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We define a nontrivial homomorphism from the Hecke algebra of type B onto a reduced algebra of the Hecke algebra of type A at roots of unity. We use this homomorphism to describe semisimple quotients of the Hecke algebra of type B at roots of unity. Using these quotients we determine subfactors obtained from the inclusion of Hecke algebra of type A into Hecke algebras of type B. We also study intermediate subfactors related to the Hecke algebra of type D.

Introduction.

In [**W1**] Wenzl found examples of subfactors of the hyperfinite II_1 factor by studying the complex Hecke algebras of type A, denoted by $H_n(q)$. In this paper we construct examples of subfactors obtained from the inclusion of the Hecke algebra of type A into the Hecke algebra of type B, denoted by $H_n(q,Q)$. To do this we must find the values of the parameters of the Hecke algebra of type B for which the inductive limit, i.e., $H_\infty(q,Q) = \bigcup_{n\geq 0} H_n(q,Q)$, has C^* representations. We show that there are C^* representations when $q = e^{2\pi i/l}$ and $Q = -q^k$ for some positive integers l and k.

In [**O**] we defined a surjective homomorphism from the specialized Hecke algebra of type B, $H_n(q, -q^{r_1+m})$, onto a reduced Hecke algebra of type A. Here we show that this homomorphism is well-defined and onto when q is a root of unity. This implies that there exist quotients of the reduced Hecke algebra of type A which are isomorphic to quotients of $H_n(q,-q^k)$ at roots of unity. These quotients are C^* algebras and we use them to construct the II_1 hyperfinite factor.

Geck and Lambropoulou [**GL**] have defined a two parameter trace, called Markov trace, on the Hecke algebra of type B. In [**O**] we showed that when $Q = -q^k$, k a positive integer, this trace can be obtained as a pull back of the Markov trace on the Hecke algebra of type A. Moreover, this trace satisfies the commuting square property needed for the construction of subfactors. We use this trace and the Hecke algebra of type B to construct the hyperfinite II_1 factor.

The subfactors obtained from the inclusion of the Hecke algebra of type A into the Hecke algebra of type B are equivalent to special cases of subfactors already obtained in [**W1**] for the Hecke algebras of type A. We compute the index and higher relative commutants for these subfactors. We found that the index is related to the Schur function of a rectangular Young diagram.

We also obtain intermediate subfactors of index two by studying the inclusion of the Hecke algebra of type D into the Hecke algebra of type B. We also consider the inclusion of the Hecke algebra of type A into the Hecke algebra of type D. We compute the index for these subfactors.

This paper is organized as follows: In the first section we introduce notations and definitions which will be used throughout the paper. In the second section we define the Hecke algebras and recall the homomorphism in [**O**]. We begin Section Three by reviewing the representation theory of the Hecke algebra of type B, [**H**]. We then proceed to the study of a quotient of this algebra when q is a root of unity and $Q = -q^k$ for $k \in \mathbb{N}$. We also give the well-defined irreducible representations for this quotient.

In Section Four we summarize the necessary results about Markov traces and give the weight vector for this trace. We also show that this trace is well-defined on the quotients at roots of unity. In Section 5 we show that there are C^* representations for $H_\infty(q, -q^k)$, i.e., representations on a Hilbert space such that the images of the generators $t, g_1, \dots, g_{n-1}, \dots$ are unitary. In Section 6 we give the index and higher relative commutants for the subfactors obtained from the inclusion of the Hecke algebra of type A into the Hecke algebra of type B. We conclude this paper with the study of intermediate subfactors involving the Hecke algebra of type D. In particular, we compute the index for these subfactors.

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1. Preliminaries.

We take the ground field to be the complex numbers $\mathbb{C}$. For convenience, an algebra A will be called *semisimple* if it is a direct sum of full matrix rings. Let $M_n(\mathbb{C})$ denote the algebra of $n \times n$ matrices over $\mathbb{C}$. If $A \subset B$ are semisimple then $A = \bigoplus A_i$ and $B = \bigoplus B_j$ with $A_i = M_{a_i}(\mathbb{C})$ and $B_j = M_{b_j}(\mathbb{C})$ for some a_i, $b_j \in \mathbb{N}$. Furthermore, any simple B_j module is also an A-module. Let g_{ij} be the number of simple A_i modules in the decomposition of B_j into simple A modules. The matrix $G = (g_{ij})$ is called the *inclusion matrix* for $A \subset B$.

The inclusion of A in B can be described by the *Bratteli diagram*. This is a graph with vertices arranged in 2 lines. In one line, the vertices are in one-to-one correspondence with the minimal direct summands A_i of A, in the other one with the summands B_j of B. Then a vertex corresponding to

A_i is joined with a vertex corresponding to B_j by g_{ij} edges. See Figure 1 for the Bratteli diagram of the inclusion of the group algebra of the symmetric group in 3 letters into the group algebra of the symmetric group in 4 letters, i.e., $\mathbb{C}S_3 \subset \mathbb{C}S_4$.

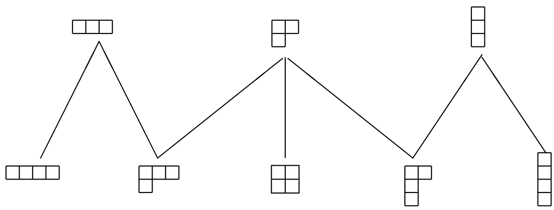

Figure 1.

A *trace* is a linear functional $\text{tr} : B \to \mathbb{C}$ such that $\text{tr}(ab) = \text{tr}(ba)$ for all $a, b \in B$. There is only one trace on $M_n(\mathbb{C})$ up to scalar multiples. Thus, any trace tr on $B = \bigoplus B_j$ is completely determined by a vector $\vec{t} = (t_j)$, where $t_j = \text{tr}(p_j)$ and p_j is a minimal idempotent in B_j, $\vec{t}$ is called the *weight vector* and the t_i are called the *weights*. A trace is *nondegenerate* if for any $b \in B$, there is a $b' \in B$ such that $\text{tr}(bb') \neq 0$. It is not hard to show that tr is nondegenerate if and only if $t_j \neq 0$ for every j.

Recall that there is an isomorphism between B and its dual B^* defined by $b \in B \to \text{tr}(b\,\cdot) \in B^*$, where $\text{tr}(b\,\cdot)$ denotes the function $x \to \text{tr}(bx)$. Assuming tr is nondegenerate on both A and B, and using the above isomorphism for A and A^*, we obtain for every $b \in B$ a unique element $\varepsilon_A(b) \in A$ such that $\text{tr}(b\,\cdot)\big|_A = \text{tr}(\varepsilon_A(b)\,\cdot)\big|_A$. The linear map $\varepsilon_A : B \to A$ defined by $b \to \varepsilon_A(b)$ is called a trace preserving *conditional expectation* from B onto A, the element $\varepsilon_A(b) \in A$ is uniquely determined by the equation

$$\text{tr}(\varepsilon_A(b)a) = \text{tr}(ba) \quad \text{for all } a \in A.$$

1.1. Young Diagrams.

In this section we use notation and terminology from [**M**].

A *partition* is a finite sequence of nonnegative integers in decreasing order: $\lambda = [\lambda_1 \geq \lambda_2 \geq \cdots]$. We make no distinction between two sequences that differ only by zeros. The number of parts is called the *length* of λ, and is denoted by $l(\lambda)$; $|\lambda| = \lambda_1 + \lambda_2 + \cdots$ is called the *weight* of λ. If $|\lambda| = n$ then λ is a partition of n, denoted $\lambda \vdash n$.

It is common to associate partitions with *Young diagrams*. The Young diagrams of λ is an array of n boxes with λ_1 boxes in the first row, λ_2 boxes in the second row, and so on. We count rows from top to bottom.

A *standard tableau* is a Young diagram with n boxes such that the boxes have been filled with numbers from 1 to n, in such a manner that the numbers increase along the rows and along the columns. Let T_λ be the set of all standard tableaux of shape λ.

If λ and μ are partitions, we shall write $\mu \subset \lambda$ to mean that the diagram of λ contains the diagram of μ, i.e., $\lambda_i \geq \mu_i$ for all $i \geq 1$.

A *double partition* of size n, (α, β), is an ordered pair of partitions α and β such that $|\alpha| + |\beta| = n$. If (γ, ρ) is another double partition we write $(\gamma, \rho) \subset (\alpha, \beta)$ if $\gamma \subset \alpha$ and $\rho \subset \beta$.

A pair of standard tableaux is a pair $\tau_{(\alpha,\beta)} = (t^\alpha, t^\beta)$ of Young diagrams filled with numbers from 1 to n such that t^α and t^β are each a standard tableau.

We say that a box in (α, β) has coordinates (i, j) if it is in the i-th row and j-th column of either α or β. Two boxes in (α, β) can have the same coordinates if they occur in the same box in α as in β, for instance the most upper-left box in α and the most upper-left box in β, they both have coordinates $(1, 1)$.

1.2. Subfactors.

In this section we recall some definitions and basic results for constructing subfactors and for computing their invariants. For details and proof of the following statements see [**Jo**]. A *von Neumann algebra* A is a $*$-subalgebra of the algebra of bounded operators on a Hilbert space. A contains 1 and is closed in the weak operator topology. A von Neumann algebra A whose center is trivial, i.e., $Z(A) = \mathbb{C}\cdot 1$, is called a *factor*. A II_1 *factor* is an infinite dimensional factor A which admits a normalized finite trace $\mathrm{tr} : A \to \mathbb{C}$ such that (i) $\mathrm{tr}(1) = 1$; (ii) $\mathrm{tr}(xy) = \mathrm{tr}(yx)$, for all $x, y \in A$; and (iii) $\mathrm{tr}(x^*x) \geq 0$, $x \in A$. This trace is unique.

An algebra is *approximately finite* (AF-algebra) if it is a C^* algebra that contains an increasing sequence $(A_n)_{n=1}^\infty$ of finite dimensional C^*-subalgebras such that $\bigcup_{n+1}^\infty A_n$ is dense in A. The *hyperfinite* II_1 factor is a separable II_1 factor which is approximately finite.

