rightarrow$ (iii) $\Rightarrow$ (i) $\Rightarrow$ (ii) is trivial, and (ii) $\Rightarrow$ (iii) is 2.5. To show (iii) $\Rightarrow$ (iv), let π be a representation, which we may assume is nondegenerate, of A on a Hilbert space $\mathbf{H}$, which we may assume is separable. Let $J = \pi^{-1}(\mathbf{K})$. Then $\pi = \pi_1 \oplus \pi_2$, where $\pi_1|_J$ is nondegenerate and $\pi_2(J) = 0$. Then π_1 is a direct sum of irreducible representations, hence quasidiagonal, and π_2 is quasidiagonal by Voiculescu's Weyl-von Neumann Theorem since $\pi_2(A)$ is a quasidiagonal C^*-algebra by (i). $\qquad\square$

Proposition 2.9. *Let A be a C^*-algebra, and J_1, J_2 ideals of A. Set $J = J_1 \cap J_2$. If A/J_1 and A/J_2 are inner quasidiagonal, then A/J is inner quasidiagonal.*

Proof. We may clearly assume $J = 0$. Let $x_1, \ldots, x_m \in A$ and $\varepsilon > 0$. Let ρ_k ($k = 1, 2$) be the quotient map from A to A/J_k. Then, for each j, $\|x_j\| = \max(\|\rho_1(x_j)\|, \|\rho_2(x_j)\|)$. By reordering we may assume $\|\rho_1(x_j)\| = \|x_j\|$ for $1 \le j \le r$ and $\|\rho_2(x_j)\| = \|x_j\| > \|\rho_1(x_j)\|$ for $r + 1 \le j \le m$. We may also assume ε is small enough that $\|\rho_1(x_j)\| < \|x_j\| - \varepsilon$ for $r + 1 \le j \le m$.

Let σ_k ($k = 1, 2$) be representations of A/J_k as in 2.3 for $\{\rho_k(x_1), \ldots, \rho_k(x_m)\}$ and the given ε, with projections p_k. Let σ_0 be the subrepresentation of σ_2 consisting of those irreducible subrepresentations σ such that $\|p\sigma(\rho_2(x_j))p\| > \|x_j\| - \varepsilon$ for at least one j, $r + 1 \le j \le m$ (where p denotes the σ-component of p_2), and p_0 the component of p_2 in σ_0. Then all of the

irreducible subrepresentations of σ_0, regarded as representations of A, are disjoint from $\sigma_1 \circ \rho_1$ since each such σ satisfies $\|\sigma(\rho_2(x_j))\| > \|\rho_1(x_j)\|$ for at least one j. Thus $\pi = \sigma_1 \circ \rho_1 \oplus \sigma_0 \circ \rho_2$ and $p = p_1 \oplus p_0$ have the desired properties from 2.2. □

Corollary 2.10. *A C^*-algebra A is inner quasidiagonal if A contains a collection $\{J_i\}$ of ideals with A/J_i inner quasidiagonal for all i and $\cap J_i = 0$.*

Proof. A direct proof can be given along the lines of the proof of 2.9. Alternatively, note that the result is immediate from the definition of inner quasidiagonality if the J_i are directed by inclusion. In general, let $J_{i_1,\ldots,i_n} = J_{i_1} \cap \ldots \cap J_{i_n}$, and use 2.9 to conclude that $A/J_{i_1,\ldots,i_n}$ is inner quasidiagonal. □

Remark 2.11. It is obvious from the definition that if $\{A_i, \phi_{ij}\}$ is an (ordinary) inductive system (indexed by any directed set) of inner quasidiagonal C^*-algebras with *injective* connecting maps ϕ_{ij}, then the inductive limit is inner quasidiagonal. It is not true that the inductive limit of an inductive system with noninjective connecting maps is necessarily inner quasidiagonal, as the the following example shows. (It is an inductive system with surjective connecting maps.) The same question can be asked about quasidiagonality, where it appears to have a positive answer; the closely related classes of MF and NF algebras are closed under (ordinary) inductive limits with noninjective connecting maps, as well as certain generalized inductive limits [**BKb**, 3.4.4 and 5.3.5].

Example 2.12. Let B be a (separable) quasidiagonal C^*-algebra which is not inner quasidiagonal, e.g., the example of [**Bn**] (2.7). Let π be a faithful representation of B of infinite multiplicity on a separable Hilbert space $\mathbf{H}_0$; then π is quasidiagonal. Let $\langle \mathbf{H}_n \rangle_{n \geq 1}$ be a sequence of separable infinite-dimensional Hilbert spaces, with distinguished unit vectors ξ_n, and set $\mathbf{H}^{(n)} = \otimes_{k \geq n}(\mathbf{H}_k, \xi_k)$. If $\mathbf{H} = \mathbf{H}_0 \otimes \mathbf{H}^{(1)}$, define C^*-subalgebras of $\mathbf{B}(\mathbf{H})$ by $C_n = \mathbf{K}(\mathbf{H}_0) \otimes 1_{\mathbf{H}_1} \otimes \cdots \otimes 1_{\mathbf{H}_{n-1}} \otimes \mathbf{K}(\mathbf{H}^{(n)})$ $[C_1 = \mathbf{K}(\mathbf{H})]$, $J_n = C_1 + \cdots + C_n$, $A_n = J_n + \rho(B)$, $J = [\cup J_n]^-$, $A = J + \rho(B) = [\cup A_n]^-$, where $\rho = \pi \otimes 1_{\mathbf{H}^{(1)}}$. Then $\langle J_n \rangle$ is an increasing sequence of ideals of A. Each A_n is inner quasidiagonal by repeated applications of 2.5; thus A is inner quasidiagonal by 2.11. A/J_n is isomorphic to A for any n, hence inner quasidiagonal; but $A/J \cong B$ is not inner quasidiagonal.

3. Variations and technicalities.

A somewhat cleaner alternative definition of inner quasidiagonality can be given using the socle of the bidual. See [**BoD**] for the general theory of socles of Banach algebras.

Definition 3.1. If B is a C^*-algebra, then a projection $p \in B$ is in the *socle* of B if pBp is finite-dimensional.

Proposition 3.2. *A C^*-algebra A is inner quasidiagonal if and only if, for any $x_1, \ldots, x_m \in A$ and $\varepsilon > 0$, there is a projection p in the socle of A^{**} with $\|px_jp\| > \|x_j\| - \varepsilon$ and $\|[p, x_j]\| < \varepsilon$ for all j.*

Proposition 3.3. *Let B be a C^*-algebra, $x \in B$, and p a projection in B. Then*

$$\|[x, p]\| = \max(\|(1 - p)xp\|, \|px(1 - p)\|)$$
$$= \max(\|px^*xp - px^*pxp\|^{1/2}, \|pxx^*p - pxpx^*p\|^{1/2}).$$

Proof. Set $a = (1 - p)xp$, $b = px(1 - p)$. Then calculation shows that $[x, p]^*[x, p] = a^*a + b^*b$, so $\|[x, p]\|^2 = \|a^*a + b^*b\| = \max(\|a^*a\|, \|b^*b\|)$ since a^*a and b^*b are orthogonal. Also, $\|[x, p]\|^2 = \max(\|a^*a\|, \|bb^*\|)$, and $a^*a = px^*xp - px^*pxp$ and $bb^* = pxx^*p - pxpx^*p$.   □

The following fact, which is a slight sharpening of the Kadison Transitivity Theorem, follows immediately from [**Pd**, 2.7.5 and 3.11.9].

Proposition 3.4. *Let A be a C^*-algebra, and p a projection in the socle of A^{**}. Set $N_p = \{x \in A : [p, x] = 0\} = \{p\}' \cap A$. Then*
  (a) *$pN_p = pN_pp = pA^{**}p \; (= pAp)$.*
  (b) *The weak closure of N_p in A^{**} is $pA^{**}p + (1 - p)A^{**}(1 - p)$.*

Corollary 3.5. *Let A, p, N_p be as in* 3.4, *and let $x \in A$. Then $d(x, N_p) = \|[x, p]\|$.*

Proof. $d(x, N_p) = d(x, N_p^{**})$ by the Hahn-Banach Theorem. We have $y = pxp + (1 - p)x(1 - p) \in N_p^{**}$ by 3.4, and

$$\|x - y\| = \|(1 - p)xp + px(1 - p)\|$$
$$= \max(\|(1 - p)xp\|, \|px(1 - p)\|) = \|[x, p]\|.$$

So $d(x, N_p) \leq \|[x, p]\|$. Conversely, if $y \in N_p$, then

$$\|[x, p]\| = \|[x - y, p]\|$$
$$= \max(\|(1 - p)(x - y)p\|, \|p(x - y)(1 - p)\|) \leq \|x - y\|.$$
                                                                                                       □

We now show that in many instances, the study of inner quasidiagonality can be reduced to the separable case.

Proposition 3.6. *Let A be an inner quasidiagonal C^*-algebra, and B a separable C^*-subalgebra of A. Then there is a separable inner quasidiagonal C^*-subalgebra E of A containing B.*

Proof. We show that if $x_1, \ldots, x_m \in B$ and $\varepsilon > 0$, then there is a separable C^*-subalgebra D of A containing B, and a projection q in the socle of D^{**} with $\|[q, x_j]\| < \varepsilon$ and $\|qx_jq\| > \|x_j\| - \varepsilon$ for $1 \leq j \leq m$. Choose a projection

p in the socle of A^{**} with $\|[p, x_j]\| < \varepsilon/2$ and $\|px_jp\| > \|x_j\| - \varepsilon/2$ for all j. Let C be a separable C^*-subalgebra of N_p with $d(x_j, C) = d(x_j, N_p)$ for all j and $pC = pN_p = pAp$, and let D be the C^*-subalgebra of A generated by B and C. Then D is separable. Regard the homomorphism $\pi : C \to pC = pAp$ as a representation of C on a finite-dimensional Hilbert space $\mathbf{H}$, and extend π to a representation p of D on a larger Hilbert space $\widetilde{\mathbf{H}}$ such that ρ is a direct sum of finitely many mutually inequivalent irreducible representations (by extending each irreducible subrepresentation of π to an irreducible representation of D and identifying together equivalent ones if necessary). If Q is the projection from $\widetilde{\mathbf{H}}$ onto $\mathbf{H}$, then Q may be regarded as a projection q in the socle of D^{**}, and $C \subseteq N_q$. We have by 3.5, for each j,

$$\|[q, x_j]\| = d(x_j, N_q) \leq d(x_j, C) = \|[p, x_j]\| < \varepsilon.$$

Also, there is an isometry ψ from qDq to pAp induced by ρ, and $\psi(qaq) = pap$ for $a \in C$. Since $d(x_j, C) < \varepsilon/2$, we have, for each j,

$$\|qx_jq\| > \|px_jp\| - \varepsilon/2 > \|x_j\| - \varepsilon.$$

As a consequence, if $B = B_1$ is a separable C^*-subalgebra of A, then there is a larger separable C^*-subalgebra B_2 of A such that, if $x_1, \ldots, x_m \in B_1$ and $\varepsilon > 0$, there is a projection q in the socle of B_2^{**} with $\|[q, x_j]\| < \varepsilon$ and $\|qx_jq\| > \|x_j\| - \varepsilon$ for $1 \leq j \leq m$. Iterating this construction, obtain an increasing sequence $\langle B_n \rangle$, and set $E = [\cup B_n]^-$. Then E is separable and inner quasidiagonal. $\square$

Separability is nice because of the following characterizations of inner quasidiagonality. By a slight extension of usual terminology, we will say an irreducible representation π of a C^*-algebra A is a *GCR representation* if $\pi(A)$ contains the compact operators.

Proposition 3.7. *Let A be a separable C^*-algebra. The following are equivalent:*

(i) *A is inner quasidiagonal.*

(ii) *Given $x_1, \ldots, x_m \in A$ and $\varepsilon > 0$, there is an irreducible representation π of A on a Hilbert space $\mathbf{H}$ and a finite-rank projection $p \in \mathbf{B}(\mathbf{H})$ such that $\|[p, \pi(x_j)]\| < \varepsilon$ for $1 \leq j \leq m$ and $\|p\pi(x_1)p\| > \|x_1\| - \varepsilon$.*

(iii) *There is a sequence of irreducible representations $\langle \pi_n \rangle$ of A on Hilbert spaces $\mathbf{H}_n$, and finite-rank projections $p_n \in \mathbf{B}(\mathbf{H}_n)$, such that $\|[p_n, \pi_n(x)]\| \to 0$ and $\limsup \|p_n\pi_n(x)p_n\| = \|x\|$ for all $x \in A$.*

(iv) *There is a sequence of irreducible representations $\langle \pi_n \rangle$ of A on Hilbert spaces $\mathbf{H}_n$, and finite-rank projections $p_n \in \mathbf{B}(\mathbf{H}_n)$, such that $\|[p_n, \pi_n(x)]\| \to 0$ and $\limsup \|p_n\pi_n(x)p_n\| = \|x\|$ for all $x \in A$, such that any representation occuring more than once (up to equivalence) in the sequence is quasidiagonal and GCR.*

Proof. (i) $\Rightarrow$ (ii) $\Leftrightarrow$ (iii) are obvious. To prove (iv) $\Rightarrow$ (i), let $x_1, \dots, x_m \in A$ and $\varepsilon > 0$, and let $\langle (\pi_n, p_n) \rangle$ be as in (iv). Choose $(\pi_{n_1}, p_{n_1}), \dots, (\pi_{n_m}, p_{n_m})$ with $\|p_{n_k} \pi_{n_k}(x_k) p_{n_k}\| > \|x_k\| - \varepsilon$ and $\|[\pi_{n_k}(x_j), p_{n_k}]\| < \varepsilon$ for all j and k. If there is a subset F of $\{1, \dots, m\}$ with $|F| > 1$ such that $\pi_{n_k} = \pi_0$ for $n \in F$, then π_0 is quasidiagonal, so there is a finite-rank projection q in $\pi_0(A)''$ with $\|q\pi_0(x_k)q\| > \|x_k\| - \varepsilon$ for all $k \in F$. Replace $\oplus_{k=1}^m (\pi_{n_k}, p_{n_k})$ by $(\pi_0, q) \oplus \oplus_{k \in F}(\pi_{n_k}, p_{n_k})$. Repeating the process finitely many times if necessary, we obtain a representation as in 2.3.

So we need only prove (ii) $\Rightarrow$ (iv). Let $\langle x_k \rangle$ be a dense sequence in A, and choose a doubly indexed sequence $\langle \rho_{ij} \rangle$ ($1 \leq k \leq j$) of irreducible representations of A on $\mathbf{H}_{jk}$, and finite-rank projections $q_{jk} \in \mathbf{B}(\mathbf{H}_{jk})$, such that $\|[q_{jk}, \rho_{jk}(x_i)]\| < j^{-1}$ for $1 \leq i \leq j$ and $\|q_{jk}\rho_{jk}(x_i)q_{jk}\| > \|x_k\| - j^{-1}$. (Note that this sequence satisfies the conditions of (iii).) The rest of the proof will consist of two parts.

(1) We first show that if infinitely many ρ_{jk} are equivalent to a single ρ, then either ρ is quasidiagonal and GCR or the sequence can be modified to a new sequence in which no ρ_{jk} is equivalent to ρ. If $\rho(A) \cap \mathbf{K} = \{0\}$, then there are infinitely (in fact, uncountably) many mutually inequivalent irreducible representations of A with the same kernel as ρ (see Appendix), and if π is any such representation, on a Hilbert space $\widetilde{\mathbf{H}}$, and $\varepsilon > 0$, then for any j, k with $\rho_{jk} \approx \rho$, by [**Vo1**, Lemma 1] there is a unitary u from $\mathbf{H}_{jk}$ to $\widetilde{\mathbf{H}}$ and a finite-rank projection $p \in \mathbf{B}(\widetilde{\mathbf{H}})$ such that

$$\|uq_{jk}\rho_{jk}(x_i)q_{jk}u^* - p\pi(x_i)p\| < \varepsilon$$

for $1 \leq i \leq j$. For sufficiently small ε, (ρ_{jk}, q_{jk}) can be replaced by (π, p), and a different π can be used for each ρ_{jk} equivalent to ρ.

Now suppose that ρ is GCR. If $J = \ker \rho$, then there is an ideal K of A with K/J essential in A/J and isomorphic to $\mathbf{K}$. By identifying $\mathbf{H}_{jk}$ with $\mathbf{H}$ (the Hilbert space on which ρ acts) for each ρ_{jk} equivalent to ρ, the projections q_{jk} become a sequence $\langle r_n \rangle$ of finite-rank projections in $\mathbf{B}(\mathbf{H})$ which asymptotically commute in norm with $\rho(A)$, and in particular with $\mathbf{K}$; thus the only possible weak operator limit points of the sequence are 0 and 1 [any limit point must be a scalar by irreducibility, and if $\lambda 1$ is a limit point and p a finite-rank projection, then $\langle pr_n p \rangle$ has a subsequence converging in norm to λp; but $pr_n p$ is approximately a projection for large n, so any norm limit point must be a projection]. If 1 is a limit point, then there is a subsequence of $\langle r_n \rangle$ converging weakly, hence strongly, to 1, so ρ is quasidiagonal.

If 1 is not a limit point, then $r_n \to 0$ weakly, hence strongly, and so $r_n a r_n \to 0$ in norm for all $a \in \mathbf{K}$, i.e., $\|r_n \rho(x) r_n\| \to 0$ for all $x \in K$. Fix $x_0 \in K$ of norm 1. We can then find a subsequence $\langle \rho_n \rangle$ of $\{\rho_{jk}\}$, with associated Hilbert spaces $\mathbf{H}_n$ and projections q_n, such that $\|[q_n, \rho_n(x)]\| \to 0$ for all $x \in A$, $\|q_n \rho_n(x_0) q_n\| \to 1$, and such that ρ_n is not equivalent to ρ

for any n. Define $\phi : A \to \prod q_n \mathbf{B}(\mathbf{H}_n) q_n$ by $\phi(x) = \prod q_n \rho_n(x) q_n$. ϕ then drops to a *-homomorphism σ from A to $(\prod q_n \mathbf{B}(\mathbf{H}_n) q_n)/(\oplus q_n \mathbf{B}(\mathbf{H}_n) q_n)$. Set $I = \ker \sigma$. Then $I \cap K \subseteq J$ since $x_0 \notin I$, so $(I + J)/J$ is an ideal of A/J orthogonal to K/J. But K/J is essential in A/J, so $(I + J)/J = 0$, $I \subseteq J$, and so $\|\sigma(x)\| \geq \|\rho(x)\|$ for all $x \in A$. Thus the subsequence of $\{\rho_{jk}\}$ consisting of those which are not equivalent to ρ still satisfies the conditions of (iii), and a smaller subsequence will have the same specific properties as the full double sequence $\{\rho_{jk}\}$ if suitably reindexed.

(2) We now construct a doubly indexed sequence $\langle (\pi_{jk}, p_{jk}) \rangle$ satisfying the conditions of (iv). Suppose $\{(\pi_{jk}, p_{jk})\}$ have been chosen from among the (ρ_{nr}, q_{nr}) for $1 \leq k \leq j < m$, satisfying the following properties:

(a) $\|[p_{jk}, \pi_{jk}(x_i)]\| < j^{-1}$ for $1 \leq i \leq j$.
(b) $\|p_{jk} \pi_{jk}(x_i) p_{jk}\| > \|x_k\| - j^{-1}$.
(c) No irreducible representation occurs more than once among the π_{jk} chosen so far unless it is quasidiagonal and GCR.

If $m = 1$, choose (π_{11}, p_{11}) to satisfy (a) and (b). If $m > 1$, the π_{jk} already chosen come from the ρ_{nr} for $n \leq n_0$ for some n_0. The tail $\{\rho_{nr} : n > n_0\}$ can be modified as in (1), and further truncated by increasing n_0 if necessary, so that none of the $\{\pi_{jk} : 1 \leq k \leq j < m\}$ occurs in the tail unless it is quasidiagonal and GCR. Then a suitable element of the modified tail satisfies (a) and (b) and can be chosen as π_{m1}. After again modifying and truncating the tail to eliminate subsequent appearances of π_{m1} if necessary, π_{m2} can be chosen. The process can be continued inductively to get the desired representations and projections.

This completes the proof. $\qquad \square$

Lemma 3.8. *Let A be a C^*-algebra. Then, for any k, A is inner quasidiagonal if and only if $M_k(A) = A \otimes \mathbf{M}_k$ is inner quasidiagonal. The projections for $A \otimes \mathbf{M}_k$ may be chosen of the form $p \otimes 1_k$, where p is in the socle of A^{**}.*

Proof. Suppose A is inner quasidiagonal. It follows from 3.6 that we may assume A is separable (or the following argument may be easily modified to apply to the nonseparable case). Let $\langle (\pi_n, p_n) \rangle$ be a sequence as in 3.7(iii). The map $\phi : A \to \prod p_n \mathbf{B}(\mathbf{H}_n) p_n$ given by $\phi(x) = \prod p_n \pi_n(x) p_n$ drops to an injective *-homomorphism σ from A to $(\prod p_n \mathbf{B}(\mathbf{H}_n) p_n)/ \oplus p_n \mathbf{B}(\mathbf{H}_n) p_n$, and hence $\sigma \otimes 1_k$ is an injective *-homomorphism from $A \otimes M_k$ to

$$
\begin{aligned}
&\left[\left(\prod p_n \mathbf{B}(\mathbf{H}_n) p_n \right) / \oplus p_n \mathbf{B}(\mathbf{H}_n) p_n \right] \otimes M_k \\
&\cong \left(\prod (p_n \otimes 1_k)[\mathbf{B}(\mathbf{H}_n) \otimes M_k](p_n \otimes 1_k) \right) / \oplus (p_n \otimes 1_k) \\
&\qquad \cdot [\mathbf{B}(\mathbf{H}_n) \otimes M_k](p_n \otimes 1_k).
\end{aligned}
$$

The converse is trivial. $\qquad \square$

Proposition 3.9. *Let A and B be C^*-algebras, with A inner quasidiagonal and B residually finite-dimensional (e.g., commutative). Then $A \otimes B = A \otimes_{\min} B$ is inner quasidiagonal.*

Proof. Let $\{J\}$ be a collection of ideals of B with intersection 0, with B/J_i finite-dimensional for all i. Then $\{A \otimes J_i\}$ is a collection of ideals of $A \otimes B$, with intersection 0, and $(A \otimes B)/(A \otimes J_i) = A \otimes (B/J_i)$ is a finite direct sum of matrix algebras over A, hence inner quasidiagonal. □

We will prove a partial converse to 3.9 in 3.10 below. We believe that a tensor product of any two inner quasidiagonal C^*-algebras is inner quasidiagonal, at least if one of the factors is nuclear. This would follow if the converse to 2.4 is true. (See note added in proof.) Note that a tensor product of strong NF algebras is strong NF (5.17).

Proposition 3.10. *Let A and C be C^*-algebras, with C commutative. If $A \otimes C$ is inner quasidiagonal, then A is also inner quasidiagonal.*

Proof. We reduce to the case where A is separable. If $A \otimes C$ is inner quasidiagonal, and S is a countable subset of A, then by 3.6 and an obvious additional construction we may construct an increasing sequence $\langle B_n \rangle$ of separable C^*-subalgebras of $A \otimes C$ such that B_n is inner quasidiagonal for n odd and $B_n = D_n \otimes E_n$ for n even, for separable C^*-subalgebras D_n of A containing S and E_n of C. Then $B = [\cup B_n]^-$ is separable, inner quasidiagonal, and equal to $D \otimes E$ for $D = [\cup D_n]^- \subseteq A$, which contains S, and $E = [\cup E_n]^- \subseteq C$.

Now suppose A is separable. Write $C = C_0(X)$ for a locally compact Hausdorff space X; then $A \otimes C = C_0(X, A)$. Suppose $A \otimes C$ is inner quasidiagonal; let $x_1, \ldots, x_m \in A$, all of norm 1, and $0 < \varepsilon < 1$. Choose $\delta > 0$ such that $\frac{\delta}{1-\delta} < \varepsilon$. Let U be an open set in X with compact closure, $g \in C_0(X)$ of norm 1 supported in U, and $f \in C_0(X)$ of norm 1 and identically 1 on U, with f and g taking values in $[0,1]$. Let $V = \{x : g(x) > 1 - \delta\} \subseteq U$. Let π be an irreducible representation of $A \otimes C = C_0(X, A)$ and p a finite-rank projection such that $\|p\pi(x_1 \otimes g)p\| > 1 - \delta$ and $[p, \pi(x_j \otimes g)] < \delta$ for all j. Then π is supported on a point x_0 of V, so π may be regarded as a representation ρ of A by $\rho(x) = \pi(x \otimes f)$. For each j, $\pi(x_j \otimes f)$ is a scalar multiple of $\pi(x_j \otimes g)$, with a scalar $\lambda = g(x_0)^{-1}$ satisfying $1 \leq \lambda \leq (1 - \delta)^{-1}$; thus $\|[q, \pi(x_j \otimes f)]\| \leq (1 - \delta)^{-1} \|[q, \pi(x_j \otimes g)]\| < \varepsilon$, and $\|p\pi(x_1 \otimes f)p\| \geq \|p\pi(x_j \otimes g)p\| > 1 - \delta > 1 - \varepsilon$. Thus ρ and p satisfy condition (ii) of 3.7, so A is inner quasidiagonal. □

Corollary 3.11. *SA is inner quasidiagonal if and only if A is inner quasidiagonal, and similarly for CA.*

Proof. Combine 3.10 with 3.9. □

Remark 3.12. 3.11 shows that unlike quasidiagonality [**Vo2**], inner quasidiagonality is not a homotopy invariant for C^*-algebras.

We have the following refinement of the notion of inner quasidiagonality:

Definition 3.13. Let A be a C^*-algebra, and z a central projection in A^{**}. Then A is *z-inner quasidiagonal* if, for any $x_1, \ldots, x_m \in A$ and $\varepsilon > 0$, there is a projection p in the socle of zA^{**} with $\|px_jp\| > \|x_j\| - \varepsilon$ and $\|[p, x_j]\| < \varepsilon$ for all j. If Π is a subset of $\widehat{A}$, A is *Π-inner quasidiagonal* if A is z-inner quasidiagonal, where z is the support projection of Π in the center of A^{**}.

A is inner quasidiagonal if and only if A is $1_{A^{**}}$-inner quasidiagonal. A is z-inner quasidiagonal if and only if A is Π_z-inner quasidiagonal, where Π_z is the set of irreducible representations of A with central support $\leq z$.

Example 3.14. Let A be a C^*-algebra with a quasidiagonal faithful irreducible representation π, and let z be the support projection of π in A^{**}. Then A is z-inner quasidiagonal, and zA^{**} is a type I factor. More generally, if $\{\pi_j\}$ is a separating family of quasidiagonal irreducible representations of A, and z is the support projection of $\oplus \pi_j$, then A is z-inner quasidiagonal.

There are versions of 3.6-3.8 for z-inner or Π-inner quasidiagonality, although 3.7(iv) must be weakened (but see 3.18). If A is a C^*-algebra, B a C^*-algebra, and Π a subset of $\widehat{A}$, let $\Pi|_B$ be the subset of $\widehat{B}$ consisting of all irreducible representations (actually, not just weakly) contained in $\pi|_B$ for some $\pi \in \Pi$.

Proposition 3.15. *Let A be a C^*-algebra and Π a subset of $\widehat{A}$. If A is Π-inner quasidiagonal and B is separable C^*-subalgebra of A, then there is a separable C^*-subalgebra E of A, containing B, which is $\Pi|_E$-inner quasidiagonal.*

Proposition 3.16. *Let A be a separable C^*-algebra, and Π a subset of $\widehat{A}$. The following are equivalent:*

(i) *A is Π-inner quasidiagonal.*
(ii) *Given $x_1, \ldots, x_m \in A$ and $\varepsilon > 0$, there is an irreducible representation $\pi \in \Pi$ on a Hilbert space $\mathbf{H}$ and a finite-rank projection $p \in \mathbf{B}(\mathbf{H})$ such that $\|[p, \pi(x_j)]\| < \varepsilon$ for $1 \leq j \leq m$ and $\|[p\pi(x_j)p]\| > \|x_1\| - \varepsilon$.*
(iii) *There is a sequence of irreducible representations $\langle \pi_n \rangle$ in Π on Hilbert spaces $\mathbf{H}_n$, and finite-rank projections $p_n \in \mathbf{B}(\mathbf{H}_n)$, such that $\|[p_n, \pi_n(x)]\| \to 0$ and $\limsup \|p_n\pi_n(x)p_n\| = \|x\|$ for all $x \in A$.*
(iv) *There is a sequence of irreducible representations $\langle \pi_n \rangle$ in Π on Hilbert spaces $\mathbf{H}_n$, and finite-rank projections $p_n \in \mathbf{B}(\mathbf{H}_n)$, such that $\|[p_n, \pi_n(x)]\| \to 0$ and $\limsup \|p_n\pi_n(x)p_n\| = \|x\|$ for all $x \in A$, such that any GCR representation occuring more than once (up to equivalence) in the sequence is quasidiagonal.*

Proposition 3.17. *Let A be a C^*-algebra, z a central projection in A^{**}. Then, for any k, A is z-inner quasidiagonal if and only if $M_k(A) = A \otimes \mathbf{M}_k$ is $(z \otimes 1)$-inner quasidiagonal. The projections for $A \otimes \mathbf{M}_k$ may be chosen of the form $p \otimes 1_k$, where p is in the socle of zA^{**}.*

Actually, z-inner or Π-inner quasidiagonality is not really a stronger condition than inner quasidiagonality for separable C^*-algebras, as the next result shows. This is a generalization of 2.5, and is closely related to 3.7.

Theorem 3.18. *Let A be a separable inner quasidiagonal C^*-algebra, and Π a separating set of mutually inequivalent irreducible representations of A. Then A is Π-inner quasidiagonal.*

Lemma 3.19. *Let A be a C^*-algebra, Π a faithful family of irreducible representations of A, $\pi_1, \dots, \pi_n \in \Pi$, p_k a finite-rank projection in $\pi_k(A)''$ for $1 \le k \le n$, X a finite subset of A, and $\eta > 0$. If ρ is an representation of A not equivalent to any $\pi \in \Pi$, and q a finite-rank projection on $\mathbf{H}_\rho$, then there is a $\pi \in \Pi$ and finite-rank projection p on $\mathbf{H}_\pi$, with $p \perp p_k$ if $\pi \sim \pi_k$, and an isometry z from $q\mathbf{H}_\rho$ onto $p\mathbf{H}_\pi$, such that $\|q\rho(x)q - z^*p\pi(x)pz\| < \eta$ for all $x \in X$.*

Proof. First suppose $\pi_0 \in \Pi$ is GCR, with kernel J, and let $K = \pi_0^{-1}(\mathbf{K})$. If $I = \cap\{\ker \pi : \pi \in \Pi, \ \pi \not\sim \pi_0\}$, we have $I \cap J = 0$, so if $L \in \mathrm{Prim}\,(A)$, $L \ne J$, then either $I \subseteq L$ or $K \subseteq L$. Thus if p is any finite-rank projection on $\mathbf{H}_{\pi_0}$, and ρ is an irreducible representation of A not equivalent to π_0, then for any $x \in A$ we have

$$\|\rho(x)\| \le \max\{\|(1-p)\pi_0(x)(1-p)\|, \ \max\{\|\pi(x)\| : \pi \in \Pi, \ \pi \not\sim \pi_0\}\}.$$

This formula also holds if π_0 is not GCR, for any ρ and p, since then the right-hand side is equal to $\|x\|$.

Now let ρ and q be as in the statement of the lemma, with $r = \mathrm{rank}\,q$. By replacing A by $M_r(A)$ and using the standard identifications of [**Pa**, §5] (cf. [**BKb**, 4.3]), we may assume $r = 1$. If $p_1, \dots, p_n$ are given, let S be the set of vector states coming from representations in Π, where only vector states from π_k coming from vectors orthogonal to p_k are included. Then by the first part of the proof, for any $x = x^*$ in A we have $\|\rho(x)\| \le \sup_{f \in S} |f(x)|$. So by the bipolar theorem, the weak-* closure of S contains all pure states of $\rho(A)$, proving the lemma. $\qquad\square$

Proof of Theorem 3.18. Let $x_1, \dots, x_m \in A$, of norm 1, and $0 < \varepsilon < 1$. Choose mutually inequivalent irreducible representations $\rho_1, \dots, \rho_n$ of A and finite-rank projections $q_1, \dots, q_n$ such that $\|[q_k, \rho_k(x_j)]\| < \varepsilon/2m$ for all j and k and such that, for each j, there is at least one k with $\|q_k\rho_k(x_j)q_k\| > \|x_j\| - \varepsilon/2$ (2.3). Define new pairs (π_k, p_k) for $1 \le k \le n$ inductively as follows, with each π_k in Π. Suppose $(\pi_1, p_1), \dots, (\pi_{k-1}, p_{k-1})$ have been defined. If ρ_k is in Π, set $(\pi_k, p_k) = (\rho_k, q_k)$. Otherwise, choose π, p, z as

in 3.19 for $\eta = (\varepsilon/2m)^2$, $X = \{x_j, x_j^* x_j, x_j x_l^* : 1 \le j \le m\}$, and $\rho = \rho_k$, $q = q_k$, such that $p \perp p_i$ for all $i < k$ for which $\pi = \pi_i$. Then $\|p\pi(x_j)p\| > \|q_k \rho_k(x_j) q_k\| - \varepsilon/2m$ for all j. Also, for each j,

$$
\begin{aligned}
&\|p\pi(x_j)^*\pi(x_j)p - p\pi(x_j)^*p\pi(x_j)p\| \\
&\le \|z^* p\pi(x_j^* x_j)pz - q\rho(x_j^* x_j)q\| \\
&\quad + \|q\rho(x_j^* x_j)q - q\rho(x_j)^*q\rho(x_j)q\| \\
&\quad + \|q\rho(x_j)^*q\rho(x_j)q - z^*p\pi(x_j)^*pzq\rho(x_j)q\| \\
&\quad + \|z^*p\pi(x_j)^*pzq\rho(x_j)q - z^*p\pi(x_j)^*p\pi(x_j)pz\| \\
&< \left(\frac{\varepsilon}{2m}\right)^2 + \left(\frac{\varepsilon}{2m}\right)^2 + \left(\frac{\varepsilon}{2m}\right)^2 + \left(\frac{\varepsilon}{2m}\right)^2 = \left(\frac{\varepsilon}{m}\right)^2
\end{aligned}
$$

(see 3.3 for the second term). Similarly, $\|p\pi(x_j x_j^*)p - p\pi(x_j)p\pi(x_j)^*p\| < (\varepsilon/m)^2$, so $\|[p, \pi(x_j)]\| < \varepsilon/m$ by 3.3. Set $(\pi_k, p_k) = (\pi, p)$.

We have now obtained a set $\{(\pi_k, p_k) : 1 \le k \le n\}$ of representations in Π and finite-rank projections such that $\|[p_k, \pi_k(x_j)]\| < \varepsilon/m$ for all j and k and such that, for each j, there is at least one k for which $\|[p_k, \pi_k(x_j)p_k]\| > \|x_j\| - \varepsilon$. The π_k are, however, not necessarily distinct. Suppose, for some set F, each π_k for $k \in F$ is equal to a representation π_0 in Π. Then $\{p_k : k \in F\}$ are mutually orthogonal, and if $p_0 = \sum_{k \in F} p_k$, then

$$
\|[p_0, \pi_0(x_j)]\| \le \sum_{k \in F} \|[p_k, \pi_k(x_j)]\| < \frac{\varepsilon |F|}{m} \le \varepsilon
$$
$$
\|p_0 \pi_0(x_j) p_0\| \ge \max_{k \in F} \|p_k \pi_k(x_j) p_k\|
$$

for all j, so $\oplus(\pi_k, p_k)$ may be replaced by $(\pi_0, p_0) \oplus \oplus_{k \in F}(\pi_k, p_k)$. After finitely many such procedures, a direct sum of mutually inequivalent irreducible representations in Π is obtained, satisfying the definition of Π-inner quasidiagonality for $x_1, \ldots, x_m, \varepsilon$. $\qquad \square$

4. The Main Theorem.

We first show that strong NF algebras are inner quasidiagonal. We begin by recalling one of the characterizations of strong NF algebras from [**BKb**]; we state a slightly refined form for later use. The proof is essentially identical to the proof of [**BKb**, 6.1.1] (note that that proof works throughout if the finite-dimensional algebras are restricted to be in a given class **B**).

Proposition 4.1. *Let* **B** *be a set of finite-dimensional C^*-algebras, and A a separable C^*-algebra. Then the following are equivalent:*

(i) *A can be written as $\varinjlim(A_n, \phi_{m,n})$ for a generalized inductive system $(A_n, \phi_{m,n})$ with each A_n isomorphic to an algebra in **B** and each $\phi_{m,n}$ a complete order embedding (completely positive complete isometry).*

(ii) *For every $x_1, \ldots, x_m \in A$ and $\varepsilon > 0$, there is a $B \in \mathbf{B}$, elements $b_1, \ldots, b_m \in B$, and a complete order embedding $\phi : B \to A$ with $\|x_j - \phi(b_j)\| < \varepsilon$ for all j.*

The strong NF algebras are exactly the A for which these conditions hold for $\mathbf{B}$ the set of all finite-dimensional C^*-algebras. We may take condition (ii) (with $\mathbf{B}$ the set of all finite-dimensional C^*-algebras) to be the definition of a strong NF algebra even in the nonseparable case.

Proposition 4.2. *Every strong* NF *algebra is inner quasidiagonal.*

Proof. Let A be a strong NF algebra, $x_1, \ldots, x_m \in A$, and $\varepsilon > 0$. Choose a finite-dimensional C^*-algebra B, elements $b_1, \ldots, b_m \in B$, and a complete order embedding $\phi : B \to A$ such that if $y_j = \phi(b_j)$, then $\|x_j - y_j\| < \varepsilon/2$ for all j. Let D be the C^*-subalgebra of A generated by $\phi(B)$. By [**CE1**, 4.1] (cf. [**BKb**, 4.2.2]), there is a *-homomorphism π from D onto B with $\pi(y_j) = b_j$ for all j. If $B = B_1 \oplus \cdots \oplus B_n$ with each B_i a full matrix algebra, and $\pi = \pi_1 \oplus \cdots \oplus \pi_n$, then π_i can be regarded as an irreducible representation of D on a finite-dimensional Hilbert space $\mathbf{H}_i$. Extend π_i to an irreducible representation $\widetilde{\pi}_i$ of A on a larger Hilbert space $\widetilde{\mathbf{H}}_i$, and let p_i be the projection of $\widetilde{\mathbf{H}}_i$ onto $\mathbf{H}_i$. The $\widetilde{\pi}_i$ are not in general inequivalent; we may assume that $\widetilde{\pi}_1, \ldots, \widetilde{\pi}_r$ are a set of representatives for the equivalence classes. Set $\widetilde{\mathbf{H}} = \widetilde{\mathbf{H}}_1 \oplus \cdots \oplus \widetilde{\mathbf{H}}_r$ and $\widetilde{\pi} = \widetilde{\pi}_1 \oplus \cdots \oplus \widetilde{\pi}_r$. For $i > r$, choose $k \leq r$ with $\widetilde{\pi}_i \approx \widetilde{\pi}_k$ and identify $\widetilde{\mathbf{H}}_i$ with $\widetilde{\mathbf{H}}_k$, and p_i with the corresponding projection on $\widetilde{\mathbf{H}}_k$. Let $p \in \mathbf{B}(\widetilde{\mathbf{H}})$ be the sum of the p_i (note that for a fixed k the p_i on $\widetilde{\mathbf{H}}_k$ are orthogonal since the π_i are disjoint). Then $p \in \widetilde{\pi}(A)''$; and for each j, $[p, \widetilde{\pi}(y_j)] = 0$. so $\|[p, \widetilde{\pi}(x_j)]\| \leq 2\|x_j - y_j\| < \varepsilon$. For each j we have $\|p\widetilde{\pi}(y_j)p\| = \|y_j\|$ (since $\|\pi(y_j)\| = \|y_j\|$); so

$$\|p\widetilde{\pi}(x_j)p\| \geq \|y_j\| - \|x_j - y_j\| > \|y_j\| - \varepsilon/2 > \|x_j\| - \varepsilon.$$

$\square$

The next proposition gives an important technical characterization of inner quasidiagonality.

Proposition 4.3. *Let A be a C^*-algebra, and z a central projection in A^{**}. Then A is z-inner quasidiagonal if and only if, for any $x_1, \ldots, x_m \in A$, completely positive contraction $\phi : A \to \mathbf{M}_n$, and $\varepsilon > 0$, there is a projection p in the socle of zA^{**} with $\|[p, x_j]\| < \varepsilon$ for all j, and a completely positive contraction $\psi : pA^{**}p \to \mathbf{M}_n$ with $\|\phi(x_j) - \psi(px_jp)\| < \varepsilon$ for all j.*

Proof. The "if" part is obvious (consider the case $n = 1$). Conversely, suppose A is z-inner quasidiagonal; we may assume A is unital. Fix $x_1, \ldots, x_m \in A$ and $\varepsilon > 0$. By 3.15 we may assume A is separable. Then we may assume there is a set Π of irreducible representations as in 3.16(iv) such that z is the support projection of Π. Because of 3.17 and the identifications described

in [**Pa**, §5] (cf. [**BKb**, 4.3]), we may assume $n = 1$. For each $\delta > 0$ let S_δ be the weak-* closure of the set of all states ω of A of the form $\omega(x) = \psi(pxp)$, where p is in the socle of zA^{**}, $\|[p, x_j]\| < \delta$ for all j, and ψ is a state on $pA^{**}p$. We want to show that S_ε is the entire state space of A. For any δ, S_δ is norming for A, i.e., for $x = x^* \in A$, $\sup\{|\phi(x)| : \phi \in S_\delta\} = \|x\|$ by Π-inner quasidiagonality. Therefore, if $\|x\| \leq 1$ and $\phi(x) \geq 0$ for all $\phi \in S_\delta$, then $\phi(1 - x) \leq 1$ for all $\phi \in S_\delta$, so $\|1 - x\| \leq 1$, $x \geq 0$. Thus S_δ contains all pure states of A by [**Dx**, 3.4.1], so it suffices to show that a convex combination of two elements of $S_{\delta/2}$ is in S_δ. So let p_1, p_2 be projections in the socle of zA^{**}, $\|[p_i, x_j]\| < \delta/2$ for $i = 1, 2$, $1 \leq j \leq n$, and ω_i states on A of the form $\omega_i(x) = \psi_i(p_i x p_i)$ for states ψ_i on $p_i A^{**} p_i$. As in 2.3, there are representations π_i of A, each of which is a direct sum of mutually inequivalent irreducible representations in Π, such that $\pi_i(p_i)\pi_i(A)\pi_i(p_i) = p_i A p_i$, and ω_i is a linear combination of vector states from vectors in the range of $\pi(p_i)$. If $0 < \lambda < 1$ is fixed, we must show that $\omega = \lambda\omega_1 + (1 - \lambda)\omega_2$ is approximately of the same form.

The difficulty comes when one or more of the irreducible subrepresentations of π_1 is equivalent to a subrepresentation of π_2. By the choice of Π, any such representation ρ is either quasidiagonal or not GCR. We will separately work within each such ρ, so fix ρ, on a Hilbert space $\mathbf{H}$.

If ρ is quasidiagonal, identify the components of $\pi(p_i)$ $(i = 1, 2)$ in ρ with $q_i \in \mathbf{B}(\mathbf{H})$. Then, for any $\eta > 0$, there is a finite-rank projection r such that $\|q_i - rq_i\| < \eta$ $(i = 1, 2)$ and $\|[r, \rho(x_j)]\| < \varepsilon$ for $1 \leq j \leq m$; the component of ω corresponding to ρ can thus be approximated within η in norm by a convex combination of vector states in the range of r, and such states are in S_ε.

Now suppose ρ is not GCR, i.e., $\rho(A) \cap \mathbf{K} = \{0\}$. Identify the subrepresentation ρ_1 of π_1 equivalent to ρ with ρ, giving a projection q_1; then the component of ω_1 corresponding to ρ is a convex combination of vector states in the range of q_1. Let ρ_2 on $\mathbf{H}_2$ be the subrepresentation of π_2 equivalent to ρ, and r_2 the corresponding projection. By [**Vo1**, Lemma 1], for any $y_1, \ldots, y_r \in A$ and $\eta > 0$, there is a unitary u from $\mathbf{H}_2$ to $\mathbf{H}$ and a finite-rank projection $q_2 \in \mathbf{B}(\mathbf{H})$ orthogonal to q_1 with $q_2 = ur_2u^*$ and $\|q_2\rho(x_j)q_2 - ur_2\rho_2(x_j)r_2u^*\| < \eta$, $\|q_2\rho(y_k)q_2 - ur_2\rho_2(y_k)r_2u^*\| < \eta$ for all j, k. Thus every weak-* neighborhood of the component of ω_2 in ρ_2 contains a state $\widetilde{\omega}_2$ which is a convex combination of vector states in the range of some such q_2, and $\widetilde{\omega}_2$ is in $S_{\delta/2}$ for sufficiently small η. Then the component of ω corresponding to ρ is approximated by a convex combination of vector states in the range of $q_1 + q_2$, and $\|[q_1 + q_2, x_j]\| \leq \|[q_1, x_j]\| + \|[q_2, x_j]\| < \delta$ for all j, so $\omega \in S_\delta$. $\qquad\square$

Theorem 4.4. *Let A be a nuclear C^*-algebra, and z a central projection in A^{**}. If A is z-inner quasidiagonal, then A satisfies condition* (ii) *of* 4.1 *with*

$\mathbf{B} = \{pAp : p \text{ is in the socle of } zA^{**}\}$. *So if A is separable, then A can be written as $\varinjlim(A_n, \phi_{m,n})$ for a generalized inductive system $(A_n, \phi_{m,n})$ where each A_n is isomorphic to $p_n A^{**} p_n$ for some p_n in the socle of zA^{**}, and each $\phi_{m,n}$ is a complete order embedding.*

Putting together 4.4 with 4.2 we obtain:

Theorem 4.5. *Let A be a separable C^*-algebra. Then A is a strong NF algebra if and only if A is nuclear and inner quasidiagonal.*

Proof of Theorem 4.4. Suppose A is nuclear and z-inner quasidiagonal. Let $x_1, \ldots, x_m \in A$ and $\varepsilon > 0$. Choose a matrix algebra $\mathbf{M}_n$ and completely positive contractive maps $\alpha : A \to \mathbf{M}_n$ and $\beta : \mathbf{M}_n \to A$ such that $\|\beta \circ \alpha(x_j) - x_j\| < \varepsilon/7$ for $1 \le j \le m$. Then by 4.3 choose p in the socle of zA^{**} with $\|[p, x_j]\| < \varepsilon/7$ and $\sigma : pAp \to \mathbf{M}_n$ a completely positive contraction such that $\|\sigma(px_jp) - \alpha(x_j)\| < \varepsilon/7$ for $1 \le j \le m$. Set $B = pAp$ and $\omega = \beta \circ \sigma : B \to A$. Then $\|\omega(px_jp) - x_j\| < \varepsilon/7$ for all j.

We have $d(x_j, N_p) = \|[p, x_j]\|$ for each j by 3.5; let $y_j \in N_p$ with $\|x_j - y_j\| < \varepsilon/7$. Then $\|\omega(py_j) - y_j\| < 4\varepsilon/7$ for all j. The map $x \to px = pxp$ is a *-homomorphism from N_p onto B; let J be the kernel, $\psi : B \to N_p$ a completely positive contractive cross section for the quotient map, and $\{e_i\}$ a quasicentral approximate identity for J in N_p. For each i, define $\phi_i : B \to A$ by

$$\phi_i(b) = (1 - e_i)^{1/2}\psi(b)(1 - e_i)^{1/2} + e_i^{1/2}\omega(b)e_i^{1/2}.$$

For each i, ϕ_i is a complete order embedding since $p\phi_i(b)p = b$ for all $b \in B$. For i sufficiently large, $\|y_j - ((1 - e_i)^{1/2}y_j(1 - e_i)^{1/2} + e_i^{1/2}y_je_i^{1/2})\| < \varepsilon/7$ for all j since $\{e_i\}$ is quasicentral. We also have

$$\|(1 - e_i)^{1/2}(y_j - \psi(py_j))(1 - e_i)^{1/2}\| < \varepsilon/7$$

for each j, for i large, since $y_j - \psi(py_j) \in J$. Thus, for i sufficiently large, we have, for all $1 \le j \le m$,

$$\|x_j - \phi_i(py_j)\|$$
$$\le \|x_j - y_j\| + \left\|y_j - \left((1 - e_i)^{1/2}y_j(1 - e_i)^{1/2} + e_i^{1/2}y_je_i^{1/2}\right)\right\|$$
$$+ \left\|(1 - e_i)^{1/2}(y_j - \psi(py_j))(1 - e_i)^{1/2}\right\| + \left\|e_i^{1/2}(y_j - \omega(py_j))e_i^{1/2}\right\|$$
$$< \frac{\varepsilon}{7} + \frac{\varepsilon}{7} + \frac{\varepsilon}{7} + \frac{4\varepsilon}{7} = \varepsilon,$$

so A satisfies condition (ii) of 4.1. If A is separable, the last statement of 4.4 follows from 4.1.

The following diagram summarizes the maps used in the proof.

$$
\begin{array}{ccc}
A & \xrightarrow{\ \ \text{id}\ \ } & A \\
{\scriptstyle p\cdot p}\Big\downarrow & \begin{array}{c}\alpha \\ \mathbf{M}_n \\ \sigma \end{array} \begin{array}{c}\beta\end{array} & \Big\uparrow{\scriptstyle \subseteq} \\
pA^{**}p & \xrightarrow{\ \ \psi\ \ } & N_p
\end{array}
\qquad \square
$$

5. Corollaries.

Corollary 5.1. *A separable nuclear C^*-algebra with a separating family of quasidiagonal irreducible representations is a strong NF algebra. In particular, every separable nuclear residually finite-dimensional C^*-algebra is a strong NF algebra, and every separable nuclear strongly quasidiagonal C^*-algebra is a strong NF algebra.*

Corollary 5.2. *Let A be a separable subhomogeneous C^*-algebra, and suppose each irreducible representation of A is of dimension $\leq k$. Then A is a strong NF algebra, and has a strong NF system $(A_n, \phi_{m,n})$ where each A_n is a (finite) direct sum of matrix algebras of size not more than $k \times k$.*

Since the class of strong NF algebras is closed under inductive limits [**BKb**, 6.1.3], we obtain:

Corollary 5.3. *Every (separable) approximately subhomogeneous C^*-algebra is a strong NF algebra.*

Corollary 5.4. *If A is separable, nuclear, and prime, then A is a strong NF algebra if and only if some (hence every) faithful irreducible representation of A is quasidiagonal.*

Corollary 5.5. *Every antiliminal prime NF algebra is a strong NF algebra. Every simple NF algebra is a strong NF algebra.*

Example 5.6. The examples of 2.7 are NF but not strong NF.

Corollary 5.7. *Let A be a separable nuclear C^*-algebra. The following are equivalent:*
  (i) *Every quotient of A is a strong NF algebra.*
 (ii) *Every primitive quotient of A is a strong NF algebra.*
(iii) *Every irreducible representation of A is quasidiagonal.*
 (iv) *A is strongly quasidiagonal.*

Corollary 5.8. *Let A be any NF algebra, and let B be a split essential extension of A by $\mathbf{K}$. Then B is a strong NF algebra. So A can be embedded as a C^*-subalgebra of a strong NF algebra B with a retraction (homomorphic conditional expectation) from B onto A. In particular, A is a quotient of a strong NF algebra. So every separable nuclear C^*-algebra is a quotient of a strong NF algebra [**BKb**, 6.1.8].*

We can obtain a refinement of 5.4-5.5.

Definition 5.9. A strong NF algebra is *of monomial type* if it can be written as $\varinjlim(A_n, \phi_{m,n})$, with each A_n a single matrix algebra and each $\phi_{m,n}$ a complete order embedding.

We have used the terminology "monomial type" instead of "UHF type" or "matroid type" since the class of AF algebras which are strong NF of monomial type is considerably larger than the class of UHF or matroid C^*-algebras. (In fact, an AF algebra is strong NF of monomial type if and only if it is prime.)

Proposition 5.10 (cf. [**Dx**, 1.9.13]). *A C^*-algebra B is prime if and only if, for every nonzero $x, y \in B$, there is an irreducible representation π of B with $\pi(x)$ and $\pi(y)$ both nonzero.*

Proof. If I and J are nonzero ideals of B with $I \cap J = 0$, then every irreducible representation of A must annihilate either I or J, so if $x \in I$ and $y \in J$ are nonzero, then no irreducible representation of B can be nonzero on both x and y. Conversely, if B is prime, and x, y are nonzero elements of B, then there is a $z \in B$ with $xzy \neq 0$ (otherwise the two-sided ideals generated by x and y annihilate each other); if π is an irreducible representation of B with $\pi(xzy) \neq 0$, then $\pi(x)$ and $\pi(y)$ are both nonzero. $\square$

Remark 5.11. The second half of the proof can be simplified if B is primitive (e.g., if B is separable). It is still an open question whether every prime C^*-algebra is primitive.

Proposition 5.12. *A strong NF algebra of monomial type is prime.*

Proof. If A is strong NF of monomial type and x_1, x_2 are nonzero elements of A, then by 4.1 there is a complete order embedding ϕ of a full matrix algebra B into A such that $\|x_j - \phi(b_j)\| < \|x_j\|/2$ for $j = 1, 2$, for some $b_j \in B$. ϕ^{-1} extends to an irreducible representation $\widetilde{\pi}$ of A as in proof of 4.2, and $\widetilde{\pi}(x_j) \neq 0$ for $j = 1, 2$. $\square$

Theorem 5.13. *Let A be a prime separable nuclear C^*-algebra. Then the following are equivalent:*
  (i) *A is a strong NF algebra.*
  (ii) *A is a strong NF algebra of monomial type.*
  (iii) *For some faithful irreducible representation π, $\pi(A)$ is a quasidiagonal C^*-algebra of operators.*
  (iv) *For every faithful irreducible representation π, $\pi(A)$ is a quasidiagonal C^*-algebra of operators.*

In particular, every antiliminal prime NF algebra and every simple NF algebra is a strong NF algebra of monomial type.

Proof. (ii) $\Rightarrow$ (i) and (iv) $\Rightarrow$ (iii) are trivial; (i) $\Rightarrow$ (iv) by 2.5 and 4.2, and (iii) $\Rightarrow$ (ii) by 4.4, letting z be the support of π in A^{**} (cf. 3.14). $\square$

We have the following versions of 2.9-2.10:

Proposition 5.14. *Let A be a (separable) C^*-algebra, and J_1, J_2 ideals of A. Set $J = J_1 \cap J_2$. If A/J_1 and A/J_2 are strong NF algebras, then A/J is strong NF.*

Proof. In light of 2.9 it suffices to note that if A/J_1 and A/J_2 are nuclear, then A/J is nuclear. This can be seen in several ways. Perhaps the easiest is to use the fact that a separable C^*-algebra B is nuclear if and only if every factor representation of B generates an injective factor, and note that every factor representation of A/J factors though either A/J_1 or A/J_2. Alternatively, A/J is an extension of A/J_1 by J_1/J, and $J_1/J \cong (J_1+J_2)/J_2$, which is nuclear since it is an ideal in A/J_2. $\square$

Corollary 5.15. *A separable nuclear C^*-algebra A is a strong NF algebra if (and only if) A contains a sequence $\langle J_n \rangle$ of ideals with A/J_n strong NF for all n and $\cap J_n = 0$.*

Note that neither of the assumptions that A be separable and nuclear follow from the other hypotheses of 5.15 (e.g., $A = \prod_n \mathbf{M}_n$, J_n the sequences vanishing in the n'th coordinate).

The situation with an increasing sequence of ideals, and hence with inductive limits with noninjective connecting maps, is quite different. Recall that an (ordinary) inductive limit, with injective connecting maps, of strong NF algebras is strong NF ([**BKb**, 6.1.3]; this is an immediate corollary of 4.1, or of 2.1 and 4.5).

Proposition 5.16. *An (ordinary) inductive limit of an inductive system of strong NF algebras with noninjective connecting maps is not necessarily strong NF.*

Proof. Example 2.12 is a counterexample. $\square$

For completeness, we note the following fact, which should have been included in [**BKb**]:

Proposition 5.17. *The class of strong NF algebras is closed under tensor products.*

Proof. By 4.1 it suffices to show that, if A_1, A_2, B_1, B_2 are C^*-algebras, with B_1, B_2 finite-dimensional, and $\phi_i : B_i \to A_i$ are complete order embeddings, then the finite-dimensional subspace $\phi_1(B_1) \otimes \phi_2(B_2)$ of $A_1 \otimes A_2$ is completely order isomorphic to a C^*-algebra. This follows immediately from [**CE2**, 3.1] (cf. [**BKb**, 4.2.1]), since if ω_i is an idempotent completely positive contraction from A_i onto $\phi_i(B_i)$, then $\omega_1 \otimes \omega_2$ is an idempotent completely positive contraction from $A_1 \otimes A_2$ onto $\phi_1(B_1) \otimes \phi_2(B_2)$. $\square$

Finally, the next proposition is an immediate consequence of 3.11 and [**BKb**, 5.3.3].

Proposition 5.18. *Let A be a separable C^*-algebra. Then A is a strong NF algebra if and only if SA is strong NF, and similarly for CA. Thus, if A is a separable nuclear C^*-algebra which is not strong NF (e.g., if A is not NF), then SA and CA are NF but not strong NF.*

Using 5.18, we get examples of NF algebras which are not strong NF which are very different from those of 5.6:

Example 5.19. SO_2 is an antiliminal NF algebra which is not strong NF.

Appendix A.

This appendix contains a "folklore" result that we have been unable to find in the literature. The arguments are slight variations of those of Glimm [**Gl**], as presented in [**Dx**]. The word "ideal" will mean "closed two-sided ideal".

If J is a primitive ideal in a C^*-algebra A, we will call J a *GCR ideal* if J is the kernel of a GCR irreducible representation (one whose image contains the compact operators). The next proposition is well known and easy to prove (cf. [**Dx**, 4.1.10]).

Proposition A.1. *Let J be a primitive ideal in a separable C^*-algebra A. Then the following are equivalent:*

(i) *J is a GCR ideal.*
(ii) *There is an ideal K of A, containing J, such that K/J is an essential ideal of A/J isomorphic to $\mathbf{K}$.*
(iii) *A/J is not antiliminal.*

Theorem A.2. *Let J be a primitive ideal in a separable C^*-algebra A. Then the following are equivalent:*

(i) *J is not GCR ideal.*
(ii) *A/J is antiliminal.*
(iii) *J is the kernel of a non-type I factor representation of A.*
(iv) *There are two inequivalent irreducible representations of A with kernel J.*
(v) *There are uncountably many mutually inequivalent irreducible representations of A with kernel J.*

Proof. (i) $\Leftrightarrow$ (ii) is A.1, (v) $\Rightarrow$ (iv) is trivial, and (iv) $\Rightarrow$ (i) follows from [**Dx**, 4.1.10].

By replacing A by A/J, we may and will assume that $J = 0$ in the rest of the proof, to simplify notation.

(iii) $\Rightarrow$ (v) by a slight modification of the argument in [**Dx**, 9.1]: if π is a faithful non-type-I factor representation of A on a separable Hilbert

space and C is a masa in $\pi(A)'$, the direct integral decomposition of π as $\int^{\oplus} \pi_x d\mu(x)$ with respect to C has almost all π_x faithful and irreducible by the argument of [**Dx**, 9.1]. If, for a set E of nonzero measure, each π_x for $x \in E$ is equivalent to a fixed representation π_0, then by [**Dx**, 8.1.7] $\int_E^{\oplus} \pi_x d\mu(x)$ is a subrepresentation of π equivalent to a multiple of π_0, a contradiction. Thus, for each x, the set $E_x = \{y : \pi_y \sim \pi_x\}$ has measure 0, so there must be uncountably many such sets.

It remains to prove (ii) $\Rightarrow$ (iii). This follows from the results of Glimm if A has a minimal nonzero ideal, but not directly otherwise. However, using the next three lemmas, Glimm's argument essentially works in our case.

Lemma A.3. *Let A be a separable primitive C^*-algebra. Then A contains a decreasing sequence $\langle J_n \rangle$ of nonzero (not necessarily proper) ideals, such that every nonzero ideal of A contains J_n for some n.*

Proof. This is an immediate consequence of the fact that $\mathrm{Prim}\,(A)$ is a second countable T_0 space and 0 is a dense point.   □

Lemma A.4 (cf. [**Dx**, 9.3.5]). *Let B be an antiliminal C^*-algebra and I an essential ideal in B. If $d \in B_+$ of norm 1 and $0 < \tau \le 1$, then there exist w, w', d' in I satisfying the conclusions of [**Dx**, 9.3.5].*

Proof. The proof is identical to the proof of [**Dx**, 9.3.5] except that π is chosen to be an irreducible representation of B which is nonzero on I (this is possible since I is essential and is itself an antiliminal C^*-algebra), so $\pi|_I$ is irreducible, and c is chosen in I. Then d_0 and hence ν are in I, so d', w, and w' are also in I.   □

Lemma A.5 (cf. [**Dx**, 9.3.7]). *Let B be a unital antiliminal C^*-algebra, $\langle J_n \rangle$ a decreasing sequence of essential ideals of B, and let $(s_0, s_1, \dots)$ be a sequence of self-adjoint elements of B. Then there exist elements $\nu(a_1, \dots, a_k)$ $(k = 1, 2, \dots)$ of B satisfying all the conditions of [**Dx**, 9.3.7], and in addition $\nu(a_1, \dots, a_k) \in J_k$ for all k and all $(a_1, \dots, a_k)$.*

Proof. The proof is identical to the proof of [**Dx**, 9.3.7], with A.4 used (with $I = J_{n+1}$) in place of [**Dx**, 9.3.5].   □

Proof of A.2 (ii) $\Rightarrow$ (iii). Let $\langle J_n \rangle$ be a sequence of ideals of A as in A.3. Choose elements $\nu(a_1, \dots, a_k)$ as in A.5 (if A is nonunital, work in $\widetilde{A}$). Choose the states f and g as in [**Dx**, 9.4]; then the representation π_g is a type II factor representation of A. If $I = \ker \pi_g$ is nonzero, then π_g is zero on J_n for some n; but this is impossible since $\nu(a_1, \dots, a_k) \in J_n$ and $\pi_g(\nu(a_1, \dots, a_k)) \ne 0$. Thus $I = 0$ and π_g is faithful.   □

The same technique can be used to give the following version of Maréchal's refinement [**Ma**, §2] of Glimm's result:

Lemma A.6. *Let A be a separable unital primitive antiliminal C^*-algebra. Then there is a unital sub-C^*-algebra B of A and ideal J of B, such that:*

(a) *B/J is isomorphic to the CAR algebra D (write $\phi : B \to D$ for the quotient map).*

(b) *For any cyclic representation π of D, there is a faithful cyclic representation ρ of A, and a projection $F \in \rho(B)'' \cap \rho(B)'$, of central support 1 in $\rho(A)''$, such that the subrepresentation ρ_1 of $\rho|_B$ defined by F is equivalent to $\pi \circ \phi$ and $\rho_1(B)'' = F\rho(A)''F$.*

Corollary A.7. *Let A be a separable C^*-algebra and J a non-GCR primitive ideal of A. If M is any properly infinite injective von Neumann algebra (in particular, any infinite injective factor) with separable predual, then there is a representation π of A with kernel J, such that $\pi(A)'' \cong M$.*

Note added in proof. The authors have recently shown that the converse of 2.4 is true: an inner quasidiagonal C^*-algebra has a separating family of quasidiagonal irreducible representations. As a consequence, if A and B are inner quasidiagonal and one of them is nuclear, then $A \otimes B$ is inner quasidiagonal (see the comment after 3.9). Some of the other arguments in this paper can be simplified using this result.

References

[BKb] B. Blackadar and E. Kirchberg, *Generalized inductive limits of finite-dimensional C^*-algebra*, Math. Ann., **307** (1997), 343-380.

[BoD] F. Bonsall and J. Duncan, *Complete Normed Algebras*, Springer-Verlag, Berlin-Heidelberg-New York-Tokyo, 1973.

[Bn] L. Brown, *The universal coefficient theorem for Ext and quasidiagonality*, in 'Operator Algebras and Group Representations', Pitman, Boston-London-Melbourne, 1983.

[BnD] L. Brown and M. Dadarlat, *Extensions of C^*-algebras and quasidiagonality*, J. London Math. Soc. (2), **53** (1996), 582-600.

[CE1] M.-D. Choi and E. Effros, *The completely positive lifting problem for C^*-algebras*, Ann. of Math., **104** (1976), 585-609.

[CE2] _____, *Injectivity and operator spaces*, J. Funct. Anal., **24** (1977), 156-209.

[Dx] J. Dixmier, *C^*-Algebras*, Gauthier-Villars, Paris, 1964; North-Holland, Amsterdam, 1977.

[Gl] J. Glimm, *Type I C^*-algebras*, Ann. of Math., **73** (1961), 572-612.

[Ma] O. Maréchal, *Topologie et structure Borélienne sur l'ensemble des algebres de von Neumann*, C.R. Acad. Sci. Paris, **276** (1973), 847-850.

[Pa] V. Paulsen, *Completely Bounded Maps and Dilations*, Pitman Research Notes in Math., **146** (1986).

[Pd] G. Pedersen, *C^*-Algebras and their Automorphism Groups*, Academic Press, London-New York-San Francisco, 1979.

[Vo1]   D. Voiculescu, *A noncommutative Weyl-von Neumann theorem*, Rev. Roum. Math.
        Pures Appl., **21** (1976), 97-113.

[Vo2]   ———— , *A note on quasidiagonal C^*-algebras and homotopy*, Duke Math. J., **62**
        (1991), 267-272.

UNIVERSITY OF NEVADA
RENO, NV 89557
E-mail address: bruceb@math.unr.edu

INSTITUT FÜR REINE MATHEMATIK
HUMBOLDT-UNIVERSITÄT
10099 BERLIN
GERMANY
E-mail address: kirchbrg@mathematik.hu-berlin.de

APPROXIMATION ENTROPIES IN CROSSED PRODUCTS WITH AN APPLICATION TO FREE SHIFTS

Nathanial P. Brown and Marie Choda

It is shown that for two dynamical approximation entropies (one C^* and one W^*) the implementing inner automorphism in a crossed product $A \rtimes_\alpha \mathbb{Z}$ has the same entropy value as the automorphism α.

Using the techniques in the proof, an example of a highly ergodic non-asymptotically abelian automorphism with topological entropy zero is also given. More specifically, it is shown that the free shifts on the Cuntz algebra $\mathcal{O}_\infty$ and the reduced free group C^*-algebra $C_r^*(\mathbb{F}_\infty)$ have topological entropy zero.

1. Introduction.

In this paper we show that for two definitions of dynamical entropy (both based on Voiculescu's *approximation approach*; [**Vo2**]) getting calculations for general automorphisms is equivalent to getting calculations for inner automorphisms. More precisely, we show that if $\alpha \in \mathrm{Aut}(A)$, $\eta \in S(A)$ is an α-invariant state and $\eta \circ E$ denotes the Adu-invariant state on $A \rtimes_\alpha \mathbb{Z}$ induced by η ($u \in A \rtimes_\alpha \mathbb{Z}$ is the implementing unitary) then the entropies of α and Adu agree with respect to η and $\eta \circ E$, respectively, for the entropy quantities defined in [**Ch3**] and [**Vo2**, Section 3]. (See [**St2**, Problem 4.2].)

One may regard $A \rtimes_\alpha \mathbb{Z}$ as the closure of the "fibers" $A_k = \{au^k : a \in A\}$. Then each A_k is globally invariant under Adu and, moreover, the action of Adu on A_k is precisely that of α. Thus it seems natural to expect the same entropy value for α and Adu, which we show by constructing explicit completely positive maps on $A \rtimes_\alpha \mathbb{Z}$ using the techniques of [**SS**] as in [**Br**].

The maps constructed on $A \rtimes_\alpha \mathbb{Z}$ can also be used to estimate entropy for some outer automorphisms of $A \rtimes_\alpha \mathbb{Z}$. Since many operator algebras can be realized as crossed products we get a large class of examples where these techniques are relevant. Indeed, similar ideas were used in [**Ch2**, **Ch3**] to obtain various entropy values for Cuntz's canonical endomorphism of the Cuntz algebra $\mathcal{O}_n$, $2 \leq n < \infty$ [**Cu**], and Longo's canonical endomorphism of type III factors. In this paper we exploit the isomorphism $\mathcal{O}_\infty \otimes \mathcal{K} \cong \mathcal{F} \rtimes_\Phi \mathbb{Z}$, where $\mathcal{K}$ denotes the algebra of compact operators and $\mathcal{F}$ is an AF algebra, to obtain the following result.

Theorem. *The free shift on the Cuntz algebra $\mathcal{O}_\infty$ has topological entropy zero in the sense of* [**Vo2**, **Br**].

The free shift is the automorphism of $\mathcal{O}_\infty = C^*(\{S_i : i \in \mathbb{Z}\})$ such that $S_i \mapsto S_{i+1}$. This is a highly ergodic non-asymptotically abelian automorphism.

There is a natural embedding $C_r^*(\mathbb{F}_\infty) \hookrightarrow \mathcal{O}_\infty$ of the reduced group C^*-algebra of the free group in infinitely many generators and hence the above theorem also holds for the free shift on $C_r^*(\mathbb{F}_\infty)$ since topological entropy decreases in subalgebras (cf. [**Br**, Prop. 2.1]). In fact, we will obtain the same results for automorphisms arising from any bijective function $\mathbb{Z} \to \mathbb{Z}$. (See also [**St1**, **St3**], [**Dy**] for related results.)

In Section 2 we observe several consequences of the construction of Sinclair and Smith [**SS**]. The reader is encouraged to first go through [**SS**] as we will be rather sketchy. In Section 3 we prove that the entropy of α and $\text{Ad}u$ agree for the entropies defined in [**Ch3**] and [**Vo2**, Section 3]. In Section 4 the topological entropy (in the sense of [**Br**]) of the free shift is shown to be zero.

2. Maps on Crossed Products.

We first observe that the techniques of [**SS**] allow one to construct maps on $A \rtimes_\alpha G$ out of maps on A in such a way that the map constructed on $A \rtimes_\alpha G$ inherits many nice properties that the map on A may have (e.g., normality, positivity, invariance with respect to an α-invariant state, approximation properties). For future reference it will be convenient to separate each of these observations into individual propositions. However, all of the results in this section are easy consequences of [**SS**] and we refer the reader to that paper for all of the details and notation which appears below.

In this section A will denote a C^*-algebra which is faithfully nondegenerately represented in $B(H)$, where H is a separable Hilbert space. We assume that an action $\alpha : G \to \text{Aut}(A)$ is given with G a countable discrete amenable group. As in [**SS**], we further assume (without loss of generality) that α is spatially implemented; i.e., that there exists a unitary representation $G \to B(H)$, $g \mapsto U_g$ such that $\alpha_g(a) = U_g a U_g^*$ for all $a \in A$ and $g \in G$. We will regard $A \rtimes_\alpha G$, the reduced (or full, since G is amenable) crossed product, as faithfully represented (via the regular representation) in $B(l^2(G) \otimes H)$ and let $\pi : A \hookrightarrow A \rtimes_\alpha G$ denote the natural inclusion. Since α is spatially implemented, the map π makes perfectly good sense on all of $B(H)$. An easy calculation shows that $\pi : B(H) \to B(l^2(G) \otimes H)$ is both ultraweak-ultraweak and ultrastrong-ultrastrong continuous. Recall that there is a natural unitary representation $g \mapsto \lambda_g$ of G into $B(l^2(G) \otimes H)$ such that $\lambda_g \pi(x) \lambda_g^* = \pi(\alpha_g(x))$ for all $x \in A$ and such that the span of $\{\pi(x)\lambda_g : g \in G, x \in A\}$ is norm dense in $A \rtimes_\alpha G$.

If $F \subset G$ is a finite set, we will let p_F denote the orthogonal projection onto the span of $\{\xi_g : g \in F\}$ (where $\{\xi_g\}_{g \in G}$ is the natural orthonormal basis of $l^2(G)$) and $P_F : B(l^2(G) \otimes H) \to (p_F \otimes I)B(l^2(G) \otimes H)(p_F \otimes I)$ be the compression map. If $f \in l^\infty(G)$ has finite support then we let $T_f : B(l^2(G) \otimes H) \to B(l^2(G) \otimes H)$ be the map constructed in [**SS**, Lem. 3.3].

Definition 2.1. If $\Lambda : A \to B(H)$ is a linear map, $f \in l^\infty(G)$ has finite support, and $F \subset G$ is a finite set we define $\Phi_{\Lambda,f,F} : A \rtimes_\alpha G \to B(l^2(G) \otimes H)$ by

$$\Phi_{\Lambda,f,F} = T_f \circ (id_F \otimes \Lambda) \circ P_F,$$

where $id_F : p_F(B(l^2(G)))p_F \to p_F(B(l^2(G)))p_F$ is the identity map.

It follows from [**SS**, Lem. 2.1] that $P_F(A \rtimes_\alpha G) \subset p_F(B(l^2(G)))p_F \otimes A$ and hence $\Phi_{\Lambda,f,F}$ is well defined. Since P_F is weakly continuous, this also shows that when A is a von Neumann algebra, the weak closure of $A \rtimes_\alpha G$ (i.e., the W^*-crossed product) also gets mapped into $p_F(B(l^2(G)))p_F \otimes A$ and hence $\Phi_{\Lambda,f,F}$ is well defined for W^*-algebras and W^*-crossed products as well.

In the following proposition, I will denote the identity operator on both $l^2(G)$ and H and hence $I \otimes I$ denotes the unit of $B(l^2(G) \otimes H)$. For each finite set $F \subset G$ we also let $\{e_{p,q}\}_{p,q \in F}$ denote the canonical matrix units of $p_F B(l^2(G))p_F$.

Proposition 2.2. *The following assertions hold.*

1) *T_f is a completely positive map (cf. [**Pa**]) with $T_f(I \otimes I) = \|f\|_2^2 I \otimes I$. Hence $\|T_f\|_{cb} = \|T_f\| = \|f\|_2^2$. Also, if F contains the support of f then $T_f(p_F \otimes I) = T_f(I \otimes I)$.*
2) *$T_f(e_{p,q} \otimes a) = f(p)\overline{f(q)}\pi(\alpha_p(a))\lambda_{pq^{-1}}$ for $e_{p,q} \otimes a \in p_F B(l^2(G))p_F \otimes A$.*
3) *If $x \in p_F B(l^2(G))p_F \otimes A$ and $\{x_i\} \subset p_F B(l^2(G))p_F \otimes A$ is a net converging to x in the ultraweak (resp. ultrastrong) topology then $T_f(x_i) \to T_f(x)$ in the ultraweak (resp. ultrastrong) topology.*
4) *If Λ is completely bounded (cf. [**Pa**]) then $\Phi_{\Lambda,f,F}$ is also completely bounded with $\|\Phi_{\Lambda,f,F}\|_{cb} \leq \|f\|_2^2 \|\Lambda\|_{cb}$. If Λ is completely positive then $\Phi_{\Lambda,f,F}$ is completely positive.*
5) *If A is a von Neumann algebra, $\Lambda(A) \subset A$ and Λ is ultraweakly (resp. ultrastrongly) continuous then $\Phi_{\Lambda,f,F}$ is ultraweakly (resp. ultrastrongly) continuous as a map $(A \rtimes_\alpha G)'' \to B(l^2(G) \otimes H)$.*
6) *If A is unital, Λ is unital, $F \supset \operatorname{supp}(f)$ and $\|f\|_2^2 = 1$ then $\Phi_{\Lambda,f,F}$ is unital.*

Proof. The first assertion is essentially [**SS**, Lem. 3.3] and it's proof. (The last statement follows easily from the definition of T_f.)

The second assertion follows from Lemmas 2.2 and 3.1 in [**SS**], together with the definition of T_f.

The third assertion follows from the second since we noted that π is continuous in both the ultraweak and ultrastrong topologies.

The fourth is immediate from the first since P_F is a completely positive contraction and $\|id_F \otimes \Lambda\|_{cb} \leq \|\Lambda\|_{cb}$.

The fifth follows from the third since $\Phi_{\Lambda,f,F} = T_f \circ (id_F \otimes \Lambda) \circ P_F$.

The final assertion follows from the first since

$$\Phi_{\Lambda,f,F}(I \otimes I) = T_f(id_F \otimes \Lambda(p_F \otimes I)) = T_f(p_F \otimes I) = T_f(I \otimes I) = I \otimes I.$$

$\square$

Proposition 2.3. *If $\Lambda(A) \subset A$ then*

$$\Phi_{\Lambda,f,F}(\pi(a)\lambda_g) = \sum_{t \in F \cap (gF)} f(t)\overline{f(g^{-1}t)}\pi(\alpha_t(\Lambda(\alpha_{t^{-1}}(a))))\lambda_g$$

for all $a \in A$ and $g \in G$. In particular, if $\Lambda(A) \subset A$ then $\Phi_{\Lambda,f,F}(A \rtimes_\alpha G) \subset A \rtimes_\alpha G$.

Proof. See Lemma 3.2 in [**SS**]. $\square$

Let 1 denote the unit of G. Then $P_{\{1\}}(A \rtimes_\alpha G) = p_{\{1\}} \otimes A$. Identifying $p_{\{1\}} \otimes A$ with $I \otimes A \cong A$ we let $E : A \rtimes_\alpha G \to A$ denote the resulting faithful normal projection such that $E(\pi(a)\lambda_g) = 0$ for all $g \neq 1$ and $E(\pi(a)) = a$ for all $a \in A$ (cf. [**Pe**, Lem. 7.11.3]). When A is a von Neumann algebra then we regard E as a map $(A \rtimes_\alpha G)'' \to A$.

Proposition 2.4. *If $\|f\|_2^2 = 1$, $F \supset \text{supp}(f)$, $\eta \in S(A)$ is an α-invariant state (i.e., $\eta \circ \alpha_g = \eta$ for all $g \in G$) and $\Lambda : A \to A$ is a completely positive map such that $\eta \circ \Lambda = \eta$ then $\eta \circ E \circ \Phi_{\Lambda,f,F} = \eta \circ E$ (as states on $A \rtimes_\alpha G$).*

If A is a von Neumann algebra, η is normal and Λ is ultraweakly continuous then $\eta \circ E \circ \Phi_{\Lambda,f,F} = \eta \circ E$ (as states on $(A \rtimes_\alpha G)''$).

Proof. Evidently Proposition 2.3 implies $\eta \circ E \circ \Phi_{\Lambda,f,F}(\pi(a)\lambda_g) = \eta \circ E(\pi(a)\lambda_g)$ for all $a \in A$, $g \in G$. Thus the proposition follows from parts 4 and 5 of Proposition 2.2. $\square$

The next proposition is immediate from the definitions and part 1 of Proposition 2.2.

Proposition 2.5. *Assume $\Lambda = \psi \circ \varphi$ where $\varphi : A \to B$, $\psi : B \to B(H)$ are linear maps. Letting $\Phi = (id_F \otimes \varphi) \circ P_F$ and $\Psi = T_f \circ (id_F \otimes \psi)$ we have:*

1) *The diagram*

$$
\begin{array}{ccc}
A \rtimes_\alpha G & \xrightarrow{\;\;\Phi_{\psi \circ \varphi, f, F}\;\;} & B(l^2(G) \otimes H) \\
& \Phi \searrow & \uparrow \Psi \\
& & M_{|F|} \otimes B,
\end{array}
$$

is commutative, where $M_{|F|} = p_F B(l^2(G)) p_F$ is isomorphic to the matrix algebra of dimension $|F|^2 = \text{cardinality}\,(F)^2$.

2) Φ (resp. Ψ) is completely positive whenever φ (resp. ψ) is completely positive.

Let $\{e_{p,q}\}_{p,q \in F}$ be the canonical matrix units of $M_{|F|} = p_F B(l^2(G)) p_F$. When $\psi(B) \subset A$ we can give an explicit formula for the map Ψ. (There is always an explicit formula for Φ.)

Proposition 2.6. *With the assumptions and notation of Proposition* 2.5 *we have:*

1. $\Phi(\pi(a)\lambda_g) = \displaystyle\sum_{t \in F \cap (gF)} e_{t,g^{-1}t} \otimes \varphi(\alpha_{t^{-1}}(a))$,

$1'$. *If φ is unital then Φ is unital,*

2. $\Psi(e_{p,q} \otimes b) = f(p)\overline{f(q)}\,\pi(\alpha_p(\psi(b)))\lambda_{pq^{-1}}$,

$2'$. *If $F \supset \text{supp}\,(f)$, $\|f\|_2 = 1$ and ψ is unital then Ψ is unital.*

Proof. The first assertion follows from [**SS**, Lem. 2.1] while 2 follows from the definition of T_f, [**SS**, Lem. 3.1] and [**SS**, Lem. 2.2]. $1'$ (resp. $2'$) is an easy calculation using 1 (resp. 2). $\qquad\square$

We will need the following proposition to compute [**Ch3**] entropy.

