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Let Ω be a smoothly bounded convex domain of finite type in $\mathbb{C}^n$. We show that a divisor in Ω satisfying the Blaschke condition (respectively associated to a current of order $a > 0$) can be defined by a function in the Nevanlinna class $N_0(\Omega)$ (respectively the Nevanlinna-Djrbachian class $N_a(\Omega)$). The proof is based on $L^1(b\Omega)$ estimates (resp. weighted $L^1(\Omega)$ estimates) for the solution of the $\bar\partial$-equation on Ω.

1. Introduction and statement of results.

Let $\Omega = \{\rho < 0\}$ be a smoothly bounded domain in $\mathbb{C}^n$ with ρ a defining function.

$N_0(\Omega)$ denotes the Nevanlinna class for Ω and $N_a(\Omega)$, where $a > 0$, denote the Nevanlinna-Djrbachian classes. Recall that a function h is in $N_a(\Omega)$ if h is holomorphic on Ω and satisfies the condition

$$\begin{aligned}\sup_{\varepsilon>0}\int_{b\Omega_\varepsilon} \ln^+ |h(z)|\, d\lambda_{2n-1}(z) &< +\infty \ \text{ if } a = 0\\ \int_\Omega |\rho(z)|^{a-1} \ln^+ |h(z)|\, d\lambda_{2n}(z) &< +\infty \ \text{ if } a > 0.\end{aligned}$$

Here, $\Omega_\varepsilon = \{\rho < -\varepsilon\}$.

For h holomorphic in Ω consider M the zero set of h and (M_j, ν_j) the divisor associated to h. The divisor (M_j, ν_j) satisfies the condition (B_a) if

$$(B_a) \qquad \int_M |\rho(z)|^{a+1}\, d\lambda_{2n-2}(z) = \sum_j \nu_j \int_{M_j} |\rho(z)|^{a+1}\, d\lambda_{2n-2}(z) < +\infty$$

where $d\lambda_{2n-k}$ denotes the $(2n-k)$-dimensional volume element; from now on we will write $d\lambda$ for $d\lambda_{2n}$ for brevity.

(B_0) is the Blaschke condition for (M_j, ν_j) (and for the associated Lelong current $[M]$); (M_j, ν_j) satisfies (B_a) means exactly that $[M]$ is a Lelong current of order a on Ω.

If h is in the class N_a, thus its associated divisor (M_j, ν_j) satisfies (B_a) (for $a > 0$ see for instance [**DH77**]) . Henkin ([**H77**]) and Skoda ([**Sk76**]) proved independently that the condition (B_0) is a sufficient condition on a divisor for it being defined by a function belonging to the class N_0, in the case of a smoothly bounded strictly pseudoconvex domain (under the topological condition $H^2(\Omega, \mathbb{C}) = 0$).

Dautov and Henkin ([**DH77**]) obtained the equivalent result for $a > 0$.

Recently Bruna, Charpentier and Dupain ([**BCD98**]) generalized the Henkin-Skoda's result to the case of smoothly bounded convex domains of finite type in $\mathbb{C}^n$ which are of strict type i.e., a domain Ω satisfies the strict-type condition if the following holds:

There exists a constant c such that for all boundary points z, all unit vectors v in the complex-tangent space $T_z^c(b\Omega)$ and all small real t, one has

$$(\star) \qquad c^{-1}\,\rho(z+tv) \le \rho(z+itv) \le c\,\rho(z+tv).$$

In solving the equation $i\partial\overline{\partial}W = T$ for T a (1,1) positive closed current satisfying the Blaschke condition, the authors need the condition ($\star$) only at the last step of their proof i.e., the step where the equation $\overline{\partial}u = f$ is solved on the domain Ω with an $L^1(b\Omega)$-estimate on u.

In this paper we treat this problem of characterization of the zero sets of functions in the classes N_a, for $a \ge 0$, in smoothly bounded convex domains of finite type in $\mathbb{C}^n$ (without the strict-type condition ($\star$)).

Bruna, Charpentier and Dupain ([**BCD98**]) have introduced a suitable non-isotropic norm $|||.|||_k$ of forms on a convex domain Ω of finite type; the definition of this norm is based on geometric quantities introduced by McNeal ([**Mc94**]) and is a bit technical; so we do not give it precisely in the introduction. In terms of this non-isotropic norm, they obtained a new necessary condition on a divisor to be defined by a function in the Nevanlinna class of Ω, as shown by the first theorem we recall below. As already mentioned, two important results in [**BCD98**] are valid without the condition ($\star$):

Theorem 1.1 ([**BCD98**, Theorem 1.1])**.** *Let $\Omega \subset\subset \mathbb{C}^n$ be a convex domain of finite type m in the d'Angelo sense with a C^∞-smooth boundary. There exists a constant C such that*

$$\int_\Omega d(z)\,|||T(z)|||_k\,d\lambda(z) \;\le\; C\int_\Omega d(z)\,||T(z)||\,d\lambda(z)$$

for all closed positive $(1,1)$-currents on Ω.

Here, $d(z)$ means $\mathrm{dist}(z, b\Omega)$.