The trace induces a Hilbert norm on A. Moreover, we can perform the GNS construction with respect to the trace and obtain a faithful representation of A on $L^2(A, \mathrm{tr})$; this Hilbert space is obtained as the closure of A in the norm induced by the trace. A acts by left multiplication operators on itself and the GNS representation is precisely this representation extended to $L^2(A, \mathrm{tr})$. Observe that the identity is the cyclic and separating vector in $L^2(A, \mathrm{tr})$. This representation is called the *standard form* of A.

From now on all factors and subfactors discussed will be II_1 factors. If A and B are a pair of factors, then A is a *subfactor* of B if A is a sub-von Neumann algebra of B, which is itself a factor and has the same identity as B, i.e., $1_A = 1_B$. The von Neumann algebra $A' \cap B$ is called *relative commutant* of A in B.

Let $A \subset B$ be the inclusion of II_1 factors with $1_A = 1_B$. If tr is the unique normalized trace on B then $\mathrm{tr}|_A$ is the unique normalized trace on A by uniqueness of the trace. We define the *orthogonal projection* $e_A : L^2(B, \mathrm{tr}) \to L^2(A, \mathrm{tr}|_A)$ by

$$e_A(\vec{x}) = \overrightarrow{\varepsilon_A(x)}, \quad \vec{x} \in L^2(B, \mathrm{tr}) \text{ and } x \in B,$$

where ε_A is the trace preserving conditional expectation. We denote by $\langle B, e_A \rangle$ the von Neumann algebra generated by B and e_A on $L^2(B, \mathrm{tr})$, this is called the *basic construction.* In particular, if A is a factor, then so is $\langle B, e_A \rangle$. If it is a finite factor, we define the index $[B : A]$ of A in B to be the number $1/\mathrm{tr}(e_A)$, where tr denotes the unique normalized trace on $\langle B, e_A \rangle$. If $\langle B, e_A \rangle$ is not finite, the index is defined to be infinite.

In this paper we will study examples of subfactors constructed using the following set-up.

(i) Let (B_n) be an ascending sequence of C^* algebras with B_n a proper subalgebra of B_{n+1} for all $n \in \mathbb{N}$. Furthermore, let tr be a positive finite extremal trace on its inductive limit $B_\infty = \bigcup_{n \geq 0} B_n$ and π_{tr} be the GNS construction with respect to tr. Then it is well-known that the weak closure B of $\pi_{\mathrm{tr}}(B_\infty)$ is isomorphic to R, the hyperfinite II_1 factor.

(ii) Let (A_n) be an ascending sequence of subalgebras such that $A_n \subset B_n$ and the weak closure A of $\pi_{\mathrm{tr}}(A_\infty)$ is a subfactor.

(iii) Consider the following square

$$\begin{array}{ccc} B_n & \overset{\varepsilon_{B_n}}{\longleftarrow}\!\!\!\!\!\!\!\!\!\!\!\!\subset\quad & B_{n+1} \\ {\scriptstyle \varepsilon_{A_n}}\downarrow & & \downarrow{\scriptstyle \varepsilon_{A_{n+1}}} \\ A_n & \subset & A_{n+1} \end{array}$$

where $\varepsilon_{A_{n+1}}$, ε_{A_n} and ε_{B_n} are the trace preserving conditional expectations onto A_{n+1}, A_n and B_n respectively. We require that this diagram commutes, i.e.,

$$\varepsilon_{A_{n+1}} \varepsilon_{B_n} = \varepsilon_{A_n}, \quad \text{for all } n \in \mathbb{N}.$$

This condition is called the *commuting square property.*

The sequence (A_n) is *periodic* with period k if there is an $n_0 \in \mathbb{N}$ such that for all $n > n_0$ the inclusion matrix for $A_{n+k} \subset A_{n+k+1}$ is the same (after relabeling of the central projections) as the one for $A_n \subset A_{n+1}$.

We say that $(A_n) \subset (B_n)$ is periodic if both (A_n) and (B_n) are periodic with same period k and if also the inclusion matrices for $A_{n+k} \subset B_{n+k}$ and $A_n \subset B_n$ are the same. If the inclusion matrices for $A_n \subset B_n$, $A_n \subset A_{n+1}$

and $B_n \subset B_{n+1}$ become periodic for $n \geq n_0$ for some n_0, then the index $[B : A]$ of the subfactor A is the square of the norm of the inclusion matrix for $A_n \subset B_n$ for all $n \geq n_0$.

There are finer invariants for the subfactor $A \subset B$ than the index. If $B^{(1)} = \langle B, e_A \rangle$ is obtained by the basic construction, then it is known by [**Jo**] that $[B : A] = [B^{(1)} : B]$. Now iterate the basic construction to obtain a tower $A \subset B \subset B^{(1)} \subset B^{(2)} \subset \cdots$ of II_1 factors. Let $C_i = A' \cap B^{(i)}$ be the relative commutant of A in $B^{(i)}$. Then the structure of the algebras $C_1, C_2, \ldots$ is an invariant of subfactors of B. The C_i's are called *higher relative commutants* of $A \subset B$.

2. Hecke algebras.

2.1. Hecke algebras of type A.

In this section we summarize results by Wenzl [**W1**] about the representation theory of the Hecke algebra of type A.

The Hecke algebra of type A_{n-1}, $H_n(q)$, is the free complex algebra with generators $g_1, g_2, \ldots, g_{n-1}$ and parameter $q \in \mathbb{C}$ with defining relations

(H1) $g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1}$, for $i = 1, 2, \ldots, n-2$;
(H2) $g_i g_j = g_j g_i$, whenever $|i - j| \geq 2$;
(H3) $g_i^2 = (q-1) g_i + q$ for $i = 1, 2, \ldots, n-1$.

It is well-known that $H_n(q) \cong \mathbb{C}S_n$ if q is not a root of unity, where $\mathbb{C}S_n$ is the group algebra of the symmetric group, S_n, (see [**Bou**, pp. 54-56]). It follows from this that $H_n(q)$ has dimension $n!$. Similarly as for the symmetric group, we can label the irreducible representations of $H_n(q)$ by Young diagrams.

The full-twist, Δ_f^2, is a central element in $H_n(q)$ and it is defined algebraically by

$$\Delta_f^2 := (g_{f-1} \cdots g_1)^f.$$

The following lemma describes the action of the full-twist on the Hecke algebra of type A_{n-1}.

Lemma 2.1. *Let α_λ be the scalar by which the full-twist acts in the irreducible Hecke algebra representation labeled by λ. Then*

$$\alpha_\lambda = q^{n(n-1) - \sum_{i<j} (\lambda_i + 1)\lambda_j}. \qquad (1)$$

For the proof of this lemma see [**W2**, p. 261].

The Hecke algebras of type A satisfy the following embedding of algebras $H_0(q) \subset H_1(q) \subset \cdots \subset H_n(q) \subset H_{n+1}(q) \subset \cdots$. The inductive limit is defined by $H_\infty(q) = \bigcup_{n>0} H_n(q)$.

The interesting case for defining subfactors is when the parameter q is a root of unity, $q \neq 1$. In what follows we will describe the semisimple quotients of $H_n(q)$ which are associated with $\mathfrak{sl}(r)$, the special linear algebra, for $1 < r < l$.

Let $r, l \in \mathbb{N}$ and $l > r$, then an (r,l)-*diagram* is a Young diagram μ with r rows such that $\mu_1 - \mu_r \leq l - r$. We denote the set of all (r,l) diagrams of size n by $\Lambda_n^{(r,l)}$. An (r,l) *tableau* of shape $\mu \in \Lambda_n^{(r,l)}$ is a standard tableau, such that if we remove the box containing n the Young subdiagram with $n-1$ boxes is an (r,l)-diagram and an (r,l) tableau. The set of (r,l) tableaux of shape μ is denoted by $T_\mu^{(r,l)}$.

For each $\mu \in \Lambda_n^{(r,l)}$ let V_μ be the vector space with basis $\{v_\tau\}$ indexed by elements of $T_\mu^{(r,l)}$. The following representations of $H_n(q)$ at roots of unity were defined in [**W1**]

$$\pi_\mu^{(r,l)}(g_i)v_\tau = b_d(q)v_\tau + c_d = v_{s_i(\tau)},$$

where $d = d_{\tau,i} = c(i+1) - c(i) + r(i) - r(i+1)$ with $c(j)$ and $r(j)$ the column and row of the box containing j, respectively. Here $b_d = \frac{q^d(1-q)}{1-q^d}$, $c_d = \frac{\sqrt{(1-q^{d+1})(1-q^{d-1})}}{1-q^d}$ and $s_i(\tau)$ is the tableau obtained from τ by interchanging the numbers i and $i+1$. Note that if the $s_i(\tau)$ is not standard then c_d is 0.

Theorem 2.2 (Wenzl [**W1**, Corollary 2.5])**.** *Let q be a primitive l-th root of unity with $l \geq 4$. Then there exists for every $\mu \in \Lambda_n^{(r,l)}$ a semisimple irreducible representation $\pi_\mu^{(r,l)}$ of $H_n(q)$. Then*

$$\pi_n^{(r,l)} : x \in H_n(q) \to \bigoplus_{\mu \in \Lambda_n^{(r,l)}} \pi_\mu^{(r,l)}(x)$$

is semisimple but generally not a faithful representation. Also representations corresponding to different (r,l) diagrams are nonequivalent.