Proposition 2.7. *If φ and ψ in Proposition* 2.5 *are unital and completely positive, $\|f\|_2^2 = 1$, $F \supset \text{supp}\,(f)$ and $\eta \in S(A)$ is an α-invariant state then*

$$\eta \circ E \circ \Psi \left(\sum_{q \in F} e_{q,q} \otimes b \right) = \eta \circ \psi(b),$$

for all $b \in B$ (i.e., under the natural identifications of B and $1 \otimes B \subset M_{|F|} \otimes B$, the states $\eta \circ \psi$ and $\eta \circ E \circ \Psi$ agree).

Proof. This is an easy calculation using the previous proposition. $\qquad\square$

Finally we observe that $\Phi_{\Lambda,f,F}$ has good approximation properties whenever Λ does. If $K \subset G$ is a finite set and $f \in l^\infty(G)$ has finite support F then we let

$$F_{K,f} = F \cup \left(\bigcup_{g \in K} g^{-1}F \right).$$

Proposition 2.8. *For each finite set $K \subset G$ and $\delta > 0$ there exists $f \in l^\infty(G)$ of finite support with $\|f\|_2 = 1$ and the following property: Let $\omega \subset A$ be a finite set with $\|x\| \leq 1$ for all $x \in \omega$. If $\|\Lambda(y) - y\| \leq \delta/2$ for all $y \in \cup_{g \in F_{K,f}} \alpha_{g^{-1}}(\omega)$ then $\|\Phi_{\Lambda,f,F_{K,f}}(\pi(x)\lambda_k) - \pi(x)\lambda_k\| \leq \delta$ for all $x \in \omega, k \in K$.*

Proof. If $\Lambda(A) \subset A$ then this is essentially contained in the proof of [**SS**, Thm. 3.4]. However, a slightly different series of estimates handles the general case (see the proof of [**Br**, Lem. 3.4]). $\qquad\square$

Assume that $\eta \in S(A)$ is an α-invariant state and consider the seminorm $\|x\|_\eta = \eta(x^*x)^{1/2}$ for all $x \in A$. Then $\eta \circ E$ is a state on $A \rtimes_\alpha G$ which is $\mathrm{Ad}\lambda_g$-invariant for all $g \in G$. An easy calculation shows $\|\pi(a)\lambda_g\|_{\eta \circ E} = \|a\|_\eta$ for all $a \in A$ and $g \in G$. Similarly one shows $\|\alpha_t(a)\|_\eta = \|a\|_\eta$ for all $a \in A$ and $t \in G$. From this it follows that if $\Lambda(A) \subset A$ then

$$\|\Lambda(\alpha_{t^{-1}}(a)) - \alpha_{t^{-1}}(a)\|_\eta = \|\pi(\alpha_t \circ \Lambda \circ \alpha_{t^{-1}}(a))\lambda_g - \pi(a)\lambda_g\|_{\eta \circ E},$$

for all $a \in A$ and $g, t \in G$. However, with this observation the estimates in the proof of [**SS**, Thm. 3.4] go through essentially without change. Hence we get the following analogue of the previous proposition.

Proposition 2.9. *For each finite set $K \subset G$ and $\delta > 0$ there exists $f \in l^\infty(G)$ of finite support with $\|f\|_2 = 1$ and the following property: Let $\eta \in S(A)$ be an α-invariant state and $\omega \subset A$ be a finite set with $\|x\|_\eta \leq 1$ for all $x \in \omega$. If $\|\Lambda(y) - y\|_\eta \leq \delta/2$ for all $y \in \cup_{g \in F_{K,f}} \alpha_{g^{-1}}(\omega)$ then $\|\Phi_{\Lambda,f,F_{K,f}}(\pi(x)\lambda_k) - \pi(x)\lambda_k\|_{\eta \circ E} \leq \delta$ for all $x \in \omega, k \in K$.*

3. Entropy and Inner Automorphisms.

We will now establish the analogue of [**Br**, Thm. 3.5] for the dynamical entropies defined in [**Ch3**] and [**Vo2**, Section 3]. In this section, α will always denote an action of a countable discrete *abelian* group G on a given operator algebra. Crossed products (both C^* and W^*) will be regarded as subalgebras of $B(l^2(G) \otimes H)$ (as in the previous section), $\pi : A \to A \rtimes_\alpha G$ is the natural inclusion and $E : A \rtimes_\alpha G \to A$ is the natural faithful normal conditional expectation.

We begin with the analogues of [**Br**, Lem. 3.4]. The next lemma is used to compute [**Ch3**] entropy in crossed products. We refer the reader to [**Ch3**] and [**Vo2**, Section 3] for the definitions and notation which appears below.

Lemma 3.1. *Let A be a unital nuclear C^*-algebra (cf. [**Wa**]) and $\eta \in S(A)$ be an α-invariant state (i.e., $\eta \circ \alpha_g = \eta$ for all $g \in G$). For each finite set $K \subset G$ and $\delta > 0$ there exists a finite set $F = F(K, \delta) \subset G$ such that if $\omega \subset A$ is a finite set with $\|x\| \leq 1$ for all $x \in \omega$ then*

$$scp_{\eta \circ E}(\omega_K, \delta) \leq scp_\eta \left(\bigcup_{g \in F} \alpha_{-g}(\omega), \delta/2 \right) + \log(|F|),$$

where $\omega_K = \{\pi(x)\lambda_k : x \in \omega, k \in K\}$ and $|F| = \mathrm{cardinality}\,(F)$.

Proof. Apply Proposition 2.8 with K, δ to get a function $f \in l^2(G)$ with finite support, $\|f\|_2 = 1$ and the property stated in that proposition. We will show that $F = F_{K,f}$ is the desired finite set.

To prove the inequality we let $\varepsilon > 0$ be arbitrary and choose unital completely positive maps $\varphi : A \to B$, $\psi : B \to A$ such that B is finite dimensional, $\|\psi \circ \varphi(y) - y\| \leq \delta/2$ for all $y \in \cup_{g \in F} \alpha_{-g}(\omega)$ and $S(\eta \circ \psi) \leq$

$scp_\eta(\bigcup_{g \in F} \alpha_{-g}(\omega), \delta/2) + \varepsilon$. Letting $\Lambda = \psi \circ \varphi$ we can factor $\Phi_{\Lambda, f, F}$ (as $\Psi \circ \Phi$) through $M_{|F|} \otimes B$ by Proposition 2.5. Note also that Φ and Ψ are unital completely positive maps by Propositions 2.5 and 2.6. Proposition 2.8 says that $\|\Phi_{\Lambda, f, F}(x) - x\| \leq \delta$ for all $x \in \omega_K$ and thus (by definition) $scp_{\eta \circ E}(\omega_K, \delta) \leq S(\eta \circ E \circ \Psi)$. By [**OP**, Prop. 1.9] we have

$$S(\eta \circ E \circ \Psi) \leq S(\eta \circ E \circ \Psi|_{1 \otimes B}) + \log(|F|).$$

Finally, from Proposition 2.7, we have $S(\eta \circ E \circ \Psi|_{1 \otimes B}) = S(\eta \circ \psi) \leq scp_\eta(\bigcup_{g \in F} \alpha_{-g}(\omega), \delta/2) + \varepsilon$, by our choice of ψ, which proves the lemma since ε was arbitrary. □

The next lemma allows one to compute the entropy of [**Vo2**, Section 3] in crossed products. The proof is similar to the previous one and will be omitted (see also [**Br**, Lem. 3.4]). Due to the definitions involved, one uses Proposition 2.9 instead of Proposition 2.8. The replacement of the inequality

$$S(\eta \circ E \circ \Psi) \leq S(\eta \circ E \circ \Psi|_{1 \otimes B}) + \log(|F|)$$

is the remark that if rank (C) denotes the dimension of a maximal abelian subalgebra of C then rank $(M_n(\mathbb{C}) \otimes B) \leq n \cdot rank(B)$. We also note that one must appeal to Proposition 2.4 to ensure that the maps used in the previous proof (i.e., $\Phi_{\Lambda, f, F}$) remain $\eta \circ E$-invariant. (Though not explicitly stated in [**Vo2**, Section 3], it follows from the assumptions that the state is faithful and the approximating maps in $CPA(M, \eta)$ are η-invariant that the maps in $CPA(M, \eta)$ are ultraweakly continuous and hence the hypotheses of Proposition 2.4 are indeed satisfied.)

Lemma 3.2. *Let M be a hyperfinite von Neumann algebra with α-invariant faithful normal state η. For each finite set $K \subset G$ and $\delta > 0$ there exists a finite set $F = F(K, \delta) \subset G$ such that if $\omega \subset M$ is a finite set with $\|x\|_\eta \leq 1$ for all $x \in \omega$ then*

$$rcp_{\eta \circ E}(\omega_K, \delta) \leq |F| rcp_\eta \left(\bigcup_{g \in F} \alpha_{-g}(\omega), \delta/2 \right),$$

where $\omega_K = \{\pi(x)\lambda_k : x \in \omega, k \in K\}$ and $|F| = $ cardinality (F).

As in the previous section, we let $\lambda_g \in A \rtimes_\alpha G$ be the unitary implementing $\alpha_g \in \text{Aut}(A)$. We also remind the reader that G is now assumed to be a discrete abelian group.

Theorem 3.3. *If A is a unital nuclear C^*-algebra with α-invariant state η, then for all $g \in G$ we have $ht_\eta(\alpha_g) = ht_{\eta \circ E}(\text{Ad}\lambda_g)$, where $ht_\eta(\cdot)$ is defined in [**Ch3**].*

Proof. We only sketch the argument as it is similar to the proof of [**Br**, Thm. 3.5]. The inequality $ht_\eta(\alpha_g) \leq ht_{\eta \circ E}(\text{Ad}\lambda_g)$ for all $g \in G$ follows from [**Ch3**, Prop. 2.2].

Let $\delta > 0$, $\omega \subset A$ a finite set with $\|x\| \leq 1$ for all $x \in \omega$, and a finite set $K \subset G$ be given. Choose a finite set $F = F(K, \delta)$ according to Lemma 3.1 and define $\Omega = \cup_{g \in F} \alpha_{-g}(\omega)$. Since G is abelian, from Lemma 3.1 one may deduce the inequality

$$scp_{\eta \circ E}(\omega_K \cup \ldots \cup \mathrm{Ad}\lambda_g^{n-1}(\omega_K), \delta) \leq scp_\eta(\Omega \cup \ldots \cup \alpha_g^{n-1}(\Omega), \delta/2) + \log(|F|),$$

as in the proof of [**Br**, Thm. 3.5]. Since this inequality holds for all $n \in \mathbb{N}$, the desired inequality follows from [**Ch3**, Prop. 2.3]. $\square$

The proof of the following theorem is similar where one uses Lemma 3.2 instead of Lemma 3.1. The analogues of [**Ch3**, Prop. 2.2] and [**Ch3**, Prop. 2.3] are [**Vo2**, Prop. 3.5] and [**Vo2**, Prop. 3.4], respectively. Of course, $\mathrm{Ad}\lambda_g$ should now be regarded as an automorphism of the W^*-crossed product.

Theorem 3.4. *Let M be a hyperfinite von Neumann algebra and η be an α-invariant faithful normal state. For all $g \in G$, we have $hcpa_\eta(\alpha_g) = hcpa_{\eta \circ E}(\mathrm{Ad}\lambda_g)$, where $hcpa_\eta(\cdot)$ is defined in [**Vo2**, Section 3].*

In particular, this theorem generalizes the results of [**Vo2**, Appendix].

4. Entropy for Automorphisms of $\mathcal{O}_\infty$.

In this section we will show that $ht(\alpha) = 0$ (cf. [**Br**]) for the free shifts on $\mathcal{O}_\infty$ and $C_r^*(\mathbb{F}_\infty)$. This will follow from a more general result concerning automorphisms of $\mathcal{O}_\infty$ induced by bijective mappings $\alpha : \mathbb{Z} \to \mathbb{Z}$. These results have also been obtained by K. Dykema (cf. [**Dy**, Thm. 1 and Example 7]) using directly the free product construction as opposed to the crossed product construction used here. As mentioned in the introduction, we use the isomorphism $\mathcal{O}_\infty \otimes \mathcal{K} \cong \mathcal{F} \rtimes_\Phi \mathbb{Z}$ and the techniques of the previous two sections to achieve our calculations.

Recall that the Cuntz algebra $\mathcal{O}_\infty$ is defined as the universal C^*-algebra generated by isometries $\{S_i\}_{i \in \mathbb{Z}}$ which satisfy the relation

$$\sum_{i=-r}^{r} S_i S_i^* \leq 1$$

for all $r \in \mathbb{N}$. If $\alpha : \mathbb{Z} \to \mathbb{Z}$ is any bijective function, then from the universality of $\mathcal{O}_\infty$ we get a well defined automorphism $\mathcal{O}_\infty \to \mathcal{O}_\infty$ defined by $S_i \mapsto S_{\alpha(i)}$. We will also use α to denote the automorphism of $\mathcal{O}_\infty$ induced by $\alpha : \mathbb{Z} \to \mathbb{Z}$. If α is the mapping $i \mapsto i + 1$ then α is called the *free shift*.

We begin with a technical lemma which should have appeared in [**Br**] and will be necessary for our calculations. If $A \subset B(H)$ we will let ι_A denote the inclusion $A \hookrightarrow B(H)$. See [**Br**, Def. 1.1] for the notation which appears below.

Lemma 4.1. *Let C, $D \subset B(H)$ be exact C^*-algebras (cf. [**Wa**]) and π : $C \to D$ be a *-monomorphism. For each finite set $\omega \subset C$ and $\delta > 0$, $rcp(\iota_C, \omega, \delta) = rcp(\iota_D, \pi(\omega), \delta)$.*

Proof. From the proofs of [**Br**, Prop. 1.3 and 2.14] we have $rcp(\iota_C, \omega, \delta) = rcp(\pi, \omega, \delta)$ and $rcp(\iota_D, \pi(\omega), \delta) = rcp(\iota_{\pi(C)}, \pi(\omega), \delta)$. Hence it is sufficient to show $rcp(\pi, \omega, \delta) = rcp(\iota_{\pi(C)}, \pi(\omega), \delta)$.

We only show $rcp(\pi, \omega, \delta) \geq rcp(\iota_{\pi(C)}, \pi(\omega), \delta)$ as the other inequality is similar. So choose $(\varphi, \psi, B) \in CPA(\pi, C)$ such that $\|\psi \circ \varphi(x) - \pi(x)\| \leq \delta$ for all $x \in \omega$ and rank $(B) = rcp(\pi, \omega, \delta)$. Then $(\varphi \circ \pi^{-1}, \psi, B) \in CPA(\iota_{\pi(C)}, \pi(C))$ and $\|\psi \circ \varphi \circ \pi^{-1}(\pi(x)) - \pi(x)\| \leq \delta$ for all $x \in \omega$. But this implies $rcp(\pi, \omega, \delta) \geq rcp(\iota_{\pi(C)}, \pi(\omega), \delta)$ as desired. □

Remark 4.2. In particular, this lemma improves [**Br**, Lem. 2.4] and hence the proofs of Propositions 2.5, 2.6 and 2.8 in [**Br**] are slightly more technical than they need to be.

Given $n \in \mathbb{N}$ and a subset $I \subset \mathbb{Z}$ we let $W(n, I) = \{\mu = (\mu_1, \dots, \mu_n) : \mu_j \in I$ for $1 \leq j \leq n\}$ and $W(0, I) = \{\emptyset\}$. If $\mu \in W(n, I)$ we define the operator $S_\mu \in \mathcal{O}_\infty$ by $S_\mu = S_{\mu_1} \cdots S_{\mu_n}$ and $S_\mu = 1$ if $n = 0$. For $m \leq n$ we let $[m, n]$ be the integer interval and $\mathcal{F}([m, n], I)$ be the C^*-subalgebra of $\mathcal{O}_\infty$ generated by

$$\bigcup_{j=m}^{n} \{S_\mu S_\nu^* : \mu, \nu \in W(j, I)\}.$$

It is known that if I is a finite subset of $\mathbb{Z}$ then $\mathcal{F}(n, I) = \mathcal{F}([n, n], I)$ is isomorphic to the matrix algebra $M_{|I|^n}(\mathbb{C})$, where $|I|$ denotes the cardinality of I, and is isomorphic to the compact operators on an infinite dimensional separable Hilbert space when I is infinite (cf. [**Cu**]). If $I \subset \mathbb{Z}$ is a finite set, we define for each $j \in \mathbb{N}$ the projection

$$P_j = \sum_{\mu \in W(j, I)} S_\mu S_\mu^*.$$

Note that $P_j \geq P_{j+1}$.

Lemma 4.3. *If $I \subset \mathbb{Z}$ is a finite set then*

$$\mathcal{F}([0, n], I) \cong \mathbb{C} \oplus M_{|I|}(\mathbb{C}) \oplus \dots \oplus M_{|I|^n}(\mathbb{C})$$

with a complete set of pairwise orthogonal minimal projections given by

$$\{S_\mu(1 - P_1)S_\mu^*, S_\nu S_\nu^* : \mu \in \cup_{j \in [0, n-1]} W(j, I) \text{ and } \nu \in W(n, I)\}.$$

In particular, rank $(\mathcal{F}([0, n], I)) = 1 + |I| + \dots + |I|^n$.

Proof. For each j, the set $\{S_\mu S_\nu^* : \mu, \nu \in W(j, I)\}$ is a complete set of matrix units for $\mathcal{F}(j, I)$ and the unit of $\mathcal{F}(j, I)$ is P_j. For each $0 \leq j \leq n - 1$ we define

$$E_j(\mu, \nu) = S_\mu(1 - P_1)S_\nu^*,$$

where $\mu,\ \nu \in W(j, I)$. For $j = n$ we let

$$E_n(\mu, \nu) = S_\mu S_\nu^*,$$

where $\mu,\ \nu \in W(n, I)$. For each $0 \leq j \leq n$ let $A_j = C^*(\{E_j(\mu, \nu) : \mu,\ \nu \in W(j, I)\})$. (Note that $A_n = \mathcal{F}(n, I)$.) Evidently we have $A_j \subset \mathcal{F}([0, n], I)$ for $0 \leq j \leq n$ and hence $C^*(\{A_j : 0 \leq j \leq n\}) \subset \mathcal{F}([0, n], I)$. Note also that for $\mu,\ \nu \in W(n - 1, I)$, $S_\mu S_\nu^* = E_{n-1}(\mu, \nu) + S_\mu P_1 S_\nu^*$ and hence $S_\mu S_\nu^* \in A_{n-1} + A_n$. Similarly one argues that $S_\mu S_\nu^* \in A_j + \cdots + A_n$ for $\mu,\ \nu \in W(j, I)$ and hence $C^*(\{A_j : 0 \leq j \leq n\}) = \mathcal{F}([0, n], I)$.

For $0 \leq j \leq n - 1$ a simple calculation shows $E_j(\mu, \nu)E_j(\mu', \nu') = \delta_{\nu, \mu'} E_j(\mu, \nu')$ and $\sum_\mu E_j(\mu, \mu) = P_j - P_{j+1}$. One also verifies that for $0 \leq j \leq n - 1$ and $\mu, \nu \in W(j, I)$, $S_\mu^* P_{j+1} = P_1 S_\mu^*$. Hence $P_{j+1} S_\mu S_\nu^* = S_\mu S_\nu^* P_{j+1}$ and

$$(P_j - P_{j+1})S_\mu S_\nu^*(P_j - P_{j+1}) = S_\mu(1 - P_1)S_\nu^* = E_j(\mu, \nu),$$

for $0 \leq j \leq n - 1$ and $\mu, \nu \in W(j, I)$. Thus $\{E_j(\mu, \nu) : \mu,\ \nu \in W(j, I)\}$ is a complete set of matrix units for $A_j = (P_j - P_{j+1})\mathcal{F}(j, I)(P_j - P_{j+1}) \cong M_{|I|^j}(\mathbb{C})$. This also shows that the A_j are pairwise orthogonal and hence

$$\mathcal{F}([0, n], I) = C^*(\{A_j : 0 \leq j \leq n\}) = A_0 \oplus \ldots \oplus A_n.$$

However, this clearly implies the lemma. $\square$

Since $\mathcal{F}([0, n], [-n, n]) \subset \mathcal{F}([0, n + 1], [-n - 1, n + 1]) \subset \mathcal{O}_\infty$, the closure $\mathcal{F}_\infty$ of $\cup_n \mathcal{F}([0, n], [-n, n])$ is an AF subalgebra containing the unit of $\mathcal{O}_\infty$.

For each $i \in \mathbb{Z}$ let $B_i = \mathcal{F}_\infty$ and define *-monomorphisms $\beta_{i,i-1} : B_i \to B_{i-1}$ by $x \mapsto S_0 x S_0^*$. Let B denote the inductive limit of the sequence

$$\cdots B_1 \overset{S_0 \cdot S_0^*}{\hookrightarrow} B_0 \overset{S_0 \cdot S_0^*}{\hookrightarrow} B_{-1} \overset{S_0 \cdot S_0^*}{\hookrightarrow} \cdots,$$

and $\rho_i : B_i \hookrightarrow B$ be the induced embeddings. As in [**Cu**], there is an automorphism Φ of B which shifts the above sequence one space to the left and satisfies the relation

$$\Phi^j \circ \rho_i = \rho_{i+j},$$

for all $i, j \in \mathbb{Z}$ (under the natural identifications $B_i \cong \mathcal{F}_\infty \cong B_{i+j}$). Another important relation that follows immediately from the construction is

$$\rho_i(x) = \rho_{i-j}(S_0^j x S_0^{*j}),$$

for all $x \in B_i \cong \mathcal{F}_\infty \cong B_{i-j}$, $i \in \mathbb{Z}$ and $j \in \mathbb{N}$.

Let u be the implementing unitary in the multiplier algebra $M(B \rtimes_\Phi \mathbb{Z})$. One readily verifies that the elements $\tilde{S}_i = \rho_0(S_i S_0^*)u \in B \rtimes_\Phi \mathbb{Z}$ are partial isometries with support projection $\rho_0(1)$ and satisfying

$$\sum_{i=-r}^{r} \tilde{S}_i \tilde{S}_i^* \leq \rho_0(1)$$

for all $r \in \mathbb{N}$. Hence there is an induced *-monomorphism $\pi : \mathcal{O}_\infty \to B \rtimes_\Phi \mathbb{Z}$ such that $\pi(S_i) = \tilde{S}_i$ for all $i \in \mathbb{Z}$. In [**Cu**], it is shown that $\pi(\mathcal{O}_\infty) = \rho_0(1)(B \rtimes_\Phi \mathbb{Z})\rho_0(1)$. However, for our calculations, it will only be necessary to observe that for every $x \in \mathcal{F}_\infty$ we have $\pi(x) = \rho_0(x)$.

We will use this map π (and Lemma 4.1) to estimate the completely positive δ-rank of certain finite sets in $\mathcal{O}_\infty$. To compute δ-ranks in crossed products (i.e., $B \rtimes_\Phi \mathbb{Z}$) we recall [**Br**, Lem. 3.4] (in a slightly different, but equivalent, form than the original).

Lemma 4.4 ([**Br**, Lem. 3.4]). *Let $A \subset B(H)$ be an exact C^*-algebra (cf. [**Wa**]) and $\alpha : G \to \mathrm{Aut}(A)$ be an action of a discrete abelian group. For each finite set $K \subset G$ and $\delta > 0$ there exists a finite set $F = F(K, \delta) \subset G$ such that if $\omega \subset A$ is a finite set with $\|x\| \leq 1$ for all $x \in \omega$ then*

$$rcp(id_{A \rtimes_\alpha G}, \omega_K, \delta) \leq |F| rcp\left(id_A, \bigcup_{g \in F} \alpha_{-g}(\omega), \delta/2\right),$$

where $\omega_K = \{\pi(x)\lambda_k : x \in \omega, k \in K\}$ and $|F| = $ cardinality (F).

Considering the case $G = \mathbb{Z}$, what this lemma roughly says is that to approximate polynomials in $A \rtimes_\alpha \mathbb{Z}$ it suffices to approximate a finite number of iterates of the coefficients.

For convenience, we assume both $\mathcal{O}_\infty$ and $B \rtimes_\Phi \mathbb{Z}$ to be faithfully represented on the same Hilbert space H and use $\iota_{\mathcal{O}_\infty}$ and $\iota_{B \rtimes_\Phi \mathbb{Z}}$ to denote the inclusions. It will also be convenient to define

$$\omega_{k,l,I} = \bigcup_{i=0}^{k} \left(\bigcup_{j=0}^{l} \{S_0^{*i} S_\mu S_\nu^*, \ S_\mu S_\nu^* S_0^i : \mu, \nu \in W(j, I)\} \right).$$

Lemma 4.5. *For each $k, l \in \mathbb{N}$ with $k \leq l$ and $\delta > 0$ there is a constant $C = C(k, l, \delta)$ such that $rcp(\iota_{\mathcal{O}_\infty}, \omega_{k,l,I}, \delta) \leq C|I|^C$ for all finite subsets $I \subset \mathbb{Z}$ containing 0.*

Proof. Note that $\pi(S_0^i) = \rho_0(S_0^i S_0^{*i})u^i$. Hence for all $i \leq k$ and $\mu, \nu \in W(j, I)$ we have

$$\pi(S_\mu S_\nu^* S_0^i) = \rho_0(S_\mu S_\nu^* S_0^i S_0^{*i})u^i = \rho_{-k}(S_0^k S_\mu S_\nu^* S_0^i S_0^{*i} S_0^{*k})u^i$$

and

$$\pi(S_0^{*i} S_\mu S_\nu^*) = u^{*i}\rho_0(S_0^i S_0^{*i} S_\mu S_\nu^*).$$

Since $u^*\rho_0(x) = \rho_{-1}(x)u^*$ we have

$$\pi(S_0^{*i}S_\mu S_\nu^*) = \rho_{-i}(S_0^i S_0^{*i}S_\mu S_\nu^*)u^{*i} = \rho_{-k}(S_0^{k-i}S_0^i S_0^{*i}S_\mu S_\nu^* S_0^{*(k-i)})u^{*i}.$$

Thus we see that the "coefficients" of $\pi(\omega_{k,l,I})$ (i.e., $\rho_{-k}(S_0^k S_\mu S_\nu^* S_0^i S_0^{*(k+i)})$ and $\rho_{-k}(S_0^k S_0^{*i}S_\mu S_\nu^* S_0^{*(k-i)})$) all come from the finite dimensional algebra $\rho_{-k}(\mathcal{F}([0,l+k],I))$ by reducing the terms $S_\nu^* S_0^i$ and $S_0^{*i}S_\mu$.

By virtue of Lemmas 4.1 and 4.4, to estimate $rcp(\iota_{\mathcal{O}_\infty},\omega_{k,l,I},\delta)$ it suffices to understand a finite number of the iterates (under Φ) of the coefficients of $\pi(\omega_{k,l,I})$. But since there is always a conditional expectation onto finite dimensional subalgebras, we only need to understand a finite number of iterates of the finite dimensional subalgebra $\rho_{-k}(\mathcal{F}([0,l+k],I))$ (since this contains the coefficients of $\pi(\omega_{k,l,I})$).

So let $m \in \mathbb{N}$ be arbitrary and consider

$$\Phi^m\big(\rho_{-k}(\mathcal{F}([0,l+k],I))\big) \cup \ldots \cup \Phi^{-m}\big(\rho_{-k}(\mathcal{F}([0,l+k],I))\big).$$

By the relations $\Phi^j \circ \rho_{-k} = \rho_{-k+j}$, and $\rho_t(x) = \rho_{t-r}(S_0^r x S_0^{*r})$ $(r \geq 0)$ we have

$$\Phi^m\big(\rho_{-k}(\mathcal{F}([0,l+k],I))\big) \cup \ldots \cup \Phi^{-m}\big(\rho_{-k}(\mathcal{F}([0,l+k],I))\big)$$

$$= \rho_{-k+m}\Big(\mathcal{F}([0,l+k],I)\Big) \cup \ldots \cup \rho_{-k-m}\Big(\mathcal{F}([0,l+k],I)\Big)$$

$$= \rho_{-k-m}\Big(S_0^{2m}\mathcal{F}([0,l+k],I)S_0^{*(2m)} \cup \ldots \cup \mathcal{F}([0,l+k],I)\Big)$$

$$\subset \rho_{-k-m}(\mathcal{F}([0,l+k+2m],I)).$$

Hence, by Lemmas 4.1, 4.4 and our observations above, there exists $m = m(k,\delta) \in \mathbb{N}$ such that

$$rcp(\iota_{\mathcal{O}_\infty},\omega_{k,l,I},\delta) \leq (2m+1)\mathrm{rank}\,(\rho_{-k-m}(\mathcal{F}([0,l+k+2m],I)))$$

$$= (2m+1)\mathrm{rank}\,(\mathcal{F}([0,l+k+2m],I))$$

$$= (2m+1)(1+|I|+\cdots+|I|^{(l+k+2m)})$$

$$\leq C|I|^C,$$

where $C = l+k+2m+1$.   □

Since $\mathcal{O}_\infty$ is nuclear, the following theorem holds for the entropy defined in [**Vo2**, Section 4] although we will be using the definition in [**Br**] (cf. [**Br**, Prop. 1.4]).

Theorem 4.6. *If $\alpha \in \mathrm{Aut}(\mathcal{O}_\infty)$ is induced by a bijective function $\alpha : \mathbb{Z} \to \mathbb{Z}$ then $ht(\alpha) = 0$.*

Proof. If $I \subset \mathbb{Z}$ contains 0, $\mu \in W(j, I)$ and $\nu \in W(j', I)$, where $j \leq j'$, then $S_\mu S_\nu^* = S_0^{*i} S_\gamma S_\nu^*$ with $\gamma \in W(j', I)$ and $i = j' - j$. From this observation and a similar remark when $j > j'$, one deduces that the span of the finite sets $\omega_{k,l,I}$ with $k \leq l$ and $0 \in I$ are norm dense in $\mathcal{O}_\infty$. Hence it suffices, by [**Br**, Prop. 2.6], to show that $ht(\iota_{\mathcal{O}_\infty}, \alpha, \omega_{k,l,I}, \delta) = 0$ for all such sets.

If $\mu, \nu \in W(j, I)$ then an easy calculation shows $\alpha(S_\mu S_\nu^* S_0^i) = S_\gamma S_\lambda^* S_0^i$ for some $\gamma, \lambda \in W(j, \{0\} \cup \alpha(I))$ if $j \geq i$ or $\gamma, \lambda \in W(i, \{0\} \cup \alpha(I))$ if $j < i$. In any case, one deduces that $\alpha(\omega_{k,l,I}) \subset \omega_{k,l,\{0\}\cup\alpha(I)}$ whenever $k \leq l$. Similarly one shows

$$\omega_{k,l,I} \cup \ldots \cup \alpha^{n-1}(\omega_{k,l,I}) \subset \omega_{k,l,I\cup\ldots\cup\alpha^{n-1}(I)},$$

for all $n \in \mathbb{N}$, whenever $k \leq l$ and $0 \in I \subset \mathbb{Z}$.

Hence

$$rcp(\iota_{\mathcal{O}_\infty}, \omega_{k,l,I} \cup \ldots \cup \alpha^{n-1}(\omega_{k,l,I}), \delta) \leq rcp(\iota_{\mathcal{O}_\infty}, \omega_{k,l,I\cup\ldots\cup\alpha^{n-1}(I)}, \delta),$$

for all n and all $\delta > 0$. However, the previous lemma shows that

$$rcp(\iota_{\mathcal{O}_\infty}, \omega_{k,l,I\cup\ldots\cup\alpha^{n-1}(I)}, \delta) \leq C(n|I|)^C$$

for some constant C depending only on k, l and δ. This implies

$$ht(\iota_{\mathcal{O}_\infty}, \alpha, \omega_{k,l,I}, \delta) \leq \limsup_{n\to\infty} n^{-1}(\log(C(n|I|)^C)) = 0.$$

$\square$

Remark 4.7. Note that we have never used the fact that $\alpha : \mathbb{Z} \to \mathbb{Z}$ is surjective. Thus the previous theorem also holds for any endomorphism of $\mathcal{O}_\infty$ which is induced by an injective function $\alpha : \mathbb{Z} \to \mathbb{Z}$.

The following recovers a special case of [**St3**, Thm. 2] in the case of CNT entropy.

Corollary 4.8. *Let $\alpha \in \mathrm{Aut}(\mathcal{O}_\infty)$ be induced by a bijective function $\alpha : \mathbb{Z} \to \mathbb{Z}$ and φ be an α-invariant state. Then $h_\varphi(\alpha) = ht_\varphi(\alpha) = 0$, where $h_\varphi(\cdot)$ and $ht_\varphi(\cdot)$ are defined in [**CNT**] and [**Ch3**], respectively.*

Proof. Since $\mathcal{O}_\infty$ is nuclear we appeal to [**Ch3**, Thm. 2.6.1] to get the inequalities $ht(\alpha) \geq ht_\varphi(\alpha) \geq h_\varphi(\alpha)$. $\square$

If $\mathbb{F}_\infty$ is the free group on generators $\{g_i\}_{i\in\mathbb{Z}}$ then $C_r^*(\mathbb{F}_\infty)$ is the C^*-algebra generated by the left regular representation $\lambda : \mathbb{F}_\infty \to B(l^2(\mathbb{F}_\infty))$. If $\alpha : \mathbb{Z} \to \mathbb{Z}$ is any bijective function then there is a natural automorphism of $C_r^*(\mathbb{F}_\infty)$, also denoted by α, such that $\lambda(g_i) \mapsto \lambda(g_{\alpha(i)})$. The free shift on $C_r^*(\mathbb{F}_\infty)$ is induced by the mapping $i \mapsto i + 1$.

Corollary 4.9. *Let $\alpha \in \operatorname{Aut}(C_r^*(\mathbb{F}_\infty))$ be induced by a bijective function $\alpha : \mathbb{Z} \to \mathbb{Z}$. Then $ht(\alpha) = 0$.*

Proof. By monotonicity of $ht(\cdot)$ (i.e., the fact that topological entropy decreases in invariant subalgebras; cf. [**Br**, Prop. 2.1]) it suffices to provide an embedding of $C_r^*(\mathbb{F}_\infty)$ into $\mathcal{O}_\infty$ such that α lifts to an automorphism of $\mathcal{O}_\infty$ of the type considered in Theorem 4.6. That such an embedding exists is known to the experts so we only sketch the proof. (See also [**BD**].)

Let $\{S_i\}_{i\in\mathbb{Z}}$ generate $\mathcal{O}_\infty$ and $\varphi \in S(\mathcal{O}_\infty)$ be the vacuum state (cf. [**VDN**, Ex. 1.5.8]). The restriction of φ to $A_i = C^*(S_i)$ is denoted by φ_i. Let $B_i \subset A_i$ be the subalgebra generated by the identity and $b_i = (S_i^* + S_i)/2$. Then the distribution of b_i with respect to φ_i is the semicircular law, $\gamma_{0,1}$ (cf. [**VDN**, Def. 2.6.1]). Thus each B_i is isomorphic to $C([-1,1])$ and

$$\varphi_i(b_i^n) = \frac{2}{\pi} \int_{-1}^{1} t^n \sqrt{1-t^2} dt,$$

for all $n \in \mathbb{N}$. The unitary $v \in C([-1,1])$ defined by $v(t) = \exp(2i(\arcsin t + \sqrt{1-t^2}))$ satisfies

$$\frac{2}{\pi} \int_{-1}^{1} v^n(t) \sqrt{1-t^2} dt = 0,$$

for all nonzero $n \in \mathbb{Z}$. Hence each B_i contains a unitary u_i with $\varphi_i(u_i^n) = 0$ for all nonzero $n \in \mathbb{Z}$. Then $C^*(\{u_i : i \in \mathbb{Z}\})$ is isomorphic to $C_r^*(\mathbb{F}_\infty)$ and α lifts to an automorphism of $\mathcal{O}_\infty$ under this identification. $\qquad\square$

References

[Av]   D. Avitzour, *Free products of C^*-algebras*, Trans. Amer. Math. Soc., **271** (1982), 423-435.

[BD]   E. Blanchard and K. Dykema, *Embeddings of reduced free products of operator algebras*, preprint.

[Br]   N.P. Brown, *Topological entropy in exact C^*-algebras*, Math. Ann., **314** (1999), 347-367.

[Ch1]   M. Choda, *Reduced free products of completely positive maps and entropy for free products of automorphisms*, Publ. RIMS, Kyoto Univ., **32** (1996), 371-382.

[Ch2]   _____, *Entropy of Cuntz's canonical endomorphism*, Pac. J. Math., **190** (1999), 235-245.

[Ch3]   _____, *A C^*-dynamical entropy and applications to canonical endomorphisms*, J. Funct. Anal., **172** (2000), to appear.

[CNT]   A. Connes, H. Narnhofer and W. Thirring, *Dynamical entropy of C^*-algebras and von Neumann algebras*, Comm. Math. Phys., **112** (1987), 691-719.

[CS]   A. Connes and E. Størmer, *Entropy of II_1 von Neumann algebras*, Acta Math., **134** (1975), 289-306.

[Cu]   J. Cuntz, *Simple C^*-algebras generated by isometries*, Comm. Math. Phys., **57** (1977), 173-185.

[De]   V. Deaconu, *Entropy estimates for some C^*-endomorphisms*, Proc. Amer. Math. Soc., **127** (1999), 3653-3658.

[Dy]   K. Dykema, *Topological entropy for some automorphisms of reduced amalgamated free product C^*-algebras*, preprint.

[Hu]   T. Hudetz, *Topological entropy for appropriately approximated C^*-algebras*, J. Math. Phys., **35** (1994), 4303-4333.

[OP]   M. Ohya and D. Petz, *Quantum Entropy and its Use*, Texts and Monographs in Physics, Springer-Verlag, Berlin, 1993.

[Pa]   V. Paulsen, *Completely Bounded Maps and Dilations*, Pitman Research Notes in Mathematics, Vol. 146, Longman, 1986.

[Pe]   G. Pedersen, *C^*-algebras and Their Automorphism Groups*, Academic Press, London, 1979.

[SS]   A.M. Sinclair and R.R. Smith, *The completely bounded approximation property for discrete crossed products*, Indiana Univ. Math. J., **46** (1997), 1311-1322.

[St1]   E. Størmer, *Entropy of some automorphisms of the II_1 factor of the free group in infinite number of generators*, Invent. Math., **110** (1992), 63-73.

[St2]   E. Størmer, *Entropy in operator algebras*, Asterisque, **232** (1995), 211-230.

[St3]   E. Størmer, *States and shifts on infinite free products of C^*-algebras*, Fields Inst. Comm., **12** (1997), 281-291.

[Th]   K. Thomsen, *Topological entropy for endomorphisms of local C^*-algebras*, Comm. Math. Phys., **164** (1994), 181-193.

[Vo1]   D. Voiculescu, *Symmetries of Some Reduced Free Product C^*-algebras*, Operator Algebras and their connections with Topology and Ergodic Theory, Lecture Notes in Math., Vol. 1132, Springer-Verlag, 1985, 556-588.

[Vo2]   _____, *Dynamical approximation entropies and topological entropy in operator algebras*, Comm. Math. Phys., **170** (1995), 249-281.

[VDN]   D. Voiculescu, K. Dykema and A. Nica, *Free Random Variables*, CRM Monograph Series, Amer. Math. Soc., 1992.

[Wa]   S. Wassermann, *Exact C*-algebras and Related Topics*, Lecture Notes Series, **19**,
        GARC, Seoul National University, 1994.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF CALIFORNIA
BERKELEY, CA 94720
E-mail address: nbrown@math.berkeley.edu

DEPARTMENT OF MATHEMATICS
OSAKA KYOIKU UNIVERSITY
KASHIWARA 582-8582
JAPAN
E-mail address: marie@cc.osaka-kyoiku.ac.jp

GEOMETRIC PROPERTIES OF JULIA SETS OF THE COMPOSITION OF POLYNOMIALS OF THE FORM $z^2 + c_n$

Rainer Brück

For a sequence (c_n) of complex numbers we consider the quadratic polynomials $f_{c_n}(z) := z^2 + c_n$ and the sequence (F_n) of iterates $F_n := f_{c_n} \circ \cdots \circ f_{c_1}$. The Fatou set $\mathcal{F}_{(c_n)}$ is by definition the set of all $z \in \widehat{\mathbb{C}}$ such that (F_n) is normal in some neighbourhood of z, while the complement of $\mathcal{F}_{(c_n)}$ is called the Julia set $\mathcal{J}_{(c_n)}$. The aim of this article is to study geometric properties, Lebesgue measure and Hausdorff dimension of the Julia set $\mathcal{J}_{(c_n)}$ provided that the sequence (c_n) is bounded.

1. Introduction.

For a sequence (c_n) of complex numbers we consider the quadratic polynomials $f_{c_n}(z) := z^2 + c_n$ and the sequence (F_n) of iterates $F_n := f_{c_n} \circ \cdots \circ f_{c_1}$. (Note that F_n depends on $c_1, \ldots, c_n$ which we do not indicate explicitly in the notation.) If $c_n = c$ for all n, we write f_c^n instead of F_n. The *Fatou set* $\mathcal{F}_{(c_n)}$ is by definition the set of all $z \in \widehat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ such that (F_n) is normal (in the sense of Montel) in some neighbourhood of z, while the complement of $\mathcal{F}_{(c_n)}$ (in $\widehat{\mathbb{C}}$) is called the *Julia set* $\mathcal{J}_{(c_n)}$. A component of the Fatou set is called a *stable domain*. For iteration theory of a fixed function we refer the reader to the books of Beardon [**Be**], Carleson and Gamelin [**CG**], Milnor [**M**] or Steinmetz [**St**]. We also mention the survey articles of Blanchard [**Bl**], Lyubich [**L2**] or Eremenko and Lyubich [**EL**].

We always assume that $|c_n| \leq \delta$ for some $\delta > 0$. Then from [**Bü2**] it is known that to some extent the sequence (F_n) behaves similar to the sequence (f_c^n). There exists a stable domain $\mathcal{A}_{(c_n)}(\infty)$ which contains the point ∞ and wherein $F_n \to \infty$ as $n \to \infty$ locally uniformly. This domain need not be *invariant* (i.e., $f_{c_k}(\mathcal{A}_{(c_n)}(\infty)) \subset \mathcal{A}_{(c_n)}(\infty)$ for all k) or *backward invariant* (i.e., $f_{c_k}^{-1}(\mathcal{A}_{(c_n)}(\infty)) \subset \mathcal{A}_{(c_n)}(\infty)$ for all k), but there exists an invariant domain $M = M_\delta \subset \mathcal{A}_{(c_n)}(\infty)$ which contains the point ∞ and which satisfies $\mathcal{A}_{(c_n)}(\infty) = \{ z \in \widehat{\mathbb{C}} : F_k(z) \in M$ for some $k \in \mathbb{N} \}$. Therefore, the *filled Julia set* $\mathcal{K}_{(c_n)} := \widehat{\mathbb{C}} \setminus \mathcal{A}_{(c_n)}(\infty)$ and the Julia set $\mathcal{J}_{(c_n)}$ are compact in $\mathbb{C}$, and $\mathcal{K}_{(c_n)}$ is the set of all $z \in \mathbb{C}$ such that $(F_k(z))_{k=1}^{\infty}$ is bounded. Furthermore, we have $\mathcal{J}_{(c_n)} = \partial \mathcal{A}_{(c_n)}(\infty) = \partial \mathcal{K}_{(c_n)}$. Also $\mathcal{J}_{(c_n)}$ and $\mathcal{K}_{(c_n)}$ are perfect sets.

Finally, $\mathcal{J}_{(c_n)}$ and $\mathcal{F}_{(c_n)}$ are invariant in the sense that $F_k^{-1}(F_k(\mathcal{J}_{(c_n)})) = \mathcal{J}_{(c_n)}$ and $F_k^{-1}(F_k(\mathcal{F}_{(c_n)})) = \mathcal{F}_{(c_n)}$ for all $k \in \mathbb{N}$. For further results we also refer to [**Brü**], [**BBR**], [**Bü1**] and [**FS**].

The *Mandelbrot set* $\mathcal{M}$ is defined as the set of all $c \in \mathbb{C}$ such that $(f_c^n(0))_{n=1}^{\infty}$ is bounded, and $\mathcal{M}$ is compact in $\mathbb{C}$. It plays an important role in iteration of a fixed quadratic polynomial f_c. We recall that the largest disk with center 0 which is contained in $\mathcal{M}$ has radius $\frac{1}{4}$.

The plan of this article is as follows. After introducing some notations and known auxiliary results (Section 2) we show that the Julia set $\mathcal{J}_{(c_n)}$ is always uniformly perfect (Section 3).

Our main result (Section 4) states that the Julia set $\mathcal{J}_{(c_n)}$ is a quasicircle provided that $|c_n| \leq \delta$ for some $\delta < \frac{1}{4}$. This is done by proving that $\mathcal{F}_{(c_n)}$ consists of two simply connected John domains $\mathcal{A}_{(c_n)}(0)$ and $\mathcal{A}_{(c_n)}(\infty)$ which have $\mathcal{J}_{(c_n)}$ as their common boundaries.

Concerning the two-dimensional Lebesgue measure $m_2(\mathcal{J}_{(c_n)})$ of Julia sets (Section 5) we show that it is almost surely zero provided that the c_n are randomly chosen in $\{ z \in \mathbb{C} : |z| \leq \delta \}$ for some $\delta > \frac{1}{4}$. For $\delta < \frac{1}{4}$ we always have $m_2(\mathcal{J}_{(c_n)}) = 0$.

Section 6 deals with Hausdorff dimension $\dim_{\mathrm{H}} \mathcal{J}_{(c_n)}$ of Julia sets. We give a lower estimate for $\dim_{\mathrm{H}} \mathcal{J}_{(c_n)}$ depending only on δ which implies that $\dim_{\mathrm{H}} \mathcal{J}_{(c_n)}$ is always positive. For that purpose we prove that the Green function of $\mathcal{A}_{(c_n)}(\infty)$ (which is known to exist) is Hölder continuous. Furthermore, for $\delta < \frac{1}{4}$ it follows that $\dim_{\mathrm{H}} \mathcal{J}_{(c_n)} < 2$.

A point $\zeta \in \mathbb{C}$ is called a repelling fixpoint of the sequence of iterates (F_n) if $F_k(\zeta) = \zeta$ for some $k \in \mathbb{N}$ and $|F_k'(\zeta)| > 1$. The set of all those points is denoted by $\mathcal{R}_{(c_n)}$. In this general setting it is not necessarily true that $\mathcal{R}_{(c_n)} \subset \mathcal{J}_{(c_n)}$. But we prove (Section 7) that if $|c_n| \leq \delta < \frac{1}{4}$, then the derived set of $\mathcal{R}_{(c_n)}$ coincides with $\mathcal{J}_{(c_n)}$. In the last section we investigate the asymptotic distribution of certain predecessors.

2. Notations and auxiliary results.

We introduce a few further notations and collect some known auxiliary results that are frequently used in the sequel. If $E \subset \mathbb{C}$, then E' denotes the derived set (that is the set of points $z \in \mathbb{C}$ such that every neighbourhood of z contains a point $w \in E \setminus \{z\}$), $\overline{E}$ the closure and E° the set of interior points of E. Furthermore, the diameter of E is defined by $\operatorname{diam} E := \sup \{ |z - w| : z, w \in E \}$, and the distance of a point $z \in \mathbb{C}$ from E by $\operatorname{dist}(z, E) := \inf \{ |z - w| : w \in E \}$. For $a \in \mathbb{C}$ and $r > 0$ we set $D_r(a) := \{ z \in \mathbb{C} : |z - a| < r \}$, $D_r := D_r(0)$, $\mathbb{D} := D_1$ and $K_r := \overline{D}_r$. Finally, for $R > 0$ let $\Delta_R := \{ z \in \widehat{\mathbb{C}} : |z| > R \}$.

If $(c_n) \in K_\delta^{\mathbb{N}}$, then the invariant domain $M \subset \mathcal{A}_{(c_n)}(\infty)$ may be chosen as $M = \Delta_R$ for any

$$R \geq R_\delta := \tfrac{1}{2}\big(1 + \sqrt{1 + 4\delta}\big).$$

More precisely, if $R > R_\delta$, then $f_c(\Delta_{R_\delta}) \subset \Delta_{R_\delta}$ and $f_c(\overline{\Delta}_R) \subset \Delta_R$ for all $c \in K_\delta$. This implies that $\mathcal{K}_{(c_n)} \subset K_{R_\delta}$. If $\delta \leq \tfrac{1}{4}$, we set

$$r_\delta := \tfrac{1}{2}\big(1 + \sqrt{1 - 4\delta}\big) \in \big[\tfrac{1}{2}, 1\big], \qquad s_\delta := \tfrac{1}{2}\big(1 - \sqrt{1 - 4\delta}\big) \in \big[0, \tfrac{1}{2}\big].$$

Then we have $f_c(D_{s_\delta}) \subset D_{s_\delta}$, $f_c(D_{r_\delta}) \subset D_{r_\delta}$ and $f_c(\overline{D}_r) \subset D_r$ for all $c \in K_\delta$ and all $r \in (s_\delta, r_\delta)$. This implies that there exists a stable domain $\mathcal{A}_{(c_n)}(0) \supset D_{r_\delta}$, and there holds $\mathcal{J}_{(c_n)} \subset K_{R_\delta} \cap \overline{\Delta}_{r_\delta}$.

From [**FS**, Theorem 2.1] it follows that $\mathcal{A}_{(c_n)}(\infty)$ is regular for logarithmic potential theory which means that the Green function of $\mathcal{A}_{(c_n)}(\infty)$ with pole at infinity exists. More precisely, the function $g_{(c_n)}$ defined by

$$(2.1) \qquad\qquad g_{(c_n)}(z) := \lim_{k \to \infty} \frac{1}{2^k} \log^+ |F_k(z)|$$

is continuous in $\mathbb{C}$, $g_{(c_n)}(z) = 0$ for $z \in \mathcal{K}_{(c_n)}$, and it is the Green function of $\mathcal{A}_{(c_n)}(\infty)$ with pole at infinity.

Furthermore, we introduce the *critical set* (or set of critical points)

$$\mathcal{C}_{(c_n)} := \{\, z \in \mathbb{C} : F_j'(z) = 0 \text{ for some } j \in \mathbb{N}_0 \,\}$$

of (F_n), where $F_0(z) := z$. This is motivated by the fact that

$$F_k'(z) = 2^k \prod_{j=0}^{k-1} F_j(z)$$

so that $F_k'(z) = 0$ if and only if $F_j(z) = 0$ for some $j \in \{0, 1, \ldots, k-1\}$. We call a point $w \in \mathbb{C}$ a *critical value* of (F_n), if $w = F_k(z)$ and $F_k'(z) = 0$ for some $k \in \mathbb{N}$ and some $z \in \mathbb{C}$. If $w \in \mathbb{C}$ is not a critical value of F_k, then in some sufficiently small disk $D_\varepsilon(w)$ there exist 2^k analytic branches of the inverse function of F_k.

Finally, we recall a result of Büger [**Bü1**] that the Julia set $\mathcal{J}_{(c_n)}$ is self-similar. This means that for any open set D meeting $\mathcal{J}_{(c_n)}$ there exists $k_0 \in \mathbb{N}$ such that $F_k(\mathcal{J}_{(c_n)} \cap D) = F_k(\mathcal{J}_{(c_n)})$ for all $k \geq k_0$.

3. Uniform perfectness of Julia sets.

An open set $A \subset \widehat{\mathbb{C}}$ is called a *conformal annulus*, if it can be mapped conformally onto an annulus $\{\, z \in \mathbb{C} : 1 < |z| < \varrho \,\}$ for some $\varrho > 1$. Then the number ϱ is uniquely determined and $\operatorname{mod} A := \frac{1}{2\pi} \log \varrho$ is called the *modulus* of A. Now, let $E \subset \widehat{\mathbb{C}}$ be a compact set. A conformal annulus A *seperates* E, if both components of $\widehat{\mathbb{C}} \setminus A$ meet E. The set E is called *uniformly perfect*, if it is not a single point and if there is a constant $\alpha > 0$ such that for any

conformal annulus A which seperates E there holds mod $A \leq \alpha$. Obviously, a uniformly perfect set is also perfect (that is $E' = E$), and every connected compact set with at least two points is uniformly perfect. Uniformly perfect sets were introduced by Beardon and Pommerenke [**BeP**] (see also [**P1**]). It is known that the Julia set of a fixed rational function is always uniformly perfect [**MR**] (see also [**CG**, p. 64]). We show that this result extends to our situation.

Theorem 3.1. *Let* $\delta > 0$ *and* $(c_n) \in K_\delta^{\mathbb{N}}$. *Then the Julia set* $\mathcal{J}_{(c_n)}$ *is uniformly perfect.*

Proof. We assume that $\mathcal{J}_{(c_n)}$ is not uniformly perfect. Then there exists a sequence of conformal annuli $A_k \subset \mathcal{F}_{(c_n)}$ which seperate $\mathcal{J}_{(c_n)}$ and mod $A_k \to \infty$ as $k \to \infty$. Let E_k be the component of $\widehat{\mathbb{C}} \setminus A_k$ with the smaller chordal diameter (which we denote by $\operatorname{diam}_\chi E_k$). Then we have $\operatorname{diam}_\chi E_k \to 0$ as $k \to \infty$. If $\lambda_k \colon \mathbb{D} \to A_k \cup E_k$ is a conformal map of $\mathbb{D}$ onto $A_k \cup E_k$ with $\lambda_k(0) \in E_k$, and if $M_k := \lambda_k^{-1}(E_k) \subset \mathbb{D}$, then M_k is compact and connected, $0 \in M_k$ and $\operatorname{diam}_\chi M_k \to 0$ as $k \to \infty$.

It is elementary to see that (f_{c_n}) satisfies a uniform Lipschitz condition with respect to the chordal metric χ, that means that there exists a constant $L > 0$ (which depends only on δ but not on n) such that $\chi(f_{c_n}(z), f_{c_n}(w)) \leq L\chi(z, w)$ for all $z, w \in \widehat{\mathbb{C}}$ and all $n \in \mathbb{N}$. From Lemma 4.1 in [**BBR**] we know that $\operatorname{diam} F_k(\mathcal{J}_{(c_n)}) \geq 1$ for all $k \in \mathbb{N}_0$ so that $\operatorname{diam}_\chi F_k(\mathcal{J}_{(c_n)}) \geq C := 2(1 + R_\delta^2)^{-1}$.

We choose $\varepsilon > 0$ with $\varepsilon < C$ and

$$\text{(3.1)} \qquad\qquad\qquad \frac{C}{3} > L\varepsilon.$$

Let $k_0 \in \mathbb{N}$ such that $\operatorname{diam}_\chi E_k < \varepsilon$ for all $k \geq k_0$. Since $(A_k \cup E_k) \cap \mathcal{J}_{(c_n)} \neq \varnothing$ and since $\mathcal{J}_{(c_n)}$ is self-similar (cf. [**Bü1**]), for every $k \geq k_0$ there exists a smallest index $m(k) \in \mathbb{N}$ such that $\operatorname{diam}_\chi F_{m(k)}(E_k) > \varepsilon$. Setting $G_k := F_{m(k)} \circ \lambda_k$ we obtain

$$\text{(3.2)} \qquad\qquad\qquad \operatorname{diam}_\chi G_k(M_k) > \varepsilon$$

for all $k \geq k_0$. By the choice of $m(k)$ we have $\operatorname{diam}_\chi F_{m(k)-1}(E_k) \leq \varepsilon$ and thus $\operatorname{diam} F_{m(k)}(E_k) = \operatorname{diam}_\chi f_{c_{m(k)}}(F_{m(k)-1}(E_k)) \leq L\varepsilon$ for all $k \geq k_0$.

Because of (3.1) there exist at least three different points $a_{1,k}, a_{2,k}, a_{3,k} \in F_{m(k)}(\mathcal{J}_{(c_n)})$ whose chordal distance is greater than $L\varepsilon$. We have $G_k(\mathbb{D} \setminus M_k) = F_{m(k)}(\lambda_k(\mathbb{D} \setminus M_k)) = F_{m(k)}(A_k) \subset F_{m(k)}(\mathcal{F}_{(c_n)})$ and $\operatorname{diam}_\chi G_k(M_k) = \operatorname{diam}_\chi F_{m(k)}(E_k) \leq L\varepsilon$ for all $k \geq k_0$. This implies that G_k omits at least two of the values $a_{1,k}, a_{2,k}, a_{3,k}$ in $\mathbb{D}$ and hence (G_k) is normal in $\mathbb{D}$ by a generalized version of Montel's theorem (cf. [**Be**, p. 57]). Since $\operatorname{diam}_\chi M_k \to 0$ as $k \to \infty$ and $0 \in M_k$ we get $\operatorname{diam}_\chi G_k(M_k) \to 0$ as $k \to \infty$ which contradicts (3.2). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \square$

4. Julia sets and quasicircles.

From iteration theory of a fixed function it is known that $\mathcal{J}(f_c)$ is a quasicircle if c is in the interior of the main cardioid of the Mandelbrot set (cf. Yakobson [**Y**], see also [**CG**, p. 103]). The goal of this section is to show that this result remains valid in our general situation provided that $\delta < \frac{1}{4}$. We do this in several steps, and we first recall some facts on quasicircles and John domains.

A *quasicircle* $\Gamma \subset \mathbb{C}$ is the image of the unit circle $\partial \mathbb{D}$ under a quasiconformal homeomorphism of $\mathbb{C}$ onto itself. An equivalent geometric definition is the three-point property, i.e., there exists a constant $a > 0$ such that if z_1, z_2, $z_3 \in \Gamma$ and z_2 is on the arc between z_1 and z_3 with the smaller diameter, then $|z_1 - z_2| + |z_2 - z_3| \le a|z_1 - z_3|$. A quasicircle may be non-rectifiable but it has no cusps. For details we refer, for example, to the books of Ahlfors [**A**] or Lehto and Virtanen [**LV**].

A domain $G \subset \widehat{\mathbb{C}}$ with $\partial G \subset \mathbb{C}$ is called a *John domain*, if there exists a constant $b > 0$ and a point $w_0 \in G$ such that for any $z_0 \in G$, there is an arc $\gamma = \gamma(z_0) \subset G$ joining z_0 and w_0 and satisfying $\operatorname{dist}(z, \partial G) \ge b|z - z_0|$ for all $z \in \gamma$. A simply connected John domain G has locally connected boundary ∂G so that by Carathéodory's theorem (cf. [**P2**, p. 20]) the Riemann map from $\mathbb{D}$ onto G extends continuously to $\overline{\mathbb{D}}$. The image of a John domain under a quasiconformal homeomorphism of $\widehat{\mathbb{C}}$ onto itself is again a John domain. Thus, the two complementary domains of a quasicircle are John domains. Conversely, if the two complementary components of a Jordan curve (a homeomorphic image of the unit circle) Γ are John domains, then Γ is a quasicircle. For this and further background material we refer to [**NV**].

For $\delta \le \frac{1}{4}$ we know that $\mathcal{J}_{(c_n)}$ is connected [**BBR**], and since $\mathcal{J}_{(c_n)} = \partial \mathcal{A}_{(c_n)}(\infty)$, the stable domain $\mathcal{A}_{(c_n)}(\infty)$ is simply connected. Furthermore, there exists a stable domain $\mathcal{A}_{(c_n)}(0)$ containing D_{r_δ}. We now show:

Theorem 4.1. *Let* $\delta \le \frac{1}{4}$, $(c_n) \in K_\delta^{\mathbb{N}}$ *and* $s_\delta \le r \le r_\delta$. *Then there holds* $\mathcal{A}_{(c_n)}(0) = \bigcup_{k=0}^{\infty} F_k^{-1}(D_r)$ *and* $\partial \mathcal{A}_{(c_n)}(0) = \mathcal{J}_{(c_n)}$. *In particular,* $\mathcal{A}_{(c_n)}(0)$ *is simply connected and* $\mathcal{F}_{(c_n)} = \mathcal{A}_{(c_n)}(0) \cup \mathcal{A}_{(c_n)}(\infty)$.

Proof. We set $A := \bigcup_{k=0}^{\infty} U_k$ with $U_k := F_k^{-1}(D_r)$. It is elementary to see that each U_k is a domain containing D_r, and since D_r is invariant, we get $U_k \subset \mathcal{F}_{(c_n)}$. Thus, A is a domain with $D_r \subset A \subset \mathcal{F}_{(c_n)}$ which gives $A \subset \mathcal{A}_{(c_n)}(0)$.

We show that $\mathcal{J}_{(c_n)} \subset \partial A$. For that purpose, let $z_0 \in \mathcal{J}_{(c_n)}$ and $D := D_\varepsilon(z_0)$ for $\varepsilon > 0$. By Montel's theorem the set $\widehat{\mathbb{C}} \setminus \bigcup_{k=0}^{\infty} F_k(D)$ contains at most two points so that there exists $w \in D_r$ such that $w \in F_m(D)$ for some $m \in \mathbb{N}_0$. Therefore, $D_r \cap F_m(D)$ is a non-empty open set, and this

implies that there exists $\zeta \in D \setminus \mathcal{J}_{(c_n)}$ with $F_m(\zeta) \in D_r$. That means $\zeta \in A$, and since $\varepsilon > 0$ was arbitrary we arrive at $z_0 \in \partial A$. Summarizing, we have $A \subset \mathcal{A}_{(c_n)}(0)$ and $\partial \mathcal{A}_{(c_n)}(0) \subset \mathcal{J}_{(c_n)} \subset \partial A$ which gives the assertion. $\qquad\square$

For $\delta < \frac{1}{4}$ and $\frac{1}{2} < r < r_\delta$ we set $V := \Delta_r \supset \mathcal{J}_{(c_n)}$. Then V is backward invariant, and V does not contain any critical value of (F_n) so that in every disk $D \subset V$ there exist 2^n analytic branches F_n^{-1} of the inverse function of F_n. We prove:

Lemma 4.2. *Let $\delta < \frac{1}{4}$, $(c_n) \in K_\delta^{\mathbb{N}}$ and $\frac{1}{2} < r < r_\delta$. Furthermore, let $\gamma \colon [0,1] \to V$ be a rectifiable curve in $V := \Delta_r$, $z := \gamma(0)$, $w := \gamma(1)$ and let F_n^{-1} be an analytic branch of the inverse function of F_n in some disk $D \subset V$ with center z. Finally, we denote the analytic continuation of F_n^{-1} along γ also by F_n^{-1}. Then there holds*

$$\left| \frac{(F_n^{-1})'(z)}{(F_n^{-1})'(w)} \right| \leq 1 + \alpha \ell(\gamma) e^{\alpha \ell(\gamma)},$$

where $\alpha := 4r(2r-1)^{-1}$ and $\ell(\gamma)$ denotes the length of γ. In particular, for any disk $D \subset V$ and any analytic branch F_n^{-1} in D there holds

$$\left| \frac{(F_n^{-1})'(z)}{(F_n^{-1})'(w)} \right| \leq 1 + \alpha e^{\alpha d} |z - w|$$

for all z, $w \in D$ and $n \in \mathbb{N}$, where $d := \operatorname{diam} D$.

Proof. For $n \in \mathbb{N}$ and $k = 0, 1, \dots, n-1$ we set $F_{n,k} := f_{c_n} \circ \cdots \circ f_{c_{k+1}}$. Since

$$(F_n^{-1})'(z) = \frac{1}{F_n'(F_n^{-1}(z))} = \frac{1}{2^n \displaystyle\prod_{j=0}^{n-1} F_j(F_n^{-1}(z))} = \frac{1}{2^n \displaystyle\prod_{j=0}^{n-1} F_{n,j}^{-1}(z)}$$

and V is backward invariant we have

$$|(F_n^{-1})'(z)| \leq q^n \qquad (z \in V),$$

or

$$|(F_{n,k}^{-1})'(z)| \leq q^{n-k} \qquad (z \in V),$$

where $q := \frac{1}{2r} < 1$. This implies

$$(4.1) \qquad |F_{n,k}^{-1}(w) - F_{n,k}^{-1}(z)| \leq \left| \int_z^w |(F_{n,k}^{-1})'(\zeta)| |d\zeta| \right| \leq q^{n-k} \ell(\gamma),$$

where we integrate over the curve γ. Furthermore, we have

$$\frac{(F_n^{-1})'(z)}{(F_n^{-1})'(w)} = \prod_{k=0}^{n-1} \frac{F_k(F_n^{-1}(w))}{F_k(F_n^{-1}(z))} = \prod_{k=0}^{n-1} \frac{F_{n,k}^{-1}(w)}{F_{n,k}^{-1}(z)}.$$

Writing

$$\frac{F_{n,k}^{-1}(w)}{F_{n,k}^{-1}(z)} = 1 + \frac{F_{n,k}^{-1}(w) - F_{n,k}^{-1}(z)}{F_{n,k}^{-1}(z)},$$

we obtain from (4.1)

$$\left| \frac{F_{n,k}^{-1}(w)}{F_{n,k}^{-1}(z)} \right| \le 1 + 2q^{n-k+1}\ell(\gamma).$$

This implies

$$\left| \frac{(F_n^{-1})'(z)}{(F_n^{-1})'(w)} \right| \le \prod_{k=0}^{n-1}(1 + 2q^{n-k+1}\ell(\gamma)) = \prod_{k=2}^{n+1}(1 + 2q^k\ell(\gamma))$$

$$\le \prod_{k=0}^{\infty}(1 + 2q^k\ell(\gamma)) = \exp\left(\sum_{k=0}^{\infty}\log\left(1 + 2q^k\ell(\gamma)\right)\right)$$

$$\le \exp\left(\sum_{k=0}^{\infty}2q^k\ell(\gamma)\right) = e^{\alpha\ell(\gamma)},$$

where $\alpha := 2(1-q)^{-1}$. Finally, this gives the assertion since $e^x \le 1 + xe^x$ for $x \ge 0$. $\qquad\square$

Theorem 4.3. *Let $\delta < \frac{1}{4}$ and $(c_n) \in K_\delta^{\mathbb{N}}$. Then $\mathcal{A}_{(c_n)}(\infty)$ is a John domain.*

Proof. We first introduce a few notations. For z_1, $z_2 \in \mathbb{C}$ let $[z_1, z_2]$ denote the line segment joining z_1 and z_2. If $\zeta \in \mathbb{C}$, $\zeta \ne 0$, and if Γ is the ray from 0 to ∞ passing through ζ, then let Γ_ζ denote that part of Γ from ζ to ∞.

Let $R > R_\delta$ such that $R^2 + \delta - R \le \frac{1}{2}$, $\varepsilon := R - R_\delta \le 1$ and $U_k := F_k^{-1}(\Delta_R)$ for $k \in \mathbb{N}$. Then we have $U_k \subset U_{k+1} \subset \mathcal{A}_{(c_n)}(\infty)$ and $\mathcal{A}_{(c_n)}(\infty) = \bigcup_{k=1}^{\infty} U_k$. Furthermore, U_k is a simply connected domain (in $\widehat{\mathbb{C}}$) bounded by an analytic Jordan curve. For $z \in \mathcal{A}_{(c_n)}(\infty)$ let $d(z) := \mathrm{dist}\,(z, \mathcal{J}_{(c_n)})$. We prove a lower estimate for $d(z)$, if $z \in U_k$ for some $k \in \mathbb{N}$. We set $w := F_k(z)$. If U denotes the component of $F_k^{-1}(D_\varepsilon(w))$ containing z, there holds $U \subset \mathcal{A}_{(c_n)}(\infty)$. Let $\varrho > 0$ such that $D_\varrho(z) \subset U$. If $z' \in D_\varrho(z)$ and $w' := F_k(z')$, then

$$w' - w = F_k(z') - F_k(z) = \int_z^{z'} F_k'(\zeta)\,d\zeta = F_k'(F_k^{-1}(w))\int_z^{z'} \frac{F_k'(\zeta)}{F_k'(F_k^{-1}(w))}\,d\zeta$$

$$= F_k'(z)\int_z^{z'} \frac{(F_k^{-1})'(w)}{(F_k^{-1})'(F_k(\zeta))}\,d\zeta,$$

where we integrate over the line segment $[z, z']$. By Lemma 4.2 we obtain

$$\left| \frac{(F_k^{-1})'(w)}{(F_k^{-1})'(F_k(\zeta))} \right| \le 1 + \alpha e^{\alpha \varepsilon} |w - F_k(\zeta)| \le 1 + \alpha e^{\alpha \varepsilon} \varepsilon \le 1 + \alpha e^{\alpha}$$

and thus

$$|w' - w| \le |F_k'(z)||z' - z|(1 + \alpha e^{\alpha}) \le |F_k'(z)|\varrho(1 + \alpha e^{\alpha}).$$

Setting

$$\varrho := \frac{\varepsilon}{|F_k'(z)|(1 + \alpha e^{\alpha})}$$

we obtain $D_\varrho(z) \subset U$ and thus

$$(4.2) \qquad d(z) \ge \frac{\varepsilon}{|F_k'(z)|(1 + \alpha e^{\alpha})} = \frac{\alpha_1}{|F_k'(z)|} \qquad (z \in U_k).$$

In order to prove the John property, let $w_0 := \infty$ and $z_0 \in \mathcal{A}_{(c_n)}(\infty)$. We may assume that $z_0 \in U_k \backslash U_{k-1}$ for some $k \in \mathbb{N}$. Then $R < |F_k(z_0)| \le R^2 + \delta$. We construct an arc in U_k joining z_0 and w_0 as follows. First, we join z_0 with ∂U_{k-1} by an arc $\gamma_k \subset U_k \setminus U_{k-1}$ such that $F_k(\gamma_k) \subset \Gamma_{F_k(z_0)}$, and we denote the endpoint of γ_k on ∂U_{k-1} by ζ_{k-1}. Then we join ζ_{k-1} with ∂U_{k-2} by an arc $\gamma_{k-1} \subset U_{k-1} \setminus U_{k-2}$ such that $F_{k-1}(\gamma_{k-1}) \subset \Gamma_{F_{k-1}(\zeta_{k-1})}$, and we denote the endpoint of γ_{k-1} on ∂U_{k-2} by ζ_{k-2}. Proceeding in this way we get an arc in $U_k \cap \overline{D}_R$ with endpoint ζ_0 on ∂D_R. Finally, we set $\gamma = \gamma(z_0) := \gamma_k \cup \cdots \cup \gamma_1 \cup \Gamma_{\zeta_0}$. We note that the line segments $F_j(\gamma_j)$ $(j = 1, \ldots, k)$ all lie in $\overline{\Delta}_R \cap \overline{D}_{R^2+\delta}$ and thus have lengths at most $\frac{1}{2}$.

We now show that the arc γ has the John property. For that purpose, let $z \in \gamma$. We may assume that $z \in D_R$. First, let $z \in U_k \setminus U_{k-1}$. We deduce an upper estimate for $|z - z_0|$. There holds

$$z - z_0 = F_k^{-1}(F_k(z)) - F_k^{-1}(F_k(z_0)) = \int_{F_k(z_0)}^{F_k(z)} (F_k^{-1})'(\zeta)\, d\zeta$$

$$= (F_k^{-1})'(F_k(z)) \int_{F_k(z_0)}^{F_k(z)} \frac{(F_k^{-1})'(\zeta)}{(F_k^{-1})'(F_k(z))}\, d\zeta,$$

where we integrate over the line segment $[F_k(z_0), F_k(z)]$. By Lemma 4.2 we obtain

$$\left| \frac{(F_k^{-1})'(\zeta)}{(F_k^{-1})'(F_k(z))} \right| \le 1 + \alpha e^{\alpha} |F_k(z) - \zeta| \le 1 + \alpha e^{\alpha} |F_k(z) - F_k(z_0)| \le 1 + \alpha e^{\alpha}$$

and thus

$$(4.3) \qquad |z - z_0| \le |(F_k^{-1})'(F_k(z))|(1 + \alpha e^{\alpha})|F_k(z) - F_k(z_0)|$$

$$\leq \frac{1 + \alpha e^{\alpha}}{|F_k'(z)|} = \frac{\alpha_2}{|F_k'(z)|} \qquad (z \in \gamma \setminus U_{k-1}).$$

Putting (4.2) and (4.3) together we arrive at

$$d(z) \geq \frac{\alpha_1}{\alpha_2}|z - z_0| = \alpha_3|z - z_0| \qquad (z \in \gamma \setminus U_{k-1}).$$

Now, let $z \in U_{k-m} \setminus U_{k-m-1}$ for some $m \in \{1, \ldots, k-1\}$. By (4.2) we have

$$d(z) \geq \frac{\alpha_1}{|F_{k-m}'(z)|}.$$

From the construction of γ and (4.3) we obtain

$$|z - z_0| \leq |z_0 - \zeta_{k-1}| + |\zeta_{k-1} - \zeta_{k-2}| + \cdots + |\zeta_{k-m+1} - \zeta_{k-m}| + |\zeta_{k-m} - z|$$

$$\leq \alpha_2 \left(\frac{1}{|F_k'(\zeta_{k-1})|} + \frac{1}{|F_{k-1}'(\zeta_{k-2})|} + \cdots \right.$$

$$\left. + \frac{1}{|F_{k-m+1}'(\zeta_{k-m})|} + \frac{1}{|F_{k-m}'(z)|} \right)$$

and thus

(4.4)
$$\frac{d(z)}{|z - z_0|} \geq \frac{\alpha_3}{1 + \sum_{j=1}^{m} \left| \dfrac{F_{k-m}'(z)}{F_{k-m+j}'(\zeta_{k-m+j-1})} \right|}.$$

In order to estimate the denominator of the right hand side we consider a single term

$$\frac{F_{k-m}'(z)}{F_{k-m+j}'(\zeta_{k-m+j-1})} = \frac{1}{2^j F_{k-m+j-1}(\zeta_{k-m+j-1}) \cdots F_{k-m}(\zeta_{k-m+j-1})}$$

$$\times \frac{F_{k-m}'(z)}{F_{k-m}'(\zeta_{k-m+j-1})}.$$

Because of $|F_{k-m+j-1}(\zeta_{k-m+j-1})| = R$ and the invariance of D_r we obtain

(4.5)
$$\left| \frac{F_{k-m}'(z)}{F_{k-m+j}'(\zeta_{k-m+j-1})} \right| \leq q^j \left| \frac{F_{k-m}'(z)}{F_{k-m}'(\zeta_{k-m+j-1})} \right|,$$

where $q := \frac{1}{2r} < 1$.

Now, we deduce an estimate of the right hand side of (4.5). For abbreviation we set $p := k - m$ and write

$$\frac{F_p'(z)}{F_p'(\zeta_{p+j-1})} = \frac{(F_p^{-1})'(F_p(\zeta_{p+j-1}))}{(F_p^{-1})'(F_p(z))}.$$

From Lemma 4.2 we get

$$(4.6) \qquad \left| \frac{F_p'(z)}{F_p'(\zeta_{p+j-1})} \right| \leq 1 + \alpha \ell(\sigma) e^{\alpha \ell(\sigma)},$$

where $\sigma = \sigma_{p,j}$ is the curve $F_p(\gamma_p' \cup \gamma_{p+1} \cup \cdots \cup \gamma_{p+j-1})$, and where γ_p' is that part of γ_p joining ζ_p with z. Hence, there holds $\ell(\sigma) \leq \ell(F_p(\gamma_p)) + \cdots + \ell(F_p(\gamma_{p+j-1}))$. We have $\ell(F_p(\gamma_p)) \leq \frac{1}{2}$ and $F_p(\gamma_{p+\nu}) = F_{p+\nu,p}^{-1}(s_{p,\nu})$, where $s_{p,\nu} := F_{p+\nu}(\gamma_{p+\nu})$ is a line segment on $\Gamma_{F_{p+\nu}(\zeta_{p+\nu})}$ of length at most $\frac{1}{2}$ for $\nu \geq 1$. Furthermore, we know that $F_p(\gamma_{p+\nu}) \subset \Delta_r$. Therefore, we obtain

$$\ell(F_p(\gamma_{p+1})) = \int_{s_{p,1}} \frac{|dw|}{2\sqrt{|w - c_{p+1}|}} \leq \frac{\ell(s_{p,1})}{2r} \leq \frac{1}{4r}.$$

By induction we get $\ell(F_p(\gamma_{p+\nu})) \leq \frac{1}{2(2r)^\nu} = \frac{1}{2}q^\nu$ and thus

$$\ell(\sigma) \leq \frac{1}{2}(1 + q + \cdots + q^{j-1}) \leq \frac{1}{2(1-q)} = \alpha_4.$$

Setting $\alpha_5 := 1 + \alpha \alpha_4 e^{\alpha \alpha_4}$ we obtain together with (4.4), (4.5) and (4.6)

$$\frac{d(z)}{|z - z_0|} \geq \frac{\alpha_3}{1 + \alpha_5 \sum_{j=1}^{m} q^j} \geq \frac{\alpha_3(1-q)}{\alpha_5}$$

which finally shows that γ has the John property. $\square$

Theorem 4.4. *Let $\delta < \frac{1}{4}$ and $(c_n) \in K_\delta^{\mathbb{N}}$. Then $\mathcal{A}_{(c_n)}(0)$ is a John domain.*

Proof. The proof is very similar to the proof of Theorem 4.3. The only difficulty that arises is that $\mathcal{A}_{(c_n)}(0)$ contains critical values which all lie in D_{s_δ}. Therefore, we only give a sketch and omit the details.

Let $\frac{1}{2} < r < r' < r_\delta$, $\varepsilon := r' - r \leq 1$ and $U_k := F_k^{-1}(D_{r'})$ for $k \in \mathbb{N}$. Then we have $U_k \subset U_{k+1} \subset \mathcal{A}_{(c_n)}(0)$ and $\mathcal{A}_{(c_n)}(0) = \bigcup_{k=1}^{\infty} U_k$. For $z \in \mathcal{A}_{(c_n)}(0)$ let $d(z) := \mathrm{dist}(z, \mathcal{J}_{(c_n)})$. If $z \in U_k \setminus U_{k-1}$ for some $k \in \mathbb{N}$, $k \geq 2$ and $w := F_{k-1}(z)$, then $|w| \geq r'$ and thus $D_\varepsilon(w) \cap D_r = \varnothing$. Therefore, we obtain

$$(4.2a) \qquad d(z) \geq \frac{\alpha_1}{|F_{k-1}'(z)|} \qquad (z \in U_k \setminus U_{k-1}).$$

In order to prove the John property, let $w_0 := 0$ and $z_0 \in \mathcal{A}_{(c_n)}(0)$. We may assume that $z_0 \in U_k \setminus U_{k-1}$ for some $k \in \mathbb{N}$, $k \geq 2$. Then $|F_{k-1}(z_0)| \geq r'$. We construct an arc in U_k joining z_0 and w_0 as follows. First, we join z_0 with ∂U_{k-1} by an arc $\gamma_k \subset U_k \setminus U_{k-1}$ such that $F_{k-1}(\gamma_k) \subset [0, F_{k-1}(z_0)]$, and we denote the endpoint of γ_k on ∂U_{k-1} by ζ_{k-1}. Then we join ζ_{k-1} with ∂U_{k-2} by an arc $\gamma_{k-1} \subset U_{k-1} \setminus U_{k-2}$ such that $F_{k-2}(\gamma_{k-1}) \subset [0, F_{k-2}(\zeta_{k-1})]$, and we denote the endpoint of γ_{k-1} on ∂U_{k-2} by ζ_{k-2}. Proceeding in this way we get an arc in $U_k \cap (\mathbb{C} \setminus U_1)$ with endpoint ζ_1 on ∂U_1. Finally, we set $\gamma = \gamma(z_0) := \gamma_k \cup \cdots \cup \gamma_2 \cup [0, \zeta_1]$. We mention that $[0, \zeta_1] \subset \overline{U}_1$, since U_1

is a starlike domain with respect to 0 bounded by an analytic Jordan curve. Furthermore, we note that the line segments $F_{j-1}(\gamma_j)$ ($j = 2, \ldots, k$) all lie in $\overline{\Delta}_{r'} \cap \overline{D}_{R_\delta}$ and thus have lengths at most one.