Theorem 1.2 ([**BCD98**, Theorem 1.2])**.** *If Ω is as in Theorem 1.1, there exists a constant C such that the solution of the equation $dw = T$ obtained by the Poincaré homotopy formula satisfies*

$$\int_\Omega |||w(z)|||_k\,d\lambda(z) \;\le\; C\int_\Omega d(z)\,|||T(z)|||_k\,d\lambda(z).$$

Our aim will thus be to solve the $\bar{\partial}$-equation with suitable estimates without using the strict-type condition. In [**Cu97**], [**Cu01**], we presented $\bar{\partial}$ solving integral operator whose kernels are well adapted to the geometry of the convex domains of finite type. Using such integral operators, we can prove:

Theorem 1.3. *Under the assumptions of Theorem 1.1 for the domain Ω, the equation $\bar{\partial}u = f$, for f a $\bar{\partial}$-closed $(0,1)$-form with coefficients in $C^1(\overline{\Omega})$ has a solution u C^1-smooth in Ω such that*

a) $\displaystyle\int_\Omega d(z)^{a-1}|u(z)|\,d\lambda(z) \leq cst(\Omega,a)\int_\Omega d(z)^a|||f(z)|||_k\,d\lambda(z),$ *for $a > 0$.*

b) $\displaystyle\int_{b\Omega} |u(z)|\,d\lambda_{2n-1}(z) \leq cst(\Omega)\int_\Omega |||f(z)|||_k\,d\lambda(z).$

From Theorem 1.1 and 1.2 and part b) of Theorem 1.3 follows:

Corollary 1.4. *Under the assumptions of Theorem 1.1 on Ω, every divisor in Ω satisfying the Blaschke condition can be defined by a function in the Nevanlinna class $N_0(\Omega)$.*

One can prove the analogous of Theorem 1.1 and Theorem 1.2 with $d(z)^{a+1}$ in place of $d(z)$ and $d(z)^a\,|||w(z)|||_k$ in place of $|||w(z)|||_k$; since $a > 0$ there are no difficulties: It just suffices to mimic the proofs given in [**BCD98**]. Taking also into account part a) of Theorem 1.3, we thus obtain:

Corollary 1.5. *Under the assumptions of Theorem 1.1 on Ω, every divisor in Ω satisfying the condition (B_a), where $a > 0$, can be defined by a function in the Nevanlinna-Djrbachian class $N_a(\Omega)$.*

The plan of the paper is as follows. Section 2 is devoted to the proof of Theorem 1.3; we also give some comment on the proof of Corollary 1.4. We use the same kernel as in [**Cu97**], [**Cu01**]. We postpone to Section 3 –as an appendix– all the geometrical notions we need regarding the convex domains of finite type.

In the sequel, we will use the standard notation $A \lesssim B$, for A and B functions of several variables, to denote that $A \leq C\,B$ for a constant C independent of certain parameters which will be clear in the context. Of course $A \approx B$ will mean $A \lesssim B$ and $B \gtrsim A$.[1]

The contents of the present paper were distributed as a preprint [**Cu98**].

2. Proof of Theorem 1.3.

Ω is a bounded convex domain in $\mathbb{C}^n$ with a C^∞-smooth boundary. Suppose every $p \in b\Omega$ is a point of finite type $\leq m$, in the sense of D'Angelo. Following

[1] I have heard that K. Diederich and E. Mazzilli have proved part b) of Theorem 1.3 and Corollary 1.4 in a preprint headed *"Zero varieties for the Nevanlinna class on all convex domains of finite type"* – they use another kernel based on the support functions recently constructed by K. Diederich and J.E. Fornaess [**DiFo99**].

[**BCD98**] we may assume that $0 \in \Omega$ and will choose as defining function for Ω the function $\rho = g - 1$, where g is the gauge function of Ω; ρ is of class C^∞ on $\Omega \setminus \{0\}$.

2.1. Preliminaries.

• First we recall the definition of the kernel introduced in [**Cu97**].

$\mathcal{B}(\zeta, z)$ will denote the Bergman kernel for the domain Ω; $\mathcal{B}(\zeta, z)$ is holomorphic in z, antiholomorphic in ζ; under the assumptions made on Ω, $\mathcal{B}(.,.)$ is of class C^∞ on $\overline{\Omega} \times \overline{\Omega} \setminus \Delta_{b\Omega}$, where $\Delta_{b\Omega}$ denotes the diagonal of $b\Omega \times b\Omega$ ([**Mc94**]).

Denote

$$\widetilde{Q} = \widetilde{Q}(\zeta, z) = \frac{1}{\mathcal{B}(\zeta, \zeta)} \int_0^1 (\partial_Z \mathcal{B})\big(\zeta, \zeta + t(z - \zeta)\big)\, dt, \tag{2.1}$$

where

$$(\partial_Z \mathcal{B})\big(\zeta, \zeta + t(z - \zeta)\big) = \sum_{j=1}^{n} \frac{\partial \mathcal{B}}{\partial Z_j}\big(\zeta, \zeta + t(z - \zeta)\big) dz_j$$

and $\partial/\partial Z_j$ denotes a derivative with respect to the second variable.

Let $N \geq 2n$ be a positive integer te be fixed later on. We define for $(\zeta, z) \in \Omega \times \overline{\Omega} \setminus \Delta$, where Δ denotes the diagonal of $\mathbb{C}^n \times \mathbb{C}^n$

$$\begin{aligned} K(\zeta, z) &= \sum_{k=0}^{n-1} c_{(k,n,N)} \left(\frac{\mathcal{B}(\zeta, z)}{\mathcal{B}(\zeta, \zeta)} \right)^{N-k} \\ &\qquad \times \frac{\big(\partial_z |\zeta - z|^2\big) \wedge (\overline{\partial}_\zeta \widetilde{Q})^k \wedge \big(d\partial_z |\zeta - z|^2\big)^{n-k-1}}{|\zeta - z|^{2n-2k}} \\ &= \sum_{k=0}^{n-1} c_{(k,n,N)} K^{(k)}(\zeta, z), \end{aligned} \tag{2.2}$$

where $c_{(k,n,N)} = -(-1)^{n(n-1)/2} \binom{N}{k}$.

