Pacific Journal of Mathematics

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Volume 199 No. 2 June 2001

PACIFIC JOURNAL OF MATHEMATICS
Vol. 199, No. 2, 2001

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Let F be a p-adic field and let O be the orthogonal group attached to a quaternary quadratic form with coefficients in F and of Witt rank one over F. We determine, up to one possible exception, which nonsupercuspidal representations of $\mathrm{O}(F)$ occur in the theta correspondences attached to $(\mathrm{SL}_2(F), \mathrm{O}(F))$.

This paper is the second in a series of papers examining in detail the local theta correspondences attached to reductive dual pairs $(\mathrm{SL}_2(F), \mathrm{O}(F))$ where F is a p-adic field of characteristic zero and O is the orthogonal group attached to a quaternary quadratic form with coefficients in F and of Witt rank one over F. In this paper we determine, up to one possible exception, which nonsupercuspidal representations of $\mathrm{O}(F)$ occur in the correspondences. The determination is explicit and in terms of parabolic inducing data.

The results we obtain are consistent with the first occurrence in towers conjecture [**KR1**] and Prasad's conjectures [**P2**]. They are sharper and more explicit than the corresponding results in Cognet's thesis [**C**] and complement the results of Roberts [**Ro2**] in the case of p odd, which are stated in terms of distinguished representations. In future papers in this series, we will examine which supercuspidal representations of $\mathrm{O}(F)$ occur in the correspondence and the explicit correspondence.

To explain our method, we first recall the general setting of theta correspondences for symplectic and orthogonal groups; see, e.g., [**MVW**], [**H**]. For $i = 1, 2$, let V_i be a finite-dimensional vector space over F equipped with a nondegenerate bilinear form $\langle\ ,\ \rangle_i$; assume that $\langle\ ,\ \rangle_1$ is skew-symmetric while $\langle\ ,\ \rangle_2$ is symmetric. Equip $W = V_1 \otimes V_2$ with the skew-symmetric form $\langle\ ,\ \rangle$ coming from tensoring the $\langle\ ,\ \rangle_i$. Let G_1, G_2 and G be the isometry groups of $\langle\ ,\ \rangle_1$, $\langle\ ,\ \rangle_2$ and $\langle\ ,\ \rangle$, respectively, and identify G_1 and G_2 with subgroups of G via their usual actions on W; then (G_1, G_2) is called a reductive dual pair in G. Let χ be a nontrivial additive character of F and let ω_χ^∞ denote the (smooth) oscillator representation of $\tilde{G}$ attached to χ where $\tilde{G}$ is the (unique) nontrivial two-fold cover of G. For H a closed subgroup of G, let $\tilde{H}$ denote the inverse image of H in G and let $\mathcal{R}_\chi(\tilde{H})$

denote the set of irreducible admissible representations of $\tilde{H}$ which occur as quotients of $\omega_\chi^\infty|_{\tilde{H}}$. Then $\tilde{G}_1$ and $\tilde{G}_2$ commute and $\mathcal{R}_\chi(\widetilde{G_1 G_2})$ gives rise to a correspondence between $\mathcal{R}_\chi(\tilde{G}_1)$ and $\mathcal{R}_\chi(\tilde{G}_2)$. These correspondences are called theta correspondences. We denote these correspondences by $\theta : \mathcal{R}_\chi(\tilde{G}_1) \to \mathcal{R}_\chi(\tilde{G}_2)$ and $\theta : \mathcal{R}_\chi(\tilde{G}_2) \to \mathcal{R}_\chi(\tilde{G}_1)$; the direction of θ will be clear from context. Theta correspondences are known in general to be bijections for p odd [**Wa**] and for all p in the cases considered in this paper [**R2**]. Furthermore, in all cases considered here, the space V_2 will be even-dimensional and thus the $\tilde{G}_1$ and $\tilde{G}_2$ are trivial covers so that we write, in an abuse of notation, $\mathcal{R}_\chi(G_1)$ and $\mathcal{R}_\chi(G_2)$ instead of $\mathcal{R}_\chi(\tilde{G}_1)$ and $\mathcal{R}_\chi(\tilde{G}_2)$. Elements of these sets will be considered as representations of G_1 and G_2, respectively.

Then our argument and organization for this paper are as follows. In the first section, we establish notation and recall briefly known results that will be necessary in what follows. These results include the parameterizations of the admissible duals of $\mathrm{GL}_2(F)$ and $\mathrm{SL}_2(F)$, the results of the first paper in this series [**M**] on which representations of $\mathrm{SL}_2(F)$ occur in $\mathcal{R}_\chi(\mathrm{SL}_2(F))$ for the pair $(\mathrm{SL}_2(F), \mathrm{O}(F))$, quadratic base change for $\mathrm{GL}_2(F)$ and $\mathrm{SL}_2(F)$ and finally the results of Cognet's thesis [**C**] that will be necessary.

We begin the second section by showing that the representations of $\mathrm{O}(F)$ that occur in $\mathcal{R}_\chi(\mathrm{O}(F))$ must restrict irreducibly to $\mathrm{SO}(F)$. This is a standard seesaw duality [**K1**] argument. We then parameterize the irreducible admissible nonsupercuspidal parameterizations of $\mathrm{O}(F)$ with this property. The parameterization is explicit and in terms of inducing data from the (unique up to conjugacy) maximal parabolic subgroup of $\mathrm{SO}(F)$. This parabolic has Levi, M say, isomorphic to $F^\times \times E^1$, where $F^\times$ denotes the multiplicative group of F and E^1 is the kernel of the norm map from $E^\times$ to $F^\times$ where E/F is the quadratic extension of F attached to the anisotropic part of the quadratic form, Q say, giving rise to O.

In the third section, we consider ω_χ^∞ in the Schroedinger model. In this model, $\mathrm{O}(F)$ acts linearly on $\mathcal{S}(V)$ where V is the space on which Q is defined and $\mathcal{S}(V)$ is the space of locally constant compactly supported function on V. We use this to determine necessary conditions on the representations of M that can be used for inducing data of representations occurring in $\mathcal{R}_\chi(\mathrm{O}(F))$. The argument is by calculation of Jacquet modules.

In the fourth and final section, we show that the necessary conditions of the third section are also sufficient with one possible exception. The argument involves our previous results, Cognet's results and results on base change. It also involves determining which representation of $\mathrm{O}(F)$ pairs with the trivial representation of $\mathrm{SL}_2(F)$. The pairing representation is also one-dimensional and the argument involves the determination of which orbits in V under $\mathrm{O}(F)$ can support this representation. We plan on returning to the possible exception, a generalized Steinberg representation, in a future paper in this series.

Finally, we would like to thank the referee of this paper for pointing out to us that we needed to modify the proof of Lemma 4.2.

1. Notation and known results.

In this section, we establish notation, recall the parameterization of the admissible dual of $G_1 = \mathrm{SL}_2(F)$ and recall some other known results necessary for this paper. We will be brief in our discussion.

Let F be a nonarchimedean local field of characteristic 0. Let p denote the residual characteristic of F and let $\mathcal{O} = \mathcal{O}_F$, $P = P_F$, $\omega = \omega_F$, $k = k_F$, $q = q_F$ and $|\ \ | = |\ \ |_F$ denote, respectively, the ring of integers, the prime ideal, a uniformizing parameter, the residue field and the absolute value on F normalized so that $|x| = q^{-\nu(x)}$ where $\nu = \nu_F$ denotes the order function on F. Let $U = U_F = \mathcal{O}_F^\times$ and $U^n = U_F^n = 1 + P_F^n$ for n a positive integer. Further, for K/F a Galois extension of fields, let $\Gamma(K/F)$ denote the associated Galois group and if, in addition, $[K : F] < \infty$, let $N_{K/F} = N$ denote the norm map and let $K^1 = K_F^1$, the norm one elements in K. Finally, fix an algebraic closure $\bar{F}$ of F and a Weil group W_F; let the associated Weil group notation be as in [**T**].