We now show that the arc γ has the John property. For that purpose, let $z \in \gamma$. We may assume that $z \notin U_1$. First, let $z \in U_k \setminus U_{k-1}$. Then we obtain the upper estimate for $|z - z_0|$

(4.3a) $$|z - z_0| \leq \frac{\alpha_2}{|F'_{k-1}(z)|} \qquad (z \in \gamma \setminus U_{k-1}).$$

Putting (4.2a) and (4.3a) together we arrive at

$$d(z) \geq \alpha_3 |z - z_0| \qquad (z \in \gamma \setminus U_{k-1}).$$

Finally, the case that $z \in U_{k-m} \setminus U_{k-m-1}$ for some $m \in \{1, \ldots, k-2\}$ is handled as in the proof of Theorem 4.3. $\qquad\square$

Corollary 4.5. *Let $\delta < \frac{1}{4}$ and $(c_n) \in K_\delta^{\mathbb{N}}$. Then $\mathcal{J}_{(c_n)}$ is a quasicircle.*

Proof. From Theorem 4.1 we know that $\mathcal{F}_{(c_n)} = \mathcal{A}_{(c_n)}(\infty) \cup \mathcal{A}_{(c_n)}(0)$. Then the assertion follows from Theorems 4.3 and 4.4 and the known results mentioned at the beginning of this section. $\qquad\square$

If $(c_n) \in K_{1/4}^{\mathbb{N}}$, then $\mathcal{J}_{(c_n)}$ need not be a quasicircle. For example, if $c_n = \frac{1}{4}$ for all n, then it is known that $\mathcal{J}(f_{1/4})$ is still a Jordan curve (see for example [**CG**, p. 97] or [**St**, p. 124]) but it has cusps. Furthermore, Corollary 4.5 does not hold true in general when all c_n are contained in the interior of the main cardioid of the Mandelbrot set. This can be seen by the simple example $c_1 = -\frac{1}{2} - \eta$ and $c_n = \frac{1}{4} - \varepsilon$ for $n \geq 2$ with $0 < \eta < \frac{1}{4}$ and $0 < \varepsilon < \eta^2$. In this case we have $F_n(0) \to \infty$ as $n \to \infty$ so that by Theorem 1.1 in [**BBR**] the Julia set $\mathcal{J}_{(c_n)}$ is even disconnected. It would be of interest whether $\mathcal{J}_{(c_n)}$ is also a Jordan curve in our more general setting provided that $(c_n) \in K_{1/4}^{\mathbb{N}}$ or what holds when $(c_n) \in D_{1/4}^{\mathbb{N}}$.

Furthermore, we consider the dynamics of (F_n) in the stable domain $\mathcal{A}_{(c_n)}(0)$ provided that $(c_n) \in K_{1/4}^{\mathbb{N}}$. We will show that $\mathcal{A}_{(c_n)}(0)$ is a *contracting domain*, that is a stable domain U such that all limit functions of (F_n) in U are constant. This property is equivalent to diam $F_n(K) \to 0$ as $n \to \infty$ for every compact set $K \subset U$.

Theorem 4.6. *Let $(c_n) \in K_{1/4}^{\mathbb{N}}$. Then $\mathcal{A}_{(c_n)}(0)$ is a contracting domain.*

Proof. Let $K \subset \mathcal{A}_{(c_n)}(0)$ be a compact set. We first assume that $(c_n) \in K_\delta^{\mathbb{N}}$ for some $\delta < \frac{1}{4}$, and we choose $r \in \left(s_\delta, \frac{1}{2}\right)$. Then by Theorem 4.1 there exists $N \in \mathbb{N}$ such that $F_N(K) \subset D_r$. If $z_1, z_2 \in K$, then $w_1 := F_N(z_1)$, $w_2 := F_N(z_2) \in D_r$ and thus $|f_{c_k}(w_1) - f_{c_k}(w_2)| = |w_1 + w_2||w_1 - w_2| \leq 2r|w_1 - w_2|$ which implies $|F_{N+k}(z_1) - F_{N+k}(z_2)| \leq (2r)^k |w_1 - w_2|$. Therefore,

we obtain $\operatorname{diam} F_{N+k}(K) \leq (2r)^k \operatorname{diam} F_N(K) \to 0$ as $k \to \infty$, and the assertion follows.

Now, let $|c_n| \leq \frac{1}{4}$ for all $n \in \mathbb{N}$. Again, by Theorem 4.1 there exists $N \in \mathbb{N}$ such that $F_N(K) \subset K_{1/2}$, and we obtain as above $\operatorname{diam} F_{N+k}(K) \leq \operatorname{diam} F_{N+k-1}(K)$ so that the sequence $(\operatorname{diam} F_{N+k}(K))$ is monotonically decreasing and thus convergent. In order to deduce $\operatorname{diam} F_{N+k}(K) \to 0$ as $k \to \infty$ we need a better estimate. If $w_1, w_2 \in F_N(K)$, we obtain

$$|f_{c_k}(w_1) - f_{c_k}(w_2)| \leq 2 \left| \int_{w_1}^{w_2} |z| \, |dz| \right|.$$

For the estimate of the right hand side we consider the worst case which can happen, that is $|w_1| = |w_2| = \frac{1}{2}$. For simplicity, we may assume that $w_2 = \overline{w}_1$, and we set $\varrho := \operatorname{Re} w_1 = \operatorname{Re} w_2 \in \left[0, \frac{1}{2}\right)$. Then with $d := \frac{1}{2}|w_1 - w_2|$ we get $\varrho^2 + d^2 = \frac{1}{4}$ and thus

$$2 \left| \int_{w_1}^{w_2} |z| \, |dz| \right| \leq 4 \int_0^d |\varrho + it| \, dt = 4 \int_0^d \sqrt{\varrho^2 + t^2} \, dt$$

$$= d + 2\varrho^2 \log \frac{2d+1}{2\varrho}$$

$$= \frac{1}{2}|w_1 - w_2| + \frac{1}{4}(1 - |w_1 - w_2|^2) \log \frac{1 + |w_1 - w_2|}{1 - |w_1 - w_2|}.$$

This implies with $d_n := \operatorname{diam} F_n(K)$

$$d_{N+k} \leq \frac{1}{2}d_{N+k-1} + \frac{1}{4}(1 - d_{N+k-1}^2) \log \frac{1 + d_{N+k-1}}{1 - d_{N+k-1}}.$$

Setting $\alpha := \lim_{k \to \infty} \operatorname{diam} F_{N+k}(K)$ we see that

$$\alpha \leq \frac{1}{2}\alpha + \frac{1}{4}(1 - \alpha^2) \log \frac{1 + \alpha}{1 - \alpha},$$

and an elementary argument shows that this is possible only for $\alpha = 0$ which gives the assertion. $\qquad\square$

If $(c_n) \in K_\delta^{\mathbb{N}}$ for some $\delta \leq \frac{1}{4}$, we denote by $L_{(c_n)}$ the set of (constant) limit functions of (F_n) in $\mathcal{A}_{(c_n)}(0)$, that is the set of all $\zeta \in \mathbb{C}$ such that for some subsequence (F_{n_k}) of (F_n) there holds $F_{n_k} \to \zeta$ as $k \to \infty$ locally uniformly in $\mathcal{A}_{(c_n)}(0)$. It is easy to see that $L_{(c_n)}$ is a compact set, and from the proof of Theorem 4.6 it follows that $L_{(c_n)} \subset K_{s_\delta} \subset K_{1/2}$. From Theorem 1.6 in [**BBR**] we know that the case $L_{(c_n)} = K_{s_\delta}$ may occur. Moreover, this phenomenon happens almost surely, that means that the product measure (cf. Section 5) of the set of these sequences (c_n) in $K_\delta^{\mathbb{N}}$ is one. In a similar way it is possible to construct sequences $(c_n) \in K_\delta^{\mathbb{N}}$ such that $L_{(c_n)} = \partial K_{s_\delta}$.

On the other hand, if $L_{(c_n)}$ consists of a single point ζ, then $F_n \to \zeta$ as $n \to \infty$ locally uniformly in $\mathcal{A}_{(c_n)}(0)$, and since $F_{n+1}(z) = (F_n(z))^2 + c_n$ we obtain $c_n \to c \in K_\delta$ as $n \to \infty$, where $c = \zeta - \zeta^2$. Therefore, the set C_δ of all these points ζ is the component of the preimage of K_δ under the map $z \mapsto z - z^2$ which is contained in K_{s_δ}. Therefore, C_δ is a proper subset of K_{s_δ} and $C_\delta \cap \partial K_{s_\delta} = \{s_\delta\}$. It would be of interest to characterize those compact sets $K \subset K_{s_\delta}$ such that $K = L_{(c_n)}$ for some sequence $(c_n) \in K_\delta^{\mathbb{N}}$.

The stable domain $\mathcal{A}_{(c_n)}(\infty)$ may be viewed as a Böttcher domain. If it is simply connected, then there exists a conformal map ϕ of $\mathcal{A}_{(c_n)}(\infty)$ onto Δ_1 normalized at infinity by

$$(4.7) \qquad \phi(z) = z + a_0 + \frac{a_1}{z} + \frac{a_2}{z^2} + \cdots .$$

Note that the capacity of $\mathcal{K}_{(c_n)}$ (cf. Section 8) is equal to one. Like in the iteration of a fixed polynomial we show that ϕ may be described dynamically.

Theorem 4.7. *Let $\delta > 0$ and $(c_n) \in K_\delta^{\mathbb{N}}$ such that $\mathcal{A}_{(c_n)}(\infty)$ is simply connected. Then the conformal map ϕ of $\mathcal{A}_{(c_n)}(\infty)$ onto Δ_1 with the normalization (4.7) is given by*

$$\phi(z) = \lim_{k \to \infty} \sqrt[2^k]{F_k(z)} = z \lim_{k \to \infty} \sqrt[2^k]{\frac{F_k(z)}{z^{2^k}}}$$

with locally uniform convergence in $\mathcal{A}_{(c_n)}(\infty)$, and where the branch of the root is determined by $\sqrt[2^k]{1} = 1$.

Proof. Let $R > R_\delta$ such that $R^2 \geq 2\delta$ and $U_m := F_m^{-1}(\Delta_R)$ for $m \in \mathbb{N}$. Then we have $U_m \subset U_{m+1} \subset \mathcal{A}_{(c_n)}(\infty)$ and $\mathcal{A}_{(c_n)}(\infty) = \bigcup_{m=1}^{\infty} U_m$. For $k \in \mathbb{N}$ we define

$$\phi_k(z) := \sqrt[2^k]{F_k(z)} = z \sqrt[2^k]{\frac{F_k(z)}{z^{2^k}}}.$$

Then ϕ_k maps U_k conformally onto Δ_{R_k}, where $R_k := \sqrt[2^k]{R}$. For $z \in U_m$ and $k \geq m$ we have

$$\left| \frac{c_k}{(F_k(z))^2} \right| \leq \frac{\delta}{R^2} \leq \frac{1}{2},$$

and the elementary inequality

$$|\sqrt[p]{1 + u} - 1| \leq \frac{1}{p} \qquad (u \in K_{1/2})$$

yields

$$\left| \frac{\phi_{k+1}(z)}{\phi_k(z)} - 1 \right| = \left| \sqrt[2^{k+1}]{\frac{F_{k+1}(z)}{(F_k(z))^2}} - 1 \right| = \left| \sqrt[2^{k+1}]{1 + \frac{c_k}{(F_k(z))^2}} - 1 \right| \leq \frac{1}{2^{k+1}}.$$

Therefore, the limit

$$\phi(z) := \lim_{k \to \infty} \phi_k(z) = z \prod_{k=0}^{\infty} \frac{\phi_{k+1}(z)}{\phi_k(z)}$$

exists uniformly in U_m, and ϕ is the desired conformal map.   □

5. Lebesgue measure of Julia sets.

From a result of Lyubich [**L2**] (see also [**CG**, p. 90] or [**St**, p. 144]) it follows that the Julia set of a hyperbolic rational function has two-dimensional Lebesgue measure (which we denote by m_2) zero. In particular, this is true for $\mathcal{J}(f_c)$ provided that c is contained in a hyperbolic component of the interior of the Mandelbrot set $\mathcal{M}$ or $c \notin \mathcal{M}$. In this section we show that this is true to a certain extent in our situation.

We begin with $\delta < \frac{1}{4}$. Then by Section 4 we know that if $(c_n) \in K_\delta^{\mathbb{N}}$, then $\mathcal{J}_{(c_n)}$ is a quasicircle, and from the differentiability properties of quasiconformal maps it follows that quasicircles always have two-dimensional Lebesgue measure zero (see for example [**LV**, p. 165]).

Corollary 5.1. *Let $\delta < \frac{1}{4}$ and $(c_n) \in K_\delta^{\mathbb{N}}$. Then $m_2(\mathcal{J}_{(c_n)}) = 0$.*

Now, we will show that $m_2(\mathcal{J}_{(c_n)})$ is almost surely zero provided that the c_n are randomly chosen in K_δ for some $\delta > \frac{1}{4}$. To be more precise, let λ_δ denote the two-dimensional Lebesgue measure on K_δ normalized by $\lambda_\delta(K_\delta) = 1$. Then the product space $K_\delta^{\mathbb{N}}$ carries the usual product measure $\tilde{\lambda}_\delta := \bigotimes_{k=1}^{\infty} \lambda_\delta$. We set

$$(5.1) \qquad \mathfrak{N}_\delta := \{ (c_n) \in K_\delta^{\mathbb{N}} : m_2(\mathcal{J}_{(c_n)}) = 0 \}.$$

Then the goal is to show that $\tilde{\lambda}_\delta(\mathfrak{N}_\delta) = 1$. In order to do this we recall:

Theorem 5.2 ([**BBR**]). *Let $\delta > \frac{1}{4}$ and $R > 0$. Then for every $z \in \widehat{\mathbb{C}}$ there exists an open set $\mathfrak{U}_z \subset K_\delta^{\mathbb{N}}$ with the following properties:*

(a) *$\tilde{\lambda}_\delta(\mathfrak{U}_z) = 1$,*
(b) *for every $(c_n) \in \mathfrak{U}_z$ there holds $|F_k(z)| > R$ for all sufficiently large k.*

Theorem 5.3. *Let $\delta > \frac{1}{4}$, and let $\mathfrak{N}_\delta \subset K_\delta^{\mathbb{N}}$ be defined by (5.1). Then $\tilde{\lambda}_\delta(\mathfrak{N}_\delta) = 1$.*

Proof. Let $M = \Delta_R$ be an invariant domain and

$$\widetilde{\mathfrak{E}} := \{ ((c_n), z) \in K_\delta^{\mathbb{N}} \times \widehat{\mathbb{C}} : F_k(z) \in M \text{ for some } k \in \mathbb{N} \}.$$

By Theorem 5.2 we have $\tilde{\lambda}_\delta(\widetilde{\mathfrak{E}}_z) = 1$ for $z \in \widehat{\mathbb{C}}$, where

$$\widetilde{\mathfrak{E}}_z := \{ (c_n) \in K_\delta^{\mathbb{N}} : ((c_n), z) \in \widetilde{\mathfrak{E}} \}.$$

If σ denotes the two-dimensional Lebesgue measure on $\widehat{\mathbb{C}}$ normalized by $\sigma(\widehat{\mathbb{C}}) = 1$, it follows

$$(\tilde{\lambda}_\delta \otimes \sigma)(\widetilde{\mathfrak{E}}) = \int_{\widehat{\mathbb{C}}} \tilde{\lambda}_\delta(\widetilde{\mathfrak{E}}_z) \, d\sigma(z) = 1.$$

Now, let

$$\mathfrak{E} := \{ (c_n) \in K_\delta^{\mathbb{N}} : \sigma(\widetilde{\mathfrak{E}}_{(c_n)}) = 1 \},$$

where

$$\widetilde{\mathfrak{E}}_{(c_n)} := \{ z \in \widehat{\mathbb{C}} : ((c_n), z) \in \widetilde{\mathfrak{E}} \}.$$

Since

$$1 = (\tilde{\lambda}_\delta \otimes \sigma)(\widetilde{\mathfrak{E}}) = \int_{K_\delta^{\mathbb{N}}} \sigma(\widetilde{\mathfrak{E}}_{(c_n)}) \, d\tilde{\lambda}_\delta((c_n))$$

we obtain $\tilde{\lambda}_\delta(\mathfrak{E}) = 1$. If $(c_n) \in \mathfrak{E}$, then

$$\sigma(\mathcal{A}_{(c_n)}(\infty)) = \sigma(\{ z \in \widehat{\mathbb{C}} : F_k(z) \in M \text{ for some } k \in \mathbb{N} \}) = \sigma(\widetilde{\mathfrak{E}}_{(c_n)}) = 1$$

which implies $\sigma(\mathcal{J}_{(c_n)}) = 0$. $\qquad\square$

It would be of interest whether Theorem 5.3 remains valid for $\delta = \frac{1}{4}$. Concerning the question whether there exists a sequence $(c_n) \in K_\delta^{\mathbb{N}}$ for some $\delta \geq \frac{1}{4}$ such that $m_2(\mathcal{J}_{(c_n)}) > 0$, the referee mentioned that, recently, a group of mathematicians around P.W. Jones at Yale University have constructed such an example. More precisely, there exists a sequence (c_n) with $c_n \in \{0, \pm\frac{1}{4}, \frac{1}{2}\}$ such that $m_2(\mathcal{J}_{(c_n)}) > 0$. This result was communicated to the author by P.W. Jones. The author is grateful to both for bringing this information to his attention.

Finally, we prove:

Theorem 5.4. *Let $\delta > 2$ (which is equivalent to $\delta > R_\delta$), and let $\varepsilon > 0$ such that $R_\delta + \varepsilon \leq |c_n| \leq \delta$ for all $n \in \mathbb{N}$. Then $m_2(\mathcal{J}_{(c_n)}) = 0$.*

Proof. We choose R such that $R_\delta < R < R_\delta + \varepsilon$ and $\eta := R_\delta + \varepsilon - R > 0$. Then we have $\mathcal{J}_{(c_n)} \subset \overline{D}_{R_\delta} \subset D := D_R$, and D is backward invariant, that is $f_{c_n}^{-1}(D) \subset D$ for all $n \in \mathbb{N}$. Furthermore, there holds $|f_{c_n}(0)| = |c_n| \geq R_\delta + \varepsilon = R + \eta$ and thus $|(f_{c_m} \circ \cdots \circ f_{c_{k+1}})(0)| \geq R + \eta$ for $k = 0, 1, \ldots, m-1$ and all $m \in \mathbb{N}$. Therefore, D does not contain any critical value of (F_n) so that in D there exist 2^k analytic branches of the inverse function of F_k which we denote by $G_{j,k}$ for $j = 1, \ldots, 2^k$ and $k \in \mathbb{N}$. Furthermore, we set $D_{j,k} := G_{j,k}(D) \subset D$ and $D_k := \bigcup_{j=1}^{2^k} D_{j,k}$. Then $D_{1,k}, \ldots, D_{2^k,k}$ are mutually disjoint simply connected domains, and $\mathcal{J}_{(c_n)} \subset D_{k+1} \subset D_k$. Finally, we set $U_k := D_k \backslash \overline{D}_{k+1}$ and $U_{j,k} := D_{j,k} \backslash \overline{D}_{k+1}$ so that $U_{1,k}, \ldots, U_{2^k,k}$ are mutually disjoint multiply connected domains, and $U_k = \bigcup_{j=1}^{2^k} U_{j,k}$.

Now, we prove that there exists a constant $q > 0$ such that

$$(5.2) \qquad \frac{m_2(U_k)}{m_2(D_k)} \geq q$$

for all $k \in \mathbb{N}$. For that purpose it is enough to show that

$$(5.3) \qquad \frac{m_2(U_{j,k})}{m_2(D_{j,k})} \geq q$$

for $j = 1, \ldots, 2^k$ and all $k \in \mathbb{N}$.

Let $V_{1,k}$ and $V_{2,k}$ denote the two components of $f_{c_{k+1}}^{-1}(D)$, and let $W_k := D \setminus (\overline{V}_{1,k} \cup \overline{V}_{2,k})$. Then $U_{j,k} = G_{j,k}(W_k)$, and we obtain

$$\frac{m_2(U_{j,k})}{m_2(D_{j,k})} = \frac{\int_{W_k} |G'_{j,k}(z)|^2 \, dm_2(z)}{\int_D |G'_{j,k}(z)|^2 \, dm_2(z)} \geq \frac{|G'_{j,k}(z_{j,k})|^2 m_2(W_k)}{|G'_{j,k}(\zeta_{j,k})|^2 m_2(D)},$$

where $z_{j,k} \in W_k \subset D$ and $\zeta_{j,k} \in D$ such that $|G'_{j,k}(z_{j,k})| = \min_{z \in W_k} |G'_{j,k}(z)|$ and $|G'_{j,k}(\zeta_{j,k})| = \max_{z \in D} |G'_{j,k}(z)|$. By the Koebe distortion theorem (see for example [**P2**, p. 9]) applied to the disk $D_{R+\eta}$ there holds

$$\left| \frac{G'_{j,k}(z)}{G'_{j,k}(\zeta)} \right| \geq \left(\frac{\eta}{\eta + 2R} \right)^4$$

for all $z, \zeta \in D$. Therefore, it remains to show that there exists a constant $\gamma > 0$ such that

$$\frac{m_2(W_k)}{m_2(D)} \geq \gamma$$

for all $k \in \mathbb{N}$.

For simplicity we write $c = c_{k+1}$, and let $V \in \{V_{1,k}, V_{2,k}\}$. Then

$$m_2(V) = \frac{1}{4} \int_D \frac{dm_2(z)}{|z - c|} = \frac{1}{4} \int_0^R \int_0^{2\pi} \frac{\varrho}{|\varrho e^{it} - c|} \, dt \, d\varrho.$$

By the Cauchy-Schwarz inequality we get

$$\int_0^{2\pi} \frac{dt}{|\varrho e^{it} - c|} \leq \sqrt{2\pi} \left(\int_0^{2\pi} \frac{dt}{|\varrho e^{it} - c|^2} \right)^{1/2},$$

and the Poisson integral formula yields

$$\int_0^{2\pi} \frac{dt}{|\varrho e^{it} - c|^2} = \frac{2\pi}{|c|^2 - \varrho^2}.$$

Therefore, we arrive at

$$m_2(V) \leq \frac{\pi}{2} \int_0^R \frac{\varrho}{\sqrt{|c|^2 - \varrho^2}} \, d\varrho = \frac{\pi}{2}(|c| - \sqrt{|c|^2 - R^2}) \leq \frac{1}{2}\pi R.$$

This implies $m_2(V_{1,k} \cup V_{2,k}) \leq \pi R$ and thus

$$\frac{m_2(W_k)}{m_2(D)} \geq 1 - \frac{1}{R} \geq \frac{1}{2}$$

which proves (5.3).

Finally, (5.2) gives $m_2(\mathcal{J}_{(c_n)}) \leq m_2(D_{k+1}) = m_2(D_k) - m_2(U_k) \leq (1 - q)m_2(D_k)$ so that $m_2(\mathcal{J}_{(c_n)}) \leq (1-q)^k m_2(D) \to 0$ as $k \to \infty$ which completes the proof. $\qquad\square$

6. Hausdorff dimension of Julia sets.

We first recall the notion of Hausdorff dimension. Let $E \subset \mathbb{C}$ be a non-empty compact set, and denote by $(D_j)_\varepsilon$ any covering of E by finitely many open sets D_j with $\operatorname{diam} D_j < \varepsilon$. Then for $t \in (0, 2]$

$$m_t(E) := \sup_{\varepsilon > 0} \inf_{(D_j)_\varepsilon} \sum_j (\operatorname{diam} D_j)^t$$

is called the *t-dimensional Hausdorff measure* of E. Obviously, $m_t(E) < \infty$ implies $m_s(E) = 0$ for $s > t$, and conversely, $m_t(E) > 0$ implies $m_s(E) = \infty$ for $s < t$. Hence, there exists a unique $\tau \in [0, 2]$ such that $m_s(E) = 0$ and $m_t(E) = \infty$ for $0 < t < \tau < s \leq 2$. This number τ is called the *Hausdorff dimension* of E and is denoted by $\dim_{\mathrm{H}} E$.

It is well-known (cf. [**G**], see also [**Be**, p. 251] or [**St**, p. 169]) that the Hausdorff dimension of the Julia set of any fixed rational function f is positive. More precisely, if $\infty \notin \mathcal{J}(f)$ and if d denotes the degree of f, then

$$\dim_{\mathrm{H}} \mathcal{J}(f) \geq \frac{\log d}{\log \max_{z \in \mathcal{J}(f)} |f'(z)|}.$$

We show that this estimate holds true in a certain sense in our situation.

Theorem 6.1. *Let* $\delta > 0$ *and* $(c_n) \in K_\delta^{\mathbb{N}}$. *Then* $\dim_{\mathrm{H}} \mathcal{J}_{(c_n)} > 0$. *More precisely, there holds*

$$\dim_{\mathrm{H}} \mathcal{J}_{(c_n)} \geq \frac{\log 2}{\log (2R_\delta)} = \frac{\log 2}{\log (1 + \sqrt{1 + 4\delta})}.$$

Proof. We show that the Green function g of $\mathcal{A}_{(c_n)}(\infty)$ is Hölder continuous with exponent

$$\alpha = \frac{\log 2}{\log 2 + \log (2R - R_\delta)}$$

for any $R > R_\delta$. Then a result of Carleson [**C**] gives $\dim_{\mathrm{H}} \mathcal{J}_{(c_n)} \geq \alpha$. For that purpose, it suffices to show that there exists a constant $\gamma > 0$ such that $g(z) \leq \gamma(d(z))^\alpha$ for all $z \in \mathcal{A}_{(c_n)}(\infty)$, where $d(z) := \operatorname{dist}(z, \mathcal{J}_{(c_n)})$. Of course, we may assume that $d(z)$ is small.

Let $R > R_\delta$ and $U_k := F_k^{-1}(\Delta_R)$ for $k \in \mathbb{N}$. Then we have $U_k \subset U_{k+1} \subset \mathcal{A}_{(c_n)}(\infty)$ and $\mathcal{A}_{(c_n)}(\infty) = \bigcup_{k=1}^\infty U_k$. The Green function g_k of U_k with pole at infinity is given by

$$g_k(z) = \frac{1}{2^k} \log \frac{|F_k(z)|}{R} \qquad (z \in U_k).$$

There holds $g_k(z) \leq g_{k+1}(z) \leq g(z)$ for $z \in U_k$ and $g_k \to g$ as $k \to \infty$ locally uniformly in $\mathcal{A}_{(c_n)}(\infty)$.

We will show that there exists some constant $C > 0$ such that $g(z) \leq g_k(z) + \frac{C}{2^k}$ for $z \in U_k$. There holds $|F_{k+1}(z)| = |(F_k(z))^2 + c_{k+1}| \leq |F_k(z)|^2 + \delta$ and this gives

$$g_{k+1}(z) \leq \frac{1}{2^{k+1}} \log \frac{|F_k(z)|^2 + \delta}{R}.$$

If a, $b > 0$, then $\log^+(a+b) \leq \log^+ a + \log^+ b + \log 2$, and thus

$$g_{k+1}(z) \leq \frac{1}{2^{k+1}} \left(\log \frac{|F_k(z)|^2}{R} + \log^+ \frac{\delta}{R} + \log 2 \right)$$

$$= \frac{1}{2^{k+1}} \left(2 \log \frac{|F_k(z)|}{R} + \log^+ \frac{\delta}{R} + \log(2R) \right)$$

$$= g_k(z) + \frac{C}{2^{k+1}},$$

where $C := \log^+ \frac{\delta}{R} + \log(2R)$. From this we obtain by induction

$$g_{k+m}(z) \leq g_k(z) + C \left(\frac{1}{2^{k+1}} + \cdots + \frac{1}{2^{k+m}} \right) \leq g_k(z) + \frac{C}{2^k}$$

for all $m \in \mathbb{N}$. Letting $m \to \infty$ we get

$$g(z) \leq g_k(z) + \frac{C}{2^k} \qquad (z \in U_k).$$

Now, let $z \in U_k \setminus U_{k-1}$ for some $k \in \mathbb{N}$. Then $|F_{k-1}(z)| \leq R$ which implies $|F_k(z)| \leq R^2 + \delta$. Hence, we have $g_k(z) \leq \frac{1}{2^k} \log \left(R + \frac{\delta}{R} \right)$ and thus

(6.1) $$g(z) \leq \frac{\Gamma}{2^k} \qquad (z \in U_k \setminus U_{k-1}),$$

where $\Gamma := C + \log \left(R + \frac{\delta}{R} \right)$.

Finally, we prove a lower estimate for $d(z)$, if $z \in U_k$ for some $k \in \mathbb{N}$. We set $w := F_k(z)$ and $\eta := |w| - R_\delta$. If U denotes the component of $F_k^{-1}(D_\eta(w))$ containing z, there holds $U \subset \mathcal{A}_{(c_n)}(\infty)$. Let $\varrho > 0$ such that $D_\varrho(z) \subset U$. Then $F_k(D_\varrho(z)) \subset D_\eta(w) \subset D_{|w|+\eta}$ which implies $F_j(D_\varrho(z)) \subset D_{|w|+\eta}$ for

$j = 0, 1, \ldots, k$ and thus $|F'_k(t)| \leq 2^k(|w| + \eta)^k$ for all $t \in D_\varrho(z)$. If $z' \in D_\varrho(z)$ and $w' := F_k(z')$, then

$$w' - w = \int_z^{z'} F'_k(t)\, dt,$$

where we integrate over the line segment joining z and z'. This yields

$$|w' - w| \leq 2^k(|w| + \eta)^k |z' - z| < 2^k(|w| + \eta)^k \varrho.$$

Setting

$$\varrho := \frac{\eta}{2^k(|w| + \eta)}$$

we obtain $D_\varrho(z) \subset U$ and thus

$$d(z) \geq \frac{\eta}{2^k(|w| + \eta)} = \frac{|w| - R_\delta}{2^k(2|w| - R_\delta)} \geq \frac{R - R_\delta}{2^k(2R - R_\delta)} \qquad (z \in U_k).$$

We choose $q := R - R_\delta$ and

$$\alpha := \frac{\log 2}{\log 2 + \log (R + q)}$$

and arrive at

(6.2) $$(d(z))^\alpha \geq \frac{q^\alpha}{2^k} \qquad (z \in U_k).$$

Finally, putting (6.1) and (6.2) together we get

$$g(z) \leq \frac{\Gamma}{q^\alpha}\, (d(z))^\alpha \qquad (z \in U_k \setminus U_{k-1})$$

which completes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Gehring and Väisälä [**GV**] have shown that quasicircles always have Hausdorff dimension less than two and thus by Corollary 4.5 we obtain:

Corollary 6.2. *Let $\delta < \frac{1}{4}$ and $(c_n) \in K_\delta^{\mathbb{N}}$. Then $\dim_{\mathrm{H}} \mathcal{J}_{(c_n)} < 2$.*

If $0 < \delta \leq \frac{1}{4}$ and $(c_n) \in K_\delta^{\mathbb{N}}$, then the Julia set $\mathcal{J}_{(c_n)}$ is connected so that its Hausdorff dimension is at least one. Moreover, Sullivan [**Su**] has shown, that if $c \neq 0$ is in the interior of the main cardioid of the Mandelbrot set, then $\dim_{\mathrm{H}} \mathcal{J}(f_c) > 1$. Furthermore, it follows by a result of Shishikura [**Sh**] that $\dim_{\mathrm{H}} \mathcal{J}(f_{1/4}) = 2$. It would be of interest whether $\dim_{\mathrm{H}} \mathcal{J}_{(c_n)}$ is almost surely (in the sense of Section 5) greater than one if $(c_n) \in K_\delta^{\mathbb{N}}$ for some $\delta < \frac{1}{4}$. In our general setting, it is clear that we can only expect such an almost surely statement.

7. Density of repelling fixpoints.

From iteration theory of a fixed rational function it is well-known that the repelling periodic points are dense in the Julia set (cf. [**Be**, p. 148], [**CG**, p. 63] or [**St**, p. 35]). In our setting we consider the set $\mathcal{R}_{(c_n)}$ of *repelling fixpoints* of the sequence of iterates (F_n), i.e.,

$$\mathcal{R}_{(c_n)} := \{\, \zeta \in \mathbb{C} : F_k(\zeta) = \zeta \text{ for some } k \in \mathbb{N} \text{ and } |F_k'(\zeta)| > 1 \,\}.$$

It is not necessarily true that $\mathcal{R}_{(c_n)} \subset \mathcal{J}_{(c_n)}$. But from a result of Fornæss and Sibony [**FS**, Theorem 2.3] it follows that if $\delta > 0$ is sufficiently small and $(c_n) \in K_\delta^{\mathbb{N}}$, then $(\mathcal{R}_{(c_n)})' = \mathcal{J}_{(c_n)}$. More precisely, we show:

Theorem 7.1. *Let* $\delta < \frac{1}{4}$ *and* $(c_n) \in K_\delta^{\mathbb{N}}$. *Then* $(\mathcal{R}_{(c_n)})' = \mathcal{J}_{(c_n)}$.

Proof. Since $\delta < \frac{1}{4}$ we have $f_c(\overline{D}_r) \subset D_r$ for all $c \in K_\delta$ and $s_\delta < r < r_\delta$. This implies that $F_k(z) \neq z$ for all $k \in \mathbb{N}$ and $s_\delta < |z| < r_\delta$. Since $F_k'(z) = 2^k \prod_{j=0}^{k-1} F_j(z)$ and $f_c(K_{1/2}) \subset K_{1/2}$, we have $\mathcal{R}_{(c_n)} \cap K_{1/2} = \varnothing$. Setting $K := K_r$ for some $r \in \left(\frac{1}{2}, r_\delta\right)$, we also have $\mathcal{R}_{(c_n)} \cap K = \varnothing$. We set $U := \mathbb{C} \setminus K$. If $z \in U$ and $F_j(z) \in U$ for all $j = 1, \ldots, k-1$, then $|F_k'(z)| \geq q^k$ with $q := 2r > 1$.

We first show that $(\mathcal{R}_{(c_n)})' \subset \mathcal{J}_{(c_n)}$. For that purpose let $F_{k_\ell}(z_\ell) = z_\ell$, $|F_{k_\ell}'(z_\ell)| > 1$ and $z_\ell \to \zeta$ as $\ell \to \infty$. If $\zeta \in \mathbb{C} \setminus \mathcal{K}_{(c_n)}$, then $F_{k_\ell} \to \infty$ as $\ell \to \infty$ uniformly in some neighbourhood of ζ. This gives $F_{k_\ell}(z_\ell) \to \infty$ as $\ell \to \infty$ which is a contradiction. Now, assume that $\zeta \in (\mathcal{K}_{(c_n)})^\circ$. If $F_j(\zeta) \in U$ for all $j \in \mathbb{N}_0$, then $|F_k'(\zeta)| \geq q^k \to \infty$ as $k \to \infty$. But this is impossible since (F_k) is normal and bounded in $(\mathcal{K}_{(c_n)})^\circ$. Therefore, we have $F_{k_0}(\zeta) \in K$ for some $k_0 \in \mathbb{N}_0$, and thus $F_k(\zeta) \in K$ for all $k \geq k_0$. By passing to a subsequence we may assume that $F_{k_\ell} \to \phi$ as $\ell \to \infty$ uniformly in some neighbourhood U_ζ of ζ, where ϕ is holomorphic in U_ζ. This implies $z_\ell = F_{k_\ell}(z_\ell) \to \phi(\zeta)$ as $\ell \to \infty$ and thus $z_\ell \in K$ for all ℓ large enough which is again a contradiction.

Now, we show that $\mathcal{J}_{(c_n)} \subset (\mathcal{R}_{(c_n)})'$. Suppose that there exists $\zeta \in \mathcal{J}_{(c_n)}$ and a neighbourhood V of ζ such that $F_k(z) \neq z$ for all $z \in V$ and $k \geq k_0 = k_0(V)$. We set

$$h_k(z) := \frac{1}{2^k} \log|F_k(z) - z|.$$

Then h_k is harmonic and uniformly bounded above in V. By Eq. (2.1) we have $h_k \to g_{(c_n)}$ as $k \to \infty$ in $V \setminus \mathcal{K}_{(c_n)}$, and thus $h_k \to h$ as $k \to \infty$ for some harmonic function h in V. Furthermore, there holds $h_k \to 0$ as $k \to \infty$ in $V \cap \mathcal{K}_{(c_n)}$ so that $h = 0$ in $V \cap \mathcal{K}_{(c_n)}$. But this is a contradiction to the minimum principle for harmonic functions.

Therefore, for every $\zeta \in \mathcal{J}_{(c_n)}$ there exists a strictly increasing sequence (k_ℓ) in $\mathbb{N}$ and $z_\ell \in U$ such that $z_\ell \to \zeta$ as $\ell \to \infty$ and $F_{k_\ell}(z_\ell) = z_\ell$. Then we

have $F_j(z_\ell) \in U$ for $j = 1, \ldots, k_\ell - 1$ which gives $|F'_{k_\ell}(z_\ell)| \geq q^{k_\ell} > 1$ so that $z_\ell \in \mathcal{R}_{(c_n)}$. □

It would be of interest whether Theorem 7.1 holds for all $\delta > 0$. However, the proof shows that we always have $(\mathcal{R}_{(c_n)})' \subset \mathcal{K}_{(c_n)}$.

8. Asymptotic distribution of predecessors.

If $(c_n) \in K_\delta^{\mathbb{N}}$ and if $a \in \Delta_{R_\delta}$, then the predecessors $F_k^{-1}(a)$ of a are all contained in $\mathcal{A}_{(c_n)}(\infty)$, and they only accumulate on the Julia set $\mathcal{J}_{(c_n)}$. In fact, this follows from the invariance of Δ_R for $R > R_\delta$ and $F_k \to \infty$ as $k \to \infty$ locally uniformly in $\mathcal{A}_{(c_n)}(\infty)$. We want to study the asymptotic distribution of $F_k^{-1}(a)$ as $k \to \infty$. For iteration of a fixed polynomial this was done by Brolin [**Bro**].

We first recall some facts from potential theory which are needed in the sequel and which can be found, for example, in the book of Tsuji [**T**]. Let $E \subset \mathbb{C}$ be an infinite compact set, and let D be its outer domain, that is the component of $\widehat{\mathbb{C}} \setminus E$ containing the point ∞. Furthermore, we denote by $\operatorname{cap} E \geq 0$ the *logarithmic capacity* (or *transfinite diameter*) of E. (We do not recall the definition of $\operatorname{cap} E$ because it will not be needed.) We suppose that the Green function g_D of D with pole at infinity exists. Then

$$g_D(z) = \log |z| + V + o(1) \quad \text{as } z \to \infty$$

and $\operatorname{cap} E = e^{-V} > 0$. Note that by Eq. (2.1) this is true for $E = \mathcal{J}_{(c_n)}$ with $\operatorname{cap} E = 1$. Now, let μ be any probability measure on E. Then the *energy integral*

$$I[\mu] := \iint_{E \times E} \log \frac{1}{|\zeta - \omega|} \, d\mu(\zeta) \, d\mu(\omega)$$

is finite, and the *logarithmic potential*

$$p_\mu(z) := \int_E \log \frac{1}{|z - \zeta|} \, d\mu(\zeta)$$

is harmonic in D. Furthermore, there exists a unique probability measure μ^* on E which minimizes the energy integral $I[\mu]$, and there holds

$$g_D(z) - V = -p_{\mu^*}(z) \qquad (z \in D).$$

This measure μ^* is called the *equilibrium measure* on E. In the following μ^* always denotes the equilibrium measure on the Julia set $\mathcal{J}_{(c_n)}$, and $\operatorname{supp} \mu^*$ denotes its support, that is the set of points $z \in \mathcal{J}_{(c_n)}$ such that $\mu^*(D_\varepsilon(z) \cap \mathcal{J}_{(c_n)}) > 0$ for every $\varepsilon > 0$. Note that $\operatorname{supp} \mu^*$ is a closed set.

In order to study the asymptotic distribution of $F_k^{-1}(a)$ for $a \in \Delta_{R_\delta}$ as $k \to \infty$ we consider the following sequence (μ_k^a) of probability measures.

If δ_z denotes the *Dirac measure* concentrated at the point $z \in \mathbb{C}$ (that is $\delta_z(E) = 1$ if $z \in E$ and $\delta_z(E) = 0$ if $z \notin E$), then let

$$(8.1) \qquad \mu_k^a := \frac{1}{2^k} \sum_{F_k(z)=a} \delta_z.$$

We will show that (μ_k^a) is weakly convergent to μ^*, that is $\mu_k^a(E) \to \mu^*(E)$ as $k \to \infty$ for every Borel set $E \subset \mathbb{C}$ with $\mu^*(E^\circ) = \mu^*(\overline{E})$. For that purpose we first collect some auxiliary results.

Lemma 8.1 ([**Bro**, Lemma 15.4]). *Let $E \subset \mathbb{C}$ be a compact set, and let f be a function defined on E such that for some constant L there holds $|f(z_1) - f(z_2)| \leq L|z_1 - z_2|$ for all $z_1, z_2 \in E$. If $\operatorname{cap} E = 0$, then $\operatorname{cap} f(E) = 0$.*

Lemma 8.2. *Let $\delta > 0$ and $(c_n) \in K_\delta^{\mathbb{N}}$. Then $\operatorname{cap}(\mathcal{J}_{(c_n)} \setminus \operatorname{supp} \mu^*) = 0$.*

Proof. Since $\mathcal{J}_{(c_n)} = \partial \mathcal{A}_{(c_n)}(\infty)$ and $\operatorname{cap} \mathcal{J}_{(c_n)} > 0$, the assertion immediately follows from Theorem III.31 in [**T**, p. 79]. $\qquad\square$

Lemma 8.3. *Let $\delta > 0$ and $(c_n) \in K_\delta^{\mathbb{N}}$. Then $\operatorname{supp} \mu^* = \mathcal{J}_{(c_n)}$.*

Proof. We assume that $\mathcal{J}^* := \mathcal{J}_{(c_n)} \setminus \operatorname{supp} \mu^* \neq \varnothing$. By Lemma 8.2 we have $\operatorname{cap} \mathcal{J}^* = 0$. Since $\mathcal{J}^*$ is an open set in $\mathcal{J}_{(c_n)}$ we may choose $z_0 \in \mathcal{J}^*$ and $\varepsilon > 0$ such that $\mathcal{J}_\varepsilon := \mathcal{J}^* \cap D_\varepsilon(z_0) \subset \mathcal{J}^*$. We also have $\operatorname{cap} \mathcal{J}_\varepsilon = 0$. But by the self-similarity of $\mathcal{J}_{(c_n)}$ (cf. [**Bü1**]) there exists $m \in \mathbb{N}$ such that $F_m(\mathcal{J}_\varepsilon) = F_m(\mathcal{J}_{(c_n)})$. Since $|f_{c_k}(z_1) - f_{c_k}(z_2)| = |z_1 + z_2||z_1 - z_2| \leq 2R_\delta|z_1 - z_2|$ for all $k \in \mathbb{N}$ and $z_1, z_2 \in \mathcal{J}_{(c_n)}$, we obtain $\operatorname{cap} F_m(\mathcal{J}_\varepsilon) = 0$ by Lemma 8.1. On the other hand there holds $F_m(\mathcal{J}_{(c_n)}) = \mathcal{J}_{(c_{n+m})}$ and thus $\operatorname{cap} F_m(\mathcal{J}_{(c_n)}) = 1$ which gives a contradiction. $\qquad\square$

Lemma 8.4 ([**Bro**, Lemma 15.5]). *Let $E, H \subset \mathbb{C}$ be compact sets with $E \subset H$ and $\operatorname{cap} E = e^{-V} > 0$. Furthermore, let (μ_n) be a sequence of probability measures on H which converges weakly to a probability measure μ on E. If u_n denotes the logarithmic potential with respect to μ_n and μ^* denotes the equilibrium measure on E, then suppose $\liminf_{n \to \infty} u_n(z) \geq V$ for $z \in E$ and $\operatorname{supp} \mu^* = E$. Then there holds $\mu = \mu^*$.*

Theorem 8.5. *Let $\delta > 0$ and $(c_n) \in K_\delta^{\mathbb{N}}$. Then for any $a \in \Delta_{R_\delta}$ the sequence (μ_k^a) of probability measures defined by (8.1) converges weakly to the equilibrium measure μ^* on $\mathcal{J}_{(c_n)}$.*

Proof. For $k \in \mathbb{N}$ let $z_{1,k}, \dots, z_{2^k,k}$ be the solutions of the equation $F_k(z) = a$. Then we have $z_{j,k} \in \mathcal{A}_{(c_n)}(\infty)$ and $z_{j,k} \in H := K_{|a|}$ for $j = 1, \dots, 2^k$ so that $\operatorname{supp} \mu_k^a \subset H$. Since $|F_k(z)| \leq R_\delta$ for $z \in \mathcal{J}_{(c_n)}$ and

$$|F_k(z) - a| = \prod_{j=1}^{2^k} |z - z_{j,k}|,$$

we obtain for $z \in \mathcal{J}_{(c_n)}$

$$\sum_{j=1}^{2^k} \log |z - z_{j,k}| = \log |F_k(z) - a| \leq \log (R_\delta + |a|) = C$$

and thus

$$u_k(z) := \frac{1}{2^k} \sum_{j=1}^{2^k} \log \frac{1}{|z - z_{j,k}|} \geq -\frac{C}{2^k}.$$

This can be written as

$$u_k(z) = \int_H \log \frac{1}{|z - \zeta|} \, d\mu_k^a(\zeta) \geq -\frac{C}{2^k}$$

so that

(8.2) $$\liminf_{k \to \infty} u_k(z) \geq 0 = \log \operatorname{cap} \mathcal{J}_{(c_n)} \qquad (z \in \mathcal{J}_{(c_n)}).$$

By the Selection Theorem (cf. [**T**, p. 34]) every sequence of probability measures on H contains a weakly convergent subsequence. Therefore, we only have to show that for every subsequence of (μ_k^a) which converges weakly to some probability measure ν there holds $\nu = \mu^*$. In fact, since the predecessors $F_k^{-1}(a)$ of a do not accumulate in $\mathcal{A}_{(c_n)}(\infty)$ we obtain $\operatorname{supp} \nu \subset \mathcal{J}_{(c_n)}$, and because of (8.2) the assertion follows from Lemma 8.3 and 8.4. $\qquad \square$

Remark 8.6. If $\delta < \frac{1}{4}$ and $(c_n) \in K_\delta^{\mathbb{N}}$, then the assertion of Theorem 8.5 also holds for any $a \in D_{r_\delta}$. This requires only a few simple modifications in the proof.

Like in the iteration of a fixed function there holds that for any $a \in \mathcal{J}_{(c_n)}$ the set $\bigcup_{k=1}^\infty F_k^{-1}(F_k(a))$ is dense in $\mathcal{J}_{(c_n)}$ (cf. [**Bü1**]). We also want to study the asymptotic distribution of $F_k^{-1}(F_k(a))$ as $k \to \infty$. For that purpose, we consider the following sequence (ν_k^a) of probability measures defined by

(8.3) $$\nu_k^a := \frac{1}{2^k} \sum_{F_k(z) = F_k(a)} \delta_z.$$

Then $\operatorname{supp} \nu_k^a \subset \mathcal{J}_{(c_n)}$, and from iteration theory of a fixed polynomial f_c it is known (cf. [**Bro**], see also [**St**, p. 148]) that (ν_k^a) converges weakly to the equilibrium measure μ^* on $\mathcal{J}(f_c)$. We show that this holds true in our situation.

Theorem 8.7. *Let* $\delta > 0$ *and* $(c_n) \in K_\delta^{\mathbb{N}}$. *Then for any* $a \in \mathcal{J}_{(c_n)}$ *the sequence* (ν_k^a) *of probability measures defined by* (8.3) *converges weakly to the equilibrium measure* μ^* *on* $\mathcal{J}_{(c_n)}$.

Proof. For $k \in \mathbb{N}$ let $z_{1,k}, \ldots, z_{2^k,k}$ be the solutions of the equation $F_k(z) = F_k(a)$. Then we have for $z \in \mathcal{A}_{(c_n)}(\infty)$

$$\frac{1}{2^k} \log |F_k(z) - F_k(a)| = \frac{1}{2^k} \sum_{j=1}^{2^k} \log |z - z_{j,k}| = \int_{\mathcal{J}_{(c_n)}} \log |z - \zeta| \, d\nu_k^a(\zeta).$$

Again, we only have to show that every weakly convergent subsequence (λ_ℓ) of (ν_k^a) has the limit μ^*. If $\lambda_\ell \to \lambda$ as $\ell \to \infty$ weakly, then for $z \in \mathcal{A}_{(c_n)}(\infty)$

$$\lim_{\ell \to \infty} \int_{\mathcal{J}_{(c_n)}} \log |z - \zeta| \, d\lambda_\ell(\zeta) = \int_{\mathcal{J}_{(c_n)}} \log |z - \zeta| \, d\lambda(\zeta).$$

On the other hand we have

$$\frac{1}{2^k} \log |F_k(z) - F_k(a)| = \frac{1}{2^k} \log \left| \frac{F_k(z) - F_k(a)}{F_k(z)} \right|$$

$$+ \frac{1}{2^k} \log |F_k(z)| \to g_{(c_n)}(z) \quad \text{as } k \to \infty.$$

This implies

$$g_{(c_n)}(z) = \int_{\mathcal{J}_{(c_n)}} \log |z - \zeta| \, d\lambda(\zeta) \qquad (z \in \mathcal{A}_{(c_n)}(\infty)),$$

and since μ^* is unique the assertion follows. $\square$

References

[A]   L.V. Ahlfors, *Lectures on Quasiconformal Mappings*, van Nostrand, Princeton, 1966.

[Be]   A.F. Beardon, *Iteration of Rational Functions*, Springer-Verlag, New York, 1991.

[BeP]   A.F. Beardon and Ch. Pommerenke, *The Poincaré metric of plane domains*, J. London Math. Soc. (2), **41** (1979), 475-483.

[Bl]   P. Blanchard, *Complex analytic dynamics on the Riemann sphere*, Bull. Amer. Math. Soc. (N.S.), **11** (1984), 85-141.

[Bro]   H. Brolin, *Invariant sets under iteration of rational functions*, Ark. Mat., **6** (1965), 103-144.

[Brü]   R. Brück, *Connectedness and stability of Julia sets of the composition of polynomials of the form* $z^2 + c_n$, J. London Math. Soc. (2), **61** (2000), 462-470.

[BBR]   R. Brück, M. Büger and S. Reitz, *Random iterations of polynomials of the form* $z^2 + c_n$: *Connectedness of Julia sets*, Ergodic Theory Dynam. Systems, **19** (1999), 1221-1231.

[Bü1] M. Büger, *Self-similarity of Julia sets of the composition of polynomials*, Ergodic Theory Dynam. Systems, **17** (1997), 1289-1297.

[Bü2] _____, *On the composition of polynomials of the form $z^2 + c_n$*, Math. Ann., **310** (1998), 661-683.

[C] L. Carleson, *Removable singularities of continuous harmonic functions in $\mathbb{R}^m$*, Math. Scand., **12** (1963), 15-18.

[CG] L. Carleson and T.W. Gamelin, *Complex Dynamics*, Springer-Verlag, New York, 1993.

[EL] A.E. Eremenko and M.Yu. Lyubich, *The dynamics of analytic transformations*, Algebra i Analiz, **1** (1989), 1-70; English transl. in Leningrad Math. J., **1** (1990), 563-634.

[FS] J.E. Fornæss and N. Sibony, *Random iterations of rational functions*, Ergodic Theory Dynam. Systems, **11** (1991), 687-708.

[G] V. Garber, *On the iteration of rational functions*, Math. Proc. Cambridge Philos. Soc., **84** (1978), 497-505.

[GV] F.W. Gehring and J. Väisälä, *Hausdorff dimension and quasiconformal mappings*, J. London Math. Soc. (2), **6** (1973), 504-512.

[LV] O. Lehto and K.I. Virtanen, *Quasiconformal Mappings in the Plane*, Springer-Verlag, Berlin, 1973.

[L1] M.Yu. Lyubich, *On typical behavior of the trajectories of a rational mapping of the sphere*, Dokl. Akad. Nauk SSSR, **268** (1983), 29-32; English transl. in Soviet Math. Dokl., **27** (1983), 22-25.

[L2] _____, *The dynamics of rational transforms*: *The topological picture*, Uspekhi Mat. Nauk, **41** (1986), 35-95; English transl. in Russian Math. Surveys, **41** (1986), 43-117.

[MR] R. Mañé and L.F. da Rocha, *Julia sets are uniformly perfect*, Proc. Amer. Math. Soc., **116** (1992), 251-257.

[M] J. Milnor, *Dynamics in One Complex Variable*, Vieweg Verlag, Braunschweig, 1999.

[NV] R. Näkki and J. Väisälä, *John disks*, Exposition. Math., **9** (1991), 3-43.

[P1] Ch. Pommerenke, *Uniformly perfect sets and the Poincaré metric*, Arch. Math. (Basel), **32** (1979), 192-199.

[P2] _____, *Boundary Behaviour of Conformal Maps*, Springer-Verlag, Berlin, 1992.

[Sh] M. Shishikura, *The Hausdorff dimension of the boundary of the Mandelbrot set and Julia sets*, Ann. of Math. (2), **147** (1998), 225-267.

[St] N. Steinmetz, *Rational Iteration* (*Complex Analytic Dynamical Systems*), Walter de Gruyter, Berlin, 1993.

[Su] D. Sullivan, *Conformal dynamical systems*, in 'Geometric Dynamics' (J. Palis, editor), Lecture Notes in Mathematics, **1007**, Springer-Verlag, New York, 1983, 725-752.

[T] M. Tsuji, *Potential Theory in Modern Function Theory*, Maruzen, Tokyo, 1959.

[Y]   M.V. Yakobson, *The boundaries of certain normality domains for rational maps*,
        Uspekhi Mat. Nauk, **39** (1984), 211-212; English transl. in Russian Math. Surveys,
        39 (1984), 229-230.

Justus-Liebig-Universität Giessen
Arndtstrasse 2
35392 Giessen
Germany
E-mail address: rainer.brueck@math.uni-giessen.de

SPECIAL VALUES OF KOECHER–MAASS SERIES OF SIEGEL CUSP FORMS

YOUNGJU CHOIE AND WINFRIED KOHNEN

A certain finiteness result for special values of character twists of Koecher-Maass series attached to Siegel cusp of genus g is proved.

1. Introduction.

Let f be an elliptic cusp form of even integral weight k on $\Gamma_1 := SL_2(\mathbf{Z})$. Let χ be a primitive Dirichlet character modulo a positive integer N and denote by $L(f, \chi, s)$ $(s \in \mathbf{C})$ the Hecke L-function of f twisted with χ, defined by analytic continuation of the series

$$\sum_{n \geq 1} \chi(n)a(n)n^{-s} \qquad (\text{Re}\,(s) \gg 0;\ a(n) = n\text{-th Fourier coefficient of } f).$$

Let $g(\chi)$ be the Gauss sum attached to χ. As is well-known, there exists a $\mathbf{Z}$-module $M_f \subset \mathbf{C}$ (depending only on f) of finite rank such that all the special values

$$i^{s+1}(2\pi)^{-s}g(\overline{\chi})L(f, \chi, s)$$

$(s \in \mathbf{N},\ 1 \leq s \leq k - 1;$

χ a primitive Dirichlet character modulo $N,\ N \in \mathbf{N})$

lie in $M_f \otimes_{\mathbf{Z}} \mathbf{Z}[\overline{\chi}]$, where $\mathbf{Z}[\overline{\chi}]$ is the $\mathbf{Z}$-module obtained from $\mathbf{Z}$ by adjoining the values of $\overline{\chi}$. In fact, if f is a Hecke eigenform, one has $\text{rk}_{\mathbf{Z}}\, M_f \leq 2$ [**1**, **7**, **8**, **10**].

The purpose of this paper is to give a generalization of the above result to the case of a Siegel cusp form f, where now $L(f, \chi, s)$ is replaced by an appropriate χ-twist of the Koecher-Maass series attached to f.

More precisely, let f be a cusp form of even integral weight $k \geq g + 1$ w.r.t. the Siegel modular group $\Gamma_g := Sp_g(\mathbf{Z})$ of genus g and write $a(T)$ (T a positive definite half-integral matrix of size g) for its Fourier coefficients. For χ as above we set

$$(1) \qquad L(f, \chi, s) := \sum_{\{T > 0\}/GL_{g,N}(\mathbf{Z})} \frac{\chi(\text{tr}\,T)a(T)}{\epsilon_N(T)(\det T)^s} \qquad (\text{Re}\,(s) \gg 0),$$

where the summation extends over all positive definite half-integral (g, g)-matrices T modulo the action $T \mapsto T[U] := U^t T U$ of the group $GL_{g,N}(\mathbf{Z}) := \{U \in GL_g(\mathbf{Z}) \mid U \equiv E_g \pmod{N}\}$ and $\epsilon_N(T) := \#\{U \in GL_{g,N}(\mathbf{Z}) \mid T[U] = T\}$ is the order of the corresponding unit group of T (note that $\epsilon_N(T) = 1$ whenever $N > 2$ by a classical result of Minkowski). Furthermore, $\operatorname{tr} T$ denotes the trace of T. Note that $\chi(\operatorname{tr} T)$ depends only on the $GL_{g,N}(\mathbf{Z})$-class of T.

In §2 (Thm. 1) we shall prove that the series $L(f, \chi, s)$ have holomorphic continuations to $\mathbf{C}$ and satisfy functional equations under $s \mapsto k - s$. The proof is fairly standard and follows the same pattern as in [6] for the case $N = 1$ (compare also [5]) and [9, §3.6] for $g = 1$.

The main result of the paper (Thm. 2) which will be proved in §3, states that all the special values

$$i^{gs + \frac{g(g+1)}{2}} \pi^{\frac{g(g-1)}{4} + [\frac{g}{2}]} (2\pi)^{-gs} g(\overline{\chi}) L(f, \chi, s)$$

$$\left(s \in \mathbf{N}, \ \frac{g+1}{2} \le s \le k - \frac{g+1}{2}; \right.$$

$$\left. \chi \ \text{a primitive Dirichlet character modulo} \, N, \, N \in \mathbf{N}\right)$$

are contained in $M_f \otimes_{\mathbf{Z}} \mathbf{Z}[\overline{\chi}]$ where $M_f \subset \mathbf{C}$ is a finite $\mathbf{Z}$-module depending only on f. Its rank is bounded by the rank of a certain singular relative homology group of a toroidal compactification of a quotient space of $\mathcal{H}_g \times \mathbf{C}^{gw}$, where $\mathcal{H}_g$ is the Siegel upper half-space of genus g and $w := k - (g+1)$. See §3 for details.

For the proof one represents the functions $L(f, \chi, s)$ (similar as in the case $g = 1$) as finite linear combinations of integrals of certain differential forms attached to f along certain $\frac{g(g+1)}{2}$-dimensional real subcycles of $\Gamma_g \backslash \mathcal{H}_g$. Our assertion then can be deduced if we use results of Hatada given in [2, 3]. More precisely, in [2] it is shown that the space of cusp forms of weight $k \ge g + 1$ w.r.t. a torsion-free congruence subgroup $\Gamma \subset \Gamma_g$ is canonically isomorphic to the space of holomorphic differential forms of highest degree on a compactification of $\Gamma \propto \mathbf{Z}^{2gw} \backslash \mathcal{H}_g \times \mathbf{C}^{gw}$, and in [3] using [2] a certain finiteness statement for a certain family of integrals of Siegel cusp forms is derived. (Actually, as we think, some of the assertions of [3] have to be slightly modified, for complete correctness' purposes; cf. §3.)

Inspecting the proof of Thm. 2, it is quite suggestive or even more or less clear that a similar finiteness statement as given there can be proved for special values of Dirichlet series of a much more general type. In fact, such a result essentially seems to be true for finite linear combinations of all the

partial series

$$\sum_{\{T>0\}/GL_g^{(S)}(\mathbf{Z})} \frac{e^{2\pi i\,\mathrm{tr}\,(TS)}a(T)}{\epsilon^{(S)}(T)(\det T)^s} \qquad (\mathrm{Re}\,(s) \gg 0),$$

where S is any rational symmetric matrix of size g, $GL_g^{(S)}(\mathbf{Z})$ is the subgroup $\{U \in GL_g(\mathbf{Z}) \,|\, S[U^t] \equiv S \pmod{\mathbf{Z}}\}$ and $\epsilon^{(S)}(T) := \#\{U \in GL_g^{(S)}(\mathbf{Z}) \,|\, T[U] = T\}$. However, we do not want to pursue this point further.

We finally remark that in [**4**] the Koecher-Maass series of a Siegel-Eisenstein series of genus g is explicitly expressed in terms of "elementary" zeta functions. In particular, if g is odd it is shown to be a sum of products of Riemann zeta functions. It would be interesting to investigate if a similar statement as given in Thm. 2 would also hold in this case. In fact, it is suggestive that such an assertion can be derived directly from the explicit formulas given in [**4**].

One can also ask similar questions in the case of a Klingen-Siegel-Eisenstein series.

Notations. If A and B are complex matrices of appropriate sizes, we put $A[B] := B^tAB$. We simply write $E = E_g$ resp. $0 = 0_g$ for the unit resp. zero matrix of size g if there is no confusion.

We often write elements of the group $GSp_g^+(\mathbf{R}) \subset GL_{2g}(\mathbf{R})$ consisting of real symplectic similitudes of size $2g$ with positive scale in the form $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$, understanding that A, B, C and D are real (g,g)-matrices.

If $Y \in \mathbf{R}^{(g,g)}$, we write $Y > 0$ if Y is symmetric and positive definite. The group $GL_g(\mathbf{R})$ operates on $\mathcal{P}_g := \{Y \in \mathbf{R}^{(g,g)} \,|\, Y > 0\}$ in the usual way from the right by $Y \mapsto Y[U]$.

If $f(Z)$ is a complex-valued function on $\mathcal{H}_g$, k a positive integer and $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in GSp_g^+(\mathbf{R})$, we set

$$(f|_k\gamma)(Z) := \det{(CZ+D)}^{-k} f((AZ+B)(CZ+D)^{-1}) \qquad (Z \in \mathcal{H}_g).$$

We often write $f|\gamma$ instead of $f|_k\gamma$ if there is no misunderstanding.

If k is a positive integer, Γ is a subgroup of Γ_g and χ is a character of Γ of finite order, we denote by $S_k(\Gamma, \chi)$ the space of Siegel cusp forms of weight k and character χ w.r.t. Γ. If $\chi = 1$ we simply write $S_k(\Gamma)$.

2. Character twists of Koecher-Maass series.

For N a natural number we define

$$\Gamma_{g,0}^*(N^2) := \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_g \,|\, C \equiv 0 \pmod{N^2}, \right.$$

$$\left. D \equiv \lambda E \pmod{N} \text{ for some } \lambda \in \mathbf{Z} \right\}$$

(note that λ must necessarily satisfy $(\lambda, N) = 1$).

It is easy to see that $\Gamma_{g,0}^*(N^2)$ is a subgroup of Γ_g. If χ is a Dirichlet character modulo N, we extend χ to a character of $\Gamma_{g,0}^*(N^2)$ by putting

$$\chi(\gamma) := \chi(\lambda) \text{ if } \gamma \equiv \begin{pmatrix} * & * \\ 0 & \lambda E \end{pmatrix} \pmod{N}.$$

Lemma 1. *Let* $f \in S_k(\Gamma_g)$ *with Fourier coefficients* $a(T)$ $(T > 0$ *half-integral). Let* χ *be a primitive Dirichlet character modulo* N. *Then the function*

$$f_\chi(Z) := \sum_{T>0} \chi(\operatorname{tr} T) a(T) e^{2\pi i \operatorname{tr}(TZ)} \qquad (Z \in \mathcal{H}_g)$$

belongs to $S_k(\Gamma_{g,0}^*(N^2), \chi^2)$.

Proof. Let

$$g(\overline{\chi}) := \sum_{\nu \pmod{N}} \overline{\chi}(\nu) e^{2\pi i \nu / N}$$

be the Gauss sum attached to $\overline{\chi}$. Since

$$\sum_{\nu \pmod{N}} \overline{\chi}(\nu) e^{2\pi i \operatorname{tr}(T) \frac{\nu}{N}} = \chi(\operatorname{tr} T) g(\overline{\chi}),$$

we obtain

$$(2) \qquad\qquad f_\chi = \frac{1}{g(\overline{\chi})} \sum_{\nu \pmod{N}} \overline{\chi}(\nu) f|\alpha_\nu,$$

where

$$\alpha_\nu := \begin{pmatrix} E & \frac{\nu}{N}E \\ 0 & E \end{pmatrix} \qquad (\nu \in \mathbf{Z}).$$

Let $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_{g,0}^*(N^2)$ and put

$$A' := A + \frac{\nu}{N}C,$$

$$B' := B + \frac{\nu}{N}(E - AD^t)D - \frac{\nu^2}{N^2}CD^tD,$$

$$D' := D - \frac{\nu}{N}CD^tD.$$

Then A', B' and D' are integral matrices, one has $D' \equiv D \pmod{N}$ and

$$\alpha_\nu \gamma = \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix} \begin{pmatrix} E & \frac{\nu}{N} D^t D \\ 0 & E \end{pmatrix};$$

in particular $\begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix} \in \Gamma_{g,0}^*(N^2)$, and it follows that

$$f_\chi|\gamma = \frac{1}{g(\overline{\chi})} \sum_{\nu \pmod N} \overline{\chi}(\nu) f| \begin{pmatrix} E & \frac{\nu}{N} D^t D \\ 0 & E \end{pmatrix}$$

$$= \chi(\lambda^2) \cdot \frac{1}{g(\overline{\chi})} \sum_{\nu \pmod N} \overline{\chi}(\nu) f|\alpha_\nu \qquad (D \equiv \lambda E \pmod N)$$

$$= \chi^2(\gamma) f.$$

This proves the claim.

Lemma 2. *Let the notations be as in Lemma* 1 *and put*

$$W_{N^2} := \begin{pmatrix} 0 & -E \\ N^2 E & 0 \end{pmatrix}.$$

Then

$$f_\chi|W_{N^2} = g(\chi)^2 N^{-gk-1} f_{\overline{\chi}}.$$

Proof. For $(\nu, N) = 1$ determine $\lambda, \mu \in \mathbf{Z}$ with $\lambda N - \mu\nu = 1$. Then

$$\alpha_\nu W_{N^2} = N \begin{pmatrix} 0 & -E \\ E & 0 \end{pmatrix} \begin{pmatrix} NE & -\mu E \\ -\nu E & \lambda E \end{pmatrix} \alpha_\mu.$$

Hence

$$g(\overline{\chi}) \cdot f_\chi|W_{N^2} = N^{-gk} \sum_{\nu \pmod N, (\nu,N)=1} \overline{\chi}(\nu) f|\alpha_\mu$$

$$= \chi(-1) N^{-gk} \sum_{\mu \pmod N, (\mu,N)=1} \chi(\mu) f|\alpha_\mu$$

$$= \chi(-1) g(\chi) N^{-gk} f_{\overline{\chi}}.$$

Since $g(\chi)g(\overline{\chi}) = \chi(-1)N$, we obtain our claim.

Theorem 1. *Let k be even and let $f \in S_k(\Gamma_g)$. Let χ be a primitive Dirichlet character modulo N and define $L(f, \chi, s)$ $(\operatorname{Re}(s) \gg 0)$ by* (1)*. Let*

$$\gamma_g(s) := (2\pi)^{-gs} \prod_{\nu=1}^{g} \pi^{(\nu-1)/2} \Gamma\left(s - \frac{\nu-1}{2}\right) \qquad (s \in \mathbf{C})$$

and set

$$L^*(f, \chi, s) := N^{gs} \gamma_g(s) L(f, \chi, s) \qquad (\operatorname{Re}(s) \gg 0).$$

Then $L^*(f, \chi, s)$ extends to a holomorphic function on $\mathbf{C}$, and the functional equation

$$L^*(f, \chi, k - s) = (-1)^{\frac{gk}{2}} g(\chi)^2 \frac{1}{N} L^*(f, \overline{\chi}, s)$$

holds, where $g(\chi)$ is the Gauss sum attached to χ.

Proof. Since

$$\left\{ \begin{pmatrix} U & 0 \\ 0 & (U^t)^{-1} \end{pmatrix} \mid U \in GL_{g,N}(\mathbf{Z}) \right\} \subset \Gamma_{g,0}^*(N^2)$$

and k is even, the function $f_\chi(iY)$ $(Y > 0)$ is invariant under $Y \mapsto Y[U]$ $(U \in GL_{g,N}(\mathbf{Z}))$. Hence it follows in the usual way that

$$(3) \qquad L^*(f, \chi, s) = \frac{1}{2} N^{gs} \int_{\mathcal{F}_{g,N}} f_\chi(iY)(\det Y)^s dv \qquad (\mathrm{Re}\,(s) \gg 0),$$

where $\mathcal{F}_{g,N}$ is any fundamental domain for the action of $GL_{g,N}(\mathbf{Z})$ on $\mathcal{P}_g$ and $dv = (\det Y)^{-(g+1)/2} dY$ is the $GL_g(\mathbf{R})$-invariant volume element on $\mathcal{P}_g$.

We fix a set of representatives $U_1, \dots, U_r$ for $GL_g(\mathbf{Z})/GL_{g,N}(\mathbf{Z})$ and now take

$$(4) \qquad \mathcal{F}_{g,N} = \bigcup_{\nu=1}^{r} \mathcal{R}_g[U_\nu],$$

where $\mathcal{R}_g$ is Minkowski's fundamental domain for the action of $GL_g(\mathbf{Z})$.

Since $GL_{g,N}(\mathbf{Z})$ is closed under transposition, also $\mathcal{F}_{g,N}^{-1}$ is a fundamental domain for $GL_{g,N}(\mathbf{Z})$.

We let

$$\mathcal{P}_{g,+} := \{ Y \in \mathcal{P}_g \mid \det Y \geq N^{-g} \}, \quad \mathcal{P}_{g,-} := \{ Y \in \mathcal{P}_g \mid \det Y \leq N^{-g} \},$$

write

$$\mathcal{F}_{g,N} = \left(\mathcal{F}_{g,N} \cap \mathcal{P}_{g,+} \right) \cup \left(\mathcal{F}_{g,N} \cap \mathcal{P}_{g,-} \right)$$

and observe that $\mathcal{F}_{g,N} \cap \mathcal{P}_{g,-}$ under the map $Y \mapsto (N^2 Y)^{-1}$ is transformed bijectively onto $\mathcal{F}_{g,N}^{-1} \cap \mathcal{P}_{g,+}$. We also observe that both $\mathcal{F}_{g,N} \cap \mathcal{P}_{g,+}$ and $\mathcal{F}_{g,N}^{-1} \cap \mathcal{P}_{g,+}$ are fundamental domains for the induced action of $GL_{g,N}(\mathbf{Z})$ on $\mathcal{P}_{g,+}$, the integral in (3) is absolutely convergent and the integrand is invariant under $GL_{g,N}(\mathbf{Z})$.

Therefore, since by Lemma 2

$$f_\chi(i(N^2 Y)^{-1}) = (-1)^{\frac{gk}{2}} g(\chi)^2 N^{gk-1} (\det Y)^k f_{\overline{\chi}}(iY),$$

we conclude that

$$(5) \qquad L^*(f,\chi,s) = \frac{1}{2} \int_{\mathcal{F}_{g,N} \cap \mathcal{P}_{g,+}} \Big(f_\chi(iY)(N^g \det Y)^s$$
$$+ (-1)^{\frac{gk}{2}} g(\chi)^2 N^{-1} f_{\overline{\chi}}(iY)(N^g \det Y)^{k-s} \Big) dv.$$

Standard arguments and estimates taking into account (4) and properties of $\mathcal{R}_g$ (compare e.g., [5, Chap. VI]) now show that the integral on the right of (5) is (absolutely) convergent for all $s \in \mathbf{C}$ and represents a holomorphic function of s.

Since

$$g(\chi)g(\overline{\chi}) = \chi(-1)N,$$

we also easily see the claimed functional equation. This concludes the proof of the Theorem.

3. Special values.

In this section we shall prove:

Theorem 2. *Let k be even, $k \geq g+1$ and let $f \in S_k(\Gamma_g)$. If χ is a primitive Dirichlet character modulo N, define $L(f,\chi,s)$ ($s \in \mathbf{C}$) by holomorphic continuation of the series (1) (Theorem 1). Let $g(\overline{\chi})$ be the Gauss sum attached to $\overline{\chi}$ and let $\mathbf{Z}[\overline{\chi}]$ be the $\mathbf{Z}$-module obtained from $\mathbf{Z}$ by adjoining the values of $\overline{\chi}$.*

Then there exists a $\mathbf{Z}$-module $M_f \subset \mathbf{C}$ depending only on f of finite rank such that all the special values

$$i^{gs + \frac{g(g+1)}{2}} \pi^{\frac{g(g-1)}{4} + [\frac{g}{2}]} (2\pi)^{-gs} g(\overline{\chi}) L(f,\chi,s)$$

where $s \in \mathbf{N}$, $\frac{g+1}{2} \leq s \leq k - \frac{g+1}{2}$ and χ runs over all primitive Dirichlet characters modulo all positive integers N, are contained in $M_f \otimes_{\mathbf{Z}} \mathbf{Z}[\overline{\chi}]$.

Proof. From (2) and (3) and the proof of Theorem 1 we find that
$$(6)$$
$$g(\overline{\chi})\gamma_g(s)L(f,\chi,s) = \frac{1}{2} \sum_{\nu \pmod N} \overline{\chi}(\nu) \int_{\mathcal{F}_{g,N}} f\left(iY + \frac{\nu}{N}E\right)(\det Y)^{s - \frac{g+1}{2}} dY$$

for all $s \in \mathbf{C}$.

Note that the individual integrands on the right of (6) are $GL_{g,N}(\mathbf{Z})$-invariant since $f(Z)$ is invariant under $\left\{ \begin{pmatrix} U & 0 \\ 0 & (U^t)^{-1} \end{pmatrix} \mid U \in GL_{g,N}(\mathbf{Z}) \right\}$ and under translations. Let $w \in \mathbf{Z}$, $w \geq 0$ and $Sp_g(\mathbf{R}) \propto \mathbf{R}^{2gw}$ be the semi-direct product of $Sp_g(\mathbf{R})$ and $\mathbf{R}^{2gw} \cong (\mathbf{R}^{2g})^w$ with multiplication given by

$$(\gamma, \lambda)(\gamma', \lambda') = (\gamma\gamma', \lambda{\gamma'}^\uparrow + \lambda')$$

where by $\gamma \mapsto \gamma^{\uparrow}$ we denote the diagonal embedding of $Sp_g(\mathbf{R})$ into $GL_{2gw}(\mathbf{R})$.

The group $Sp_g(\mathbf{R}) \propto \mathbf{R}^{2gw}$ acts on $\mathcal{H}_g \times \mathbf{C}^{gw}$ (with $\mathbf{C}^{gw} \cong (\mathbf{C}^g)^w$) from the left by

$$
(\gamma, \lambda) \circ \big(Z, (\zeta_1, \dots, \zeta_w)\big)
$$
$$
= \bigg((AZ + B)(CZ + D)^{-1}, \bigg(\zeta_1 + (\mu_1, \nu_1)\begin{pmatrix} Z \\ E_g \end{pmatrix}(CZ + D)^{-1},
$$
$$
\dots, \zeta_w + (\mu_w, \nu_w)\begin{pmatrix} Z \\ E_g \end{pmatrix}(CZ + D)^{-1}\bigg)\bigg)
$$

where $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ and $\lambda = \big((\mu_1, \nu_1), \dots, (\mu_w, \nu_w)\big)$ with $\mu_j, \nu_j \in \mathbf{R}^g$ for all j. The discrete subgroup $\Gamma_g \propto \mathbf{Z}^{2gw}$ acts properly discontinuously.

Let $\Gamma \subset \Gamma_g$ be any congruence subgroup acting without fixed points on $\mathcal{H}_g$ (e.g., the principal congruence subgroup $\Gamma_g(\ell)$ with $\ell \geq 3$) and view f as an element of $S_k(\Gamma)$.

Put $w := k - (g + 1)$. It was shown in [2] that the map

$$
h(Z) \mapsto h(Z)dZd\zeta
$$

gives an isomorphism between $S_k(\Gamma)$ and the space of holomorphic differential forms of degree $\frac{g(g+1)}{2} + gw$ of (any) non-singular compactification of the quotient space $\Gamma \propto \mathbf{Z}^{2gw} \backslash \mathcal{H}_g \times \mathbf{C}^{gw}$.

Using toroidal compactifications, in [3] from this a certain finiteness statement for certain cycle integrals attached to h was derived which we now want to describe in the special case we need.

Let S be a given rational symmetric matrix of size g and let n be an integer with $0 \leq n \leq w$. Define

$$
T_g(S; n) := \bigcup_{Y \in \mathcal{P}_g} \{S + iY\}
$$
$$
\times \Big((\mathbf{R}^g)^{w-n} \times \{(\mu_1 iY, \dots, \mu_n iY)\,|\,\mu_1, \dots, \mu_n \in \mathbf{R}^g\}\Big)
$$
$$
\subset \mathcal{H}_g \times \mathbf{C}^{gw}.
$$

Then $T_g(S; n)$ is a real submanifold of $\mathcal{H}_g \times \mathbf{C}^{gw}$ of dimension $\frac{g(g+1)}{2} + gw$.

(In the notation of [3, §6] we have taken $a_1 = a_2 = \dots = a_{w-n} \in \{g + 1, \dots, 2g\}$ and $a_{w-n+1} = \dots a_w \in \{1, \dots, g\}$. Also note that in the definition of $T_g(a_1, \dots, a_w; X)$ in [3, p. 401] we have replaced the "Z" in $W(a_1, \dots, a_w)[Z]$ by "iY". We think that this is the correct definition, since otherwise the corresponding integrals in [3, Lemma 6.2 and Thm. 5] in general would not be convergent.)

Put

$$\mathcal{U}_g := \left\{ \begin{pmatrix} U & 0 \\ 0 & (U^t)^{-1} \end{pmatrix} \mid U \in GL_g(\mathbf{R}) \right\} \subset Sp_g(\mathbf{R}),$$

$$V_{g,n} := \{(\lambda_1, \dots, \lambda_{w-n}, (\mu_1, 0), \dots, (\mu_n, 0)) \mid \lambda_1, \dots,$$
$$\lambda_{w-n} \in \mathbf{R}^{2g}, \mu_1, \dots, \mu_n \in \mathbf{R}^g\}$$

and

$$H_{g,n} := \mathcal{U}_g \propto V_{g,n} \subset Sp_g(\mathbf{R}) \propto \mathbf{R}^{2gw}.$$

Let

$$\alpha^{(S)} := \begin{pmatrix} E & S \\ 0 & E \end{pmatrix}.$$

Then one easily checks that the conjugate subgroup

$$H_{g,n}^{(S)} := (\alpha^{(S)}, 0) \cdot H_{g,n} \cdot (\alpha^{(S)}, 0)^{-1}$$

leaves $T_g(S; n)$ stable.

Note that $H_{g,n}^{(S)}$ consists of all pairs

$$\left(\begin{pmatrix} U & S(U^t)^{-1} - US \\ 0 & (U^t)^{-1} \end{pmatrix}, (\lambda_1, \dots, \lambda_{w-n}, (\mu_1, -\mu_1 S), \dots, (\mu_n, -\mu_n S)) \right)$$

with $\lambda_1, \dots, \lambda_{w-n} \in \mathbf{R}^{2g}$ and $\mu_1, \dots, \mu_n \in \mathbf{R}^g$.

Let

$$H_{g,n,\Gamma}^{(S)} := H_{g,n}^{(S)} \cap \Gamma \propto \mathbf{Z}^{2gw}.$$

Write $M := \Gamma \propto \mathbf{Z}^{2gw} \backslash \mathcal{H}_g \times \mathbf{C}^{gw}$ and denote by $\overline{M}$ a fixed toroidal compactification of M. Let $\partial M = \overline{M} \setminus M$. Then according to [**3**, Lemma 6.1] the closure of the image of $H_{g,n,\Gamma}^{(S)} \backslash T_g(S; n)$ in $\overline{M}$ w.r.t. the usual complex topology is the support of a singular relative $\frac{g(g+1)}{2} + gw$-cycle with integral coefficients w.r.t. $(\overline{M}, \partial M)$.

Since $H_{\frac{g(g+1)}{2} + gw}(\overline{M}, \partial M, \mathbf{Z})$ is of finite rank, one concludes that for any given $h \in S_k(\Gamma)$ all the numbers

$$\int_{H_{g,n,\Gamma}^{(S)} \backslash T_g(S;n)} h(Z) dZ d\zeta \qquad (S \in \mathbf{Q}^{(g,g)}, S = S^t)$$

are contained in a finite $\mathbf{Z}$-module (depending only on h) whose rank is bounded by the rank of the above cohomology group ([**3**, Thm. 5], compare our above remark).

On the other hand (compare [**3**, Lemma 6.2]) one has the equality

$$
(7) \qquad \int_{H_{g,n,\Gamma}^{(S)} \backslash T_g(S;n)} h(Z) dZ d\zeta
$$
$$
= \int_{\alpha^{(S)} \cdot \mathcal{U}_g \cdot (\alpha^{(S)})^{-1} \cap \Gamma \backslash \{S+iY \,|\, Y \in \mathcal{P}_g\}} h(Z) \det (Z - S)^n dZ.
$$

In particular, now take $\Gamma = \Gamma_g(\ell)$ with some fixed $\ell \geq 3$. Then the integral on the right of (7) is equal to

$$
i^{gn+\frac{g(g+1)}{2}} \int_{\mathcal{P}_g/GL_{g,\ell}^{(S)}(\mathbf{Z})} h(S + iY)\,(\det Y)^n dY,
$$

where

$$
GL_{g,\ell}^{(S)}(\mathbf{Z}) := \{U \in GL_{g,\ell}(\mathbf{Z}) \,|\, S[U^t] \equiv S \pmod{\ell \mathbf{Z}}\}.
$$

Let $S = \frac{\nu}{N} E$ with $\nu \in \mathbf{Z}$ (so $\alpha^{(S)} = \alpha_\nu$ in the notation of §2). Then we see that $GL_{g,\ell N}(\mathbf{Z})$ is contained in $GL_{g,\ell}^{(S)}(\mathbf{Z})$. Since the index of $GL_{g,\ell N}(\mathbf{Z})$ in $GL_{g,N}(\mathbf{Z})$ is bounded by a number depending only on ℓ, the assertion of Thm. 2 now follows taking into account (6) and the fact that $\Gamma(\frac{1}{2}+\nu) \in \mathbf{Q}\sqrt{\pi}$ for $\nu = 0, 1, 2 \ldots$.