Proposition 2.1 ([**Cu97**, Proposition 2.1])**.** *If f is a (n, q)-form with coefficients in $C^1(\overline{\Omega})$, $q \geq 1$, then $\forall z \in \Omega$,*

$$f(z) = cst(n, q) \left[\overline{\partial}_z \int_\Omega f(\zeta) \wedge K(\zeta, z) - \int_\Omega \overline{\partial} f(\zeta) \wedge K(\zeta, z) \right].$$

A classical approximation argument reduces the proof of Theorem 1.3 to the case of forms which have coefficients in $C^1(\overline{\Omega})$. Define, for F a (n,.)-form

$$\Theta F(z) = \int_\Omega F(\zeta) \wedge K(\zeta, z). \tag{2.3}$$

ΘF is a solution of the equation $\bar{\partial} v = F$ for F a $\overline{\partial}$-closed form in $C^1_{n,q}(\overline{\Omega})$ (cf. Proposition 2.1).

Let f be a $(0,1)$-form which is assumed to be C^1-smooth in $\bar\Omega$.

(2.4) We denote $\omega(\zeta) = d\zeta_1 \wedge ... \wedge d\zeta_n$ and define $\tilde f(\zeta) = f(\zeta) \wedge \omega(\zeta)$. Thus $\Theta \tilde f(z) = u(z)\omega(z)$.

We will choose this solution u of the equation $\bar\partial u = f$ and prove that it satisfies the estimates of Theorem 1.3. Of course, the integer N involved in the definition of K will be chosen in terms of the constant a for part a) of Theorem 1.3.

• Now let us recall the definition of the norm $|||.|||_k$ on forms (cf [**BCD98**]).

For $\eta > 0$ and $v \in \mathbb{C}^n$, $|v| = 1$, McNeal has introduced the quantity $\sigma(z, v, \eta)$ (where $z \in \overline{\Omega}$, z close to $b\Omega$), which measures the radius of the largest complex disc, centered at z, in the direction v, which lies entirely in the domain $\{\rho < \rho(z) + \eta\}$. More precisely

$$\sigma(z, v, \eta) = \sup\{r > 0 \mid \rho(z + \lambda v) - \rho(z) \leq \eta, |\lambda| \leq r\}. \tag{2.5}$$

Bruna, Charpentier and Dupain ([**BCD98**]) introduced the norm $k(z, v, \eta)$ $= d(z)/\sigma(z, v, \eta)$; they write $k(z, v)$ when η is d(z)/2 (up to a uniform constant multiple).

With respect to this norm $k(z, .)$, they defined a non-isotropic norm on forms as follows:

If T is a smooth 2-form on $\bar\Omega$, $|||T(z)|||_k$ is the smooth function

$$|||T(z)|||_k = \sup\left\{ \frac{|T(z)(u, v)|}{k(z, u)k(z, v)}; u \neq 0, v \neq 0 \right\}.$$

If w is a smooth 1-form on Ω, $|||w(z)|||_k$ is the smooth function

$$|||w(z)|||_k = \sup\left\{ \frac{|w(z)(u)|}{k(z, u)}; u \neq 0 \right\}. \tag{2.6}$$

• We want to prove, for f a $(0,1)$-form smooth in $\bar\Omega$

$$a \int_\Omega d(z)^{a-1} |u(z)|\, d\lambda(z) \leq \, cst \int_\Omega d(z)^a |||f(z)|||_k \, d\lambda(z), \quad \text{for} \;\; a > 0,$$

where the constant remains bounded if $a \to 0$. Part b) of Theorem 1.3 will be obtained by letting $a \to 0$.

Using Fubini's Theorem, we have thus to obtain the estimate

$$a \int_\Omega d(z)^{a-1} |K(\zeta, z) \wedge f(\zeta) \wedge \omega(\zeta)|\, d\lambda(z) \leq \, C\, d(\zeta)^a |||f(\zeta)|||_k \tag{2.7}$$

for $a > 0$ and a constant C uniformly bounded with respect to a if $0 < a < 1$.

Remark 2.2. For the case $a = 0$ there is no problem of regularity for the solution u defined in (2.4); if f has coefficients in $C^1(\bar\Omega)$, so one can extend

u continuously up to the boundary since we have proved in [**Cu01**] the following

$$\|\Theta\tilde{f}\|_{\Lambda^{1/m}(\Omega)} \lesssim \|f\|_{L^\infty(\Omega)},$$

where $\Lambda^{1/m}(\Omega)$ is the usual Lipschitz space and $\Theta\tilde{f}$ is defined in (2.3) and (2.4).

Remark 2.3. If one has to solve in Ω the equation $i\partial\bar{\partial}W = T$ with W in the Nevanlinna class for T a $(1,1)$ closed positive current satisfying the Blaschke condition, it suffices to assume T is smooth up to the boundary and to prove the estimate

$$\int_{b\Omega} |W(z)|\, d\lambda_{2n-1}(z) \lesssim \int_{\Omega} d(z)|T(z)|\, d\lambda(z).$$

Then, a classical argument of approximation permits to get a solution W in the Nevanlinna class of Ω.

Therefore Theorems 1.1, 1.2 and 1.3 b) yield Corollary 1.4. □

2.2. Estimates on the term $G = K^{(n-1)}$. Recall the expression of G:

$$G(\zeta, z) = \left(\frac{\mathcal{B}(\zeta, z)}{\mathcal{B}(\zeta, \zeta)}\right)^{N'} \frac{\left(\partial_z|\zeta - z|^2\right)\,(\overline{\partial}_\zeta \widetilde{Q})^{n-1}}{|\zeta - z|^2} \tag{2.8}$$

where $N' = N - n + 1$.

We refer to Section 3 for the definition of the quantities and notions under question below (see (3.3), (3.9) for $\mathcal{M}(z,\zeta)$ and $P(z,\eta)$).