Using this theorem Wenzl defines a representation for the inductive limit $H_\infty(q)$ at roots of unity, [**W1**]. By definition one has the following inclusion of algebras:

$$H_0(q) \subset H_1(q) \subset \cdots H_n(q) \subset H_{n+1}(q) \subset \cdots .$$

Furthermore, one has by definition the inclusion of the representations $\pi_n^{(r,l)}(H_n(q)) \subset \pi_{n+1}^{(r,l)}(H_{n+1}(q))$ which is compatible with the inclusion $H_n(q) \subset H_{n+1}(q)$ for all n. This is equivalent to saying that the following diagram commutes

$$\begin{array}{ccc} H_n(q) & \longrightarrow & \pi_n^{(r,l)}(H_n(q)) \\ \cap \big\downarrow & & \cap \big\downarrow \\ H_{n+1}(q) & \longrightarrow & \pi_{n+1}^{(r,l)}(H_{n+1}(q)) \end{array}$$

The representation of the inductive limit is denoted by $\pi^{(r,l)}$ with the understanding that $\pi^{(r,l)}(x) = \pi_n^{(r,l)}(x)$ if $x \in H_n(q)$. Furthermore, the representation $\pi^{(r,l)}$ is a unitary representation, i.e., the image of the generators under this representation are unitary. Wenzl also showed that the ascending sequence of finite dimensional C^*-algebras (A_n) is periodic with period r.

Thus when $q = e^{2\pi i/l}$ Wenzl obtains from the Hecke algebras, $H_n(q)$, an AF algebra with periodic Bratteli diagram for the sequence $(\pi^{(r,l)}(H_n(q)))$.

Let t^μ be a Young tableau with n boxes and $(t^\mu)'$ be the Young tableau obtained from t^μ by removing the box containing n. The map $t^\mu \to (t^\mu)'$ defines a bijection between T_μ and $\bigcup_{\mu' \subset \mu} T_{\mu'}$, where T_μ denotes the set of all standard tableaux of shape μ. Therefore, we have the following decomposition of modules

$$V_\mu\big|_{H_{n-1}(q)} = \bigoplus_{\mu' \subset \mu} V_{\mu'}. \tag{2}$$

2.2. Hecke algebra of type B.

In this context, we will mean by the Hecke algebra $H_n(q, Q)$ of type B_n the free complex algebra with generators $t, \tilde{g}_1, \dots, \tilde{g}_{n-1}$ and parameters $q, Q \in \mathbb{C}$ the generators $\tilde{g}_i$'s satisfy (H1)-(H3) as in the definition of the Hecke algebra of type A and the following relations:

(B1) $t^2 = (Q-1)t + Q$;
(B2) $t\tilde{g}_1 t\tilde{g}_1 = \tilde{g}_1 t\tilde{g}_1 t$;
(B3) $t\tilde{g}_i = \tilde{g}_i t \quad$ for $i \geq 2$.

Hoefsmit [**H**] has written down explicit irreducible representations of $H_n(q, Q)$ indexed by ordered pairs of Young diagrams. It is clear that there exists an inclusion $H_n(q) \subset H_n(q, Q)$.

The Hecke algebras of type B satisfy the following embedding of algebras $H_0(q, Q) \subset H_1(q, Q) \subset H_2(q, Q) \subset \cdots$. The inductive limit of the Hecke algebra of type B is defined by

$$H_\infty(q, Q) := \bigcup_{n \geq 0} H_n(q, Q).$$

Observe that (H3) and (B1) imply that t and $\tilde{g}_i$ have at most 2 eigenvalues each, hence also at most 2 projections corresponding to these eigenvalues. There exists an alternate presentation for the Hecke algebra in terms of these projections.

For $q \neq -1$ and $Q \neq -1$ let

$$e_t = \frac{(Q - t)}{(Q+1)}, \qquad e_i = \frac{(q - g_i)}{(q+1)} \quad \text{for } i = 1, \dots, n-1$$

be the projections corresponding to the eigenvalue -1. Then $g_i = q(1 - e_i) - e_i = q - (q+1)e_i$. So $\langle 1, t, g_1, g_2, \dots, g_{n-1} \rangle = \langle 1, e_t, e_1, e_2, \dots, e_{n-1} \rangle$ and the defining relations (H1)-(H3) and (B1)-(B3) of $H_n(q, Q)$ translate to

(PH1) $e_ie_{i+1}e_i - q/(q+1)^2e_i = e_{i+1}e_ie_{i+1} - q/(q+1)^2e_{i+1}$ for $i = 1, 2, \dots, n-2$;
(PH2) $e_ie_j = e_je_i$, whenever $| i - j | \geq 2$;
(PH3) $e_i^2 = e_i$ for $i = 1, 2, \dots, n-1$;
(PH4) $e_t^2 = e_t$;
(PH5) $e_te_1e_te_1 - (Q+q)/(q+1)(Q+1)e_te_1$
$= e_1e_te_1e_t - (Q+q)/(q+1)(Q+1)e_1e_t$;
(PH6) $e_te_i = e_ie_t$ for $i \geq 2$.

2.3. Representations of the Hecke algebra of type B onto a reduced Hecke algebra of type A.

In this section we describe a way to obtain representations of the Hecke algebra of type B onto a reduced Hecke algebra of type A. First, we introduce some necessary background.

Let $p \in H_n(q)$ be an idempotent then the *reduced algebra* with respect to p is defined by $pH_n(q)p = \{pap \,|\, a \in H_n(q)\}$. In [**W1**] Wenzl defines a set of minimal idempotents of the Hecke algebra of type A. These idempotents are indexed by the standard tableaux. So if $\lambda \vdash f$, then we denote by t^λ a standard tableau of shape λ. And accordingly p_{t^λ} will denote a minimal idempotent in $H_f(q)$ corresponding to t^λ.

Fix an integer n, and let $m, r_1 \in \mathbb{N}$. Throughout this section we assume $n < m$ and $n < r_1$. Set $\lambda = [m^{r_1}]$ and $\gamma = [m^{r_1}, 1]$. Then $p_{t^\lambda}H_{n+f}(q)p_{t^\lambda}$ is the reduced algebra associated to p_{t^λ}. We define a map $\rho_{f,n}$ from the generators of $H_n(q, -q^{r_1+m})$ into the reduced Hecke algebra $p_{t^\lambda}H_{n+f}(q)p_{t^\lambda}$ as follows:

$$\rho_{f,n}(1) = p_{t^\lambda}, \quad \rho_{f,n}(t) = -\frac{\alpha_\lambda}{\alpha_\gamma}p_{t^\lambda}\Delta_f^{-2}\Delta_{f+1}^2, \quad \text{and}$$
$$\rho_{f,n}(\widetilde{g}_i) = p_{t^\lambda}g_{f+i} \qquad \text{for } i = 1, \dots, n-1$$

where α_λ is the scalar by which the full-twist acts on the irreducible module V_λ of the Hecke algebra of type A, see Lemma 2.1, and $\Delta_f^{-2}\Delta_{f+1}^2 = g_f \cdots g_2g_1^2g_2 \cdots g_f$.

Theorem 2.3. *With the conditions stated above we have that $\rho_{f,n}$ extends to a well-defined surjective homomorphism from $H_n(q, -q^{r_1+m})$ onto $p_{t^\lambda}H_{n+f}(q)p_{t^\lambda}$.*

This theorem was proved in [**O**].

3. Representations of the Hecke algebra of type B.

In this section we give some of the representation theory of $H_n(q, Q)$. Then we use the homomorphism of Section 2.3 to find simple representations of $H_n(q, Q)$ when q is a root of unity and $Q = -q^{m+r_1}$, where $m, r_1 \in \mathbb{N}$.

We briefly describe the semi-orthogonal representations of $H_n(q,Q)$. Hoefsmit [**H**] constructed for each double partition (α,β) of n an irreducible representation $(\pi_{(\alpha,\beta)}, V_{(\alpha,\beta)})$ of $H_n(q,Q)$ of degree $\binom{n}{|\alpha|} f^\alpha f^\beta$ where f^α is the number of standard tableaux of shape α.

Let $T_{(\alpha,\beta)}$ denote the set of pairs of standard tableaux of shape (α,β). We define the complex vector space $V_{(\alpha,\beta)}$ with orthonormal basis given by $\{v_\tau \mid \tau \in T_{(\alpha,\beta)}\}$. In what follows we describe the action of the generators of $H_n(q,Q)$ on $V_{(\alpha,\beta)}$. The following notations and definitions are needed to define this action.

Let (α,β) be a pair of Young diagrams and $\tau = (t^\alpha, t^\beta)$ be a pair of standard tableaux of shape (α,β). Define the *content* of a box b as follows:

$$ct(b) = \begin{cases} Qq^{j-i} & \text{if } b \text{ is in position } (i,j) \text{ in } t^\alpha \\ -q^{j-i} & \text{if } b \text{ is in position } (i,j) \text{ in } t^\beta. \end{cases}$$

Now define for each $1 \le i \le n-1$

$$(\widetilde{g}_i)_\tau = \frac{q-1}{1 - \frac{ct(\tau(i))}{ct(\tau(i+1))}}$$

where $\tau(i)$ denotes the coordinates of the box containing the number i. Notice that $(\widetilde{g}_i)_\tau$ depends only on the position of i and $i+1$. We are now ready to define the action of the generators on $V_{(\alpha,\beta)}$.

$$\begin{aligned} tv_\tau &= ct(\tau(1))v_\tau \\ \widetilde{g}_i v_\tau &= (\widetilde{g}_i)_\tau v_\tau + (q - (\widetilde{g}_i)_\tau) v_{s_i(\tau)}, \quad \text{for } i = 1, \ldots, n-1 \end{aligned} \tag{3}$$

where $s_i(\tau)$ is the standard tableau obtained from τ by switching i and $i+1$ in τ. If i and $i+1$ do not occur in the same row or column of t^α or t^β, then $s_i(t^\alpha, t^\beta)$ is again a pair of standard tableaux. Let V be the span of $\{v_\tau, v_{s_i(\tau)}\}$. Obviously, V is $\widetilde{g}_i$-invariant. The action of $\widetilde{g}_i\big|_V$ is given by the following 2×2 matrix

$$\begin{pmatrix} (\widetilde{g}_i)_\tau & (q - (\widetilde{g}_i)_\tau) \\ (q - (\widetilde{g}_i)_{s_i(\tau)}) & (\widetilde{g}_i)_{s_i(\tau)} \end{pmatrix}. \tag{4}$$

Finally, we have that if i and $i+1$ occur in the same row then $\widetilde{g}_i v_\tau = qv_\tau$; and if i and $i+1$ occur in the same column then $\widetilde{g}_i v_\tau = -v_\tau$.