References

[1] M. Eichler, *Eine Verallgemeinerung der abelschen Integrale*, Math. Ann., **67** (1957), 267-298.

[2] K. Hatada, *Siegel cusp forms as holomorphic differential forms on certain compact varieties*, Math. Ann., **262** (1983), 503-509.

[3] ———, *Homology Groups, Differential Forms and Hecke Rings on Siegel Modular Varieties*, in 'Topics in Mathematical Analysis' (ed. Th.M. Rassias), World Scientific Publ. Co., (1989), 371-409.

[4] T. Ibukiyama and H. Katsurada, *An explicit form of Koecher-Maass series of Siegel-Eisenstein series*, preprint 2000.

[5] H. Klingen, *Introductory Lectures on Siegel Modular Forms*, Cambridge Univ. Press, Cambridge, 1990.

[6] H. Maass, *Siegel's modular forms and Dirichlet series*, Lect. Not. in Mathematics, **216**, Springer, Berlin, 1971.

[7] J. Manin, *Periods of parabolic forms and p-adic Hecke series*, Math. Sb., **21** (1973), 371-393.

[8] G. Shimura, *Sur les intégrales attachées aux formes automorphes*, J. Math. Soc. Japan, **11**(**4**) (1959), 291-311.

[9] G. Shimura, *Introduction to the Arithmetic Theory of Automorphic Functions*, Iwanami Shoten Publ. and Princeton Univ. Press, 1971.

[10] V.V. Šokurov, *Shimura integrals of cusp forms*, Math. USSR Izvestija, **16**(**3**) (1981), 603-646.

DEPARTMENT OF MATHEMATICS
POHANG INSTITUTE OF SCIENCE & TECHNOLGY
POHANG 790-784
KOREA
E-mail address: yjc@yjc.postech.ac.kr

UNIVERSITÄT HEIDELBERG
MATHEMATISCHES INSTITUT,
INF 288, D-69120 HEIDELBERG
GERMANY
E-mail address: winfried@mathi.uni-heidelberg.de

AN ABSTRACT
VOICULESCU–BROWN–DOUGLAS–FILLMORE
ABSORPTION THEOREM

George A. Elliott and Dan Kucerovsky

A common generalization is given of what are often referred to as the Weyl–von Neumann theorems of Voiculescu, Kasparov, Kirchberg, and, more recently, Lin. (These in turn extend a result of Brown, Douglas, and Fillmore.)

More precisely, an intrinsic characterization is obtained of those extensions of one separable C*-algebra by another—the first, i.e., the ideal, assumed to be stable, so that Brown-Douglas-Fillmore addition of extensions can be carried out—which are absorbing in a certain natural sense related to this addition, a sense which reduces to that considered by earlier authors if either the ideal or the quotient is nuclear. The specific absorption theorems referred to above can be deduced from this characterization.

1. Let B be a C*-algebra, and let C be a C*-algebra containing B as a closed two-sided ideal. Let us say that C is purely large with respect to B if for every element c of C which is not in B, the C*-algebra $\overline{cBc^*}$ (the intersection with B of the hereditary sub-C*-algebra of C generated by cc^*) contains a sub-C*-algebra which is stable (i.e., isomorphic to its tensor product with the C*-algebra $\mathcal{K}$ of compact operators on an infinite-dimensional separable Hilbert space) and is full in B (i.e., not contained in any proper closed two-sided ideal of B).

2. Let A and B be C*-algebras, and let

$$0 \rightarrow B \rightarrow C \rightarrow A \rightarrow 0$$

be an extension of B by A (i.e., a short exact sequence of C*-algebras). Let us say that the extension is purely large if the C*-algebra of the extension, C, is purely large with respect to the image of B in it, in the sense described above.

Note that, if B is non-zero, a purely large extension of B by A is essential (that is, the image of B in the C*-algebra of the extension is an essential closed two-sided ideal—every non-zero closed two-sided ideal has non-zero intersection with it).

3.   Let A and B be C*-algebras, with A unital. An extension $0 \rightarrow B \rightarrow C \rightarrow A \rightarrow 0$ will be said to be unital if C is unital.

In this paper we shall consider primarily the context of unital extensions (although we shall indicate how to modify our main result, Theorem 6, to be valid in the non-unital setting).

4.   Recall that an extension of B by A is determined by its Busby map— the naturally associated map from A to the quotient multiplier algebra, or corona, of B, $\mathrm{M}(B)/B$. (The C*-algebra of the extension is the pullback of the Busby map and the canonical quotient map $\mathrm{M}(B) \rightarrow \mathrm{M}(B)/B$.)

Recall (see e.g., [6]) that, if B is stable, so that the Cuntz algebra $\mathcal{O}_2$ may be embedded unitally in $\mathrm{M}(B)$, then the Brown-Douglas-Fillmore addition of extensions, defined by

$$\tau_1 \oplus \tau_2 \ := \ s_1\tau_1 s_1^* + s_2\tau_2 s_2^*,$$

where τ_1 and τ_2 are (the Busby maps of) two extensions of B by A, and s_1 and s_2 are (the images in $\mathrm{M}(B)/B$ of) the canonical generators of $\mathcal{O}_2$ (which are isometries with range projections summing to 1), is compatible with Brown-Douglas-Fillmore equivalence (defined as unitary equivalence with respect to the unitary group of $\mathrm{M}(B)$—or, rather, the image of this group in $\mathrm{M}(B)/B$), and the resulting binary operation on equivalence classes is independent of the embedding of $\mathcal{O}_2$.

With respect to this operation, the equivalence classes of extensions of the stable C*-algebra B by the C*-algebra A form an abelian semigroup.

Recall that an extension of B by A is said to be trivial if, considered as a short exact sequence of C*-algebra maps, it splits. In other words, the map $C \rightarrow A$ in the sequence $0 \rightarrow B \rightarrow C \rightarrow A \rightarrow 0$ should have a left inverse, $C \leftarrow A$. (Equivalently, the Busby map $A \rightarrow \mathrm{M}(B)/B$ should lift to a C*-algebra homomorphism $A \rightarrow \mathrm{M}(B)$.)

In the setting of unital extensions, we shall understand triviality of an extension to mean that the splitting can be chosen to be unital.

Recall, furthermore, that, in [8], Kasparov called an extension absorbing if, in the Brown-Douglas-Fillmore semigroup, it is equal to its sum with any trivial extension. (Briefly, if it absorbs every trivial extension.) Of course, a unital extension cannot be absorbing in this sense (unless the quotient algebra is zero); let us say that a unital extension is absorbing if—in the subsemigroup of unital extensions—it is equal to its sum with any trivial unital extension. (Trivial in the sense of admitting a unital splitting.)

5.   In order to be able to formulate our main result (Theorem 6, below) for arbitrary (separable) C*-algebras A and B (with B stable and A unital)— i.e., without assuming A or B to be nuclear—we must restrict the notion of trivial extension as follows.

Let us say that an extension of C*-algebras $0 \to B \to C \to A \to 0$ is trivial in the nuclear sense if the splitting homomorphism $A \to C$ may be chosen to be weakly nuclear as defined by Kirchberg in [**9**]: The splitting homomorphism $\pi\colon A \to C$ will be said to be weakly nuclear if, for every $b \in B \subseteq C$, the map

$$A \ni a \;\mapsto\; b\pi(a)b^* \in B \subseteq C$$

is nuclear. (Recall that a C*-algebra map is said to be nuclear if it factors approximately through finite-dimensional C*-algebras, by means of completely positive contractions, in the sense of convergence in norm.)

Let us say, correspondingly, that an extension is absorbing in the nuclear sense if it absorbs every extension which is trivial in the nuclear sense. Again, let us say that a unital extension is absorbing in the nuclear sense to mean that this holds within the semigroup of (equivalence classes of) unital extensions. (With triviality in the nuclear sense the existence of a unital weakly nuclear splitting.)

6.

Theorem. *Let A and B be separable C*-algebras, with B stable and A unital. A unital extension of B by A is absorbing, in the nuclear sense, if, and only if, it is purely large.*

7.   Purely large algebras have an approximation property similar to that of purely infinite algebras. (This is the fundamental ingredient in the proof of our main result, that an extension that is purely large is absorbing—either in the unital setting, as in Theorem 6, or, if the extension is non-unital, as in Corollary 16.)

Lemma. *Let C be a C*-algebra that is purely large with respect to a closed two-sided ideal B, in the sense of Section 1. Then, for any positive element c of C which is not in B, any $\epsilon > 0$, and any positive element b of B, there exists $b_0 \in B$ with*

$$\|b - b_0 c b_0^*\| < \epsilon.$$

If b is of norm one, and if the image of c in C/B is of norm one, then b_0 may be chosen to have norm one.

Proof. Let $c \in C^+ \setminus B$, $b \in B^+$, and $\epsilon > 0$ be given. Multiplying c by a positive element of the sub-C*-algebra it generates, and changing notation, we may suppose that the hereditary sub-C*-algebra C_c of C on which c acts as a unit is not contained in B.

By hypothesis, there exists a full, stable sub-C*-algebra D of B contained in C_c.

Since D is full in B, there exist $d \in D^+$ and $b_1, \dots, b_n$ in B such that

$$\left\| b - \sum b_i d b_i^* \right\| < \epsilon.$$

(The set S of such elements $\sum b_i db_i^*$ is closed under the map $x \mapsto yxy^*$ for any $y \in B$, and therefore the closure of S is a hereditary subset of B^+— recall that if $0 \leq s \leq t$ in B then $s^{\frac{1}{2}} = \lim y_n t^{\frac{1}{2}}$ with $y_n = s^{\frac{1}{2}}(t + \frac{1}{n})^{-\frac{1}{2}} \in B$, so that $s = \lim y_n t y_n^*$. In particular, the closure of S is a subcone of B^+, as the sum of two elements of S, associated with d_1 and d_2, say, is majorized by an element associated with the single element $d_1 + d_2$, and therefore is a limit of elements associated with $d_1 + d_2$. The closure of S is thus a closed subcone of B^+ closed under the map $x \mapsto yxy^*$ for any $y \in B$. Such a subset is known to be the positive part of a closed two-sided ideal: as above it must be a hereditary subset, and it is then the positive part of an ideal by Theorem 2.7(ii) of [**4**].)

(Alternatively, to obtain the assertion of the preceding paragraph, approximate $b^{\frac{1}{2}}$ by $\sum b_{1i} d_0 b_{2i}^*$ for some $d_0 \in D^+$ and $b_{1i}, b_{2i} \in B$. Replacing $\sum b_{1i} d_0 b_{2i}^*$ by its self-adjoint part (which has a similar form), and changing notation, we may suppose that $\sum b_{1i} d_0 b_{2i}^*$ is self-adjoint. (In any case, this element is almost self-adjoint, which is sufficient.) Write

$$\sum b_{1i} d_0 b_{2i}^* = b_1 (d_0) b_2^*$$

where b_1 and b_2 denote the row vectors (b_{1i}) and (b_{2i}), and (d_0) denotes the square matrix of appropriate size with d_0 repeated down the diagonal and 0 elsewhere. Note that $b_1 (d_0) b_2^* = b_2 (d_0) b_1^*$. We then have

$$\|b - b_1 (d_0) b_2^* b_2 (d_0) b_1^*\|$$

$$= \|b - b^{\frac{1}{2}} b_1 (d_0) b_2^* + b^{\frac{1}{2}} b_2 (d_0) b_1^* - b_1 (d_0) b_2^* b_2 (d_0) b_1^*\|$$

$$\leq \|b^{\frac{1}{2}}\| \, \|b^{\frac{1}{2}} - b_1 (d_0) b_2^*\| + \|b^{\frac{1}{2}} - b_1 (d_0) b_2^*\| \, \|b_2 (d_0) b_1^*\|,$$

and the right side is arbitrarily small. Finally, noting that $(d_0) b_2^* b_2 (d_0)$ belongs to the hereditary sub-C*-algebra generated by $(d_0^{\frac{1}{2}})$, we may approximate this element by the element

$$((d_0^{\frac{1}{2}}) c (d_0^{\frac{1}{2}}))((d_0^{\frac{1}{2}}) c (d_0^{\frac{1}{2}}))^* = (d_0^{\frac{1}{2}}) c(d_0) c^* (d_0^{\frac{1}{2}})$$

for some matrix c over B, and then with $b_1' = b_1 (d_0^{\frac{1}{2}}) c$, the element

$$b - b_1' (d_0) b_1'^*$$

is small, i.e., $b - \sum b_{1i}' d_0 b_{1i}'^*$ is small, as desired.)

Since D is stable, we may suppose, changing d by a small amount, that there exists a multiplier projection e of D such that $ed = d$ and such that for multipliers $u_1, \ldots, u_n$,

$$u_i u_j^* = \delta_{ij} e.$$

Hence with $d_i = d^{\frac{1}{2}} u_i$,

$$d_i d_j^* = \delta_{ij} d.$$

Set
$$\sum b_i d_i = b_0.$$
Then, on the one hand,
$$b_0 b_0^* = \sum b_i d_i d_j^* b_j^* = \sum b_i d b_i^*,$$
and, on the other hand, as $d_i c = d_i$ and so $b_0 c = b_0$,
$$b_0 b_0^* = b_0 c b_0^*.$$
We now have
$$\|b - b_0 c b_0^*\| = \|b - b_0 b_0^*\| = \left\| b - \sum b_i d b_i^* \right\| < \epsilon.$$

Now suppose that
$$\|b\| = \|c + B\| = 1,$$
and let us show that b_0 may be chosen with norm one. The modification of c in the above construction may then be arbitrarily small, and so, as b_0 will be chosen with norm one (see below), we may again suppose that the hereditary sub-C*-algebra C_c of C on which c acts as a unit is not contained in B. Repeating the construction above with $\epsilon/2$ in place of ϵ, we have $\|b - b_0 b_0^*\| \le \epsilon/2$, and so (as $\|b\| = 1$),
$$1 - \frac{\epsilon}{2} \le \|b\| - \|b - b_0 b_0^*\| \le \|b_0 b_0^*\| \le \|b\| + \|b_0 b_0^* - b\| \le 1 + \frac{\epsilon}{2}.$$
Hence,
$$\left\| b_0 b_0^* \left(1 - \frac{1}{\|b_0 b_0^*\|} \right) \right\| = \left| \|b_0 b_0^*\| - 1 \right| \le \frac{\epsilon}{2},$$
and so
$$\left\| b - \frac{b_0}{\|b_0\|} c \frac{b_0^*}{\|b_0\|} \right\| = \left\| b - \frac{b_0 b_0^*}{\|b_0 b_0^*\|} \right\| \le \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

8. Let us recall the generalization of Glimm's Lemma due to Akemann, Anderson, and Pedersen (Proposition 2.2 of [**1**]). Because we shall only need the unital case, and that case is much easier, let us give a proof in that case.

Lemma. *Let C be a separable unital C*-algebra and let ρ be a pure state of C. There exists $c_0 \in C^+$ with $\|c_0\| = 1$ such that $\rho(c_0) = 1$ and*
$$\lim_{n \to \infty} \|c_0^n (c - \rho(c)) c_0^n\| = 0, \quad c \in C.$$

Proof. With $N_\rho = \{c \in C; \ \rho(c^* c) = 0\}$, recall that, as ρ is pure, $\mathrm{Ker}\rho = N_\rho + N_\rho^*$. Hence,
$$C = \mathbb{C}1 + N_\rho + N_\rho^*.$$
Choose a strictly positive element h of $N_\rho \cap N_\rho^*$ of norm at most one, and set $1 - h = c_0$. The desired convergence—which is additive—holds obviously for $c \in \mathbb{C}$, and it holds for $c \in N_\rho$ or N_ρ^* because $h c_0^n = h(1-h)^n \to 0$ (as $t(1-t)^n \to 0$ uniformly for $t \in [0,1]$).

9.   The following consequence of Lemmas 7 and 8 is the main step in the proof of Theorem 6.

Corollary. *Let C be a separable unital C^*-algebra that is purely large with respect to the closed two-sided ideal B. Let ρ be a pure state of C that is zero on B, let $c_1 = (c_{11}, \dots, c_{1n})$ be a row vector over C and let $b_1 = (b_{11}, \dots, b_{n1})$ be a row vector over B. Denote the tensor product of ρ with the identity on $\mathrm{M}_n(\mathbb{C})$,*

$$\mathrm{id} \otimes \rho \colon \mathrm{M}_n(C) = \mathrm{M}_n(\mathbb{C}) \otimes C \to \mathrm{M}_n(\mathbb{C}),$$

by ρ_n. The map

$$C \to B$$
$$c \mapsto b_1 \rho_n(c_1^* c c_1) b_1^*$$

can be approximated on finite sets by the maps

$$c \mapsto bcb^*, \quad b \in B.$$

Proof. It is immediate to reduce to the case $n = 1$. (Considering b_1 and c_1 as elements of $\mathrm{M}_n(C)$, and C as the subalgebra of upper left corner matrices, extend ρ to a pure state of $\mathrm{M}_n(C)$—necessarily unique, and concentrated in the upper left corner—and denote this again by ρ. If $(b_{ij}) \in \mathrm{M}_n(B)$ gives an approximating map for the map

$$\mathrm{M}_n(C) \to \mathrm{M}_n(B)$$
$$c \mapsto b_1 \rho(c_1^* c c_1) b_1^*,$$

which fulfils the hypotheses of the Corollary with $n = 1$ and with C and B replaced by $\mathrm{M}_n(C)$ and $\mathrm{M}_n(B)$ (note that in this case $\rho_1 = \rho$), then $b_{11} \in B$ gives an approximating map for the given map.)

Let a finite subset $F \subseteq C$ be given. By Lemma 8, there exists $c_0 \in C^+$ such that $\|c_0\| = 1$, $\rho(c_0) = 1$, and $c_0 c_1^* c c_1 c_0$ is arbitrarily close to $c_0 \rho(c_1^* c c_1) c_0$ for each $c \in F$. Namely, c_0 may be taken to be a power of the c_0 of Lemma 8; note that $0 \le c_0 \le 1$ and $\rho(c_0) = 1$ imply that $\rho(c_0^k) = 1$ for any k.

Since $\rho(c_0^2) = 1$ and $\rho(B) = 0$, the element c_0^2 does not belong to B and so by Lemma 7 there exists $b_0 \in B$ such that $b_0 c_0^2 b_0^*$ is arbitrarily close to any given positive element of B. In particular, approximating an approximate unit for B, we may choose $b_0 \in B$ such that $b_1(b_0 c_0^2 b_0^*)$ is arbitrarily close to b_1. Since the image of c_0 in C/B is of norm one, by Lemma 7 we may suppose that $\|b_0\| = 1$. Then, with $b = b_1 b_0 c_0 c_1^*$, for each $c \in F$, the element

$$bcb^* = b_1 b_0 (c_0 c_1^* c c_1 c_0) b_0^* b_1^*$$

is (by choice of c_0), arbitrarily close to

$$b_1 b_0 c_0 \rho(c_1^* c c_1) c_0 b_0^* b_1^* = b_1 b_0 c_0^2 b_0^* b_1^* \rho(c_1^* c c_1),$$

which in turn (by choice of b_0) is arbitrarily close to

$$b_1 b_1^* \rho(c_1^* c c_1) \; = \; b_1 \rho(c_1^* c c_1) b_1^*.$$

In other words, the desired approximation holds.

10. The following lemma, incorporating techniques of Kirchberg, brings Corollary 9 to bear in the nuclear setting.

Lemma. *Let C be a separable unital C^*-algebra and let B be a closed two-sided ideal of C. Suppose that C is purely large with respect to B. Let ψ be a completely positive map from C to B which is zero on B. If the map from the quotient C/B to B determined by ψ is nuclear, and if B is stable, then ψ can be approximated on finite sets by the maps*

$$c \; \mapsto \; b^* c b, \quad b \in B.$$

Proof. First, without assuming that C is purely large, let us show, using ideas of Kirchberg presented in [**9**], that if the map $C/B \to B$ determined by ψ is nuclear then ψ can be approximated on finite sets by sums of maps of the kind considered in Corollary 9 (each one corresponding to a row vector over C, a pure state of C zero on B, and a column vector over B).

By the nuclearity hypothesis, which implies that ψ is the limit of a sequence of products of two completely positive maps, the first from C to M_k for some k, and zero on B, and the second from M_k to B, we may suppose that ψ itself is the product of two such maps—i.e., a completely positive map $C \to \mathrm{M}_k$, zero on B, and a completely positive map $\mathrm{M}_k \to B$.

As B is stable, so that $\mathcal{O}_k$ is unitally contained in $\mathrm{M}(B)$—unless $B = 0$ in which case the assertion is vacuous—, by Lemma 1.1 of [**9**] a completely positive map $\mathrm{M}_k \to B$ is necessarily of the form $x \mapsto R x R^*$ where R is a row vector over B. (As shown in [**9**] this holds with R the transpose of the matrix $(e_1, \dots, e_k)^*$—i.e., for $R = (e_1^*, \dots, e_k^*)$, where

$$(e_1, \dots, e_k) \; = \; (s_1, \dots, s_k) G^{\frac{1}{2}}$$

with $s_1, \dots, s_k$ the canonical generating isometries of $\mathcal{O}_k$ and G the image in $\mathrm{M}_k \otimes B$ of the positive element (e_{ij}) of $\mathrm{M}_k(\mathrm{M}_k)$ corresponding to the canonical system of matrix units for M_k.)

It remains to show—in order to verify the assertion above—that a completely positive map $C \to \mathrm{M}_k$, zero on B, can be approximated on finite sets by sums of maps of the form

$$c \mapsto \rho(F^* c F)$$

where F is a row vector over C of length k and ρ is a pure state of C zero on B. Replacing C by C/B, we see that it is enough to establish this in the case $B = 0$. In this case, we may proceed as follows (in a way somewhat similar to the proof of Lemma 1.2 of [**9**]—which concerns the special case

that C is simple and not elementary). By the Krein-Milman theorem, we may suppose (since we are allowing sums) that the given completely positive map $C \to \mathrm{M}_k$ belongs to an extremal ray within the cone of all such maps. (Consider a compact base for this cone.) By Stinespring's theorem, the given map may be expressed as a representation of C on a Hilbert space, followed by cutting down to a generating subspace of dimension k—with a specified orthonormal basis identifying the operators on this subspace with the elements of M_k. By extremality of the ray containing the given map (just as in the case of a positive linear functional), this representation must be irreducible. By the Kadison transitivity theorem the specified basis then has the form $F\eta$ where η is an arbitrary nonzero vector in the space of the representation and F is a row vector over C. With ρ the pure state of C determined by η, the given completely positive map is now equal to the map $c \mapsto \rho(F^*cF)$.

This completes the proof that ψ can be approximated on finite sets by sums of maps of the kind considered in Corollary 9, say in particular, on the given finite subset S of C, by the sum

$$\psi_1 + \cdots + \psi_n$$

where each ψ_i is as in Corollary 9 (and in particular is zero on B, which of course is no longer necessarily zero). Then—as C is purely large—by Corollary 9, on a given finite family of elements of C, say S, the map ψ_1 can be approximated by the map $c \mapsto b_1 c b_1^*$ for some $b_1 \in B$. By Corollary 9 again, the map ψ_2 can be approximated by the map $c \mapsto b_2 c b_2^*$ for some $b_2 \in B$, not only on S but on any larger finite subset of C, and in particular on the set

$$S_2 \ := \ S \cup \{cb_1^*b_1c^*; \quad c \in S \cup S^*\}.$$

Since ψ_2 is zero on $cb_1^*b_1c^*$, $c \in S \cup S^*$, it follows that $b_2cb_1^*b_1c^*b_2^*$ is small for $c \in S \cup S^*$, i.e., the norms $\|b_2cb_1^*\| = \|b_2cb_1^*b_1c^*b_2^*\|^{\frac{1}{2}}$ and $\|b_1cb_2^*\| = \|b_2c^*b_1^*b_1cb_2^*\|^{\frac{1}{2}}$ are small for each $C \in S$. Hence, for each $c \in S$,

$$(b_1 + b_2)c(b_1 + b_2)^*$$

is close to $b_1cb_1^* + b_2cb_2^*$, and so to $(\psi_1 + \psi_2)(c)$. Proceeding in this way (as, for instance, in [2]), we obtain $b_1, \cdots, b_n \in B$ such that $\psi_1 + \cdots + \psi_n$—and hence ψ—is approximated (arbitrarily closely) on S by the map

$$c \ \mapsto \ (b_1 + \cdots + b_n)c(b_1 + \cdots + b_n)^*.$$

11.   The following technique is basic in some form to all earlier absorption results. It was formulated more or less explicitly in special cases in [2], [8], and [9], and expressed in the following abstract form in a later version of the preprint [9].

Lemma (Kirchberg). *Let C be a unital separable C^*-algebra and let B be an essential closed two-sided ideal of C, so that we may view C as a unital subalgebra of $\mathrm{M}(B)$:*

$$B \subseteq C \subseteq \mathrm{M}(B); \quad 1 \in C.$$

Let $\phi : C \to \mathrm{M}(B)$ be a completely positive map which is zero on B, and suppose that, for every $b_0 \in B$, the map

$$b_0^* \phi b_0 : C \to B$$
$$c \mapsto b_0^* \phi(c) b_0$$

can be approximated (on finite sets) by the maps

$$c \mapsto b^* c b, \quad b \in B.$$

It follows that there exists $v \in \mathrm{M}(B)$ such that

$$\phi(c) - v^* c v \in B, \quad c \in C.$$

The element v may be chosen so that the map $c \mapsto v^* c v$ also approximates ϕ on a given finite subset of C.

Proof. Let us recall, for the convenience of the reader, the argument of (the extended version of) [**9**].

First, by a slight reformulation of Theorem 2 of [**2**] (and its proof), there exist positive elements $w_1, w_2, \ldots$ of B of norm one such that the series $\sum w_i x_i w_i$ converges strictly in $\mathrm{M}(B)$ for any bounded sequence (x_i) in $\mathrm{M}(B)$, and such that the sum $\sum w_i^2 \phi(c) w_i^2 \in \mathrm{M}(B)$ is equal to $\phi(c)$ modulo B for every c, and approximately equal to $\phi(c)$ in $\mathrm{M}(B)$ (in norm) for each c in a given finite subset of C. The sequence $w_1, w_2, \ldots$ may be chosen furthermore such that the sequence $(\sum_1^n w_i^4)$ is an approximate unit for B, and such that w_{n+2}^4 is orthogonal to $\sum_1^n w_i^4$ for each n.

One now proceeds very much as in the proof of Lemma 10 above (which dealt with a finite sum of maps from C to B) to show that the infinite sum $w_1^2 \phi w_1^2 + w_2^2 \phi w_2^2 + \cdots$ of maps from C to B (convergent pointwise in the strict topology of $\mathrm{M}(B)$ to a map from C to $\mathrm{M}(B)$—equal to ϕ modulo B and equal to ϕ approximately on the given finite set), each of which is zero on B and is determined approximately by an element of B, is determined approximately on the given finite set by a strictly converging sum of elements of B, and determined by this multiplier exactly modulo B.

More explicitly, one chooses $b_1 \in B$ such that $b_1^* c b_1$ is close to $w_1 \phi(c) w_1$ for c in a finite set S_1, to be specified, one then chooses $b_2 \in B$ such that $b_2^* c b_2$ is close to $w_2 \phi(c) w_2$ for c in a finite set S_2, also to be specified—and depending in addition on the choice of b_1, as in the proof of Lemma 10—and one continues in this way. As we shall show, with suitable choices of the finite sets $S_1, S_2, \cdots$ and of the approximations at each stage, the series

$$b_1 w_1 + b_2 w_2 + \cdots$$

converges strictly in $\mathrm{M}(B)$ to an element v with the desired properties (it determines the sum of maps $w_1^2 \phi w_1^2 + w_2^2 \phi w_2^2 + \cdots$, and hence also the map ϕ, to within a specified approximation on a given finite set, and exactly, modulo B, on all of C).

The sets $S_1, S_2, \ldots$ should of course all contain the given finite set, say S, and their union should be dense in C. They should also all contain the unit of C ($1 \in \mathrm{M}(B)$); then for each i the element $b_i^* b_i$ ($= b_i^* 1 b_i$) is close to $w_i \phi(1) w_i$ and in particular the sequence $b_1, b_2, \ldots$ is bounded. In order for the series $b_1 w_1 + b_2 w_2 + \cdots$ to be strictly convergent, it would be sufficient in view of the properties of the sequence $w_1, w_2, \ldots$ and the boundedness of the sequence $b_1, b_2, \ldots$ to ensure that

$$\sum \|b_i - w_i^4 b_i\| \ < \ \infty,$$

as then convergence of the series $b_1 w_1 + b_2 w_2 + \cdots$ (in the strict topology) follows from convergence of the series $w_1^4 b_1 w_1 + w_2^4 b_2 w_2 + \cdots$, which holds as the sequence $w_1^3 b_1, w_2^3 b_2, \ldots$ is bounded.

It would also be sufficient to arrange that, instead of convergence of $\sum \|b_i - w_i^4 b_i\|$, one has convergence of the series

$$\sum \|b_i - z_i^4 b_i\|$$

where $z_1, z_2, \ldots$ is some other sequence of positive elements of B with the last property mentioned for (w_i) (namely, that $\sum_1^n z_i^4$ is an approximate unit for B, and $z_{n+2}^4 \sum_1^n z_i^4 = 0$ for each n). Indeed, this property (for both (w_i) and (z_i)) is enough for the series

$$\sum z_i x_i w_i$$

to converge strictly in $\mathrm{M}(B)$ for any bounded sequence x_i in $\mathrm{M}(B)$. While z_i^4 may be taken to be the sum of a consecutive group of elements w_j^4, it would not appear to be possible to choose $z_i = w_i$.

Let us now elaborate on the choice of the finite sets $S_1, S_2, \ldots$, and on the choice of a partition of $\mathbf{N}$ into consecutive subsets $J_1, J_2, \ldots$ such that, with

$$\sum_{j \in J_i} w_j^4 \ =: \ z_i^4,$$

the necessary approximations can be made. (Namely, for $\sum b_i w_i$ to exist and have the desired properties; note that the introduction of z_i is purely to ensure convergence of the sum.)

The finite set S_i should contain, as well as the given finite set S and the unit, $1 \in C \subseteq \mathrm{M}(B)$, the first i elements of a fixed dense sequence $(c_1, c_2, \ldots)$ in C. Let us choose

$$S_1 \ = \ S \cup \{c_1\} \cup \{1\}.$$

In order to ensure convergence of $v = \sum b_i w_i$, and negligibility of the cross terms in the product $v^* cv$, for $c \in S$ or, when working modulo B, for $c \in C$ (it is enough to consider $c \in \{c_1, c_2, \dots\}$), we must choose

$$S_2 = (S_1 \cup \{c_2\}) \cup \left\{ \left(\sum_1^{n_1} w_k^4 \right)^2 \right\} \cup \{cb_1 b_1^* c^*; \quad c \in S_1 \cup S_1^*\},$$

where b_1 is such that $b_1^* c b_1$ is close to $w_1 \phi(c) w_1$ for $c \in S_1$, and n_1 is such that the difference

$$\left(\sum_1^{n_1} w_k^4 \right) b_1 - b_1$$

is small; proceeding in this way, for each $i \geq 2$ we must choose

$$S_{i+1} = (S_i \cup \{c_{i+1}\}) \cup \left\{ \left(\sum_1^{n_i} w_k^4 \right)^2 \right\} \cup \{cb_i b_i^* c^*; \quad c \in S_i \cup S_i^*\},$$

where b_i is such that $b_i^* c b_i$ is close to $w_i \phi(c) w_i$ for $c \in S_i$, and n_i is such that the difference

$$\left(\sum_1^{n_i} w_k^4 \right) b_i - b_i$$

is small. By "close", and "small", we mean that the sum of all the tolerances in question should be finite, and smaller than a certain single number (small enough that the desired approximation of ϕ occurs on the set S).

Note that, as $\phi(B) = 0$, the element

$$b_{i+1}^* \left(\sum_1^{n_i} w_k^4 \right)^2 b_{i+1},$$

is small, i.e., $(\sum_1^{n_i} w_k^4) b_{i+1}$ is small. As $(\sum_1^{n_{i+1}} w_k^4) b_{i+1} - b_{i+1}$ is small (by the choice of n_{i+1}), also

$$\left(\sum_{n_i+1}^{n_{i+1}} w_k^4 \right) b_{i+1} - b_{i+1} \quad \text{is small.}$$

In other words, with

$$\{n_i + 1, \dots, n_{i+1}\} = J_{i+1},$$

$i = 1, 2, \dots$, and with, say, $J_1 = \{1, \dots, n_i\}$, setting $\sum_{j \in J_i} w_j^4 = z_i^4$ (with $z_i \geq 0$), we have a sequence (z_i) with the desired properties (including that $z_i^4 b_i - b_i$ is small, in the sense of being summable).

For each i, $(b_i w_i)^* c (b_i w_i)$ is close to $w_i^2 \phi(c) w_i^2$ for $c \in S \cup \{1, c_1, \dots, c_i\}$—in the summable sense described above. The cross terms in the expression

$$v^* cv = \left(\sum b_i w_i \right)^* c \left(\sum b_i w_i \right)$$

are negligible in the sense described above by the choice of the sequence $S_1, S_2, \ldots$ (to correlate with the choice of $b_1, b_2, \ldots$; cf. proof of Lemma 10).

12. In order to prove that an arbitrary extension (of a stable separable C*-algebra by a separable C*-algebra) which is absorbing in the nuclear sense is purely large, we must first establish the existence of some purely large extension, and in fact one which is trivial in the nuclear sense—so that we can use the absorbing hypothesis. (It follows from the other implication of Theorem 6 that, in the unital setting, such an extension is necessarily unique—up to equivalence.)

An extension with these properties (purely large, and trivial in the nuclear sense) was constructed by Kasparov in [**8**]—although Kasparov did not establish these properties. (What Kasparov proved, in terms of our terminology, was that his extension was absorbing in the nuclear sense.) Let us now verify the asserted properties.

Lemma. *Let A and B be separable C*-algebras, with B stable and A unital. There exists a purely large unital extension of B by A which is trivial in the nuclear sense (as a unital extension).*

Proof. We may suppose that both A and B are non-zero. Kasparov in [**8**] considered the extension of $B \otimes \mathcal{K}(H)$ by A with splitting

$$A \hookrightarrow 1 \otimes B(H) \hookrightarrow \mathrm{M}(B \otimes \mathcal{K}(H)),$$

where $A \hookrightarrow B(H)$ is a faithful unital representation of A on the separable infinite-dimensional Hilbert space H. Choosing such a representation π of A, and choosing an isomorphism of $B \otimes \mathcal{K}(H)$ with B, we obtain an extension of B by A—obviously trivial (but a priori depending on the choices made). Let us denote this extension by τ_0.

To show that τ_0 is trivial in the nuclear sense, it is sufficient to show that the given splitting,

$$A \xrightarrow{\pi} 1 \otimes B(H) \hookrightarrow \mathrm{M}(B \otimes \mathcal{K}(H)) \cong \mathrm{M}(B),$$

is weakly nuclear. In other words, given $d \in B \otimes \mathcal{K}(H)$, it is enough to show that the map

$$d\pi d^* : \quad A \ni a \ \mapsto \ d\pi(a)d^* \in B \otimes \mathcal{K}(H)$$

is nuclear, i.e., factorizes approximately through a finite-dimensional C*-algebra by means of completely positive maps. With (e_n) an approximate unit for $\mathcal{K}(H)$ consisting of projections of finite rank, note that for each n the completely positive map

$$(e_n d e_n)\pi(e_n d e_n)^*,$$

where we write e_n again for $1 \otimes e_n \in \mathrm{M}(B \otimes \mathcal{K}(H))$, factors through the finite-dimensional C*-algebra $e_n \mathcal{K}(H) e_n$ (as the composition of the completely

positive maps $e \mapsto e_n \pi(a) e_n \in e_n \mathcal{K}(H) e_n$ and $x \mapsto (e_n d e_n) x (e_n d e_n)^* \in B \otimes \mathcal{K}(H))$. Since $e_n = 1 \otimes e_n$ converges to 1 in $\mathrm{M}(B \otimes \mathcal{K}(H))$ in the strict topology, in the topology of pointwise convergence

$$(e_n d e_n) \pi (e_n d e_n)^* \; \to \; d\pi d^*.$$

We shall prove below, in Theorem 17(iii), that a considerably more general construction than Kasparov's also gives rise to a purely large extension (trivial, but not necessarily in the nuclear sense). Therefore, rather than duplicating this proof—or omitting it in the more general case, which includes the interesting class of extensions considered by Lin in [**10**]—, we shall omit it in the present case.

13.

Lemma. *The sum of any two C*-algebra extensions one of which is purely large is again purely large.*

Proof. Recall that, by definition, an extension is purely large when the associated C*-algebra is purely large, with respect to the canonical closed two-sided ideal. Recall also, that, in this case, the canonical closed two-sided ideal is essential—so that the C*-algebra of the extension may be considered as a subalgebra of the multiplier algebra of the ideal. It is sufficient to show, then, that if B is a C*-algebra, if C is a sub-C*-algebra of $\mathrm{M}(B)$ containing B, and if there exists a projection e in $\mathrm{M}(B)$ commuting with C modulo B, such that the C*-algebra eCe is purely large with respect to the ideal eBe, such that if $c \in C$ and $ece \in B$ then $c \in B$, and such that eBe is full in B, then the C*-algebra C is purely large with respect to the ideal B.

With B and C (and e) as above, let c be an element of C not contained in B, and let us show that $\overline{cBc^*}$ contains a stable sub-C*-algebra which is full in B.

Since $c \notin B$, by hypothesis $ece \notin eBe$, and so $(eceBec^*e)^-$ contains a stable sub-C*-algebra which is full in eBe, and hence also full in B. Hence, as

$$eceBec^*e \; \subseteq \; ecBc^*e,$$

the sub-C*-algebra $(ecBc^*e)^-$ of B contains a stable sub-C*-algebra which is full in B.

While $(ecBc^*e)^-$ may not be contained in $(cBc^*)^-$, there is a natural isomorphism of the C*-algebra $(ecBc^*e)^-$ with $(c^*eBec)^-$, which is contained in the algebra $(c^*Bc)^-$. Furthermore, as this isomorphism consists of the restriction to $(ecBc^*e)^-$ of the map

$$B^{**} \ni b \; \mapsto \; w^* b w \in B^{**},$$

where w denotes the partially isometric part of $ec \in C \subseteq B^{**}$, and its inverse is the restriction to $(c^*eBec)^-$ of the map

$$B^{**} \ni b \ \mapsto \ wbw^* \in B^{**},$$

the subalgebras $(ecBc^*e)^-$ and $(c^*eBec)^-$ of B generate the same closed two-sided ideal. This shows that $(c^*Bc)^-$ contains a stable sub-C*-algebra which is full in B. It follows by a similar argument (or just by replacing c by c^*) that $(cBc^*)^-$ does, too.

14.

Lemma. *Any C*-algebra extension equivalent to a purely large one is purely large.*

Proof. The property in question is, by definition, a property of the C*-algebra of the extension, together with the distinguished ideal, not of the extension itself. Equivalence of extensions preserves the isomorphism class of the associated C*-algebra, with its canonical ideal.

15. *Proof of Theorem* 6. Let τ be a unital extension of B by A. (We shall identify τ with its Busby map $A \to \mathrm{M}(B)/B$.)

Suppose that τ is purely large, and let us show that τ is absorbing in the nuclear sense.

Given a unital extension τ' of B by A which is trivial in the nuclear sense, i.e., which has a unital weakly nuclear splitting, we must show that

$$\tau \ \sim \ \tau \oplus \tau',$$

i.e., that τ and $\tau \oplus \tau'$, considered as maps from A to $\mathrm{M}(B)/B$, are unitarily equivalent, by means of the image in $\mathrm{M}(B)/B$ of a unitary element of $\mathrm{M}(B)$.

As in the case of earlier absorption theorems, it is sufficient to prove (for arbitrary τ' as above) that

$$\tau \ \sim \ \sigma \oplus \tau'$$

for some unital extension σ, not necessarily equal to τ. Indeed, as in [**2**] (which systematizes [**13**], and is the model for later absorption proofs, including the present one)—see also below—one may construct a trivial extension τ''—trivial also in the nuclear sense, and as a unital extension—such that

$$\tau'' \oplus \tau' \ \sim \ \tau''.$$

Hence, with σ such that

$$\tau \sim \sigma \oplus \tau'',$$

it follows that

$$
\begin{aligned}
\tau \oplus \tau' &\sim (\sigma \oplus \tau'') \oplus \tau' \\
&\sim \sigma \oplus (\tau'' \oplus \tau') \\
&\sim \sigma \oplus \tau'' \\
&\sim \tau.
\end{aligned}
$$

A unital extension τ'' such that $\tau'' \oplus \tau' \sim \tau''$, which is trivial in the nuclear sense—as a unital extension—, is obtained by forming the infinite multiplicity sum $(\pi')^\infty$ of a unital weakly nuclear splitting π' of τ' (cf. [2]). This is defined first as just the map

$$
\begin{aligned}
\pi' \oplus 1 : \quad A &\rightarrow \mathrm{M}(B \otimes \mathcal{K}) \\
a &\mapsto \pi'(a) \otimes 1.
\end{aligned}
$$

This map then is transformed into a (unital) map

$$
\pi'' : \quad A \rightarrow \mathrm{M}(B)
$$

by identifying B with $B \otimes e_{11} \subseteq B \otimes \mathcal{K}$, and then transforming $B \otimes e_{11}$ onto $B \otimes \mathcal{K}$ by means of an isometry in $\mathrm{M}(B \otimes \mathcal{K})$, which we shall denote by s_2, such that $s_2 s_2^* = 1 \otimes e_{11}$. (Such an isometry exists because B is stable; more explicitly, with $B = B_0 \otimes \mathcal{K}$, we may choose $s_2 = 1 \otimes t_2 \in \mathrm{M}(B_0 \otimes (\mathcal{K} \otimes \mathcal{K}))$ where t_2 is an isometry in $\mathrm{M}(\mathcal{K} \otimes \mathcal{K})$ with range $1 \otimes e_{11}$.) Choose an isometry t_1 in $\mathrm{M}(\mathcal{K})$ with range $1 - e_{11}$, and set $1 \otimes t_1 = s_1$. The (desired) equivalence

$$
\pi'' \oplus \pi' \sim \pi''
$$

(unitary equivalence of maps from A to $\mathrm{M}(B)$) then reduces (by transformation by s_2) to the equivalence

$$
(\pi' \otimes 1) \oplus s_2^*(\pi' \otimes e_{11})s_2 \sim \pi' \otimes 1
$$

(unitary equivalence of maps from A to $\mathrm{M}(B \otimes \mathcal{K})$), which may be seen by using the Cuntz isometries s_1 and s_2 to compute the left-hand side:

$$
\begin{aligned}
s_1(\pi' \otimes 1)s_1^* + s_2(s_2^*(\pi' \otimes e_{11})s_2)s_2^* &= \pi' \otimes (1 - e_{11}) + \pi' \otimes e_{11} \\
&= \pi' \otimes 1.
\end{aligned}
$$

With τ'' the unital extension with splitting π'', we then have $\tau'' \oplus \tau' \sim \tau''$; it remains only to note that τ'' is trivial in the nuclear sense, as $\pi' \otimes 1$ and hence π'' are weakly nuclear.

To show that $\tau \sim \sigma \oplus \tau'$, for some unital extension σ, with τ' as given—a unital extension with a weakly nuclear unital splitting—we shall in fact not use that this splitting is a C*-algebra homomorphism, but only that it is completely positive (and unital, and weakly nuclear, in the sense described for a homomorphism).

Since the C*-algebra, C, of the extension τ is purely large with respect to the closed two-sided ideal B (canonically contained in it), in particular B is essential, and so we may write

$$B \subseteq C \subseteq M(B),$$

and aim to apply Lemma 11 to the completely positive map $\phi : C \to M(B)$ obtained by composing the canonical quotient map from C to A with a weakly nuclear, unital, completely positive map from A to $M(B)$ lifting τ'. (Note that the existence of such a map is clearly equivalent to the existence of a splitting map with these properties from A to the C*-algebra of the extension τ', namely, the pullback of A and its preimage in $M(B)$.)

In order to apply Lemma 11, we must verify that for every $b_0 \in B$, the map

$$b_0^* \phi b_0 : \quad C \ni c \mapsto b_0^* \phi(c) b_0 \in B$$

can be approximated by the maps

$$c \mapsto b^* c b, \quad b \in B.$$

Fix $b_0 \in B$, and set $b_0^* \phi b_0 = \psi$. Since, by construction, the map ψ from C to B is zero on B, and the associated map from C/B to B is nuclear, the approximibility of ψ by maps $c \mapsto b^* c b$ with $b \in B$ is ensured by Lemma 10.

By Lemma 11, there exists $v \in M(B)$ such that

$$\phi(c) - v^* c v \in B, \quad c \in C,$$

and such that also $v^* c v$ is close to $\phi(c)$ for c belonging to any given finite set, and in particular for $c = 1$. As ϕ is unital, $v^* v$ is close to 1, and equal to 1 modulo B. Hence, replacing v by $v(v^* v)^{-\frac{1}{2}}$, we may suppose that $v^* v = 1$.

The first property of v may be rewritten as

$$\tau' = v^* \tau v$$

(i.e., $\tau'(a) = v^* \tau(a) v, \quad a \in A$, where v denotes the image of $v \in M(B)$ in $M(B)/B$).

Since τ' is multiplicative this in particular implies that the projection $vv^* \in M(B)/B$ commutes with $\tau(A)$. (As v is an isometry, also $v\tau'v^*$ is multiplicative, and therefore also $(vv^*)\tau(vv^*)$; with $vv^* = e$ we then have $e\tau(a^*)\tau(a)e = e\tau(a^*a)e = e\tau(a^*)e\tau(a)e$, whence $e\tau(a^*)(1-e)\tau(a)e = 0$, i.e., $(1-e)\tau(a)e = 0$; since a is arbitrary, also $(1-e)\tau(a^*)e = 0$, and so $\tau(a)e = e\tau(a)$.)

Since Brown-Douglas-Fillmore addition of (equivalence classes of) extensions is independent of the choice of the unital copy of $\mathcal{O}_2$ in $M(B)$ (cf. above), to show that

$$\tau \sim \sigma \oplus \tau'$$

it would be sufficient to know that the projection $1 - vv^*$ is Murray-von Neumann equivalent to 1 in $M(B)$. Indeed, with s_1 an isometry with range

$1 - vv^*$, and $s_2 = v$,

$$\begin{aligned}
\tau &= (1 - vv^*)\tau + vv^*\tau \\
&= (1 - vv^*)\tau(1 - vv^*) + vv^*\tau vv^* \\
&= s_1 s_1^* \tau s_1 s_1^* + s_2 s_2^* \tau s_2 s_2^* \\
&= s_1 \sigma s_1^* + s_2 \tau' s_2^* \\
&= \sigma \oplus \tau'
\end{aligned}$$

where $\sigma = s_1^* \tau s_1$ (recall that $\tau' = v^* \tau v$).

Instead of showing directly that it is possible to choose v above with $1 - vv^*$ equivalent to 1, let us choose v with respect to $\tau' \oplus \tau'$ instead of τ'—and call this w. (Note that $\tau' \oplus \tau'$ has a unital weakly nuclear completely positive splitting if τ' does.) Then

$$\begin{aligned}
\tau &= (1 - ww^*)\tau + ww^*\tau \\
&= (1 - ww^*)\tau + e_1\tau + e_2\tau,
\end{aligned}$$

where e_1 and e_2 are projections equivalent to 1 with $e_1 + e_2 = ww^*$, commuting with $\tau(A)$ modulo B, and $e_2\tau$ is equivalent (by means of an isometry with range e_2) to τ'. Provided we show that also $(1 - ww^*) + e_1$ is equivalent to 1, this says that $\tau = \sigma \oplus \tau'$.

Let us show, then, using that B is stable, that if e is a projection in $\mathrm{M}(B)$ equivalent to 1, and f is any projection orthogonal to e, then $e + f$ is equivalent to 1. We shall deduce this from the well known fact that $\mathrm{M}(\mathcal{K})$, and hence $\mathrm{M}(B)$, contains an infinite sequence of mutually orthogonal projections, say $e_1, e_2, \dots$, equivalent to 1 and with sum 1 (in the strict topology). It follows that any sequence of projections (f_i) in $\mathrm{M}(B)$ with $f_i \leq e_i$ also has convergent sum. Clearly, the projection $e_2 + e_3 + \cdots$ is equivalent to $e_1 + e_2 + \cdots = 1$. If f_1 is any subprojection of e_1, choose a subprojection f_i of e_i for $i \geq 2$ equivalent to f_1, and note that, also, $f_1 + f_2 + \cdots$ is equivalent to $f_2 + f_3 + \cdots$. Therefore, by additivity of equivalence, on adding the single projection $(e_2 - f_2) + (e_3 - f_3) + \cdots$ to both of these projections we obtain that $f_1 + e_2 + e_3 + \cdots$ is equivalent to $e_2 + e_3 + \cdots$, as desired.

(The preceding considerations are superfluous in the case $B = \mathcal{K}$, considered in [**13**] and [**2**].)

Now assume that τ is absorbing, in the nuclear sense, and let us show that τ is purely large.

By Lemma 12, there exists a purely large unital extension τ_0 of B by A which is trivial in the nuclear sense. By hypothesis,

$$\tau \sim \tau \oplus \tau_0.$$

By Lemma 13, $\tau \oplus \tau_0$ is purely large. Hence by Lemma 14, τ is purely large, as desired.

16. It follows immediately from Theorem 6 that, if one considers the non-unital setting (i.e., extensions which are not necessarily unital, or with a non-unital quotient), then one has the following criterion for an extension τ of a stable separable C*-algebra B by a separable C*-algebra A to be absorbing in the nuclear sense:

The unital extension $\tilde{\tau}$ of B by $\tilde{A}$, the C*-algebra A with unit adjoined (i.e., a new unit if A is already unital), naturally corresponding to τ (with Busby map extending that of τ), should be purely large.

(To see this, note that by Theorem 6, the preceding condition is equivalent to the condition that $\tilde{\tau}$ be absorbing in the nuclear sense, as a unital extension. This, on the other hand, is equivalent to the condition that τ be absorbing in the nuclear sense (in the non-unital setting): the extensions of B by A which are trivial in the nuclear sense are in bijective correspondence with the unital extensions of B by $\tilde{A}$ which are trivial in the nuclear sense, in the unital setting, by the map $\sigma \mapsto \tilde{\sigma}$. Finally it is clear that $\tau + \sigma \sim \tau$ if, and only if, $\tilde{\tau} + \tilde{\sigma} \sim \tilde{\tau}$.)

Let us note that, for the extension $\tilde{\tau}$ of B by $\tilde{A}$ to be purely large, it is necessary and sufficient for τ itself to be purely large, and non-unital. (If C is purely large with respect to B, and non-unital, we must show that also $\tilde{C}$ is purely large with respect to B. (Clearly, if $\tilde{C}$ is purely large then C is purely large and non-unital.) In other words, we must show that for any $c \in C$, $((1 + c)B(1 + c)^*)^-$ contains a full stable sub-C*-algebra of B. If $(1 + c)C \subseteq B$, then the image of $-c$ in $\mathrm{M}(B)/B$ is a unit for the image of C in $\mathrm{M}(B)/B$; hence, the image of C in $\mathrm{M}(B)$ contains $1 \in \mathrm{M}(B)$; as τ is essential the map $C \to \mathrm{M}(B)$ is injective, and hence C is unital, contrary to hypothesis. This shows that there exists $c' \in C$ with $(1 + c)c'$ not in B. Hence, the subalgebra

$$((1 + c)c'B((1 + c)c')^*)^- \subseteq ((1 + c)B(1 + c)^*)^-$$

contains a stable sub-C*-algebra which is full in B.) (As a consequence, Corollary 9 and Lemma 10 hold also in the non-unital case—but we will not use this.)

Let us summarize:

Corollary. *Let B be a stable separable C*-algebra, and let A be a separable C*-algebra. Let τ be an extension of B by A.*

The extension τ is absorbing, in the nuclear sense, if and only if τ is purely large and non-unital.

In particular, if A is non-unital (i.e., does not have a unit element), then τ is absorbing if and only if τ is purely large.

17. Let us now show directly that those extensions previously known to be absorbing (in the nuclear sense) are purely large. (We refer to the absorption theorems of [**3**], [**13**], [**11**], [**8**], [**9**], and [**10**].)

On the one hand, this yields a new proof—via Theorem 6—of the absorption property. On the other hand, as pointed out in Section 12, the proof that an arbitrary extension which is absorbing in the nuclear sense is purely large depends on first knowing the existence of at least one purely large extension—which is also trivial in the nuclear sense. This is proved in Lemma 12, using Theorem 17(iii) below.

Concerning the notion of absorbing extension, note that if an extension is absorbing in the sense that it absorbs every trivial extension (in the class of unital extensions, say), then it is certainly absorbing in the nuclear sense; on the other hand, so far the only known examples of true absorbing extensions are in the case that either the ideal or the quotient is nuclear, so that the true sense and the nuclear sense coincide (every trivial extension is trivial in the nuclear sense).

Theorem. *Let A and B be separable C^*-algebras, with B stable. Let τ be a C^*-algebra extension of B by A. Suppose that τ is essential (i.e., that the Busby map $A \to M(B)/B$ is injective; see Section 2). In each of the following cases, τ is purely large (in the sense of Section 2).*

(i) *$B = \mathcal{K}$. (Cf. [**3**], [**13**].)*

(i)′ *$B = C_0(X) \otimes \mathcal{K}$ where X is a finite-dimensional locally compact Hausdorff space, and the map from A to the canonical quotient $M(\mathcal{K})/\mathcal{K}$ of $M(B)/B$ corresponding to each point of X is injective (in other words, τ is homogeneous in the sense introduced for such a B in [**11**]). (Cf. [**11**].) (In [**11**], X is restricted to be compact.)*

(ii) *B is simple and purely infinite. (Cf. [**9**].)*

(iii) *τ is trivial, with a splitting*

$$\pi: \quad A \to 1 \otimes M(B_1) \hookrightarrow M(B_0 \otimes B_1) = M(B)$$

*for some tensor product decomposition $B = B_0 \otimes B_1$, with B_0 stable, such that, for any non-zero $a \in A$, the closed two-sided ideal of B_1 generated by $\pi(a)B_1$ is equal to B_1. (This last property is automatic if B_1 is simple—for instance, as in [**10**], or as in the case $B_1 = \mathcal{K}$ considered in [**8**] and in Lemma 12 above. It is also automatic if A is simple and $\pi(A)B_1$ is dense in B_1—as considered also in [**10**].)*

Proof. As in the proof of Theorem 6, since the map $A \to M(B)/B$ is injective, we may suppose that the C^*-algebra of the extension is a subalgebra of $M(B)$.

Ad (i). For any $c \in M(\mathcal{K})$ which is not in $\mathcal{K}$, the hereditary sub-C^*-algebra $(c\mathcal{K}c^*)^-$ of $\mathcal{K}$ is infinite-dimensional and hence, as it is equal to $e\mathcal{K}e$ for some projection $e \in M(\mathcal{K})$ ($M(\mathcal{K})$ being the bidual of $\mathcal{K}$), it is isomorphic to $\mathcal{K}$ and in particular is stable and full.

404 GEORGE A. ELLIOTT AND DAN KUCEROVSKY

Ad (i)'. By hypothesis, for any $c \in \mathrm{M}(B)$ belonging to the C*-algebra of the extension, but not to B, the hereditary sub-C*-algebra $(cBc^*)^-$ of B is full. (This is a simple reformulation of the hypothesis of homogeneity.)

Let us show, that for any such c the C*-algebra $(cBc^*)^-$ is stable. By hypothesis, for each point of X, not only is the image of $(cBc^*)^-$ in the quotient $\mathcal{K}$ of B at this point non-zero, but (since the C*-algebra of the extension contains B, and this property holds with c replaced by $c + b$ for any $b \in B$) also this image is stable.

Let us show that, more generally, any hereditary sub-C*-algebra of B the image of which in each primitive quotient of B is stable (possibly equal to zero) is itself stable. Here, B is still as above. Let D be such a hereditary sub-C*-algebra of B. Note that D is a C*-algebra with continuous trace—as B has continuous trace, and this property is preserved (as is easily seen) under passage to a hereditary sub-C*-algebra. By Theorem 10.9.5 of [5], D is determined up to isomorphism, among the class of all separable C*-algebras with continuous trace, with all primitive quotients equal to $\mathcal{K}$ and with the same spectrum as D (note that this space has finite dimension), by its Dixmier-Douady invariant. By inspection of the construction of this invariant (see 10.7.14 of [5]), one sees that it is unchanged by tensoring by $\mathcal{K}$. It follows that D is isomorphic to $D \otimes \mathcal{K}$, as desired.

Ad (ii). For any $c \in \mathrm{M}(B)$ which is not in B, the hereditary sub-C*-algebra $(cBc^*)^-$ of B is non-zero and therefore (by the definition of purely infinite simple C*-algebra that we shall use) contains an infinite projection. In other words, $(cBc^*)^-$ contains a partial isometry v such that $vv^* < v^*v$. The partial isometries $v^n(v^*v - vv^*)$, $n = 1, 2, \ldots$, generate a sub-C*-algebra of $(cBc^*)^-$ isomorphic to $\mathcal{K}$, full in B as B is simple.

(In fact, in the present case, as $(cBc^*)^-$ cannot be unital, by [14] this algebra itself is stable.)

Ad (iii). Recall that, as shown by Hjelmborg and Rørdam in [7], using the criterion for stability that they established, as B is separable and stable the hereditary sub-C*-algebra $((1+b)B(1+b)^*)^-$ is stable for any $b \in B$.

Let us begin by noting that a similar, but rather simpler, argument shows that, also, the hereditary sub-C*-algebra $((1+b)B(1+b)^*)^-$ is full in B for each $b \in B$. (We are indebted to M. Rørdam for this argument.) With (u_n) a sequence of unitary elements of $\mathrm{M}(B)$ such that

$$b_1 u_n b_2 \to 0 \quad \text{for all} \quad b_1, b_2 \in B,$$

as exists by [7] if B is stable (u_n may be chosen to be $1 \otimes v_n$ with (v_n) such a sequence in $\mathrm{M}(\mathcal{K})$, in particular a sequence of unitaries corresponding to finite permutations of an orthonormal basis), one has for each fixed $b \in B$,

$$u_n(1+b)u_n^* \to 1 \quad \text{strictly in} \quad \mathrm{M}(B).$$

Hence, for each $b' \in B$,

$$(u_n(1+b)u_n^*)b'(u_n(1+b)u_n^*)^* \; \to \; b'.$$

This shows in particular that the closed two-sided ideal generated by $(1 + b)B(1 + b)^*$ is dense in B, i.e., $((1 + b)B(1 + b)^*)^-$ is full in B, as asserted.

Now let us show that for any element c of C, the C*-algebra of the extension, not contained in B, the C*-algebra $(cBc^*)^-$ contains a stable sub-C*-algebra which is full in B. We shall base our argument on the case $c = 1+b$, considered above.

The special nature of the present setting may be expressed as follows:

In a certain decomposition of B as $B_1 \otimes \mathcal{K}$, with B_1 stable (and hence isomorphic to B), the given element $c \in \mathrm{M}(B)$ is decomposed as $c_1 + b$ where $c_1 \in 1 \otimes \mathrm{M}(\mathcal{K})$ and $b \in B_1 \otimes \mathcal{K}$.

This may then be exploited as follows:

Write B_1 as $B_2 \otimes \mathcal{K}$, so that $c = c_1 + b$ with

$$c_1 \in 1 \otimes 1 \otimes \mathrm{M}(\mathcal{K}) \; \subseteq \; \mathrm{M}(B_2 \otimes \mathcal{K} \otimes \mathcal{K})$$

and $b \in B_2 \otimes \mathcal{K} \otimes \mathcal{K}$. As in [7] (see also above), choose a sequence of unitaries (u_n) in $\mathrm{M}(B_2 \otimes \mathcal{K} \otimes \mathcal{K})$ with

$$b_1 u_n b_2 \; \to \; 0 \quad \text{for all} \quad b_1, b_2 \in B_2 \otimes \mathcal{K} \otimes \mathcal{K},$$

such that, in addition,

$$u_n \; = \; 1 \otimes v_n \otimes 1 \quad \text{with} \quad v_n \in \mathrm{M}(\mathcal{K}).$$

Then, as $c_1 \in 1 \otimes 1 \otimes \mathrm{M}(\mathcal{K})$,

$$u_n c_1 u_n^* \; = \; c_1.$$

Hence (cf. above),

$$u_n c u_n^* \; \to \; c_1 \quad \text{strictly in} \quad \mathrm{M}(B_2 \otimes \mathcal{K} \otimes \mathcal{K}) \; = \; \mathrm{M}(B).$$

(This holds as $c = c_1 + b$ with $u_n c_1 u_n^* = c_1$ and $u_n b u_n^* \to 0$ strictly.)

Note also that $(c_1 b c_1^*)^-$ is stable, and full in B, as $c_1 \in 1 \otimes \mathrm{M}(\mathcal{K}) \subseteq \mathrm{M}(B_1 \otimes \mathcal{K}) = \mathrm{M}(B)$ and $c_1 \notin B$. (See proof of Case (i).)

Let us first show that $(cBc^*)^-$ is full in B—this is the simpler step. Since

$$(u_n c u_n^*)b'(u_n c u_n^*)^* \; \to \; c_1 b' c_1^* \quad \text{for all} \quad b' \in B,$$

the closed two-sided ideal of B generated by cBc^* contains $(c_1 B c_1^*)^-$, and hence is equal to B, as desired.

We are unable to prove that $(cBc^*)^-$ is stable, for arbitrary c as above, i.e., for c equal to $c_1 + b$, with c_1 fixed as above, and b arbitrary in B. Nevertheless, we shall show that, for arbitrary such c, the algebra $(cBc^*)^-$ contains a stable sub-C*-algebra which is full in B, which is all that is required. (The subalgebra will be constructed to be $(c'Bc'^*)^-$ for some $c' \in C \setminus B$; such a subalgebra is full in B by the preceding paragraph.)

Note that for any $x \in \mathrm{M}(B)$ the sub-C*-algebra $(xBx^*)^-$ is equal to $(xx^*Bxx^*)^-$, so that the problem reduces to considering the case that c is positive—and dividing by B we see that c_1 is then positive, too. Of course, we may also suppose that c and c_1 have norm at most one.

Now, set $c^{\frac{1}{2}}c_1 c^{\frac{1}{2}} = c'$ and $c_1^{\frac{1}{2}}cc_1^{\frac{1}{2}} = c''$, and note that, first,

$$0 \le c' \le c, \quad 0 \le c'' \le c_1,$$

so that

$$(c'Bc')^- \subseteq (cBc)^-, \quad (c''Bc'')^- \subseteq (c_1 Bc_1)^-,$$

and, second, the hereditary sub-C*-algebras $(c'Bc')^-$ and $(c''Bc'')^-$ are isomorphic. (As shown in the proof of Lemma 13, $(xBx^*)^-$ is isomorphic to $(x^*Bx)^-$ for any $x \in \mathrm{M}(B)$, and applying this with $x = c_1^{\frac{1}{2}}c^{\frac{1}{2}}$ yields

$$(c'Bc')^- = (xBx^*)^- \cong (x^*Bx)^- = (c''Bc'')^-,$$

as asserted.)

It now suffices, to complete the proof, to show that $(c''bc'')^-$ is stable—as then $(c'Bc')^-$ is a stable sub-C*-algebra of $(cBc)^-$, full in B by the first part of the proof.

To simplify notation, let us assume that already $c \le c_1$, and let us show that, at least in this case, $(cBc)^-$ is stable. We shall essentially repeat the proof of Corollary 4.3 of [7].

Recall that $b_1 u_n b_2 \to 0$ for all $b_1, b_2 \in B$. Let us verify the criterion (b) of Proposition 2.2 of [7], shown in Proposition 2.2 and Theorem 2.1 of [7] to be equivalent to stability for a C*-algebra with countable approximate unit (in particular, for a separable C*-algebra), with $(cBc)^-$ in place of A. Fix $0 \le a \in (cBc)^-$. Since $(cBc)^- \subseteq (c_1 Bc_1)^-$, there exists a continuous function (a root) d_1 of c_1 such that $d_1 a$ is arbitrarily close to a. Since $c = c_1 + b$, also $d - d_1 \in B$ where d is the corresponding function of c. Therefore, for large n, $du_n a^{\frac{1}{2}}$ is arbitrarily close to $d_1 u_n a^{\frac{1}{2}} = u_n d_1 a^{\frac{1}{2}}$ and hence also to $u_n a^{\frac{1}{2}}$. Since

$$(u_n a^{\frac{1}{2}})^*(u_n a^{\frac{1}{2}}) = a,$$

with $a_n = du_n a^{\frac{1}{2}} \in (cBc)^-$ we have that, if n is sufficiently large, the element $a_n^* a_n$ is close to a, and the product of equivalent elements

$$(a_n^* a_n)(a_n a_n^*) = a_n^*(du_n a^{\frac{1}{2}} du_n a^{\frac{1}{2}})a_n$$

is close to zero (as $a^{\frac{1}{2}}du_n a^{\frac{1}{2}} \to 0$), as required in the criterion 2.2(b) of Hjelmborg and Rørdam.

18.   Questions.   A number of questions arise naturally in connection with the notion of purely large extension.

For instance, is the obvious stronger form of the property that an extension is purely large in fact the same thing? In other words, if the C*-algebra C is purely large with respect to the closed two-sided ideal B, i.e., if $(cBc^*)^-$ always contains a stable sub-C*-algebra which is full in B for any $c \in C$ not in B, must the subalgebra $(cBc^*)^-$ always be stable itself (for such c)?

Again, is it possible to characterize when an extension is purely large in terms of the image of the Busby map in the corona of B, the quotient $M(B)/B$? (Remembering also that the extension is essential—equivalently, that the Busby map is injective.) Of course, this must mean by some intrinsic property of the image, which makes sense more generally—perhaps in an arbitrary C*-algebra. (As the image of the Busby map in the corona—if this is given as the corona—is already enough to reconstruct the C*-algebra associated with an essential extension—which by definition contains sufficient information to determine whether the extension is purely large.) For instance, is it sufficient that every non-zero element of the image be full (i.e., not contained in any proper closed two-sided ideal)? This condition is at least necessary—at least in the separable case—as can be seen by Theorem 6, together with the (obvious) fact that Kasparov's extension (17(iii) above) satisfies this condition—and, as shown in Lemma 12, is trivial in the nuclear sense (and so, by Theorem 6, is absorbed by a purely large extension).

Note that as (by Theorem 17(iii)) Kasparov's extension is also absorbing in the nuclear sense, an extension of one separable C*-algebra by another is purely large—equivalently, absorbing in the nuclear sense—precisely when it absorbs Kasparov's extension. One might ask whether this characterization of purely large extensions can be extended to the non-separable case. The difficulty with this is that Kasparov's extension, being based on an extension of $\mathcal{K}$, does not exist if the quotient has too large a cardinality. On the other hand, the characterization of purely large extensions simply as those which are absorbing in the nuclear sense (either among unital extensions, if the extension is unital, or among all extensions if it is not unital—Theorem 6 and Corollary 16), although it is proved using Kasparov's extension, makes sense and could conceivably still hold in the non-separable case.

One thing the notion of purely large extension—or, more precisely, the notion of extension which is absorbing in the nuclear sense (cf. Theorem 6 and Corollary 16)—makes possible is a generalization of Kasparov's semigroup description of $\mathrm{Ext}(A, B)$ in the setting of nuclear (separable) C*-algebras. Namely, for arbitrary (separable) C*-algebras A and B, with B stable, the extensions of B by A which are absorbing in the nuclear sense form, as we have shown, a semigroup with zero element. The invertible elements of this semigroup are seen—on using Kasparov's Stinespring Theorem, [8]—to be precisely the weakly nuclear extensions which are absorbing in the nuclear sense.

Here, by a weakly nuclear extension of B by A we mean an extension for which the Busby map $A \to \mathrm{M}(B)/B$ lifts to a completely positive contraction $A \to \mathrm{M}(B)$ which is weakly nuclear, in the sense described in Section 5 for homomorphisms. (Recall that if A is exact, then by Corollary 5.11 of [**9**], any weakly nuclear map with domain A is nuclear.) One should note that the proof of Kasparov's Stinespring theorem preserves weak nuclearity: a weakly nuclear completely positive map dilates to a weakly nuclear homomorphism. The group of invertible elements of this semigroup with zero (the semigroup of absorbing extensions in the nuclear sense, i.e., those extensions absorbing every trivial extension with a weakly nuclear splitting) therefore maps into the group, which we shall denote by $\mathrm{Ext}_{\mathrm{nuc}}(A, B)$, of all Brown-Douglas-Fillmore equivalence classes of weakly nuclear extensions of B by A, modulo extensions trivial in the nuclear sense. Since Kasparov's extension is weakly nuclear, and, what is more, trivial in the nuclear sense, and so zero in $\mathrm{Ext}_{\mathrm{nuc}}(A, B)$, and since the sum of this with any extension is absorbing in the nuclear sense, this mapping is onto $\mathrm{Ext}_{\mathrm{nuc}}(A, B)$. Since any two extensions which are both absorbing and trivial in the nuclear sense are equivalent, this map is injective, and therefore an isomorphism.

It is interesting to consider whether the group $\mathrm{Ext}_{\mathrm{nuc}}(A, B)$ defined above —and realized as a subset of the Brown-Douglas-Fillmore semigroup—is isomorphic in the natural way to the group $\mathrm{KK}_{\mathrm{nuc}}(A, B)$ defined by Skandalis in [**12**]. (With the appropriate dimension shift.) This amounts to the following, perhaps surprising, question:

As pointed out above, any extension which is trivial in the nuclear sense— i.e., has a weakly nuclear splitting—is weakly nuclear. Is every weakly nuclear trivial extension trivial in the nuclear sense?

References

[1] C.A. Akemann, J.H. Anderson, and G.K. Pedersen, *Excising states of* C*-*algebras*, Canad. J. Math., **38** (1986), 1239-1260.

[2] W.B. Arveson, *Notes on extensions of* C*-*algebras*, Duke Math. J., **44** (1977), 329-355.

[3] L.G. Brown, R.G. Douglas, and P.A. Fillmore, *Extensions of* C*-*algebras and K-homology*, Ann. of Math., **105** (1977), 265-324.

[4] F. Combes, *Sur les faces d'une* C*-*algèbre*, Bull. Sci. Math., **93** (1969), 37-62.

[5] J. Dixmier, *Les* C*-*algèbres et leurs représentations,* Gauthier-Villars, Paris, 1964.

[6] G.A. Elliott and D.E. Handelman, *Addition of* C*-*algebra extensions*, Pacific J. Math., **137** (1989), 87-121.

[7] J.v.B. Hjelmborg and M. Rørdam, *On stability of* C*-*algebras*, J. Funct. Anal., **155** (1998), 153-170.

[8] G.G. Kasparov, *Hilbert* C*-*modules: Theorems of Stinespring and Voiculescu*, J. Operator Theory, **4** (1980), 133-150.

[9] E. Kirchberg, *The classification of purely infinite* C*-*algebras using Kasparov's theory*, preprint (3rd draft, December 1994).

[10] H. Lin, *Stable approximate unitary equivalence of homomorphisms*, J. Operator Theory, to appear.

[11] M. Pimsner, S. Popa, and D. Voiculescu, *Homogeneous* C*-*extensions of* C$(X) \otimes \mathcal{K}(H)$, Part I, J. Operator Theory, **1** (1979), 55-108.

[12] G. Skandalis, *Une notion de nucléarité en K-théorie*, K-Theory, **1** (1998), 549-574.

[13] D. Voiculescu, *A non-commutative Weyl-von Neumann theorem*, Rev. Roum. Math. Pures Appl., **21** (1976), 97-113.

[14] S. Zhang, *Certain* C*-*algebras with real rank zero and their corona and multiplier algebras*, Pacific J. Math., **155** (1992), 169-197.

Department of Mathematics
University of Copenhagen
Copenhagen, Denmark;

Department of Mathematics
University of Toronto
Toronto, M5S 3G3
Canada

The Fields Institute for Research in Mathematical Sciences
222 College Street
Toronto, M5T 3J1
Canada
E-mail address: elliott@math.toronto.edu

The Fields Institute for Research in Mathematical Sciences
222 College Street
Toronto, M5T 3J1
Canada

Department of Mathematics
University of New Brunswick
Fredericton, New Brunswick E3B 5A3
Canada
E-mail address: dan@math.unb.ca

ON THE DEFORMATION QUANTIZATION OF COADJOINT ORBITS OF SEMISIMPLE GROUPS

R. Fioresi and M.A. Lledó

To the memory of Moshe Flato

In this paper we consider the problem of deformation quantization of the algebra of polynomial functions on coadjoint orbits of semisimple Lie groups. The deformation of an orbit is realized by taking the quotient of the universal enveloping algebra of the Lie algebra of the given Lie group, by a suitable ideal. A comparison with geometric quantization in the case of SU(2) is done, where both methods agree.

1. Introduction.

A system in classical mechanics is given by a symplectic manifold X which we call phase space and a function on X, H, which we call Hamiltonian. The points in X represent possible states of the system, the commutative algebra $C^\infty(X)$ is the set of classical observables, corresponding to possible measurements on the system, and the integral curves of the Hamiltonian vector field X_H represent the time evolution of the classical system.

A quantization of the classical system X has three ingredients [**Be**]:
1. A family of noncommutative complex algebras $\mathbf{A}_h$ depending on a real parameter h, which we will identify with Planck's constant, satisfying

$$\mathbf{A}_h \mapsto \mathbf{A} = C^\infty(X)^{\mathbf{C}} \quad \text{when} \quad h \mapsto 0,$$

or a suitable subalgebra of $C^\infty(X)^{\mathbf{C}}$ determined by physical requirements, but enough to separate the points of X. $C^\infty(X)^{\mathbf{C}}$ denotes the complexification of $C^\infty(X)$.
2. A family of linear maps $Q_h : \mathbf{A} \mapsto \mathbf{A}_h$, called the *quantization maps* satisfying

$$\frac{Q_h(F) *_h Q_h(G) - Q_h(G) *_h Q_h(F)}{h} \mapsto \{F, G\} \quad \text{when} \quad h \mapsto 0,$$

where $\{,\}$ is the Poisson bracket in $\mathbf{A}$ (extended by linearity).
3. A representation of $\mathbf{A}_h$ on a Hilbert space $\mathcal{H}_X$, $R : \mathbf{A}_h \mapsto \text{End}(\mathcal{H}_X)$. The real functions in $\mathbf{A}_h$ (belonging to $C^\infty(X)$) are mapped into hermitian operators.

The elements of $\mathbf{A}_h$ are the quantum observables and the rays in $\mathcal{H}_X$ are the states of the quantum system. Not every possible realization of $\mathbf{A}_h$ on a Hilbert space $\mathcal{H}_X$ satisfies the physical requirements for the quantum system, since the set of rays of $\mathcal{H}_X$ should be in one to one correspondence with the quantum physical states. So further requirements should be imposed on $\mathcal{H}_X$.

A first step to find a quantization of a physical system is the construction of a formal deformation of the Poisson algebra classical observables [**BFFLS**]. In general, formal deformations do not present a closed solution to the quantization problem. One needs to see if it is possible to specialize the deformation to an interval of values of the formal parameter h (including 0, so the limit $h \mapsto 0$ is smooth), besides constructing the Hilbert space where this algebra is represented. Nevertheless having a formal deformation is a powerful technical tool in the process of quantization.

A first approach to this problem appears in [**Be**]. Berezin explicitly computes $*$-products for Kähler manifolds that are homogeneous spaces. His approach provides an explicit integral formula for a $*$-product where h is a real number. In [**RCG**] a geometric construction of Berezin's quantization is performed.

Later De Wilde and Lecomte [**DL**] and Fedosov [**Fe**] separately, constructed and classified formal $*$-products on generic symplectic manifolds. Etingof and Kazhdan [**EK**] proved the existence of a formal deformation for another class of Poisson manifolds, the Poisson-Lie groups. Finally, Kontsevich [**Ko**] proved the existence of an essentially unique formal $*$-product on general Poisson manifolds.

More recently Reshetekhin and Taktajan [**RT**], starting from Berezin's construction, were able to give an explicit integral formula for the formal $*$-product on Kähler manifolds.

It is our purpose to study the deformation quantization of coadjoint orbits of semisimple Lie groups. In [**ALM**] it has been proven that a covariant $*$-product exists on the orbits of the coadjoint orbit that admit a polarization. We will consider the algebra of polynomials on coadjoint orbits. In the above mentioned works $*$-products are given on C^∞ functions, however there is no guarantee that there is a subalgebra of functions that is closed under it. Instead, we will obtain both a formal deformation and a deformation for any real value of h for the subalgebra of polynomial functions.

In [**Ko**] Kontsevich briefly describes the algebra of polynomials over the dual of the Lie algebra (a Poisson manifold) as a special case of his general formula for $*$-product on Poisson manifolds (this special case was known long before [**Gu**]). He does not however consider the restriction of those polynomials to a coadjoint orbit submanifold and, as he points out later, the knowledge of $*$-product on a certain domain is far from giving knowledge

of $*$-product on subdomains of it. The formulation of a star product on some coadjoint orbits using this deformation of the polynomial algebra was investigated in the series of works [**CG**], [**ACG**] and [**Ho**] (and references inside).

Our approach starts also from the fact that the universal enveloping algebra of a complex semisimple Lie algebra is the deformation quantization of the polynomial algebra on the dual Lie algebra. By quotienting by a suitable ideal we get a deformation quantization of the polynomial algebra on a regular coadjoint orbit. Using some known facts on real and complex orbits this gives us a deformation quantization on the regular orbits of compact semisimple Lie groups. No selection of ordering rule is needed for the proof, which means that we obtain a whole class of star products on the orbit. A proof of the analiticity of the deformation in the deformation parameter is provided here, and the convergence of the deformed product for polynomials on the orbit is obtained. More general cases, as regular orbits of non-compact Lie groups, involve some subtleties that are partially explored in Section 2. Further developments will be given in a subsequent paper. Also, the extension of the proof to non-regular (although still semisimple) orbits is non-trivial.

Our construction has the advantage that it is given in a coordinate independent way. Also the symmetries and its possible representations are better studied in this framework. The formal deformation is realized using a true deformation of the polynomials on the complex orbit. We obtain the deformation quantization as a non-commutative algebra depending on a formal parameter h containing a subalgebra in which h can be specialized to any real value.

Geometric quantization is another approach to the problem. The elements of the quantum system are constructed using the geometric elements of the classical system. (For an introduction to geometric quantization, see for example [**Pu**] and references inside.) In the case when the phase space is $\mathbf{R}^{2n}$, a comparison between both procedures, deformation and geometric quantization has been established [**GV**]. Less trivial systems, as coadjoint orbits, have been the subject of geometric quantization. The guiding principle is the preservation of the symmetries of the classical system after the quantization. The idea of finding a unitary representation of the symmetry group naturally attached to the coadjoint orbit is known as the Kirillov-Kostant orbit principle. The action of the group on the Hilbert space of the representation should be induced by the action of the group on the phase space as symplectomorphisms. The algebra of classical observables should be substituted by a non-commutative algebra and the group should act also naturally by conjugation on this algebra.

The procedure we used in constructing the formal deformation, that is assigning an ideal in the enveloping algebra to the coadjoint orbit, makes the comparison with geometric quantization easier. In Section 4 we show that in the special case of SU(2) there is an isomorphism between our deformation quantization and the algebra of twisted differential operators that appears in geometric quantization.

The organization of the paper is as follows. In Section 2 we make a review of the algebraic properties of the coadjoint orbits on which our method of deformation is based. In Section 3 we prove the existence of the deformation and describe it explicitly in terms of a quotient of the enveloping algebra by an ideal. In Section 4 we make a comparison of our results with the results of geometric quantization for a particularly simple case, the coadjoint orbits of SU(2).

2. Algebraic Structure of Coadjoint Orbits of Semisimple Lie Groups.

Let G_R be a real Lie group and $\mathcal{G}_R$ its Lie algebra. The coadjoint action of G_R on $\mathcal{G}_R{}^*$ is given by

$$\langle \mathrm{Ad}^*(g)\lambda, Y \rangle = \langle \lambda, \mathrm{Ad}(g^{-1})Y \rangle \quad \forall \, g \in G_R, \quad \lambda \in \mathcal{G}_R{}^*, \quad Y \in \mathcal{G}_R.$$

We will denote by $C_{G_R}(\lambda)$ (or simply C_λ if G_R can be suppressed without confusion) the orbit of the point $\lambda \in \mathcal{G}_R{}^*$ under the coadjoint action of G_R.