There exists a constant $\beta > 0$ such that

$$\forall\zeta \in \Omega \cap \mathcal{U}, \forall\eta,\, 0 < \eta \ll 1, \mathcal{M}(z,\zeta) < \eta \Rightarrow z \in P(\zeta, \beta\eta).$$

Define

$$\begin{aligned} \mathcal{C}_0 = \mathcal{C}_0(\zeta) \quad &:= \quad P(\zeta, \beta\delta(\zeta)) \cap \mathcal{W} \cap \Omega, \text{ where } \delta(\zeta) = \operatorname{dist}(\zeta, b\Omega), \\ \mathcal{C}_\ell = \mathcal{C}_\ell(\zeta) \quad &:= \quad \{z \in \Omega \cap \mathcal{W}\,|2^{\ell-1}\delta(\zeta) \le \mathcal{M}(z,\zeta) < 2^\ell\delta(\zeta)\} \quad \text{for } \ell \ge 1, \end{aligned}$$

where $\mathcal{W} = 1/2\,\mathcal{U}$ and $\mathcal{U}$ is some $\mathcal{U}(p)$ defined in (3.7).

Notation. From now on we will often use the *short-hand notations* $d = d(z) = \operatorname{dist}(z, b\Omega)$ and $\delta = \delta(\zeta) = \operatorname{dist}(\zeta, b\Omega)$.

Let $(e_j^{(\ell)})_j = (e_j^{(\ell)}(\zeta))_j$ be a $2^\ell\beta\delta(\zeta)$-extremal basis at ζ (cf. Definition 3.1); if $(w_1, \ldots, w_n)$ is the new system of coordinates with respect to this basis, we denote $L_j^{(\ell)} = \partial/\partial w_j$ and $(L_j^{(\ell)*})_j$ is the basis of $T^*_\zeta\mathbb{C}^n$ which is the dual basis of $(L_j^{(\ell)})_j$.

We will use the basis $(L_j^{(\ell)*(z)})_j$ (resp. $(\overline{L}_j^{(\ell)*(\zeta)})_j$) in order to express the forms dz_k (resp. $d\overline{\zeta}_k$) in $G(\zeta,z) \wedge f(\zeta) \wedge \omega(\zeta)$, for $z \in \mathcal{C}_\ell(\zeta)$. (We have already used this process in [**Cu97**] and [**Cu01**, Subsection 4.2.2] in order to estimate derivatives of G.)

Convention. *In any ambiguous case, $L^{(z)}\mathcal{Y}(.,.)$ means a derivative with respect to the second variable of $\mathcal{Y}(.,.)$; $\mathcal{Y}$ will be essentially $\mathcal{B}$ or some derivative of $\mathcal{B}$.*

Thus we get from (2.8), (2.1) for $z\in\mathcal{C}_\ell(\zeta)$, $\zeta\in\mathcal{U}$

$$G(\zeta,z)\wedge f(\zeta)\wedge\omega(\zeta)=\frac{\mathcal{B}(\zeta,z)^{N'}}{\mathcal{B}(\zeta,\zeta)^{N'+n-1}\,|\zeta-z|^2}\,[H_1+H_2], \tag{2.9}$$

with

$$\begin{aligned}&H_1(\zeta,z)\\&=\frac{1}{\mathcal{B}(\zeta,\zeta)}\sum_{|I|=n,|J|=n} cst\,L_{i_0}^{(z)}\big(|\zeta-z|^2\big)\big(\overline{L}_{j_1}^{(\zeta)}\mathcal{B}\big)(\zeta,\zeta)\int_0^1\big(L_{i_1}^{(z)}\mathcal{B}\big)(\zeta,z_t)\,dt\\&\quad\times\prod_{k=2}^{n-1}\Big(\int_0^1\big(L_{i_k}^{(z)}\overline{L}_{j_k}^{(\zeta)}\mathcal{B}\big)(\zeta,z_t)\,dt\Big)\Big[f(\zeta)(e_{j_n}^{(\ell)})\Big]\,A_S(\zeta)L_I^{*(z)}\wedge\overline{L}_J^{*(\zeta)}\wedge L_S^{*(\zeta)},\\&H_2(\zeta,z)\\&=\sum_{|I|=n,|J|=n} cst\,L_{i_0}^{(z)}\big(|\zeta-z|^2\big)\\&\quad\times\prod_{k=1}^{n-1}\Big(\int_0^1\big(L_{i_k}^{(z)}\overline{L}_{j_k}^{(\zeta)}\mathcal{B}\big)(\zeta,z_t)\,dt\Big)\Big[f(\zeta)(e_{j_n}^{(\ell)})\Big]\,A_S(\zeta)L_I^{*(z)}\wedge\overline{L}_J^{*(\zeta)}\wedge L_S^{*(\zeta)},\end{aligned}$$

where $z_t=\zeta+t(z-\zeta)$, $I=(i_0,i_1,\dots,i_{n-1})$, $J=(j_1,\dots,j_n)$ and $S=(s_1,\dots,s_n)\in\mathbb{N}^{*n}$; $L_I^*=L_{i_0}^*\wedge\cdots\wedge L_{i_{n-1}}^*$, $\overline{L}_J^*=\overline{L}_{j_1}^*\wedge\cdots\wedge\overline{L}_{j_n}^*$, $L_S^*=L_{s_1}^*\wedge\cdots\wedge L_{s_n}^*$, $A_S(\zeta)$ is uniformly bounded in Ω , $\forall S\in\mathbb{N}^{*n}$.

Using (3.3), (3.9), (3.4) and (3.5) we have

$$\begin{aligned}&\varepsilon(\zeta,z_t)\approx|\rho(\zeta)|+|\rho(z_t)|+\mathcal{M}(\zeta,z_t)\gtrsim\delta(\zeta)\ ,\\&\inf_{0\le t\le 1}\operatorname{Vol}T_{\zeta,z_t}\gtrsim\operatorname{Vol}T_\zeta\approx\operatorname{Vol}P\big(\zeta,\delta(\zeta)\big),\end{aligned}\tag{2.10}$$

where T_ζ is a smallest tent (i.e., of smallest volume) containing ζ.