For G a group and σ a representation (all representations assumed smooth unless stated otherwise) of a subgroup H of G, let $\mathrm{Ind}(G, H; \sigma)$ denote the representation of G induced by σ (form of induction determined by context) and for g in G, let σ^g denote the representation of $H^g = gHg^{-1}$ defined by $\sigma^g(h) = \sigma(g^{-1}hg)$ for h in H^g. If J is a subgroup of H, we let $\sigma|_J$ denote the restriction of σ to J. Further, if $J \lhd H$ and σ is a representation of H/J, then we also view σ as a representation of H via inflation. If σ and τ representations of G, then we let $\mathrm{Hom}_G(\sigma, \tau)$ denote the set of G-intertwining operators from σ to τ with the category, once again, specified by context. Finally, we let $G^\wedge$ denote the admissible dual of G.

By a character, we mean a (not necessarily unitary) one-dimensional representation. If χ is a character of $F^\times$, we also view χ as a character of W_F via local class field theory and as a character of $\mathrm{GL}_2(F)$ by composition with det, the determinant map. Further, if K/F is a finite-dimensional Galois extension, we view χ as a character χ_K of $K^\times$ via composition with $N_{K/F}$. If χ is a character of F and a is an element of F, we let χ_a denote the character of F defined by $\chi_a(y) = \chi(ay)$. Finally, we say representations π_1 and π_2 of $\mathrm{GL}_2(F)$ are twist equivalent if there exists a character η of $F^\times$ such that $\pi_1 \cong \pi_2 \otimes \eta$.

We now briefly recall the parameterization of the admissible dual of $G_1(F) = \mathrm{SL}_2(F)$ in [**LL**]. To do this we first recall the parameterization of the admissible dual of $G_1'(F) = \mathrm{GL}_2(F)$ in [**JL**] in a form suitable for our purposes. If μ and ν are characters of $F^\times$ such that $\mu(x)\nu^{-1}(x) \neq |x|$ or $|x|^{-1}$, let $\pi(\mu,\nu)$ denote the irreducibly induced (normalized induction) principal series representation of G_1' attached to μ and ν. Note that $\pi(\mu,\nu) \cong \pi(\nu,\mu)$. If $\mu(x)\nu^{-1}(x) = |x|$, write $\mu = \chi|\ |^{1/2}$ and $\nu = \chi|\ |^{-1/2}$ and let $\sigma(\mu,\nu)$ denote the special representation corresponding to the unique invariant subspace of the space of the associated induced representation from the Borel subgroup of G_1' and let $\pi(\mu,\nu)(\cong \chi)$ denote the corresponding quotient. Similarly, if $\mu(x)\nu^{-1}(x) = |x|^{-1}$, let $\sigma(\mu,\nu)$ denote the corresponding special representation (now the quotient) and $\pi(\mu,\nu)$ the corresponding one-dimensional. Note that $\sigma(\mu,\nu) \cong \sigma(\nu,\mu)$ and $\pi(\mu,\nu) \cong \pi(\nu,\mu)$. Further, if K/F is quadratic and θ is a character of $K^\times$, let $\pi(\theta) = \pi(\rho)$ denote the corresponding irreducible representation of G_1' associated to $\rho = \mathrm{Ind}(\mathrm{W}_F, \mathrm{W}_K; \theta)$; note that $\pi(\theta) \cong \pi(\theta^{-1})$ and note also that π is supercuspidal if and only if θ does not factor through $N_{K/F}$ which in turn happens if and only if ρ is irreducible. We call representations of the form $\pi(\theta)$ Weil representations. The irreducible representations of G_1' not of one of the above forms are called exceptional and occur only if $p = 2$. These representations are supercuspidal and can be parameterized naturally in terms of the primitive (i.e., not induced from a proper subgroup) two-dimensional representations of W_F [**Ku**]; for σ such a representation of W_F, we write $\pi(\sigma)$ for the corresponding exceptional representation. Finally, we note that the representations enjoy no other equivalences with the exception that if μ and ν are characters of $F^\times$ with $\mu\nu^{-1}$ of order two, then $\pi(\mu,\nu) \cong \pi(\mu_K)$ where K/F is the quadratic extension of F associated to $\mu\nu^{-1}$ by local class field theory.

Now let $G_1 = G_1(F) = \mathrm{SL}_2(F)$ viewed as a subgroup of G_1'. Then we have:

Theorem 1.1 ([**LL**]). *Let π_1 be an irreducible representation of G_1; then there exists an irreducible representation π of G_1', unique up to twist equivalence, which contains π_1 upon restriction to G_1. The L-packet of π_1 is of the form $\{\pi_1, \ldots, \pi_s\}$ where the π_i are distinct irreducible representations of G_1 and the restriction of π to G_1 decomposes as $\bigoplus_{i=1}^s \pi_i$. Further, given $1 \leq i, j \leq s$ there exists g in G_1' such that $\pi_i^g \cong \pi_j$. Moreover, with χ a character of $F^\times$:*

(i) *If π is not a Weil representation, then $s = 1$. Further, if $\pi \otimes \chi \cong \pi$, then χ is trivial.*
(ii) *If $\pi = \pi(\theta)$ with θ a character of $K^\times$ such that $\theta|_{K^1}$ is not of order two, then $s = 2$ and $\pi_i^g \cong \pi_i$ if and only $\det g$ is a norm from $K^\times$. Further, $\pi \otimes \chi \cong \pi$ if and only if χ is trivial or $\chi = \omega_{K/F}$, the character of $F^\times$ associated to K by local class field theory. If π is supercuspidal in this setting, then ρ is singly imprimitive (i.e., can only be induced nontrivially from W_K).*

(iii) *If $\pi = \pi(\theta)$ with θ a character of $K^\times$ such that $\theta|_{K^1}$ is of order two, then $s = 4$. In this case, ρ is triply imprimitive and if K_i, $i = 1, 2, 3$, are the fields such that ρ may be induced from W_{K_i} and L is their composite, then $\Gamma(L/F) \cong \mathbb{Z}/2\mathbb{Z} \otimes \mathbb{Z}/2\mathbb{Z}$ and $\pi_i^g \cong \pi_i$ if and only if* $\det g$ *is a norm from $L^\times$. Further, $\pi \cong \pi \otimes \chi$ if and only if χ is trivial or $\chi = \omega_{K_i/F}$ for some i.*

(iv) *The collection of distinct L-packets partitions $G_1^\wedge$. Further, another representation π' in $(G_1')^\wedge$ gives rise to the same L-packet as π if and only if π and π' can be realized as follows:*
   (a) *$\pi = \pi(\mu, \nu)$ and $\pi' = \pi(\mu', \nu')$ with $\mu\nu^{-1} = (\mu')(\nu')^{-1}$;*
   (b) *$\pi = \sigma(\mu, \nu)$ and $\pi' = \sigma(\mu, \nu')$ with $\mu\nu^{-1} = (\mu')(\nu')^{-1}$;*
   (c) *$\pi = \pi(\theta)$ and $\pi = \pi'(\theta')$ with θ and θ' on $K^\times$ with $\theta(\theta')^{-1}|_{K^1} = 1$;*
   (d) *$\pi = \pi(\sigma)$ and $\pi' = \pi(\sigma')$ with σ and σ' primitive projectively equivalent representations.*

In what follows we will distinguish among the π_i by their Whittaker models. In particular, recall that if π is an infinite-dimensional irreducible representation of G' and η is a nontrivial character of F, then π has, up to scaling, a unique Whittaker model with respect to η and, of course, if π is finite-dimensional, then it has no Whittaker models. Thus, for infinite-dimensional π, we let $\pi(\mu, \nu; \eta)$ denote the component of $\pi(\mu, \nu)$ with η-Whittaker model and similarly for $\sigma(\mu, \nu), \pi(\theta)$ and $\pi(\sigma)$. The only remaining representation is the trivial representation which we denote by 1. Finally, for a in $F^\times$ and π an irreducible representation of G_1, let $\pi^a = \pi^g$ where g is an element of G_1' with $\det g = a$. Then one checks that if π has an η-Whittaker model then π^a has an η_a-Whittaker model.