Consider now the algebra of C^∞ functions on $\mathcal{G}_R{}^*$, $C^\infty(\mathcal{G}_R{}^*)$. We can turn it into a Poisson algebra with the so called Lie-Poisson structure

$$\{f_1, f_2\}(\lambda) = \langle [(df_1)_\lambda, (df_2)_\lambda], \lambda \rangle, \qquad f_1, f_2 \in C^\infty(\mathcal{G}_R{}^*), \quad \lambda \in \mathcal{G}_R{}^*.$$

If $f \in C^\infty(\mathcal{G}_R{}^*)$, $(df)_\lambda$ is a map from $\mathcal{G}_R{}^*$ to $\mathbf{R}$, so it can be regarded as an element of $\mathcal{G}_R$ and $[\,,\,]$ is the Lie bracket in $\mathcal{G}_R$. By writing the Poisson bracket in linear coordinates, it is clear that $\mathbf{R}[\mathcal{G}_R{}^*]$, the ring of polynomials on $\mathcal{G}_R{}^*$, is closed under the Poisson bracket.

The Hamiltonian vector fields define an integrable distribution on $\mathcal{G}_R{}^*$ whose integral manifolds (the symplectic leaves) are precisely the orbits of the coadjoint action. So all the coadjoint orbits are symplectic manifolds with the symplectic structure inherited from the Poisson structure on $\mathcal{G}_R{}^*$.

Let G be a connected complex, semisimple Lie group and $\mathcal{G}$ its Lie algebra. We wish to describe the coadjoint orbits of different real forms of G. We can identify $\mathcal{G}$ and $\mathcal{G}^*$ by means of the Cartan-Killing form, so we will work with the adjoint action instead. We denote by G_R an arbitrary real form of G, and $\mathcal{G}_R$ its Lie algebra.

We start with the adjoint orbits of the complex group G itself. Let $Z_s \in \mathcal{G}_R \subset \mathcal{G}$ be a semisimple element. The orbit of Z_s in $\mathcal{G}$ under G will be denoted by $C_G(Z_s)$. It is well known that this orbit is a smooth

complex algebraic variety defined over $\mathbf{R}$ [**Bo**]. That means that the real form of $C_G(Z_s)$, $C_G(Z_s)(\mathbf{R}) = C_G(Z_s) \cap \mathcal{G}_R$ is a real algebraic variety. If G_R is compact, $C_G(Z_s)(\mathbf{R})$ coincides with the real orbit $C_{G_R}(Z_s)$. In general $C_G(Z_s)(\mathbf{R})$ is the union of several real orbits $C_{G_R}(X_i), i \in I$ for some finite set of indices I [**Va2**]. Hence the real orbits are not always algebraic varieties. We will give one of such examples later. Still, the algebraic structure of the closely related manifold $C_G(Z_s)(\mathbf{R})$ will be useful for the quantization.

The algebra that we want to deform is the polynomial ring on the complex orbit. When $C_G(Z_s)(\mathbf{R})$ consists of one real orbit, the complex polynomial ring is the complexification of the polynomial ring on the real orbit. In this case, giving a formal deformation defined over $\mathbf{R}$ of the polynomial ring of the complex orbit is completely equivalent to give a formal deformation of the polynomial ring of the real orbit.

In general I will have many elements. One can always consider the algebra of polynomials on $C_G(Z_s)(\mathbf{R})$ and restrict it to each of the connected components. The $*$-product we obtain can also be defined on the algebra of restricted polynomials without ambiguity, so we have a deformation of certain algebra of functions on the real orbit. Interesting subalgebras of the restricted polynomials that still separate the points of the real orbit could be found, being also closed under the $*$-product. We will see such kind of construction in an example.

We summarize now the classification of real coadjoint orbits [**Va2**], [**Vo**]. The easiest situation is when G_R is a compact group. In this case the orbits are real algebraic varieties defined by the polynomials on $\mathcal{G}$, invariant with respect to the coadjoint action. These invariant polynomials (or Casimir polynomials) are in one to one correspondence with polynomials on the Cartan subalgebra that are invariant under the Weyl group. So every point in a Weyl chamber determines a value of the invariant polynomials, and hence, an adjoint orbit.

The general case is a refinement of this particular one. We will consider only orbits that contain a semisimple element $Z_s \in \mathcal{G}_R$. There are two special cases: The elliptic orbits, when the minimal polynomial of the element Z_e has only purely imaginary eigenvalues, and the hyperbolic orbits, when the minimal polynomial of Z_h has only real eigenvalues. The general case $Z_s = Z_h + Z_e$ can be understood in terms of the special cases.

Let us denote by U a compact real form of G and $\mathcal{U}$ its Lie algebra, while G_0 and $\mathcal{G}_0$ denote a non-compact form and its Lie algebra. The involution $\theta : \mathcal{G}_0 \mapsto \mathcal{G}_0$ induces the Cartan decomposition $\mathcal{G}_0 = \mathcal{L}_0 + \mathcal{P}_0$, and $\mathcal{U} = \mathcal{L}_0 + i\mathcal{P}_0$. K is a maximal compact subgroup of G_0 with Lie algebra $\mathcal{L}_0$. We denote by $\mathcal{H}_{\mathcal{P}_0}$ the maximal abelian subalgebra of $\mathcal{P}_0$ and by $\mathcal{H}_{\mathcal{L}_0}$ a CSA of $\mathcal{L}_0$. $W(G_0, \mathcal{H}_{\mathcal{L}_0})$ and $W(G_0, \mathcal{H}_{\mathcal{P}_0})$ will denote the Weyl groups corresponding to

the root systems of K ($W(G_0, \mathcal{H}_{\mathcal{L}_0})$) and the restricted root system of G_0 ($W(G_0, \mathcal{H}_{\mathcal{P}_0})$).

The set of hyperbolic orbits is in one to one correspondence with the set of orbits of $W(G_0, \mathcal{H}_{\mathcal{P}_0})$ on $\mathcal{H}_{\mathcal{P}_0}$, while the set of elliptic orbits is in one to one correspondence with the set of orbits of $W(G_0, \mathcal{H}_{\mathcal{L}_0})$ on $\mathcal{H}_{\mathcal{L}_0}$. In summary, each point in the Weyl chamber of the corresponding root system determines a unique semisimple orbit and vice versa.

Example 2.1 (Orbits of $SO(2,1)$). We want to show explicitly an example where the real form of the complex orbit is the union of two real orbits. The value of the invariant polynomials in this case doesn't completely determine a real orbit.

Consider the connected component containing the identity of the non-compact orthogonal group $SO(2,1) = \{3 \times 3 \text{ real matrices } \Lambda/\Lambda^T \eta \Lambda = \eta\}$, where

$$\eta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

The Lie algebra so(2,1) is given by so(2,1)= span$\{G, \tilde{E}, \tilde{F}\}$, where

$$G = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \tilde{E} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \tilde{F} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

with commutation relations

$$[G, \tilde{E}] = \tilde{F}, \quad [G, \tilde{F}] = -\tilde{E}, \quad [\tilde{E}, \tilde{F}] = -G.$$

The involutive automorphism associated to this non-compact form of so(3) is $\sigma(X) = \eta X \eta$ so the Cartan decomposition is given by $\mathcal{L}_0 = \text{span}\{G\}$ and $\mathcal{P}_0 = \text{span}\{\tilde{E}, \tilde{F}\}$. $\mathcal{L}_0$ is the Lie algebra of $SO(2)$, the maximal compact subgroup, which in this case is abelian.

The only Casimir polynomial is given in the coordinates $X = x\tilde{E} + y\tilde{F} + zG$ by $P(X) = x^2 + y^2 - z^2$. The elliptic orbits are classified by the elements $\{tG, t \in \mathbf{R} - \{0\}\}$, so the equation describing this orbit is

$$x^2 + y^2 - z^2 = -t^2.$$

Notice that t and $-t$ define the same equation (the same value for the Casimir), but they define different orbits. In fact, the solution of the equation above is a double sheeted hyperboloid, each of the sheets being a different orbit (inside the past and future cone respectively).

Consider now the following automorphism of so(2,1) (in the ordered basis we gave before)

$$A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

A can in fact be written as $A = \mathrm{Ad}(g)$ with g an element in the complexification of SO(2,1). In fact,

$$g = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

belongs to SO(3), the compact real form. Acting on the CSA, span$\{G\}$, it gives the only Weyl reflection (the Weyl group of SO(3) is $\{\mathrm{Id}, -\mathrm{Id}\}$), so g is a representative of the non-trivial element in the Weyl group of SO(3).

Notice that the CSA of the maximal compact subgroup SO(2) and of SO(3) have the same dimension, but the automorphism A is just the Weyl reflection of SO(3) that is "missing" in SO(2). A takes a point in one sheet of the hyperboloid and sends it to the other sheet, so A is a diffeomorphism between the two real orbits.

Consider now the subalgebra of polynomials on $\mathcal{G}$ that are invariant under A (since $A^2 = \mathrm{Id}$, $\{\mathrm{Id}, A\}$ is a subgroup of automorphisms of so(2,1)). It is easy to see that it is also a Poisson subalgebra. Moreover, since the Casimir polynomial is invariant under A, it is also possible to define a subalgebra of the polynomial algebra of the complex orbit. It is defined over $\mathbf{R}$, since A leaves the real form so(2,1) invariant. This algebra is contained as subalgebra in the algebra of polynomial functions over the real orbit (by polynomial functions we mean polynomials in the ambient space restricted to the orbit).

The implementation of such kind of procedure for more general cases is still under study and will be written elsewhere.

Hyperbolic orbits are classified by the Weyl chamber of the restricted root system. One can take $\mathcal{H}_{\mathcal{P}_0} = \mathrm{span}\{\tilde{E}\}$, then $\mathcal{H}_0 = \mathrm{span}\{\tilde{E}\}$ so the only root is the restricted root. The Weyl chamber is $\{t\tilde{E}, t \in \mathbf{R}^+\}$, so the hyperbolic orbits are given by

$$x^2 + y^2 - z^2 = t^2.$$

This is a single sheeted hyperboloid, so in this case the orbit is an algebraic manifold.

Finally we have the orbits in the light cone (nilpotent orbits) satisfying

$$x^2 + y^2 - z^2 = 0.$$

There are three of them, one for z=0, others for $z > 0$ and $z < 0$, but we are not studying nilpotent orbits here.

3. Deformation of the polynomial algebra of regular coadjoint orbits of semisimple groups.

Definition 3.1. Given a real Poisson algebra $\mathbf{P}$, a *formal deformation* of $\mathbf{P}$ is an associative algebra $\mathbf{P}_h$ over $\mathbf{R}[h]$, where h is a formal parameter, with the following properties:

    a. $\mathbf{P}_h$ is isomorphic to $\mathbf{P}[[h]]$ as a $\mathbf{R}[[h]]$-module.

    b. The multiplication $*_h$ in $\mathbf{P}_h$ reduces $\mathrm{mod}(h)$ to the one in $\mathbf{P}$.

    c. $\tilde{F} *_h \tilde{G} - \tilde{G} *_h \tilde{F} = h\{F,G\} \mathrm{\ mod\ } (h^2)$, where $\tilde{F}, \tilde{G} \in \mathbf{P}_h$ reduce to $F, G \in \mathbf{P} \mathrm{\ mod}(h)$ and $\{\,,\,\}$ is the Poisson bracket in $\mathbf{P}$.

If X is a Poisson manifold and $\mathbf{P} = C^\infty(X)$ we call $\mathbf{P}_h$ a *formal deformation* of X. Some authors also use the term *deformation quantization* of X.

We can also speak of the formal deformation of the complexification $\mathbf{A}$ of a real Poisson algebra. The formal deformation of $\mathbf{A}$ will be an associative algebra $\mathbf{A}_h$ with the same properties (a), (b) and (c) where $\mathbf{R}$ has been replaced by $\mathbf{C}$. We want to note here that this doesn't convert the complexification of the symplectic manifold X in a real Poisson manifold of twice the dimension.

We are going to describe first the formal deformation of the polynomial algebra on the complex orbit.

In the first place we will consider $\mathbf{C}[h]$-modules, that is, we will restrict the modules appearing on Definition 3.1 to be modules over $\mathbf{C}[h]$, the algebra of the polynomials in the indeterminate h. This will give us immediately the formal deformation by extending to $\mathbf{C}[[h]]$. Notice that our formal deformation will contain a subalgebra that can be specialized to any value of $h \in \mathbf{R}$.

Let G be a complex semisimple Lie group of dimension n, $\mathcal{G}$ its Lie algebra and U the enveloping algebra of $\mathcal{G}$. Let's denote by $T_A(V)$ the full tensor algebra of a complex vector space V over a $\mathbf{C}$-algebra A. Consider the proper two sided ideal in $T_{\mathbf{C}[h]}(\mathcal{G})$

$$\mathcal{L}_h = \sum_{X,Y \in \mathcal{G}} T_{\mathbf{C}[h]}(\mathcal{G}) \otimes (X \otimes Y - Y \otimes X - h[X,Y]) \otimes T_{\mathbf{C}[h]}(\mathcal{G}).$$

We define $U_h =_{def} T_{\mathbf{C}[h]}(\mathcal{G})/\mathcal{L}_h$. U_h can be interpreted in the following way:

Let $\mathcal{G}_h$ be the Lie algebra over $\mathbf{C}[h]$ $\mathcal{G}_h = \mathbf{C}[h] \otimes_{\mathbf{C}} \mathcal{G}$ with Lie bracket

$$[p(h)X, q(h)Y]_h = p(h)q(h)[X,Y]$$

where $[\,,\,]$ and $[\,,\,]_h$ denote the brackets in $\mathcal{G}$ and $\mathcal{G}_h$ respectively. Then, U_h is the universal enveloping algebra of the algebra $\mathcal{G}_h$.

We will denote with capital letters elements of the tensor algebras and of U_h, while we will use lower case letters for the elements of the polynomial algebra over $\mathcal{G}^*$, $\mathbf{C}[\mathcal{G}^*]$. The product of two elements $A, B \in U_h$ will be written AB.

Proposition 3.2 (Poincaré-Birkhoff-Witt theorem for U_h). *Let* $\{X_1, \ldots,$ $X_n\}$ *be a basis for* $\mathcal{G}$. *Then*

$$1, X_{i_1} \cdots X_{i_k} \qquad 1 \leq i_1 \leq \cdots \leq i_k \leq n$$

form a basis for U_h *as* $\mathbf{C}[h]$-*module.*

U_h is a free $\mathbf{C}[h]$-module. In particular, U_h is torsion free.

Definition 3.3. Let $S(\mathcal{G}) = T_{\mathbf{C}}(\mathcal{G})/\mathcal{L}$, with

$$\mathcal{L} = \sum_{X,Y \in \mathcal{G}} T_{\mathbf{C}}(\mathcal{G}) \otimes (X \otimes Y - Y \otimes X) \otimes T_{\mathbf{C}}(\mathcal{G}),$$

be the symmetric algebra of $\mathcal{G}$. The natural homomorphism from $T_{\mathbf{C}}(\mathcal{G})$ to $S(\mathcal{G})$ is an isomorphism if restricted to the symmetric tensors. Let λ be the inverse of such isomorphism.

The canonical isomorphism $\mathcal{G}^{**} \cong \mathcal{G}$, can be extended to an algebra isomorphism $\mathbf{C}[\mathcal{G}^*] \cong S[\mathcal{G}]$ where $\mathbf{C}[\mathcal{G}^*]$ denotes the polynomial algebra over $\mathcal{G}^*$. The composition of such isomorphism with λ will be called the *symmetrizer map*.

Let $\{X_1, \ldots, X_n\}$ be a basis for $\mathcal{G}$ and $\{x_1, \ldots, x_n\}$ the corresponding basis for $\mathcal{G}^{**} \subset \mathbf{C}[\mathcal{G}^*]$. Then the symmetrizer map $\mathrm{Sym} : \mathbf{C}[\mathcal{G}^*] \longrightarrow T_{\mathbf{C}}(\mathcal{G})$ is given by

$$\mathrm{Sym}(x_1 \cdots x_n) = \frac{1}{p!} \sum_{s \in S_p} X_{s(1)} \otimes \cdots \otimes X_{s(p)}$$

where S_p is the group of permutations of order p.

Let $I \subset \mathbf{C}[\mathcal{G}^*]$ be the set of polynomials on $\mathcal{G}^*$ invariant under the coadjoint action,

$$I = \{p \in \mathbf{C}[\mathcal{G}^*] \mid p(\mathrm{Ad}^*(g)\xi) = p(\xi) \quad \forall \xi \in \mathcal{G}^*, \, g \in G\}.$$

By Chevalley theorem we have that $I = \mathbf{C}[p_1, \ldots, p_m]$, where $p_1, \ldots, p_m$ are algebraically independent homogeneous polynomials and m is the rank of $\mathcal{G}$.

Definition 3.4. We define a *Casimir element* in $T_{\mathbf{C}}(\mathcal{G})$ as the image of an invariant polynomial under the symmetrizer map. Since $T(\mathcal{G}) \subset T_{\mathbf{C}[h]}(\mathcal{G})$ Casimirs are also elements of $T_{\mathbf{C}[h]}(\mathcal{G})$. We call *Casimir element* in U (respectively U_h) an element which is the image of a *Casimir element* in $T(\mathcal{G})$ (respectively in $T_{\mathbf{C}[h]}(\mathcal{G})$) under the natural projection.

It is well known that the Casimir elements lie in the center of U. We want now to prove that they also lie in the center of U_h.

Let's denote by $\tilde{U}_{h_0}$ the algebra $U_h/((h - h_0)\mathbf{1})$, where $h_0 \in \mathbf{C}$, and by ev_{h_0} the natural projection $U_h \longrightarrow \tilde{U}_{h_0}$.

Lemma 3.5. *Let P be a Casimir in U_h. Then $ev_{h_0}(P)$ is in the center of $\tilde{U}_{h_0}$.*

Proof. This is because $\tilde{U}_{h_0}$ is the universal enveloping algebra of $\mathcal{G}_{h_0}$, where $\mathcal{G}_{h_0}$ is the complex Lie algebra coinciding with $\mathcal{G}$ as vector space and with bracket $[X, Y]_{h_0} = h_0[X, Y]$ where $[\ ,\ ]$ is the bracket in $\mathcal{G}$.

Theorem 3.6. *The Casimir elements lie in the center of U_h.*

Proof. Let P be a Casimir element and let $X_1, \ldots, X_n$ be generators for $\mathcal{G}$ hence for $\mathcal{G}_h$. We need to show: $PX_i = X_i P$ for all $1 \le i \le n$.

$$PX_i - X_i P = \sum_{1 \le i_1 \le \cdots \le i_k \le n} u_{i_1 \ldots i_k}(h) X_{i_1} \cdots X_{i_k}.$$

Let us apply the ev_{h_0} map.

$$ev_{h_0}\left(PX_i - X_i P - \sum_{1 \le i_1 \le \cdots \le i_k \le n} u_{i_1 \ldots i_k}(h) X_{i_1} \cdots X_{i_k} \right)$$
$$= - \sum_{1 \le i_1 \le \cdots \le i_k \le n} u_{i_1 \ldots i_k}(h_0) X_{i_1} \cdots X_{i_k} = 0$$

because by Lemma 3.5 $ev_0(PX_i - X_i P) = 0$. Since there are no relations among the standard monomials $X_{i_1} \cdots X_{i_k}$ (Proposition 3.2) we have that $u_{i_1 \ldots i_k}(h_0) = 0$. Since this is true for infinitely many h_0 and since $u_{i_1 \ldots i_k}(h)$ is a polynomial we have that $u_{i_1 \ldots i_k}(h) \equiv 0$.

We now restrict our attention to the regular coadjoint orbits, that is the orbits of regular elements. We recall here the definition of a regular element in $\mathcal{G}^*$. Consider the characteristic polynomial of $\mathrm{ad}^*(\xi)$, $\xi \in \mathcal{G}^*$,

$$\det(T \cdot \mathbf{1} - \mathrm{ad}^*(\xi)) = \sum_{i \ge m} q_i(\xi) T^i$$

where $m = \mathrm{rank}\mathcal{G}^*$. The q_i's are invariant polynomials. An element $\xi \in \mathcal{G}^*$ is regular if $q_m(\xi) \ne 0$. The regular elements are dense in $\mathcal{G}^*$ and they are semisimple. In particular the regular elements in a Cartan subalgebra form the interior of the Weyl chambers.

The orbits of regular elements are orbits of maximal dimension $n - m$. Observe also that the 0-eigenspace coincides with the centralizer of ξ, Z_ξ. A semisimple element ξ is regular if and only if $\dim(Z_\xi) = m$.

Let us fix the coadjoint orbit C_ξ of a regular element $\xi \in \mathcal{G}^*$. The ideal of polynomials vanishing on C_ξ is given by

$$I_0 = (p_i - c_{i0}, \ i = 1, \ldots, m), \qquad c_{i0} \in \mathbf{C},$$

where the p_i have been defined above (see after Definition 3.3). I_0 is a prime ideal or equivalently the orbit C_ξ is an irreducible algebraic variety. (In fact, the orbit of any semisimple element, regular or not, is an irreducible algebraic variety [**Ks**]).

Let's consider the Casimirs $\hat{P}_i = \mathrm{Sym}(p_i)$, where the $p_1, \ldots, p_m$ are generators for I that satisfy Chevalley theorem. Let P_i be the image of $\hat{P}_i$ in U_h. Define the two sided ideal generated by the relations $P_i - c_i(h)$, $i = 1, \ldots, m$:

$$I_h = (P_i - c_i(h), \ i = 1, \ldots, m) \subset U_h$$

for $c_i(h) = \sum_j c_{ij} h^j$, $c_{ij} \in \mathbf{C}$ ($c_i(0) = c_{i0}$, the constants appearing in the definition of I_0).

It is our goal to give a basis of the algebra U_h/I_h as $\mathbf{C}[h]$-module. We need first a couple of lemmas.

Lemma 3.7. *Let $\xi \in \mathcal{G}^*$ be a regular element of $\mathcal{G}^*$ (or equivalently a point in which the centralizer has dimension equal to the rank of $\mathcal{G}^*$). Then $(dp_1)_\xi$, $\ldots$, $(dp_m)_\xi$ are linearly independent.*

Proof. See [**Va3**].

Lemma 3.8. *Let r be a fixed positive integer and let all the notation be as above. Let*

$$\sum_{1 \leq i_1 \leq \cdots \leq i_r \leq m} a_{i_1 \ldots i_r} (p_{i_1} - k_{i_1}) \ldots (p_{i_r} - k_{i_r}) = 0$$

with $a_{i_1 \ldots i_r} \in \mathbf{C}[\mathcal{G}^*]$, $k_{i_1} \ldots k_{i_r} \in \mathbf{C}$. Then $a_{i_1 \ldots i_r} \in (p_1 - k_1, \ldots, p_m - k_m) \subset \mathbf{C}[\mathcal{G}^*]$.

Proof. By Lemma 3.7 we can choose local coordinates $(z_1, \ldots, z_n)$ in a neighborhood of ξ so that $z_i = p_i - k_i$, $i = 1, \ldots, m$. Since $a_{i_1 \ldots i_r}(z_1, \ldots, z_n)$ are analytic functions, we can represent them as power series in $z_1, \ldots, z_n$:

$$a_{i_1 \ldots i_r}(z_1, \ldots, z_n) = \sum_{\substack{1 \leq j_1 \leq \cdots j_s \leq n \\ 0 \leq s}} a_{i_1 \ldots i_r, j_1 \ldots j_s} z_{j_1} \cdots z_{j_s}.$$

This can be rewritten as:

$$a_{i_1 \ldots i_r}(z_1, \ldots, z_n) = \sum_{\substack{m+1 \leq j_1 \leq \cdots j_s \leq n \\ 0 \leq s}} a_{i_1 \ldots i_r, j_1 \ldots j_s} z_{j_1} \cdots z_{j_s}$$

$$+ \sum_{\substack{1 \leq l_1 \leq \cdots l_t \leq n \\ l_1 < m, \ 1 \leq t}} a_{i_1 \ldots i_r, l_1 \ldots l_t} z_{l_1} \cdots z_{l_t}.$$

By substituting into the given equation we get:

$$\sum_{1 \leq i_1 \leq \cdots i_r \leq m} \sum_{\substack{m+1 \leq j_1 \leq \cdots j_s \leq n \\ 0 \leq s}} a_{i_1 \ldots i_r, j_1 \ldots j_s} z_{j_1} \cdots z_{j_s} z_{i_1} \cdots z_{i_r}$$

$$+ \sum_{1 \leq i_1 \leq \cdots i_r \leq m} \sum_{\substack{1 \leq l_1 \leq \cdots l_t \leq n \\ l_1 < m, 1 \leq t}} a_{i_1 \ldots i_r, l_1 \ldots l_t} z_{l_1} \cdots z_{l_t} z_{i_1} \cdots z_{i_r} = 0.$$

Notice that, by the way the sums are defined, and being r fixed, both terms in the above equations have no monomials in common. This implies that

$$\sum_{1 \leq i_1 \leq \cdots i_r \leq m} \sum_{m+1 \leq j_1 \ldots j_s \leq n} a_{i_1 \ldots i_r, j_1 \ldots j_s} z_{j_1} \ldots z_{j_s} z_{i_1} \ldots z_{i_r} = 0$$

from which

$$a_{i_1 \ldots i_r, j_1 \ldots j_s} = 0 \quad \forall\, 1 \leq i_1 \ldots i_r \leq m, \quad m+1 \leq j_1 \cdots j_s.$$

This implies

$$a_{i_1 \ldots i_r}(z_1 \ldots z_m) \in (z_1 \ldots z_r).$$

That is, locally

$$a_{i_1 \ldots i_r} = \sum b_{i_1 \ldots i_r j}(p_j - k_j).$$

So we have obtained that for all η in a neighbourhood of ξ:

$$a_{i_1 \ldots i_r}(\eta) - \sum b_{i_1 \ldots i_r}^j(\eta)(p_j(\eta) - k_j) = 0.$$

But since this function is algebraic and C_ξ is irreducible this means that this function is identically 0 on C_ξ. Hence the Lemma is proven.

Let's consider the projection $\pi : U_h \longrightarrow U_h/(h\mathbf{1}) \cong S(\mathcal{G}) \cong \mathbf{C}[\mathcal{G}^*]$. We have that $\pi(A) = \pi(B)$ if and only if $A \equiv B \mod h$. To simplify the notation we will denote the element of $\mathbf{C}[\mathcal{G}^*]$ corresponding to $\pi(A)$ by a (same letter, but lower case), as we did for the Casimirs P_i before.

Lemma 3.9. *Let k be a fixed integer and let*

$$\sum_{i_1 \leq \cdots i_k \leq m} A_{i_1 \ldots i_k}(P_{i_1} - c_{i_1}(h)) \cdots (P_{i_k} - c_{i_k}(h)) \equiv 0 \qquad \mod h$$

where $A_{i_1 \ldots i_k} \in U_h$ and the P_i's and $c_i(h)$'s have been defined above. Then

$$\sum_{i_1 \leq \cdots i_k \leq m} A_{i_1 \ldots i_k}(P_{i_1} - c_{i_1}(h)) \cdots (P_{i_k} - c_{i_k}(h))$$

$$= h \sum_{i_1 \leq \cdots i_k \leq m} B_{j_1 \ldots j_l, i_1 \ldots i_k}(P_{j_1} - c_{j_1}(h)) \cdots$$

$$(P_{j_l} - c_{j_l}(h))(P_{i_1} - c_{i_1}(h)) \cdots (P_{i_k} - c_{i_k}(h)).$$

Proof. By induction on $N = \max_{i_1 \ldots i_k} \deg a_{i_1 \ldots i_k}$, where, using the the convention above, $a_{i_1 \ldots i_k} = \pi(A_{i_1 \ldots i_k})$. Let $N = 0$. We have:

$$\sum a_{i_1 \ldots i_k}(p_{i_1} - c_{i_1 0}) \cdots (p_{i_k} - c_{i_k 0}) = 0$$

with $a_{i_1 \ldots i_k} \in \mathbf{C}$. By Lemma (3.8) $a_{i_1 \ldots i_k} \in I_0$ hence $a_{i_1 \ldots i_k} = 0$. This implies that $A_{i_1 \ldots i_k} = h B_{i_1 \ldots i_k}$.

Let's now consider a generic N,

$$\sum a_{i_1 \ldots i_k}(p_{i_1} - c_{i_1 0}) \cdots (p_{i_k} - c_{i_k 0}) = 0.$$

By Lemma (3.8)

$$a_{i_1 \ldots i_k} = \sum_j a_{i_1 \ldots i_k j}(p_j - c_{j0})$$

with $\max_{i_1 \ldots i_k} \deg a_{i_1 \ldots i_k j} < N$. Again we have that

$$A_{i_1 \ldots i_k} = \sum_j A_{i_1 \ldots i_k j}(P_j - c_j(h)) + h C_{i_1 \ldots i_k}.$$

Let's substitute $A_{i_1 \ldots i_k}$

$$\sum A_{i_1 \ldots i_k j}(P_j - c_j(h))(P_{i_1} - c_{i_1}(h)) \cdots (P_{i_k} - c_{i_k}(h)) \equiv 0 \qquad \mod h.$$

By induction we have our result.

Lemma 3.10. *If $hF \in I_h$ then $F \in I_h$.*

Proof. Since $hF \in I_h$ and since the P_i are central elements:

$$hF = \sum A_i(P_i - c_i(h)).$$

We have $\sum A_i(P_i - c_i(h)) \equiv 0 \mod h$. Hence, by Lemma 3.9 and also by the fact that U_h is torsion free we have our result.

We have shown that U_h/I_h is a $\mathbf{C}[h]$-module without torsion. We are ready now to show that it is a free module by explicitly constructing a basis. Let's fix a basis $\{X_1, \ldots, X_n\}$ of $\mathcal{G}$ and let $x_1, \ldots, x_n$ be the corresponding elements in $\mathbf{C}[\mathcal{G}^*]$. With this choice $\mathbf{C}[\mathcal{G}^*] \cong \mathbf{C}[x_1, \ldots, x_n]$. Let $\{x_{i_1}, \ldots, x_{i_k}\}_{(i_1, \ldots, i_k) \in \mathbf{A}}$ be a basis in of $\mathbf{C}[\mathcal{G}^*]/I_0$ as $\mathbf{C}$-module, where $\mathbf{A}$ is a set of multiindices appropriate to describe the basis. In particular, we can take them such that $i_1 \leq \cdots \leq i_k$.

Proposition 3.11. *The monomials $\{X_{i_1} \cdots X_{i_k}\}_{(i_1, \ldots, i_k) \in \mathbf{A}}$ are linearly independent in U_h/I_h.*

Proof. Suppose that there exists a linear relation among the $X_{i_1}, \cdots X_{i_k}$'s, $(i_1, \ldots, i_k) \in \mathbf{A}$ and let $G \in I_h$ be such relation,

$$G = G_0 + G_1 h + \cdots, \qquad G_i \in \mathrm{span}_{\mathbf{C}}\{X_{i_1} \cdots X_{i_k}\}_{(i_1 \ldots i_k) \in \mathbf{A}}.$$

Assume $G_i = 0$, $i < k$, $G_k \neq 0$. We can write $G = h^k F$, with

$$F = F_0 + F_1 h + \cdots, \qquad F_0 \neq 0.$$

Since $h^k F \in I_h$ by hypothesis, using Lemma (3.10) we have that $F \in I_h$, that is

$$F = \sum A_i(P_i - c_i(h)),$$

and reducing mod h,

$$f = \sum a_i(p_i - c_{i0}).$$

This would mean that f represents a non-trivial relation among the monomials $\{x_{i_1} \cdots x_{i_k}\}_{(i_1 \ldots i_k) \in \mathbf{A}}$ in $C[\mathcal{G}^*]/I_0$, which is a contradiction, so the linear independence is proven.

We want to give a procedure to construct a basis on $\mathbf{C}[\mathcal{G}^*]/I_0$ starting from a set of generators of $\mathbf{C}[\mathcal{G}^*]$, $S = \{x_{i_1} \cdots x_{i_k}\} \ \forall \ 1 \leq i_1 \leq \cdots i_k \leq n$. As a linear space $I_0 = \text{span}_{\mathbf{C}}\{x_{i_1} \cdots x_{i_k}(p_i - c_i)\}$. Every element of the set that spans I_0 will provide one relation that will allow us to eliminate at most one element of the set S. We can choose to eliminate successively the greatest element with respect to lexicographic ordering. This means that any monomial in S will be expressed in terms of monomials of degree less or equal to its degree.

Remarks 3.12. We want to make two remarks that will be used later.

1. An arbitrary monomial $x_{j_1} \cdots x_{j_r}$ in $\mathbf{C}[\mathcal{G}^*]$ can be written as:

$$x_{j_1} \cdots x_{j_r} = \sum_{\substack{k \leq r \\ (m_1, \ldots, m_k) \in \mathbf{A}}} a_{m_1 \ldots m_k}^{j_1 \ldots j_r} x_{m_1} \cdots x_{m_k} + \sum_{i, d_i + g_i \leq r} b_i(p_i - c_i)$$

where b_i is polynomial of degree g_i, $d_i = \deg p_i$ and $a_{m_1 \ldots m_k}^{j_1 \ldots j_r} \in \mathbf{C}$.

2. Let $A \in U_h$, $A \neq 0$, $A \in \text{span}_{\mathbf{C}}\{X_{j_1} \cdots X_{j_p}\}_{p \leq r}$, $j_1 \ldots j_p$ not necessarily ordered. If $A \equiv 0 \bmod h$, then $A = hB$, $B \in \text{span}_{\mathbf{C}}\{X_{i_1} \cdots X_{i_p}\}_{\substack{p < r \\ i_1 \leq \cdots \leq i_p}}$.

Next proposition will show the generation, so we will have a basis.

Proposition 3.13. *The standard monomials* $\{X_{i_1} \cdots X_{i_k}\}$ *with* $(i_1, \ldots, i_k)$ $\in \mathbf{A}$ *generate* U_h/I_h *as* $\mathbf{C}[h]$-*module.*

Proof. By Proposition 3.2 (PBW theorem in U_h) it is sufficient to prove that

$$X_{j_1} \cdots X_{j_r} \in \text{span}_{\mathbf{C}[h]}\{X_{i_1} \cdots X_{i_k}\}_{(i_1, \ldots, i_k) \in \mathbf{A}}$$

where $1 \leq j_1 \leq \cdots j_r \leq n$ and $X_{j_1} \cdots X_{j_r}$ denotes also the projection onto U_h/I_h of the standard monomial.

We proceed by induction on r. For $r = 0$ it is clear. For generic r we write (see Remark 3.12)

$$x_{j_1} \cdots x_{j_r} = \sum_{\substack{k \leq r \\ (m_1, \ldots, m_k) \in \mathbf{A}}} a_{m_1 \ldots m_k}^{j_1 \ldots j_r} x_{m_1} \cdots x_{m_k} + \sum_{i, d_i + g_i \leq r} b_i(p_i - c_i).$$

Lifting this equation from the symmetric algebra to the enveloping algebra we have

$$X_{j_1} \cdots X_{j_r} - \sum_{\substack{k \leq r \\ (m_1, \ldots, m_k) \in \mathbf{A}}} a_{m_1 \ldots m_k}^{j_1 \ldots j_r} X_{m_1} \cdots X_{m_k} - \sum_i B_i(P_i - c_i(h)) = hB$$

where, by the Remark 2 in 3.12, $B \in \text{span}\{X_{i_1} \cdots X_{i_p}\}_{p < r}$. Applying the induction hypothesis, we have our result.

Let $\mathbf{C}_h[\mathcal{G}^*] = \mathbf{C}[h] \otimes \mathbf{C}[\mathcal{G}^*]$, $I_0' = \mathbf{C}[h] \otimes I_0$. We are now ready to prove the following theorem:

Theorem 3.14. *Let the notation be as above. We have that U_h/I_h has the following properties:*

1. *U_h/I_h is isomorphic to $\mathbf{C}_h[\mathcal{G}^*]/I_0'$ as a $\mathbf{C}[h]$-module.*
2. *The multiplication in U_h/I_h reduces $mod(h)$ to the one in $\mathbf{C}[\mathcal{G}^*]/I_0'$.*
3. *If $FG - GF = hP$, $F, G, P \in U_h/I_h$, then $p = \{f, g\}$, where $\{,\}$ is the Poisson bracket on the orbit defined by I_0. (We are using the same convention, $f = \pi(F)$.)*

Proof.
1. It is a consequence of Propositions 3.11 and 3.13.
2. It is trivial.
3. This property is satisfied by the multiplication in U_h and the Poisson bracket in $\mathbf{C}[\mathcal{G}^*]$ (see [**Ko**], [**CP**], [**Ki**]). The Poisson bracket in the $\mathbf{C}[\mathcal{G}^*]/I_0$ is induced from the one in $\mathbf{C}[\mathcal{G}^*]$, it is enough to see that p will not depend on the representative chosen in U_h/I_h, which is trivial.

It is now immediate to obtain the properties of Definition 3.1 when we consider the extension of $\mathbf{C}[h]$ to $\mathbf{C}[[h]]$. We define

$$\mathbf{C}_{[h]}[\mathcal{G}^*] = \mathbf{C}[[h]][\mathcal{G}^*] \quad I_{[0]} \subset \mathbf{C}_{[h]}[\mathcal{G}^*]$$

$$U_{[h]} = T_{\mathbf{C}[[h]]}(\mathcal{G})/\mathcal{L}_{[h]} \qquad I_{[h]} \subset U_{[h]}$$

being $I_{[0]}$ and $I_{[h]}$ the ideals obtained by extending I_0 and I_h to $C_{[h]}[\mathcal{G}^*]$ and $U_{[h]}$ respectively.

Theorem 3.15. *$U_{[h]}/I_{[h]}$ is a formal deformation (or a deformation quantization) of $\mathbf{C}_{[h]}[\mathcal{G}^*]/I_{[0]}$.*

We want to note here that whatever is the real form chosen, the deformed algebra is defined over $\mathbf{R}$, provided $c_{ij} \in \mathbf{R}$. Care should be taken, nevertheless, in choosing the appropriate generators of I_0 with real coefficients and this is always possible ([**Bo**]).

Finally we want to come back to Example 2.1 and exhibit the deformed algebra.

Example 3.16. Let $G = SL_2(\mathbf{C})$. The standard basis for $\mathcal{G} = sl_2(\mathbf{C})$ is $\{H, X, Y\}$ with commutation relations

$$[H, X] = 2X \quad [H, Y] = -2Y \quad [X, Y] = H.$$

We identify $\mathcal{G}$ and $\mathcal{G}^*$ via the Cartan Killing form. The only independent invariant polynomial is:

$$p = \frac{1}{4}h^2 + xy$$

or, in terms of the compact generators

$$E = \frac{1}{2}(X - Y) \quad F = i/2(X + Y) \quad G = i/2H$$
$$p = -(e^2 + f^2 + g^2).$$

The orbit C_ξ of the regular semisimple element $\xi = \begin{pmatrix} ia/2 & 0 \\ 0 & -ia/2 \end{pmatrix}$ (see the fundamental representation in the next section) has coordinate ring $\mathbf{C}[h, x, y]/(e^2 + f^2 + g^2 - a^2)$. So we have that

$$U_{[[h]]}/(E^2 + F^2 + G^2 - a^2 + c_1 h + \cdots + c_l h^l)$$

is a formal deformation of C_ξ. If one chooses $a, c_1, \ldots, c_l$ to be real, then it becomes the complexification of a formal deformation of the real orbit $C_\xi \cap su(2)$.

To go to the non-compact form it is enough to take the basis $\{\tilde{E} = iE, \tilde{F} = iF, G\}$. The deformed algebra is

$$U_{[[h]]}/(-\tilde{E}^2 - \tilde{F}^2 + G^2 - a^2 + c_1 h + \cdots + c_l h^l).$$

A basis for U/I_0 is

$$\{g^m \tilde{e}^n \tilde{f}^\mu\}_{\substack{m,n=0,1,2\ldots \\ \mu=0,1}}.$$

The subalgebra invariant under the automorphism A of Example 2.1, has instead a basis

$$\{g^m \tilde{e}^{2n-m} \tilde{f}^\mu\}_{\substack{m,n=0,1,2\ldots \\ \mu=0,1}}.$$

We can also express this algebra in terms of the set of commutative generators

$$v_1 = g^2, \quad v_2 = \tilde{e}^2, \quad v_3 = g\tilde{e}, \quad v_4 = \tilde{f}$$

with relations

$$v_3^2 = v_1 v_2, \quad v_1 - v_2 - v_4^2 = a^2.$$

It is clear that this algebra separates the points of the real orbit. Since the Casimir element is invariant under the automorphism A (extended to U_h), it restricts to an automorphism of U_h/I_h. Analogously to the commutative case, the subalgebra of U_h/I_h invariant under A can be given in terms of the generators

$$V_1 = G^2, \quad V_2 = \tilde{E}^2, \quad V_3 = G\tilde{E}, \quad V_4 = \tilde{F}$$

and relations

$$V_3^2 = V_1 V_2 - h V_3 V_4 - h^2 V_1, \quad V_1 - V_2 - V_4^2 = c(h),$$

in addition to the commutation relations

$$V_4V_1 - V_1V_4 = h(2V_3) - h^2V_4, \qquad V_4V_2 - V_2V_4 = h(2V_3) - h^2V_4,$$

$$V_4V_3 - V_3V_4 = h(V_1 + V_2), \qquad V_3V_1 - V_1V_3 = -h(2V_1V_4) - h^2V_3,$$

$$V_3V_2 - V_2V_3 = h(V_4V_2 + V_2V_4) + h^2V_3 - h^3V_4,$$

$$V_2V_1 - V_1V_2 = -h(2V_3V_4) + h^2(V_4^2 - V_2 - V_1).$$

4. Geometric quantization of S^2.

The subject of geometric quantization is a very vast one and we do not intend to make a review here. Many excellent reviews exist in the literature (see for example [**Pu**], [**Vo**]). We will try to explain only what is needed to understand the geometric quantization of our particular case, S^2. Some of the results we exhibit here date back to [**So**]. We will follow closely the scheme of [**Vo**], because there the importance of constructing the algebra of observables is emphasized.

Consider a classical system with phase space X and a group G of symmetries. This means that G is a group of symplectomorphisms of the symplectic manifold X,

$$g \in G, \quad g : X \mapsto X \quad \text{satisfying} \quad g^*\omega = \omega,$$

where ω is the symplectic form on X. The Hamiltonian is a G-invariant function, that is, $gH = H$, so G is a group of symmetries of the equations of motion.

We want to find a quantization of the classical system that preserves the symmetry under the group G. The goal of geometric quantization is to construct the Hilbert space $\mathcal{H}_X$ and the algebra of quantum observables $\mathbf{A}_h$ acting on $\mathcal{H}_X$ using only the geometrical elements of the classical system. This construction should be "natural", that is, the action of G on X as symplectomorphisms should induce a unitary representation of G on $\mathcal{H}_X$ and an action of G on $\mathbf{A}_h$. This action should reduce to the conjugation by the unitary representation on the operators on $\mathcal{H}_X$ representing the elements of $\mathbf{A}_h$.

Integral orbit data.

Let $\xi \in \mathcal{G}_0^*$ and let G_ξ the isotropy group of ξ and $\mathcal{G}_{0\xi}$ the corresponding Lie algebra. It is clear that for $Z \in \mathcal{G}_{0\xi}$, $\mathrm{ad}_Z^*\xi = 0$, which implies

$$(4.1) \qquad \qquad \xi([Z,Y]) = 0, \quad \forall\, Y \in \mathcal{G}_0.$$

Suppose that we have a character τ of G_ξ satisfying

$$\tau(\mathrm{e}^X) = \mathrm{e}^{i\xi(X)}, \quad Z \in \mathcal{G}_{0\xi}.$$

Such character is called an integral orbit datum. Notice that property (4.1) is essential. Also, ξ must be such that $\xi(Z) = 2\pi m$, $m \in \mathbf{Z}$ whenever $e^Z = \text{Id}$.

From an integral orbit datum we can construct a unitary representation of G by induction. We consider the induced vector bundle $E(G/G_\xi, \mathbf{C}_\tau) = (G \times \mathbf{C})/\tau$, where the equivalence relation is given by

$$(g, v) \quad \approx \quad (gh^{-1}, \tau(h)v), \quad h \in G_\xi.$$

We can describe the sections on this bundle by functions $f : G \mapsto \mathbf{C}$ satisfying

(4.2) $$f(gh) = \tau(h)^{-1}f(g).$$

By considering the compactly supported sections, and from the fact that there is a a G-invariant measure on G/G_ξ the construction of the Hilbert space is straightforward, with bilinear form

$$\langle f_1, f_2 \rangle = \int_{G/G_\xi} f_1 \bar{f}_2.$$

The problem is that this representation is not necessarily irreducible. Nevertheless, in many cases (like for elliptic orbits), it is possible to restrict naturally the space of sections (4.2) to an irreducible component. We are then interested in computing the integral orbit data for SU(2).

The Lie algebra of SU(2) is spanned by the matrices

$$G = \frac{i}{2}\sigma_3, \quad E = \frac{i}{2}\sigma_2, \quad F = \frac{i}{2}\sigma_1$$

with

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and commutation relations[1]

$$[E, F] = G, \quad [F, G] = E, \quad [G, E] = F.$$

Consider $\xi_a \in \mathcal{G}_0{}^*$ such that $\xi_a(xE + yF + zG) = az$. The isotropy group is

$$G_{\xi_a} = \{e^{zG}, z \in \mathbf{R}\} = \left\{ \begin{pmatrix} e^{iz/2} & 0 \\ 0 & e^{-iz/2} \end{pmatrix}, z \in \mathbf{R} \right\}$$

with Lie algebra $\mathcal{G}_{0\xi_a} = \text{span}\{G\}$. If $z = 4\pi n$, $n \in \mathbf{Z}$, then $e^{zG} = \text{Id}$, so in order to have an integral orbit datum,

$$\xi_a(4\pi n G) = 4\pi n a \in 2\pi \mathbf{Z} \quad \forall\, n,$$

which is possible if and only if $a \in \mathbf{Z}/2$.

---

[1]The spin operators which are used in physics are given by $G' = -i\hbar G$, $E' = -i\hbar E$, $F' = -i\hbar F$. We can reintroduce $\hbar = h/2\pi$ in the analysis with this rescaling, the multiplication by $-i$ changing a representation by antihermitian operators of SU(2) to hermitian operators.

The Cartan-Killing form allows the identification of $\mathcal{G}_0$ and $\mathcal{G}_0{}^*$, also intertwining the adjoint and coadjoint representations. It is given by

$$\langle X, Y \rangle = -\frac{1}{2} \mathrm{Tr}(\mathrm{ad}X \mathrm{ad}Y), \quad X, Y \in \mathcal{G}_0$$

that is,

$$\langle E, E \rangle = \langle F, F \rangle = \langle G, G \rangle = 1$$

and the rest 0. So $\xi_a \approx aG$, and the orbit is given by the Casimir polynomial

$$C = x^2 + y^2 + z^2 = a^2.$$

We conclude that only orbits with half integer radius have integral orbit data. We will denote by τ_m the corresponding integral orbit datum, $\tau_m(e^{zG}) = (e^{iz/2})^m$.

It is easy to convince oneself that the representation in the space of functions (4.2) is far too large to be irreducible. To overcome this problem we need to further restrict the space of sections. We will do that with the help of a *complex polarization*.

Complex polarization and Hilbert space.

Elliptic orbits have a G-invariant complex structure. We define this complex structure following [**Vo**]. From now on we use the identification between $\mathcal{G}_0$ and $\mathcal{G}_0{}^*$ given by the Cartan-Killing form, so we will use alternatively $\xi = \xi_X \in \mathcal{G}_0{}^*$ with $X \in \mathcal{G}_0$.

Theorem 4.1. *Let $X \in \mathcal{G}_0$ be such that ad_X has only imaginary eigenvalues. Let $\mathcal{G}$ be the complexified Lie algebra of $\mathcal{G}_0$ and let $\mathcal{G}^t$ ($t \in \mathbf{R}$) be the t-eigenspace of ad_{iX}. Then*

$$\mathcal{G} = \sum_{t \in \mathbf{R}} \mathcal{G}^t, \quad (\mathcal{G}_{0X})_c = \mathcal{G}_X = \mathcal{G}^0$$

is a gradation of $\mathcal{G}$. We define

$$\mathcal{P}_X = \sum_{t \geq 0} \mathcal{G}^t, \quad \mathcal{N}_X = \sum_{t > 0} \mathcal{G}^t.$$

The following properties are satisfied

    a. *$\mathcal{G}^s$ and $\mathcal{G}^t$ are orthogonal unless $s = -t$.*
    b. *$\bar{\mathcal{G}}^s = \mathcal{G}^{-s}$. (Bar means complex conjugation with respect to the real form $\mathcal{G}_0$.)*
    c. *The adjoint action of G_X preserves $\mathcal{G}^t$.*

$\mathcal{G}/\mathcal{G}_X \approx T_{\xi_X}(G \cdot \xi_X)_c$ is the complexified tangent space at the identity coset. The G-invariant complex structure can be characterized by requiring that $\mathcal{P}_X/\mathcal{G}_X$ is the antiholomorphic tangent space at the identity coset.

Let us write down the standard complex structure on S^2 to relate it with this formalism. Let $V = xE + yF + zG = x\partial_x + y\partial_y + z\partial_z \in \mathcal{G}_0$. We take a representative aG for the orbit of radius a,

$$x^2 + y^2 + z^2 = a^2.$$

Stereographic coordinates are given in terms of the embedding coordinates by

$$V_1 = S^2 - \{(0,0,-a)\}, \quad u_1 = \frac{ax}{z+a}, \quad v_1 = \frac{ay}{z+a}$$

$$V_2 = S^2 - \{(0,0,a)\}, \quad u_2 = \frac{ax}{z-a}, \quad v_2 = \frac{ay}{z-a}.$$

The action of SU(2) is the one induced by the adjoint representation of SU(2).

Let $x_1 : U_1 \longrightarrow \mathbf{C}$, $x_2 : U_2 \longrightarrow \mathbf{C}$ be the projective coordinates for the complex projective space $\mathbf{P}^1 = U_1 \cup U_2$. If we identify

$$x_1 \equiv -v_1 + iu_1, \quad x_2 \equiv -v_2 - iu_2,$$

we obtain a diffeomorphism $S^2 \approx \mathbf{P}^1$. This gives to S^2 the complex structure mentioned above. For this particular choice, the action of SU(2) obtained from the three dimensional representation restricted to S^2 coincides with the one obtained from the fundamental representation with the projective structure.

We write now the complexification of su(2), sl(2,$\mathbf{C}$), in the standard basis

$$H = -i2G, \quad X = E - iF, \quad Y = -E - iF.$$

The eigenvalues of iaG are $-a, 0, +a$ and the corresponding eigenspaces are

$$\mathcal{G}^0 = \mathrm{span}\{G\}, \quad \mathcal{G}^a = \mathrm{span}\{Y\}, \quad \mathcal{G}^{-a} = \mathrm{span}\{X\}.$$

The tangent space at the North pole ($x = y = 0, z = a$) is spanned by $\partial_x, \partial_y \in \mathcal{G}_0/\mathcal{G}_{0aG}$ and in terms of the stereographic coordinates,

$$\partial_x = \frac{1}{2}\partial_{u_1}, \quad \partial_y = \frac{1}{2}\partial_{v_1}.$$

In the complexified tangent space,

$$X = \partial_x - i\partial_y = \frac{i}{2}(-\partial_{v_1} - i\partial_{u_1}), \quad Y = -\partial_x - i\partial_y = \frac{i}{2}(-\partial_{v_1} + i\partial_{u_1}),$$

and since the complex coordinate is $x_1 = -v_1 + iu_1$,

$$\mathcal{G}^a = \mathrm{span}\{Y\} = \mathrm{span}\{\partial_{\bar{x}_1}\}.$$

Definition 4.2. A *G-invariant complex polarization* is a lagrangian subspace of the complexified tangent bundle at ξ, $T_\xi(G \cdot \xi)_c \approx \mathcal{G}/\mathcal{G}_\xi$.

We remind that a subspace is a lagrangian subspace if the symplectic form is 0 on that subspace and its dimension is half the dimension of the symplectic manifold. Because of property a in Theorem 4.1, $\mathcal{P}_X/\mathcal{G}_X$ is a lagrangian subspace and then a complex polarization.

Consider now an integral orbit datum, τ. One can prove that $d\tau$ extends to a representation ϕ of $\mathcal{P}_X$. This extension satisfies $\phi|_{\mathcal{N}_X} = 0$. The induced bundle associated to the character τ, $E(G/G_X, \mathbf{C}_\tau)$ has also a complex structure and the holomorphic sections are characterized by

$$(4.3) \qquad\qquad Z.f = -\phi(Z)f \qquad Z \in \mathcal{P}_X$$

where $f : G \mapsto \mathbf{C}$ satisfies $f(gh) = \tau(h)^{-1}f(g), \;\; g \in G, \;\; h \in G_X$. We will see that in our case this construction gives directly the Hilbert space. For other groups, further corrections are needed.

It is easy to see that for SU(2) the principal bundle $E(\mathrm{SU}(2)/\mathrm{U}(1), \mathrm{U}(1))$ is only a reduction of the principal bundle given by the natural projection

$$\pi : \mathbf{C}^2 - \{0\} \mapsto S^2 \approx \mathbf{P}_1$$

that we call $\Theta(S^2, \mathbf{C}^*)$. The corresponding associated bundles by the representation τ_m (extended to $\mathbf{C}^*$), will be denoted by $E(m)$, $\Theta(m)$. $\Theta(m)$ is an holomorphic vector bundle, whose sections satisfy (4.3), which in this case is simply

$$\partial_{\bar{x}_1}f = 0.$$

Line bundles over S^2 are well studied. A holomorphic section on $\Theta(m)$

$$s : \mathbf{P}_1 \mapsto \Theta(m)/ \quad \pi \circ s = \mathrm{id}_{\mathbf{P}_1},$$

can be given in terms of a function

$$\tilde{s} : \mathbf{C}^2 - \{0\} \mapsto \mathbf{C}_m$$

$((\lambda, \rho) \in \mathbf{C}^2 - \{0\})$ satisfying $\tilde{s}(\lambda \cdot \gamma, \lambda \cdot \rho) = \lambda^m \tilde{s}(\gamma, \rho)$ where $\tilde{s}$ is a homogeneous polynomial in two variables of degree m. The group SU(2) naturally acts on this space of sections, constituting the $(m+1)$-dimensional (unitary) irreducible representation of SU(2).

We see that geometric quantization associates quite naturally to the orbit a Hilbert space where the group G acts. The last step now is to find the algebra of quantum observables.

Quantum observables.

Following [**Vo**], the algebra of observables is the algebra of "twisted differential operators" [**Vo**] on sections of the bundle given by the polarization (real or complex). These operators are endomorphisms of the space of sections of the bundle satisfying certain conditions (which make plausible the name of "differential operators"). We will not give here the general definition, but we will work with the SU(2)-bundles using the description given above.

Consider the space of functions $f : \mathbf{C}^2 - \{0\} \longrightarrow \mathbf{C}$, and (γ, ρ) global coordinates on $\mathbf{C}^2 - \{0\}$. Consider the algebra of differential operators generated by the elements

$$\gamma\partial_\gamma, \quad \gamma\partial_\rho, \quad \rho\partial_\gamma, \quad \rho\partial_\rho.$$

We denote this algebra by $\mathcal{D}$. It is a filtered algebra (each of the elements above has degree 1).

The algebra of twisted differential operators on $\Theta(m)$ is

$$\mathcal{D}_m = \mathcal{D}/(D - m\mathrm{Id})$$

where $D = \gamma\partial_\gamma + \rho\partial_\rho$ is an element in the center of $\mathcal{D}$.

We want to give a presentation for $\mathcal{D}_m$ and compare it to the algebra U_h/I_h obtained in Section 3.

Consider now $\mathcal{U}$ the universal enveloping algebra of the Lie algebra su(2)$^{\mathbf{C}}$ $\approx$ sl(2,$\mathbf{C}$). Let $\{X, Y, H\}$ be the standard basis of sl(2,$\mathbf{C}$) (Example 3.16).

Lemma 4.3. *The filtered algebra homomorphism* $p : \mathcal{U} \longrightarrow \mathcal{D}$, *given by*

$$p(X) = -\gamma\partial_\rho, \quad p(Y) = -\rho\partial_\gamma, \quad p(H) = -\gamma\partial_\gamma + \rho\partial_\rho$$

is injective.

Proof. Notice that $\mathcal{D}$ acts on the space $P_m = \{$homogeneous polynomials of degree $m\}$. We denote by $R_m : \mathcal{D} \longrightarrow \mathrm{End}(P_m)$ this representation. Notice that $\tilde{R}_m = R_m \circ p$ is the $m + 1$-dimensional irreducible representation of su(2). Since we have that $\tilde{R}_m(Z) = 0$ $Z \in \mathcal{U}$ $\forall m \Rightarrow Z = 0$ [**HC**], it follows that p is an injective map.

Lemma 4.4.

$$\mathcal{D} \cong \mathcal{U} \otimes \mathrm{span}\,\{D\}/\left(C - \frac{D}{2}\left(\frac{D}{2} + 1\right)\right)$$

where $C = \frac{1}{2}(XY + YX + \frac{1}{2}H^2)$ *is the Casimir element in* $\mathcal{U}$.

Proof. Define the Lie algebra homomorphism

$$\mathcal{U} \otimes \mathrm{span}\,\{D\} \xrightarrow{S} \mathcal{D}$$

as $S(W \otimes D) = p(W)D$. Since $\{p(X), p(Y), p(H), D\}$ generate $\mathcal{D}$, S is surjective. We want to show that ker$S=I$, where $I = (C - D/2(D/2 + 1))$. One can check directly that $I \subset$ ker$S$. We prove ker $S \subset I$ by contradiction.

Observe first that any element $P \in \mathcal{U} \otimes \mathrm{span}\{D\}/(C - \frac{D}{2}(\frac{D}{2} + 1))$ can be written as $AD + B$. In fact, let $P = \sum_{k=0}^{N} A_k D_k$. By induction on N. The cases of $N = 0, 1$ are obvious. Let $N > 1$.

$$P = A_N D^{N-2}(4C - 2D) + \sum_{k=0}^{N-1} A_k D_k.$$

By induction we have our result.

Let $P_{N-1} = B_1 D + B_0$ be a non zero element in $\ker S$ that is not in I. Let us construct the combination

$$P'_{N-1} = B_1 P_1 + \frac{1}{4} P_{N-1} = \left(\frac{1}{4} B_0 - \frac{1}{2} B_1\right) D + B_1 C$$

it is clear that P'_{N-1} doesn't belong to I unless it is identically 0, that is, $B_0 = B_1 = 0$. In this case P_{N-1} is also 0, against the hypothesis. So P'_{N-1} is in $\ker S$ and not in I. Let us construct now the combination

$$P_N = \left(\frac{1}{4} B_0 - \frac{1}{2} B_1\right) P_{N-1} - P'_{N-1} B_1 = \frac{1}{4} B_0^2 - \frac{1}{2} B_1 B_0 - B_1^2 C.$$

Since $P_N \in \ker S$ and P_N does not contain D, by the injectivity of p we must have $P_N = 0$, that is

$$\frac{1}{4} B_0^2 - B_1^2 C = \frac{1}{2} B_1 B_0.$$

Similarly if we construct

$$P'_N = P_{N-1} \left(\frac{1}{4} B_0 - \frac{1}{2} B_1\right) - P'_{N-1} B_1 = \frac{1}{4} B_0^2 - \frac{1}{2} B_0 B_1 - B_1^2 C.$$

P'_N must also be 0, so we have that

$$\frac{1}{4} B_0^2 - B_1^2 C = \frac{1}{2} B_0 B_1.$$

It follows that B_1 and B_0 commute. Lets us rewrite any of these two relations as

(4.4) $$(B_0 - B_1)^2 = (4C + 1) B_1^2.$$

We show that this relation cannot be satisfied unless $B_0 = B_1 = 0$ and this will be a contradiction. Consider the homomorphism from the (filtered) enveloping algebra to the (graded) symmetric algebra, given by the natural projections

$$\pi_n : \mathcal{U}^{(n)} \longrightarrow S^n = \mathcal{U}^{(n)} / \mathcal{U}^{(n-1)}$$

and project (4.4) to the symmetric algebra (isomorphic to the polynomial algebra). It is obvious that the polynomial $\pi_n(4C + 1)$ is not the square of another polynomial. It follows that (4.4) cannot be satisfied unless $B_0 = B_1 = 0$.

Theorem 4.5.

$$\mathcal{D}_m = \mathcal{U} / \left(C - \frac{m}{2}\left(\frac{m}{2} + 1\right) \mathrm{Id}\right).$$

Proof. Immediate from the definition of $\mathcal{D}$ and the Lemma 4.4.

We now want to make an explicit comparison with the result of deformation quantization, let us make the rescaling

(4.5) $\tilde{X} \mapsto \hbar X, \quad \tilde{Y} \mapsto \hbar Y, \quad \tilde{H} \mapsto \hbar H, \quad \tilde{D} = \hbar D.$

In what follows, $\hbar$ is a number, not an indeterminate; so we are comparing the geometric quantization with the specialization for a value of $\hbar$ of the deformation of the polynomial algebra obtained in Section 3. Notice that with this rescaling we obtain a family of isomorphic Lie algebras

$$[\tilde{H}, \tilde{X}] = \hbar 2\tilde{X}, \quad [\tilde{H}, \tilde{Y}] = -\hbar 2\tilde{Y}, \quad [\tilde{X}, \tilde{Y}] = \hbar \tilde{H}$$

(and $\tilde{D}$ in the center) except for $\hbar \mapsto 0$ (while keeping the generators constant) in which the algebra becomes abelian. $\mathcal{U}_\hbar$ is the enveloping algebra of the Lie algebra for each value of $\hbar$.

The Casimir operator is

$$\tilde{C} = \frac{1}{2}\left(\tilde{X}\tilde{Y} + \tilde{Y}\tilde{X} + \frac{1}{2}\tilde{H}^2 \right).$$

Using (4.4), the corresponding ideal in $\mathcal{U}_\hbar$ is

$$(\tilde{C} - l(l + \hbar)), \qquad l = \hbar m/2.$$

It is enough to take $c(\hbar) = l(l + \hbar)$ to obtain the result of Section 3.

Since l is the eigenvalue of the central element $D/2$ in the corresponding representation, taking the limit $\hbar \mapsto 0$ and keeping the generators constant (abelian Lie algebra) is equivalent to take $m \mapsto \infty$. In the physical picture one says that the classical limit corresponds to large quantum numbers.

We want to make the following observations. By choosing different polynomials $c(h)$ and different values of h we obtain that the specialized **C**-algebras in general are not isomorphic. In fact, it is a known result (see [**Va1**]) that $\mathcal{U}/(C - \mu 1)$ has no finite dimensional representations when μ is not rational, hence different values of μ (that is of $c(h)$) may give non-isomorphic algebras.

We also want to remark that our deformation quantization not only gives a subalgebra that can be specialized for any value of h (namely the subalgebra of elements that have coefficients that are polynomials in h), but in the special case of $SU(2)$, $SL(2, \mathbf{C})$, when h is taking certain values, realizes the subalgebra as a concrete algebra of differential operators on the space of sections described above.

Finally, comparing with the approach of [**BBEW**], it is easy to see that the subalgebra of observables with converging star product is the same as the one we obtain, that is, the algebra of polynomials on the algebraic manifold.

References

[ACG]   D. Arnal, M. Cahen and S. Gutt, *Deformations on coadjoint orbits*, J. Geom. Phys., **3** (1987), 327-351.

[ALM]   D. Arnal, J. Ludwig and M. Masmoudi, *Déformations covariantes sur les orbites polarisées d'un group de Lie*, J. Geom. Phys., **14** (1994), 309-331.

[BFFLS] F. Bayen, M. Flato, C. Fronsdal, A. Lichnerowicz and D. Sternheimer, *Deformation theory and quantization*, I *and* II, Ann. Phys., **111**(**1**) (1978), 61-151.

[Be]   F.A. Berezin, *Quantization*, Math. USSR Izvestija, **8** (1974), 1109-1165.

[BBEW] M. Bordemann, M. Brischle, C. Emmrich and S. Waldmann, *Subalgebras with converging star products in deformation quantization*, preprint q-alg/9512019, (1995).

[Bo]   A. Borel, *Linear Algebraic Groups*, Springer Verlag, (1991).

[C]   B. Cahen, *Deformation program for principal series representations*, Lett. Math. Phys., **36** (1996), 65-75.

[CG]   M. Cahen and S. Gutt, *Produits ∗ sur les orbites des groupes semi-simples de rang* 1, C.R. Acad. Sc. Paris, **296** (1983), série I, 821-823; *An algebraic construction of ∗ product on the regular orbits of semisimple Lie groups*, in 'Gravitation and Cosmology', Monographs and Textbooks in Physical Sciences. A volume in honor of Ivor Robunian, Bibliopolis, Eds W. Rundler and A. Trautman, (1987); *Non-localité d'une déformation symplectique sur la sphère S^2*, Bull. Soc. Math. Belg., **36 B** (1987), 207-221.

[CGR]   M. Cahen, S. Gutt and J. Rawnsley, *On tangential star product for the coadjoint Poisson Structure*, Comm. Math. Phys., **180** (1996), 99-108.

[CP]   V. Chari and A. Pressley, *A Guide to Quantum Groups*, Cambridge University Press, (1994).

[DL]   M. De Wilde and P.B.A. Lecomte, *Existence of star-products and of formal deformations in Poisson Lie algebras of arbitrary symplectic manifolds*, Lett. Math. Phys., **7** (1983), 487-496.

[EK]   P. Etingof and D.A. Kazhdan, *Quantization of Lie bialgebras*, I, Selecta Math., New Series **2**(**1**) (1996), 1-41.

[Fe]   B. Fedosov, *A simple geometric construction of deformation quantization*, J. Diff. Geom., **40**(**2**) (1994), 213-238.

[GV]   J.M. García Bondía and Joseph C. Varilly, *From geometric quantization to Moyal quantization*, J. Math. Phys., **36** (1995), 2691-2701.

[Gu]   S. Gutt, *An explicit ∗-product on the cotangent bundle of a Lie group*, Lett. Math. Phys., **7** (1983), 249-258.

[HC]   Harish-Chandra, *Representations of a semisimple Lie group* I, II, II, Trans. Amer. Math. Soc., **75** (1953), 185-243; **76** (1954), 26-65; **76** (1954), 234-253.

[Ho]   J. Hoppe, *On the deformation of time harmonic flows*, in 'Deformation Theory and Symplectic Geometry', proceedings of the Ascona meeting, June 1996, Eds. D. Sternheimer, J. Rawnsley and S. Gutt, Kluwer Academic Publishers, (1997).

[Ki]   A. Kirillov, *Elements of the Theory of Representations*, Springer Verlag, (1975).

[Ko]   M. Kontsevich, *Deformation Quantization of Poisson Manifolds*, preprint q-alg/9709040, (1997).

[Ks]   B. Kostant, *Lie group representations on polynomial rings*, Amer. J. of Math., **86** (1964), 271-309.

[Pu]   M. Puta, *Hamiltonian Mechanical Systems and Geometric Quantization*, Kluwer Academic Publishers, (1993).

[RCG]   J.Rawsnsley, M. Cahen and S. Gutt, *Quantization of Kähler manifolds* I: *Geometric interpretation of Berezin's quantization*, J. Geom. Phys., **7**(**1**) (1990), 45-62.

[RT]   N. Reshetekhin and L. Takhtajan, *Deformation Quantization of Kähler Manifolds*, preprint, (1999).

[So]   Sorieau, *Structure des Systémes Dynamiques*, Dunod, Paris, (1970).

[Va1]   V.S. Varadarajan, *Lie Groups, Lie Algebras and their Representations*, Springer Verlag, (1984).

[Va2]   ———— , *Harmonic Analysis on Real Reductive Groups*, Lecture Notes in Mathematics, **576**, Springer-Verlag, (1977).

[Va3]   ———— , *On the ring of invariant polynomials on a semisimple Lie algebra*, Amer. J. Math, **90** (1968), 308-317.

[Vo]   D. Vogan, *The Orbit Method and Unitary Representations for Reductive Lie Groups*, in 'Algebraic and Analytic Methods in Representation Theory', Perspectives in Mathematics, Vol. 17, Academic Press, (1996).

Universita Di Bologna
Piazza Porta San Donato 5
40125 Bologna
Italy
E-mail address: fioresi@dm.unibo.it

Dipartimento di Fisica
Politecnico di Torino
Corso Duca degli Abruzzi, 24
10129 Torino
Italy

Istituto Nazionale di Fisica Nucleare (INFN)
Sezione di Torino
Italy
E-mail address: lledo@athena.polito.it

NONEXISTENCE OF LOCAL MINIMA OF SUPERSOLUTIONS FOR THE CIRCULAR CLAMPED PLATE

Hans-Christoph Grunau and Guido Sweers

In general, superbiharmonic functions do not satisfy a minimum principle like superharmonic functions do, i.e., functions u with $0 \not\equiv \Delta^2 u \geq 0$ may have a strict local minimum in an interior point. We will prove, however, that when a superbiharmonic function is defined on a disk and additionally subject to Dirichlet boundary conditions, it cannot have interior local minima. For the linear model of the clamped plate this means that a circular plate, which is pushed from below, cannot bend downwards even locally.

The simple biharmonic function $u(x) := |x|^2$ shows that there are no classical *local* maximum/minimum principles for the biharmonic operator Δ^2 (and for higher order elliptic operators at all). On the other hand it is known that *boundary value problems* like the clamped plate equation

$$(1) \qquad \begin{cases} \Delta^2 u = f \text{ in } \Omega, \\ u|\partial\Omega = \dfrac{\partial u}{\partial \nu}|\partial\Omega = 0, \end{cases}$$

enjoy some positivity properties. Here $\Omega \subset \mathbb{R}^2$ is a bounded smooth domain with exterior unit normal ν at $\partial\Omega$. This boundary value problem has an obvious physical interpretation. The solution u gives the deflection of a plate of shape Ω from the equilibrium $u \equiv 0$, which is clamped horizontally and which is subject to the vertical force f. In this context there exist some positivity results: If the domain Ω is the unit disk $B = \{x \in \mathbb{R}^2 : |x| < 1\}$ (see [**Bo**]) or if Ω is close to the disk B in a suitable sense (see [**GS1**]), then it is known that $0 \not\equiv f \geq 0$ implies $u > 0$, i.e., upwards pushing yields (globally) upwards bending. So, at least in these domains, nonconstant supersolutions of the clamped plate equation (1) are strictly positive. Here we call a function $u \in C^4(\Omega) \cap C^1(\bar{\Omega})$ a supersolution of (1), if it solves (1) with some $f \geq 0$. It should be stressed that, in spite of the seemingly quadratic structure of (1), the so called Dirichlet boundary conditions (1.b) prevent us from iterating second order methods.

Although existence/nonexistence of local minima is of no use for proving positivity of solutions, it would be interesting to have a more precise information about the shape of supersolutions to (1). So an interesting question to ask in this context would be:

> May a solution u to the clamped plate boundary value problem (1) have a local minimum in Ω, although the right hand side is nonnegative: $f \geq 0$ and $f \not\equiv 0$?

There are (highly nontrivial) examples even for arbitrarily smooth convex domains Ω where the answer is affirmative, the first of which is due to Duffin [**Du**]. Subsequently Garabedian could prove a striking result (see [**Ga**]): In a long thin ellipse Ω there exists a right hand side $0 \not\equiv f \geq 0$ such that the corresponding solution of (1) also has negative values and hence a (global and local) negative minimum in Ω. For a more extensive survey on positivity results/nonpositivity examples we refer to [**GS1**].

In contrast to the examples above, we shall show that it is actually possible to exclude the existence of local minima, when Ω is a disk:

Theorem 1. *Let $\Omega = B \subset \mathbb{R}^2$ be the unit disk. Assume that $u \in C^4(B) \cap C^1(\bar{B})$ is a solution of the clamped plate equation (1) with some $0 \not\equiv f \geq 0$. Then u has no local minimum in B.*

We remark that this is not a one dimensional result, as neither u nor f are assumed to be radially symmetric. The radial analogue of Theorem 1 can be found in [**So**, Proposition 1], cf. also [**Da**, Theorem 2.4].

To prove the result we reduce our nonsymmetric problem to a radial one: A possible minimum will be moved to the origin by means of a suitable Moebius transform. After radialization one would obtain a radial nonconstant supersolution, which has an interior local minimum at the origin. Due to Soranzo's result just mentioned this is impossible.

We feel that positivity and absence of local minima should be related. Could one perhaps show that in those domains Ω, where the Green function for the clamped plate equation (1) is positive, solutions with $0 \not\equiv f \geq 0$ are not only strictly positive, but don't have any local minimum in Ω, too?

However, here we are restricted to the disk. Neither the proof of Theorem 1 below nor the proof of our positivity result for domains close to the disk in [**GS1**] seems to give any indication on how to treat such a conjecture.

We finish the introduction with a brief description of a further physical interpretation of Theorem 1. Let the velocity field (v_1, v_2) and the pressure p be a solution of the linear Stokes system in $B \subset \mathbb{R}^2$ subject to zero boundary

conditions and an exterior force field (F_1, F_2):

(2)
$$\begin{cases} -\Delta v_j + p_{x_j} = F_j & \text{in } B, \ j = 1, 2, \\ v_{1,x_1} + v_{2,x_2} = 0 & \text{in } B, \\ v_j | \partial B = 0 & j = 1, 2. \end{cases}$$

Then for the stream funtion $u : \bar{B} \to \mathbb{R}, \quad u_{x_1} = -v_2, \ u_{x_2} = v_1$, normalized by the condition $u | \partial B = 0$, Theorem 1 yields: If the vorticity $F_{2,x_1} - F_{1,x_2}$ of the force field is nonnegative (and not identically zero), then the stream function cannot have a local minimum in B. That means that around an interior rest point of the velocity field the fluid cannot rotate clockwise provided the vorticity of the force field is nonnegative. According to the above mentioned example of Garabedian [**Ga**] this could actually happen e.g., in a long thin ellipse.

Moebius transforms. In what follows we will identify $\mathbb{R}^2$ and $\mathbb{C}$ and use real and complex notation simultaneously: $x = (x_1, x_2) = x_1 + i x_2$. The dot "$\cdot$" denotes the multiplication in $\mathbb{C}$: $a \cdot x = a_1 x_1 - a_2 x_2 + i(a_1 x_2 + a_2 x_1)$, while we use brackets for the scalar product in $\mathbb{R}^2$: $\langle a, x \rangle = a_1 x_1 + a_2 x_2$. In case of holomorphic mappings h we denote the complex derivative by h'.

For $a \in B$ we consider the biholomorphic Moebius transform

(3)
$$h : \bar{B} \to \bar{B}, \quad h(x) = \frac{a - x}{1 - \bar{a} \cdot x}$$

and its inverse $h^{-1} = h$. We have $h(B) = B$, $h(\partial B) = \partial B$, $h(0) = a$, $h(a) = 0$. We know from Loewner [**Loe**] that Moebius transforms as in (3) and suitable simultaneous transformations of the dependent variable u leave the biharmonic *equation* invariant. As we are interested in biharmonic *inequalities* we need a slightly more precise information:

Lemma 1. *Let $u \in C^4(B, \mathbb{R})$. For some $a \in B$ we consider the Moebius transform h from (3). For the C^4-function v defined by*

(4)
$$v(x) := \frac{1}{|h'(x)|} u(h(x)), \quad x \in B,$$

we have

(5)
$$\Delta^2 v(x) = |h'(x)|^3 \left(\Delta^2 u\right)(h(x)), \quad x \in B.$$

Proof. Instead of the real variables x_1, x_2, we use the complex variables $x, \bar{x}$ and also $z = h(x)$, $\bar{z} = \overline{h(x)} = \frac{\bar{a} - \bar{x}}{1 - a \cdot \bar{x}}$.

One has $\frac{\partial}{\partial \bar{x}} h(x) = 0$, $\frac{\partial}{\partial x} \overline{h(x)} = 0$, $h' = \frac{\partial}{\partial x} h(x) = \frac{|a|^2 - 1}{(1 - \bar{a} \cdot x)^2}$ and

$$h''(x) = \frac{|a|^2 - 1}{(1 - \bar{a} \cdot x)^3} 2\bar{a} = \frac{2\bar{a}}{1 - \bar{a} \cdot x} h'(x).$$

In complex notation we have

$$\left(1 - |a|^2\right) \ v\left(x, \bar{x}\right) = \left(1 - \bar{a} \cdot x\right)\left(1 - a \cdot \bar{x}\right) u\left(h\left(x\right), \overline{h\left(x\right)}\right).$$

Then

$$\left(1 - |a|^2\right) v_{xx}\left(x, \bar{x}\right)$$
$$= -2\bar{a}\left(1 - a \cdot \bar{x}\right) h' u_z + \left(1 - \bar{a} \cdot x\right)\left(1 - a \cdot \bar{x}\right)\left(\left(h'\right)^2 u_{zz} + h'' u_z\right)$$
$$= \left(1 - \bar{a} \cdot x\right)\left(1 - a \cdot \bar{x}\right)\left(h'\right)^2 u_{zz}$$

and similarly

$$\left(1 - |a|^2\right) v_{xx\bar{x}\bar{x}}\left(x, \bar{x}\right) = \left(1 - \bar{a} \cdot x\right)\left(1 - a \cdot \bar{x}\right)\left(h'\right)^2 \left(\overline{h'}\right)^2 u_{zz\bar{z}\bar{z}}$$
$$= \left(1 - |a|^2\right) \left|h'\right|^3 u_{zz\bar{z}\bar{z}}$$

and the relation (5) follows by $\Delta^2 u = 16 u_{zz\bar{z}\bar{z}}$. $\qquad\square$

Proof of Theorem 1.

Let $u \in C^4(B) \cap C^1(\bar{B})$ be a solution of the clamped plate equation (1) with $\Omega = B$ and $0 \not\equiv f \geq 0$. We assume by contradiction that u has a local minimum at $a \in B$.

From u we want to construct a radial superbiharmonic function with homogeneous Dirichlet boundary conditions, which would also have a local minimum. According to [**So**] that will be impossible.

Before we may radialize the solution, we will move the point a, where u has a local minimum, into the origin. For this purpose we consider the Moebius transform (3) and define $v \in C^4(B) \cap C^1(\bar{B})$ according to (4):

$$v(x) := \frac{1}{|h'(x)|} u(h(x)), \quad x \in \bar{B}.$$

By means of (5) from Lemma 1 we see that v solves a related clamped plate equation

$$\begin{cases} \Delta^2 v = |h'|^3 \left(f \circ h\right) \text{ in } B, \\ v|\partial B = \dfrac{\partial v}{\partial \nu}|\partial B = 0. \end{cases}$$

Now we radialize. As radialization and the Laplace operator commute, we see that the radially symmetric function

$$(6) \qquad\qquad w(x) := \frac{1}{2\pi} \int_{|\xi|=1} v\left(|x|\xi\right) d\omega(\xi)$$

is also in $C^4(B) \cap C^1(\bar{B})$ and solves the Dirichlet problem

$$(7) \qquad\qquad \begin{cases} \Delta^2 w = g \text{ in } B, \\ w|\partial B = \dfrac{\partial w}{\partial \nu}|\partial B = 0. \end{cases}$$

Here we have set

$$g(x) := \frac{1}{2\pi} \int_{|\xi|=1} \left(|h'|^3 \left(f \circ h \right) \right) (|x|\xi)\, d\omega(\xi).$$

Obviously we have $0 \not\equiv g \geq 0$.

Due to the dilation factor $\frac{1}{|h'|}$ in the definition (4) of v it is not clear whether or not v has a local minimum in 0. But for the radialization w this is indeed the case. Using that u has a local minimum in a, we conclude for $|x|$ small enough:

$$
\begin{aligned}
w(x) &= \frac{1}{2\pi} \int_{|\xi|=1} \frac{1}{|h'\left(|x|\xi\right)|} u \left(h \left(|x|\xi \right) \right)\, d\omega(\xi) \\
&\geq u(a) \left\{ \frac{1+|a|^2|x|^2}{1-|a|^2} - \frac{|x|}{\pi \left(1-|a|^2\right)} \int_{|\xi|=1} \langle a, \xi \rangle\, d\omega(\xi) \right\} \\
&= \frac{1+|a|^2|x|^2}{1-|a|^2} u(a) \geq \frac{1}{1-|a|^2} u(a) = \frac{1}{|h'(0)|}\, u(h(0)) = w(0).
\end{aligned}
$$

Here we used that $u(a) \geq 0$, which follows from $f \geq 0$ and from the positivity of the corresponding Green's function in B, see [**Bo**].