Hence it follows from (3.7), (3.2), (3.1) and (2.10)

$$\begin{aligned}\Big|\int_0^1\big(L_{i_1}^{(z)}\mathcal{B}\big)(\zeta,z_t)\,dt\Big|&\lesssim[\operatorname{Vol}P(\zeta,\delta)\ \sigma(\zeta,e_{i_1},\delta)]^{-1}.\\\Big|\int_0^1\big(L_{i_k}^{(z)}\overline{L}_{j_k}^{(\zeta)}\mathcal{B}\big)(\zeta,z_t)\,dt\Big|&\lesssim[\operatorname{Vol}P(\zeta,\delta)\ \sigma(\zeta,e_{j_k},\delta)\ \sigma(\zeta,e_{i_k},\delta)]^{-1}.\end{aligned}$$

From the definition (2.6) of $|||f|||_k$ we have

$$|f(\zeta)(e_j^{(\ell)}(\zeta))|\le\frac{|||f|||_k\ \delta(\zeta)}{\sigma(\zeta,e_j^{(\ell)}(\zeta),\delta)}.\tag{2.11}$$

Using moreover (3.7), (3.10) and (3.11) we can write

$$|G(\zeta,z)\wedge f(\zeta)\wedge\omega(\zeta)| \lesssim \frac{\delta^{N'+1}\,|||f(\zeta)|||_k}{\varepsilon(\zeta,z)^{N'}\,|\zeta-z|}\left[\prod_{j=1}^{n}\sigma(\zeta,e_j^{(\ell)},\delta)\right]^{-1}\sum_I\left[\prod_{k=1}^{n-1}\sigma(\zeta,e_{i_k}^{(\ell)},\delta)\right]^{-1}$$

where $I=(i_1,\dots,i_{n-1})$ are multi-indices with $1\le i_1<\dots<i_{n-1}\le n$.

For all I, let i_n be chosen such that $(1,\dots,n)=(i_1,\dots,i_n)$. We will need the following estimates (they are easily deduced respectively from the definition of $\mathcal{C}_\ell$, (3.9); (3.2); Definition 3.1).

$$\varepsilon(\zeta,z)\approx 2^\ell\delta(\zeta)\ \text{ if } z\in\mathcal{C}_\ell(\zeta)\ , \tag{2.12}$$

$$\sigma(\zeta,e_j^{(\ell)}(\zeta),\delta)\gtrsim 2^{-\ell/2}\sigma(\zeta,e_j^{(\ell)}(\zeta),\beta2^\ell\delta)\text{ uniformly in }\zeta,\ j\ ,\ell, \tag{2.13}$$
$$\sigma(\zeta,e_{i_n}^{(\ell)}(\zeta),\beta2^\ell\delta)\le\sigma(\zeta,e_2^{(\ell)}(\zeta),\beta2^\ell\delta).$$

Denoting $\tau_j^{(\ell)}(\zeta)=\sigma(\zeta,e_j^{(\ell)}(\zeta),\beta2^\ell\delta(\zeta))$, we obtain for $z\in\mathcal{C}_\ell(\zeta)$

$$|G(\zeta,z)\wedge f(\zeta)\wedge\omega(\zeta)|\lesssim\frac{|||f(\zeta)|||_k\,\tau_2^{(\ell)}(\zeta)\,\delta(\zeta)}{2^{(N'-n)\ell}\,|\zeta-z|\,\prod_{j=1}^n\left(\tau_j^{(\ell)}(\zeta)\right)^2}\,. \tag{2.14}$$

Using (3.4) and (3.6) we have thus to estimate

$$\mathcal{J}_\ell:=\frac{\tau_2^{(\ell)}(\zeta)\,\delta(\zeta)}{\mathrm{Vol}\,P(\zeta,2^\ell\delta)}\int_{P(\zeta,\beta2^\ell\delta)}\frac{d(z)^{a-1}\,d\lambda(z)}{|\zeta-z|}\,. \tag{2.15}$$

We use the system of coordinates associated to the basis $(e_k^{(\ell)})_k$ and denote

$$w_k=\langle\zeta-z,e_k^{(\ell)}\rangle,\ \text{ for } 1\le k\le n,\ \ t_1=-\rho(z),\ \ t_2=\Im m\,w_1 \tag{2.16}$$
$$\text{and for } 2\le k\le n,\ \ t_{2k-1}=\Re e\,w_k,\ \ t_{2k}=\Im m\,w_k.$$

We integrate with respect to $(t_1+it_2,w_2,\dots,w_n)$.

Using (3.6), the estimate $\tau_1^{(\ell)}(\zeta)\approx2^\ell\delta$ and the fact that $d(z)\lesssim2^\ell\delta$ if $z\in P(\zeta,\beta2^\ell\delta)$ we obtain

$$\mathcal{J}_\ell\lesssim\frac{\tau_2^{(\ell)}\,\delta(\zeta)}{\mathrm{Vol}\,P(\zeta,2^\ell\delta)}\int_{|t_1|+|t_2|<2^\ell\delta,\,|w_j|<\tau_j^{(\ell)},\,j\ge2}\frac{t_1^{a-1}\,dt_1\,dt_2\,d\lambda(w_2,\dots,w_n)}{|w_2|}$$

$$\mathcal{J}_\ell\lesssim\ a^{-1}\,(\delta(\zeta))^a\ 2^{\ell(a-1)}. \tag{2.17}$$

We choose $N'\ge n+a$; (2.14), (2.15), and (2.17) yield

$$\sum_\ell a\int_{\mathcal{C}_\ell(\zeta)}d(z)^{a-1}\,|G(\zeta,z)\wedge f(\zeta)\wedge\omega(\zeta)|\,d\lambda(z)\le C\ (\delta(\zeta))^a\ |||f(\zeta)|||_k,$$

where C remains bounded as $a\to0$.