We continue by recalling the result of [**M**] that will be necessary for this paper. To this end, let E/F be a quadratic extension and set $V = \{A \in M_2(E) \mid \bar{A}^t = A\}$ where $\bar{A}$ denotes the matrix obtained from A by applying σ to each entry where $\Gamma(E/F) = \langle\sigma\rangle$. Now the negative of the determinant map $\det : M_2(E) \to E$ when restricted to V maps to F and defines a quadratic form, Q say, on V viewed as an F vector space. Let H_1 denote the isometry group of this form. Further, for χ a nontrivial additive character of F, let $\mathcal{R}_\chi(G_1)$ denote the representations in the admissible dual of G_1 that occur in the theta correspondence attached to χ and the reductive dual pair (G_1, H_1); see [**M**] and [**MVW**] for more details concerning theta correspondences.

Theorem 1.2 ([**M**]). *If π is an irreducible representation of G_1 such that either, for some b in $N_{E/F}(E^\times), \pi^b$ has a Whittaker model with respect to χ, or π is trivial, then π is in $\mathcal{R}_\chi(G_1)$.*

Theorem 1.1 and Theorem 1.2 have the following consequences: If π' is an irreducible representation of G_1' that cannot be realized as a $\pi(\theta)$ with θ a character of $E^\times$, then the entire L-packet for G_1 associated to π occurs in

$\mathcal{R}_\chi(G_1)$. On the other hand if π can be realized as a $\pi(\theta)$ with θ a character of $E^\times$, then at least half of the representations in the associated L-packet occur.

We now recall the results on base change from $\mathrm{SL}_2(F)$ to $\mathrm{SL}_2(E)$ [**LL**] that will be necessary in what follows. To begin, we first recall base change from $G_1'(F) = \mathrm{GL}_2(F)$ to $G_1'(E) = \mathrm{GL}_2(E)$ [**L**]. In particular, if π is an irreducible representation of $G_1'(F)$, let Π denote its (Langlands-Saito-Shintani) base change to $G_1'(E)$.

Theorem 1.3. *Let π be an irreducible representation of $G_1'(F)$.*

(i) *If $\pi \cong \pi(\mu, \nu)$ with μ and ν characters of $F^\times$, then $\Pi \cong \pi(\mu_E, \nu_E)$.*
(ii) *If $\pi \cong \sigma(\mu, \nu)$ with μ and ν characters of $F^\times$, then $\Pi \cong \sigma(\mu_E, \nu_E)$.*
(iii) *If $\pi \cong \pi(\theta) = \pi(\rho)$ with θ a character of $K^\times$, K/F quadratic, then $\Pi \cong \Pi(\rho|_{W_E})$. In particular, if $K \neq E$, then $\Pi \cong \pi(\theta_{KE})$, and if $K = E$, then $\Pi \cong \pi(\theta, \theta^\sigma)$ where $\langle\sigma\rangle = \Gamma(E/F)$.*
(iv) *If π is exceptional, then so is Π.*

Proof. (i) through (iii) follow directly from [**L**] (see also [**GL**] for a convenient summary).

(iv) Suppose Π is not exceptional. Then there exists a nontrivial character, η say, of $E^\times$ such that $\Pi \otimes \eta \cong \Pi$, by Theorem 1.1. First assume η factors through $N_{E/F}$, i.e., that $\eta = \chi_E$ for some nontrivial character χ of $F^\times$. Then the base change of $\pi \otimes \chi$ to $G_1'(E)$ is $\Pi \otimes \chi_E \cong \Pi$ [**L**] and thus, also by [**L**], $\pi \cong \pi \otimes \chi$ or $\pi \cong \pi \otimes \chi \otimes \omega_{E/F}$ where $\omega_{E/F}$ is the character of $F^\times$ associated to E/F by local class field theory. But then since π is exceptional, either χ is trivial or $\chi \otimes \omega_{E/F}$ is. By assumption χ is nontrivial so $\chi = \omega_{E/F}$ but then η is trivial, a contradiction. Thus, assume η does not factor through $N_{E/F}$, i.e., that $\eta^\sigma \neq \eta$ where $\langle\sigma\rangle = \Gamma(E/F)$ and η^σ is the character of $E^\times$ defined by $\eta^\sigma(x) = \eta(x^\sigma)$. Now define the representation Π^σ of $G_1'(E)$ by $\Pi^\sigma(g) = \Pi(g^\sigma)$ where σ acts coordinatewise. Then, by [**L**] since Π is in the image of base change, $\Pi \cong \Pi^\sigma$. Thus, $\Pi \otimes \eta \cong \Pi$ implies that $\Pi \otimes \eta^\sigma \cong \Pi$ whence $\Pi \otimes \eta\eta^\sigma \cong \Pi$. Now, $\eta\eta^\sigma$ does factor through $N_{E/F}$ so it follows from the first part of this argument that $\eta\eta^\sigma = 1$. But η is of order two (see Theorem 1.1) and thus $\eta = \eta^\sigma$, a contradiction. $\square$

Remark 1.4. *We only include a proof of* (iv) *above since we know of no place where it occurs in the literature. We, however, make no claim to the result.*

Quadratic base change for SL_2 is then at the level of L-packets and can be summarized by the following theorem. Note that, with notation as in the theorem, the representations $\{\Pi_i\}_{i=1}^S$ actually factor to $\mathrm{PSL}_2(E)$ since the central character of Π is that of π composed with $N_{E/F}$ [**L**].

Theorem 1.5. *If $\{\pi_i\}_{i=1}^s$ is an L-packet for $\mathrm{SL}_2(F)$, then the base change of $\{\pi_i\}_{i=1}^s$ to $\mathrm{SL}_2(E)$ is the packet $\{\Pi_i\}_{i=1}^S$ obtained by restricting Π to $\mathrm{SL}_2(E)$*

where Π *is the base change of* π*, a representation of* $\mathrm{GL}_2(F)$ *restricting to* $\mathrm{SL}_2(F)$ *to give the L-packet* $\{\pi_i\}_{i=1}^s$.

Finally, we need to recall a result of Cognet. To this end, we first recall the structure of the orthogonal group H_1. Let H_1' denote the generalized orthogonal group attached to Q and V. Then (see [**D**] for further details of this discussion) the map $\Psi : G_1'(E) \times F^\times \to \mathrm{End}_F(V)$ defined by

$$\Psi(g,u)(A) = \begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix} gA\bar{g}^t,$$

where $^-$ denotes Galois conjugation coordinatewise, is a homomorphism into H_1'. It has kernel

$$\left\{ \left(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}, N(a)^{-1} \right) \middle| \, a \in E^\times \right\}$$

and image of index two. Further, $H_1' \cong \mathrm{Im}\,\Psi \rtimes \langle\sigma\rangle$ where σ is the element of V corresponding to the isometry of V given by conjugation and $\mathrm{Im}\,\Psi$ consists of those elements in H_1' whose determinant is the square of their similitude factor. Now consider the restriction of Ψ to those elements of the form (g,u) with $N(\det g)u^2 = 1$; call this group H. Then $\Psi(H)$ is the subgroup, H_1^0 say, of H_1 consisting of those elements of determinant one and $H_1 = H_1^0 \rtimes \langle\sigma\rangle$. We map $G_1(E)$ to H_1 via the map $k(g) = \Psi((g,1))$ for g in G_1. The kernel of k is $\pm I$ and thus, in a slight abuse of notation, we can use k to identify $\mathrm{PSL}_2(E)$ with a subgroup of H_1. Then $k(\mathrm{PSL}_2(E))$ is the commutator subgroup of H_1 and has index 2^{n+3} where $n = 0$ unless $p = 2$ in which case $n = [F : \mathbb{Q}_2]$. Indeed identifying $F^\times$ with a subgroup of H_1^0 via the map $i : F^\times \to H_1^0$ defined by

$$i(a) = \Psi\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}, a^{-1} \right)$$

for a in $F^\times$ we get that $i(F^\times)k(\mathrm{PSL}_2(E)) \cong H_1^0$ and $H_1^0/k(\mathrm{PSL}_2(E)) \cong i(F^\times)/i((F^\times)^2)$.