Let us sum up what we have shown. From our assumption that u has a local minimum in some point $a \in B$ we could conclude that there is a radial supersolution w of the clamped plate equation (7) with $0 \not\equiv g \geq 0$, which has a local minimum in 0. (If $a \neq 0$, this minimum would be even strict.) We obtain a contradiction by a result of Soranzo [**So**, Proposition 1] (cf. also [**Da**, Theorem 2.4]), according to which w is strictly radially decreasing in $|x| \in (0,1)$. $\qquad\square$

Remark. The same method applies to solutions of the clamped plate equation

$$
(8) \qquad
\begin{cases}
\Delta^2 u = f \text{ in } B \subset \mathbb{R}^2, \\
u|\partial B = 0, \quad \left(-\dfrac{\partial u}{\partial \nu} \right)|\partial B = \varphi,
\end{cases}
$$

where the boundary datum φ, as well as the right hand side f, is assumed to be nonnegative (and one of these two not identically zero). It may seem unsatisfactory that the solution itself has to be prescribed homogeneously on ∂B, but also in [**GS2**] this boundary datum played a special role. There we were concerned with a perturbation theory for positivity in generalized clamped plate equations under inhomogeneous Dirichlet boundary conditions.

References

[Bo] T. Boggio, *Sulle funzioni di Green d'ordine m*, Rend. Circ. Mat. Palermo, **20** (1905), 97-135.

[Da] R. Dalmasso, *A priori estimates for some semilinear elliptic equations of order* $2m$, Nonlinear Anal., **29** (1997), 1433-1452.

[Du] R.J. Duffin, *On a question of Hadamard concerning super-biharmonic functions*, J. Math. Phys., **27** (1949), 253-258.

[Ga] P.R. Garabedian, *A partial differential equation arising in conformal mapping*, Pacific J. Math., **1** (1951), 485-524.

[GS1] H.-Ch. Grunau and G. Sweers, *Positivity for perturbations of polyharmonic operators with Dirichlet boundary conditions in two dimensions*, Math. Nachr., **179** (1996), 89-102.

[GS2] _____, *The role of positive boundary data in generalized clamped plate equations*, Z. angew. Math. Phys. ZAMP, **49** (1998), 420-435.

[Loe] Ch. Loewner, *On generation of solutions of the biharmonic equation in the plane by conformal mappings*, Pacific J. Math., **3** (1953), 417-436.

[So] R. Soranzo, *A priori estimates and existence of positive solutions of a superlinear polyharmonic equation*, Dyn. Syst. Appl., **3** (1994), 465-487.

FACHGRUPPE MATHEMATIK
UNIVERSITÄT BAYREUTH
D-95440 BAYREUTH
GERMANY
E-mail address: Hans-Christoph.Grunau@uni-bayreuth.de

UNIVERSITY OF UTRECHT
NL-3508 TA UTRECHT
THE NETHERLANDS

AFDELING TOEGEPASTE WISKUNDIGE ANALYSE - FACULTEIT ITS
TU DELFT
MEKELWEG 4
NL-2628 CD DELFT
THE NETHERLANDS
E-mail address: G.H.Sweers@its.tudelft.nl

THE SPECTRUM OF AN INTEGRAL OPERATOR IN WEIGHTED L_2 SPACES

Mourad E.H. Ismail and Plamen C. Simeonov

We find the spectrum of the inverse operator of the q-difference operator $D_{q,x}f(x) = (f(x) - f(qx))/(x(1-q))$ on a family of weighted L_2 spaces. We show that the spectrum is discrete and the eigenvalues are the reciprocals of the zeros of an entire function. We also derive an expansion of the eigenfunctions of the q-difference operator and its inverse in terms of big q-Jacobi polynomials. This provides a q-analogue of the expansion of a plane wave in Jacobi polynomials. The coefficients are related to little q-Jacobi polynomials, which are described and proved to be orthogonal on the spectrum of the inverse operator. Explicit representations for the little q-Jacobi polynomials are given.

1. Introduction.

The q-difference operator $D_{q,x}$ is defined by

$$(1.1) \qquad D_{q,x}f(x) := \frac{f(x) - f(qx)}{x(1-q)}.$$

We shall use the following notations for finite and infinite products:

$$(z;q)_0 := 1, \quad (z;q)_n := \prod_{j=0}^{n-1}(1 - q^j z), \quad n > 0 \quad \text{or} \quad n = \infty,$$

$$(z_1, z_2, \dots, z_s; q)_n := \prod_{k=1}^{s}(z_k;q)_n, \quad n \geq 0 \quad \text{or} \quad n = \infty.$$

The infinite product is defined for $|q| < 1$.

A basic hypergeometric series is defined by

$$
\begin{aligned}
(1.2) \qquad {}_r\phi_s\left(\begin{array}{c} a_1, \dots, a_r \\ b_1, \dots, b_s \end{array} \middle|\, q, z \right) &= {}_r\phi_s(a_1, \dots, a_r; b_1, \dots, b_s; q, z) \\
&:= \sum_{n=0}^{\infty} \frac{(a_1, \dots, a_r; q)_n}{(q, b_1, \dots, b_s; q)_n} z^n \left((-1)^n q^{n(n-1)/2}\right)^{1+s-r}.
\end{aligned}
$$

Let $\mathbf{t} = (t_1, t_2) \in \mathbf{R}^2$. The big q-Jacobi polynomials of Andrews and Askey, [2], are defined by

(1.3)
$$p_n(x, \mathbf{t}) = p_n(x; t_1, t_2) := {}_3\phi_2\left(\begin{array}{c} q^{-n},\, at_1t_2q^{n-1},\, t_1x \\ t_1,\, at_1 \end{array}\,\middle|\, q, q\right), \quad 0 < q < 1$$

and are orthogonal with respect to the measure $\mu(x, \mathbf{t})$, [3, p. 594], [6], defined by

(1.4)
$$\mu(x, \mathbf{t}) := \frac{\mu^{(a)}(x)}{(t_1x, t_2x; q)_\infty},$$

where for $a < 0$, $\mu^{(a)}$ is the discrete probability measure

(1.5)
$$\mu^{(a)} := \sum_{n=0}^{\infty}\left[\frac{q^n}{(q, q/a; q)_n(a; q)_\infty}\delta_{q^n} + \frac{q^n}{(q, aq; q)_n(1/a; q)_\infty}\delta_{aq^n}\right].$$

In (1.5), δ_z denotes the unit measure supported on $\{z\}$. The orthogonality relation is, [3], [6],

(1.6)
$$\int_{\mathbf{R}} p_m(x, \mathbf{t})p_n(x, \mathbf{t})\, d\mu(x, \mathbf{t}) = \delta_{m,n}\xi_n(\mathbf{t}),$$

where

(1.7)
$$\xi_n(\mathbf{t}) = \frac{(q, t_2, at_2, at_1t_2q^{n-1}; q)_n(at_1t_2q^{2n}; q)_\infty}{(t_1, at_1, t_2, at_2; q)_\infty(t_1, at_1; q)_n}(-at_1^2)^n q^{n(n-1)/2}.$$

Furthermore, (1.3), (1.6), (1.7), and the symmetry of μ in t_1 and t_2 imply the symmetry relation

(1.8)
$$p_n(x; t_1, t_2) = \frac{t_1^n(t_2, at_2; q)_n}{t_2^n(t_1, at_1; q)_n}p_n(x; t_2, t_1).$$

We shall use the following q-analogue of the Chu-Vandermonde sum, [4, (II.6)],

(1.9)
$${}_2\phi_1(q^{-n}, b; c; q, q) = \frac{(c/b; q)_n}{(c; q)_n}b^n,$$

and its special case ($b = t_2x$, $c = t_2q^{1-n}/t_1$)

(1.10)
$$(t_1x; q)_n = (t_1/t_2)_n {}_2\phi_1(q^{-n}, t_2x; t_2q^{1-n}/t_1; q, q).$$

Letting $t_1 \to \infty$ in (1.10) with $t_2 = t$ we obtain

(1.11)
$$t^n x^n = \sum_{j=0}^{n}\begin{bmatrix} n \\ j \end{bmatrix}_q (-1)^j q^{\binom{j}{2}-j(n-1)}(tx; q)_j,$$

where

$$\begin{bmatrix} n \\ j \end{bmatrix}_q := \frac{(q; q)_n}{(q; q)_j(q; q)_{n-j}}, \quad j = 0, \ldots, n$$

are the so called q-binomial coefficients.

We shall use Euler's identities, [**6**],

$$(1.12) \qquad e_q(z) := \frac{1}{(z;q)_\infty} = \sum_{j=0}^\infty \frac{z^j}{(q;q)_j}, \quad |z| < 1,$$

$$(1.13) \qquad E_q(z) := (z;q)_\infty = \sum_{j=0}^\infty \frac{q^{\binom{j}{2}}(-z)^j}{(q;q)_j},$$

and the terminating version of the q-binomial theorem, [**6**],

$$(1.14) \qquad (z;q)_n = \sum_{j=0}^n \begin{bmatrix} n \\ j \end{bmatrix}_q q^{\binom{j}{2}}(-z)^j.$$

We shall also use the following identity

$$(1.15) \qquad (q^{1-n}/A;q)_k = (-1/A)^k q^{\binom{k}{2}+k(1-n)} \frac{(A;q)_n}{(A;q)_{n-k}}.$$

The following theorem of H. Schwartz, [**13**], plays an important role in the spectral analysis in Section 2.

Theorem 1.1. *Let $\{p_{n,\nu}(x)\}$ be a family of monic polynomials generated by*

$$(1.16) \qquad p_{0,\nu}(x) = 1, \quad p_{1,\nu}(x) = x + B_\nu,$$

$$(1.17) \qquad p_{n+1,\nu}(x) = (x + B_{n+\nu})p_{n,\nu}(x) + C_{n+\nu}p_{n-1,\nu}(x), \quad n \geq 1.$$

If both

$$(1.18) \qquad \sum_{n=0}^\infty |B_{n+\nu} - \beta| < \infty \quad and \quad \sum_{n=0}^\infty |C_{n+\nu}| < \infty$$

hold, then $x^n p_{n,\nu}(-\beta + 1/x)$ converges uniformly on compact subsets of the complex plane to an entire function of x.

The paper is organized in six sections. In Section 2, we study the eigenfunctions of the right inverse operator of $D_{q,x}$ on the space $L_2(\mu(\cdot, q\mathbf{t}))$. This right inverse operator will be denoted by $\mathcal{T}_\mathbf{t}$. The operator $\mathcal{T}_\mathbf{t}$ is defined first on the big q-Jacobi polynomials and then extended by linearity to $L_2(\mu(\cdot, q\mathbf{t}))$, in which the big q-Jacobi polynomials are complete. The operator $\mathcal{T}_\mathbf{t}$ is also a discrete integral operator. The rest of the paper is devoted to the study of the properties of $\mathcal{T}_\mathbf{t}$ and its eigenfunctions. It turns out that the matrix representation of $\mathcal{T}_\mathbf{t}$ in the basis formed by the big q-Jacobi polynomials is tridiagonal. This gives a three-term recurrence relation for the coefficients in the expansion of the eigenfunctions of $\mathcal{T}_\mathbf{t}$ in big q-Jacobi polynomials.

In Section 3, we find the polynomial solution of the recurrence relation for the coefficients in the expansion of the eigenfunctions of the operator $\mathcal{T}_\mathbf{t}$

from Section 2. The recurrence relation is identified with that of the little q-Jacobi polynomials.

In Section 4, we find the expansion of the formal eigenfunctions of the operator $D_{q,x}$ on the space $L_2(\mu(\cdot, \mathbf{t}))$ in terms of $\{p_n(x, \mathbf{t})\}$. The first coefficient of the expansion is used to determine the spectrum of the inverse operator $\mathcal{T}_\mathbf{t}$. This eigenfunction expansion gives a discrete q-analogue (on a q-linear lattice) of the well-known expansion of a plane wave $\exp(i\lambda x)$ in Jacobi polynomials.

Section 5 contains asymptotic properties of the orthogonal polynomials found in Section 3 and a formula for the Stieltjes transform of the measure of orthogonality. We prove that the measure of orthogonality is purely discrete and we identify the location of its masses with the discrete zero-set of an entire function and show how it is related to the spectrum of the operator $\mathcal{T}_\mathbf{t}$.

The paper concludes with Section 6, where we find the connection coefficients between families of big q-Jacobi polynomials corresponding to different values of the parameter $\mathbf{t}$.

2. The Spectral Analysis.

Let $\mathbf{t} = (t_1, t_2)$. By $q\mathbf{t}$ we shall denote the pair (qt_1, qt_2). The Hilbert space $L_2(\mu(\cdot, \mathbf{t}))$ is defined through the standard dot-product

$$\langle f, g \rangle = \int_{\mathbf{R}} f(x)\overline{g(x)}\, d\mu(x, \mathbf{t}),$$

and

$$L_2(\mu(\cdot, \mathbf{t})) = \left\{ f \ : \ ||f||_{L_2(\mu(\cdot, \mathbf{t}))} = \left(\int_{\mathbf{R}} |f(x)|^2\, d\mu(x, \mathbf{t}) \right)^{1/2} < \infty \right\}.$$

Let $\mathcal{T}_\mathbf{t}$ denote the right inverse operator of $D_{q,x}$ on $L_2(\mu(x, \mathbf{t}))$, that is, $\mathcal{T}_\mathbf{t}$ is a linear operator from $L_2(\mu(x, q\mathbf{t}))$ to $L_2(\mu(x, \mathbf{t}))$ such that $D_{q,x}\mathcal{T}_\mathbf{t}$ is the identity operator on the range of $D_{q,x}$ acting on $L_2(\mu(x, \mathbf{t}))$. From (1.3) and (1.1) we find

$$(2.1) \qquad D_{q,x}p_n(x, \mathbf{t}) = \frac{t_1 q^{1-n}(1 - q^n)(1 - at_1 t_2 q^{n-1})}{(1 - q)(1 - t_1)(1 - at_1)} p_{n-1}(x, q\mathbf{t})$$
$$=: \sigma_n(\mathbf{t}) p_{n-1}(x, q\mathbf{t}).$$

Thus we require $\mathcal{T}_\mathbf{t}$ to satisfy

$$(2.2) \qquad \mathcal{T}_\mathbf{t} g(x) \sim \sum_{n=0}^{\infty} (g_n / \sigma_{n+1}(\mathbf{t}))\, p_{n+1}(x, \mathbf{t})$$

if

$$g(x) \sim \sum_{n=0}^{\infty} g_n p_n(x, q\mathbf{t}).$$

The operator $\mathcal{T}_{\mathbf{t}}$ can be expressed as an integral operator as well. From the orthogonality relation (1.6) we get

$$g_n = \frac{1}{\xi_n(q\mathbf{t})} \int_{\mathbf{R}} g(u) p_n(u, q\mathbf{t}) \, d\mu(u, q\mathbf{t}).$$

Substituting in (2.2) and formally interchanging the order of integration and summation we get

$$(2.3) \qquad \mathcal{T}_{\mathbf{t}} g(x) = \int_{\mathbf{R}} g(u) \left\{ \sum_{n=0}^{\infty} \frac{p_n(u, q\mathbf{t}) p_{n+1}(x, \mathbf{t})}{\xi_n(q\mathbf{t}) \sigma_{n+1}(\mathbf{t})} \right\} d\mu(u, q\mathbf{t}).$$

The sum in (2.3) is the kernel of the integral operator $\mathcal{T}_{\mathbf{t}}$.

We now consider the eigenvalue problem for the operator $\mathcal{T}_{\mathbf{t}}$, namely

$$(2.4) \qquad \mathcal{T}_{\mathbf{t}} g = \lambda g, \quad g(x) = \sum_{n=0}^{\infty} a_n(\lambda, \mathbf{t}) p_n(x, \mathbf{t}),$$

with $g \in L_2(\mu(\cdot, \mathbf{t})) \cap L_2(\mu(\cdot, q\mathbf{t}))$. The function g can be expanded in terms of the polynomials $\{p_n(x, \mathbf{t})\}$ since they are dense in $L_2(\mu(\cdot, \mathbf{t}))$. Furthermore, (2.2) implies $a_0(\lambda, \mathbf{t}) = 0$ since g is in the range of $\mathcal{T}_{\mathbf{t}}$.

We will need a connection coefficient formula of the form

$$p_n(x, \mathbf{t}) = \sum_{j=0}^{n} c_{n,j}(\mathbf{t}) p_j(x, q\mathbf{t}).$$

Such formula we can get using a simple duality theorem, [**15**, Theorem 2.5]. Let μ be a measure, w and ρ, weight functions, and $\{p_n\}$ and $\{q_n\}$, polynomials orthogonal with respect to $w\mu$ and $\rho\mu$, respectively. Let $\alpha_n = \int |p_n|^2 w d\mu$ and $\beta_n = \int |q_n|^2 \rho d\mu$. If

$$w(x) p_n(x) \sim \rho(x) \sum_{j=n}^{\infty} c_{n,j} q_j(x), \quad \text{then} \quad q_n(x) = \sum_{k=0}^{n} (\beta_n/\alpha_k) c_{k,n} p_k(x).$$

Indeed, if $q_n = \sum_{k=0}^{n} d_{n,k} p_k$, then $d_{n,k} = (1/\alpha_k) \int q_n p_k w d\mu = (\beta_n/\alpha_k) c_{k,n}$. Moreover, if w/ρ is a polynomial of degree s, then $c_{n,j} = 0$ for $j > n + s$. In our case with $w(x) = 1/(xqt_1, xqt_2; q)_{\infty}$, $\rho(x) = 1/(xt_1, xt_2; q)_{\infty}$, and $\mu = \mu^{(a)}$ we have $w(x)/\rho(x) = (1 - t_1 x)(1 - t_2 x)$. Thus

$$(2.5) \qquad p_n(x, \mathbf{t}) = \sum_{m=n-2}^{n} c_{m,n}(\mathbf{t}) \left(\xi_n(\mathbf{t})/\xi_m(q\mathbf{t}) \right) p_m(x, q\mathbf{t}), \quad n \geq 0,$$

where the coefficients $\{c_{m,n}(\mathbf{t})\}$ satisfy the equation

$$(2.6) \qquad (1 - t_1 x)(1 - t_2 x)p_n(x, q\mathbf{t}) = \sum_{k=n}^{n+2} c_{n,k}(\mathbf{t})p_k(x, \mathbf{t}), \quad n \geq 0.$$

The coefficients in (2.6) can be computed explicitly. Comparing the coefficients of x^{n+2} in (2.6) we get

(2.7)

$$c_{n,n+2}(\mathbf{t}) = \frac{(q^{-n}, at_1 t_2 q^{n+1}; q)_n}{(q^{-n-2}, at_1 t_2 q^{n+1}; q)_{n+2}} \frac{(t_1, at_1, q; q)_{n+2}}{(qt_1, aqt_1, q; q)_n} \frac{q^n(-qt_1)^n q^{\binom{n}{2}}}{q^{n+2}(-t_1)^{n+2}q^{\binom{n+2}{2}}} t_1 t_2$$

$$= \frac{(1 - t_1)(1 - at_1)(1 - t_1 q^{n+1})(1 - at_1 q^{n+1})}{(1 - at_1 t_2 q^{2n+1})(1 - at_1 t_2 q^{2n+2})}(t_2/t_1)q^n.$$

From (1.3) and (1.8) we have $p_m(1/t_1; t_1, t_2) = 1$ and

$$p_m(1/t_2; t_1, t_2) = t_1^m(t_2, at_2; q)_m/(t_2^m(t_1, at_1; q)_m).$$

Substituting in (2.6) x first by $1/t_1$ and then by $1/t_2$ we obtain

$$(2.8) \qquad\qquad c_{n,n}(\mathbf{t}) + c_{n,n+1}(\mathbf{t}) + c_{n,n+2}(\mathbf{t}) = 0,$$
$$(2.9) \qquad\qquad c_{n,n}(\mathbf{t}) + \alpha_n c_{n,n+1}(\mathbf{t}) + \alpha_n \alpha_{n+1} c_{n,n+2}(\mathbf{t}) = 0,$$

where

$$\alpha_n := \frac{t_1(1 - t_2 q^n)(1 - at_2 q^n)}{t_2(1 - t_1 q^n)(1 - at_1 q^n)}.$$

From (2.8) and (2.9) we obtain

$$(2.10) \qquad c_{n,n}(\mathbf{t}) = \frac{\alpha_n(\alpha_{n+1} - 1)}{\alpha_n - 1} c_{n,n+2}(\mathbf{t})$$

$$= \frac{(1 - t_1)(1 - at_1)(1 - t_2 q^n)(1 - at_2 q^n)}{(1 - at_1 t_2 q^{2n})(1 - at_1 t_2 q^{2n+1})}q^n.$$

Applying $D_{q,x}$ to both sides of (2.4) and using (2.1) and (2.5) we obtain

$$D_{q,x}(\lambda g(x)) = \lambda \sum_{n=1}^{\infty} a_n(\lambda, \mathbf{t})\sigma_n(\mathbf{t})p_{n-1}(x, q\mathbf{t}) = g(x)$$

$$= \sum_{m=1}^{\infty} a_m(\lambda, \mathbf{t})p_m(x, \mathbf{t})$$

$$= \sum_{m=1}^{\infty} a_m(\lambda, \mathbf{t}) \sum_{j=m-2}^{m} (\xi_m(\mathbf{t})/\xi_j(q\mathbf{t}))\, c_{j,m}(\mathbf{t})p_j(x, q\mathbf{t}).$$

The above identity implies that the coefficients $\{a_n(\lambda, \mathbf{t})\}$ are generated by

(2.11)
$$\lambda a_n(\lambda, \mathbf{t})\sigma_n(\mathbf{t}) = \sum_{m=n-1}^{n+1} (\xi_m(\mathbf{t})/\xi_{n-1}(q\mathbf{t}))\, c_{n-1,m}(\mathbf{t})a_m(\lambda, \mathbf{t}), \quad n \geq 1,$$

$a_0(\lambda, \mathbf{t}) = 0$ and $a_1(\lambda, \mathbf{t}) \neq 0$ is arbitrary. Formula (2.11) and the initial conditions show that $a_n(\lambda, \mathbf{t})/a_1(\lambda, \mathbf{t})$ is a polynomial of degree $n-1$. We set

$$\tilde{a}_{m-1}(\lambda, \mathbf{t}) := \xi_m(\mathbf{t})a_m(\lambda, \mathbf{t})/(\xi_1(\mathbf{t})a_1(\lambda, \mathbf{t})), \quad m \geq 1,$$

$\tilde{a}_{-1}(\lambda, \mathbf{t}) := 0$ and $c_{n,m}(\mathbf{t}) := (1 - t_1)(1 - at_1)\tilde{c}_{n,m}(\mathbf{t})$. Since

$$\frac{\sigma_n(\mathbf{t})\xi_{n-1}(q\mathbf{t})}{\xi_n(\mathbf{t})} = \frac{(1 - t_1)(1 - at_1)}{-at_1(1 - q)},$$

formula (2.11) can be written in the form

(2.12)
$$\lambda\tilde{a}_n(\lambda, \mathbf{t}) = -at_1(1 - q)\sum_{m=n}^{n+2} \tilde{c}_{n,m}(\mathbf{t})\tilde{a}_{m-1}(\lambda, \mathbf{t}), \quad n \geq 0.$$

In terms of the variable η and the functions $\tilde{b}_m(\eta, \mathbf{t})$ defined by

$$\eta := -\lambda/(at_1(1 - q)) \quad \text{and} \quad \tilde{b}_m(\eta, \mathbf{t}) := \tilde{a}_m(-at_1(1 - q)\eta, \mathbf{t}),$$

formula (2.12) can be written in the form

(2.13)
$$\eta\tilde{b}_n(\eta, \mathbf{t}) = \sum_{m=n}^{n+2} \tilde{c}_{n,m}(\mathbf{t})\tilde{b}_{m-1}(\eta, \mathbf{t}), \quad n \geq 0$$

with the initial conditions

$$\tilde{b}_{-1}(\eta, \mathbf{t}) = 0, \quad \tilde{b}_0(\eta, \mathbf{t}) = 1.$$

From (2.10), (2.7) and (2.8) for the coefficients we obtain

(2.14)
$$\tilde{c}_{n,n} = \tilde{c}_{n,n}(\mathbf{t}) = \frac{(1 - t_2q^n)(1 - at_2q^n)}{(1 - at_1t_2q^{2n})(1 - at_1t_2q^{2n+1})}q^n,$$

(2.15)
$$\tilde{c}_{n,n+2} = \tilde{c}_{n,n+2}(\mathbf{t}) = \frac{(1 - t_1q^{n+1})(1 - at_1q^{n+1})t_2}{(1 - at_1t_2q^{2n+1})(1 - at_1t_2q^{2n+2})t_1}q^n,$$

(2.16)
$$\tilde{c}_{n,n+1} = \tilde{c}_{n,n+1}(\mathbf{t}) = -(\tilde{c}_{n,n}(\mathbf{t}) + \tilde{c}_{n,n+2}(\mathbf{t})).$$

It is convenient to have (2.13) written in monic form. Let

$$\tilde{b}_m(\eta, \mathbf{t}) := G_m(\mathbf{t})b_m(\eta, \mathbf{t}).$$

From (2.13) we have

$$
\eta b_n(\eta, \mathbf{t}) = \frac{\tilde{c}_{n,n} G_{n-1}(\mathbf{t})}{G_n(\mathbf{t})} b_{n-1}(\eta, \mathbf{t}) + \tilde{c}_{n,n+1} b_n(\eta, \mathbf{t})
$$
$$
+ \frac{\tilde{c}_{n,n+2} G_{n+1}(\mathbf{t})}{G_n(\mathbf{t})} b_{n+1}(\eta, \mathbf{t}), \quad n \geq 0,
$$

which is a monic equation if the coefficient of $b_{n+1}(\eta, \mathbf{t})$ is 1, that is, if $G_{n+1}(\mathbf{t}) = G_n(\mathbf{t})/\tilde{c}_{n,n+2}$. In this case

$$
G_n(\mathbf{t}) = G_0(\mathbf{t}) / \prod_{j=0}^{n-1} \tilde{c}_{j,j+2}, \quad n \geq 1, \quad G_0(\mathbf{t}) \neq 0,
$$

and (2.13) takes the form

(2.17)
$$
\eta b_n(\eta, \mathbf{t}) = b_{n+1}(\eta, \mathbf{t}) + \tilde{c}_{n,n+1} b_n(\eta, \mathbf{t}) + \tilde{c}_{n,n} \tilde{c}_{n-1,n+1} b_{n-1}(\eta, \mathbf{t}), \quad n \geq 0.
$$

We set $b_0(\eta, \mathbf{t}) = G_0(\mathbf{t}) = 1$ and $b_{-1}(\eta, \mathbf{t}) = 0$. When the coefficients in (2.17) are written explicitly in terms of t_1, t_2, and n, we obtain

$$
\text{(2.18)} \quad b_{n+1}(\eta, \mathbf{t}) = \left(\eta + \frac{(1 - t_2 q^n)(1 - a t_2 q^n)}{(1 - a t_1 t_2 q^{2n})(1 - a t_1 t_2 q^{2n+1})} q^n \right.
$$
$$
+ \left. \frac{(1 - t_1 q^{n+1})(1 - a t_1 q^{n+1}) t_2}{(1 - a t_1 t_2 q^{2n+1})(1 - a t_1 t_2 q^{2n+2}) t_1} q^n \right) b_n(\eta, \mathbf{t})
$$
$$
- \frac{(1 - t_1 q^n)(1 - a t_1 q^n)(1 - t_2 q^n)(1 - a t_2 q^n) t_2}{(1 - a t_1 t_2 q^{2n-1})(1 - a t_1 t_2 q^{2n})^2 (1 - a t_1 t_2 q^{2n+1}) t_1} q^{2n-1} b_{n-1}(\eta, \mathbf{t}).
$$

We now apply Schwartz's theorem (Theorem 1.1). With $B_n(\mathbf{t}) = -t_1 \tilde{c}_{n,n+1}(\mathbf{t})$, $C_n(\mathbf{t}) = -t_1^2 \tilde{c}_{n,n}(\mathbf{t}) \tilde{c}_{n-1,n+1}(\mathbf{t})$, and $\hat{b}_n(\eta, \mathbf{t}) = t_1^n b_n(\eta/t_1, \mathbf{t})$, (2.18) takes the form

$$
\hat{b}_{n+1}(\eta, \mathbf{t}) = (\eta + B_n(\mathbf{t})) \hat{b}_n(\eta, \mathbf{t}) + C_n(\mathbf{t}) \hat{b}_{n-1}(\eta, \mathbf{t}), \quad n \geq 0.
$$

Furthermore, by (2.14)-(2.16) we have $B_n(q^\nu \mathbf{t}) = B_{n+\nu}(\mathbf{t})$, $C_n(q^\nu \mathbf{t}) = C_{n+\nu}(\mathbf{t})$,

$$
\sum_{n=0}^{\infty} |B_n(\mathbf{t})| < \infty, \quad \text{and} \quad \sum_{n=0}^{\infty} |C_n(\mathbf{t})| < \infty.
$$

Then by Theorem 1.1, $\eta^n \hat{b}_n(1/\eta, \mathbf{t}) = (t_1 \eta)^n b_n(1/(t_1 \eta), \mathbf{t})$, or equivalently, $\eta^n b_n(1/\eta, \mathbf{t})$ converges locally uniformly in the complex plane to an entire function of η.

3. The Polynomial Solution of the Recurrence Equation for the Coefficients.

In this section we solve recurrence equation (2.18):

$$(3.1) \qquad b_{n+1}(\eta, \mathbf{t}) = (\eta + \beta_n q^n) b_n(\eta, \mathbf{t}) - \gamma_n q^{2n-1} b_{n-1}(\eta, \mathbf{t}), \quad n \geq 0,$$

with $b_{-1}(\eta, \mathbf{t}) = 0$, $b_0(\eta, \mathbf{t}) = 1$, and

$$(3.2)$$
$$\beta_n = \frac{(1 - t_2 q^n)(1 - a t_2 q^n)}{(1 - a t_1 t_2 q^{2n})(1 - a t_1 t_2 q^{2n+1})} + \frac{(1 - t_1 q^{n+1})(1 - a t_1 q^{n+1}) t_2}{(1 - a t_1 t_2 q^{2n+1})(1 - a t_1 t_2 q^{2n+2}) t_1},$$

$$(3.3)$$
$$\gamma_n = \frac{(1 - t_1 q^n)(1 - a t_1 q^n)(1 - t_2 q^n)(1 - a t_2 q^n) t_2}{(1 - a t_1 t_2 q^{2n-1})(1 - a t_1 t_2 q^{2n})^2 (1 - a t_1 t_2 q^{2n+1}) t_1}.$$

The coefficients β_n can be simplified. We have

$$\begin{aligned}
t_1 (a t_1 t_2 q^{2n}; q)_3 \beta_n &= t_1 (1 - t_2 q^n)(1 - a t_2 q^n)(1 - a t_1 t_2 q^{2n+2}) \\
&\quad + t_2 (1 - t_1 q^{n+1})(1 - a t_1 q^{n+1})(1 - a t_1 t_2 q^{2n}) \\
&= t_1 \left(1 - (1+a) t_2 q^n + a t_2^2 q^{2n} - a t_1 t_2 q^{2n+2} \right. \\
&\qquad \left. + a(1+a) t_1 t_2^2 q^{3n+2} - a^2 t_1 t_2^3 q^{4n+2} \right) \\
&\quad + t_2 \left(1 - (1+a) t_1 q^{n+1} + a t_1^2 q^{2n+2} - a t_1 t_2 q^{2n} \right. \\
&\qquad \left. + a(1+a) t_1^2 t_2 q^{3n+1} - a^2 t_1^3 t_2 q^{4n+2} \right) \\
&= (t_1 + t_2)(1 - a^2 t_1^2 t_2^2 q^{4n+2}) - (1+a) t_1 t_2 q^n (1+q) \\
&\quad + a(1+a) t_1^2 t_2^2 q^{3n+1} (1+q) \\
&= (1 - a t_1 t_2 q^{2n+1}) \left((t_1 + t_2)(1 + a t_1 t_2 q^{2n+1}) \right. \\
&\qquad \left. - (1+a)(1+q) t_1 t_2 q^n \right).
\end{aligned}$$

Therefore,

$$(3.4) \qquad \beta_n = \frac{(t_1 + t_2)(1 + a t_1 t_2 q^{2n+1}) - (1+a)(1+q) t_1 t_2 q^n}{(1 - a t_1 t_2 q^{2n})(1 - a t_1 t_2 q^{2n+2}) t_1}.$$

Recurrence relation (3.1) can be identified with the recurrence relation of the associated little q-Jacobi polynomials, [5]. The latter work gets the little q-Jacobi polynomials as limiting cases of the associated big q-Jacobi polynomials and does not give an explicit representation for the polynomials. Theorem 4.1 below provides an explicit representation for the little q-Jacobi polynomials.

We shall use (3.1)–(3.4) to find the coefficient $\beta_{n,n-1}$ of η^{n-1} in $b_n(\eta, \mathbf{t})$ and to guess the structure of the polynomials $b_n(\eta, \mathbf{t})$. From (3.1) we get

$\beta_{n+1,n} = \beta_n q^n + \beta_{n,n-1}$ which implies $\beta_{n,n-1} = \sum_{j=0}^{n-1} \beta_j q^j$. From (3.4) we have

$$a(1-q)t_1\beta_j q^j = a(t_1+t_2)q^j \left(\frac{1}{1-at_1t_2q^{2j}} - \frac{q}{1-at_1t_2q^{2j+2}} \right)$$

$$-(1+a)\left(\frac{1}{1-at_1t_2q^{2j}} - \frac{1}{1-at_1t_2q^{2j+2}} \right) =: R(q^j) - R(q^{j+1}),$$

where

$$R(z) = \frac{a(t_1+t_2)z - (1+a)}{1-at_1t_2z^2}.$$

Then $\beta_{n,n-1}$ becomes a telescoping sum and we find

$$a(1-q)t_1\beta_{n,n-1}$$
$$= (R(1)+1) - (R(q^n)+1)$$
$$= -\frac{a(1-t_1)(1-t_2)}{(1-at_1t_2)} + \frac{a(1-t_1q^n)(1-t_2q^n)}{(1-at_1t_2q^{2n})}$$
$$= \frac{a(1-t_1q^n)(1-t_2q^n)}{(1-at_1t_2q^{2n})} \left(1 - \frac{(1-t_1)(1-t_2)(1-at_1t_2q^{2n})}{(1-at_1t_2)(1-t_1q^n)(1-t_2q^n)} \right).$$

The coefficient $\beta_{n,n-1}$ can be written in a form that resembles similar formulas for the Wimp polynomials from [16] and their q-analogue from [8]:

$$(3.5) \qquad \beta_{n,n-1} = -\frac{(1-q^{-n}/t_1)(1-q^{-n}/t_2)}{(1-q)(1-q^{-2n}/(at_1t_2))}(at_1)^{-1}$$

$$\times \left(1 + \frac{(q^{-1}, t_1, t_2, at_1t_2q^{2n}; q)_1 q}{(at_1t_2, t_1q^n, t_2q^n, q; q)_1} \right).$$

The analogue of these polynomials that solves (3.1) is defined below.

Theorem 3.1. *The polynomials $b_n(\eta, \mathbf{t})$ defined by*

$$(3.6) \qquad b_n(\eta, \mathbf{t}) = \sum_{j=0}^{n} \frac{(q^{-n}/t_1, q^{-n}/t_2; q)_j}{(q, q^{-2n}/(at_1t_2); q)_j}(-at_1)^{-j}\eta^{n-j}$$

$$\times {}_4\phi_3\left(\begin{matrix} q^{-j}, \ t_1, \ t_2, \ at_1t_2q^{2n+1-j} \\ t_1q^{n+1-j}, \ t_2q^{n+1-j}, \ at_1t_2 \end{matrix} \ \middle| \ q, q \right), \quad n \geq 0$$

are the solutions of the recurrence relation (3.1)-(3.4).

Proof. The proof is similar to the proof of Theorem 4.1 in [9]. Let $\tilde{b}_n(\eta, \mathbf{t})$ denote the polynomial on the right-hand side of (3.6). We shall demonstrate that $\tilde{b}_n(\eta, \mathbf{t})$ satisfies (3.1).

The polynomials $\tilde{b}_n(\eta, \mathbf{t})$ can be written in the form

$$(3.7) \quad \tilde{b}_n(\eta, \mathbf{t}) = \sum_{j=0}^{n} \eta^{n-j} \sum_{k=0}^{j} \frac{(-1)^j}{(at_1)^j} \frac{(t_1, t_2; q)_k}{(q, at_1 t_2; q)_k}$$

$$\times \frac{(q^{-n}/t_1, q^{-n}/t_2; q)_j}{(t_1 q^{n+1-j}, t_2 q^{n+1-j}; q)_k} \frac{(at_1 t_2 q^{2n+1-j}; q)_k}{(q^{-2n}/(at_1 t_2); q)_j} \frac{(q^{-j}; q)_k q^k}{(q; q)_j}$$

$$= \sum_{j=0}^{n} \eta^{n-j} \sum_{k=0}^{j} \frac{(-1)^j}{(at_1)^j} \frac{(t_1, t_2; q)_k}{(q, at_1 t_2; q)_k} a^k \frac{(q^{-n}/t_1, q^{-n}/t_2; q)_{j-k}}{(q, q^{-2n}/(at_1 t_2); q)_{j-k}}$$

$$=: \sum_{j=0}^{n} \sum_{k=0}^{j} A_k B_{j-k}^{(n)} (-1/t_1)^j a^{k-j} \eta^{n-j},$$

where we applied formula (1.15) with $A = 1/(t_1 q^n)$, $1/(t_2 q^n)$, $1/(at_1 t_2 q^{2n})$, and q, and we defined

$$A_k := \frac{(t_1, t_2; q)_k}{(q, at_1 t_2; q)_k},$$

$$B_s^{(n)} := \frac{(q^{-n}/t_1, q^{-n}/t_2; q)_s}{(q, q^{-2n}/(at_1 t_2); q)_s}.$$

We separate the leading term η^{n+1} and write $\tilde{b}_{n+1}(\eta, \mathbf{t})$ in the form

$$(3.8) \quad \tilde{b}_{n+1}(\eta, \mathbf{t}) = \eta \left(\tilde{b}_n(\eta, \mathbf{t}) - \sum_{j=1}^{n} \sum_{k=0}^{j} A_k B_{j-k}^{(n)} a^{k-j} (-1/t_1)^j \eta^{n-j} \right)$$

$$+ \sum_{j=1}^{n+1} \sum_{k=0}^{j} A_k B_{j-k}^{(n+1)} a^{k-j} (-1/t_1)^j \eta^{n+1-j}$$

$$= (\eta + \beta_n q^n) \tilde{b}_n(\eta, \mathbf{t}) - \tilde{r}_n(\eta, \mathbf{t}),$$

where

$$(3.9) \qquad \tilde{r}_n(\eta, \mathbf{t}) := \sum_{j=1}^{n} \sum_{k=0}^{j} A_k B_{j-k}^{(n)} a^{k-j} (-1/t_1)^j \eta^{n+1-j}$$

$$- \sum_{j=1}^{n+1} \sum_{k=0}^{j} A_k B_{j-k}^{(n+1)} a^{k-j} (-1/t_1)^j \eta^{n+1-j}$$

$$+ \beta_n q^n \sum_{j=0}^{n} \sum_{k=0}^{j} A_k B_{j-k}^{(n)} a^{k-j} (-1/t_1)^j \eta^{n-j}$$

$$= \sum_{j=0}^{n-1} \sum_{k=0}^{j+1} A_k B_{j+1-k}^{(n)} a^{k-j-1} (-1/t_1)^{j+1} \eta^{n-j}$$

$$- \sum_{j=0}^{n} \sum_{k=0}^{j+1} A_k B_{j+1-k}^{(n+1)} a^{k-j-1} (-1/t_1)^{j+1} \eta^{n-j}$$

$$+ \beta_n q^n \sum_{j=0}^{n} \sum_{k=0}^{j} A_k B_{j-k}^{(n)} a^{k-j} (-1/t_1)^j \eta^{n-j}.$$

The coefficient of η^n in $\tilde{r}_n(\eta, \mathbf{t})$ equals

$$\sum_{k=0}^{1} A_k \left(B_{1-k}^{(n)} - B_{1-k}^{(n+1)} \right) a^{k-1} (-1/t_1) + \beta_n q^n$$

$$= - \left\{ B_1^{(n)} - B_1^{(n+1)} \right\} / (at_1) + \beta_n q^n$$

since $A_0 = 1$ and $B_0^{(n)} = 1$. Furthermore,

$$B_1^{(n)} - B_1^{(n+1)} = -\frac{a}{1-q} \left(\frac{(1 - t_1 q^n)(1 - t_2 q^n)}{1 - at_1 t_2 q^{2n}} - \frac{(1 - t_1 q^{n+1})(1 - t_2 q^{n+1})}{1 - at_1 t_2 q^{2n+2}} \right)$$

and then

$$- (at_1 t_2 q^{2n}; q^2)_2 ((1-q)/a) \left\{ B_1^{(n)} - B_1^{(n+1)} \right\}$$

$$= (1 - t_1 q^n)(1 - t_2 q^n)(1 - at_1 t_2 q^{2n+2})$$

$$- (1 - t_1 q^{n+1})(1 - t_2 q^{n+1})(1 - at_1 t_2 q^{2n})$$

$$= -q^n \left((t_1 + t_2)(1-q)(1 + at_1 t_2 q^{2n+1}) - (1+a)(1 - q^2) t_1 t_2 q^n \right).$$

Hence in view of (3.4) we get

$$(3.10) \qquad - \left\{ B_1^{(n)} - B_1^{(n+1)} \right\} / (at_1) = -\beta_n q^n,$$

which shows that the coefficient of η^n in $\tilde{r}_n(\eta, \mathbf{t})$ is zero. Then in (3.9) we can replace the lower bound of the range of j by 1 and then replace j by

$j + 1$. We obtain

$$(3.11) \qquad \tilde{r}_n(\eta, \mathbf{t}) = \beta_n q^n \sum_{j=0}^{n-1} \sum_{k=0}^{j+1} A_k B_{j+1-k}^{(n)} a^{k-j-1} (-1/t_1)^{j+1} \eta^{n-j-1}$$

$$+ \sum_{j=0}^{n-2} \sum_{k=0}^{j+2} A_k B_{j+2-k}^{(n)} a^{k-j-2} (-1/t_1)^{j+2} \eta^{n-j-1}$$

$$- \sum_{j=0}^{n-1} \sum_{k=0}^{j+2} A_k B_{j+2-k}^{(n+1)} a^{k-j-2} (-1/t_1)^{j+2} \eta^{n-j-1}.$$

Note that in (3.11) we can first separate the two constant terms, and then in the last two double sums we can replace the upper bound $j + 2$ of the range of k by $j + 1$ using that $B_{j+2-k}^{(n)} - B_{j+2-k}^{(n+1)} = 0$ if $k = j + 2$. Then we can write

$$(3.12) \qquad \tilde{r}_n(\eta, \mathbf{t}) = K_n + \sum_{j=0}^{n-2} \sum_{k=0}^{j+1} A_k a^{k-j-2} (-1/t_1)^{j+2} \triangle_{j+2-k}^{(n)} \eta^{n-j-1},$$

where

$$(3.13) \qquad \triangle_s^{(n)} = B_s^{(n)} - B_s^{(n+1)} - at_1 \beta_n q^n B_{s-1}^{(n)}, \quad s = 1, \ldots, n,$$

and K_n is the constant term of $\tilde{r}_n(\eta, \mathbf{t})$.

From (3.10) we get $\triangle_1^{(n)} = 0$.

For $s \in \{2, \ldots, n\}$ we have

$$(3.14)$$

$$\triangle_s^{(n)} = \frac{(q^{-n}/t_1, q^{-n}/t_2; q)_s}{(q, q^{-2n}/(at_1 t_2); q)_s} - \frac{(q^{-n-1}/t_1, q^{-n-1}/t_2; q)_s}{(q, q^{-2n-2}/(at_1 t_2); q)_s}$$

$$- \frac{q^{-n-1}}{t_1 t_2} \frac{\left((t_1 + t_2)(1 + q^{-2n-1}/(at_1 t_2)) - (1 + a)(1 + q)q^{-n-1}/a \right)}{(1 - q^{-2n}/(at_1 t_2))(1 - q^{-2n-2}/(at_1 t_2))}$$

$$\times \frac{(q^{-n}/t_1, q^{-n}/t_2; q)_{s-1}}{(q, q^{-2n}/(at_1 t_2); q)_{s-1}}$$

$$= \frac{(q^{-n}/t_1, q^{-n}/t_2; q)_{s-1}}{(q; q)_s (q^{-2n-2}/(at_1 t_2); q)_{s+2} (1 - q^{-2n}/(at_1 t_2))}$$

$$\times \Big\{ \big[(1 - q^{-n-1+s}/t_1)(1 - q^{-n-1+s}/t_2)(q^{-2n-2}/(at_1 t_2); q)_2$$

$$- (1 - q^{-n-1}/t_1)(1 - q^{-n-1}/t_2)(q^{-2n-2+s}/(at_1 t_2); q)_2 \big]$$

$$\times (1 - q^{-2n}/(at_1 t_2)) - (q^{-n-1}/(t_1 t_2))(1 - q^s)$$

$$\times \big((t_1 + t_2)(1 + q^{-2n-1}/(at_1 t_2)) - (1 + a)(1 + q)q^{-n-1}/a \big)$$

$$\times (1 - q^{-2n-1}/(at_1 t_2))(1 - q^{-2n-1+s}/(at_1 t_2)) \Big\}.$$

We set $x = q^{-n-1}/t_1$, $y = q^{-n-1}/t_2$, $\alpha = q^s$, $\beta = 1/a$, $S = x + y$, and $P = xy$. Let E_1 and E_2 denote the expressions inside $\{\ \}$ and $[\ ]$ in (3.14), respectively. Then

$$
\begin{aligned}
E_1 = {}& E_2(1 - q^2\beta P) \\
& - (1 - \alpha)\left(S(1 + q\beta P) - (1 + \beta)(1 + q)P\right)(1 - q\beta P)(1 - q\alpha\beta P)
\end{aligned}
$$

and

$$
\begin{aligned}
E_2 = {}& (1 - \alpha x)(1 - \alpha y)(1 - \beta P)(1 - q\beta P) \\
& - (1 - x)(1 - y)(1 - \alpha\beta P)(1 - q\alpha\beta P).
\end{aligned}
$$

We now simplify E_2. We have

(3.15)
$$
\begin{aligned}
E_2 = {}& 1 - \alpha S + \alpha^2 P - (1 + q)\beta P(1 - \alpha S + \alpha^2 P) + q\beta^2 P^2(1 - \alpha S + \alpha^2 P) \\
& - 1 + S - P + (1 + q)\alpha\beta P(1 - S + P) - q\alpha^2\beta^2 P^2(1 - S + P) \\
= {}& (1 - \alpha)\big[S - (1 + \alpha)P - (1 + q)\beta P \\
& + (1 + q)\alpha\beta P^2 + q(1 + \alpha)\beta^2 P^2 - q\alpha\beta^2 P^2 S\big] \\
= {}& (1 - \alpha)\left[S(1 - q\alpha\beta^2 P^2) - (1 + \alpha)(1 - q\beta^2 P)P - (1 + q)\beta(1 - \alpha P)P\right].
\end{aligned}
$$

From (3.15) and the definition of E_1 and E_2 we obtain

(3.16)

$$
\begin{aligned}
E_1/(1 - \alpha) = {}& S\left[(1 - q\alpha\beta^2 P^2)(1 - q^2\beta P) - (1 - q^2\beta^2 P^2)(1 - q\alpha\beta P)\right] \\
& - P\big[\left((1 + \alpha)(1 - q\beta^2 P) + (1 + q)\beta(1 - \alpha P)\right)(1 - q^2\beta P) \\
& - (1 + q)(1 + \beta)(1 - q\beta P)(1 - q\alpha\beta P)\big] \\
=: {}& B_1 S - B_2 P,
\end{aligned}
$$

where B_1 and B_2 denote the expressions inside the brackets. We factor B_1 and B_2:

(3.17)
$$
\begin{aligned}
B_1 = {}& 1 - q\alpha\beta^2 P^2 - q^2\beta P + q^3\alpha\beta^3 P^3 - 1 + q^2\beta^2 P^2 + q\alpha\beta P - q^3\alpha\beta^3 P^3 \\
= {}& q\beta P(\alpha + q\beta P - q - \alpha\beta P) = q\beta(\alpha - q)(1 - \beta P)P,
\end{aligned}
$$

(3.18)
$$B_2 = (1 + \alpha + (1+q)\beta - \beta (q\beta(1+\alpha) + (1+q)\alpha) P) (1 - q^2\beta P)$$
$$- (1+q)(1+\beta) (1 - q\beta(1+\alpha)P + q^2\alpha\beta^2 P^2)$$
$$= \alpha - q + \beta ((1+\alpha)(q(1+q)(1+\beta) - q\beta - q^2) - (1+q)(\alpha + q^2\beta)) P$$
$$+ q^2\beta^2 (q(1+\alpha)\beta + (1+q)\alpha - (1+q)\alpha(1+\beta)) P^2$$
$$= \alpha - q + \beta ((1+\alpha)(q + \beta q^2) - (1+q)(\alpha + \beta q^2)) P + q^2(q - \alpha)\beta^3 P^2$$
$$= -(q - \alpha) + \beta(q - \alpha)(1 - \beta q^2)P + q^2(q - \alpha)\beta^3 P^2$$
$$= -(q - \alpha)(1 - \beta P)(1 + q^2\beta^2 P).$$

From (3.16)-(3.18) we obtain

(3.19)
$$\frac{E_1}{(1 - \alpha)(q - \alpha)(1 - \beta P)} = (-q\beta S + (1 + q^2\beta^2 P)) P$$
$$= (-q\beta(x + y) + 1 + q^2\beta^2 xy) xy = (1 - q\beta x)(1 - q\beta y)xy.$$

We recall that $\alpha = q^s$, $\beta = 1/a$, $x = q^{-n-1}/t_1$, $y = q^{-n-1}/t_2$, and $s \in \{2, \dots, n\}$.

From (3.14)-(3.19) we get

(3.20)
$$\triangle_s^{(n)}$$
$$= \frac{(q^{-n}/t_1, q^{-n}/t_2; q)_{s-1} q}{(1 - \alpha)(q - \alpha)(q; q)_{s-2}(q^{-2n-2}/(at_1t_2); q)_{s+2}(1 - q^{-2n}/(at_1t_2))}$$
$$\times (1 - \alpha)(q - \alpha)(1 - \beta xy)(1 - q\beta x)(1 - q\beta y)xy$$
$$= \frac{(q^{-n+1}/t_1, q^{-n+1}/t_2; q)_{s-2}}{(q, q^{-2n+2}/(at_1t_2); q)_{s-2}}$$
$$\times \frac{(1 - q^{-n}/t_1)(1 - q^{-n}/t_2)(1 - q^{-n}/(at_1))(1 - q^{-n}/(at_2))q^{-2n-1}/(t_1t_2)}{(q^{-2n-1}/(at_1t_2); q)_3(1 - q^{-2n}/(at_1t_2))}$$
$$= a^2 t_1^2 \gamma_n q^{2n-1} B_{s-2}^{(n-1)}.$$

At the end we used formula (3.3) for γ_n.

From (3.12) and (3.20) we obtain

(3.21)
$$\tilde{r}_n(\eta, \mathbf{t}) = \gamma_n q^{2n-1} \sum_{j=0}^{n-2} \sum_{k=0}^{j} A_k B_{j-k}^{(n-1)} a^{k-j}(-1/t_1)^j \eta^{n-1-j} + K_n,$$

where by (3.8), K_n is the constant term of

(3.22)
$$\tilde{r}_n(\eta, \mathbf{t}) = (\eta + \beta_n q^n)\tilde{b}_n(\eta, \mathbf{t}) - \tilde{b}_{n+1}(\eta, \mathbf{t}).$$

We recall that $\tilde{b}_n(\eta, \mathbf{t})$ denotes the polynomial on the right-hand side of (3.6).

To complete the proof it remains to show that K_n equals $\gamma_n q^{2n-1}$ times the constant term of $\tilde{b}_{n-1}(\eta, \mathbf{t})$. Let f_n denotes the constant term of $\tilde{b}_n(\eta, \mathbf{t})$. From (3.22) we have $K_n = \beta_n q^n f_n - f_{n+1}$. Therefore, it is enough to verify that

$$(3.23) \qquad f_{n+1} = \beta_n q^n f_n - \gamma_n q^{2n-1} f_{n-1}.$$

From (3.6) and (1.15) we have

$$(3.24) \qquad f_n = \frac{(q^{-n}/t_1, q^{-n}/t_2; q)_n (-1)^n}{(q, q^{-2n}/(at_1t_2); q)_n a^n t_1^n} \phi_n = \frac{(t_1 q, t_2 q; q)_n q^{\binom{n}{2}}}{(q, at_1 t_2 q^{n+1}; q)_n t_1^n} \phi_n,$$

with

$$(3.25) \qquad \phi_n = {}_4\phi_3 \left(\begin{array}{c} q^{-n}, \, at_1t_2 q^{n+1}, \, t_1, \, t_2 \\ qt_1, \, qt_2, \, at_1t_2 \end{array} \middle| \, q, \, q \right).$$

We shall use a recurrence formula for the Askey-Wilson polynomials (see [**8**] or [**12**]). The Askey-Wilson polynomials $p_n(x; A, B, C, D \mid q)$ are defined by

$$(3.26)$$
$$p_n(x; A, B, C, D|q) := {}_4\phi_3 \left(\begin{array}{c} q^{-n}, \, ABCDq^{n-1}, \, Ae^{i\theta}, \, Ae^{-i\theta} \\ AB, \, AC, \, AD \end{array} \middle| \, q, \, q \right),$$

where $x = \cos\theta = (e^{i\theta} + e^{-i\theta})/2$. They satisfy the recurrence equation

$$(3.27) \qquad xp_n(x; A, B, C, D|q) = \frac{A_n}{2} p_{n+1}(x; A, B, C, D|q)$$

$$+ \frac{B_n}{2} p_n(x; A, B, C, D \mid q) + \frac{C_n}{2} p_{n-1}(x; A, B, C, D \mid q), \quad n \geq 0,$$

with $p_{-1}(x; A, B, C, D \mid q) = 0$, $p_0(x; A, B, C, D \mid q) = 1$, and coefficients

$$(3.28) \qquad A_n = \frac{(1 - ABq^n)(1 - ACq^n)(1 - ADq^n)(1 - ABCDq^{n-1})}{A(1 - ABCDq^{2n-1})(1 - ABCDq^{2n})},$$

$$(3.29) \qquad C_n = \frac{A(1 - q^n)(1 - BCq^{n-1})(1 - BDq^{n-1})(1 - CDq^{n-1})}{(1 - ABCDq^{2n-2})(1 - ABCDq^{2n-1})},$$

$$(3.30) \qquad B_n = A + 1/A - A_n - C_n.$$

As in Section 2 recurrence equation (3.27) can be written in monic form in terms of the polynomials

$$q_m = q_m(x; A, B, C, D \mid q) = p_m(x; A, B, C, D \mid q) 2^{-m} \prod_{j=0}^{m-1} A_j.$$

The monic equation is

$$(3.31) \qquad xq_n = q_{n+1} + (B_n/2)q_n + (C_n A_{n-1}/4)q_{n-1}, \quad n \geq 0,$$

$q_{-1} = 0$, $q_0 = 1$.

We select $A = \sqrt{t_1 t_2}$, $B = q\sqrt{t_1/t_2}$, $C = q\sqrt{t_2/t_1}$, $D = a\sqrt{t_1 t_2}$ and $e^{i\theta} = \sqrt{t_1/t_2}$. Then $x = \cos\theta = (t_1 + t_2)/(2\sqrt{t_1 t_2})$, and from (3.24)-(3.31) we obtain

$$(3.32) \qquad A_n = \frac{(1 - t_1 q^{n+1})(1 - t_2 q^{n+1})(1 - at_1 t_2 q^n)(1 - at_1 t_2 q^{n+1})}{\sqrt{t_1 t_2}(1 - at_1 t_2 q^{2n+1})(1 - at_1 t_2 q^{2n+2})},$$

$$(3.33) \qquad C_n = \frac{\sqrt{t_1 t_2}(1 - q^n)(1 - q^{n+1})(1 - at_1 q^n)(1 - at_2 q^n)}{(1 - at_1 t_2 q^{2n})(1 - at_1 t_2 q^{2n+1})},$$

$$(3.34) \qquad B_n = \sqrt{t_1 t_2} + 1/\sqrt{t_1 t_2} - A_n - C_n,$$

and

$$(3.35) \quad q_n = 2^{-n} \left(\prod_{j=0}^{n-1} A_j \right)$$
$$\times p_n \left((t_1 + t_2)/(2\sqrt{t_1 t_2}); \sqrt{t_1 t_2}, q\sqrt{t_1/t_2}, q\sqrt{t_2/t_1}, a\sqrt{t_1 t_2} \mid q \right)$$
$$= \frac{(t_1 q, t_2 q, at_1 t_2, at_1 t_2 q; q)_n}{2^n (t_1 t_2)^{n/2} (at_1 t_2 q; q)_{2n}} \phi_n = \frac{(t_1 q, t_2 q, at_1 t_2; q)_n}{2^n (t_1 t_2)^{n/2} (at_1 t_2 q^{n+1}; q)_n} \phi_n$$
$$= \frac{(q, at_1 t_2; q)_n (t_1/t_2)^{n/2}}{2^n q^{\binom{n}{2}}} f_n.$$

Furthermore, from (3.31) and (3.34) we have

$$(3.36) \qquad q_{n+1} = \left(\frac{t_1 + t_2}{2\sqrt{t_1 t_2}} - \frac{t_1 t_2 + 1}{2\sqrt{t_1 t_2}} + \frac{A_n}{2} + \frac{C_n}{2} \right) q_n - \frac{C_n A_{n-1}}{4} q_{n-1}$$
$$=: (R_n/(2\sqrt{t_1 t_2})) q_n - (C_n A_{n-1}/4) q_{n-1}.$$

From (3.35), (3.36), (3.32), (3.33), and (3.3) we get

$$(3.37) \quad f_{n+1} = \frac{R_n q^n}{(1 - q^{n+1})(1 - at_1 t_2 q^n) t_1} f_n - \frac{4q^{2n-1}}{(q^n, at_1 t_2 q^{n-1}; q)_2 t_1/t_2}$$
$$\times \frac{1}{4} \frac{(1 - q^n)(1 - q^{n+1})(1 - at_1 q^n)(1 - at_2 q^n)}{(1 - at_1 t_2 q^{2n})(1 - at_1 t_2 q^{2n+1})}$$
$$\times \frac{(1 - t_1 q^n)(1 - t_2 q^n)(1 - at_1 t_2 q^{n-1})(1 - at_1 t_2 q^n)}{(1 - at_1 t_2 q^{2n-1})(1 - at_1 t_2 q^{2n})} f_{n-1}$$
$$= \frac{R_n q^n}{(1 - q^{n+1})(1 - at_1 t_2 q^n) t_1} f_n - \gamma_n q^{2n-1} f_{n-1}.$$

To complete the proof of (3.23) and the theorem we have to show that the coefficient of f_n in (3.37) equals $\beta_n q^n$. From (3.36), (3.32), and (3.33) we

get

$$(at_1t_2q^{2n};q)_3 R_n = -(1-t_1)(1-t_2)(at_1t_2q^{2n};q)_3$$
$$+(1-t_1q^{n+1})(1-t_2q^{n+1})(1-at_1t_2q^n)(1-at_1t_2q^{n+1})(1-at_1t_2q^{2n})$$
$$+(1-q^n)(1-q^{n+1})(1-at_1q^n)(1-at_2q^n)(1-at_1t_2q^{2n+2})t_1t_2.$$

Setting $S = t_1 + t_2$, $P = t_1t_2$, and $\alpha = q^n$ we obtain

(3.38)
$$
\begin{aligned}
(at_1t_2q^{2n};q)_3 R_n &= -(1-S+P)(1-a\alpha^2 P)(1-aq\alpha^2 P)(1-aq^2\alpha^2 P) \\
&\quad + (1-q\alpha S+q^2\alpha^2 P)(1-a\alpha P)(1-aq\alpha P)(1-a\alpha^2 P) \\
&\quad + (1-\alpha)(1-q\alpha)(1-a\alpha S+a^2\alpha^2 P)(1-aq^2\alpha^2 P)P \\
&=: E_1 S + E_2 P + E_3 P.
\end{aligned}
$$

The expressions $\{E_j\}_{j=1}^3$ are defined and factored below:

(3.39)
$$
\begin{aligned}
E_1 &:= (1-a\alpha^2 P)(1-aq\alpha^2 P)(1-aq^2\alpha^2 P) \\
&\quad - q\alpha(1-a\alpha P)(1-aq\alpha P)(1-a\alpha^2 P) \\
&\quad - a\alpha(1-\alpha)(1-q\alpha)(1-aq^2\alpha^2 P)P \\
&= (1-a\alpha^2 P)\left[1-aq\alpha^2 P-aq^2\alpha^2 P+a^2q^3\alpha^4 P^2\right. \\
&\quad \left. -q\alpha+aq\alpha^2 P+aq^2\alpha^2 P-a^2q^2\alpha^3 P^2\right] \\
&\quad - a\alpha(1-\alpha)(1-q\alpha)(1-aq^2\alpha^2 P)P \\
&= (1-q\alpha)\left((1-a\alpha^2 P)(1-a^2q^2\alpha^3 P^2)\right. \\
&\quad \left. -a\alpha(1-\alpha)(1-aq^2\alpha^2 P)P\right) \\
&= (1-q\alpha)(1+a^3q^2\alpha^5 P^3-a\alpha P-a^2q^2\alpha^4 P^2) \\
&= (1-q\alpha)(1-a\alpha P)(1-a^2q^2\alpha^4 P^2),
\end{aligned}
$$

(3.40)
$$
\begin{aligned}
E_2 &:= \left(-(1-a\alpha^2 P)(1-aq\alpha^2 P)(1-aq^2\alpha^2 P)\right. \\
&\quad \left. + (1-a\alpha P)(1-aq\alpha P)(1-a\alpha^2 P)\right)/P \\
&= (1-a\alpha^2 P)a\alpha(\alpha q+\alpha q^2-aq^3\alpha^3 P-1-q+aq\alpha P) \\
&= a\alpha(1-q\alpha)(1-a\alpha^2 P)(-(1+q)+(1+q\alpha)aq\alpha P) \\
&= -a\alpha(1-q\alpha)\left[1+q-a\alpha((1+q)\alpha+q(1+q\alpha))P+a^2q\alpha^3(1+q\alpha)P^2\right],
\end{aligned}
$$

and

(3.41)
$$E_3 := -(1 - a\alpha^2 P)(1 - aq\alpha^2 P)(1 - aq^2\alpha^2 P)$$
$$+ q^2\alpha^2(1 - a\alpha P)(1 - aq\alpha P)(1 - a\alpha^2 P)$$
$$+ (1 - \alpha)(1 - q\alpha)(1 + a^2\alpha^2 P)(1 - aq^2\alpha^2 P)$$
$$= (1 - a\alpha^2 P)\left(-1 + aq(1 + q)\alpha^2 P + q^2\alpha^2 - aq^2(1 + q)\alpha^3 P\right)$$
$$+ (1 - \alpha)(1 - q\alpha)(1 + a(a - q^2)\alpha^2 P - a^3 q^2\alpha^4 P^2)$$
$$= (1 - q\alpha)(1 - a\alpha^2 P)(-1 - q\alpha + aq(1 + q)\alpha^2 P)$$
$$+ (1 - q\alpha)(1 - \alpha)\left(1 + a(a - q^2)\alpha^2 P - a^3 q^2\alpha^4 P^2\right)$$
$$= (1 - q\alpha)\left[-(1 + q)\alpha + a\alpha^2\left(q(1 + q) + 1 + q\alpha + (1 - \alpha)(a - q^2)\right)P\right.$$
$$\left. - a^2 q\alpha^4(1 + q + aq(1 - \alpha))P^2\right].$$

Combining (3.40) and (3.41) we obtain

(3.42)
$$(E_2 + E_3)/(1 - q\alpha) = -(1 + q)(1 + a)\alpha$$
$$+ a\alpha^2\left(a\alpha + aq\alpha + aq + aq^2\alpha + q\right.$$
$$\left. + 1 + q\alpha + a - a\alpha + q^2\alpha\right)P$$
$$- a^2 q\alpha^4(a(1 + q\alpha) + 1 + q + aq(1 - \alpha))P^2$$
$$= -(1 + q)(1 + a)\alpha + a\alpha^2(1 + q)(aq\alpha + a + 1 + q\alpha)P$$
$$- a^2 q\alpha^4(1 + q)(1 + a)P^2$$
$$= -(1 + q)(1 + a)\alpha\left(1 - a\alpha(1 + q\alpha)P + a^2 q\alpha^3 P^2\right)$$
$$= -(1 + q)(1 + a)\alpha(1 - a\alpha P)(1 - aq\alpha^2 P).$$

Then from (3.38), (3.39), and (3.42) we get

(3.43)
$$(at_1 t_2 q^{2n}; q)_3 R_n = (1 - q\alpha)(1 - a\alpha P)(1 - aq\alpha^2 P)$$
$$\times \left((1 + aq\alpha^2 P)S - (1 + q)(1 + a)\alpha P\right).$$

In terms of $\alpha = q^n$, $S = t_1 + t_2$, and $P = t_1 t_2$, $(at_1 t_2 q^{2n}; q)_3 = (a\alpha^2 P; q)_3$, hence (3.43) implies

(3.44)
$$\frac{R_n}{(1 - q^{n+1})(1 - at_1 t_2 q^n)t_1} = \frac{R_n}{(1 - q\alpha)(1 - a\alpha P)t_1}$$
$$= \frac{(1 + aq\alpha^2 P)S - (1 + q)(1 + a)\alpha P}{(1 - a\alpha^2 P)(1 - aq^2\alpha^2 P)t_1} = \beta_n$$

in view of (3.4). The proof of Theorem 3.1 is complete. □

4. The Spectrum of the Inverse Operator.

To find the spectrum of the operator $\mathcal{T}_{\mathbf{t}}$, that is, to solve eigenvalue problem (2.4) we consider the eigenfunctions of the q-difference operator $D_{q,x}$. It is easy to see that for every λ the equation $D_{q,x}f(x) = \lambda f(x)$ has solution $f_\lambda(x) = 1/(\lambda(1-q)x;q)_\infty = e_q(\lambda(1-q)x)$. The eigenfunctions of $\mathcal{T}_{\mathbf{t}}$ are also eigenfunctions for $D_{q,x}$, in fact if g_λ is such that $\mathcal{T}_{\mathbf{t}}g_\lambda = \lambda g_\lambda$, then $g_\lambda = D_{q,x}\mathcal{T}_{\mathbf{t}}g_\lambda = \lambda D_{q,x}g_\lambda$ and therefore, $g_\lambda(x) = f_{1/\lambda}(x) = e_q((1-q)x/\lambda)$. Hence, the eigenvalues of $\mathcal{T}_{\mathbf{t}}$ are the reciprocals of the numbers λ such that

$$(4.1) \qquad f_\lambda(x) = \sum_{n=0}^{\infty} c_n(\lambda, \mathbf{t})p_n(x, \mathbf{t}) \in L_2(\mu(\cdot, \mathbf{t})) \cap L_2(\mu(\cdot, q\mathbf{t})).$$

From equations $\mathcal{T}_{\mathbf{t}}g_\lambda = \lambda g_\lambda$, $g_\lambda = f_{1/\lambda}$, (2.2), and (4.1) we get $c_0(1/\lambda, \mathbf{t}) = 0$. It turns out that the condition $c_0(1/\lambda, \mathbf{t}) = 0$ completely characterizes the spectrum of the operator $\mathcal{T}_{\mathbf{t}}$.

We proceed with computing the coefficients $\{c_n(\lambda, \mathbf{t})\}$ in (4.1). Let λ be small enough so that $|\lambda(1-q)| < 1$. Using the orthogonality relation (1.6) and applying Euler's identity (1.12) to $f_\lambda(x) = e_q(\lambda(1-q)x)$ we obtain

$$(4.2) \quad c_n(\lambda, \mathbf{t})\xi_n(\mathbf{t}) = \int_{\mathbf{R}} f_\lambda(x)p_n(x, \mathbf{t}) \, d\mu(x, \mathbf{t}) = \sum_{s=0}^{\infty} \frac{(\lambda(1-q))^s}{(q;q)_s} d_{n,s}(\lambda, \mathbf{t}),$$

where we have defined

$$(4.3) \qquad d_{n,s}(\lambda, \mathbf{t}) := \int_{\mathbf{R}} x^s p_n(x, \mathbf{t}) \, d\mu(x, \mathbf{t}).$$

By (1.6), $d_{n,s} = 0$ if $s < n$, hence we may assume that $s \geq n$.

From (1.11) and (1.3) we have

$$x^s = \frac{1}{t_2^s} \sum_{j=0}^{s} a_{s,j}(t_2x;q)_j, \quad a_{s,j} := (-1)^j q^{\binom{j}{2}-j(s-1)} \begin{bmatrix} s \\ j \end{bmatrix}_q,$$

$$p_n(x, \mathbf{t}) = \sum_{k=0}^{n} b_{n,k}(t_1x;q)_k, \qquad b_{n,k} := \frac{(q^{-n}, at_1t_2q^{n-1};q)_k}{(t_1, at_1, q;q)_k}q^k.$$

Then for $d_{n,s}(\lambda, \mathbf{t})$ we get

$$(4.4) \qquad d_{n,s}(\lambda, \mathbf{t}) = \sum_{j=0}^{s}\sum_{k=0}^{n} a_{s,j}b_{n,k}\frac{1}{t_2^s}\int_{\mathbf{R}} \frac{d\mu^{(a)}(x)}{(q^k t_1 x, q^j t_2 x;q)_\infty}$$

$$= \sum_{j=0}^{s}\sum_{k=0}^{n} a_{s,j}b_{n,k}\frac{1}{t_2^s}\frac{(at_1t_2q^{k+j};q)_\infty}{(q^k t_1, q^j t_2, aq^k t_1, aq^j t_2;q)_\infty},$$

where (6.4) was used to evaluate the integrals. From the formulas for $a_{s,j}$ and $b_{n,k}$, and (4.4) we obtain

(4.5)

$$
d_{n,s} = \frac{(at_1t_2;q)_\infty(q;q)_s}{(t_1,at_1,t_2,at_2;q)_\infty t_2^s}
$$

$$
\times \sum_{j=0}^{s}\sum_{k=0}^{n} \frac{(-1)^j q^{\binom{j}{2}-j(s-1)+k}(q^{-n},at_1t_2q^{n-1};q)_k(t_2,at_2;q)_j}{(q;q)_j(q;q)_{s-j}(q;q)_k(at_1t_2;q)_{k+j}}
$$

$$
= c^{(1)} \sum_{j=0}^{s} \frac{(-1)^j q^{\binom{j}{2}-j(s-1)}(t_2,at_2;q)_j}{(q;q)_j(q;q)_{s-j}(at_1t_2;q)_j}\,{}_2\phi_1\left(\begin{array}{c} q^{-n},\,at_1t_2q^{n-1} \\ at_1t_2q^j \end{array}\middle|\,q,q\right),
$$

where $c^{(1)}$ denotes the coefficient of the double sum. By (1.9) the ${}_2\phi_1$ sum is equal to

$$
\frac{(q^{j+1-n};q)_n(at_1t_2q^{n-1})^n}{(at_1t_2q^j;q)_n},
$$

which is 0 for $j < n$. Then for $s \geq n$ we obtain

(4.6) $$
d_{n,s} = c^{(2)} \sum_{j=n}^{s} \frac{(-1)^j q^{\binom{j}{2}-j(s-1)}(t_2,at_2;q)_j}{(q;q)_{j-n}(q;q)_{s-j}(at_1t_2;q)_{n+j}},
$$

where $c^{(2)} = (at_1t_2q^{n-1})^n c^{(1)}$ and we used the identity $(q^{j+1-n};q)_n = (q;q)_j/(q;q)_{j-n}$. Replacing j by $n+l$ we get

(4.7) $$
d_{n,s} = c^{(3)} \sum_{l=0}^{s-n} \frac{(-1)^l q^{\binom{n+l}{2}-(n+l)(s-1)}(t_2q^n,at_2q^n;q)_l}{(q;q)_l(q;q)_{(s-n)-l}(at_1t_2q^{2n};q)_l}, \quad s \geq n,
$$

with $c^{(3)} = (-1)^n(t_2,at_2;q)_n/(at_1t_2;q)_{2n}c^{(2)}$. Next with $p = s - n$ we have

$$
\frac{(q;q)_p}{(q;q)_{p-l}} = \prod_{j=p-l+1}^{p}(1-q^j) = (-1)^l q^{\binom{p+1}{2}-\binom{p-l+1}{2}}(q^{-p};q)_l
$$

and

$$
\binom{n+l}{2} - (n+l)(s-1) + \binom{p+1}{2} - \binom{p-l+1}{2} = \binom{n}{2} - n(s-1) + l.
$$

Substituting these identities in (4.7) we obtain

(4.8) $$
d_{n,s} = c^{(4)}\,{}_3\phi_2\left(\begin{array}{c} q^{-(s-n)},\,t_2q^n,\,at_2q^n \\ at_1t_2q^{2n},\,0 \end{array}\middle|\,q,q\right), \quad s \geq n,
$$

where

(4.9)

$$
c^{(4)} = c^{(4)}_{n,s} = \frac{(at_1t_2q^{2n};q)_\infty(t_2,at_2;q)_n(at_1t_2)^n(q;q)_s q^{\binom{n}{2}+n(n-s)}(-1)^n}{(t_1,at_1,t_2,at_2;q)_\infty(q;q)_{s-n}t_2^s}.
$$

From formula (1.7) for $\xi_n(\mathbf{t})$ and (4.9) we get

$$(4.10) \qquad c_{n,s}^{(4)} = \frac{(t_1, at_1; q)_n (q; q)_s q^{n(n-s)} t_2^{n-s}}{(at_1 t_2 q^{n-1}, q; q)_n (q; q)_{s-n} t_1^n} \xi_n(\mathbf{t}).$$

Then from (4.2), (4.8), (4.9), and (4.10) we get

$$(4.11) \qquad c_n(\lambda, \mathbf{t}) = \sum_{s=n}^{\infty} \frac{((1-q)\lambda)^s}{(q; q)_s} (d_{n,s}/\xi_n(\mathbf{t})) = \frac{(t_1, at_1; q)_n (1-q)^n}{(at_1 t_2 q^{n-1}, q; q)_n t_1^n} \lambda^n$$

$$\times \sum_{k=0}^{\infty} \frac{((1-q)/(t_2 q^n))^k}{(q; q)_k} \lambda^k \, _3\phi_2 \left(\begin{array}{c} q^{-k}, t_2 q^n, at_2 q^n \\ at_1 t_2 q^{2n}, 0 \end{array} \middle| q, q \right),$$

with $k = s - n$. Applying the transformation, [6],

$$(4.12) \qquad _3\phi_2 \left(\begin{array}{c} q^{-k}, a, b \\ c, 0 \end{array} \middle| q, q \right) = \frac{(b; q)_k a^k}{(c; q)_k} \, _2\phi_1 \left(\begin{array}{c} q^{-k}, c/b \\ q^{1-k}/b \end{array} \middle| q, q/a \right)$$

to the $_3\phi_2$ expression in (4.11) we get

$$_3\phi_2 \left(\begin{array}{c} q^{-k}, t_2 q^n, at_2 q^n \\ at_1 t_2 q^{2n}, 0 \end{array} \middle| q, q \right)$$

$$= \frac{(at_2 q^n; q)_k (t_2 q^n)^k}{(at_1 t_2 q^{2n}; q)_k} \, _2\phi_1 \left(\begin{array}{c} q^{-k}, t_1 q^n \\ q^{1-k-n}/(at_2) \end{array} \middle| q, q^{1-n}/t_2 \right).$$

Then the second sum in (4.11) can be written in the form

$$(4.13) \qquad \sum_{k=0}^{\infty} \frac{(at_2 q^n; q)_k (1-q)^k}{(at_1 t_2 q^{2n}, q; q)_k} \lambda^k \sum_{j=0}^{k} \frac{(q^{-k}; q)_j (t_1 q^n; q)_j (q^{1-n}/t_2)^j}{(q^{1-k-n}/(at_2); q)_j (q; q)_j}.$$

Using (1.15) we obtain

$$\frac{(q^{-k}; q)_j}{(q^{1-k-n}/(at_2); q)_j} = \frac{(q; q)_k (at_2 q^n; q)_{k-j}}{(q; q)_{k-j} (at_2 q^n; q)_k} (at_2 q^{n-1})^j.$$

Hence the double sum in (4.13) equals

$$(4.14) \qquad \sum_{k=0}^{\infty} \frac{((1-q)\lambda)^k}{(at_1 t_2 q^{2n}, q; q)_k} \sum_{j=0}^{k} \begin{bmatrix} k \\ j \end{bmatrix}_q (t_1 q^n; q)_j (at_2 q^n; q)_{k-j} a^j.$$

The above formulas hold when $|(1-q)\lambda| < 1$ since in this range (1.12) can be applied. To extend the formulas to arbitrary λ we need a meromorphic continuation of the function in (4.14). We set

$$(4.15) \qquad \alpha_k(a, t_1, t_2) := \frac{1}{(at_1 t_2, q; q)_k} \sum_{j=0}^{k} \begin{bmatrix} k \\ j \end{bmatrix}_q (t_1; q)_j (at_2; q)_{k-j} a^j,$$

$$(4.16) \qquad A(z; a, t_1, t_2) := \sum_{k=0}^{\infty} \alpha_k(a, t_1, t_2) z^k, \quad |z| < 1.$$

Note that the sum in (4.14) equals $A((1-q)\lambda; a, q^n\mathbf{t})$. For $|z| < 1$ we consider the product of the functions $(z, az; q)_\infty$ and $A(z; a, t_1, t_2)$. Using (1.13) we get

$$(4.17) \quad (z; q)_\infty A(z; a, t_1, t_2) = \sum_{n=0}^{\infty} z^n \left(\sum_{k=0}^{n} \alpha_k(a, t_1, t_2) \frac{(-1)^{n-k} q^{\binom{n-k}{2}}}{(q; q)_{n-k}} \right).$$

The coefficient of z^n in (4.17) equals

(4.18)
$$\sum_{k=0}^{n} \left(\sum_{j=0}^{k} \begin{bmatrix} k \\ j \end{bmatrix}_q \frac{(t_1; q)_j (at_2; q)_{k-j} a^j}{(at_1 t_2, q; q)_k} \right) \frac{(-1)^{n-k} q^{\binom{n-k}{2}}}{(q; q)_{n-k}}$$

$$= \sum_{j=0}^{n} \frac{(t_1; q)_j a^j}{(at_1 t_2, q; q)_j (q; q)_{n-j}} \left(\sum_{\nu=0}^{n-j} \begin{bmatrix} n-j \\ \nu \end{bmatrix}_q \frac{(at_2; q)_\nu (-1)^{n-j-\nu} q^{\binom{n-j-\nu}{2}}}{(at_1 t_2 q^j; q)_\nu} \right)$$

with $\nu = k - j$. The sum over ν in (4.18) has the form

$$\sum_{\nu=0}^{m} \begin{bmatrix} m \\ \nu \end{bmatrix}_q \frac{(\alpha; q)_\nu}{(\beta; q)_\nu} (-1)^{m-\nu} q^{\binom{m-\nu}{2}}$$

$$= (-1)^m q^{\binom{m}{2}} \sum_{\nu=0}^{m} \frac{(q^{-m}, \alpha; q)_\nu}{(\beta, q; q)_\nu} q^\nu$$

$$= (-1)^m q^{\binom{m}{2}} {}_2\phi_1 \left(\begin{array}{c} q^{-m}, \alpha \\ \beta \end{array} \middle| q, q \right)$$

$$= (-1)^m q^{\binom{m}{2}} \frac{(\beta/\alpha; q)_m}{(\beta; q)_m} \alpha^m,$$

where we first used the identity

$$(4.19) \qquad (q; q)_m / (q; q)_{m-\nu} = (-1)^\nu q^{\binom{m+1}{2} - \binom{m-\nu+1}{2}} (q^{-m}; q)_\nu,$$

and then (1.9). Then the coefficient of z^n in (4.17) given with (4.18) equals

$$(4.20) \qquad \sum_{j=0}^{n} \frac{(t_1; q)_j a^j}{(at_1 t_2, q; q)_j (q; q)_{n-j}} (-1)^{n-j} q^{\binom{n-j}{2}} \frac{(t_1 q^j; q)_{n-j} (at_2)^{n-j}}{(at_1 t_2 q^j; q)_{n-j}}$$

$$= \frac{(t_1; q)_n a^n}{(at_1 t_2, q; q)_n} \sum_{j=0}^{n} \begin{bmatrix} n \\ n-j \end{bmatrix}_q q^{\binom{n-j}{2}} (-t_2)^{n-j} = \frac{(t_1, t_2; q)_n a^n}{(at_1 t_2, q; q)_n},$$

where we applied (1.14). From (4.17) and (4.20) we get

$$(4.21) \qquad (z; q)_\infty A(z; a, t_1, t_2) = {}_2\phi_1 \left(\begin{array}{c} t_1, t_2 \\ at_1 t_2 \end{array} \middle| q, az \right), \quad |az| < 1.$$

From (1.13) and (4.21) for z such that $\max\{|z|, |az|\} < 1$ we have

(4.22) $(z, az; q)_\infty A(z; a, t_1, t_2)$

$$
= \sum_{m=0}^{\infty} \left(\sum_{k=0}^{m} \frac{(t_1, t_2; q)_k}{(at_1 t_2, q; q)_k} \frac{(-1)^{m-k} q^{\binom{m-k}{2}}}{(q; q)_{m-k}} \right) a^m z^m
$$

$$
= \sum_{m=0}^{\infty} \frac{(-1)^m q^{\binom{m}{2}}}{(q; q)_m} {}_3\phi_2 \left(\begin{array}{c} q^{-m}, t_1, t_2 \\ at_1 t_2, 0 \end{array} \middle| q, q \right) a^m z^m
$$

$$
= \sum_{m=0}^{\infty} \frac{(-t_1)^m q^{\binom{m}{2}} (t_2; q)_m}{(at_1 t_2, q; q)_m} {}_2\phi_1 \left(\begin{array}{c} q^{-m}, at_1 \\ q^{1-m}/t_2 \end{array} \middle| q, q/t_1 \right) a^m z^m,
$$

where we applied (4.19) and (4.12). Next, from (1.15) we have

$$
\frac{(q^{-m}; q)_j}{(q^{1-m}/t_2; q)_j} = \frac{(q; q)_m (t_2; q)_{m-j}}{(q; q)_{m-j} (t_2; q)_m} (t_2/q)^j,
$$

which combined with (4.22) yields

(4.23)

$$
(z, az; q)_\infty A(z; a, t_1, t_2) = \sum_{m=0}^{\infty} \frac{(-t_1)^m q^{\binom{m}{2}}}{(at_1 t_2; q)_m}
$$

$$
\times \left(\sum_{j=0}^{m} \frac{(at_1; q)_j (t_2; q)_{m-j}}{(q; q)_j (q; q)_{m-j}} (t_2/t_1)^j \right) a^m z^m.
$$

Clearly $(z, az; q)_\infty$ is an entire function of z. Furthermore, the right-hand side of (4.23) is an entire function of z, and in an open neighborhood of $z = 0$ it coincides with the function $(z, az; q)_\infty A(z; a, t_1, t_2)$. Hence a meromorphic extension of $A(z; a, t_1, t_2)$ to the complex plane can be found by dividing the right-hand side of (4.23) by $(z, az; q)_\infty$.