2.3. . We now deal with the other terms of our kernel. Since we have already treated the case of $G = K^{(n-1)}$, it suffices to give the estimates for $K^{(0)}$ and $K^{(1)}$ (cf. (2.2)).

Recall N is chosen such that $N > 2n + a$.

- $|K^{(0)}(\zeta, z) \wedge f(\zeta) \wedge \omega(\zeta)| \leq |||f(\zeta)|||_k \, |K^{(0)}(\zeta, z)|$.

We have from (2.2) and (3.11)

$$\begin{aligned}&\int_{\Omega\cap\mathcal{U}} |\rho(z)|^{a-1} |K^{(0)}(\zeta, z) \wedge f(\zeta) \wedge \omega(\zeta)| \, d\lambda(z) \\ &\lesssim \delta^N \, |||f(\zeta)|||_k \int_{\Omega\cap\mathcal{U}} \frac{|\rho(z)|^{a-1} \, d\lambda(z)}{\varepsilon(\zeta, z)^N \, |\zeta - z|^{2n-1}}.\end{aligned}$$

We choose a system of coordinates with respect to a basis $(e_1, \dots, e_n)$ where $e_1 = \nabla\rho(\zeta)/||\nabla\rho(\zeta)||$; from (3.8), (3.9), (3.3) we have

$$\begin{aligned}\varepsilon(\zeta, z) &\approx |\rho(\zeta)| + |\rho(z)| + \mathcal{M}(\zeta, z) \\ &\gtrsim |\rho(z)| + |\rho(\zeta)| + |\zeta_1 - z_1| \quad \zeta, z \in \mathcal{U} \cap \Omega.\end{aligned}$$

We integrate in $t_1 = \rho(z) - \rho(\zeta)$, $t_2 = \Im m(\zeta_1 - z_1)$ and the remaining variables as done in [**DH77**, Lemma 2.2 a)] (we give some details about this lemma in Subsection 3.2); thus we get

$$\int_{\Omega\cap\mathcal{U}} \frac{|\rho(z)|^{a-1} \, d\lambda(z)}{(\varepsilon(\zeta, z))^N \, |\zeta - z|^{2n-1}} \leq C \, a^{-1} \, (\delta(\zeta))^{-N+a}$$

where C is bounded (with respect to a) for $0 < a < 1$.

- The term involving $K^{(1)}$ is less regular than the term with $K^{(0)}$. In [**Cu01**], we have given results for isotropic norms. Since the norm $|||.|||_k$ is non-isotropic, we have to handle the term with $K^{(1)}$ the same way we have done to estimate $|G(\zeta, z) \wedge f(\zeta) \wedge \omega(\zeta)|$; we thus proceed as in Section 2.2, the notations of which we use again.

$$K^{(1)}(\zeta, z) = \left(\frac{\mathcal{B}(\zeta, z)}{\mathcal{B}(\zeta, \zeta)}\right)^{N-1} \frac{\left(\partial_z|\zeta - z|^2\right) \, (\overline{\partial}\widetilde{Q}) \, \left(d\partial_z|\zeta - z|^2\right)^{n-2}}{|\zeta - z|^{2n-2}}.$$

The form $\bar{\partial}_\zeta \widetilde{Q}(\zeta, z) \wedge f(\zeta)$ is of bidegree $(0, 2)$ in ζ, of bidegree $(1, 0)$ in z; we express the forms dz_j and $d\bar{\zeta}_j$ in other bases as in Section 2.2; moreover we use (2.1), (3.7), (3.10), (3.11) in a first step, (2.11) and (2.12) in a second step; we thus obtain for $z \in \mathcal{C}_\ell(\zeta)$

$$\begin{aligned}
&|K^{(1)}(\zeta,z)\wedge f(\zeta)\wedge\omega(\zeta)| \\
&\lesssim \sum_{\substack{i,j,p=1\\ i\neq p}}^{n} \frac{|f(\zeta)(e_p^{(\ell)})|\ (\delta(\zeta))^{N-1}}{\sigma(\zeta,e_i^{(\ell)},\delta)\,\sigma(\zeta,e_j^{(\ell)},\delta)\,\varepsilon(\zeta,z)^{N-1}\,|\zeta-z|^{2n-3}} \\
&\lesssim \sum_{\substack{i,j,p=1\\ i\neq p}}^{n} \frac{|||f(\zeta)|||_k\,\delta(\zeta)\ 2^{-\ell(N-1)}}{\sigma(\zeta,e_i^{(\ell)},\delta)\,\sigma(\zeta,e_j^{(\ell)},\delta)\,\sigma(\zeta,e_p^{(\ell)},\delta)\,|\zeta-z|^{2n-3}}.
\end{aligned}$$

Using once again the notation $\tau_j^{(\ell)} = \tau_j^{(\ell)}(\zeta) = \sigma(\zeta, e_j^{(\ell)}(\zeta), \beta 2^\ell \delta(\zeta)), j = 1, \dots, n$, we deduce from (3.6) and (2.13) for $z \in \mathcal{C}_\ell(\zeta)$

(2.18)

$$|K^{(1)}(\zeta,z)\wedge f(\zeta)\wedge\omega(\zeta)| \lesssim \sum_{I,J} \frac{|||f(\zeta)|||_k\ \delta(\zeta)\,\tau_{j_2}^{(\ell)}\,\prod_{k=3}^n \tau_{i_k}^{(\ell)}\tau_{j_k}^{(\ell)}}{2^{\ell(N-5/2)}\ \mathrm{Vol}\,P(\zeta,\beta 2^\ell\delta)\ |\zeta-z|^{2n-3}},$$

where $I = (i_1, \dots, i_n)$, $J = (j_1, \dots, j_n)$ are such that $\{i_1, \dots, i_n\} = \{j_1, \dots, j_n\} = \{1, \dots, n\}$.