Theorem 1.6 ([**C**])**.** *If* π *is an irreducible infinite-dimensional representation of* $G_1 = \mathrm{SL}_2(F)$*, then there exists* π' *in the L-packet of* π *such that* π' *occurs in the theta correspondence attached to* χ *and* (G_1, H_1) *and such that the corresponding representation of* H_1*, upon restriction to* $\mathrm{PSL}_2(E)$*, decomposes as a sum of representations in the L-packet for* $\mathrm{SL}_2(E)$ *obtained from that of* π *by base change.*

Proof. Cognet's statements are at the level of the similitude groups G_1' and H_1'. However, it is a straightforward argument using results relating similitude theta correspondences to regular theta correspondences (see, e.g., [**B**] or [**Ro1**]) to obtain the result above from [**C**]. $\square$

2. Nonsupercuspidal Galois-invariant representations of H_1.

Although the emphasis in this section will be on representations which are not supercuspidal, our first result will apply to all representations in $\mathcal{R}_\chi(H_1)$. To begin, let $\det : H_1 \to \mathbb{C}^\times$ be the representation of H_1 defined by the determinant map. Then our first result is fairly standard but we provide a complete proof since we know of no good reference to the literature.

Lemma 2.1. *If π is an irreducible admissible representation of H_1 which is in $\mathcal{R}_\chi(H_1)$, then $\pi \otimes \det$ is not in $\mathcal{R}_\chi(H_1)$. In particular, π and $\pi \otimes \det$ are not isomorphic and $\pi|_{H_1^0}$ is irreducible with $(\pi|_{H_1^0})^\sigma \cong \pi|_{H_1^0}$.*

Proof. It suffices to show that $\pi \otimes \det$ does not occur. Suppose the contrary. Let W' be a four-dimensional symplectic vector space over F with form $\langle\ ,\ \rangle'$ and identify W' with two transverse copies of the space giving rise to G_1. Then in the language of [**K1**] we have the following seesaw reductive dual pair:

$$\begin{array}{ccc} G(W') & & H_1 \times H_1 \\ | & \times & | \\ G_1 \times G_1 & & H_1. \end{array}$$

Then, since both π and $\pi \otimes \det$ are in $\mathcal{R}_\chi(H_1)$ for the reductive dual pair (G_1, H_1), it follows that $\pi \otimes (\pi \otimes \det)$ occurs in $\mathcal{R}_\chi(H_1 \times H_1)$ (defined relative to the pair $(G_1 \times G_1, H_1 \times H_1)$). But then since irreducible representations of orthogonal groups are self-contragredient [**N**], it follows that $\pi \otimes (\pi \otimes \det)$ restricted to H_1 has det as a quotient. Then from the reciprocity formula for seesaw reductive dual pairs (see, e.g., [**P1**] or [**M**]), it follows that det is in $\mathcal{R}_\chi(H_1)$ relative to the pair $(G(W'), H_1)$. But this contradicts [**R1**, Appendix] since $\dim\,(W')/2 \geq \dim V_1$ does not hold. $\square$

An immediate consequence of the above lemma is that representations in $\mathcal{R}_\chi(H_1)$ are parameterized by their restriction, which is Galois invariant, to H_1^0. Our next step is to parameterize such representations in the nonsupercuspidal case.

Recall that we have $i(F^\times)k(\mathrm{PSL}_2(E)) \cong H_1^0$. Let $j : E^1 \to H_1^0$ be the imbedding of E^1 in H_1^0 defined by

$$j(a) = \Psi\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}, 1\right)$$

with Ψ also as in the previous section. We note that $i(-1) \neq j(-1)$ and that $i(-1)j(-1) = -I$, the nontrivial element of the center of H_1^0. We note that for a in $E^\times$

$$i(N(a))j(a/\bar{a}) = k\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, \tag{2.1}$$

as is easily checked.

Now let V' denote the subspace of V consisting of those matrices which are zero with the possible exception of the $(1,1)$ entry. Let P denote the parabolic subgroup of H_1^0 which stabilizes V'. Then one checks that $P = MN$ where

$$N = \left\{ k \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \middle| a \in E \right\}$$

and $M = i(F^\times)j(E^1)$. Moreover, all proper parabolic subgroups of H_1^0 are conjugate to P and thus all irreducible nonsupercuspidal representations of H_1^0 may be realized as subrepresentations (or subquotients) of representations induced from P. We parametrize these representations below. Since we use normalized induction, we note that the modulus function of P, δ_P say, is given by $\delta_P(i(a)j(b)) = |a|_F^2$, as is easily checked. Also, note that

$$\delta_P\left(k\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}\right) = \delta_P(i(N(a))) = |N(a)|_F^2 = |a|_E^2 .$$

Let T denote the subgroup of $\mathrm{PSL}_2(E)$ obtained by considering the diagonal matrices in $\mathrm{SL}_2(E)$ modulo $\pm I$. We identify $E^\times/\langle -1\rangle$ with T via the map $j' : E^\times \to T$ defined by

$$j'(a) = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}.$$

We view characters of M as characters of P, as usual, by inflation. We also note that $M/k(T) \cong H_1^0/k(\mathrm{PSL}_2(E)) \cong F^\times/(F^\times)^2$ where the first isomorphism is via the map induced by inclusion as can be checked using (2.1) and the second isomorphism is as was noted above. We will thus view characters of these groups interchangeably. Further, for any character of M, we view $\lambda|_{k(T)}$ as a character of $E^\times$ via pullback along $j' \circ k$.

Lemma 2.2. *Let λ be a character of M.*

(i) *If $\lambda|_{k(T)}$ is not $|\ |_E$ or $|\ |_E^{-1}$ and is not of order two, then* $\mathrm{Ind}(H_1^0, P; \lambda)$ *is irreducible. It is Galois invariant if and only if λ^2 is trivial upon restriction to $j(E^1)$ or $i(F^\times)$; in these cases, set $\pi(\lambda) = \mathrm{Ind}(H_1^0, P; \lambda)$.*

(ii) *If $\lambda|_{k(T)} = |\ |$, then* $\mathrm{Ind}(H_1^0, P; \lambda)$ *has a unique irreducible subrepresentation, $\sigma(\lambda)$ say, and unique irreducible quotient, $\pi(\lambda)$ say, both of which are Galois invariant. Further, $\pi(\lambda) = \lambda|\ |^{-1}$.*

(iii) *Similarly, if $\lambda|_{k(T)} = |\ |^{-1}$, then* $\mathrm{Ind}(H_1^0, P; \lambda)$ *has a unique irreducible subrepresentation, $\pi(\lambda)$ say, and unique irreducible quotient, $\sigma(\lambda)$ say, both of which are Galois invariant. Further, $\pi(\lambda) = \lambda \cdot |\ |$.*

(iv) *Assume $\lambda|_{k(T)}$ is of order two. Let ω_λ be the associated character of $E^\times$ and let $E(\lambda)/E$ be the quadratic extension associated to ω_λ by local class field theory. Then if $E(\lambda)/F$ is biquadratic, then* $\mathrm{Ind}(H_1^0, P; \lambda)$

is the direct sum of two distinct irreducible Galois-invariant representations, $\pi^+(\lambda)$ and $\pi^-(\lambda)$ say, each of which remains irreducible upon restriction to $k(\mathrm{PSL}_2(E))$ with the sign being determined by requiring that $\pi^+(\lambda) \cong \pi(\lambda, 1; \chi \circ \mathrm{tr}_{E/F})$ as a representation of $\mathrm{PSL}_2(E)$. Furthermore, in this case λ itself is of order two. If $E(\lambda)/F$ is cyclic, then $\pi(\lambda) = \mathrm{Ind}(H_1^0, P; \lambda)$ is irreducible and Galois invariant. In this case λ is not of order two but $\lambda^2(j(E^1)) = \lambda^2(i(N_{E/F}(E^\times))) = 1$. Finally, if $E(\lambda)/F$ is not Galois, then $\mathrm{Ind}(H_1^0, P; \lambda)$ is either irreducible or the direct sum of two distinct irreducible representations. In either case, none of the irreducible representations obtained is Galois invariant.