The main results of this section can be described with the following two theorems.

Theorem 4.1. *The coefficients in the expansion formula for the eigenfunction $f_\lambda(x) = e_q(\lambda(1 - q)x)$ in terms of big q-Jacobi polynomials $\{p_n(x, \mathbf{t})\}$ are given by*

(4.24)

$$
c_n(\lambda, \mathbf{t}) = \frac{1}{((1-q)\lambda, a(1-q)\lambda; q)_\infty} \frac{(t_1, at_1; q)_n (1-q)^n}{(at_1 t_2 q^{n-1}, q; q)_n t_1^n} \lambda^n
$$

$$
\times \sum_{m=0}^{\infty} \frac{(-t_1 q^n)^m q^{\binom{m}{2}}}{(at_1 t_2 q^{2n}; q)_m} \left(\sum_{j=0}^{m} \frac{(at_1 q^n; q)_j (t_2 q^n; q)_{m-j}}{(q; q)_j (q; q)_{m-j}} \left(\frac{t_2}{t_1} \right)^j \right) a^m (1-q)^m \lambda^m.
$$

Furthermore, the coefficients $\{c_n(\lambda, \mathbf{t})\}$ *satisfy the recurrence equation*

(4.25)
$$\sigma_n(\mathbf{t})c_n(\lambda, \mathbf{t}) = \lambda \sum_{m=n-1}^{n+1} (\xi_m(\mathbf{t})/\xi_{n-1}(q\mathbf{t}))\, c_{n-1,m}(\mathbf{t})c_m(\lambda, \mathbf{t}), \quad n \geq 1,$$

where $\{c_{n,m}(\mathbf{t})\}_{m=n}^{n+2}$ *are the coefficients defined with* (2.7)-(2.10).

The spectrum of the inverse operator $\mathcal{T}_{\mathbf{t}}$ is described in Theorem 4.2.

Theorem 4.2. *The function* $f_\lambda(x)$ *belongs to the space* $L_2(\mu(\cdot, \mathbf{t}))$ *for every* λ *that is not a zero of* $((1-q)\lambda, a(1-q)\lambda; q)_\infty$.

The spectrum of $\mathcal{T}_{\mathbf{t}}$, *the inverse operator of the q-difference operator* $D_{q,x}$, *acting on the space* $L_2(\mu(\cdot, \mathbf{t})) \cap L_2(\mu(\cdot, q\mathbf{t}))$ *is the set of the reciprocals of the zeros of the meromorphic function* $c_0(\lambda, \mathbf{t})$.

Proof. From (1.6), (1.7), (4.1), and (4.24) it immediately follows that

$$||f_\lambda||^2_{L_2(\mu(\cdot, \mathbf{t}))} = \sum_{n=0}^\infty c_n(\lambda, \mathbf{t})^2 \xi_n(\mathbf{t}) < \infty$$

for all λ such that $1/((1-q)\lambda) \notin \mathrm{supp}\left(\mu^{(a)}\right)$ and all parameters $\mathbf{t}$ for which the function $c_0(\lambda, \mathbf{t})$ is well-defined. This is due to the presence of the factor $q^{\binom{n}{2}}$ in $\xi_n(\mathbf{t})$. Furthermore, $\mathcal{T}_{\mathbf{t}}g_\lambda = \lambda g_\lambda$ implies $g_\lambda = f_{1/\lambda}$ and $c_0(1/\lambda, \mathbf{t}) = 0$. Hence $c_0(1/\lambda, \mathbf{t}) = 0$ is a necessary and sufficient condition for λ to be in the spectrum of the operator $\mathcal{T}_{\mathbf{t}}$.   □

5. Asymptotic Properties of the Polynomials $\{b_n(\eta, \mathbf{t})\}$.

In Section 2 we applied Schwartz's theorem to prove that the sequence $\{\eta^n b_n(1/\eta, \mathbf{t})\}$ converges locally uniformly in the complex plane to an entire function. The recurrence relation (2.17) has bounded coefficients, hence the polynomials $\{b_n(\eta, \mathbf{t})\}$ are orthogonal with respect to a unique measure $\varphi(\cdot, \mathbf{t})$ with compact support, [1], [14]. From Markov's theorem, [15], the Stieltjes transform of $\varphi(\cdot, \mathbf{t})$ is given by

(5.1)
$$\int_{\mathbf{R}} \frac{d\varphi(u, \mathbf{t})}{z - u} = \lim_{n \to \infty} \frac{b_n^*(z, \mathbf{t})}{b_n(z, \mathbf{t})}, \quad z \notin \mathbf{R},$$

where $\{b_n^*(\eta, \mathbf{t})\}$ is the solution of (2.17) or equivalently, (3.1) satisfying the initial conditions

$$b_0^*(\eta, \mathbf{t}) = 0, \quad b_1^*(\eta, \mathbf{t}) = 1.$$

We observe that $\beta_n(\mathbf{t})$ and $\gamma_n(\mathbf{t})$ defined with (3.4) and (3.3) have the property

(5.2)
$$\beta_{n-1}(q\mathbf{t}) = \beta_n(\mathbf{t}) \quad \text{and} \quad \gamma_{n-1}(q\mathbf{t}) = \gamma_n(\mathbf{t}).$$

From (3.1) with n, $\mathbf{t}$ and η replaced by $n-1$, $q\mathbf{t}$ and η/q, respectively, we get

$$(5.3) \qquad b_n(\eta/q, q\mathbf{t}) = \left(\eta/q + \beta_{n-1}(q\mathbf{t})q^{n-1}\right) b_{n-1}(\eta/q, q\mathbf{t})$$
$$- \gamma_{n-1}(q\mathbf{t})q^{2n-3}b_{n-2}(\eta/q, q\mathbf{t}).$$

Multiplying (5.3) by q^n and using (5.2) we see that

$$(5.4) \qquad b_n^*(\eta, \mathbf{t}) = q^{n-1}b_{n-1}(\eta/q, q\mathbf{t}), \quad n \geq 0.$$

We now study the limiting behavior of $\eta^n b_n(1/\eta, \mathbf{t})$ as $n \to \infty$. For each fixed j the $_4\phi_3$ expression in (3.6) is bounded by $M_1(\mathbf{t})_1\phi_0(q^{-j}; -; q, -q)$ and the coefficient of the $_4\phi_3$ is bounded by $M_2(\mathbf{t})|\eta/t_1|^j q^{j(j-1)/2}/(q;q)_j$. Here both $M_1(\mathbf{t})$ and $M_2(\mathbf{t})$ are positive and depend only on $\mathbf{t}$. The following estimate

$$\sum_{j=0}^{\infty} \left|\frac{\eta}{t_1}\right|^j \frac{q^{\binom{j}{2}}}{(q;q)_j} {}_2\phi_1(q^{-j}; -; q, -q) \leq \sum_{j=0}^{\infty} \frac{|\eta/t_1|^j}{(q;q)_j} \sum_{\nu=0}^{j} \frac{q^{\binom{j-\nu}{2}}}{(q;q)_{j-\nu}} \leq \frac{(-1;q)_\infty}{(|\eta/t_1|;q)_\infty}$$

holds for η with $|\eta| < |t_1|$. In the last inequality we used Euler's identities (1.12) and (1.13). Hence for $|\eta| < |t_1|$, Tannery's theorem (the discrete version of the Lebesgue dominated convergence theorem) can be applied. Using formula (3.6) we get

$$(5.5) \qquad \lim_{n\to\infty} \eta^n b_n(1/\eta, \mathbf{t}) = \sum_{j=0}^{\infty} \frac{q^{\binom{j}{2}}(\eta/t_1)^j}{(q;q)_j} {}_3\phi_2 \left(\begin{array}{c} q^{-j}, t_1, t_2 \\ at_1t_2, 0 \end{array} \middle| q, q \right)$$

$$= \sum_{k=0}^{\infty} \frac{(t_1, t_2; q)_k q^k}{(at_1t_2, q; q)_k} \sum_{j=k}^{\infty} \frac{(q^{-j}; q)_k q^{\binom{j}{2}}(\eta/t_1)^j}{(q;q)_j}$$

$$= \sum_{k=0}^{\infty} \frac{(t_1, t_2; q)_k (-\eta/t_1)^k}{(at_1t_2, q; q)_k} \sum_{j=k}^{\infty} \frac{q^{\binom{j-k}{2}}(\eta/t_1)^{j-k}}{(q;q)_{j-k}}$$

$$= (-\eta/t_1; q)_\infty {}_2\phi_1 \left(\begin{array}{c} t_1, t_2 \\ at_1t_2 \end{array} \middle| q, -\eta/t_1 \right),$$

where we also used (1.13).

From (5.1), (5.5), and (5.4) for $z \notin \mathbf{R}$ we get

$$(5.6) \qquad \int_{\mathbf{R}} \frac{d\varphi(u, \mathbf{t})}{z-u} = \lim_{n\to\infty} \frac{z^{-n}q^{n-1}b_{n-1}(z/q, q\mathbf{t})}{z^{-n}b_n(z, \mathbf{t})}$$

$$= \frac{{}_2\phi_1(qt_1, qt_2; at_1t_2q^2; q, -1/(t_1z))}{z {}_2\phi_1(t_1, t_2; at_1t_2; q, -1/(t_1z))}, \quad |z| > 1/|t_1|.$$

An analytic continuation of (5.6) can be found using the Heine transformation [**4**, (III.1)],

(5.7)
$$_2\phi_1(A, B; C; q, Z) = \frac{(B, AZ; q)_\infty}{(C, Z; q)_\infty} {}_2\phi_1(C/B, Z; AZ; q, B), \quad |Z|, |B| < 1.$$

From (5.6) and (5.7) we obtain

(5.8)
$$\int_{\mathbf{R}} \frac{d\varphi(u, \mathbf{t})}{z - u} = \frac{(1 - at_1 t_2)(1 - at_1 t_2 q)}{(1 - t_2)(z + 1)} \frac{{}_2\phi_1(aqt_1, -1/(t_1 z); -q/z; q, qt_2)}{{}_2\phi_1(at_1, -1/(t_1 z); -1/z; q, t_2)}.$$

The Heine transformation also provides an analytic continuation of formula (5.5).

Formulas (4.11)-(4.16), (4.22), and (5.5) imply

$$(5.9) \quad c_0(\lambda, \mathbf{t}) = A((1 - q)\lambda; a, \mathbf{t}) = ((1 - q)\lambda, a(1 - q)\lambda; q)_\infty^{-1}$$

$$\times \sum_{j=0}^\infty \frac{q^{\binom{j}{2}}(-a(1 - q)\lambda)^j}{(q; q)_j} {}_3\phi_2(q^{-j}, t_1, t_2; at_1 t_2, 0; q, q)$$

$$= \frac{1}{((1 - q)\lambda; q)_\infty} {}_2\phi_1 \left(\begin{array}{c} t_1, t_2 \\ at_1 t_2 \end{array} \middle| q, a(1 - q)\lambda \right).$$

From (5.6) and (5.9) we obtain

$$(5.10) \qquad \int_{\mathbf{R}} \frac{d\varphi(u, \mathbf{t})}{z - u} = \frac{c_0(-1/(a(1 - q)t_1 z), q\mathbf{t})}{zc_0(-1/(a(1 - q)t_1 z), \mathbf{t})}.$$

The c_0-functions in (5.10) have no common zeros. This can be seen as follows: Assume that $c_0(\lambda_0, \mathbf{t}) = c_0(\lambda_0, q\mathbf{t}) = 0$ for some λ_0. Formula (4.11) implies

$$c_n(\lambda, q\mathbf{t}) = \frac{(1 - at_1 t_2 q^n)(1 - q^{n+1})t_1}{(1 - t_1)(1 - at_1)(1 - q)\lambda} c_{n+1}(\lambda, \mathbf{t}).$$

Then our assumption implies $c_1(\lambda_0, \mathbf{t}) = 0$ and from the three term recurrence equation (4.25) we get $c_n(\lambda_0, \mathbf{t}) = 0$ for all $n \geq 0$. But then by Theorem 4.1,

$$e_q(\lambda_0(1 - q)x) = (\lambda_0(1 - q)x; q)_\infty^{-1} = \sum_{n=0}^\infty c_n(\lambda_0, \mathbf{t})p_n(x, \mathbf{t}) \equiv 0,$$

which is impossible.

The Perron-Stieltjes inversion formula is

$$(5.11) \qquad F(z) = \int_{\mathbf{R}} \frac{d\mu(t)}{z - t} \quad \text{if} \quad \text{and} \quad \text{only} \quad \text{if}$$

$$\mu(x) - \mu(y) = \lim_{\epsilon \to 0^+} \int_y^x \frac{F(t - i\epsilon) - F(t + i\epsilon)}{2\pi i} dt.$$

This inversion formula shows that φ is a purely discrete measure. It is clear that an isolated point mass m of φ located at $x = x_0$ contributes $m/(z - x_0)$ to the left-hand side of (5.8). Thus the isolated point masses of φ coincide with the isolated poles of the right-hand side of (5.8) and the masses are the corresponding residues. Below we will show that $x = 0$, which is the only essential singularity of the right-hand side of (5.8) does not support a discrete mass, so let us assume this for the time being. Formula (5.10), Theorem 4.2, and the above discussion describe the relationship between the support of the measure $\varphi(\cdot, \mathbf{t})$ and the spectrum of the operator $\mathcal{T}_{\mathbf{t}}$.

Theorem 5.1. *The support of the measure of orthogonality $\varphi(\cdot, \mathbf{t})$ is the set of the elements of the spectrum of the operator $\mathcal{T}_{\mathbf{t}}$ multiplied by $-1/(a(1 - q)t_1)$.*

It remains to show that $x = 0$ is not a mass point for φ. From the theory of the moment problem [1], [14], it is known that when the measure φ is unique then $x = x_0$ is a mass point for the measure if and only if $\sum_{n=0}^{\infty} p_n(x_0)^2$ converges, $\{p_n(x)\}$ being the orthonormal polynomials. From Theorem 3.1 we find

(5.12)
$$b_n(0, \mathbf{t}) = \frac{(q^{-n}/t_1, q^{-n}/t_2; q)_n}{(q, q^{-2n}/(at_1t_2); q)_n}(-at_1)^{-n}{}_4\phi_3\left(\begin{array}{c} q^{-n}, \, at_1t_2q^{n+1}, \, t_1, \, t_2 \\ t_1q, \, t_2q, \, at_1t_2 \end{array} \,\Big|\, q, \, q\right).$$

Applying (1.15) to (5.12) we get

(5.13)
$$b_n(0, \mathbf{t}) = \frac{(qt_1, qt_2; q)_n}{(q, at_1t_2q^{n+1}; q)_n}\frac{q^{\binom{n}{2}}}{t_1^n}{}_4\phi_3\left(\begin{array}{c} q^{-n}, \, at_1t_2q^{n+1}, \, t_1, \, t_2 \\ t_1q, \, t_2q, \, at_1t_2 \end{array} \,\Big|\, q, \, q\right).$$

Ismail and Wilson, [10], determined the asymptotic behavior of the Askey-Wilson polynomials. They proved that

(5.14)
$$\lim_{n \to \infty} (z/A)^n {}_4\phi_3\left(\begin{array}{c} q^{-n}, \, ABCDq^{n-1}, \, Az, \, A/z \\ AB, \, AC, \, AD \end{array} \,\Big|\, q, \, q\right)$$
$$= \frac{(Az, Bz, Cz, Dz; q)_\infty}{(z^2, AB, AC, AD; q)_\infty},$$

for $|z| < 1$, and that the left-hand side of (5.14) is bounded if $|z| = 1$ but $z \neq \pm 1$. If $z = \pm 1$ then the left-hand side of (5.14) is $O(n)$. The ${}_4\phi_3$ quantity in (5.13) corresponds to the ${}_4\phi_3$ function in (5.14) with $A = \sqrt{t_1t_2}$, $B = q\sqrt{t_1/t_2}$, $C = q\sqrt{t_2/t_1}$, $D = a\sqrt{t_1t_2}$, and $z = \sqrt{t_1/t_2}$ if $|t_1| \leq |t_2|$ and $z = \sqrt{t_2/t_1}$ otherwise. The orthonormal polynomials associated with the b_n's are

$$p_m(\eta, \mathbf{t}) = b_m(\eta, \mathbf{t})\xi_0^{-1/2}(\mathbf{t})\prod_{n=1}^{m} u_n^{-1/2},$$

where $-u_n$ denotes the coefficient of $b_{n-1}(\eta, \mathbf{t})$ in (2.18). From (2.18) we get

$$\prod_{k=1}^{n} u_k = \frac{(qt_1, aqt_1, qt_2, aqt_2; q)_n}{(at_1t_2q, at_1t_2q^2; q)_{2n}} (t_2/t_1)^n q^{2\binom{n+1}{2}-n}.$$

Combining the above formulas we obtain

(5.15)

$$q^n \xi_0(\mathbf{t}) z^{2n} p_n(0, \mathbf{t})^2 = \frac{(qt_1, qt_2; q)_n}{(aqt_1, aqt_2; q)_n} \frac{(at_1t_2q; q)_n^2 (1 - at_1t_2q^{2n+1})}{(q; q)_n^2 (1 - at_1t_2q)} \frac{z^{2n}}{(t_1t_2)^n}$$

$$\times {}_4\phi_3 \left(\begin{array}{c} q^{-n}, \, at_1t_2q^{n+1}, \, t_1, \, t_2 \\ qt_1, \, qt_2, \, at_1t_2 \end{array} \,\bigg|\, q, \, q \right)^2$$

$$\to \frac{(qt_1, qt_2; q)_\infty (at_1t_2q; q)_\infty^2}{(aqt_1, aqt_2; q)_\infty (1 - at_1t_2q)}$$

$$\times \frac{(z\sqrt{t_1t_2}, qz\sqrt{t_1/t_2}, qz\sqrt{t_2/t_1}, az\sqrt{t_1t_2}; q)_\infty^2}{(z^2, qt_1, qt_2, at_1t_2, q; q)_\infty^2}, \quad n \to \infty,$$

if $|z| \leq 1$ and $z \neq \pm 1$. If $z = 1$ the right-hand side of (5.15) becomes unbounded. Since $|z| \leq 1$ and $|q| < 1$, (5.15) clearly implies that $\sum_{n=0}^{\infty} p_n(0, \mathbf{t})^2$ diverges.

Equations (3.1)–(3.3) show that the polynomials $\{b_n(\eta, \mathbf{t})\}$ are constant multiples of birth and death process polynomials associated with a process with birth and death rates

$$\frac{(1 - t_1q^{n+1})(1 - at_1q^{n+1})t_2q^n}{(1 - at_1t_2q^{2n+1})(1 - at_1t_2q^{2n+2})} \quad \text{and} \quad \frac{(1 - t_2q^n)(1 - at_2q^n)t_1q^n}{(1 - at_1t_2q^{2n})(1 - at_1t_2q^{2n+1})},$$

respectively. An exposition of the theory of birth and death processes and orthogonal polynomials can be found in [**7**].

6. Connection Coefficients for the Big q-Jacobi Polynomials.

In this section we will compute the connection coefficients in the formula

(6.1) $$p_n(x, \mathbf{t}) = \sum_{l=0}^{n} a_{n,l}(\mathbf{t}, \mathbf{s}) p_l(x, \mathbf{s}),$$

where $\mathbf{t} = (t_1, t_2)$ and $\mathbf{s} = (s_1, s_2)$. From (6.1), (1.6), and (1.10) we get

(6.2) $a_{n,l}(\mathbf{t}, \mathbf{s}) \xi_l(\mathbf{s})$

$$
= \int_{\mathbf{R}} p_n(x, \mathbf{t}) p_l(x, \mathbf{s}) \, d\mu(x, \mathbf{s})
$$

$$
= \int_{\mathbf{R}} \sum_{k=0}^{n} \frac{(q^{-n}, at_1 t_2 q^{n-1}; q)_k q^k}{(t_1, at_1, q; q)_k} \left((t_1/s_2; q)_k \sum_{\nu=0}^{k} \frac{(q^{-k}, s_2 x; q)_\nu q^\nu}{(s_2 q^{1-k}/t_1, q; q)_\nu} \right)
$$

$$
\times \sum_{j=0}^{l} \frac{(q^{-l}, as_1 s_2 q^{l-1}; q)_j q^j}{(s_1, as_1, q; q)_j} (s_1 x; q)_j \, d\mu(x, \mathbf{s}).
$$

Changing the order of summation in (6.2) we obtain

(6.3)

$$
a_{n,l}(\mathbf{t}, \mathbf{s}) \xi_l(\mathbf{s}) = \sum_{\nu=0}^{n} \left(\sum_{k=\nu}^{n} \frac{(q^{-n}, at_1 t_2 q^{n-1}; q)_k q^k}{(t_1, at_1, q; q)_k} \frac{(t_1/s_2; q)_k (q^{-k}; q)_\nu}{(s_2 q^{1-k}/t_1; q)_\nu} \right) \frac{q^\nu}{(q; q)_\nu}
$$

$$
\times \sum_{j=0}^{l} \frac{(q^{-l}, as_1 s_2 q^{l-1}; q)_j q^j}{(s_1, as_1, q; q)_j} \int_{\mathbf{R}} \frac{d\mu^{(a)}(x)}{(xs_1 q^j, xs_2 q^\nu; q)_\infty}.
$$

The last integral is evaluated using the q-beta integral evaluation from [6]

(6.4) $$
\int_{\mathbf{R}} \frac{d\mu^{(a)}(x)}{(xt_1, xt_2; q)_\infty} = \frac{(at_1 t_2; q)_\infty}{(t_1, at_1, t_2, at_2; q)_\infty}.
$$

To evaluate the last sum in (6.3) we use (1.9). We get

(6.5) $$
\frac{(as_1 s_2 q^\nu; q)_\infty}{(s_1, as_1, s_2 q^\nu, as_2 q^\nu; q)_\infty} \sum_{j=0}^{l} \frac{(q^{-l}, as_1 s_2 q^{l-1}; q)_j}{(as_1 s_2 q^\nu, q; q)_j} q^j
$$

$$
= \frac{(as_1 s_2 q^\nu; q)_\infty}{(s_1, as_1, s_2 q^\nu, as_2 q^\nu; q)_\infty} \frac{(q^{\nu+1-l}; q)_l}{(as_1 s_2 q^\nu; q)_l} (as_1 s_2 q^{l-1})^l.
$$

Since $(q^{\nu+1-l}; q)_l$ vanishes for $\nu < l$, the first sum in (6.3) is over $\nu \in \{l, \dots, n\}$.

Let $S_{n,\nu}(\mathbf{t}, \mathbf{s})$ denote the sum over k in equation (6.3). Applying (1.15) to $(q^{-k}; q)_\nu$ and $(s_2 q^{1-k}/t_1; q)_\nu$ we obtain

(6.6)
$$S_{n,\nu}(\mathbf{t}, \mathbf{s}) = \frac{(q^{-n}, at_1 t_2 q^{n-1}; q)_\nu q^\nu}{(t_1, at_1; q)_\nu}$$

$$\times \sum_{k=\nu}^n \frac{(q^{-(n-\nu)}, at_1 t_2 q^{n+\nu-1}; q)_{k-\nu} q^{k-\nu}}{(t_1 q^\nu, at_1 q^\nu; q)_{k-\nu}} \frac{q^{-\nu}(t_1/s_2; q)_{k-\nu}}{(s_2/t_1)^\nu (q; q)_{k-\nu}}$$

$$= \frac{(q^{-n}, at_1 t_2 q^{n-1}; q)_\nu (t_1/s_2)^\nu}{(t_1, at_1; q)_\nu} {}_3\phi_2 \left(\begin{array}{c} q^{-(n-\nu)}, \, at_1 t_2 q^{n+\nu-1}, \, t_1/s_2 \\ t_1 q^\nu, \, at_1 q^\nu \end{array} \middle| \, q, \, q \right).$$

From (6.3), (6.5), and (6.6) using the identities $(q^{\nu+1-l}; q)_l/(q; q)_\nu = 1/(q; q)_{\nu-l}$ and

$$\frac{(as_1 s_2 q^\nu; q)_\infty}{(as_1 s_2 q^\nu; q)_l} = (as_1 s_2 q^{\nu+l}; q)_\infty = \frac{(as_1 s_2 q^{2l}; q)_\infty}{(as_1 s_2 q^{2l}; q)_{\nu-l}}$$

we obtain the following result.

Theorem 6.1. *The connection coefficients in the expansion of the polynomial $p_n(x, \mathbf{t})$ in terms of the polynomials $\{p_l(x, \mathbf{s})\}$ are given by the formula*

(6.7)
$$a_{n,l}(\mathbf{t}, \mathbf{s}) = \frac{1}{\xi_l(\mathbf{s})} \frac{(as_1 s_2 q^{2l}; q)_\infty (at_1 s_1 q^l)^l}{(s_1, as_1, s_2 q^l, as_2 q^l; q)_\infty} \frac{(q^{-n}, at_1 t_2 q^{n-1}; q)_l}{(t_1, at_1; q)_l}$$

$$\times \sum_{\nu=l}^n \frac{(q^{-(n-l)}, at_1 t_2 q^{n+l-1}, s_2 q^l, as_2 q^l; q)_{\nu-l} (t_1 q/s_2)^{\nu-l}}{(t_1 q^l, at_1 q^l, as_1 s_2 q^{2l}, q; q)_{\nu-l}}$$

$$\times {}_3\phi_2 \left(\begin{array}{c} q^{-(n-\nu)}, \, at_1 t_2 q^{n+\nu-1}, \, t_1/s_2 \\ t_1 q^\nu, \, at_1 q^\nu \end{array} \middle| \, q, \, q \right), \quad l = 0, \dots, n.$$

The connection coefficient formula (6.7) can be used to find the connection coefficients in certain special cases. In Section 2, we computed these coefficients for the case $\mathbf{s} = q\mathbf{t}$.

We will now use (6.7) to give another proof of the fact that $a_{n,l}(\mathbf{t}, q\mathbf{t}) = 0$ for $l < n - 2$. In view of (1.8) it is enough to consider the case $\mathbf{s} = (qt_2, qt_1)$ since for every $l \in \mathbf{N}$, $p_l(x; t_1, t_2)$ and $p_l(x; t_2, t_1)$, and therefore the coefficients $a_{n,l}(\mathbf{t}, q\mathbf{t})$ and $a_{n,l}(\mathbf{t}, q\mathbf{t}^*)$ are linearly dependent, where $\mathbf{t}^* = (t_2, t_1)$. So let $\mathbf{s} = q\mathbf{t}^*$. In view of (6.7) it is enough to show that $\tilde{S}_{n,l}(\mathbf{t}) = 0$ for $l < n - 2$, where $\tilde{S}_{n,l}(\mathbf{t})$ denotes the sum in (6.7) with $\mathbf{s} = q\mathbf{t}^*$. For this

sum by (6.7) we have

$$(6.8) \quad (1 - t_1 q^l)(1 - a t_1 q^l) \tilde{S}_{n,l}(\mathbf{t}) = \sum_{\nu=l}^{n} \frac{(q^{-(n-l)}, a t_1 t_2 q^{n+l-1}; q)_{\nu-l}}{(a t_1 t_2 q^{2l+2}, q; q)_{\nu-l}}$$

$$\times (1 - t_1 q^\nu)(1 - a t_1 q^\nu) \left(1 - \frac{(1 - q^{-(n-\nu)})(1 - a t_1 t_2 q^{n+\nu-1})}{(1 - t_1 q^\nu)(1 - a t_1 q^\nu)}\right)$$

$$= \sum_{i=0}^{n-l} \frac{(q^{-(n-l)}, a t_1 t_2 q^{n+l-1}; q)_i}{(a t_1 t_2 q^{2l+2}, q; q)_i}$$

$$\times \left[(1 - t_1 q^{l+i})(1 - a t_1 q^{l+i}) - (1 - q^{-(n-l)+i})(1 - a t_1 t_2 q^{n+l+i-1})\right]$$

$$= q^l(-t_1 - a t_1 + q^{-n} + a t_1 t_2 q^{n-1}) {}_2\phi_1 \left(\begin{array}{c} q^{-d}, \, a t_1 t_2 q^{d+2l-1} \\ a t_1 t_2 q^{2l+2} \end{array} \middle| \, q, \, q \right)$$

$$+ (a t_1^2 q^{2l} - a t_1 t_2 q^{2l-1}) {}_2\phi_1 \left(\begin{array}{c} q^{-d}, \, a t_1 t_2 q^{d+2l-1} \\ a t_1 t_2 q^{2l+2} \end{array} \middle| \, q, \, q^2 \right),$$

with $d = n - l$. We now show that the ${}_2\phi_1$ expressions in (6.8) vanish.

For integer numbers $d > 0$, $d_1 \geq 0$, and $d_2 > 0$ we consider

$$(6.9) \quad A_{d,d_1,d_2}(\alpha, q) := {}_2\phi_1 \left(\begin{array}{c} q^{-d}, \, \alpha q^{d_1} \\ \alpha \end{array} \middle| \, q, \, q^{d_2} \right).$$

We claim that $A_{d,d_1,d_2}(\alpha, q) \equiv 0$ if $d_1 + d_2 < d$. Indeed we have

$$(6.10) \quad A_{d,d_1,d_2}(\alpha, q) = \sum_{j=0}^{d} \frac{(q^{-d}, \alpha q^{d_1}; q)_j}{(\alpha, q; q)_j} q^{d_2 j}$$

$$= \sum_{j=0}^{d} \begin{bmatrix} d \\ j \end{bmatrix}_q \frac{(\alpha q^{d_1}; q)_j}{(\alpha; q)_j} \left(-q^{-d+d_2}\right)^j q^{\binom{j}{2}}$$

$$= \sum_{j=0}^{d} \begin{bmatrix} d \\ j \end{bmatrix}_q \frac{(\alpha q^j; q)_{d_1}}{(\alpha; q)_{d_1}} \left(-q^{-(d-d_2)}\right)^j q^{\binom{j}{2}},$$

where we used the identity

$$(\alpha q^j; q)_{d_1}(\alpha; q)_j = (\alpha; q)_{d_1+j} = (\alpha q^{d_1}; q)_j(\alpha; q)_{d_1}.$$

By the q-binomial theorem (1.14) we have $(z;q)_l = \sum_{s=0}^{l} a_{l,s}(q)z^s$ with coeffi-

cients $a_{l,s}(q) = (-1)^s q^{\binom{s}{2}} \begin{bmatrix} l \\ s \end{bmatrix}_q$. Then (6.10) can be continued as follows

(6.11)

$$A_{d,d_1,d_2}(\alpha,q) = \frac{1}{(\alpha;q)_{d_1}} \sum_{s=0}^{d_1} a_{d_1,s}(q)\alpha^s \left(\sum_{j=0}^{d} \begin{bmatrix} d \\ j \end{bmatrix}_q \left(-q^{s-(d-d_2)}\right)^j q^{\binom{j}{2}} \right)$$

$$= (1/(\alpha;q)_{d_1}) \sum_{s=0}^{d_1} a_{d_1,s}(q)\alpha^s (q^{s-(d-d_2)};q)_d = 0, \quad d - d_2 > d_1,$$

where at the end we used (1.14). From (6.8) and (6.11) we get $\tilde{S}_{n,l}(\mathbf{t}) = 0$ for $d = n-l > 2$, since in this case it is a linear combination of $A_{d,d-3,1}(\alpha,q)$ and $A_{d,d-3,2}(\alpha,q)$ with $\alpha = at_1t_2q^{2l+2}$. Then $a_{n,l}(\mathbf{t},q\mathbf{t}) = 0$ for $l < n-2$.

References

[1] N.I. Akhiezer, *The Classical Moment Problem and Some Related Questions in Analysis*, Oliver and Boyd, Edinburgh, 1965 (English translation).

[2] G.E. Andrews and R.A. Askey, *Classical Orthogonal Polynomials*, Polynomes Orthogonaux et Applications (C. Breziniski et al., eds.), Lecture Notes in Mathematics, **1171**, Springer-Verlag, Berlin, (1984), 36-63.

[3] G.E. Andrews, R.A. Askey and R. Roy, *Special Functions*, Cambridge University Press, Cambridge, 1999.

[4] G. Gasper and M. Rahman, *Basic Hypergeometric Series*, Cambridge University Press, Cambridge, 1989.

[5] D.P. Gupta, M.E.H. Ismail and D.R. Masson, *Contiguous relations, basic hypergeometric functions, and orthogonal polynomials,* II: *Associated big q-Jacobi polynomials*, J. Math. Anal. Appl., **171** (1992), 477-497.

[6] M.E.H. Ismail, *Continuous and Discrete q-Orthogonal Polynomials*, Monograph-manuscript, 1998.

[7] M.E.H. Ismail, J. Letessier, D.R. Masson and G. Valent, *Birth and Death Processes and Orthogonal Polynomials*, Orthogonal Polynomials: Theory and Practice (P. Nevai, ed.), NATO, ASI on Orthogonal Polynomials and Their Applications, Kluwer Academic Publishers, (1990), 225-229.

[8] M.E.H. Ismail and M. Rahman, *The associated Askey-Wilson polynomials*, Trans. Amer. Math. Soc., **328** (1991), 201-237.

[9] M.E.H. Ismail, M. Rahman and R. Zhang, *Diagonalization of certain integral operators* II, J. Comp. Applied Math., **68** (1996), 163-196.

[10] M.E.H. Ismail and J.A. Wilson, *Asymptotic and generating relations for the q-Jacobi and $_4\phi_3$ polynomials*, J. Approx. Theory, **36** (1982), 43-54.

[11] M.E.H. Ismail and R. Zhang, *Diagonalization of certain integral operators*, Adv. Math., **109** (1994), 1-33.

[12] R. Koekoek and R. Swarttouw, *The Askey-scheme of Hypergeometric Orthogonal Polynomials and its q-analogue*, Technical Report, **94-05**, Delft University of Technology, Delft, The Netherlands, 1994.

[13] H.M. Schwartz, *A class of continued fractions*, Duke Math. J., **6** (1940), 48-65.

[14] J. Shohat and J.D. Tamarkin, *The Problem of Moments* (Revised edition), Amer. Math. Soc., Providence, RI, 1950.

[15] G. Szegő, *Orthogonal Polynomials*, Amer. Math. Soc., Providence, RI, 4th edition, 1975.

[16] J. Wimp, *Some explicit Padé approximants for the function ϕ'/ϕ and a related quadrature formula involving Bessel functions*, SIAM J. Math. Anal., **16** (1985), 887-895.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF SOUTH FLORIDA
TAMPA, FL 33620-5700
E-mail address: ismail@math.usf.edu

DEPARTMENT OF MATHEMATICS & COMPUTER SCIENCE
UNIVERSITY OF HOUSTON-DOWNTOWN
HOUSTON, TX 77002-1094
E-mail address: simeonovp@zeus.dt.uh.edu

CANONICAL BUNDLES FOR HAMILTONIAN LOOP GROUP MANIFOLDS

E. Meinrenken and C. Woodward

We construct canonical bundles for Hamiltonian loop group actions with proper moment maps. As an application, we show that for certain moduli spaces of flat connections on Riemann surfaces with boundary, the first Chern class is a multiple of the cohomology class of the symplectic form.

1. Introduction.

One of the simplest invariants of a symplectic manifold is the isomorphism class of the canonical line bundle. Suppose (M, ω) is a symplectic manifold. For any ω-compatible almost complex structure J one defines the canonical line bundle K_M as the dual to the top exterior power of the tangent bundle TM,

$$K_M = \det_{\mathbb{C}}(TM)^*.$$

Since the space of ω-compatible almost complex structures on M is contractible, the isomorphism class of K_M is independent of this choice. If a compact Lie group G acts by symplectomorphisms on M, we can take J to be G-invariant, and K_M is a G-equivariant line bundle.

The canonical bundle behaves well under symplectic quotients. If the G-action is Hamiltonian, with moment map $\Phi : M \to \mathfrak{g}^*$, the symplectic quotient of M is defined by

$$M /\!\!/ G := \Phi^{-1}(0)/G.$$

We assume that 0 is a regular value, so that $M /\!\!/ G$ is a symplectic orbifold. The canonical line bundle for the reduced space (symplectic quotient) $M /\!\!/ G = \Phi^{-1}(0)/G$ is related to the canonical bundle on M by

$$(1) \qquad K_{M /\!\!/ G} = K_M /\!\!/ G := (K_M | \Phi^{-1}(0))/G.$$

The canonical bundle also behaves well under inductions. Let T be a maximal torus of G with Lie algebra $\mathfrak{t}$. Suppose that N is a Hamiltonian T-manifold with moment map $\Psi : N \to \mathfrak{t}^*$. The symplectic induction $M := G \times_T N$ has a unique closed two-form and moment map extending the given data on N. If the image of Ψ is contained in the interior of a positive

chamber $\mathfrak{t}_+^*$, then M is symplectic and K_M is induced from K_N, after a ρ-shift:

$$K_M \cong G \times_T (K_N \otimes \mathbb{C}_{-2\rho}).$$

Here $\mathbb{C}_{-2\rho}$ is the T-representation with weight given by the sum -2ρ of the negative roots.

In this paper we develop a notion of canonical line bundle for (infinite-dimensional) Hamiltonian loop group manifolds with proper moment maps. The idea is to use the property of the canonical bundle under inductions as the definition in the infinite-dimensional setting. Just as in the finite dimensional situation, the canonical bundle of the (finite dimensional) reduced spaces are obtained from the canonical bundle K_M upstairs. For the fundamental homogeneous space $\Omega G = LG/G$, our definition agrees with Freed's computation [4] of the regularized first Chern class of ΩG.

As an application, we prove the following fact about moduli spaces of flat G-connections on compact oriented surfaces Σ. Suppose G is simple and simply connected, and let c be the dual Coxeter number. Suppose Σ has b boundary components $B_1, \ldots, B_b$, and let $\mathcal{C}_1, \ldots, \mathcal{C}_b$ be a collection of conjugacy classes. Let $\mathcal{M}(\Sigma, \mathcal{C})$ be the (finite dimensional) moduli space of flat G-connections on Σ with holonomy around B_j contained in $\mathcal{C}_j$. The subset $\mathcal{M}(\Sigma, \mathcal{C})_{\mathrm{irr}}$ of irreducible connections is a smooth symplectic manifold. Let $[\omega]$ be the cohomology class of the basic symplectic form on $\mathcal{M}(\Sigma, \mathcal{C})_{\mathrm{irr}}$.

Theorem 1.1. *If the conjugacy classes $\mathcal{C}_j$ consist of central elements, then the first Chern class of $K_{\mathcal{M}(\Sigma, \mathcal{C})_{\mathrm{irr}}}$ is equal to $-2c[\omega]$.*

This was first proved in the special case of $SU(2)$ by Ramanan [11]. In general it is a consequence of the local family index theorem (Quillen [10], Zograf and Takhtadzhyan [13]). See also Beauville, Laszlo, and Sorger [3], and Kumar and Narasimhan [5]. Our application, Theorem 4.2 below, expands the list of conjugacy classes for which this result holds. It would be interesting to know which of these are Kähler-Einstein. Our main application of the canonical bundle will be given in a forthcoming paper [2], where it enters a fixed point formula for Hamiltonian loop group actions.

2. Hamiltonian loop group manifolds.

2.1. Notation. Let $\mathfrak{g}$ be a simple Lie algebra, and G the corresponding compact, connected, simply connected Lie group. Choose a maximal torus $T \subset G$, with Lie algebra $\mathfrak{t}$, and let $\Lambda \subset \mathfrak{t}$ resp. $\Lambda^* \subset \mathfrak{t}^*$ denote the integral resp. (real) weight lattice. Let $\mathfrak{R}$ be the set of roots and $\mathfrak{R}_+$ the subset of positive roots, for some choice of positive Weyl chamber $\mathfrak{t}_+$. We will identify $\mathfrak{g} \cong \mathfrak{g}^*$ and $\mathfrak{t} \cong \mathfrak{t}^*$, using the normalized inner product $\cdot$ for which the long roots have length $\sqrt{2}$. The highest root is denoted α_0, and the half-sum of

positive roots $\rho = \frac{1}{2} \sum_{\alpha \in \mathfrak{R}_+} \alpha$. The integer

$$c = 1 + \rho \cdot \alpha_0$$

is called the dual Coxeter number of G. The fundamental alcove for G is the simplex

$$(2) \qquad \mathfrak{A} = \{\xi \in \mathfrak{t}_+, \, \alpha_0 \cdot \xi \le 1\} \subset \mathfrak{t} \subset \mathfrak{g}.$$

It parametrizes the set of conjugacy classes of G, in the sense that every conjugacy class contains an element $\exp(\xi)$ for a unique $\xi \in \mathfrak{A}$. The centralizer $G_{\exp(\xi)}$ depends only on the open face σ containing ξ and will be denoted G_σ. Introduce a partial ordering on the set of open faces of $\mathfrak{A}$ by setting $\sigma \prec \tau$ if $\sigma \subset \bar{\tau}$. Then $\sigma \prec \tau \Rightarrow G_\sigma \supset G_\tau$.

A similar discussion holds for semi-simple simply-connected groups, with the alcove replaced by the product of the alcoves for the simple factors.

2.2. Loop groups. Let LG denote the loop group of maps $S^1 \to G$ of some fixed Sobolev class $s > 1$, $L\mathfrak{g} = \Omega^0(S^1, \mathfrak{g})$ its Lie algebra, and $L\mathfrak{g}^* \in \Omega^1(S^1, \mathfrak{g})$ the space of Lie algebra valued 1-forms of Sobolev class $s - 1$. Integration over S^1 defines a non-degenerate pairing between $L\mathfrak{g}^*$ and $L\mathfrak{g}$. One defines the (affine) coadjoint action of LG on $L\mathfrak{g}^* \in \Omega^1(S^1, \mathfrak{g})$ by

$$(3) \qquad g \cdot \mu = \mathrm{Ad}_g \, \mu - \mathrm{d}g \, g^{-1}$$

where $\mathrm{d}g \, g^{-1}$ is the pull-back of the right-invariant Maurer-Cartan form on G. Let $\widehat{LG}$ be the basic central extension [9] of LG, defined infinitesimally by the cocycle $(\xi_1, \xi_2) \mapsto \oint \mathrm{d}\xi_1 \cdot \xi_2$ on $L\mathfrak{g}$. The adjoint action of $\widehat{LG}$ on $\widehat{L\mathfrak{g}}$ descends to an action of LG since the central circle acts trivially, and for the coadjoint action of LG on $\widehat{L\mathfrak{g}}^* = \Omega^1(S^1, \mathfrak{g}) \oplus \mathbb{R}$ one finds

$$(4) \qquad g \cdot (\mu, \lambda) = (\mathrm{Ad}_g(\mu) + \lambda \mathrm{d}g g^{-1}, \lambda).$$

This identifies $L\mathfrak{g}^*$ with the affine hyperplane $\Omega^1(S^1, \mathfrak{g}) \times \{1\} \subset \widehat{L\mathfrak{g}}^*$.

There is a natural smooth map $\mathrm{Hol} : L\mathfrak{g}^* \to G$ sending $\mu \in L\mathfrak{g}^*$, viewed as a connection on the trivial bundle $S^1 \times G$, to its holonomy around S^1. This map sets up a 1-1 correspondence between the sets of G-conjugacy classes and coadjoint LG-orbits, hence both are parametrized by points in the alcove.

More explicitly this parametrization is given as follows. View $\mathfrak{A}$ as a subset of $L\mathfrak{g}^*$ by the embedding $\xi \mapsto \xi \mathrm{d}\theta/(2\pi)$. Then every coadjoint LG-orbit passes through a unique point $\xi \in \mathfrak{A}$. The stabilizer group $(LG)_\xi$ depends only on the open face $\sigma \subset \mathfrak{A}$ containing ξ and will be denoted $(LG)_\sigma$. The evaluation map $LG \to G$, $g \mapsto g(1)$ restricts to an isomorphism $(LG)_\sigma \cong G_\sigma$; in particular $(LG)_\sigma$ is compact and connected. If $\sigma \prec \tau$ then $(LG)_\sigma \supset (LG)_\tau$. In particular, every $(LG)_\sigma$ contains $T = (LG)_{\mathrm{int}\,\mathfrak{A}}$.

2.3. Hamiltonian LG-manifolds. We begin by reviewing the definition of a symplectic Banach manifold. A two-form ω on a Banach manifold M is weakly non-degenerate if the map $\omega^\sharp : T_m^M \to T_m^* M$ is injective, for all $m \in M$. A *Hamiltonian LG-manifold* is a Banach manifold M together with an LG-action, an invariant, weakly non-degenerate closed two-form ω and an equivariant moment map $\Phi : M \to L\mathfrak{g}^*$. Equivalently, one can think of M has a Hamiltonian $\widehat{LG}$-manifold, where the central circle acts trivially with constant moment map $+1$.

Example 2.1.   1) For any $\mu \in L\mathfrak{g}^*$, the coadjoint orbit $LG \cdot \mu$ is a Hamiltonian LG-manifold, with moment map the inclusion.

  2) Let Σ be a compact oriented surface with boundary $\partial\Sigma \cong (S^1)^b$. Let $\mathcal{G}(\Sigma) = \mathrm{Map}(\Sigma, G)$ be the gauge group, and $\mathcal{G}_\partial(\Sigma)$ be the gauge transformation that are trivial on the boundary.

  The space $\Omega^1(\Sigma, \mathfrak{g})$ of connections carries a natural symplectic structure, and the action of $\mathcal{G}_\partial(\Sigma)$ is Hamiltonian with moment map the curvature. The symplectic quotient $\mathcal{M}(\Sigma)$ is the moduli space of flat connection up to based gauge transformations. It carries a residual action of LG^b, with moment map induced by the pull-back of connections to the boundary.

2.4. Symplectic cross-sections. In the case where the moment map Φ is *proper*, a Hamiltonian LG-space with proper moment map behaves very much like a compact Hamiltonian space for a compact group.[1] The reason for this is that the coadjoint LG-action on $L\mathfrak{g}^*$ has finite dimensional slices, and the pre-images of these slices are finite dimensional symplectic submanifolds. To describe these slices, we view the alcove as a subset of $L\mathfrak{g}^*$ as explained above. Let

$$\mathfrak{A}_\sigma := \bigcup_{\tau \succeq \sigma} \tau.$$

Then the flow-out under the action of the compact group $(LG)_\sigma$,

$$U_\sigma = (LG)_\sigma \cdot \mathfrak{A}_\sigma \subset L\mathfrak{g}^*$$

is a slice for the LG-action at points in σ.

  For example, if $G = SU(2)$, then the alcove may be identified with the interval

$$\mathfrak{A} = [0, 1/2].$$

For the three faces $\{0\}, (0, 1/2), \{1/2\}$ we have

$$\mathfrak{A}_{\{0\}} = [0, 1/2), \quad \mathfrak{A}_{(0,1/2)} = (0, 1/2), \quad \mathfrak{A}_{\{1/2\}} = (0, 1/2].$$

---

[1]In fact, there is a 1-1 correspondence between Hamiltonian LG-spaces with proper moment map and compact Hamiltonian G-spaces with G-valued moment maps [**1**].

The slice $Y_{(0,1/2)} = (0, 1/2)$, since $LG_{(0,1/2)} = T$. The other slices $Y_{\{0\}}, Y_{\{1/2\}}$ are open balls of radius $1/2$ in $L\mathfrak{g}^*_{\{0\}}$, resp. $L\mathfrak{g}^*_{\{1/2\}}$. Note that although $L\mathfrak{g}^*_{\{0\}}, L\mathfrak{g}^*_{\{1/2\}}$ are isomorphic as G-modules to the Lie algebra $\mathfrak{g}$, the intersection $L\mathfrak{g}^*_{\{0\}} \cap L\mathfrak{g}^*_{\{1/2\}} = L\mathfrak{g}^*_{(0,1/2)}$.

If M is a symplectic Hamiltonian LG-space with proper moment map Φ, the *symplectic cross-sections*

$$Y_\sigma = \Phi^{-1}(U_\sigma)$$

are finite-dimensional symplectic submanifolds. In fact, they are Hamiltonian $\widehat{(LG)}_\sigma$-manifolds, where the central S^1 acts trivially. The moment maps are the restrictions $\Phi_\sigma = \Phi|Y_\sigma : Y_\sigma \to U_\sigma \subset (L\mathfrak{g})^*_\sigma \subset \widehat{L\mathfrak{g}}^*$. Here $(L\mathfrak{g})^*_\sigma$ is identified with the unique $(LG)_\sigma$-invariant complement to the annihilator of $(L\mathfrak{g})_\sigma$ in $L\mathfrak{g}^*$, or equivalently with the span of U_σ.

For a proof of the symplectic cross-section theorem for loop group actions, see [8]. The flowouts $LG \cdot Y_\sigma = LG \times_{(LG)_\sigma} Y_\sigma$ form an open covering of M. Therefore, the Hamiltonian LG-space (M, ω, Φ) can be reconstructed from its collection of symplectic cross-sections $(Y_\sigma, \omega_\sigma, \Phi_\sigma)$ and the inclusions $Y_\tau \hookrightarrow Y_\sigma$ for $\sigma \prec \tau$.

3. Construction of the canonical bundle.

Suppose (M, ω, Φ) is a Hamiltonian LG-manifold with proper moment map. In this section we contruct an $\widehat{LG}$-equivariant line bundle $K_M \to M$ which will play the role of a canonical line bundle.

For any $\widehat{LG}$-equivariant line bundle $L \to M$, the (locally constant) weight of the action of the central circle $S^1 \subset \widehat{LG}$ is called the *level* of L. Any $\widehat{LG}$-bundle $L \to M$ is determined by the collection of $(\widehat{LG})_\sigma$-equivariant line bundles $L_\sigma \to Y_\sigma$ over the cross-sections, together with $(LG)_\tau$-equivariant isomorphisms $\varphi_{\sigma,\tau} : L_\sigma|Y_\tau \cong L_\tau$ for all $\sigma \prec \tau$, such that

$$(5) \qquad \varphi_{\sigma,\tau} \circ \varphi_{\tau,\nu} = \varphi_{\sigma,\nu}$$

if $\sigma \preceq \tau \preceq \nu$.

Let $K_\sigma \to Y_\sigma$ be the canonical line for some invariant compatible almost complex (a.c.) structure on Y_σ. There exist $(LG)_\tau$-equivariant isomorphisms

$$(6) \qquad K_\sigma|Y_\tau \cong K_\tau \otimes \det_{\mathbb{C}}(\nu^\sigma_\tau)^*$$

where $\nu^\sigma_\tau \to Y_\tau$ is the symplectic normal bundle to Y_τ inside Y_σ. We will therefore begin by describing the complex structure on ν^σ_τ.

3.1. The normal bundle of Y_τ in Y_σ. Suppose $\sigma \prec \tau$ so that Y_τ is an $(LG)_\tau$-invariant submanifold of $(LG)_\sigma$. Since $(LG)_\sigma \times_{(LG)_\tau} Y_\tau$ is an open subset of Y_σ, the normal bundle of Y_τ in Y_σ is $(LG)_\tau$-equivariantly

isomorphic to the trivial bundle $(L\mathfrak{g})_\sigma/(L\mathfrak{g})_\tau$. It carries a unique $(LG)_\tau$-invariant complex structure compatible with the symplectic structure. In terms of the root space decomposition this complex structure is given as follows. Given a face σ of $\mathfrak{A}$, define the positive Weyl chamber $\mathfrak{t}_{+,\sigma}$ for $(LG)_\sigma$ as the cone over $\mathfrak{A} - \mu$, for any $\mu \in \sigma$. Similarly define $\mathfrak{t}_{+,\tau}$. Let $\mathfrak{R}_{+,\sigma} \supset \mathfrak{R}_{+,\tau}$ the corresponding collections of positive roots.

As complex $(\widehat{LG})_\tau$-representations,

$$(L\mathfrak{g})_\sigma/(L\mathfrak{g})_\tau = \bigoplus_{\alpha \in \mathfrak{R}_{+,\sigma} \setminus \mathfrak{R}_{+,\tau}} \mathbb{C}_\alpha.$$

In particular,

(7) $$\det{}_{\mathbb{C}}(\nu_\tau^\sigma)^* = \bigotimes_{\alpha \in \mathfrak{R}_{+,\sigma} \setminus \mathfrak{R}_{+,\tau}} \mathbb{C}_\alpha = \mathbb{C}_{-2(\rho_\sigma - \rho_\tau)},$$

where ρ_σ, ρ_τ are the half-sums of positive roots of $\mathfrak{R}_{+,\sigma}, \mathfrak{R}_{+,\tau}$ respectively.

3.2. Compatibility condition. Our candidate for $L_\sigma = (K_M)|Y_\sigma$ will be of the form $K_\sigma \otimes \mathbb{C}_{\gamma_\sigma}$, for suitable weights $\gamma_\sigma \in \Lambda^* \times \mathbb{Z}$. The key point which makes the problem non-trivial is that in order for $\mathbb{C}_{\gamma_\sigma}$ to give $\widehat{LG}_\sigma$-representations, the weight γ_σ should be fixed under the $(\widehat{LG})_\sigma$-action on $\widehat{L\mathfrak{g}}^*$. According to (6) and (7) these weights should satisfy

$$\gamma_\sigma - \gamma_\tau = 2(\rho_\sigma - \rho_\tau)$$

for all faces $\sigma \prec \tau$.

The following Lemma gives a solution to this system of equations.

Lemma 3.1. *For all faces $\sigma \subset \mathfrak{A}$, the difference $2\rho - 2\rho_\sigma \in \Lambda^*$ is the orthogonal projection of 2ρ to the affine span of the dilated face $2c\sigma$. In particular the weight*

$$\gamma_\sigma := -(2\rho - 2\rho_\sigma, 2c) \in \Lambda^* \times \mathbb{Z}$$

is fixed under $(\widehat{LG})_\sigma$.

Proof. The weight $2\rho_\sigma$ is characterized by the property

$$2\rho_\sigma \cdot \alpha = \alpha \cdot \alpha$$

for every simple root α of $(LG)_\sigma$. Letting $\{\alpha_1, \dots, \alpha_l\}$ be the simple roots for G, the simple roots for $(LG)_\sigma$ are precisely those roots in the collection $\{\alpha_1, \dots, \alpha_l, -\alpha_0\}$ which are perpendicular to the span of $\sigma - \mu$ (where $\mu \in \sigma$). In particular $-\alpha_0$ is a simple root for $(LG)_\sigma$ precisely if $0 \notin \overline{\sigma}$.

If $\alpha \in \{\alpha_1, \dots, \alpha_l\}$ is a simple root of $(LG)_\sigma$ then $2\rho \cdot \alpha = 2\rho_\sigma \cdot \alpha = \alpha \cdot \alpha$ so that $(2\rho - 2\rho_\sigma) \cdot \alpha = 0$. If $0 \notin \overline{\sigma}$ so that $-\alpha_0$ is among the set of simple roots for $(LG)_\sigma$, we also have

$$(2\rho - 2\rho_\sigma) \cdot \alpha_0 = 2(c - 1) + \alpha_0 \cdot \alpha_0 = 2c,$$

as required. □

The solution given by the lemma is unique, since for $\sigma = \{0\}$ the group $LG_\sigma = G$ has the unique fixed point $\gamma_0 = (0, -2c)$.

3.3. Gluing. Let $L_\sigma = K_\sigma \otimes \mathbb{C}_{\gamma_\sigma}$. We still have to construct isomorphisms $\varphi_{\sigma,\tau} : L_\sigma|Y_\tau \to L_\tau$ satisfying the cocycle condition. If the compatible a.c. structures on Y_σ can be chosen in such a way that for $\sigma \prec \tau$, Y_τ is an a.c. submanifold of Y_σ, the isomorphisms would be canonically defined and the cocycle condition would be automatic. Unfortunately, it is in general impossible to choose the a.c. structures to have this property.

To get around this difficulty we replace the sets Y_σ with smaller open subsets. The compact set M/LG is covered by the collection of sets $Y_\sigma/(LG)_\sigma$ with σ a vertex of $\mathfrak{A}$, since $\mathfrak{A}$ is covered by the (relative) open subsets $\mathfrak{A}_\sigma$. It is therefore possible to choose for each vertex σ of $\mathfrak{A}$, an $(LG)_\sigma$-invariant, open subset $Y'_\sigma \subset Y_\sigma$, such that the collection of these subsets has the following two properties:

  a. The collection of all $Y'_\sigma/(LG)_\sigma$ covers M/LG.
  b. The closure of Y'_σ is contained in Y_σ.

Given such a collection of subsets $\{Y'_\sigma\}$ we define, for any open face τ of $\mathfrak{A}$,

$$Y'_\tau = \bigcap_{\sigma \preceq \tau, \, \dim \sigma = 0} Y'_\sigma.$$

Then Y'_τ is an $(LG)_\tau$-invariant open subset of Y_τ, with the property that its closure in M is contained in Y_τ.

Lemma 3.2. *There exists a collection of $(LG)_\sigma$-invariant compatible a.c. structures on the collection of Y'_σ, with the property that for all $\sigma \preceq \tau$, the embedding $Y'_\tau \hookrightarrow Y'_\sigma$ is a.c.. Moreover, any two a.c. structures on the disjoint union $\coprod_\sigma Y'_\sigma$ with the required properties are homotopic.*

Proof. We construct a.c. structures J_σ on Y'_σ with the required properties by induction over dimension of the faces σ, starting from the interior of the alcove $\mathfrak{A}$ and ending at vertices.

Given $k \geq 0$, suppose that we have constructed compatible a.c. structures on all Y_σ with $\dim \sigma > \dim \mathfrak{t} - k$, in such a way that if $\sigma \preceq \tau$, the embedding $Y_\tau \hookrightarrow Y_\sigma$ is a.c. on some open neighborhood of the closure of Y'_τ. Let ν be a face of dimension $\dim \mathfrak{t} - k$. Each of the a.c. structures on Y_τ with $\tau \succ \nu$ defines an invariant compatible a.c. structure on Y_ν, and by hypothesis these complex structures match on some open neighborhood of $\bigcup_{\nu \prec \tau}(LG)_\nu \cdot \overline{Y'_\tau}$. We choose an invariant a.c. structure on Y_ν such that it matches with the given a.c. structures over a possibly smaller open neighborhood of $\bigcup_{\nu \prec \tau}(LG)_\nu \cdot \overline{Y'_\tau}$. This can be done by choosing a Riemannian metric on Y_ν which matches the given one in a possibly smaller neighborhood, and

taking the compatible almost complex structure defined by the metric in the standard way (see e.g. [**6**]).

Now let $\{J_\sigma^0\}, \{J_\sigma^1\}$ be two collections of a.c. structures with the required properties. They define Riemannian metrics g_σ^0, g_σ^1. Let $g_\sigma^t = (1-t)g_\sigma^0 + t\, g_\sigma^1$, and let J_σ^t be the compatible a.c. structure which it defines. For $\sigma \prec \tau$, the metric g_τ^t on Y_τ' is the restriction of g_σ^t and the symplectic normal bundle of Y_τ' in Y_σ' coincides with the Riemannian normal bundle. This implies that the embedding $Y_\tau' \to Y_\sigma'$ is a.c..   □

Choose a.c. structures on Y_σ as in the Lemma, and define $(\widehat{LG})_\sigma$-equivariant line bundles $L_\sigma' = K_\sigma' \otimes \mathbb{C}_{\gamma_\sigma}$. We then have canonical isomorphisms

$$\phi_{\sigma,\tau} : \ L_\sigma' | Y_\tau' = L_\tau'$$

and they automatically satisfy the cocycle condition. It follows that there is a unique $\widehat{LG}$-equivariant line bundle $K_M \to M$ with $K_M | Y_\sigma' = L_\sigma'$. By construction, the collection of line bundles L_σ', hence also K_M, is independent of the choice of a.c. structures up to homotopy.

Lemma 3.3. *The isomorphism class of K_M is independent of the choice of "cover" Y_σ'.*

Proof. Given two choices Y_σ^1 and Y_σ^2 labeled by the vertices of $\mathfrak{A}$, let $Y_\sigma^3 = Y_\sigma^1 \cup Y_\sigma^2$. Given a.c. structures J_σ^j on Y_σ^j and the canonical line bundles K_M^j constructed from them, we have an equivariant homotopy $K_M^1 \sim K_M^3 \sim K_M^2$ (because J_σ^3 restricts to a.c. structures on Y_σ^1 and Y_σ^2).   □

This completes our construction of the canonical bundle. The central circle in $\widehat{LG}$ acts with weight $-2c$, that is, K_M is a line bundle at level $-2c$.

3.4. Examples.

3.4.1. Coadjoint orbits. Let $M = LG \cdot \mu$ be the coadjoint orbit through $\mu \in \mathfrak{A}$, and let $\sigma \subset \mathfrak{A}$ denote the open face containing μ. Thus $M \cong LG/(LG)_\sigma$. Since $Y_\sigma = \{\mu\}$, the canonical line bundle K_M is the associated bundle

$$(8) \qquad\qquad K_{LG/(LG)_\sigma} := \widehat{LG} \times_{(\widehat{LG})_\sigma} \mathbb{C}_{-2(\rho-\rho_\sigma,c)}.$$

This definition of canonical bundle agrees with Freed's computation [**4**] of a regularized first Chern class of the fundamental homogeneous space $\Omega G = LG/G$. In this paper, Freed provides further evidence for this being the correct definition of a first Chern class, the simplest being that since $\hat{\rho} = (\rho, c)$ is the sum of fundamental affine weights (cf. [**9**]), the canonical bundle for LG/T is expected to be $K_{LG/T} = \widehat{LG} \times_{\widehat{T}} \mathbb{C}_{-2\hat{\rho}}$. and that for LG/G should be $\widehat{LG} \times_{\widehat{G}} \mathbb{C}_{-2(0,c)}$.

Since $LG/(LG)_\sigma$ is a homogeneous space the canonical line bundle carries a unique $\widehat{LG}$-invariant connection. Its curvature equals $-2\pi i$ times the symplectic form for the coadjoint orbit (at level $-2c$) through $-2(\rho-\rho_\sigma,\, c) = -\gamma_\sigma$. Recall that $(\rho-\rho_\sigma)/c \in \mathfrak{A}$ is the orthogonal projection of ρ/c onto the affine subspace spanned by σ. Therefore:

Lemma 3.4. *If (M,ω) is the coadjoint LG-orbit (at level 1) through the orthogonal projection μ of ρ/c onto some face σ of $\mathfrak{A}$, the curvature of the canonical line bundle is given by $\frac{i}{2\pi}\operatorname{curv}(K_M) = -2c\omega$. In particular, this is true for $\mu = \rho$ and for μ a vertex of $\mathfrak{A}$.*

3.4.2. Moduli spaces of flat connections. Let Σ be a compact, oriented surface with boundary $\partial\Sigma \cong (S^1)^b$ and $(\mathcal{M}(\Sigma),\omega)$ the corresponding moduli space. From now on, we assume that $b = 1$, although the more general case is only more difficult notationally. By Corollary 3.12 of [**7**] there is a unique $\widehat{LG}$-equivariant line bundle at each level, so that every $\widehat{LG}$-equivariant line bundle over $\mathcal{M}(\Sigma)$ at level k is isomorphic to the kth tensor power of the pre-quantum line bundle $L(\Sigma)$.[2] In particular the canonical bundle $K_{\mathcal{M}(\Sigma)} \to \mathcal{M}(\Sigma)$ carries an invariant connection such that $\frac{i}{2\pi}\operatorname{curv}(K_{\mathcal{M}(\Sigma)}) = -2c\omega$.

4. Quotients of canonical bundles.

In this section, we show that the bundles K_M behave well under symplectic quotients, that is, that the symplectic quotient of K_M is the usual canonical bundle on the quotient. For any Hamiltonian LG-space (M,ω,Φ) with proper moment map, and any coadjoint LG-orbit $\mathcal{O} \subset L\mathfrak{g}^*$, the reduced space $M_\mathcal{O}$ at level $\mathcal{O}$ is a compact space defined as the quotient

$$M_\mathcal{O} := \Phi^{-1}(\mathcal{O})/LG.$$

Let $\mu \in \mathfrak{A}$ is the point of the alcove through which $\mathcal{O}$ passes, σ the open face containing μ, and

$$\mathcal{O}_\sigma := \mathcal{O} \cap U_\sigma = (LG)_\sigma \cdot \mu.$$

Then

$$M_\mathcal{O} = \Phi^{-1}(\mu)/(LG)_\sigma = (Y_\sigma)_{\mathcal{O}_\sigma}$$

which identifies $M_\mathcal{O}$ as a reduced space of the symplectic cross-section $(Y_\sigma, \omega_\sigma, \Phi_\sigma)$. It follows that the standard theory of symplectic reduction applies: If μ is a regular value then $M_\mathcal{O}$ is a finite dimensional symplectic orbifold, and in general it is a finite dimensional stratified symplectic space in the sense of Sjamaar-Lerman [**12**].

---

[2]A sketch of the argument is as follows: Two line bundles at the same level differ by a line bundle at level 0, which descends to the quotient $\mathcal{M}(\Sigma)/\Omega G$ by the based loop group. From the holonomy description of the moduli space we have $\mathcal{M}(\Sigma)/\Omega G \cong G^{2g}$. Since $H^2_G(G^{2g})$ is trivial, the descended line bundle is trivial, so the two line bundles are isomorphic.

Over the level set $\Phi^{-1}(\mathcal{O})$ we have two line bundles at level $-2c$, the restriction of the canonical bundle of M and the pull-back by Φ of the canonical bundle $K_\mathcal{O}$ on the coadjoint orbit. They differ by an LG-equivariant line bundle (that is an $\widehat{LG}$-bundle at level 0),

$$K_M|_{\Phi^{-1}(\mathcal{O})} \otimes K_\mathcal{O}^*.$$

Proposition 4.1. *Suppose $\mathcal{O}$ consists of regular values of Φ. The canonical line bundle for the reduced space $M_\mathcal{O}$ is the quotient,*

$$\left(K_M|_{\Phi^{-1}(\mathcal{O})} \otimes \Phi^* K_\mathcal{O}^*\right)/LG.$$

Proof. Since

$$K_M = \widehat{LG} \times_{(\widehat{LG})_\sigma} (K_\sigma \otimes \mathbb{C}_{\gamma_\sigma}), \quad K_\mathcal{O} = \widehat{LG} \times_{(\widehat{LG})_\sigma} (K_{\mathcal{O}_\sigma} \otimes \mathbb{C}_{\gamma_\sigma})$$

we have

$$K_M|_{\Phi^{-1}(\mathcal{O})} \otimes \Phi^* K_\mathcal{O}^* = \widehat{LG} \times_{(\widehat{LG})_\sigma} (K_\sigma \otimes \Phi_\sigma^* K_{\mathcal{O}_\sigma}^*).$$

Taking the quotient by LG we obtain

$$\left(K_M|_{\Phi^{-1}(\mathcal{O})} \otimes \Phi^* K_\mathcal{O}^*\right)/LG = \left(K_\sigma|_{\Phi_\sigma^{-1}(\mathcal{O}_\sigma)} \otimes \Phi_\sigma^* K_{\mathcal{O}_\sigma}^*\right)/(LG)_\sigma$$

which is the canonical bundle for the reduced space $(Y_\sigma)_{\mathcal{O}_\sigma} = M_\mathcal{O}$. $\qquad\square$

Theorem 4.2. *Let $\mathcal{M}(\Sigma)$ be the moduli space of flat connections on a compact oriented surface with boundary, and $\mathcal{C}_\mu$ the conjugacy class corresponding to the projection μ of ρ/c onto σ for some face σ. Suppose μ is a regular value for the moment map $\mathcal{M}(\Sigma)$, so $\mathcal{M}(\Sigma, \mathcal{C}_\mu)$ the moduli space of flat connections with holonomy in $\mathcal{C}_\mu$ is a compact symplectic orbifold. Then the Chern class $c_1(K_M)$ for $M = \mathcal{M}(\Sigma, \mathcal{C}_\mu)$ is $-2c$ times the cohomology class of the reduced symplectic form.*

Proof. Let $\mathcal{O}$ be the a coadjoint orbit through the element ρ_σ/c. By Section 3.4, $K_{M(\Sigma)}$ resp. $K_\mathcal{O}$ are isomorphic to the $-2c$-th tensor power of the pre-quantum line bundles on $\mathcal{M}(\Sigma)$ resp. $\mathcal{O}$. By Proposition 4.1, the canonical line bundle on the quotient is isomorphic to the $-2c$-th power of the quotient of the pre-quantum line bundle on the product, which is a pre-quantum line bundle on the quotient. $\qquad\square$

References

[1] A. Alekseev, A. Malkin and E. Meinrenken, *Lie group valued moment maps*, J. Differential Geom., **48**(**3**) (1998), 445-495.

[2] A. Alekseev, E. Meinrenken and C. Woodward, *A fixed point formula for loop group actions*, Journal of Symplectic Geometry, to appear.

[3] A. Beauville, Y. Laszlo and Ch. Sorger, *The Picard group of the moduli of G-bundles on a curve*, Compositio Math., to appear.

[4] D.S. Freed, *The geometry of loop groups*, J. Differential Geom., **28** (1988), 223-276.

[5] S. Kumar and M.S. Narasimhan, *Picard group of the moduli space of G-bundles*, Math. Ann., **308** (1997), 155-173.

[6] D. McDuff and D. Salamon, *Introduction to Symplectic Topology*, Oxford Mathematical Monographs, Oxford University Press, New York, 1995.

[7] E. Meinrenken and C. Woodward, *Cobordism for Hamiltonian loop group actions and flat connections on the punctured two-sphere*, Mathematische Zeitschrift, **231** (1999), 133-168.

[8] _____ , *Hamiltonian loop group actions and Verlinde factorization*, Journal of Differential Geometry, **50** (1999), 417-470.

[9] A. Pressley and G. Segal, *Loop Groups*, Oxford University Press, Oxford, 1988.

[10] D. Quillen, *Determinants of Cauchy Riemann operators on Riemann surfaces*, Funktsional. Anal. i Prilozhen., **19**(**1**) (1985), 37-41, 96.

[11] S. Ramanan, *The moduli spaces of vector bundles over an algebraic curve*, Math. Ann., **200** (1973), 69-84.

[12] R. Sjamaar and E. Lerman, *Stratified symplectic spaces and reduction*, Ann. of Math. (2), **134** (1991), 375-422.

[13] P.G. Zograf and L.A. Takhtadzhyan, *The geometry of moduli spaces of vector bundles over a Riemann surface*, Izv. Akad. Nauk SSSR Ser. Mat., **53**(**4**) (1989), 753-770, 911.

UNIVERSITY OF TORONTO
DEPARTMENT OF MATHEMATICS
100 ST GEORGE STREET, TORONTO, ONTARIO M5S 3G3
CANADA
E-mail address: mein@math.toronto.edu

MATHEMATICS-HILL CENTER
RUTGERS UNIVERSITY
110 FRELINGHUYSEN ROAD, PISCATAWAY NJ 08854-8019
E-mail address: ctw@math.rutgers.edu

SUPERSINGULAR PRIMES AND p-ADIC L-FUNCTIONS

Luis Manuel Navas Vicente

We discuss the problem of finding a p-adic L-function attached to an elliptic curve with complex multiplication over an imaginary quadratic field K, for the case of a prime where the curve has *supersingular* reduction. While the case of primes of ordinary reduction has been extensively studied and is essentially understood, yielding many deep and interesting results, basic questions remain unanswered in the case of supersingular reduction. We will discuss a conjecture, related to another in Rubin, 1987, and some ideas related to the problem in general. The basic tools originate with the work of J. Coates and A. Wiles in 1977 and 1978, and are developed in the work of K. Rubin.

1. Set-up.

The analytic theory of L-functions and arithmetic properties of their special values goes back to the 19th-century work of Kummer on the arithmetic of cyclotomic fields. His congruences for Bernoulli numbers were re-cast more than a century later as the p-adic interpolation of Riemann's Zeta Function and Dirichlet L-series, whose known special values are basically Bernoulli numbers. Kummer himself introduced logarithmic differentiation modulo a prime p and the use of cyclotomic units as a method of uncovering the rich arithmetic structure of cyclotomic fields. In the modern theory, these classical p-adic L-functions arise as a relation between the $\mathbb{Z}_p[[t]]$-module of cyclotomic units and that of local p-adic units. The element relating them is essentially the interpolating L-function, and the precise interpolation result is obtained by a suitable logarithmic differentiation homomorphism. The theory generalizes to the arithmetic of abelian extensions of imaginary quadratic fields via the consideration of an elliptic curve as the arithmetic object. Technical complications arise at primes p which do not split in the quadratic extension, and relatively few results are known compared to the ordinary split case. The main objective of this paper is to suggest a way (§2) by which interesting two-variable p-adic L-functions may arise from an elliptic curve with CM, at primes of supersingular reduction.

The relative complexity of the method hinges on the relation between the arithmetic "elliptic" units and p-adic local units in the supersingular case,

to the author's knowledge as yet unclarified, and perhaps worthy of separate interest in itself. Propositions 2.1 and 2.2 contain preliminary suggestions on this problem. Theorem 5.5 expresses the L-values which we believe should be interpolated by a "supersingular" p-adic L-function, in terms of p-adic logarithmic derivatives on elliptic units. These are values twisted by a character of p-power order. §6 generalizes this to higher-order derivations of a two-variable formal power series, showing how the p-character and the local grossencharacter act together. Finally, Theorems 7.4 and 7.6 are local computations with logarithmic derivatives analogous to those done by Coates and Wiles in [1, 2] for primes of ordinary reduction, hopefully of use to those who may wish to obtain explicit results on the p-adic growth properties of L-values, for example. We prove the relevant properties of the logarithmic differentiation homomorphisms used for these computations. The form of these results given supersingular reduction is similar to, but rather less transparent than in the ordinary case, as far as taking p-adic valuations is concerned.

Let E be an elliptic curve over an imaginary quadratic field K, with complex multiplication by the ring of integers $\mathcal{O}_K$. The following notation is standard. Let ψ be the Hecke grossencharacter attached to E, and $\mathfrak{f}$ its conductor. Pick a prime $\mathfrak{p}$ of K not dividing $6\mathfrak{f}$, and let $p \neq 2, 3$ be the prime of $\mathbb{Z}$ below $\mathfrak{p}$. Assume that p remains prime in K. This implies that E has good supersingular reduction at $\mathfrak{p}$. Let $\pi = \psi(\mathfrak{p})$. This is the unique generator of $\mathfrak{p}$ that reduces to Frobenius modulo $\mathfrak{p}$. Note that p and π differ only by a unit of $\mathcal{O}_K$.

Consider for $n \geq 0$ the abelian extensions $K(E_{\pi^{n+1}})/K$ obtained by adjoining the coordinates of the $\mathfrak{p}^{n+1}$-division points on E. Define $E_{\pi^\infty} = \bigcup_{n\geq 0} E_{\pi^{n+1}}$ and consider the Galois groups $G_n = G(K(E_{\pi^{n+1}})/K)$ $G_\infty = G(K(E_{\pi^\infty})/K)$. Denote by $K_\mathfrak{p}$ the completion of K at $\mathfrak{p}$ and by $\mathcal{O}_\mathfrak{p}$ its local ring of integers. We use the same symbol $\mathfrak{p}$ for the prime ideal of $\mathcal{O}_\mathfrak{p}$. Let $\bar{K}_\mathfrak{p}$ be a fixed algebraic closure of $K_\mathfrak{p}$. Let $K_n = K_\mathfrak{p}(E_{\pi^{n+1}})$ and $K_\infty = \bigcup_{n=0}^\infty K_n = K_\mathfrak{p}(E_{\pi^\infty})$. One has canonically $G_n = G(K_n/K_\mathfrak{p})$ and $G_\infty = G(K_\infty/K_\mathfrak{p})$. The structure of these extensions is described by the theory of Lubin-Tate formal groups. This is a very useful fact, since all Lubin-Tate formal groups over $\mathcal{O}_\mathfrak{p}$ are isomorphic and one can choose among them one well suited for computations. This idea is illustrated in [1, 2].

In our case, the hypothesis of supersingular reduction is equivalent to this formal group having height 2 and not 1 as in the case of ordinary reduction.

The p-part of the grossencharacter corresponds to the character $\kappa : G_\infty \to \mathcal{O}_\mathfrak{p}^*$ which gives the action of G_∞ on p-power division points of any of the Lubin-Tate formal groups associated to π over $\mathcal{O}_\mathfrak{p}$. If $\mathcal{E}$ is such a group, then

$$(1) \qquad\qquad \omega^\sigma = [\kappa(\sigma)](\omega) \quad \forall \omega \in \mathcal{E}_{p^\infty}, \quad \sigma \in G_\infty,$$

where $[\alpha]$ is the $\mathcal{O}_{\mathfrak{p}}$-endomorphism of $\mathcal{E}$ corresponding to $\alpha \in \mathcal{O}_{\mathfrak{p}}$. κ establishes isomorphisms $G_n \cong \mathcal{O}_{\mathfrak{p}}^*/(1+\mathfrak{p}^{n+1}\mathcal{O}_{\mathfrak{p}}) \cong \mu_{q-1} \times (1+\mathfrak{p}\mathcal{O}_{\mathfrak{p}})/(1+\mathfrak{p}^{n+1}\mathcal{O}_{\mathfrak{p}})$, and $G_\infty \cong \mathcal{O}_{\mathfrak{p}}^* \cong \mu_{q-1} \times (1 + \mathfrak{p}\mathcal{O}_{\mathfrak{p}})$ where $q = p^2$ in the supersingular case. These correspond to the decompositions $G_n \cong \Delta \times \Gamma_n$ $G_\infty \cong \Delta \times \Gamma_\infty$ where $\Delta = G_0 = G(K_{\mathfrak{p}}(E_{\mathfrak{p}})/K_{\mathfrak{p}}), \Gamma_n = G(K_n/K_0), \Gamma_\infty = G(K_\infty/K_0)$. In the case of supersingular reduction we have $\kappa : \Gamma_\infty \cong 1 + \mathfrak{p}\mathcal{O}_{\mathfrak{p}} \cong \mathbb{Z}_p^2$. These are therefore Iwasawa $\mathbb{Z}_p^2$ extensions, not $\mathbb{Z}_p$ extensions as in the ordinary case, which complicates matters. We let κ_0 be the restriction of κ to $\Delta = G_0$. It establishes an isomorphism $\Delta \cong \mu_{q-1}$.

Let $\Lambda = \mathbb{Z}_p[[G_\infty]] = \varprojlim \mathbb{Z}_p[G_n]$ be the Iwasawa algebra. Let ρ be, as in [5], the $\mathbb{Z}_p$-representation of Δ that reduces modulo p to the $\mathbb{F}_p$-representation of Δ giving the action on $E_{\mathfrak{p}}$. Lemma 11.5 of [5] shows that this is an irreducible representation, and in the supersingular case, its degree is 2. In particular, $\Lambda^\rho = \mathcal{O}_{\mathfrak{p}}[[\Gamma_\infty]] \cong \mathcal{O}_{\mathfrak{p}}[[S,T]]$ since $\Gamma_\infty \cong \mathbb{Z}_p^2$, although the isomorphism is not canonical, depending on a choice of topological generators for Γ_∞. For this reason p-adic L-functions in the supersingular case will be 2-variable L-functions.