We have thus to estimate

$$\mathcal{L}_{\ell,I,J}(\zeta) := \frac{\delta\,\tau_{j_2}^{(\ell)}\,\prod_{k=3}^n \tau_{i_k}^{(\ell)}\,\tau_{j_k}^{(\ell)}(\zeta)}{\mathrm{Vol}\,P(\zeta,\beta 2^\ell\delta)} \int_{\mathcal{C}_\ell(\zeta)} \frac{d(z)^{a-1}\,d\lambda(z)}{|\zeta-z|^{2n-3}}. \tag{2.19}$$

Without loss of generality we can suppose that $i_1 < i_2$; we let $\nu = \min(i_1, j_1)$ and $\mu = \max(i_1, j_1)$; we thus have $\nu < i_2$.

We use the notations (2.16). On the one hand, we consider t_1, $t_{2\nu}$, t_{2i_2-1}, t_{2i_2} and on the other hand we call t' the $(2n-4)$-tuple of the remaining variables.

We denote $r_{i_2} = (t_{2i_2-1}^2 + t_{2i_2}^2)^{1/2}$.

Since $\tau_1^{(\ell)} \approx 2^\ell\delta$ and $\delta(\zeta) \lesssim 2^{-\ell}\tau_\mu^{(\ell)}$, we obtain

$$\begin{aligned}
&\delta(\zeta)\int_{\mathcal{C}_\ell(\zeta)} \frac{d(z)^{a-1}\,d\lambda(z)}{|\zeta-z|^{2n-3}} \\
&\lesssim \delta \int_{|t_{2k-1}|+|t_{2k}|<\tau_k^{(\ell)}} \frac{t_1^{a-1}\,dt_1\dots dt_{2n}}{|t|^{2n-3}} \\
&\lesssim \delta \int_{0<t_1<\tau_1^{(\ell)}} t_1^{a-1}\,dt_1 \int_{r_{i_2}<\tau_{i_2}^{(\ell)},\,|t_{2\nu}|<\tau_\nu^{(\ell)},\,|t'|<1} \frac{dt_{2\nu}\,r_{i_2}\,dr_{i_2}\,d\lambda(t')}{(r_{i_2}+|t'|)^{2n-3}} \\
&\le C\,a^{-1}2^{(a-1)\ell}\,\delta^a\,\tau_{i_2}^{(\ell)}\,\tau_\nu^{(\ell)}\tau_\mu^{(\ell)} = C\,a^{-1}2^{(a-1)\ell}\,\delta(\zeta)^a\,\tau_{i_2}^{(\ell)}\,\tau_{i_1}^{(\ell)}\tau_{j_1}^{(\ell)}
\end{aligned}$$

where C is uniformly bounded with respect to a if $0 < a < 1$.

The latest estimate, (2.18), (2.19) and (3.6) yield the required results (cf. (2.7)).

Remark. In order to prove part b) of Theorem 1.3, we could have integrated directly on the boundary. We give some precision in Subsection 3.2.

We have chosen another type of proof for Theorem 1.3 b); it allows us to treat together the case of weighted $L^1(\Omega)$ estimates and the case of $L^1(b\Omega)$ estimates.

3. Appendix.

3.1. . Ω is a smoothly bounded convex domain of finite type m in $\mathbb{C}^n$.

• We will need some properties of the quantities $\sigma(z, v, \eta)$ the definition of which has been recalled in (2.5).

We have uniformly in z, v

$$\sigma(z, v, \eta) = O\big(\eta^{1/m}\big) \text{ and } \sigma(z, v, \eta) \gtrsim \eta. \tag{3.1}$$

(3.2)

$$\text{For } \eta_1 \leq \eta_2,\ \ \big(\eta_1/\eta_2\big)^{1/2}\sigma(z, v, \eta_2) \lesssim \sigma(z, v, \eta_1) \lesssim \big(\eta_1/\eta_2\big)^{1/m}\sigma(z, v, \eta_2).$$

• We recall now the notion of η-extremal basis of McNeal as done in [**BCD98**]. Let $z \in \overline{\Omega}$ close to $b\Omega$ and $\eta > 0$ be fixed.

Definition 3.1. A orthonormal basis $\{v_1, \dots, v_n\}$ of the tangent space $T_z(\mathbb{C}^n)$ is a *η-extremal basis of McNeal at z* if it is chosen as follows: The first vector v_1 is the unit vector of the direction of the gradient vector at z; chosen $v_1, \dots, v_{i-1}$, so v_i is a unit vector realizing the maximum of $\sigma(z, v, \eta)$ among the unit vectors orthogonal in $\mathbb{C}^n$ to $v_1, \dots, v_{i-1}$.

• The polydisc $P(z, \eta)$ of McNeal centered at z, with radius η is defined as follows:

$$P(z, \eta) = \left\{ w = z + \sum_{j=1}^{n} w_j v_j,\ \text{where } |w_j| \leq c\, \sigma(z, v_j, \eta) \right\} \tag{3.3}$$

where the constant $c = c(n)$ is chosen such that $w \in P(z, \eta) \implies |\rho(w) - \rho(z)| \leq \eta$.

The construction of McNeal's polydiscs makes Ω a space of "homogeneous type" with the usual properties.