(v) *The $\pi(\lambda), \pi^\pm(\lambda)$ and $\sigma(\lambda)$ constructed above exhaust the nonsupercuspidal Galois-invariant portion of the admissible dual of H_1^0. Further, $\pi(\lambda) \cong \pi(\lambda')$ if and only if $\lambda' = \lambda$ or $\lambda' = \lambda^{-1}$ and similarly for $\sigma(\lambda)$. Finally, the representations enjoy no other equivalences.*

Proof. The composition series statements in (i), (ii) and (iii) follow readily from the background material of the first section, in particular, Theorem 1.1, since $i(F^\times)k(\mathrm{PSL}_2(E)) = H_1^0$ and

$$i(a)k(g)i(a^{-1}) = k\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} g \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}^{-1}\right)$$

for g in $\mathrm{PSL}_2(E)$ and a in $F^\times$. Now consider the composition series when $\lambda|_{K(T)}$ is of order two. In this case, it follows from the material of the first section that $\mathrm{Ind}(H_1^0, P; \lambda)$ is irreducible or the direct sum of two irreducible representations with the latter occurring if and only if $F^\times$ is contained in $N_{E(\lambda)/E}(E(\lambda)^\times)$. If $E(\lambda)/F$ is biquadratic, then the decomposition follows since $N_{E(\lambda)/E}(E(\lambda)^\times) = \{x \in E^\times \mid N_{E/F}(x) \in N_{E'/F}((E')^\times)\}$ where E' is any quadratic extension of F such that $EE' = E(\lambda)$, see, e.g., [**I**, Theorem 7.6]. On the other hand if $E(\lambda)/F$ is cyclic, then $F^\times/N_{E(\lambda)/F}(E(\lambda)^\times)$ is cyclic of order four by local class field theory and thus $N_{E(\lambda)/F}(E(\lambda)^\times)$ cannot contain $(F^\times)^2$ whence $N_{E(\lambda)/E}(E(\lambda)^\times)$ cannot contain $F^\times$. Finally, if $E(\lambda)/F$ is not Galois, then the composition series statement follows from Theorem 1.1.

We now consider equivalences amongst the $\pi(\lambda)$, $\pi^\pm(\lambda)$ and $\sigma(\lambda)$. We consider here only the infinite-dimensional representations, the other cases being trivial. Restricting to $k(\mathrm{PSL}_2(E))$, we see that $\pi(\lambda)$ can only be isomorphic to some $\pi(\lambda')$ and similarly for $\pi^+(\lambda), \pi^-(\lambda)$ and $\sigma(\lambda)$. Suppose $\pi(\lambda) \cong \pi(\lambda')$. Then, also by restricting to $k(\mathrm{PSL}_2(E))$, we get that $\lambda'|_{k(T)} = \lambda|_{k(T)}$ or $\lambda'|_{k(T)} = \lambda^{-1}|_{k(T)}$. Suppose the former. Write $\lambda' = \lambda\lambda''$ with λ'' a character of M trivial on $k(T)$. Now it follows from Frobenius reciprocity that $\pi(\lambda) \otimes \lambda'' \cong \pi(\lambda\lambda'')$ with λ'' on the left-hand side viewed as a character of H_1^0. Thus, $\pi(\lambda) \otimes \lambda'' \cong \pi(\lambda)$. This either implies $\lambda'' = 1$ or, by Theorem 1.1,

$\lambda|_{k(T)}$ is of order two and $\lambda'|_{i(F^\times)} \circ i = \omega_{E(\lambda)/E}\,|_{F^\times}$. If $\lambda'' = 1$, we are done. Thus, assume the latter. In this case, λ'' is completely determined by (2.1) since $\lambda''|_{k(T)}$ is trivial. Now, also by (2.1), since $\lambda|_{k(T)} = \lambda^{-1}|_{k(T)}$, it suffices to show that $\lambda|_{i(F^\times)} = (\lambda')^{-1}|_{i(F^\times)}$. But for a in $F^\times$, we have

$$\lambda^2(i(a)) = \lambda\left(k\begin{pmatrix} a & 0 \\ 0 & a^{-1}\end{pmatrix}\right)$$

as is easily checked and thus

$$\begin{aligned}\lambda\lambda'(i(a)) &= \lambda^2\lambda''(i(a)) \\ &= \lambda^2(i(a))\lambda''(i(a)) \\ &= \lambda\left(k\begin{pmatrix} a & 0 \\ 0 & a^{-1}\end{pmatrix}\right)\omega_{E(\lambda)/E}(a) \\ &= \omega_{E(\lambda)/E}(a)\omega_{E(\lambda)/E}(a) \\ &= 1.\end{aligned}$$

Now in general considering

$$\pi(\lambda)^{\Psi\left(\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0\end{smallmatrix}\right),1\right)}$$

one sees from Frobenius Reciprocity that $\pi(\lambda) \cong \pi(\lambda^{-1})$. Thus, the case $\lambda'|_{k(T)} \cong \lambda^{-1}|_{k(T)}$ follows from the previous case. The arguments for the remaining equivalences are similar, using that the relevant composition series are multiplicity free.

Finally, we consider Galois invariance. Let π be a nonsupercuspidal irreducible representation of H_1^0. Then $\mathrm{Hom}_{H_1^0}(\pi, \mathrm{Ind}(H_1^0, P; \lambda)) \neq 0$ for some λ, a one-dimensional representation of M. Now one checks that $\mathrm{Ind}(H_1^0, P; \lambda)^\sigma \cong \mathrm{Ind}(H_1^0, P; \lambda^\sigma)$ since the modulus character is invariant. Then it follows from a similar argument to that for the equivalences (invariance was not used) that $\lambda = \lambda^\sigma$ or $\lambda = (\lambda^{-1})^\sigma$. Now one checks that $\lambda = \lambda^\sigma$ if and only if $\lambda^2(j(E^1)) = 1$ and $\lambda = (\lambda^{-1})^\sigma$ if and only if $\lambda^2(i(F^\times)) = 1$. The invariance portions of (i), (ii), (iii) then follow. Thus, assume $\lambda|_{k(T)}$ is of order two. Now by local class field theory $E(\lambda)/F$ is Galois if and only if $E(\lambda) = E(\lambda^\sigma)$. Then since $\lambda|_{k(T)}$ is of order two, it follows that $E(\lambda) = E(\lambda^\sigma)$ if and only if $\lambda|_{k(T)} = \lambda^\sigma|_{k(T)}$ and $\lambda|_{k(T)} = (\lambda^{-1})^\sigma|_{k(T)}$. But these are equivalent to $\lambda^2(j(E^1)) = 1$ and $\lambda^2(i(N_{E/F}(E^\times))) = 1$. Finally, since

$$\lambda^2(i(a)) = \lambda\left(k\begin{pmatrix} a & 0 \\ 0 & a^{-1}\end{pmatrix}\right)$$

for all a in $F^\times$, our determination of the order of λ follows from our discussion of reducibility. $\square$

3. Jacquet modules.

We now turn to Jacquet modules to find restrictions, in addition to Galois invariance, on the nonsupercuspidal representations of H_1 that can occur in $\mathcal{R}_\chi(H_1)$. We realize ω_χ on $\mathcal{S}(V)$, the space of locally constant compactly supported functions on V with the action of H_1 being the natural linear action, i.e., a Schrodinger model. Set $\mathcal{S}(V)_N = \mathcal{S}(V)/\langle f - \omega_\chi(n)f \mid f \in \mathcal{S}(V), n \in N\rangle$ and view $\mathcal{S}(V)_N$ as an M-module (unnormalized) as usual. Finally, for f in $\mathcal{S}(V)$, let $\bar{f}$ denote the image of f in $\mathcal{S}(V)_N$.

For l an integer, let U_l denote the neighborhood of 0 in V consisting of those v such that v_{ii} is in P_F^l for $i = 1, 2$ and v_{12} is in $\mathcal{O}_E P_F^l$ where v_{ij} denotes the (i,j)-entry of v. Then the U_l form a neighborhood basis of 0 in V. For A in V and l an integer, define $f_{A,l}$ in $\mathcal{S}(V)$ by setting

$$f_{A,l}(v) = \begin{cases} 1 & \text{if } A - v \text{ is in } U_l, \\ 0 & \text{otherwise.} \end{cases}$$

Then the $f_{A,l}$ span $\mathcal{S}(V)$.