We need the following facts. If $*$ denotes the action of the non-trivial automorphism of $K_{\mathfrak{p}}/\mathbb{Q}_p$, then $\rho \cong \kappa_0 \oplus \kappa_0^*$ over $K_{\mathfrak{p}}$ and $\kappa_0^* = \kappa_0^p$ because $*$ gives the Frobenius element of $K_{\mathfrak{p}}/\mathbb{Q}_p$, and p is inert.

2. Iwasawa Structure of Local Units.

Let U_n be the group of units of K_n congruent to 1 modulo the unique prime ideal above p, and C_n the closure in K_n of the group of Robert elliptic units of K_n. One has $C_n \subseteq U_n$. Let $U_\infty = \varprojlim U_n$ and $C_\infty = \varprojlim C_n$, where the limits are with respect to the norm maps. In [5], Lemma 11.9, it is shown that $U_\infty^\rho \cong (\Lambda^\rho)^2$ and $C_\infty^\rho \cong \Lambda^\rho$. Furthermore, one can decompose U_∞^ρ into a direct sum

$$(2) \qquad\qquad U_\infty^\rho = U_1 \oplus U_2$$

such that $\delta(U_1) = \mathcal{O}_{\mathfrak{p}}$ and $\delta(U_2) = 0$, where δ is the "reciprocity law map" $\delta : U_\infty^\rho \to \mathcal{O}_{\mathfrak{p}}$. δ is a "κ-homomorphism," meaning $\delta(u^\sigma) = \kappa(\sigma)\delta(u) \quad \forall \sigma \in G_\infty$, and δ maps Λ^ρ-submodules of U_∞^ρ to ideals of $\mathcal{O}_{\mathfrak{p}}$. (See [5], Prop. 11.7.)

We come now to a problem of central interest. In [8] it was stated that one could choose a decomposition as in (2) in which C_∞^ρ would be contained in one of the two free components U_1, U_2. The truth of this statement seems still not to be known at this time. We will refer to this conjecture as (C).

If (C) is true, then a generator c of C_∞^ρ and a generator u of the free component that C_∞^ρ would lie in are related by $c = f \cdot u$, where f is an element of Λ^ρ; now f can be viewed as a power series in two variables with

coefficients in $\mathcal{O}_\mathfrak{p}$. This would be a natural candidate for a two-variable $\mathfrak{p}$-adic L-function, since this procedure is completely analogous to the way one-variable p-adic L-functions arise in the ordinary case and in the "classical" case over $\mathbb{Q}$.

The question (see [8]) would then also be to find a generator u of sufficiently explicit form that the Coates-Wiles logarithmic differentiation map and its generalizations, which yield L-values when applied to elliptic units, yield a sufficiently explicit factor when applied to u. This is what Coates and Wiles do in the ordinary case [1, 2], using the basic Lubin-Tate formal group. This in turn leads to an understanding of the p-adic interpolation properties of those L-values. Over $\mathbb{Q}$ this theory gives the classical congruences of Kummer, Clausen and Von Staudt.

We study to what extent a decomposition of U_∞^ρ as in (2) can be "perturbed."

Proposition 2.1. *If $u \in U_\infty^\rho$, then:*
  (i) *$\delta(\Lambda^\rho u) = \mathcal{O}_\mathfrak{p}$ if and only if $\delta(u) \not\equiv 0 \bmod \pi$.*
  (ii) *If $u_1, u_2 \in U_\infty^\rho$ with $\Lambda^\rho u_2 \subseteq \Lambda^\rho u_1$, and $\delta(u_1), \delta(u_2) \neq 0$, then $\Lambda^\rho u_2 = \Lambda^\rho u_1$ if and only if $\mathrm{ord}_p(\delta(u_2)) = \mathrm{ord}_p(\delta(u_1))$.*

Proof. (i) is straightforward from the properties of δ. In general, $\delta(\Lambda^\rho u) = \mathcal{O}_\mathfrak{p}\delta(u)$. For (ii), write $u_2 = f \cdot u_1$ and apply δ. Since $\delta(u_2) = f(\kappa(\gamma_1) - 1, \kappa(\gamma_2) - 1)\delta(u_1)$, (see [1]), $\Lambda^\rho u_2 = \Lambda^\rho u_1$ if and only if f is a unit in Λ^ρ, and this is so if and only if $f(0,0)$ is a unit at p. Since $f(\kappa(\gamma_1) - 1, \kappa(\gamma_2) - 1) \equiv f(0,0) \bmod \pi$, we see that this is the case if and only if the quotient of $\delta(u_1), \delta(u_2)$ is a unit at p. $\qquad\square$

Proposition 2.2. *Suppose we have $U_\infty^\rho = U_1 \oplus U_2$ with $\delta(U_2) = 0$, and hence $\delta(U_1) = \mathcal{O}_\mathfrak{p}$. Let $u \in U_\infty^\rho$ such that $\delta(u) \in \mathcal{O}_\mathfrak{p}^*$. Using additive notation, let $u = u_1 + u_2$, with $u_1 \in U_1, u_2 \in U_2$. Then $U_1 = \Lambda^\rho u_1$ and $U_\infty^\rho = \Lambda^\rho u \oplus U_2$.*

Proof. For the first part, note that $\delta(u_1) = \delta(u) \in \mathcal{O}_\mathfrak{p}^*$ and, since $\Lambda^\rho u_1 \subseteq U_1$, by the remarks above, equality must hold. As for the second, clearly $u_1 = u - u_2 \in \Lambda^\rho u + U_2$, therefore $U_1 \subset \Lambda^\rho u + U_2$, and hence $U_\infty^\rho = \Lambda^\rho u + U_2$. The sum is direct: If $v \in \Lambda^\rho u \cap U_2$ then $v = f \cdot u = v_2$ for some $f \in \Lambda^\rho$ and $v_2 \in U_2$. Thus $f \cdot u_1 = v_2 - f \cdot u_2 \in U_1 \cap U_2 = 0$, and so $f = 0$ and $v_2 = 0$. $\qquad\square$

Hence if we find an element u in U_∞^ρ such that $\delta(u)$ is a unit, and we start with a given decomposition $U_\infty^\rho = U_1 \oplus U_2$, where $\delta(U_2) = 0$, then we can replace our U_1 with $\Lambda^\rho u$ (i.e., assume that U_1 is generated by u) without changing U_2. There is then a relation $c = f \cdot u + \widetilde{f} \cdot v$, where c generates C_∞^ρ, $\delta(v) = 0$ and $f, \widetilde{f}$ are two-variable power series with coefficients in $\mathcal{O}_\mathfrak{p}$.

A natural u having a particularly "simple" form was already used by Wiles in [2] for the ordinary case, and works also in the supersingular case.

The more explicit the evaluation of the Coates-Wiles derivations on u, v is, the more explicit the relation becomes.

If (C) holds, then the second term with $\tilde{f}$ and v disappears. If (C) is false, then one must also study the "extra factor" v. We know that $\delta(v) = 0$, but the generalized δ-maps need not vanish at v. The nature of these is connected with explicit reciprocity laws. If (C) is true, this would raise the further question (C′) of whether the elliptic units C_∞^ρ lie in the free component U_1 having as generator the "special" sequence of local units u discovered by Coates and Wiles, in which case we would get an explicit relation of the form $f(*, *) = (L - \text{value}) \cdot (\text{explicit factors})$, but this may be too good to be true. Nevertheless, see [9] for a different approach to this problem and evidence that in any case makes investigation of the problem interesting.

3. The Basic Lubin-Tate Formal Group.

See [3] for details or proofs of the following facts. The basic Lubin-Tate formal group associated to π is the formal group $\mathcal{E}$ in which multiplication by π is given by the polynomial $[\pi](X) = \pi X + X^q$. It is the simplest series over $\mathcal{O}_\mathfrak{p}$ satisfying the Lubin-Tate conditions $f(X) \equiv X \mod X^2$ and $f(X) \equiv \pi X \mod \mathfrak{p}$, and is simpler to work with computationally. In general we let $[\alpha]$ denote the power series representing the $\mathcal{O}_\mathfrak{p}$-endomorphism of $\mathcal{E}$ given by the action of α.

Let $\mathbf{N}_{m,n}, \mathbf{T}_{m,n}, \mathbf{N}_n, \mathbf{T}_n$ represent the norm and trace maps from K_m to K_n and from K_n to $K_\mathfrak{p}$ respectively. Let $\dotplus$ denote addition in $\mathcal{E}$ and λ the logarithm (normalized isomorphism with the additive formal group $\mathbb{G}_a$).

We fix a generator (ω_n) of the Tate module, that is, a sequence with ω_n in the ring of integers of K_n such that $[\pi](\omega_{n+1}) = \omega_n$ for all $n \geq 0$. Then $K_n = K_\mathfrak{p}(\omega_n)$ and in fact this sequence is also norm compatible: $N_{n+1,n}(\omega_{n+1}) = \omega_n$.

If $u = (u_n)_{n \geq 0} \in U_\infty$, denote by g_u the Coleman power series associated to u, that is, the unique series $g_u \in \mathcal{O}_\mathfrak{p}[[T]]^*$ such that $g_u(\omega_n) = u_n$ for all $n \geq 0$. For σ in G_∞, given the definition of κ, we have the relation $g_{u^\sigma} = g_u \circ [\kappa(\sigma)]$.

4. L-values.

Over the complex numbers $\mathbb{C}$, special values of Hecke L-functions at the integers may be expressed as logarithmic derivatives of theta functions. One may obtain an analogous p-adic relationship. Details of these facts may be found in [6], which draws from [1, 2]. To get L-values, one uses the Robert elliptic units, which are defined by picking a suitable theta function Θ. One can find a sequence $c = (c_n)$ of elliptic units whose projection onto the ρ-eigenspace generates C_∞^ρ over the Iwasawa algebra Λ. ([6] Theorem 12.11.)

Let Φ be the Coleman power series corresponding to c. Let Ω be an $\mathcal{O}_K$-generator for the period lattice of a suitable Weierstrass model of $E/\mathbb{C}$. The central relation is contained in the following result of Rubin [**6**], §12.

Theorem 4.1. *Let $Q \in E(\bar{K})$ be of exact order $\mathfrak{f}\mathfrak{p}^{n+1}$. Then for $k \geq 1$, and χ a character of G_n of p-power order,*

$$\sum_{\sigma \in G_n} \chi(\sigma) \mathcal{D}^k \log \Phi^\sigma \big|_{T = \omega_n{}^\sigma} = \sum_{\sigma \in G_n} \chi(\sigma) \left(\frac{d}{dz} \right)^k \log \Theta(z) \big|_{z = Q^\sigma}$$

$$= B \cdot \pi^{n+1} \cdot \Omega^{-1} L_{\mathfrak{f}\mathfrak{p}} \left(\bar{\psi}^k \chi, k \right)$$

*where $\mathcal{D}f = \frac{1}{\lambda'} \frac{f'}{f}$ is the Coates-Wiles logarithmic derivation ([**9**], §2) and $B = B_k$ may be chosen to be a unit over p, at least for $1 \leq k \leq q - 1$.*

5. Formal logarithmic derivatives.

Definition 5.1. Let $\mathcal{L}$ denote the "formal logarithmic derivative" on $\mathcal{O}[[T]]$, given by $\mathcal{L}f = \frac{1}{\lambda'(T)} D \log(f) = \frac{1}{\lambda'(T)} \frac{f'}{f}$ for f in $\mathcal{O}[[T]]$.

It is easily seen to satisfy $\mathcal{L}f_1 f_2 = \mathcal{L}f_1 + \mathcal{L}f_2$ for $f_1, f_2 \in \mathcal{O}[[T]]$ and $\mathcal{L}(f \circ [\alpha]) = \alpha \cdot (\mathcal{L}f \circ [\alpha])$ for $f \in \mathcal{O}[[T]]$ and $\alpha \in \mathcal{O}_\mathfrak{p}$.

Definition 5.2. Let $u = (u_n)_{n \geq 0} \in U_\infty$, and define $\delta_m(u) = \pi^{-m} \mathbf{T}_m \mathcal{L} g_u(\omega_m)$. Then $\delta_m(u) = \delta_n(u)$ for all $m, n \geq 0$. Let $\delta(u)$ be the common value.

Lemma 5.3. *We have $\delta(u) = (\pi - 1)\mathcal{L} g_u(0)$ for all $n \geq 0$. Thus $\delta(u) \in \mathcal{O}_\mathfrak{p}$.*

Proof. See [**3**], §8. □

Definition 5.4. For a character χ of G_n, taking values in $\bar{K}_\mathfrak{p}{}^*$, define a map $\delta_{n,\chi} : U_\infty \to \bar{K}_\mathfrak{p}$ by the formula $\delta_{n,\chi}(u) = \sum_{\sigma \in G_n} \chi(\sigma) \mathcal{L} g_u(\omega_n^\sigma)$.

We list the basic properties of the maps $\delta_{n,\chi}$ from [**9**], §2, and some others.
1) $\delta_{n,\chi}(u_1 \cdot u_2) = \delta_{n,\chi}(u_1) + \delta_{n,\chi}(u_2)$ for $u_1, u_2 \in U_\infty$.
2) By continuity, $\delta_{n,\chi}(u^a) = a\delta_{n,\chi}(u)$ if $a \in \mathcal{O}_\mathfrak{p}$.
3) If $\chi = 1$, then $\delta_{n,\chi} = \pi^n \delta$.
4) If χ is a character of G_n, and τ is any element of G_∞, then lifting χ to G_∞, one has $\delta_{n,\chi}(u^\tau) = \kappa\chi^{-1}(\tau)\delta_{n,\chi}(u)$.
5) Let γ_1, γ_2 be $\mathbb{Z}_p$-generators of Γ_∞. Then for all χ of p-power order, $u \in U_\infty^\rho$ and $f \in \Lambda^\rho$, $\delta_{n,\chi}(f \cdot u) = f\big(\kappa\chi^{-1}(\gamma_1) - 1, \kappa\chi^{-1}(\gamma_2) - 1\big)\delta_{n,\chi}(u)$ (this is slightly different from [**9**] but is proved similarly using that the character values are congruent to 1 modulo the prime above $\mathfrak{p}$ in $\bar{K}_\mathfrak{p}$).

In light of this definition and 4.1 we have the following:

Theorem 5.5. *For $n \geq 0$, $\delta_{n,\chi}(c) = B \cdot \pi^{n+1} \cdot \Omega^{-1} L_{\mathfrak{g}\mathfrak{p}}(\bar{\psi}\chi, 1)$, where B is a p-unit.*

We determine the action of $\delta_{n,\chi}$ on an element in the ρ-eigenspace U_∞^ρ and prove some additional properties of these maps. If u is any element in U_∞, let u^ρ denote the ρ-component of u in U_∞^ρ. Note $\mathrm{Tr}(\rho) = \kappa_0 + \kappa_0^* = \kappa_0 + \kappa_0^p$.

Proposition 5.6. *If $\chi \in \widehat{G_n}$ and $\chi = 1$ on Δ, then $\delta_{n,\chi}(u^\rho) = \delta_{n,\chi}(u)$.*

Proof.

$$\delta_{n,\chi}(u^\rho) = \delta_{n,\chi}\left(u^{\frac{1}{q-1}\sum_{\sigma\in\Delta}\mathrm{Tr}(\rho(\sigma^{-1}))\sigma}\right)$$

$$= \frac{1}{q-1}\sum_{\sigma\in\Delta}\kappa_0\chi^{-1}(\sigma)\left(\kappa_0^{-1}(\sigma) + \kappa_0^{-p}(\sigma)\right)\delta_{n,\chi}(u)$$

$$= \frac{1}{q-1}\left(\sum_{\sigma\in\Delta}\chi^{-1}(\sigma) + \sum_{\sigma\in\Delta}\chi^{-1}\kappa_0^{1-p}(\sigma)\right)\delta_{n,\chi}(u).$$

From the above we see that

(3)
$$\delta_{n,\chi}(u^\rho) = \begin{cases} \delta_{n,\chi}(u) & \text{if } \chi = 1 \text{ or } \kappa_0^{1-p} \text{ on } \Delta \\ 0 & \text{otherwise.} \end{cases}$$

$\square$

Note that the condition $\chi = 1$ on Δ is equivalent to χ having p-power order, and in fact to really being a character on Γ_n. This is clear from the decomposition $G_n \cong \Delta \times \Gamma_n$ and $\#\Delta = p^2 - 1, \#\Gamma_n = p^{2n}$. From now on, let us assume that the characters χ have p-power order.

Proposition 5.7. *Let $\Gamma_{m,n} = G(K_m/K_n)$ for $m \geq n$. If $\chi \in \Gamma_{m,n}^\perp$ (i.e., $\chi = 1$ on $\Gamma_{m,n} \subseteq \Gamma_n$), then $\delta_{m,\chi} = \pi^{m-n}\delta_{n,\chi}$.*

Proof. Using the basic properties of $\mathcal{L}$ and g_u as in [**3**],

(4)
$$\delta_{m,\chi}(u) = \sum_{\sigma\in\Gamma_m/\Gamma_{m,n}}\sum_{\tau\in\Gamma_{m,n}}\chi(\sigma\tau)\mathcal{L}g_u(\omega_m^{\sigma\tau})$$

$$= \sum_{\sigma\in\Gamma_m/\Gamma_{m,n}}\chi(\sigma)\sum_{\tau\in\Gamma_{m,n}}\mathcal{L}g_u(\omega_m^\tau)^\sigma$$

$$= \sum_{\sigma\in\Gamma_m/\Gamma_{m,n}}\chi(\sigma)\left(\mathbf{T}_{m,n}\mathcal{L}g_u(\omega_m)\right)^\sigma$$

$$= \pi^{m-n}\sum_{\sigma\in\Gamma_n}\chi(\sigma)\mathcal{L}g_u(\omega_n^\sigma)$$

$$= \pi^{m-n}\delta_{n,\chi}(u).$$

$\square$

Corollary 5.8. *For $\chi \in \widehat{\Gamma_{n+1}}$, $\delta_{n+1,\chi^p} = \pi\delta_{n,\chi^p}$.*

Proof. From the structure of the local extensions K_n one sees immediately that $\Gamma_{n+1,n}^{\perp}$ is the subgroup of p-th powers. It follows that for any $\chi \in \hat{\Gamma}_{n+1}$, we have $\chi^p \in \Gamma_{n+1,n}^{\perp}$, and so we can view χ^p as a character of Γ_n.   □

Thus in calculations we can assume that the character χ has maximum order.

6. Higher Derivatives.

We may easily generalize the maps $\delta_{n,\chi}$ so that we obtain information concerning the values $L_{\mathfrak{fp}}(\bar{\psi}^k\chi, k)$.

Definition 6.1. For $u \in U_\infty, n \geq 0, k \geq 1$, define
$$\delta_{n,\chi}^k(u) = \sum_{\sigma \in G_n} \chi(\sigma)\mathcal{D}^{k-1}\mathcal{L}g_u(\omega_n^\sigma),$$
where $\mathcal{D}$ is the derivation $\frac{1}{\lambda'(X)}\frac{d}{dx}$.

By Theorem 4.1, we have $\delta_{n,\chi}^k(c) = \pi^{n+1} \cdot B \cdot \Omega^{-1}L_{\mathfrak{fp}}(\bar{\psi}^k\chi, k)$, where B is a unit, if $1 \leq k \leq q - 1$. In his paper [4], Katz has shown that a family of derivations $\mathcal{D}_n$ may be defined by the formula $f(X \dotplus Y) = \sum_{n=0}^\infty \mathcal{D}_n f(X)Y^n$ and in addition, if $0 \leq m \leq q - 1$, then $\mathcal{D}_m = \frac{1}{m!}\mathcal{D}^m$. Since $g_u \equiv 1$ mod (π, X), $\log g_u$ converges formally, and we may write $\mathcal{D}^{k-1}\mathcal{L} = \mathcal{D}^k \log$. Substituting $f = \log g_u$ above gives $\log g_u(t \dotplus s) = \sum_{k=0}^\infty \mathcal{D}_k \log g_u(t)s^k$. We may then define a power series, given a character χ of G_n and a sequence of units $u \in U_\infty$, by
$$g(u, \chi, t, s) = \sum_{\sigma \in G_n} \chi(\sigma)\log g_u([\kappa(\sigma)](t) \dotplus s)$$
$$= \sum_{k=0}^\infty \left(\sum_{\sigma \in G_n}\chi(\sigma)(\mathcal{D}_k \log g_u) \circ [\kappa(\sigma)](t)\right)s^k.$$
It is readily seen from the above remarks that
$$\delta_{n,\chi}^k(u) = \left(\frac{d}{ds}\right)^k g(u, \chi, \omega_n, s)\big|_{s=0} \text{ if } 1 \leq k \leq q - 1.$$
In particular, $g(c, \chi, \omega_n, s)$ yields L-values.

7. Special Local Units.

As was done in [2] for the ordinary case, we now describe a sequence of local units which will give elements of U_∞ with simple Coleman power series. As usual, $q = p^2$. Let β in $\mathcal{O}_{\mathfrak{p}}$ be such that $\beta^{q-1} = 1-\pi$ and $\beta \equiv 1$ mod π. Such a β exists by Hensel's Lemma applied to the polynomial $f(X) = X^{q-1}-(1-\pi)$. If ζ is any one of the $q - 1$ roots of unity in $K_\mathfrak{p}$, then $f(\zeta) = \pi \equiv 0 \mod \mathfrak{p}$

and $f'(\zeta) = (q-1)\zeta^{-1} \not\equiv 0 \mod \mathfrak{p}$, so that there is a lifting of ζ to a root in $\mathcal{O}_{\mathfrak{p}}$.

Lemma 7.1. $\mathbf{N}_{K_{n+1}/K_n}(\beta - \omega_{n+1}) = (\beta - \omega_n)$ *for all* $n \geq 0$.

Proof. The minimal polynomial of ω_{n+1} over K_n is $P(X) = X^q + \pi X - \omega_n$, and hence the minimal polynomial of $\beta - \omega_{n+1}$ over K_n is $-P(\beta - X)$. It follows that $N_{n+1,n}(\beta - \omega_{n+1}) = -(-1)^q P(\beta) = P(\beta) = \beta^q + \pi\beta - \omega_n = \beta - \omega_n$. $\qquad\square$

Theorem 7.2. *For each* d *dividing* $q - 1$, *we have* $N_{n+1,n}(\beta^d - \omega_{n+1}^d) = (\beta^d - \omega_n^d)$.

Proof. The lemma is valid for any β such that $\beta^{q-1} = \pi$, in particular with β changed to $\zeta\beta$ where $\zeta^{q-1} = 1$. Taking the product over $\zeta^d = 1$ gives the result. $\qquad\square$

We obtain a sequence of units $u^{(d)} = (u_n^{(d)}) \in U_\infty$ for $d | q - 1$ whose Coleman power series is $\beta^d - X^d$.

Corollary 7.3. $\delta(u^{(d)}) = 0$ *if* $d \neq 1$. $\delta(u^{(1)}) = (1 - \pi)\beta^{-1} \not\equiv 0 \mod \pi$.

Proof. Explicit calculation, using Lemma 5.3. $\qquad\square$

Theorem 7.4. $u^{(d)\rho} = 1$ *unless* $d = 1$.

Proof. We calculate the Coleman power series of the projections. First we compute the ρ-part of the unit $u^{(d)} = (u_n^{(d)})$:

$$u_n^{(d)\rho} = \prod_{\sigma \in \Delta} u_n^{(d)\frac{1}{q-1}\mathrm{Tr}(\rho(\sigma^{-1}))\sigma} = \prod_{\sigma \in \Delta} (\beta^d - \kappa_0^d(\sigma)\omega_n^d)^{\frac{1}{q-1}\mathrm{Tr}(\rho(\sigma^{-1}))}.$$

We have used the fact that $[\kappa_0(\sigma)](X) = \kappa_0(\sigma)X$ in the basic Lubin-Tate formal group. The Coleman power series for $u^{(d)\rho}$ must then be

$$\mathcal{G}(X) = \prod_{\sigma \in \Delta} (\beta^d - \kappa_0^d(\sigma)X^d)^{\frac{1}{q-1}\mathrm{Tr}(\rho(\sigma^{-1}))}.$$

Note that $\frac{1}{q-1}\mathrm{Tr}(\rho(\sigma^{-1}))$ is an element of $\mathbb{Z}_p$ and that $\beta^d - \kappa_0^d(\sigma)X^d \equiv 1 \mod (\pi, X)$, so this expression indeed defines a power series in $\mathcal{O}_{\mathfrak{p}}[[X]]$, satisfying $\mathcal{G}(\omega_n) = u_n^{e(\rho)}$ for all $n \geq 0$. Furthermore, $\mathcal{G}(X) \equiv 1 \mod (\pi, X)$. Writing $(\beta^d - \kappa_0^d(\sigma)X^d) = \beta^d \cdot (1 - (\kappa_0(\sigma)X/\beta)^d)$ we compute

$$\log \mathcal{G}(X) = \sum_{\sigma \in \Delta} \frac{1}{q-1}\mathrm{Tr}\rho(\sigma^{-1})\log_p(\beta^d)$$

$$+ \sum_{\sigma \in \Delta} \frac{1}{q-1}\mathrm{Tr}\rho(\sigma^{-1})\log\left(1 - \frac{\kappa_0^d(\sigma)}{\beta^d}X^d\right)$$

where $\log_p$ is the p-adic logarithm, and the logarithm of a power series which is congruent to 1 modulo (π, X) is given by the usual series expansion for $\log(1 + X)$. Then

$$(5) \qquad \log \mathcal{G}(X) = -\sum_{\sigma \in \Delta} \frac{1}{q-1} \mathrm{Tr}\rho(\sigma^{-1}) \sum_{k=1}^{\infty} \frac{\kappa_0^{dk}(\sigma)}{\beta^{dk}} \frac{X^{dk}}{k}$$

$$= -\sum_{k=1}^{\infty} \left(\frac{1}{q-1} \sum_{\sigma \in \Delta} \mathrm{Tr}\rho(\sigma^{-1})\kappa_0^{dk}(\sigma) \right) \frac{X^{dk}}{k\beta^{dk}}.$$

We have $\mathrm{Tr}\rho(\sigma^{-1}) \cdot \kappa_0^{dk}(\sigma) = \left(\kappa_0(\sigma^{-1}) + \kappa_0^*(\sigma^{-1})\right) \kappa_0^{dk}(\sigma)$ and, since $\kappa_0^* = \kappa_0^p$, when we sum over Δ the result is $\sum_{\sigma \in \Delta}(\kappa_0^{dk-1}(\sigma) + \kappa_0^{dk-p}(\sigma))$, which is 0 unless $dk - 1 \equiv 0 \bmod q - 1$ or $dk - p \equiv 0 \bmod q - 1$, in which cases it is equal to $q - 1$. However, since $d|q-1$, we see that unless $d = 1$ these congruences are impossible, and hence $\log \mathcal{G}(X) = 0$, so that $\mathcal{G}(X) = 1$ and thus u^d projects trivially. $\square$

For $d = 1$, we have $\log \mathcal{G}(X) = -\sum_{k\equiv 1, p \bmod q-1} X^k/k\beta^k$. It is easy to compute

$$\mathcal{Q} = \mathcal{L}\mathcal{G} = \frac{1}{\lambda'(X)} \frac{d}{dx} \log \mathcal{G}(X) = -\frac{1}{\lambda'(X)} \sum_{k \equiv 1, p} \frac{X^{k-1}}{\beta^k}$$

$$= -\frac{1}{\lambda'(X)} \beta^{-1} \sum_{k \equiv 0, p-1} \frac{X^k}{\beta^k}.$$

Compare this to the result in the ordinary case in [**1, 2**]. We may sum the series,

$$\mathcal{Q}(X) = -\frac{1}{\lambda'(X)} \beta^{-1} \left(1 + \left[\frac{X}{\beta}\right]^{p-1} \right) \frac{1}{1 - \dfrac{X^{q-1}}{1 - \pi}}.$$

We could further modify this, by employing the definition of β and the formula $(1 - X^{2ma})(1 + X^a)^{-1} = \sum_{n=0}^{2m-1}(-1)^n X^{an}$. This gives

$$\mathcal{Q}(X) = -\frac{1}{\lambda'(X)} \cdot \frac{\beta^{-1}}{1 - \left(\frac{X}{\beta}\right)^{p-1} + \left(\frac{X}{\beta}\right)^{2(p-1)} + \cdots - \left(\frac{X}{\beta}\right)^{p(p-1)}}.$$

Note that $\delta_{n,\chi}(u^\rho) = \sum_{\sigma \in G_n} \chi(\sigma)\mathcal{Q}(\omega_n^\sigma)$, although this does not simplify the expression $\delta_{n,\chi}(u^\rho) = \delta_{n,\chi}(u) = -\sum_{\sigma \in G_n} \chi(\sigma)\frac{1}{\lambda'(\omega_n^\sigma)} \cdot \frac{1}{\beta - \omega_n^\sigma}$. We compute $\lambda'(\omega_n)$.

Lemma 7.5. *For all $n \geq 0$, we have $\lambda'(\omega_n) = \prod_{k=0}^{n} \left(1 + \frac{q}{\pi}\omega_k^{q-1}\right)$.*

Proof. By differentiating the relation $\lambda \circ [\pi^{n+1}](X) = \pi^{n+1}\lambda(X)$ we obtain $[\pi^{n+1}]'(X)\lambda' \circ [\pi^{n+1}] = \pi^{n+1}\lambda'(X)$. Substitute $X = \omega_n$ and $\lambda'(0) = 1$ to get $[\pi^{n+1}]'(\omega_n) = \pi^{n+1}\lambda'(\omega_n)$. If f is a function and $f_n = f \circ \overset{\text{n times}}{\cdots} \circ f$ then $f_m'(X) = \prod_{n=0}^{m-1} f'(f_n(X))$ for every $m \geq 1$. Applying this to $f = [\pi]$ gives

$$[\pi^{n+1}](\omega_n) = \prod_{k=0}^{n}[\pi]'([\pi^k](\omega_n)) = \prod_{k=0}^{n}[\pi]'(\omega_{n-k}) = \prod_{k=0}^{n}[\pi]'(\omega_k).$$

Since $[\pi]'(X) = \pi + qX^{q-1}$ we conclude

$$\lambda'(\omega_n) = \pi^{-(n+1)}[\pi^{n+1}]'(\omega_n) = \pi^{-(n+1)}\prod_{k=0}^{n}(\pi+q\omega_k^{q-1}) = \prod_{k=0}^{n}\left(1 + \frac{q}{\pi}\omega_k^{q-1}\right).$$

$\square$

We finish by mentioning a connection to sums $S_n(\chi, k) = \sum_{\sigma \in \Gamma_n} \chi(\sigma)(\omega_n^\sigma)^k$. A straightforward calculation gives:

Theorem 7.6. *Let $\lambda'(X)^{-1} = \sum_{i=0}^{\infty} b_i X^{(q-1)i}$, with $b_i \in \mathcal{O}_{\mathfrak{p}}$. Then $\delta_{n,\chi}(u^\rho) = (1-q)\beta^{-1}\sum_{m=0}^{\infty} c_m S_n\left(\chi, (q-1)m\right)$, where $c_m = \sum_{i+j=m} \frac{b_i}{(1-\pi)^j}$.*

References

[1] J. Coates and A. Wiles, *On the conjecture of Birch and Swinnerton-Dyer*, Inventiones Mathematicae, **39** (1977), 223-251.

[2] ———, *On p-adic L-functions and elliptic units*, J. Aust. Math. Soc., **26**, (1978) 1-25.

[3] K. Iwasawa, *Local Class Field Theory*, Oxford University Press (1986).

[4] N. Katz, *Divisibilities, congruences, and Cartier duality*, J. Fac. Sci. Univ. Tokyo, **28** (Sec 1a) (1982), 667-678.

[5] K. Rubin, *The "main conjectures" of Iwasawa theory for imaginary quadratic fields*, Inventiones Mathematicae, **103** (1991), 25-68.

[6] ———, *Tate-Shafarevitch groups and L-functions of elliptic curves with complex multiplication*, Inventiones Mathematicae, **89** (1987), 527-560.

[7] ———, *Congruences for special values of L-functions of elliptic curves with complex multiplication*, Inventiones Mathematicae, **71** (1983), 339-368.

[8] ———, *Iwasawa Theory and Elliptic Curves: Supersingular Primes*, Journées Arithmétiques, 1980, London Math. Soc. LN series, **56**.

[9] ———, *Local units, elliptic units, Heegner points and elliptic curves*, Inventiones Mathematicae, **88** (1987), 405-422.

[10] A. Wiles, *Higher explicit reciprocity laws*, Ann. Math., **107** (1978), 235-254.

Departamento de Matemáticas
Universidad de Salamanca
Plaza de la Merced, 1-4
37008 Salamanca Spain
E-mail address: navas@gugu.usal.es

THE MODULE OF DERIVATIONS FOR AN ARRANGEMENT OF SUBSPACES

Jonathan Wiens

This paper examines the module of derivations for a subspace arrangement. In particular, we consider those subspace arrangements consisting of elements of the intersection lattice of a generic hyperplane arrangement. We determine generators for the associated module of derivations. These generators are indexed by certain elements of the intersection lattice.

1. Introduction.

Let V be a linear space of dimension ℓ over a field $\mathbf{K}$. By an arrangement we shall mean a finite collection of affine subspaces of V. If all of the subspaces in an arrangement $\mathcal{A}$ have codimension k then we say that $\mathcal{A}$ is an (ℓ, k)-arrangement. If $k = 1$ and so $\mathcal{A}$ is a hyperplane arrangement then we shall say that $\mathcal{A}$ is an ℓ-arrangement.

Let $\mathcal{A}$ be an arrangement and S the coordinate ring for V. For each $H \in \mathcal{A}$ let $I_H = V(H)$, the ideal of S which vanishes on H, and call it the defining ideal for H. If H is a hyperplane, then we can choose a linear functional $\alpha_H \in S$ such that $I_H = (\alpha_H)$.

We now introduce the main character of this paper. If $\mathcal{A}$ is an arrangement then the module of $\mathcal{A}$-derivations is $D(\mathcal{A})$, the set of all $\mathbf{K}$-linear derivations of S which map each defining ideal to itself. Equivalently, one could define $D(\mathcal{A})$ to be the set of all polynomial vector fields which, at each subspace, are parallel to that subspace. [1] contains an extensive review of the properties of $D(\mathcal{A})$ for hyperplane arrangements, especially for free arrangements. We shall review the situation for generic arrangements in Section 3.

Recently interest has arisen in arrangements of subspaces of codimension greater than one. The goal of this paper is to examine $D(\mathcal{A})$ in this case. In particular we investigate subspace arrangements consisting of elements of the intersection lattice of a generic hyperplane arrangement, where we find generators for $D(\mathcal{A})$ as an S-module.

In Section 2 we list several elementary properties of $D(\mathcal{A})$. In Section 3 we find generators of $D(\mathcal{A})$ for generic arrangements. In Section 4 we discuss

subspace arrangements arising from hyperplane arrangements and in Section 5 we find generators for $\mathrm{D}(\mathcal{A})$ for those subspace arrangements arising from generic hyperplane arrangements.

The author would like to thank Sergey Yuzvinsky for valuable suggestions during the preparation of this paper.

2. Subspace Arrangements.

In this section we define our terminology and give some elementary results. Let $\mathcal{A} = \{\mathrm{H}_1, \ldots, \mathrm{H}_n\}$ be an arrangement with defining ideals $\{I_{\mathrm{H}_1}, \ldots, I_{\mathrm{H}_n}\}$. An element H of $\mathcal{A}$ is *maximal* if it is not contained in any other subspace of $\mathcal{A}$. $\mathcal{A}$ is *central* if $T = \cap_{i=1}^{n} \mathrm{H}_i$ is non-empty. In this case we choose coordinates so that $0 \in T$. We say that $\mathcal{A}$ is *essential* if $T = 0$. We shall primarily concern ourselves with central arrangements.

Let S be the coordinate ring for V. Let $\mathrm{Der}_{\mathbf{K}}(S)$ denote the set of $\mathbf{K}$-linear derivations of S, that is, $\mathbf{K}$-linear maps $\theta : S \to S$ such that $\theta(fg) = f\theta(g) + g\theta(f)$ for all $f, g \in S$.

Definition 2.1. Let $\mathcal{A}$ be an arrangement in V. The module of $\mathcal{A}$-derivations is
$$\mathrm{D}(\mathcal{A}) = \{\theta \in \mathrm{Der}_{\mathbf{K}}(S) \,|\, \theta(I_{\mathrm{H}}) \subseteq I_{\mathrm{H}} \ \forall \ \mathrm{H} \in \mathcal{A}\}.$$

One obvious consequence of the definition of $\mathrm{D}(\mathcal{A})$ is the following lemma.

Lemma 2.2. *If $\mathcal{B} \subseteq \mathcal{A}$ are a pair of arrangements then $\mathrm{D}(\mathcal{A}) \subseteq \mathrm{D}(\mathcal{B})$.*

If $\mathcal{A}$ is central, then the defining ideals of $\mathcal{A}$ are all homogeneous with degree one generators and hence $\mathrm{D}(\mathcal{A})$ is a graded S-module. In this case let V^* denote the dual space of the vector space V and S^+ denote the maximal graded ideal of S. We shall abbreviate $\frac{\partial}{\partial x_i}$ by D_i. The Euler derivation is $\theta_E = \sum_{i=1}^{\ell} x_i D_i$ and has the property that if $f \in S$ is homogeneous of degree n then $\theta_E(f) = nf$. As a result we have:

Lemma 2.3. *If $\mathcal{A}$ is a central arrangement then $\theta_E \in \mathrm{D}(\mathcal{A})$.*

The Euler derivation plays a deeper role in some arrangements.

Lemma 2.4. *If $\mathcal{A}$ is central and contains a hyperplane then $S\theta_E$ is a direct summand of $\mathrm{D}(\mathcal{A})$.*

Proof. If $\mathrm{H} \in \mathcal{A}$ is a hyperplane choose a linear functional $\alpha \in V^*$ so that $\mathrm{H} = V(\alpha)$. If $\theta \in \mathrm{D}(\mathcal{A})$ then α divides $\theta(\alpha)$ and hence we can define the function $\phi : \mathrm{D}(\mathcal{A}) \to S$ by $\phi(\theta) = \theta(\alpha)/\alpha$. Since $\theta_E \in \mathrm{D}(\mathcal{A})$, ϕ is surjective with section $s : S \to \mathrm{D}(\mathcal{A})$ given by $s(f) = f\theta_E$. This shows $S\theta_E$ is a direct summand of $\mathrm{D}(\mathcal{A})$. $\qquad\square$

While freeness is an important property for hyperplane arrangements, it rarely occurs in more general arrangements.

Theorem 2.5. *If $\mathcal{A}$ contains a maximal subspace of codimension greater than 1, then $\mathrm{D}(\mathcal{A})$ is not a free S-module.*

Proof. Suppose that $\mathrm{D}(\mathcal{A})$ is free, with basis $\theta_1, \ldots, \theta_m$. Let $\mathrm{H} \in \mathcal{A}$ be maximal with codimension $k > 1$. Choose a basis $\{x_1, \ldots, x_\ell\}$ for V so that $I_{\mathrm{H}} = (x_1, \ldots, x_k)$. Since H is maximal, for each $\mathrm{K} \neq \mathrm{H}$ in $\mathcal{A}$ there exists $\beta_{\mathrm{K}} \in I_{\mathrm{K}}$ such that $\beta_{\mathrm{K}} \notin I_{\mathrm{H}}$. Let $\beta = \prod\{\beta_{\mathrm{K}} \mid \mathrm{K} \neq \mathrm{H}\}$ and note that $\beta \notin I_{\mathrm{H}}$. Hence $\beta D_1 \notin \mathrm{D}(\mathcal{A})$, but $x_1 \beta D_1, x_2 \beta D_1 \in \mathrm{D}(\mathcal{A})$. Now $x_1 \beta D_1 = \sum_{i=1}^{m} p_i \theta_i$ and $x_2 \beta D_1 = \sum_{i=1}^{m} q_i \theta_i$ and hence $x_2 p_i = x_1 q_i$ for all i. This shows that $x_1 | p_i$ for all i and hence $\beta D_1 \in \mathrm{D}(\mathcal{A})$. This is a contradiction, and hence $\mathrm{D}(\mathcal{A})$ is not free. $\qquad\square$

Different arrangements can yield the same module of derivations, as the next theorem shows.

Theorem 2.6. *If $\mathcal{A}$ is any arrangement, $\mathrm{H}_1, \ldots, \mathrm{H}_k \in \mathcal{A}$ and $\mathcal{B} = \mathcal{A} \cup \{\mathrm{H}_1 \cap \cdots \cap \mathrm{H}_k\}$ then $\mathrm{D}(\mathcal{A}) = \mathrm{D}(\mathcal{B})$.*

Proof. This result follows since if $\mathrm{J} = \mathrm{H}_1 \cap \cdots \cap \mathrm{H}_k$ then $I_{\mathrm{J}} = I_{\mathrm{H}_1} + \cdots + I_{\mathrm{H}_k}$. $\qquad\square$

Hence one could routinely assume that an arrangement is closed under intersections, as some authors do. In this paper, however, we shall not make this assumption.

3. Generic hyperplane arrangements.

In this section we review the case where $\mathcal{A}$ is a generic hyperplane arrangement. In particular, we find a minimal list of generators for $\mathrm{D}(\mathcal{A})$ if $\mathcal{A}$ is a generic hyperplane arrangement, and then compute the projective dimension of $\mathrm{D}(\mathcal{A})$ as an S-module. Much of what is found here can be gleaned from [**2**] and [**4**]. Here we give a straight-forward derivation of those results.

An essential ℓ-arrangement $\mathcal{A}$ is *generic* if $\ell > 1$ and every collection of ℓ hyperplanes from $\mathcal{A}$ is also essential. A *boolean* arrangement is a generic arrangement with exactly ℓ hyperplanes. Every 2-arrangement is generic.

For the rest of this section we shall assume that $\mathcal{A}$ is generic. Furthermore, since each hyperplane is determined by an element of V^* we shall describe each arrangement by listing functionals corresponding to each hyperplane. To this end we shall always choose a basis $\{x_1, \ldots, x_\ell\}$ for V^* such that $\mathcal{A} = \{\alpha_1 = x_1, \ldots, \alpha_\ell = x_\ell, \alpha_{\ell+1}, \ldots, \alpha_n\}$ and we let $Q = \prod \alpha_{\mathrm{H}}$ and call it a defining polynomial for $\mathcal{A}$. If $\ell < 5$ we shall often use x, y, z and w for $x_1, \ldots, x_4$.

The following two results are well-known.

Lemma 3.1. *If $\mathcal{A}$ is a 2-arrangement then $\mathrm{D}(\mathcal{A})$ is a free S-module with basis $\left\{\theta_E, \frac{Q}{x} D_y\right\}$.*

Lemma 3.2. *If $\mathcal{A}$ is a boolean ℓ-arrangement then* $\mathrm{D}(\mathcal{A})$ *is a free S-module with basis $\{x_1 D_1, \ldots, x_\ell D_\ell\}$.*

If $\ell > 3$ and $\mathcal{A}$ is generic but not boolean, then $\mathrm{D}(\mathcal{A})$ is not free. To find generators of $\mathrm{D}(\mathcal{A})$ in this case we consider the intersection poset $L(\mathcal{A})$. $L(\mathcal{A})$ consists of all intersections of the elements of V, including the empty intersection V. Let $L(\mathcal{A})_k$ be the elements of $L(\mathcal{A})$ of dimension k. If $X \in L(\mathcal{A})$ let $\mathcal{A}_X = \{\mathrm{H} \in \mathcal{A} \mid X \subseteq \mathrm{H}\}$, Q_X be a defining polynomial for $\mathcal{A}_X$ and $\pi_X = \frac{Q}{Q_X}$. If $X \in L(\mathcal{A})_1$ choose $\gamma_X = \sum b_i D_i$ non-zero with $b_i \in \mathbf{K}$ such that $\gamma_X(\alpha_{\mathrm{H}}) = 0$ for all $\mathrm{H} \in \mathcal{A}_X$. This derivation may be identified with a vector parallel to X and is projectively unique. Let $\theta_X = \pi_X \gamma_X$. One can easily see that $\theta_X \in \mathrm{D}(\mathcal{A})$. These derivations, together with θ_E, will be generators of $\mathrm{D}(\mathcal{A})$. To prove this we need to introduce the concept of deletion and restriction.

Let $\mathcal{A}$ be any hyperplane arrangement and choose $\mathrm{H} \in \mathcal{A}$. Let $\mathcal{A}' = \mathcal{A} \setminus \{\mathrm{H}\}$ and call it the *deletion* of $\mathcal{A}$ with respect to H. Let $\mathcal{A}''$ be the hyperplane arrangement in H with hyperplanes $\{\mathrm{H}' \cap \mathrm{H} \mid \mathrm{H}' \in \mathcal{A}'\}$ and call it the *restriction* of $\mathcal{A}$ to H. $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$ is called a triple of arrangements.

If $\mathcal{A}$ is generic and non-boolean then $\mathcal{A}'$ is also generic. If $\ell > 2$ and $\mathcal{A}$ is generic then $\mathcal{A}''$ is also generic. Furthermore, if $\mathcal{A}$ is generic then $L(\mathcal{A}'')$ may be identified with those elements of $L(\mathcal{A})$ contained in H.

Next, we recall the short exact sequence of [**1**, Prop. 4.45]. If we choose coordinates for V so that the functional associated with H is x_1, then multiplication by x_1 yields an injective homomorphism $\mu : \mathrm{D}(\mathcal{A}') \to \mathrm{D}(\mathcal{A})$. We can also restrict derivations to H. We identify the coordinate ring of H with $S'' = \mathbf{K}[x_2, \ldots, x_\ell]$. The canonical surjection $S \to S''$ then provides an S-module structure for $\mathrm{D}(\mathcal{A}'')$. If $\theta \in \mathrm{D}(\mathcal{A})$ and $f \in S''$ then let $r(\theta)(f) = \theta(f)|_{x_1=0}$. It was shown in [**1**] that $r(\mathrm{D}(\mathcal{A})) \subseteq \mathrm{D}(\mathcal{A}'')$ and that the sequence $0 \to \mathrm{D}(\mathcal{A}') \to \mathrm{D}(\mathcal{A}) \to \mathrm{D}(\mathcal{A}'')$ is an exact sequence of S-modules. We shall show that the last map is surjective in the case of generic arrangements, and in the process find a minimal generating set of $\mathrm{D}(\mathcal{A})$.

Let $\mathcal{A}$ be a hyperplane arrangement and fix $\mathrm{K} \in \mathcal{A}$. Let $F(\mathcal{A})$ denote the submodule of $\mathrm{D}(\mathcal{A})$ generated by θ_E and $\{\theta_X \mid X \in L(\mathcal{A})_1, \ X \subseteq \mathrm{K}\}$ and $F(\mathcal{A}'')$ the submodule of $\mathrm{D}(\mathcal{A}'')$ generated by $\theta_E'' = r(\theta_E)$ and $\{\theta_{X''} \mid X'' \in L(\mathcal{A}'')_1, \ X'' \subseteq \mathrm{K} \cap \mathrm{H}\}$.

Lemma 3.3. *Let $\mathcal{A}$ be a generic arrangement and $\mathrm{K} \in \mathcal{A}$. If $\mathrm{H} \in \mathcal{A}$ and $\mathrm{H} \neq \mathrm{K}$ then $r(F(\mathcal{A})) = F(\mathcal{A}'')$.*

Proof. The θ_X with $X \subseteq \mathrm{K}$ fall into two categories. If $X \not\subseteq \mathrm{H}$ then $x_1 | \pi_X$ and so $r(\theta_X) = 0$. If $X \subseteq \mathrm{H}$ then $\mathrm{H} \in \mathcal{A}_X$ and x_1 is not a factor of π_X. If $X'' = \cap \{\mathrm{J} \mid \mathrm{J} \in \mathcal{A}_X\}$ then $X'' \in L(A'')$ and since $\mathcal{A}$ and $\mathcal{A}''$ are generic, one can see that $r(\theta_X) = \theta_{X''}$. Furthermore, since $\mathcal{A}$ is generic, each element of $L(A'')$ containing $\mathrm{K} \cap \mathrm{H}$ arises in this fashion; and hence, r is surjective. $\qquad\square$

Theorem 3.4. *If $\mathcal{A}$ is a non-boolean generic ℓ-arrangement then*

$$0 \longrightarrow D(\mathcal{A}') \xrightarrow{\mu} D(\mathcal{A}) \xrightarrow{r} D(\mathcal{A}'') \longrightarrow 0$$

is a short exact sequence of S-modules. Furthermore, for any $K \in \mathcal{A}$, $D(\mathcal{A})$ is generated by $\theta_E \cup \{\theta_X \mid X \in L(\mathcal{A})_1, \ X \subseteq K\}$.

Proof. We shall prove the theorem by induction on ℓ. The base case $\ell = 2$ is Lemma 3.1. Now let $\ell > 2$. We shall prove this by induction on the number of hyperplanes of $\mathcal{A}$. For this inner induction, we shall prove the base case together with the inductive step.

Let $\mathcal{A}$ be a non-boolean generic ℓ-arrangement. Since $\mathcal{A}$ is generic and non-boolean, $\mathcal{A}'$ and $\mathcal{A}''$ are generic and by induction, Lemma 3.1 or Lemma 3.2, have generators listed by the theorem. Since

$$D(\mathcal{A}'') = r(F(\mathcal{A})) \subseteq r(D(\mathcal{A})) \subseteq D(\mathcal{A}''),$$

we see that r is surjective.

Now suppose that $\theta \in D(\mathcal{A})$. Since $D(\mathcal{A}'') = F(\mathcal{A}'')$ we can, by Lemma 3.3, choose $\eta \in F(\mathcal{A})$ so that $r(\eta) = r(\theta)$. By exactness we have $\theta - \eta \in \mu(D(\mathcal{A}'))$. But by induction $D(\mathcal{A}')$ is generated by the forms θ'_X where $X' \subseteq K$. Now $\mathcal{A}$ is generic so each $X' \in L(\mathcal{A}')$ is also an element of $L(\mathcal{A})$ where we denote it by X. By definition we then have $x_1\theta_{X'} = \theta_X$ and so $\mu D(\mathcal{A}') \in F(\mathcal{A})$ and hence $\theta \in F(\mathcal{A})$. $\qquad\square$

The above theorem shows that if $|\mathcal{A}| = n$ and $\mathcal{A}$ is generic, then at most $\binom{n-1}{\ell-2} + 1$ generators are needed. The short exact sequence given above allows us to compute the projective dimension of $D(\mathcal{A})$.

Corollary 3.5. *If $\mathcal{A}$ is generic and non-boolean then the projective dimension of $D(\mathcal{A})$ as an S-module is $\ell - 2$.*

Proof. We proceed by induction on ℓ. If $\ell = 2$ then $D(\mathcal{A})$ is free, so the result holds. Now assume $\ell > 2$. We proceed by induction on the number of hyperplanes in $\mathcal{A}$. If $|\mathcal{A}| = \ell + 1$ then $\mathcal{A}'$ is boolean and so $D(\mathcal{A}')$ is free. Now consider the exact sequence of Theorem 3.4. $\mathcal{A}''$ is generic and non-boolean, thus $\mathrm{pdim}_{S''}D(\mathcal{A}'') = \ell - 3$. But $S'' \simeq S/\alpha_H S$ and hence, by [**3**, Theorem 4.3.3], $\mathrm{pdim}_S D(\mathcal{A}'') = \ell - 2$. As a result, by [**3**, Exercise 4.1.2], if $\ell \neq 3$ we have $\mathrm{pdim}_S D(\mathcal{A}) = \ell - 2$. If $\ell = 3$ then our arrangement is that of [**1**, Example 4.34], which was shown to not be free, which implies that $\mathrm{pdim}_S D(\mathcal{A})$ in this case is also $\ell - 2$.

If $|\mathcal{A}| > \ell + 1$ then $\mathrm{pdim}_S D(\mathcal{A}') = \mathrm{pdim}_S D(\mathcal{A}'') = \ell - 2$ and again [**3**, Exercise 4.1.2] shows that $\mathrm{pdim} D(\mathcal{A}) = \ell - 2$. $\qquad\square$

[**4**] provides a minimal projective resolution of $D(\mathcal{A})$ which also shows that the minimal number of generators of $D(\mathcal{A})$ is exactly $\binom{n-1}{\ell-2} + 1$.

4. Arrangements arising from $L(\mathcal{A})$.

In this section we discuss arrangements which consist of a subset of $L(\mathcal{A})$ for a hyperplane arrangement $\mathcal{A}$. In particular, choose $k \geq 2$ and let $\mathcal{A}_k$ be the (ℓ, k)-arrangement consisting of those elements of $L(\mathcal{A})$ of codimension k. Note that $\mathcal{A}_1 = \mathcal{A}$. The next result gives a filtration which may be an interesting object of study.

Theorem 4.1. *If $\mathcal{A}$ is a hyperplane arrangement, then*

$$\mathrm{D}(\mathcal{A}) \subseteq \mathrm{D}(\mathcal{A}_2) \subseteq \cdots \subseteq \mathrm{D}(\mathcal{A}_\ell).$$

If $\mathcal{A}$ is essential then $\mathrm{D}(\mathcal{A}_\ell) = S^+ \mathrm{Der}_{\mathbf{K}}(S)$.

Proof. If $\theta \in \mathrm{D}(\mathcal{A})_k$ then $\theta(I(X)) \subseteq SI(X)$ for each $X \in L(\mathcal{A})_{\ell-k}$. If $Y \in L(\mathcal{A})_{\ell-k-1}$ then $Y = X_1 \cap X_2$ where X_1 and X_2 are elements of $L(\mathcal{A})_{\ell-k}$; but then $V(Y) = V(X_1) + V(X_2)$ and hence $\theta(Y) \subseteq Y$ and $\theta \in \mathrm{D}(\mathcal{A})_{k+1}$. If $\mathcal{A}$ is essential then $L(\mathcal{A})_0 = \{0\}$ and since $V(0) = S^+$ it is clear that $\mathrm{D}(\mathcal{A})_\ell = S^+ \mathrm{Der}_{\mathbf{K}}(S)$. $\square$

Next we apply the notion of deletion and restriction to these modules. One easy result is the following:

Lemma 4.2. *If $\mathcal{A}$ is an ℓ-arrangement and $\mathrm{H} \in \mathcal{A}$ with defining functional α_{H} then $\alpha_{\mathrm{H}} \mathrm{D}(\mathcal{A}'_k) \subseteq \mathrm{D}(\mathcal{A}_k)$.*

Next we consider the restriction map of Section 3.

Lemma 4.3. *If $\mathcal{A}$ is a hyperplane arrangement, $\mathrm{H} \in \mathcal{A}$, and $\mathcal{A}''$ is the restriction of $\mathcal{A}$ to H, then $r\left(\mathrm{D}(\mathcal{A}_k)\right) \subseteq \mathrm{D}(\mathcal{A}''_{k-1})$ for every $2 \leq k \leq \ell$.*

Proof. First choose coordinates so that $\alpha_{\mathrm{H}} = x_1 = x$. We identify S'' with $\mathbf{K}[x_2, \ldots, x_\ell]$. Let $\theta \in \mathrm{D}(\mathcal{A}_k)$. If $X \in L(A'')_{\ell-k}$ then $I(X) = (\beta_1, \ldots, \beta_{k-1})$ where the β_i are linear functionals associated to the elements of $\mathcal{A}''$. There exist $a_1, \ldots, a_{k-1} \in \mathbf{K}$ such that $\alpha_i = a_i x + \beta_i$ are functionals defining elements of $\mathcal{A}$. But note that there exists $Y \in \mathcal{A}_k$ such that $I_Y = (x, \alpha_1, \ldots, \alpha_{k-1}) = (x, \beta_1, \ldots, \beta_k)$ and so $\theta(I(X)) \subseteq (x, \beta_1, \ldots, \beta_{k-1})$ and hence $r(\theta)(I(X)) = \theta(I(X))|_{x=0} \subseteq (\beta_1, \ldots, \beta_{k-1})$. Since this holds true for all $X \in L(\mathcal{A}'')$ we see that $r(\theta) \in \mathrm{D}(\mathcal{A}''_{k-1})$. $\square$

These two results will be used later when we find generators of $\mathrm{D}(\mathcal{A}_k)$ when $\mathcal{A}$ is generic. If $\mathcal{A}$ is boolean, the $\mathrm{D}(\mathcal{A}_k)$ are easily described. Let V be a vector space of dimension ℓ. Let us say that a subset T of V^* is generic if every subset of T of size at most ℓ is linearly independent.

Lemma 4.4. *Let V be a vector space of dimension ℓ, $S = SV^*$ and $\mathcal{A} = \{\alpha_1, \ldots, \alpha_n\}$ be a generic subset of V^*. Let $Q = \prod_{i=1}^{n} \alpha_i$ and if $X \subseteq$*

$\{1,\ldots,n\}$ let $\pi_X = \prod_{j\in X}\alpha_j$. For each $1 \le k < \ell$ the ideal $\bigcap_{i_1<\cdots<i_k}(\alpha_{i_1}, \ldots, \alpha_{i_k})$ is generated by

$$\left\{ \frac{Q}{\pi_X} \,\middle|\, X \subseteq \{1,\ldots,n\},\ |X| = k-1 \right\}$$

and the ideal $\bigcap_{1<i_2<\cdots<i_k}(\alpha_1,\ldots,\alpha_{i_k})$ is generated by

$$\{\alpha_1\} \cup \left\{ \frac{Q}{\alpha_1\pi_X} \,\middle|\, X \subseteq \{2,\ldots,n\},\ |X| = k-2 \right\}.$$

Proof. Let $N = \bigcap_{2\le i_2<\cdots<i_k}(\alpha_1,\alpha_{i_2},\ldots,\alpha_{i_k})$ and $L = \bigcap_{2\le i_1<\cdots<i_k}(\alpha_{i_1}, \ldots, \alpha_{i_k})$. Our goal is to show that $I = N \cap L$. To prove this we shall induct on k. If $k = 1$ then the result is clear. Now assume $k > 1$. To prove the inductive step we shall induct on n. If $n = k$, then again the result is clear. Now assume $n > k$.

Let $\phi : S \to S/(\alpha_1)$ and denote $\phi(f)$ by $\overline{f}$. Since ϕ is surjective and $\ker(\phi)$ is a subset of every $(\alpha_1, \alpha_{i_2}, \ldots, \alpha_{i_k})$ we have

$$\phi(N) = \bigcap \phi(\alpha_1,\alpha_{i_2},\ldots,\alpha_{i_k}) = \bigcap(\overline{\alpha_{i_2}},\ldots,\overline{\alpha_{i_k}}).$$

Since $\mathcal{A}$ is generic $Q'' = \phi(Q/\alpha_1)$ is square free and so, by induction on k, $\phi(N)$ is generated by the $\frac{Q''}{\overline{\pi_X}} = \phi\left(\frac{Q}{\alpha_1\pi_X}\right)$ where $X \subseteq \{2,\ldots,n\}$ and $|X| = k-2$. This shows that

$$N = (\alpha_1) + \left(\left\{ \frac{Q}{\alpha_1\pi_X} \,\middle|\, X \subseteq \{2,\ldots,n\}, |X| = k-2 \right\} \right).$$

Denote the first ideal on the right by J and the second by K. By induction on n we see that

$$L = \left(\left\{ \frac{Q}{\alpha_1\pi_X} \,\middle|\, X \subseteq \{2,\ldots,n\}, |X| = k-1 \right\} \right).$$

Thus I is an intersection of the form $(J + K) \cap L$ with $K \subseteq L$ and whence $I = J \cap L + K$.

We claim that

$$J \cap L + K = \left(\left\{ \frac{Q}{\pi_X} \,\middle|\, X \subseteq \{1,\ldots,n\},\ |X| = k-1 \right\} \right)$$

and denote the latter ideal by M. Clearly $M \subseteq I$ and $K \subseteq M$. It remains to show that $J \cap L \subseteq M$. If $g \in J \cap K$ then $g = h\alpha_1 = \sum_X f_X \frac{Q}{\alpha_1\pi_X}$. If $Y \subseteq \{2,\ldots,n\}$ with $|Y| = k-1$ and $I_Y = (\{\alpha_i \mid i \in Y\})$ then consider $\psi : S \to S/I_Y$ and again denote $\psi(f)$ by $\overline{f}$. Since $k < \ell$, this is an integral domain. Note that $\overline{g} = \overline{h\alpha_1} = \overline{f_Y}\frac{Q}{\alpha_1\pi_Y}$ and hence there exists h_Y such that

$\overline{f_Y} = \overline{\alpha_1}\overline{h_Y}$ and so $f_Y = \alpha_1 h_Y + g_Y$ where $g_Y \in I_Y$, but then $f_Y \frac{Q}{\alpha_1 \pi_Y} \in M$. Since this holds for all Y we see that $J \cap L \subseteq M$ and the result is proven. $\square$

We can use the above result to determine generators for $\mathrm{D}(\mathcal{A}_k)$ if $\mathcal{A}$ is essential and boolean.

Theorem 4.5. *Let $\mathcal{A} = \{\mathrm{H}_1, \ldots, \mathrm{H}_\ell\}$ be a boolean ℓ-arrangement with $\mathrm{H}_i = V(x_i)$ and let $Q = \prod_{i=1}^{\ell} x_i$ then, for each $k > 1$, $\mathrm{D}(\mathcal{A}_k)$ is generated as an S-module by*

$$\{x_i D_i\}_{i=1}^{\ell} \bigcup \left\{ \frac{Q}{Q_X} D_i \,\middle|\, X \in L(\mathcal{A})_{\ell-k+1}, X \subseteq \mathrm{H}_i, 1 \le i \le \ell \right\}.$$

Proof. Let $\theta \in \mathrm{D}(\mathcal{A}_k)$ and write $\theta = \sum p_i D_i$. Choose $1 \le j \le \ell$ and let $J = \{1, \ldots, n\} \setminus \{j\}$. Now $p_j = \theta(x_j) \in \bigcap_{\substack{Y \subseteq J \\ |Y|=k-1}} (\alpha_j, \{\alpha_m\}_{m \in Y}) = \left(\alpha_j, \left\{ \frac{Q}{\alpha_j \pi_Y} \,\middle|\, Y \subseteq J, |Y| = k-2 \right\} \right)$. But each $\alpha_j \pi_Y$ is the defining polynomial of a subarrangement of $\mathcal{A}$ of size $k-1$ corresponding to an element of $L(\mathcal{A})$ of dimension $\ell - k + 1$. The result follows. $\square$

Note that as k increases the modules pick up smaller and smaller "factors" of the QD_i. This pattern will also hold for generic arrangements. The above result also allows us to compute the projective dimensions of $\mathrm{D}(\mathcal{A}_2)$ for a boolean arrangement.

Theorem 4.6. *If $\mathcal{A}$ is boolean then the $\mathrm{pdim}_S(\mathrm{D}(\mathcal{A}_2)) = 1$.*

Proof. It suffices to consider the case where $\mathcal{A}$ is essential. Let P_0 be the free S-module with generators $\alpha_1, \ldots, \alpha_\ell, \beta_1, \ldots, \beta_\ell$. Let $\psi : P_0 \to \mathrm{D}(\mathcal{A})$ be the map define by $\psi(\alpha_i) = x_i D_i$ and $\psi(\beta_i) = \frac{Q}{x_i} D_i$. We will show that $\ker\psi$ is a free S-module.

Suppose $\psi(\sum f_i \alpha_i + \sum g_i \beta_i) = 0$, then for each i we have $x_i f_i + \frac{Q}{x_i} g_i = 0$. Thus there exists h_i so that $f_i = \frac{Q}{x_i} h_i$ and $g_i = -x_i h_i$. Thus $\ker\psi = \left\{ \sum \left(\frac{Q}{x_i} \alpha_i - x_i \beta_i \right) h_i \,\middle|\, h_i \in S \right\}$ is a free S-module of rank ℓ. $\square$

5. Generic (l, k) arrangements.

In this section we assume that $\mathcal{A}$ is a generic ℓ-arrangement and work with the associated (ℓ, k) arrangements. We begin by finding generating sets for $\mathrm{D}(\mathcal{A}_2)$ using elements of $L(\mathcal{A})$. It is useful to consider some examples of $(\ell, 2)$-arrangements arising in this fashion. It is convenient to describe the subspaces using their corresponding ideals.

Example 5.1. In $\mathbf{K}^3$ let $\mathcal{A} = \{x, y, z, x + y + z\}$, so that

$$\mathcal{A}_2 = \{(x, y), (x, z), (x, x + y + z), (y, z), (y, x + y + z), (z, x + y + z)\}.$$

$D(\mathcal{A}_2)$ has generators

$$\theta_E, \qquad yz(x+y+z)D_x, \qquad xy(D_x - D_y), \qquad xyzD_z$$

$$y(x+y+z)D_y, \qquad xz(x+y+z)D_y, \qquad z(x+y+z)D_z, \qquad xy(x+y+z)D_z.$$

In $\mathbf{K}^3$ if $\mathcal{B} = \{x, y, x+y, z, x+y+z\}$ then $D(\mathcal{B})$ is a free S-module but $\mathcal{B}_2$ is the same arrangement as $\mathcal{A}_2$ listed above. Hence a free arrangement and a generic arrangement may yield identical $(\ell, 2)$-arrangements.

Example 5.2. In $\mathbf{K}^4$ let $\mathcal{A} = \{w, x, y, z, w+x+y+z\}$, so that

$$\mathcal{A}_2 = \{(w, x), (w, y), (w, z), (w, w+x+y+z), (x, y),$$

$$(x, z), (x, w+x+y+z), (y, z), (y, w+x+y+z), (z, w+x+y+z)\}$$

$D(\mathcal{A}_2)$ has generators

$$\theta_E, \qquad xyz(w+x+y+z)D_w, \qquad wyz(w+x+y+z)D_x$$

$$wxz(w+x+y+z)D_y, \qquad wxy(w+x+y+z)D_z, \qquad xy(D_x - D_y)$$

$$yz(D_y - D_z), \qquad wx(D_w - D_x), \qquad xyzwD_y$$

$$y(w+x+y+z)D_y, \qquad z(w+x+y+z)D_z.$$

These examples motivate the following definition. As usual, choose a basis $\{x_1, \ldots, x_\ell\}$ of V^*. Let $H \in \mathcal{A}$ with α_H its defining functional and write $\alpha_H = \sum a_i x_i$. Let $\eta_H = \frac{Q}{\alpha_H} \sum a_i D_{x_i}$. One can easily see that $\eta_H \in D(\mathcal{A}_2)$. Our goal is to show that the θ_X and η_H together with θ_E generate $D(\mathcal{A}_2)$ for a generic arrangement.

To prove this we will induct on the number of hyperplanes of $\mathcal{A}$ using the method of deletion and restriction. An examination of the examples listed above leads us to the following lemma.

Lemma 5.3. *If $\mathcal{A}$ is generic, then $r : D(\mathcal{A}_2) \to D(\mathcal{A}'')$ is surjective.*

Proof. The result is clear if $\ell = 2$. If $\ell > 2$ and $\mathcal{A}$ is generic, then so is $\mathcal{A}''$. One then notes that $\{r(\theta_X)\}_{X \in L(\mathcal{A})_1} \cup r(\theta_E)$ is the generating set of $D(\mathcal{A}'')$ given in Theorem 3.4. $\square$

We can now state one of our main results:

Theorem 5.4. *Let $\mathcal{A}$ be a generic ℓ-arrangement with $\ell > 2$ and $H \in \mathcal{A}$. The sequence*

$$D(\mathcal{A}_2') \oplus S \xrightarrow{\phi} D(\mathcal{A}_2) \xrightarrow{r} D(\mathcal{A}'') \to 0$$

is exact where $\phi(\theta, f) = \alpha_H \theta + f\eta_H$.

Proof. Since $r(\eta_{\mathrm{H}}) = 0$ we only need to show that $\ker(r) \subseteq \operatorname{im}\phi$. Let $\mathcal{A} = \{\mathrm{H}, \mathrm{H}_2, \dots, \mathrm{H}_n\}$ and choose coordinates so $\mathrm{H} = V(x_1)$. Let $\theta \in \ker(r)$ and write $\theta = \sum_{i=1}^{\ell} p_i D_i$. Since $r(\theta) = 0$, $p_i = x_i q_i$ for $i > 1$.

Since $\mathcal{A}$ is generic the ideals $\{(x_1, \alpha_i)\}_{i>1}$ are distinct, hence $\theta(x_1) \in \cap_{i=2}^{n}(x_1, \alpha_i) = (x_1, \prod_{i=2}^{n} \alpha_i) = (x_1, \frac{Q}{x_1})$. So write $p = x_1 q_1 + \frac{Q}{x_1} s$, and $\tau = \sum_{i=1}^{\ell} q_i D_{x_i}$ so that $\theta = x_1 \tau + s \eta_X$. It suffices to show $\tau \in \mathrm{D}(\mathcal{A}_2')$. Let $X \in L(\mathcal{A}')_2$, then since $\frac{Q}{x_1} \in I_X$ we see that $\theta(I_X) \subset I_X$ iff $x_1 \mu(I_X) \subseteq I_X$. But since $x_1 \notin I_X$ (as $\mathcal{A}$ is generic), we see that $x_1 \mu(I_X) \subseteq I_X$ iff $\mu(I_X) \subseteq I_X$, hence $\mu \in \mathrm{D}(\mathcal{A}_2')$. $\square$

With the above exact sequence we can prove the following result.

Theorem 5.5. *Let $\mathcal{A}$ be a generic arrangement, $\ell > 2$ and $K \in \mathcal{A}$. $\mathrm{D}(\mathcal{A}_2)$ is generated by $\{\theta_E\} \cup \{\theta_X \mid X \in L(\mathcal{A})_1, X \subseteq K\} \cup \{\eta_{\mathrm{H}} \mid \mathrm{H} \in \mathcal{A}\}$.*

Proof. The proof here is very similar to that of Theorem 3.4. We induct on ℓ. We shall prove the base case together with the inductive step. To show these we induct on $|\mathcal{A}|$. If $\mathcal{A}$ is boolean then the result follows from Theorem 4.5. If $|\mathcal{A}| > \ell$ then we use the exact sequence of the previous theorem. If $\theta \in \mathrm{D}(\mathcal{A}_2)$ then since r is surjective and $\mathcal{A}''$ is generic we can choose $\eta \in \tilde{\mathrm{D}}(\mathcal{A})$ so that $r(\eta) = r(\theta)$. Hence $\theta - \eta$ is in the image of ϕ. But $\mathcal{A}'$ is generic or boolean, so by induction its generators are given by the theorem. As in the proof of Theorem 3.4 one sees that when the generators of $\mathrm{D}(\mathcal{A}')$ are multiplied by α_{H} they become the generators postulated for $\mathrm{D}(\mathcal{A})$. The result follows. $\square$

Now assume $k > 2$. For each $\mathrm{H} \in \mathcal{A}$ write $\alpha_{\mathrm{H}} = \sum a_i x_i$. Now let $Y \in L(\mathcal{A})_{\ell-k+1}$ with $Y \subseteq \mathrm{H}$ and let $\eta_{\mathrm{H},Y} = \frac{Q}{\pi_Y} \sum a_i D_{x_i}$. Let $F(\mathcal{A}_k)$ be the S-submodule of $\mathrm{D}(\mathcal{A}_k)$ generated by θ_E together with the θ_X and the set of all $\eta_{\mathrm{H},Y}$.

The rest of this section is devoted to the proof of the following theorem.

Theorem 5.6. *Let $\ell > 2$, $\mathcal{A}$ be a generic ℓ-arrangement, $2 \leq k < \ell$ and $K \in \mathcal{A}$. $\mathrm{D}(\mathcal{A}_k)$ is generated as an S-module by*

$$\{\theta_E\} \cup \{\theta_X \mid X \in L(\mathcal{A})_1, \ X \subseteq K\}$$
$$\cup \{\eta_{\mathrm{H},Y} \mid \mathrm{H} \in \mathcal{A}, \ Y \in L(\mathcal{A})_{\ell-k+1}, Y \subseteq \mathrm{H}\}.$$

We shall prove this by induction on ℓ. The base case $\ell = 3$ is Theorem 5.5. Now assume $\ell > 3$. To prove this we induct on k. The case $k = 2$ is also Theorem 5.5, so assume $k > 2$. To show this we induct on the number of hyperplanes of $\mathcal{A}$. If $|\mathcal{A}| = \ell$, then the result follows from Theorem 4.5. Hence, we assume that $|\mathcal{A}| > \ell$.

We begin by choosing $\mathrm{H} \in \mathcal{A}$ with $\mathrm{H} \neq \mathrm{K}$, and choose coordinates so that $\alpha_{\mathrm{H}} = x_1$. Let $\mathcal{A}'$ be the deletion of $\mathcal{A}$ with respect to H and $\mathcal{A}''$

the restriction of $\mathcal{A}$ to H. Let $F(\mathcal{A}_k)$ be the S-module generated by the derivations given in the theorem. Our goal is to show that $F(\mathcal{A}_k) = \mathrm{D}(\mathcal{A}_k)$.

Lemma 5.7. *If $\mathcal{A}$ is generic, then $r : F(\mathcal{A}_k) \to \mathrm{D}(\mathcal{A}''_{k-1})$ is surjective.*

Proof. Since $\mathcal{A}$ is generic, so is $\mathcal{A}''$ and, by induction on ℓ, $\mathrm{D}(\mathcal{A}''_{k-1})$ has generators given in the theorem. One then notes that restriction of the generators of $F(\mathcal{A}_k)$ are either zero or precisely the generators given for $\mathrm{D}(\mathcal{A}''_{k-1})$. In particular, let $\mathrm{H}'' \in \mathcal{A}''$ and $Y'' \in L(\mathcal{A}'')_{\ell-k+1}$ then there is a unique $Y \in L(\mathcal{A})$ so that $Y = Y''$. Since $Y \subset \mathrm{H}$ we have $x_1 | \pi_Y$. Now $\eta_{\mathrm{H},Y} = \frac{Q}{\pi_Y} \sum_{k=1}^{\ell} a_i D_i = \frac{Q/\alpha_{\mathrm{H}}}{\pi_Y/\alpha_{\mathrm{H}}} \sum_{i=1}^{\ell} a_i D_i$ and since $\frac{Q}{\alpha_{\mathrm{H}}}|_{x_1=0} = Q''$ and $\frac{\pi_Y}{\alpha_{\mathrm{H}}}|_{x_1=0} = \pi_{Y''}$ we see that $r(\eta_{\mathrm{H},Y}) = \frac{Q''}{\pi_{Y''}} \sum_{i=2}^{\ell} a_i Dx_i = \eta_{\mathrm{H}'',Y''}$. $\square$

Lemma 5.8. *Let $\mathcal{A}$ be a generic arrangement and $\mathrm{H} \in \mathcal{A}$, then the sequence*

$$\mathrm{D}(\mathcal{A}'_k) \bigoplus_{\substack{Y \in L(\mathcal{A})_{k-1} \\ Y \subseteq \mathrm{H}}} S \xrightarrow{\phi} \mathrm{D}(\mathcal{A}_k) \xrightarrow{r} \mathrm{D}(\mathcal{A}''_{k-1}) \longrightarrow 0$$

is exact where $\phi\left(\theta, (f_Y)\right) = \alpha_{\mathrm{H}}\theta + \sum_Y f_Y \eta_{\mathrm{H},Y}$.

Proof. We only need to show that $\ker(r) \subseteq \mathrm{im}\phi$. Let $\mathcal{A} = \{\mathrm{H}, \mathrm{H}_2, \dots, \mathrm{H}_n\}$ and choose coordinates so that $\mathrm{H} = V(x_1)$. Let $\theta \in \ker(r)$ and write $\theta = \sum_{i=1}^{\ell} p_i D_i$. If $r(\theta) = 0$ then $p_i = x_1 q_i$ for each $i > 1$.

Since $\mathcal{A}$ is generic the ideals $\{(x_1, \alpha_{i_2}, \dots, \alpha_{i_k}) | 2 \leq i_2 < \cdots < i_k\}$ are distinct, hence by Lemma 4.4

$$\theta(x_1) \in \bigcap_{2 \leq i_2 < \cdots < i_k} (x_1, \alpha_{i_2}, \dots \alpha_{i_k})$$

$$= \left(x_1, \left\{\frac{Q}{x_1 \pi_X} \middle| X \subset \{2, \dots n\} \, |X| = k - 2\right\}\right).$$

Write $p_1 = x_1 q_1 + \sum_X s_X \frac{Q}{x_1 \pi_X} s$, and let $\tau = \sum_{i=1}^{\ell} q_i D_i$ so that $\theta = x_1 \tau + \sum s_X \eta_X$. Hence, it suffices to show $\tau \in \mathrm{D}(\mathcal{A}'_k)$. Let $Y \in L(\mathcal{A}')_k$, then since $\frac{Q}{x_1 \pi_X} \in I_Y$ for each $X \neq Y$ we see that $\theta(I_Y) \subset I_Y$ iff $x_1 \tau(I_Y) \subseteq I_Y$. But since $x_1 \notin I_Y$ (as $\mathcal{A}$ is generic), we see that $x_1 \tau(I_Y) \subseteq I_Y$ iff $\tau(I_Y) \subseteq I_Y$, hence $\tau \in \mathrm{D}(\mathcal{A}'_k)$. $\square$

With the above exact sequence we can finish the proof our main result. If $\theta \in \mathrm{D}(\mathcal{A}_k)$ then choose $\eta \in F(\mathcal{A}_k)$ so that $r(\theta) = r(\eta)$, in which case $\theta - \eta \in \mathrm{im}\phi$. But by induction $\mathrm{D}(\mathcal{A}'_k) = F(A'_k)$ and since ϕ sends the generators of $\mathrm{D}(\mathcal{A}'_k)$ to generators of $F(\mathcal{A}_k)$, we see that $\theta \in F(\mathcal{A}_k)$.

An interesting result would be to compute minimal resolutions of these modules.

References

[1] P. Orlik and H. Terao, *Arrangements of Hyperplanes*, Springer-Verlag, 1992.

[2] L. Rose and H. Terao, *A free resolution of the module of logarithmic forms of a generic arrangement*, J. of Algebra, **136** (1991), 376-400.

[3] C. Weibel, *An Introduction to Homological Algebra*, Cambridge University Press, 1994.

[4] S. Yuzvinsky, *A free resolution of the module of derivations for generic arrangement*, J. of Algebra, **136** (1991), 432-438.

DEPARTMENT OF MATHEMATICAL SCIENCES
UNIVERSITY OF ALASKA FAIRBANKS
FAIRBANKS, AK 99775
E-mail address: ffjjw@uaf.edu

CONTENTS

Volume 198, no. 1 and no. 2

Guidelines for Authors

Authors may submit manuscripts at pjm.math.berkeley.edu/about/journal/submissions.html and choose an editor at that time. Exceptionally, a paper may be submitted in hard copy to one of the editors; authors should keep a copy.

By submitting a manuscript you assert that it is original and is not under consideration for publication elsewhere. Instructions on manuscript preparation are provided below. For further information, visit the web address above or write to pacific@math.berkeley.edu or to Pacific Journal of Mathematics, University of California, Los Angeles, CA 90095–1555. Correspondence by email is requested for convenience and speed.

Manuscripts must be in English, French or German. A brief abstract of about 150 words or less in English must be included. The abstract should be self-contained and not make any reference to the bibliography. Also required are keywords and subject classification for the article, and, for each author, postal address, affiliation (if appropriate) and email address if available. A home-page URL is optional.

Authors are encouraged to use LaTeX, but papers in other varieties of TeX, and exceptionally in other formats, are acceptable. At submission time only a PDF file is required; follow the instructions at the web address above. Carefully preserve all relevant files, such as LaTeX sources and individual files for each figure; you will be asked to submit them upon acceptance of the paper.

Bibliographical references should be listed alphabetically at the end of the paper. All references in the bibliography should be cited in the text. Use of BibTeX is preferred but not required. Any bibliographical citation style may be used but tags will be converted to the house format (see a current issue for examples).

Figures, whether prepared electronically or hand-drawn, must be of publication quality. Figures prepared electronically should be submitted in Encapsulated PostScript (EPS) or in a form that can be converted to EPS, such as GnuPlot, Maple or Mathematica. Many drawing tools such as Adobe Illustrator and Aldus FreeHand can produce EPS output. Figures containing bitmaps should be generated at the highest possible resolution. If there is doubt whether a particular figure is in an acceptable format, the authors should check with production by sending an email to pacific@math.berkeley.edu.

Each figure should be captioned and numbered, so that it can float. Small figures occupying no more than three lines of vertical space can be kept in the text ("the curve looks like this:"). It is acceptable to submit a manuscript will all figures at the end, if their placement is specified in the text by means of comments such as "Place Figure 1 here". The same considerations apply to tables, which should be used sparingly.

Forced line breaks or page breaks should not be inserted in the document. There is no point in your trying to optimize line and page breaks in the original manuscript. The manuscript will be reformatted to use the journal's preferred fonts and layout.

Page proofs will be made available to authors (or to the designated corresponding author) at a Web site in PDF format. Failure to acknowledge the receipt of proofs or to return corrections within the requested deadline may cause publication to be postponed.