Theorem 3.1 (Hoefsmit [**H**, Thm. 2.2.7])**.** *The modules $V_{(\alpha,\beta)}$, where (α,β) runs over all double partitions of n, form a complete set of non-isomorphic irreducible modules of $H_n(q,Q)$.*

Remark. Let $q \neq -1$ and $Q \neq -1$. One can easily obtain representations for the spectral projections defined in Section 2.2. Recall the equations $e_t = \frac{Q-t}{Q+1}$ and $e_i = \frac{q-\widetilde{g}_i}{q+1}$ for $i = 1, \ldots, n-1$. The matrix representation of these projections is obtained via the substitution $(\widetilde{g}_i)_\tau = q - (q+1)(e_i)_\tau$.

Let the numbers i and j be contained in (r_i, c_i) and (r_j, c_j) respectively, then define

$$d_{\tau,i,j} = c_i - c_j + r_j - r_i$$

we will refer to $d_{\tau,i,j}$ as the *axial distance* from j to i in τ. Note that $d_{\tau,j,i} = -d_{\tau,i,j}$. Let $d = d_{\tau,i,i+1}$. We have two possibilities for the denominator of $(\widetilde{g_i})_\tau$

$$\text{(5)}\quad \text{denominator}((\widetilde{g_i})_\tau) = \begin{cases} 1 - q^d & \text{if } i \text{ and } i+1 \text{ are both in } t^\alpha \text{ or } t^\beta \\ 1 + Qq^d & \text{otherwise.} \end{cases}$$

Observe that $\pi_{(\alpha,\beta)}(\widetilde{g_i})$ is undefined if and only if $(\widetilde{g_i})_\tau$ is undefined. This implies that if $Q \neq -q^k$ for $k \in \mathbb{Z}$ and if q is not an l-th root of unity for $1 \leq l \leq n-1$ then $\pi_{(\alpha,\beta)}(\widetilde{g_i})$ is well-defined in $V_{(\alpha,\beta)}$ for all pairs (α, β) and $i = 1, 2, \ldots, n-1$. Notice that $(\widetilde{g_i})_\tau$ is also undefined when $d_{\tau,i,i+1} = 0$ and both i and $i+1$ are in t^α or t^β, but this never happens if τ is a pair of standard tableaux, see [**W1**, Lemma 2.11].

Observe that the map $\tau \to \tau'$ (where τ' is obtained from τ by removing the box containing n) defines a bijection between $T_{(\alpha,\beta)}$ and $\bigcup_{(\alpha,\beta)' \subset (\alpha,\beta)} T_{(\alpha,\beta)'}$. So, in particular, we have

$$\text{(6)}\qquad V_{(\alpha,\beta)}\big|_{H_{n-1}(q,Q)} \cong \bigoplus_{(\alpha,\beta)' \subset (\alpha,\beta)} V_{(\alpha,\beta)'}$$

where $(\alpha, \beta)'$ is a pair of Young tableaux obtained by removing one box from either α or β. From the definition of $\pi_{(\alpha,\beta)}$ and $\pi_{(\alpha,\beta)'}$ we see that this equation yields the decomposition of $V_{(\alpha,\beta)}$ as an $H_{n-1}(q, Q)$-module

$$\text{(7)}\qquad \pi_{(\alpha,\beta)}\Big|_{H_{n-1}(q,Q)} \cong \bigoplus_{(\alpha,\beta)' \subset (\alpha,\beta)} \pi_{(\alpha,\beta)'}.$$

3.1. The Hecke algebra of type B at roots of unity.

In the previous section we observed that the irreducible representations of $H_n(q, Q)$ depend on rational functions with denominator $(Qq^d + 1)$ or $(1 - q^d)$, $d \in \mathbb{Z}$. Thus some of the representations will be undefined when $Q = -q^k$ for some $k \in \mathbb{Z}$ or when q is a root of unity. It is the objective of this section to describe the simple decomposition of quotients of the Hecke algebra of type B when $Q = -q^k$ and q is an l-th root of unity.

In Section 2.3 we defined for $r_1, m \in \mathbb{N}$ such that $r_1 > n$ and $m > n$ an onto homomorphism from the specialized Hecke algebra of type B, $H_n(q, -q^{r_1+m})$, onto a reduced Hecke algebra of type A, $p_{t^\lambda} H_{n+f}(q) p_{t^\lambda}$, where p_{t^λ} is an idempotent indexed by t^λ, a standard tableau corresponding to $\lambda = [m^{r_1}]$, i.e.,

$$\rho_{f,n} : H_n(q, -q^{r_1+m}) \longrightarrow p_{t^\lambda} H_{n+f}(q) p_{t^\lambda}.$$

In what follows we show that there is a well-defined surjective homomorphism when q is a root of unity and $Q = -q^{m+r_1}$ if we map onto a well-defined quotient of $p_{t^\lambda} H_{n+f}(q) p_{t^\lambda}$.

By Theorem 2.2 when q is an l-th root of unity then $\pi_n^{(r,l)}(H_n(q))$ is a well-defined quotient of the Hecke algebra of type A which is semisimple. The simple components are indexed by (r,l)-diagrams. We will denote this quotient by $H_n^{(r,l)}(q)$.

In [**W1**] Wenzl showed that there exist well-defined minimal idempotents of $H_n^{(r,l)}(q)$ for every (r,l) tableau. We denote these idempotents by $p_{t^\lambda}^{(l)}$. In particular, we have the following well-defined reduced algebra $p_{t^\lambda}^{(l)} H_n^{(r,l)}(q) p_{t^\lambda}^{(l)}$. Throughout the sequel we will only be interested in the case when $\lambda = [m^{r_1}]$. Notice that λ is an (r,l)-diagram if $m \leq l-r$. Now we choose a Young tableau $t^\lambda \in T_\lambda^{(r,l)}$ such that $p_{t^\lambda}^{(l)}$ is well-defined. Define a map from the generators of $H_n(q, -q^{r_1+m})$ into the reduced algebra $p_{t^\lambda}^{(l)} H_{n+f}^{(r,l)}(q) p_{t^\lambda}^{(l)}$ as follows:

$$\tilde{\rho}_{f,n}(1) = p_{t^\lambda}^{(l)}, \quad \tilde{\rho}_{f,n}(t) = -\frac{\alpha_\lambda}{\alpha_\gamma} p_{t^\lambda}^{(l)} \Delta_f^{-2} \Delta_{f+1}^2 \qquad \text{and}$$

$$\tilde{\rho}_{f,n}(\widetilde{g}_i) = p_{t^\lambda}^{(l)} g_{f+i} \qquad \text{for } i = 1, \ldots, n-1.$$

Theorem 3.2. *Let m, r_1, r_2, $l \in \mathbb{N}$, $l \geq 4$ and $r = r_1 + r_2 < l$. Assume q is a primitive l-th root of unity and $Q = -q^{m+r_1}$ with $r_1 < m + r_1 \leq l - r_2$. Then $\tilde{\rho}_{f,n}$ as defined above is a nontrivial onto homomorphism.*

Proof. $\tilde{\rho}_{f,n}$ is well-defined at roots of unity. Thus the proof that $\tilde{\rho}_{f,n}$ is a homomorphism is the same as in [**O**].

To show that $\tilde{\rho}_{f,n}$ is onto, it suffices to show that every irreducible representation of the reduced algebra is an irreducible representation of $H_n(q, -q^{r_1+m})$. The proof is by induction on n. For $n = 1$, we have $\tilde{\rho}_{f,1} : H_1(q, -q^{r_1+m}) \to p_{t^\lambda}^{(l)} H_{f+1}^{(r,l)}(q) p_{t^\lambda}^{(l)}$. Since $\lambda \vdash f$ is a rectangular diagram there are only two Young diagrams with $f+1$ boxes which contain λ, i.e., $[m+1, m^{r_1-1}]$ and $[m^{r_1}, 1]$. Note that $V_{[m+1,m^{r_1-1}]}^{(l)}$ is well-defined as long as $m+1 \leq l-r$ and $V_{[m^{r_1},1]}^{(l)}$ is well-defined as long as $r_2 > 0$. The action of $\tilde{\rho}_{f,1}(t)$ on the representation indexed by $[m+1, m^{r_1-1}]$ (resp. $[m^{r_1}, 1]$) is $-q^{r_1+m}$ (resp. -1). And both representations are 1 dimensional.

The algebra $H_1(q, -q^{r_1+m})$ has two irreducible representations indexed by $([1], \emptyset)$ and $(\emptyset, [1])$. Both representations are 1 dimensional and $t \in H_1(q, -q^{r_1+m})$ acts by a scalar on these representations. The action of t on $V_{([1],\emptyset)}$ (resp. $V_{(\emptyset,[1])}$) is $-q^{r_1+m}$ (resp. -1). Notice that $l \neq r_1 + m$, since we assumed that $m \leq l - r_1 - r_2$. Therefore, these representations are irreducible and nonequivalent. This shows that $\pi_{([1],\emptyset)} \cong \pi_{[m+1,m^{r_1-1}]}$ and

$\pi_{(\emptyset,[1])} \cong \pi_{[m^{r_1},1]}$ whenever the representations $\pi_{[m+1,m^{r_1-1}]}$ and $\pi_{[m^{r_1},1]}$ in the reduced algebra are well-defined.

In what follows if $\lambda \subset \nu$, then ν/λ will be identified with the pair (α, β) of Young diagrams which remain after removing λ, see Figure 2.

Assume that for $n > 1$ we have $\tilde{\rho}_{f,n}$ is onto. If $\nu \vdash n+f$ is an (r,l)-diagram containing λ, then $V^{(l)}_{\nu/\lambda}$ is an irreducible module of $H_n(q, -q^{r_1+m})$. Now let $\mu \vdash n+f+1$ be an (r,l)-diagram which contains λ, then $V^{(l)}_{\mu/\lambda}\big|_{H_n(q,-q^{r_1+m})} \cong \bigoplus_{\lambda\subset\mu'\subset\mu} V_{\mu'/\lambda}$, as in Equation (6). Clearly $V^{(l)}_{\mu/\lambda}$ is a representation of $H_{n+1}(q, -q^{r_1+m})$.