Theorem 3.1. *Suppose λ in $M^\wedge$ occurs as a quotient of $(\omega_\chi)_N$. Set $\lambda' = \lambda \cdot |\ |^{-1}$. Then either $\lambda'|_{i(F^\times)}$ is trivial or $\lambda'|_{j(E^1)}$ is trivial. Further, if $\lambda'|_{i(F^\times)}$ is trivial, then λ' is determined by $\lambda'|_{k(T)}$. Finally, if $\lambda'|_{j(E^1)}$ is trivial, then λ' is determined by $\lambda'|_{k(T)}$ up to (possibly) twisting by $\omega_{E/F}$.*

Proof. Let $T : \mathcal{S}(V)_N \to \mathbb{C}$ be a nonzero element of $\mathrm{Hom}_M((\omega_\chi)_N, \lambda)$. Then there exist A and l such that $T\bar{f}_{A,l} \neq 0$.

We proceed by cases. First, suppose $\nu_F(a_{22}) < l$ and that $\nu_E(a_{12}a_{22}^{-1}) \geq 0$. Let

$$n = \begin{pmatrix} 1 & -a_{12}a_{22}^{-1} \\ 0 & 1 \end{pmatrix}.$$

Then one checks that $\omega_\chi(k(n))f_{A,l} = f_{A_1,l}$ with

$$A_1 = \begin{pmatrix} a_{11} & 0 \\ 0 & a_{22} \end{pmatrix}.$$

It follows that $T\bar{f}_{A_1,l} \neq 0$ and thus $\lambda(j(E^1)) = 1$ since

$$T\omega_\chi(j(\alpha))_N \bar{f}_{A_1,l} = \lambda(j(\alpha))T\bar{f}_{A_1,l}$$

for all α in E^1 as can be checked. Then by (2.1),

$$\lambda(i(N(a))) = \lambda\left(k\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}\right)$$

for a in $F^\times$ and the result follows.

Now suppose that $\nu(a_{22}) < l$ but $\nu_E(a_{12}a_{22}^{-1}) < 0$. Choose m such that

$$\nu_E((a_{12}a_{22}^{-1})\omega_F^m) \geq 0.$$

Let $U_{l,m} = \left\{A \in U_l \mid a_{12} \in \mathcal{O}_E P_F^{l+m}, a_{22} \in P_F^{l+2m}\right\}$. For B in U_l, set

$$f_{A,l,B}(v) = \begin{cases} 1 & \text{if } A + B - v \text{ is in } U_{l,m} \\ 0 & \text{otherwise.} \end{cases}$$

Then since $T\bar{f}_{A,l} \neq 0$, it follows that there exists B in U_l such that $T\bar{f}_{A,l,B} \neq 0$. Let

$$n = \begin{pmatrix} 1 & -(a_{12} + b_{12})(a_{22} + b_{22})^{-1} \\ 0 & 1 \end{pmatrix}.$$

Then one checks that $\omega_\chi(k(n))f_{A,l,B} = f_{A_1,l,B_1}$ where $(A_1 + B_1)_{12} = 0$ and $(A_1 + B_1)_{ii} = (A + B)_{ii}$. Then as in the previous case, one checks that $\lambda(j(E^1)) = 1$ and

$$\lambda(i(N(a))) = \lambda\left(k\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}\right)$$

for a in $F^\times$, whence the result.

Finally, suppose that $\nu(a_{22}) \geq l$. In this case, we may assume $a_{22} = 0$. Now if a_{12} is in $\mathcal{O}_E P_F^l$, then arguing as above we obtain that $\lambda(j(E^1)) = 1$ and

$$\lambda(i(N(a))) = \lambda\left(k\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}\right)$$

for a in $F^\times$ and we are done. Thus, suppose a_{12} is not in $\mathcal{O}_E P_F^l$. Now suppose a_{11} is in P_F^l. Then since E/F is separable, there exists an integer l' such that $\mathrm{tr}_{E/F}(P_E^{l'} a_{12}) = P_F^l$. Similarly, if a_{11} is not in P_F^l, then there exists l' such that $\mathrm{tr}_{E/F}(P_E^{l'} a_{12}) = P_F^{\nu(a_{11})}$. Let $m' \geq l$ be an integer such that $N(P_E^{l'})\omega_F^{m'} \subseteq P_F^{l+1}$ and $\mathrm{tr}(P_E^{l'} P_F^{m'}) \subset P_F^{l+1}$. Further, let $m \geq m'$ be an integer such that $P_E^{l'} P_F^m \subseteq \mathcal{O}_E P_F^{m'}$. Finally, let U' denote the neighborhood of 0 in V consisting of all v such that $\nu_F(v_{11}) \geq l, \nu_F(v_{22}) > m$ and v_{12} is in $\mathcal{O}_E P_F^{m'}$. Then by an argument similar to the above in the case $a_{22} = 0$, we may assume that $T\bar{f}'_A$ is nonzero where A' is in V such that $A - A'$ is in U_l, and for any B in V, f'_B in $\mathcal{S}(V)$ is defined by

$$f'_B(v) = \begin{cases} 1 & \text{if } v - B \text{ is in } U', \\ 0 & \text{otherwise.} \end{cases}$$

Without loss of generality, we assume $A' = A$ so that $T\bar{f}'_A \neq 0$.

Now, by our choice of l', there exists x in $P_E^{l'}$ such that $\mathrm{tr}_{E/F}(xa_{12}) = a_{11}$. Let

$$n = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}.$$

Then one checks that, by virtue of our choices of m and m', $\omega_\chi(k(n))f'_A = f'_B$ with $b_{11} = 0$, $b_{12} = a_{12}$ and $b_{22} = a_{22} = 0$. Thus, we may assume that $a_{11} =$

0 in considering $T\bar{f}'_A \neq 0$. It then follows that, for b in $\mathcal{O}_F^\times, \omega_\chi(i(b))f'_A = f'_A$ and thus $\lambda(i(b)) = \lambda'(i(b)) = 1$. Hence to complete the proof, it suffices to show that $\lambda(i(\omega_F^{-1})) = q$.

For B in V, define f''_B in $\mathcal{S}(V)$ by

$$f''_B(v) = \begin{cases} 1 & \text{if } v - A \text{ is in } i(\omega_F^{-1})U', \\ 0 & \text{otherwise.} \end{cases}$$

Then it is clear that $\omega_\chi(i(\omega_F^{-1}))f'_A = f''_A$. Now

$$f''_A = \sum_{B \in R} g_{A+B}$$

where R is a set of coset representatives for $U''/i(\omega_F^{-1})U'$ with $U'' = \{v \in U' : \nu_F(v_{22}) \geq m+1\}$ and g_{A+B} is defined as follows:

$$g_{A+B}(v) = \begin{cases} 1 & \text{if } v - A - B \text{ is in } U'', \\ 0 & \text{otherwise.} \end{cases}$$

Then arguing as above in the case $a_{11} \neq 0$, one shows that $\bar{g}_{A+B} = \bar{g}_A$ for all B in R. Thus,

$$\omega_\chi\left(i(\bar{\omega}_F^{-1})\right)_N \bar{f}'_A = q\bar{g}_A$$

so that

$$T\omega_\chi\left(i(\bar{\omega}_F^{-1})\right)_N \bar{f}'_A = qT\bar{g}_A$$

with both sides nonzero. But now

$$g_A = \sum_{C \in S} f'_{A+C}$$

with S a set of coset representatives for U'/U''. Then by an argument similar to that in case $a_{22} \neq 0$, the result of the theorem follows if $T\bar{f}'_{A+C}$ is nonzero for any C not in U''. Thus, we may assume $T\bar{g}_A = T\bar{f}'_A$ and then the theorem follows. □

As an immediate consequence of the above results, the exactness of the Jacquet functor and the adjointness of the Jacquet functor and induction, we have the following.

Corollary 3.2. *Let π be an irreducible nonsupercuspidal representation of H_1 which is in $\mathcal{R}_\chi(H_1)$. Then $\pi_0 = \pi|_{H_1^0}$ is irreducible and π, as an element of $R_\chi(H_1)$, is determined by π_0. Moreover, π_0 is of the form $\pi(\lambda), \sigma(\lambda)$, or $\pi^\pm(\lambda)$ for some λ with $\lambda|_{i(F^\times)}$ trivial or $\lambda|_{j(E^1)}$ trivial. In particular, with λ as above, λ is determined by $\lambda|_{k(T)}$ if $\lambda|_{i(F^\times)}$ is trivial and is determined by $\lambda|_{k(T)}$ up to a (possible) twist of $\omega_{E/F}$ if $\lambda|_{j(E^1)}$ is trivial.*

To close this section we prove an elementary lemma that will be useful in what follows. The statements on restriction in the lemma can easily be made more precise but we leave that to the reader since the following suffices, for our purposes.