Let us just recall what we need about these polydiscs (cf. [**Mc94**], [**BCD98**] for details).

$$\forall C > 0, \quad \operatorname{Vol} P(z, C\eta) \approx \operatorname{Vol} P(z, \eta). \tag{3.4}$$

We have, with uniform constants,

$$\operatorname{Vol} P(\zeta, \eta) \approx \operatorname{Vol} P(z, \eta) \ \text{ if } \ P(z, \eta) \cap P(\zeta, \eta) \neq \emptyset. \tag{3.5}$$

If $(v_j)_{1\leq j\leq n}$ is an η-extremal basis of McNeal at z then

$$\operatorname{Vol} P(z,\eta) \approx \prod_{i=1}^{n} \big(\sigma(z,v_i,\eta)\big)^2. \tag{3.6}$$

• Using the reformulation given by McNeal and Stein in [**McS94**], we give the McNeal's estimates of the Bergman kernel that we need.

Definition 3.2. For $z\in b\Omega$, $\eta>0$, $T(z,\eta):=P(z,\eta)\cap\Omega$ is called the tent at z of radius η.

For v a unit vector in $\mathbb{C}^n$, $\varphi\in C^\infty(\Omega)$, let $D_v\varphi$ denote the directional derivative of φ in the direction v.

For every $p\in b\Omega$, there exists a neighborhood $\mathcal{U}(p)$ such that for ζ, z in $\mathcal{U}\cap\Omega$, k, $s\in\mathbb{N}$, v and v' unit vectors

$$|\overline{D}_v^k D_{v'}^s \mathcal{B}(\zeta,z)| \leq \operatorname{cst}(k,s)\ \sigma(\zeta,v,\varepsilon)^{-k}\,\sigma(\zeta,v',\varepsilon)^{-s}(\operatorname{Vol} T_{\zeta,z})^{-1} \tag{3.7}$$

where $\operatorname{Vol} T_{\zeta,z}$ is the volume of the "smallest" tent containing both ζ and z, $\varepsilon=\varepsilon(\zeta,z)$ the radius of this tent ("smallest" means of smallest volume).

$$\varepsilon=\varepsilon(\zeta,z)\approx|\rho(\zeta)|+|\rho(z)|+\mathcal{M}(\zeta,z) \tag{3.8}$$

where $\mathcal{M}(\zeta,z)$ is the quasi-distance of McNeal; up to uniform constant multiples:

(3.9)

$$\mathcal{M}(z,\zeta)\approx\mathcal{M}(\zeta,z)=\inf\{\eta,\zeta\in P(z,\eta)\}\ \text{ for } |\zeta-z|\ll 1,\ \zeta\ \text{ close to }\ b\Omega.$$

For every $p\in b\Omega$, there exists a neighborhood $\mathcal{U}'(p)$ of p (we may assume $\mathcal{U}'(p)=\mathcal{U}(p)$) such that

(3.10)

$$\mathcal{B}(\zeta,\zeta)\gtrsim(\operatorname{Vol} P(\zeta,\delta))^{-1},\quad \zeta\in\mathcal{U}'(p)\cap\Omega,\ \text{ where } \delta=\delta(\zeta)=|\rho(\zeta)|/2.$$

• The estimate given below is implicit in [**Mc94**] and is easy to get (cf. [**Cu01**]).

$$\frac{|\mathcal{B}(\zeta,z)|}{\mathcal{B}(\zeta,\zeta)}\lesssim\frac{\operatorname{Vol} P(\zeta,\delta)}{\operatorname{Vol} P(\zeta,\varepsilon(\zeta,z))}\lesssim\frac{\delta(\zeta)}{\varepsilon(\zeta,z)},\quad \zeta,z\in\mathcal{U}\cap\Omega. \tag{3.11}$$

3.2. . We give here some technical results.

• In [**DH77**, Lemma 2.2], the authors estimate integrals like

$$\mathcal{I}_{x,\alpha,M}(\delta):=\int_{t_1+\delta>0,\,|t|<1}\frac{(t_1+\delta)^\alpha\,dt_1\,\cdots\,dt_{2n}}{(\delta+|t_1|+|t_2|+|t|^2)^M\,|t|^{2n-1-x}},$$

where $x=0,2$ or 3.

If $x = 0$ (part a) of the lemma), they use the inequality $\delta + |t_1| + |t_2| + |t|^2 \geq \delta + |t_1| + |t_2|$; so we can apply in our context their result i.e.,

$$\mathcal{I}_{0,\alpha,M}(\delta) \leq C\,(\alpha+1)^{-1}\delta^{1-M+\alpha}, \text{ if } M - \alpha > 1.$$

• If one chooses to integrate directly on the boundary in order to prove part b) of Theorem 1.3 – taking into account Remark 2.2 –, the terms involving G or $K^{(1)}$ can be handled without difficulty as in Section 2.2.

Thanks to the weight $(\mathcal{B}(\zeta,z)/\mathcal{B}(\zeta,\zeta))^N$, the integration of $|K^{(0)}|$ on $\{z \in b\Omega, |z-\zeta| \ll 1\}$ is less problematic than it looks:

If $z \in b\Omega$, we have $\delta(\zeta) \approx |\rho(\zeta)| = |\rho(\zeta) - \rho(z)| \lesssim |\zeta - z|$; thus we can prove using (3.11) that

$$\begin{aligned}\int_{b\Omega\cap\{|z-\zeta|\ll 1\}} |K^{(0)}(\zeta,z)| d\lambda_{2n-1}(z) &\lesssim \sum_{\ell=0}^{\infty} \int_{b\Omega\cap\mathcal{C}_\ell(\zeta)} |K^{(0)}(\zeta,z)| d\lambda_{2n-1}(z) \\ &\lesssim C\end{aligned}$$

uniformly with respect to ζ.

Note added in proof: Just before correcting the galley proofs of this paper, I have heard that the paper in question in Footnote 1 will appear in the Nagoya Journal of Mathematics (cf. [**DiM01**]).

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