The irreducibility can be shown exactly as in [**W1**, Theorem 2.2 and Corollary 2.5]. The fact that representations belonging to different Young diagrams are inequivalent is also shown as in [**W1**, Theorem 2.2 and Lemma 2.11]. □

This theorem constructs a semisimple quotient of $H_n(q, -q^{r_1+m})$, which we denote by $H^{(r,l)}_n(q, -q^{r_1+m})$.

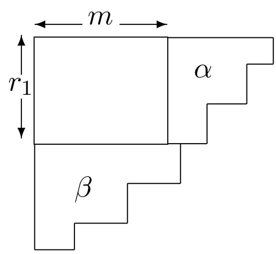

Figure 2.

Observation. There is a 1-1 correspondence between pairs of Young diagrams (α, β) satisfying the condition $\alpha_{r_1} - \beta_1 \geq -m$ with $l(\alpha) \leq r_1$ and Young diagrams containing a rectangular diagram $[m^{r_1}]$, see Figure 2.

Now we define a subset $\Gamma_n(l, m, r)$ of the set of double partitions. We will show that the quotient $H^{(r,l)}_n(q, -q^{r_1+m})$ which is isomorphic to the image of $\tilde{\rho}_{f,n}$ is indexed by the ordered pairs of Young diagrams which we now define.

Definition. Let $m, l, r \in \mathbb{N}$ with $r \leq l-1$. A pair of Young diagrams (α, β) such that $l(\alpha) \leq r_1$ and $l(\beta) \leq r_2$ is called a (m, l, r)-*diagram* if

(1) $\alpha_1 - \beta_{r_2} \leq l - r - m$ and

(2) $\alpha_{r_1} - \beta_1 \geq -m$.

Let $\Gamma_n(l, m, r)$ denote the set of all (m, l, r)-diagrams with n boxes.

We have the following corollary of Theorem 3.2.

Corollary 3.3. (i) *Let $\mu = [m+\alpha_1, \ldots, m+\alpha_{r_1}, \beta_1, \ldots, \beta_{r_2}]$ be an (r,l) diagram. Then there exists a 1-1 correspondence between $\mu \in \Lambda^{(r,l)}_{n+f}$ and $(\alpha,\beta) \in \Gamma(l,m,r)$.*

(ii) *If the representation indexed by (α,β) is well-defined, then the bijection in (i) is compatible with the homomorphism, $\tilde{\rho}_{f,n}$.*

Proof. (i) Recall that $\mu \in \Lambda^{(r,l)}_{n+f}$ implies that $\mu_1 - \mu_r \leq l - r$, where $l(\mu) \leq r = r_1 + r_2$. By substituting $\mu_1 = \alpha_1 + m$ and $\mu_r = \beta_{r_2}$ one gets $\alpha_1 - \beta_{r_2} + r_2 \leq l - r_1 + m$ which is condition (1) in the definition of the elements in $\Gamma(l,m,r)$. The other condition is easily seen by the definition of a Young diagram. $\mu_{r_1} \leq \mu_{r_1+1}$ implies condition (2) $\alpha_{r_1} - \beta_1 > -m$. Clearly, having $(a,\beta) \in \Gamma(l,m,r)$ one can construct μ by adjoining the box $[m^{r_1}]$.

(ii) By (i) we have two indexing sets for the irreducible representations of $H^{(r,l)}_n(q,-q^{r_1+m})$. If $(\pi^{(r,l)}_\mu, V^{(r,l)}_\mu)$ is a well-defined irreducible representation then we can also index it with a pair $(\alpha,\beta) \in \Gamma(l,m,r)$. Furthermore, if we restrict $V^{(r,l)}_\mu$ to $H^{(r,l)}_{n-1}(q,-q^{r_1+m})$ we obtain the decomposition

$$V^{(r,l)}_\mu \Big|_{H^{(r,l)}_{n-1}(q,-q^{r_1+m})} = \bigoplus_{\mu' \subset \mu} V^{(r,l)}_{\mu'}$$

where $\mu' \in \Lambda^{(r,l)}_{n-1}$ and $\mu' \supset \lambda$ by Theorem 3.2. Note that μ' can be associated with a pair $(a,\beta)' \in \Gamma_{n-1}(l,m,r)$ and $V^{(r,l)}_{(\alpha,\beta)}$ can be associated with $V^{(r,l)}_{\mu'}$ whenever the representations are well-defined. Therefore, the bijection in (i) is compatible with the homomorphism $\hat{\rho}_{f,n}$. □

In Figure 3 we show the Bratteli diagrams for the example, $l = 5$, $m = 2$, $r_1 = 1$ and $r_2 = 2$. In this case $\lambda = [2]$.

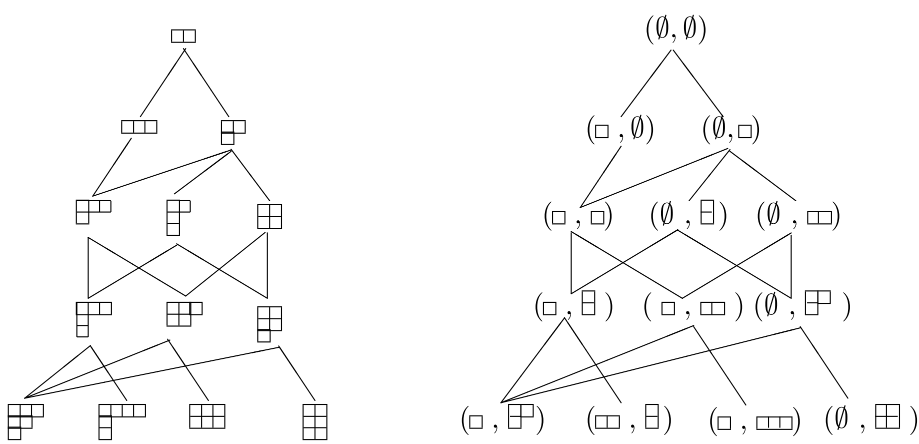

Figure 3. Bratteli Diagrams for $p^{(5)}_{t^{[2]}} H^{(3,5)}_n(q) p^{(5)}_{t^{[2]}}$ and $H^{(3,5)}_n(q,-q^3)$.

Let us fix $m, l, r \in \mathbb{N}$ with $l \geq 4$ and let $q = e^{2\pi i/l}$. Set

$$B_n = \bigoplus_{(\alpha,\beta)\in\Gamma_n(l,m,r)} \pi^{(l)}_{(\alpha,\beta)}(H_n(q, -q^{r_1+m})) \tag{8}$$

where $\pi^{(l)}_{(\alpha,\beta)}$ denotes the representation indexed by (α, β) when q is an l-th root of unity obtained through the homomorphism $\hat{\rho}_{f,n}$. By definition of the $\pi^{(l)}_{(\alpha,\beta)}$'s, the restriction of this representation to $H_{n-1}(q, -q^{r_1+m})$ is isomorphic to B_{n-1}. With this identifications we can define the representation

$$\pi^{(l)} : H_\infty(q, -q^{r_1+m}) \longrightarrow B_\infty$$

of the corresponding inductive limits by

$$\pi^{(l)}(x) = \bigoplus_{(\alpha,\beta)\in\Gamma_n(m,l,r)} \pi^{(l)}_{(\alpha,\beta)}(x) \tag{9}$$

for all $x \in H_n(q, -q^{r_1+m})$.

If $q = e^{2\pi i/l}$ then Wenzl [**W1**] showed that the inclusion diagrams for the Hecke algebras of type A eventually become periodic with period r (the maximum number of rows allowed).

Lemma 3.4. (a) *If the inclusion diagram for* $\cdots \subset H_{n-1}(q) \subset H_n(q) \subset \cdots$ *has period* r*, then the inclusion diagram for* $\cdots \subset p_{t^\lambda} H_{n-1}(q) p_{t^\lambda} \subset p_{t^\lambda} H_n(q) p_{t^\lambda} \subset \cdots$ *has period* r*.*

(b) *The inclusion diagram for* $\cdots \subset H^{(r,l)}_{n-1}(q, -q^{r_1+m}) \subset H^{(r,l)}_n(q, -q^{r_1+m}) \subset \cdots$ *has period* r *whenever* $\cdots \subset p_{t^\lambda} H^{(r,l)}_{n-1}(q) p_{t^\lambda} \subset p_{t^\lambda} H^{(r,l)}_n(q) p_{t^\lambda} \subset \cdots$ *has period* r*.*

Proof. The proof of (a) follows immediately from the definition of reduced algebra.

(b) We have shown above that the quotient $H^{(r,l)}_n(q, -q^{r_1+m})$ of the Hecke algebra of type B is isomorphic to the reduced algebra $p^{(l)}_{t^\lambda} H^{(r,l)}_{n+f}(q) p^{(l)}_{t^\lambda}$. Thus periodicity follows from this isomorphism. □

4. Markov Traces.

In this section we define special traces which will help us define II_1 factors. These traces satisfy the commuting square property, which is needed for the construction of subfactors. The existence of these traces on the Hecke algebra of type B has been proven by Geck and Lambropoulou in [**GL**].

Definition. A trace, tr, on $H_\infty(q, Q)$ is called a *Markov trace* if there is a $z \in \mathbb{C}(q, Q)$ such that $\mathrm{tr}(x g_n) = z\,\mathrm{tr}(x)$ for all $n \in \mathbb{N}$ and $x \in H_n(q, Q)$.

All generators g_i, $i = 1, 2, \ldots$ are conjugate in $H_\infty(q, Q)$. Thus, any trace function on $H_\infty(q, Q)$ must have the same value on these elements.

In particular, this implies that the parameter z is independent of n in the definition of Markov trace.