Lemma 3.3. *Let π be an irreducible representation of H_1 such that $\pi_0 = \pi|_{H_1^0}$ is irreducible. Moreover, assume π_0 is of the form $\pi(\lambda), \sigma(\lambda)$ or $\pi^{\pm}(\lambda)$ for some λ with $\lambda|_{i(F)}$ or $\lambda|_{j(E^1)}$ trivial.*

(i) *Suppose $\lambda|_{i(F^\times)}$ is trivial. Then π_0 is of the form $\pi(\lambda)$ or $\pi^{\pm}(\lambda)$ and when restricted to $k(\mathrm{PSL}_2(E))$ decomposes as a sum of representations in the L-packet associated to $\pi(\rho, \rho^\sigma)$ where ρ is any character of $E^\times$ such that $\rho(a/\bar{a}) = \lambda(j(a/\bar{a}))$ for a in $E^\times$.*

(ii) *Suppose $\lambda|_{j(E^1)}$ is trivial. Then if π_0 is of the form $\sigma(\lambda)$, then π_0 restricts to $\sigma(\lambda \circ i \circ N)$ on $k(\mathrm{PSL}_2(E))$. If π_0 is of the form $\pi(\lambda)$ or $\pi^{\pm}(\lambda)$, then, when restricted to $\mathrm{PSL}_2(E)$, π_0 decomposes as a sum of representations in the L-packet attached to $\pi(\lambda \circ i \circ N, 1)$.*

Proof. Let π_1 be an irreducible representation of $\mathrm{PSL}_2(E)$ that appears in the restriction of π to $k(\mathrm{PSL}_2(E))$. Then since $i(F^\times)k(\mathrm{PSL}_2(E)) = H_1^0$, any other representation appearing in the restriction of π to $k(\mathrm{PSL}_2(E))$ must be of the form $\pi_1^{i(a)}$ for a in $F^\times$. But then

$$i(a)k(g)i(a^{-1}) = k\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} g \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}^{-1}\right)$$

and

$$k\begin{pmatrix} b & 0 \\ 0 & b^{-1} \end{pmatrix} = i(N(b))j(b/\bar{b})$$

imply the result. □

4. More on occurrence.

In the previous section, we found some necessary conditions, Corollary 3.2, for nonsupercuspidal representations of H_1 to occur in the correspondence. In this section, we will show these conditions are also sufficient, with one possible exception. The one possible exception is a generalized Steinberg representation as is explained in (ii) and (iii) of the following theorem.

Theorem 4.1. *Let π_0 be an irreducible representation of H_1^0.*

(i) *If π_0 is of the form $\pi(\lambda)$ or $\pi^{\pm}(\lambda)$ with $\lambda|_{i(F^\times)}$ trivial or $\lambda|_{j(E^1)}$ trivial, then π_0 has a unique extension to H_1 which occurs in $\mathcal{R}_\chi(H_1)$.*

(ii) *If $\pi_0 = \sigma\left(|\ |\right)$ or $\sigma\left(\omega_{E/F}|\ |\right)$ then at most one extension of π_0 to H_1 occurs in $\mathcal{R}_\chi(H_1)$.*

(iii) *At least one extension of* $\sigma\left(|\ |\right)$ *or* $\sigma\left(\omega_{E/F}|\ |\right)$ *occurs in* $\mathcal{R}_\chi(H_1)$ *and pairs with the Steinberg representation* $\sigma\left(|\ |\right)$ *of* G_1*.*
(iv) *No other nonsupercuspidal representations of* H_1 *can occur.*

Proof. It suffices to show (i) through (iii) since then (iv) would follow from Corollary 3.2.

First consider (i) and suppose that $\lambda|_{i(F^\times)}$ is trivial. Let θ be a character of $E^\times$ such that $\theta|_{E^1} = \lambda|_{j(E^1)} \circ j$. Now suppose further that λ is not of order two. Then it follows from Theorem 1.2 that $\pi(\theta;\chi)$ occurs in $\mathcal{R}_\chi(G_1)$. Further, by Kudla's perseverence result [**K2**], it must pair with $\pi(\lambda)$. Likewise, if $\lambda|_{i(F^\times)}$ is trivial but λ is of order two, then $\pi(\theta;\chi)$ and $\pi(\theta;\chi_b)$ occur in $\mathcal{R}_\chi(G_1)$ and pair with $\pi^\pm(\lambda)$ where b is in $N_{E/F}(E^\times)$ such that $\pi(\theta;\chi)$ and $\pi(\theta;\chi_b)$ are distinct. Therefore, in proving the theorem, we may assume that $\lambda|_{i(F^\times)}$ is nontrivial.

Now consider λ with $\lambda|_{i(F^\times)}$ nontrivial. Let μ be a character of $F^\times$. Assume for the moment that $\mu_E \neq |\ |^{\pm 1}$ and that μ_E^2 is not trivial. Then μ^2 is also nontrivial and thus $\pi(\mu,1)$ and $\pi(\mu_E,1)$ restrict irreducibly to $G_1(F)$ and $G_1(E)$, respectively, with $\pi(\mu_E,1)$ the base change of $\pi(\mu,1)$. Now by Theorem 1.2, $\pi(\mu,1)$ is in $\mathcal{R}_\chi(G_1)$. Then since $\mu \neq |\ |^{\pm 1}$, $\pi(\mu,1)$ is infinite-dimensional and thus, by Theorem 1.6, the image of $\pi(\mu,1)$ under the theta correspondence, $\theta(\pi(\mu,1))$ say, must restrict to $k(\mathrm{PSL}_2(E))$ as a sum of copies of $\pi(\mu_E,1)$. Now since μ^2 is nontrivial, $\pi(\mu,1)$ does not occur in the theta correspondence attached to χ and (G_1,H_0) where H_0 is the orthogonal group attached to the anisotropic part of (V,Q). Further, since the correspondence attached to χ and (G_1,H_1) is a bijection, even in p $=2$ [**R2**], it follows from the argument of the previous paragraph that a λ giving rise to $\theta(\pi(\mu,1))$ must satisfy $\lambda|_{j(E^1)}$ is trivial. Then it follows from Lemma 3.3 that we may assume λ must satisfy $\mu_E = \lambda \circ i \circ N$, whence $\lambda = \lambda_1$ or λ_2 where

$$\lambda_l(i(a)j(b/\bar{b})) = \mu(a)\omega_{E/F}^l(a)$$

for $l = 1,2$, $b \in E^\times$ and a in $F^\times$. It follows that both $\pi(\mu,1)$ and $\pi(\mu\omega_{E/F},1)$ occur in $\mathcal{R}_\chi(G_1)$ and pair with either $\pi(\lambda_1)$ or $\pi(\lambda_2)$ in $\mathcal{R}_\chi(H_1)$. Then, once again, since the correspondence is a bijection, we get that the theorem holds for all $\pi(\lambda)$ with $(\lambda|_{i(N(E^\times))})^2$ nontrivial and $\lambda|_{i(N(E^\times))} \neq |\ |$.

If λ is trivial on $j(E^1)$ and $\lambda(i(a)) = |a|$, then $\pi(\lambda)$ is the trivial representation and, as is well-known, $\pi(\lambda)$ occurs in $\mathcal{R}_\chi(H_1)$ and pairs with $\pi\left(\omega_{E/F}|\ |,1\right)$ in $\mathcal{R}_\chi(G_1)$, see [**KR2**]. Further, by Theorem 1.6 and Lemma 3.3, $\sigma\left(|\ |\right)$ is in $\mathcal{R}_\chi(G_1)$ and pairs with $\sigma\left(|\ |\right)$ or $\sigma\left(\omega_{E/F}|\ |\right)$ in $\mathcal{R}_\chi(H_1)$.