Geck and Pfeiffer [**GP**] showed that tr is uniquely determined on elements of minimal length of the form $d_1 \cdots d_n$ where $d_i = g_{i-1}$ or $d_i = t'_{i-1} = g_{i-1} \cdots g_1 t g_1^{-1} \cdots g_{i-1}^{-1}$. So if tr is a Markov trace then

$$\mathrm{tr}(d_1 \cdots d_n) = z^a \mathrm{tr}(t'_0 \cdots t'_{b-1})$$

where a is the number of d_i's which equal g_{i-1} and b is the number which equal t'_{i-1}.

Geck and Lambropoulou have shown that given $z, y_1, y_2, \ldots \in \mathbb{C}(q, Q)$ then there is a unique Markov trace for $H_\infty(q, Q)$ such that $\mathrm{tr}(t'_0 t'_1 \cdots t'_{k-1}) = y_k$ for all $k \geq 1$. The case in which we are interested is described in the following proposition which is proved in [**GL**].

Proposition 4.1. *Let $z, y \in \mathbb{C}(q, Q)$ be a Markov trace with parameter z such that $\mathrm{tr}(t'_0 t'_1 \cdots t'_{k-1}) = y^k$ for all $k \geq 1$, then*

$$\mathrm{tr}(h t'_{n,0}) = y \mathrm{tr}(h) \quad \textit{for all } n \geq 0 \textit{ and } h \in H_n(q, Q),$$

where $t'_{n,0} = g_n \cdots g_1 t g_1^{-1} \cdots g_n^{-1}$ or $g_n^{-1} \cdots g_1^{-1} t g_1 \cdots g_n$.

In [**O**] we computed the weight vector for the trace described in Proposition 4.1. The components of this vector are indexed by double partitions and are given by the following formula:

$$W_{(\alpha,\beta)}(q, Q) = q^{n(\alpha)+n(\beta)} \left(\frac{1-q}{1-q^r} \right)^{|\alpha|+|\beta|} \prod_{1 \leq i < j \leq r_1} \frac{1 - q^{\alpha_i - \alpha_j + j - i}}{1 - q^{j-i}} \cdot \prod_{1 \leq i < j \leq r_2} \frac{1 - q^{\beta_i - \beta_j + j - i}}{1 - q^{j-i}} \prod_{i=1}^{r_1} \prod_{j=1}^{r_2} \frac{Q q^{\alpha_i - i} + q^{\beta_j - j}}{Q q^{-i} + q^{-j}} \tag{10}$$

where $n(\alpha) = \sum_{i>1} (i-1)\alpha_i$ and $r = r_1 + r_2$.

The weights for the Markov trace on the Hecke algebra of type A can be found in [**W1**]. They are given by the following formula:

$$s_{\alpha,r}(q) = q^{n(\alpha)} \left(\frac{1-q}{1-q^r} \right)^{|\alpha|} \prod_{1 \leq i < j \leq r} \frac{1 - q^{\alpha_i - \alpha_j + j - i}}{1 - q^{j-i}}. \tag{11}$$

In [**O**] we also observed that when $Q = -q^{r_1+m}$ we obtain

$$W_{(\alpha,\beta)}(q, -q^{r_1+m}) = \frac{s_{\mu,r}(q)}{s_{[m^{r_1}],r}(q)}$$

where $\mu = [m + \alpha_1, \ldots m + \alpha_{r_2}, \beta_1, \ldots \beta_{r_1}]$.

In [**W1**] Lemma 3.5, Wenzl showed that if $l(\mu) > r$ then $s_{\mu,r}(q) = 0$. Also he showed that $s_{\mu,r}(q)$ is well-defined when q is a primitive l-th root of unity with $l > 1$ if $\mu_1 - \mu_r \leq l - r + 1$, and $s_{\mu,r}(q) = 0$ if and only if

$\mu_1 - \mu_r = l - k + 1$. In particular, $s_{\mu,r}(q) > 0$ for all (r,l) diagrams. Thus, he obtained a well-defined, positive trace on $H_n^{(r,l)}(q)$. Furthermore, he showed that the weight vector for the restriction of tr to $H_n^{(r,l)}(q)$ is given by the vector $(s_{\mu,r}(q))_{\mu \in \Lambda_n^{(r,l)}}$.

Proposition 4.2. *The Markov trace defined by the weights in Equation (10) factors over the quotients of the Hecke algebra of type B, $H_{n+f}^{(r,l)}(q, -q^{r_1+m})$.*

Proof. This proposition is a direct consequence of the results cited before the statement of this proposition. We know that the quotient $H_n^{(r,l)}(q, -q^{r_1+m})$ is isomorphic to a quotient of the reduced algebra $p_{t^\lambda}^{(l)} H_{n+f}^{(r,l)}(q) p_{t^\lambda}^{(l)}$ which has weight vectors given by $\frac{s_{\mu,r}(q)}{s_{[m^{r_1}],r}(q)}$. Since $[m^{r_1}]$ is an (r,l) diagram we have that $s_{[m^{r_1}],r}(q) \neq 0$. So the weights are well-defined. Furthermore, they will be zero exactly when $s_{\mu,r}(q) = 0$. □

5. C^*-Representations of $H_n(q,Q)$.

In order to use the results in [**W1**], we need to find nontrivial C^* representations of the inductive limit of the Hecke algebra of type B, $H_\infty(q,Q)$. That is, we need to find the values of the parameters q and Q for which the generators e_i are self-adjoint. From now on $V_{(\alpha,\beta)}$ is assumed to be a complex Hilbert space with orthonormal basis $\{v_\tau \mid \tau \in T_{(\alpha,\beta)}\}$.

In what follows we show that there are C^* representations of $H_\infty(q,Q)$ when $Q = -q^k$ and q is an l-th root of unity.

Definition. A representation ρ of $H_n(q,Q)$ or $H_\infty(q,Q)$ on a Hilbert space is called a C^* representation if $\rho(e_t)$ and $\rho(e_i)$ for $i = 1, 2, \ldots n-1$ or for all $i \in \mathbb{N}$ are self-adjoint projections.

Wenzl in [**W1**] showed that there are nontrivial C^* representations of $H_\infty(q)$, if q is real and positive or if $q = e^{2\pi i/l}$, where l is a positive integer greater than or equal to 4. Since $H_\infty(q) \subset H_\infty(q,Q)$ it follows that to obtain C^* representations of $H_\infty(q,Q)$ it is necessary for q to be real and positive or an l-th root of unity. Unfortunately, this is not sufficient, we also need a condition for Q.

Proposition 5.1. *If q and Q are both real and positive there are faithful C^* representations of $H_n(q,Q)$ for all $n \in \mathbb{N}$. If $q = e^{\pm 2\pi i/l}$ and $Q = -q^{r_1+m}$ for $l, m, r_1, r_2 \in \mathbb{N}$ with $l \geq 4$, and $r_1 \leq m + r_1 \leq l - r_2$ then $\pi_{(\alpha,\beta)}^{(l)}$ is a C^* representation for all (m,l,r)-diagrams (α,β).*

Proof. Theorem 3.2 shows that there exists an onto homomorphism from $H_n^{(r,l)}(q, -q^{r_1+m})$ onto $p_{t^\lambda}^{(l)} H_{n+f}^{(r,l)}(q) p_{t^\lambda}^{(l)}$. Thus, it will suffice to show that the reduced algebra has C^* representations. In [**W1**] Wenzl showed that the quotient $H_{n+f}^{(r,l)}(q)$ is a C^* algebra when q is a root of unity. Furthermore, this quotient is semisimple and the irreducible modules are indexed by (r,l)-diagrams.

Since $p_{t^\lambda}^{(l)}$ is an orthogonal projection in $H_{n+f}^{(r,l)}(q)$ and it is well-defined when q is a root of unity then it is self-adjoint. Then it follows that $p_{t^\lambda}^{(l)} H_{n+f}^{(r,l)}(q) p_{t^\lambda}^{(l)}$ has a C^* representation and the irreducible modules are indexed by (r,l)-diagrams which contain the diagram λ. Thus, by Corollary 3.3 the irreducibles are indexed by (m,l,r)-diagrams. □

6. Subfactors, Index and Commutants.

In the previous section we showed that there are quotients of the Hecke algebra of type B, $H_n(q, -q^{r_1+m})$, which are C^* algebras. We also showed that these quotients are isomorphic to $p_{t^\lambda}^{(l)} H_{n+f}^{(r,l)}(q) p_{t^\lambda}^{(l)}$, when q is an l-th root of unity. We are now in position to construct the subfactors which arise from the inclusion of the Hecke algebra of type A into the Hecke algebra of type B. We will give the index and relative commutants for these subfactors.

In order to construct the subfactors we will use the following two sequences of algebras. Let $l,\ m,\ r_1,\ r_2 \in \mathbb{N}$ with $l > r$ and set $r = r_1 + r_2$:

(i) Let $B_n = p_{t^\lambda}^{(l)} H_{n+f}^{(r,l)}(q) p_{t^\lambda}^{(l)}$ be the finite dimensional C^* algebras in the previous section. Then we have the sequence given by the proper inclusion of $B_n \subset B_{n+1}$ for all $n \in \mathbb{N}$.

(ii) Let $A_n = p_{t^\lambda}^{(l)} H_{f,n+f}^{(r,l)}(q) p_{t^\lambda}^{(l)}$, where $H_{f,n+f}^{(r,l)}(q)$ is the finite dimensional C^* algebra generated by $g_{f+1}, \dots, g_{f+n}$ in $H_{n+f}^{(r,l)}(q)$. Furthermore, we have that $A_n \subset B_n$.

Thus, we have two sequences (A_n) and (B_n) of C^* algebras such that $A_n \subset B_n$. $p_{t^\lambda}^{(l)}$ is the identity in A_n and B_n.

From the work of Jimbo [Ji] and Drinfel'd [D] we know that if q is not a root of unity and V is the fundamental representation of $U_q(\mathfrak{sl}(r))$ (quantum group of $\mathfrak{sl}(r)$). Then there is an isomorphism

$$\phi :\ H_n^{(r)}(q) \to \mathrm{End}_{U_q(\mathfrak{sl}(r))}(V^{\otimes n})$$

where $H_n^{(r)}(q)$ is the quotient of $H_n(q)$ with irreducible representations indexed by Young diagrams with at most r rows.

By the onto homomorphism $\rho_{f,n} : H_n(q, -q^{r_1+m}) \to p_{t^\lambda} H_{n+f}(q) p_{t^\lambda}$; and the fact that the image of $\phi(p_{t^\lambda})$ is $V_{[m^{r_1}]}$, We have the following representation

$$H_n^{(r)}(q, -q^{r_1+m}) \to \mathrm{End}_{U_q(\mathfrak{sl}(r))}(V_{[m^{r_1}]} \otimes V^{\otimes n})$$

where $V_{[m^{r_1}]}$ is the $U_q(\mathfrak{sl}(r))$-module of highest weight $[m^{r_1}]$. This isomorphism is given by composing the surjective homomorphism $\rho_{n,f}$ and the isomorphism ϕ.

The Markov trace defined in Section 4, tr, on the Hecke algebras defines a Markov trace on the centralizer algebras $\mathrm{End}_{U_q(\mathfrak{sl}(r))}(V_{[m^{r_1}]} \otimes V^{\otimes n})$. We will denote this Markov trace by tr again.

We now let q be an l-th root of unity. We outline some of the results and definitions about $U_q(\mathfrak{sl}(r))$-modules as necessary for our purpose, see [**W3**] and [**A**] for the details. A *tilting module* of $U_q(\mathfrak{sl}(r))$ is a direct summand of a tensor power of the fundamental module V, or if it is a direct sum of such modules.

Tilting modules satisfy the following properties:

(1) Tensor products of tilting modules are tilting modules.
(2) Any tilting module is isomorphic to a direct sum of indecomposable tilting modules.

Each indecomposable tilting module has a q-dimension. If q is a root of unity, this q-dimension can be zero. An indecomposable tilting module with 0 q-dimension will be called *negligible*. The indecomposable negligible modules generate a tensor ideal, which we will denote by $\mathcal{N}eg(T)$.

Thus, let W_1 and W_2 be two tilting modules, we now define a tensor product $\bar{\otimes}$ as follows:

$$W_1 \bar{\otimes} W_2 = (W_1 \otimes W_2)/\mathcal{N}eg(W_1 \otimes W_2)$$

where $\mathcal{N}eg(W_1 \otimes W_2)$ is defined as follows: First decompose $W_1 \otimes W_2$ into indecomposable tilting modules then throw away the negligible ones. Using this tensor product we have the following representation of the Hecke algebras at roots of unity. Let V be the fundamental module of $U_q(\mathfrak{sl}(r))$

$$H_n^{(r,l)}(q) \to \mathrm{End}_{U_q(\mathfrak{sl}(r))}(V^{\bar{\otimes} n})$$

and also

$$H_n^{(r,l)}(q, -q^{r_1+m}) \to \mathrm{End}_{U_q(\mathfrak{sl}(r))}\left(V_{[m^{r_1}]} \bar{\otimes} V^{\bar{\otimes} n}\right).$$

Thus we have that

$$A_n \cong \mathrm{End}_{U_q(\mathfrak{sl}(r))}(1 \bar{\otimes} V^{\bar{\otimes} n})$$

and

$$B_n \cong \mathrm{End}_{U_q(\mathfrak{sl}(r))}(V_{[m^{r_1}]} \bar{\otimes} V^{\bar{\otimes} n})$$

since ϕ is surjective and injective when one restricts to Young diagrams with at most r rows.

The Markov trace on the Hecke algebras factors over the quotient $H_n^{(r,l)}(q, -q^{r_1+m})$, thus the Markov trace on the centralizer algebras also factors over the corresponding quotients.

By Proposition 4.3 in [**W3**] if we take the above identifications for A_n and B_n then we have that the sequence of algebras (A_n) and (B_n) satisfy the commuting square property. And the sequences (A_n) and (B_n) are periodic.

It is well-known that under the periodicity assumption there exists at most one normalized trace on $B_\infty = \bigcup_n B_n$, which must be a factor trace, that is, the weak closure of $\pi_{\mathrm{tr}}(\bigcup_n B_n)$ must be a factor. Similarly, for $A_\infty = \bigcup_n A_n$. Therefore, one obtains a pair of hyperfinite II_1 factors

$$A = \pi^{(r,l)}(A_\infty)'' \subset B = \pi^{(r,l)}(B_\infty)''.$$

In [**W1**] Wenzl showed that if a factor is generated by a ladder of commuting squares and if the Bratteli diagrams are periodic, then the index is given as a quotient of the weight vectors of the unique normalized trace, tr, i.e., if $\vec{s}_n$ is the weight vector on A_n and $\vec{t}_n$ is the weight vector on B_n then the index is given by the following formula whenever n is big enough:

$$[B : A] = \frac{||\vec{s}_n||^2}{||\vec{t}_n||^2}. \tag{12}$$

Proposition 6.1. *Let* $r_1,\ r_2 \in \mathbb{N}$ *and set* $r = r_1+r_2$*. For each pair* $m,\ l \in \mathbb{N}$ *such that* $m \leq l - r$*,* $Q = -q^{r_1+m}$ *and* $q = e^{2\pi i/l}$*, there is a subfactor of the hyperfinite* II_1 *factor obtained from the inclusion* $A \subset B$ *with index given by the following formula:*

$$\prod_{i=1}^{r_1}\prod_{j=1}^{r_2} \frac{\sin^2\left((r_1+m+j-i)\pi/l\right)}{\sin^2\left((r_1+j-i)\pi/l\right)}.$$

Proof. In [**O**] we showed that the weights of the Markov trace on $H_n(q, -q^{r_1+m})$ are given by

$$\frac{s_{\mu,r}(q)}{s_{[m^{r_1}],r}(q)}$$

where μ is an (r,l)-diagram with $n+f$ boxes containing $[m^{r_1}]$. Thus the norm of the weight vector is

$$||\vec{t}_n|| = \sum_{[m^{r_1}]\subset\mu\vdash n+f} \frac{(s_{\mu,r}(q))^2}{(s_{[m^{r_1}],r}(q))^2}.$$

Now, note that $H_{f,n+f}(q)$ commutes with $p^{(l)}_{t^\lambda}$, thus $p^{(l)}_{t^\lambda} H_{f,n+f}(q) p^{(l)}_{t^\lambda} = p^{(l)}_{t^\lambda} H_{f,n+f}(q)$, thus the weight vector is given by

$$||\vec{s}_n|| = \sum_{[m^{r_1}]\subset\nu\vdash n+f} (s_{\nu,r}(q))^2.$$

Therefore, we have by Wenzl's index formula in [**W1**] that the index for this subfactors is

$$\begin{aligned}[B:A] &= (s_{[m^{r_1}],r}(q))^2 \\ &= \prod_{i=1}^{r_1}\prod_{j=1}^{r_2} \frac{\sin^2((r_1+m+j-i)\pi/l)}{\sin^2((r_1+j-i)\pi/l)}\end{aligned}$$

recall that $s_{[m^{r_1}],r}(q)$ is the Schur function defined in Section 4. The last equality is obtained by the substitution $q = e^{2\pi i/l}$ into this Schur function. □

Remark. This proposition can also be proved in much more generality using the machinery introduced before the statement of the proposition, see [**W3**].

It is well-known that the index is not a complete invariant of II_1 factors. A finer invariant is given by the higher relative commutants. Consider the following tower of II_1 factors associated to $A \subset B$

$$A \subset B \subset B^{(1)} = \langle B, e_1\rangle \subset B^{(2)} = \langle B^{(1)}, e_2\rangle \subset B^{(3)} = \langle B^{(2)}, e_3\rangle \cdots$$

where $B^{(i)} = \langle B, e_i\rangle$ is obtained by the basic construction applied to $B_{i-2} \subset B_{i-1}$. Since $[B^{(i)} : A] = [B : A]^{i+1} < \infty$, $[B^{(i)} : B] = [B : A]^i < \infty$, the higher relative commutants $A' \cap B^{(i)}$ are all finite dimensional algebras.

The subfactors we have obtained are special cases of the subfactors obtained in [**W3**] by Wenzl. The higher relative commutants for the subfactors obtained in this paper are given by the following proposition.

Proposition 6.2. *Let m, r_1, $r_2 \in \mathbb{N}$. Let $A \subset B$ be the pair of factors constructed above with index as described in the previous proposition, then the higher relative commutants are given by*

$$(13)\quad A' \cap B^{(i)} = \mathrm{End}_{U_q(\mathfrak{sl}(r))}\left(\cdots V_{[m^{r_1}]}\bar{\otimes} V_{[m^{r_2}]}\bar{\otimes} V_{[m^{r_1}]}\right) \quad (i+1 \ \ \textit{factors})$$

where $U_q(\mathfrak{sl}(r))$ is the quantum group of $\mathfrak{sl}(r)$ and $V_{[m^{r_2}]} \cong (V_{[m^{r_1}]})^*$.

Proof. The proof of this proposition follows from the proof of Theorem 4.4 in [**W3**]. We will outline the proof for the reader's convenience. To compute the higher relative commutants we need to compute the i-th extension $B^{(i)}$ via Jones' basic construction. The computation of $B^{(i)}$ is done by induction on i. Let $B_n^{(1)} = \mathrm{End}_{U_q(\mathfrak{sl}(r))}(V_{[m^{r_2}]}\bar{\otimes} V_{[m^{r_1}]}\bar{\otimes} V^{\bar{\otimes} n})$. We define the embedding $B_n^{(1)} \to B_{n+1}^{(1)}$ via $B_n^{(1)} \cong B_n^{(1)}\bar{\otimes} Id_V \subset B_{n+1}^{(1)}$. Let $B_\infty^{(1)}$ be the inductive limit of the $B_n^{(1)}$'s. Then define $\tilde{B}^{(1)}$ as the weak closure of the GNS construction with respect to the normalized Markov trace, tr, i.e., $\tilde{B}^{(1)} = \overline{\pi_{\mathrm{tr}}(B_\infty^{(1)})}^{\mathrm{w}}$. Then one can check that there is an isomorphism $\tilde{B}^{(1)} \to B^{(1)}$, where $B^{(1)}$ is the basic construction $A \subset B$. This isomorphism leaves B fixed and sends e_1 to the projection onto the trivial representation contained in $V_{[m^{r_2}]}\bar{\otimes}V_{[m^{r_1}]}$.