Now let μ be a character of $F^\times$ of order two with μ_E nontrivial. Then the L-packets for $G_1(F)$ and $G_1(E)$ associated to $\pi(\mu,1)$ and $\pi(\mu_E,1)$, respectively, each have two components as does the L-packet for $G_1(F)$ attached to $\pi(\omega_{E/F}\mu, 1)$. By Theorem 1.2, the four representations in the L-packets associated to $\pi(\mu, 1)$ and $\pi(\mu\omega_{E/F}, 1)$ occur. Then, by arguments similar to those above, they must pair with the four representations of H_1, as in Lemma 3.3, attached to λ and $\omega_{E/F}\lambda$ where λ is the character of M defined by

$$\lambda(i(a)j(b/\bar{b})) = \mu(a).$$

Further, consider the representation $\pi(1, 1)$ of $G_1'(F)$. It restricts irreducibly to $G_1(F)$ and by arguments also similar to those above it occurs and pairs with the representation $\pi(\lambda)$ of H_1 with

$$\lambda(i(a)j(b/\bar{b})) = \omega_{E/F}(a).$$

Finally, consider those nontrivial characters of μ on $F^\times$ such that μ^2 is nontrivial while μ_E^2 is trivial. Such a character has order four and by local class field theory is associated to a cyclic extension of degree four of F with the quadratic subfield being E. Further, if μ is such a character, then so is $\mu\omega_{E/F}$. Then arguments such as those above show that $\pi(\mu, 1)$ and $\pi(\mu\omega_{E/F}, 1)$ are in $\mathcal{R}_\chi(G_1)$ and pair with $\pi(\lambda_1)$ and $\pi(\lambda_2)$ in $\mathcal{R}_\chi(H_1)$ where

$$\lambda_l(i(a)j(b/\bar{b})) = \mu(a)\omega_{E/F}^l(a),$$

$l = 1, 2$.

To summarize, at this point we have shown that the theorem holds for all representations of H_1^0 with the possible exception of the one-dimensional representation $\pi\left(\omega_{E/F}|\ |\right)$. Let $\pi_+\left(\omega_{E/F}|\ |\right)$ denote the extension of $\pi\left(\omega_{E/F}|\ |\right)$ to H_1 with σ acting as the identity. Then the following lemma completes the proof of the theorem. $\square$

Lemma 4.2. *The one-dimensional representation $\pi_+\left(\omega_{E/F}|\ |\right)$ occurs in $\mathcal{R}_\chi(H_1)$ and pairs with the trivial representation of G_1.*

Proof. It suffices to show that $\mathrm{Hom}_{H_1}\left(\omega_\chi^\infty, \pi_+\left(\omega_{E/F}|\ |\right)\right)$ is one-dimensional. Write $\pi_+\left(|\ |\omega_{E/F}\right) = \omega_{E/F}^+$. Let

$$x = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

in V. Then one checks that the stabilizer of x in H_1 is $H_x = j(E^1)k(N) \rtimes \langle\sigma\rangle$. One checks further that H_x is unimodular and that $\omega_{E/F}^+|_{H_x}$ is trivial since the image of the spinor norm restricted to $H_x \cap H_1^0$ is $N(E^\times)$ because

$$k\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} = i(N(a))j(a/\bar{a}).$$

Thus, it follows from [**W**] that there exists a unique, up to scalar, linear map $T : \mathcal{S}(H_1/H_x) \to \mathbb{C}$ such that

$$Tf^g = \omega_{E/F}^+(g)Tf \tag{4.1}$$

where $f^g(x) = f(gx)$. Now let $Y = Q^{-1}(0)$ and $Y' = Y - \{0\}$. Then by Witt's Theorem, Y' can be identified with H_1/H_x and thus we can view T as a distribution on Y'.

We claim further that T can be extended to a distribution on $\mathcal{S}(Y)$ satisfying (4.1) for all f in $\mathcal{S}(Y)$. To see this we recall that, up to a nonzero scalar, T can be written

$$Tf = \int_{H_1/H_x} \omega_{E/F}^+(y)f(y)\,dy$$

for all f in $\mathcal{S}(H_1/H_x)$ where $\int_{H_1/H_x} dy$ is a left-invariant positive regular Borel measure on $C_c(H_1/H_x)$, the space of compactly supported functions on H_1/H_x and $\omega_{E/F}^+(y)$ is well-defined since $\omega_{E/F}^+$ is trivial on H_x. Moreover, the measure is finite on compact sets and may be normalized so that

$$\int_{H_1/H_x}\int_{H_x} f(yh)\,dhdy = \int_{H_1} f(g)\,dg \tag{4.2}$$

where $\int_{H_1} dg$ and $\int_{H_x} dh$ denote Haar measure on H_1 and H_x respectively and f is any function in $C_c(H_1)$. We claim that T can be extended to $\mathcal{S}(Y)$ by setting $Tf = Tf|_Y$, for all f in $\mathcal{S}(Y)$. To show this it suffices to show if U is a compact neighborhood of 0 in Y then $U \cap Y'$ is of finite volume with respect to $\int_{H_1/H_x} dy$. To this end, using a Bruhat decomposition, H_1/H_x can be identified with

$$i(F^\times) \cup k(N)k(w)i(F^\times)$$

where w is the standard Weyl element $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ for $\mathrm{SL}_2(F)$. Now consider the effect of conjugation by $i(a), a \in F^\times$, on (4.2). Since H_1 is unimodular we have

$$\begin{aligned} \int_{H_1} f(y)\,dg &= \int_{H_1} f(i(a)g(i(a))^{-1})\,dg \\ &= \int_{H_1/H_x}\int_{H_x} f(i(a)y(i(a))^{-1}i(a)h(i(a))^{-1})\,dhdy \\ &= \int_{H_1/H_x} |a|^{-1}\int_{H_x} f(y(i(a))^{-1}h)\,dhdy \end{aligned} \tag{4.3}$$

where the last equality follows from explicit realization of the measure $\int_{H_x} dh$ and left-invariance of $\int_{H_1/H_x} dy$. Finite volume then follows from explicitly realizing $\int_{H_1/H_x} dy$ taking into account the $|a|^{-1}$. It is immediate that the extension satisfies (4.1) since Y' is H_1 invariant.

Now let $T : S(Y) \to \mathbb{C}$ denote the extension constructed above. Then since Y is closed in V we can extend T to $S(V)$ by setting $Tf = T(f|_Y)$. Further since Y is H_1 invariant T still satisfies (4.1), whence T is a nonzero intertwining map. Furthermore, T is the unique, up to scalar, nonzero intertwining map with image in $\mathcal{D}(V - Y)$, the space of distributions on $\mathcal{S}(V - Y)$, equal to zero since $\{0\}$ only supports the trivial representation.

Now suppose $S : \mathcal{S}(V) \to \mathbb{C}$ is an arbitrary nonzero element in $\mathrm{Hom}_{H_1}(\omega_\chi^\infty, \omega_{E/F}^+)$. Then it suffices to show that the image of S in $\mathcal{D}(V-Y)$ is zero. Suppose that the image is nonzero. Then by [**BZ**], there exists an orbit $Q^{-1}(a)$ in $V - Y, a \in F^\times$ which supports a nontrivial intertwining operator. Let $X = Q^{-1}(a)$, $a \in F^\times$, be such an orbit and let $y \in X$. Then since $Q(X) \neq 0$, we can write $V = \langle y \rangle \oplus W$ with $W = \langle y \rangle^\perp$. Now let $\mathrm{O}(W)$ be the orthogonal group associated to W and $Q|_W$. Then $X = H_1/\mathrm{O}(W)$. Now since W is three-dimensional, the spinor norm restricted to $\mathrm{O}(W)$ is onto $F^\times$. Further, both H_1 and $\mathrm{O}(W)$ are unimodular and thus any intertwining operator which is trivial when restricted to $\mathrm{O}(W)$ must have been trivial on H_1 [**W**]—contradicting the surjectivity of the spinor norm. $\square$

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Received June 1, 1999 and revised July 11, 2000. This research was supported in part by NSF grant DMS-9003213 and NSA grant MDA904-97-1-0046.

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