For $i > 1$ take the inductive limit of

$$B_n^{(i)} = \mathrm{End}_{U_q(\mathfrak{sl}(r))}((\underbrace{\cdots V_{[m^{r_2}]}\bar{\otimes}V_{[m^{r_1}]}\bar{\otimes}V_{[m^{r_2}]}}_{i\text{ factors}})\bar{\otimes}V_{[m^{r_1}]}\bar{\otimes}V^{\bar{\otimes}n}),$$

with the embedding defined as before. Thus, the i-th extension $B^{(i)} \cong \overline{\pi_{\mathrm{tr}}(B_\infty^{(i)})}^{\mathrm{w}}$ is the weak closure of the GNS costruction with respect to the Markov trace. And e_i corresponds to the projection onto the trivial representation contained in the two left-most factors $V_{[m^{r_1}]}\bar{\otimes}V_{[m^{r_2}]}$ if i is even, and in $V_{[m^{r_2}]}\bar{\otimes}V_{[m^{r_1}]}$ if i is odd.

We now compute the higher relative commutants. Let $\mathbf{1}$ denote the trivial representation. If p is the minimal projection onto a chosen copy of the trivial representation contained in $V^{\bar{\otimes}n}$, then we have that

$$\begin{aligned} &p\mathrm{End}_{U_q(\mathfrak{sl}(r))}(\cdots V_{[m^{r_2}]}\bar{\otimes}V_{[m^{r_1}]}\bar{\otimes}V^{\bar{\otimes}n})p \\ &\cong \mathrm{End}_{U_q(\mathfrak{sl}(r))}(\underbrace{\cdots V_{[m^{r_2}]}\bar{\otimes}V_{[m^{r_1}]}}_{i+1}\bar{\otimes}\mathbf{1}) \\ &\cong \mathrm{End}_{U_q(\mathfrak{sl}(r))}(\underbrace{\cdots V_{[m^{r_2}]}\bar{\otimes}V_{[m^{r_1}]}}_{i+1}) \\ &\subseteq A' \cap B^{(i)}. \end{aligned}$$

But Theorem 1.6 in [**W1**] implies that the dimension of the left-hand-side is greater than or equal to the dimension of the right-hand-side. Thus we have

$$\mathrm{End}_{U_q(\mathfrak{sl}(r))}(\underbrace{\cdots V_{[m^{r_2}]}\bar{\otimes}V_{[m^{r_1}]}}_{i+1}) = A' \cap B^{(i)}.$$

□

Remarks. (1) This proposition can also be shown by computing all the iterations of Jones' basic construction by adding generators on the "left" and then reducing by the appropriate projection, for details on this construction see [**E**]. She obtains all the higher relative commutants as a corollary of this construction for some subfactors of the Hecke algebra of type A.

(2) One can use the generalization of the Littlewood-Richardson [**GW**] rule to obtain a direct sum decomposition of the tensor product of two simple tilting modules with nonzero q-dimension when q is a root of unity.

7. Subfactors via Hecke algebra of type D.

In this section we denote by H_n^D, the Hecke algebra of type D, and by H_n^B the Hecke algebra of type B. The easiest way to study the Hecke algebras of type D is by embedding them into the Hecke algebras of type B, and then use known results for type B. Hoefsmit [**H**] observed that in order to obtain an embedding of $H_n^D(q)$ into $H_n^B(q,Q)$ we have to set the parameter Q equal to 1, thus from now on we assume that $Q=1$. Notice that in this case we have $t^2=1$. The Hecke algebra of type D is generated by $u=t\hat{g}_1 t,\ \hat{g}_1,\ldots,\hat{g}_{n-1}$ satisfying the following relations:

(D1) $\hat{g}_i\hat{g}_{i+1}\hat{g}_i=\hat{g}_{i+1}\hat{g}_i\hat{g}_{i+1}$ for $i=1,\ldots n-2$;
(D2) $\hat{g}_i\hat{g}_j=\hat{g}_j\hat{g}_i$ whenever $|i-j|\geq 2$;
(D3) $\hat{g}_i^2=(q-1)\hat{g}_i+q$ for all i;
(D4) $\hat{g}_i u=u\hat{g}_i$ for all i;
(D5) $u^2=(q-1)u+q$.

We have $H_n^D\subset H_n^B$ for all n; then $H_\infty^D=\bigcup_{n\geq 1}H_n^D\subset H_\infty^B$. Geck [**G**] has shown that the restriction of a Markov trace on H_∞^B is a Markov trace on H_∞^D and both have the same parameter. Furthermore, he shows that every Markov trace on H_∞^D can be obtained in this way.

From Hoefsmit [**H**] we know that the simple components for H_n^D are indexed by double partitions (α,β). If $\alpha\neq\beta$ we have that the H_n^B-modules $V_{(\alpha,\beta)}$ and $V_{(\beta,\alpha)}$ are simple, equivalent H_n^D-modules. And if $\alpha=\beta$ we have that the H_n^B-module $V_{(\alpha,\alpha)}$ decomposes into two simple nonequivalent H_n^D-modules, i.e., $V_{(\alpha,\alpha)_i}$ with $i=1,2$. Using Bratteli diagrams we have the following relations for simple modules of H_n^B and H_n^D

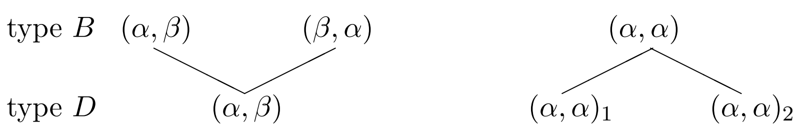

Figure 4.

We now state the proposition about the weights of the Markov trace for the Hecke algebra of type D. This proposition was proved in [**O**].

Proposition 7.1. *Let $r_1,r_2\in\mathbb{N}$ and set $r=r_1+r_2$. Then the weight formula for the Markov trace on the Hecke algebra of type D with parameters $z=q^r\frac{(1-q)}{(1-q^r)}$ and $y=\frac{(Qq^{r_2}+1)(1-q^{r_1})}{(1-q^r)}-1$ is given as follows:*

$$W^D_{(\alpha,\beta)}(q)=W_{(\alpha,\beta)}(q,1)+W_{(\beta,\alpha)}(q,1),\quad \textit{if } \alpha\neq\beta$$

and

$$W^D_{(\alpha,\alpha)_i}(q)=W_{(\alpha,\alpha)}(q,1),\ \textit{for } i=1,2\quad \textit{if }\ \alpha=\beta$$

where $W_{(\alpha,\beta)}(q,1)$ denote the weight in Section 4 Equation (10) evaluated at $Q=1$ of the Hecke algebra of type B.

Denote the Hecke algebra of type A by H_n^A. Now let $r_1, m \in \mathbb{N}$ and assume that q is a primitive $2(r_1+m)$-root of unity. This implies that $Q = -q^{r_1+m} = 1$. Observe that we have the following inclusion of algebras $H_n^A \subset H_n^D \subset H_n^B$. In the previous section we described subfactors obtained from the inclusion $H_n^A(q) \subset H_n^B(q, -q^{r_1+m})$. In what follows we would like to consider the subfactors obtained from the inclusions $H_n^D \subset H_n^B$ and $H_n^A \subset H_n^D$.

In Section 5 we showed that there exist C^*-representations for $H_n^B(q, -q^{r_1+m})$ with $r_1 + m < l - r_2$ which holds true for $l = 2(r_1+m)$ and $r_2 < r_1 + m$. Therefore, we have the following inclusion of hyperfinite II_1 factors

$$D = \pi^{(l)}(H_\infty^D)'' \subset B = \pi^{(l)}(H_\infty^B)''.$$

Proposition 7.2. *Let $r_1,\ r_2 \in \mathbb{N}$. The index for the inclusion of $D \subset B$ is given as follows:*

$$[B : D] = \begin{cases} 1 & \text{if } r_1 \neq r_2 \\ 2 & \text{if } r_1 = r_2. \end{cases}$$

Proof. Choose $n \gg r_1 + r_2$ and assume $r_1 \neq r_2$. Without loss of generality we may assume $r_1 < r_2$, then for sufficiently large n we have pairs of Young diagrams such that $l(\alpha) = r_1$ and $l(\beta) = r_2$. In this case $W_{(\beta,\alpha)}(q, 1) = 0$. This implies that the weight vector for type B is equal to the weight vector of type D. Thus by Wenzl's index formula $[B : D] = 1$.

If $r_1 = r_2$, then from Equation (10) in Section 4 we see that $W^B_{(\alpha,\beta)}(q, 1) = W^B_{(\beta,\alpha)}(q, 1)$ for any pair (α, β). Thus by the previous proposition we have that $W^D_{(\alpha,\beta)}(q) = 2W_{(\alpha,\beta)}(q, 1)$. This implies that

$$[B : D] = \frac{\sum_{\alpha\neq\beta} (2W^B_{(\alpha,\beta)}(q, 1))^2 + \sum_{\alpha=\beta} W^D_{(\alpha,\alpha)_1}(q)^2 + W^D_{(\alpha,\alpha)_2}(q)^2}{\sum_{\alpha\neq\beta} (W^B_{(\alpha,\beta)}(q, 1))^2 + \sum_{\alpha=\beta} W^B_{(\alpha,\alpha)}(q)^2} = 2$$

since $W^D_{(\alpha,\alpha)_i}(q) = W^B_{(\alpha,\alpha)}(q, 1)$. □

Corollary 7.3. *Let $r_1 = r_2 \in \mathbb{N}$. The index for the inclusion of $A \subset D$ is given as follows:*

$$[D : A] = [B : A]/2.$$

Proof. By Proposition 2.18 in [**Jo**] we have that if we have an inclusion of three II_1 factors, $A \subset D \subset B$ then $[B : A] = [D : A][B : D]$. By the previous proposition we have our result. □

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