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The Pacific Journal of Mathematics (ISSN 0030-8730) at the University of California, c/o Department of Mathematics, 969 Evans Hall, Berkeley, CA 94720-3840 is published monthly except July and August. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.
ON MIXED PRODUCTS OF COMPLEX CHARACTERS OF THE DOUBLE COVERS OF THE SYMMETRIC GROUPS

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In this article, two families of (almost) homogeneous mixed Kronecker products of non-faithful and of spin characters of the double covers of the symmetric groups are described. This is then applied to classify the irreducible mixed products, thus completing the classification of all irreducible Kronecker products of characters of the double covers of the symmetric groups.

1. Introduction.

Kronecker products of complex characters of the symmetric group $S_n$ have been studied in many papers. Information on special products and on the coefficients of special constituents have been obtained but there is no efficient combinatorial algorithm in sight for computing these products. In [1], products of $S_n$-characters with few homogeneous components and homogeneous products of characters of the alternating group $A_n$ have been classified. In particular, there are no non-trivial homogeneous Kronecker products for $S_n$, but there are such products for $A_n$, when $n$ is a square number (these are even irreducible).

For the double covers $\tilde{S}_n$ of the symmetric groups, information about products of characters is even more sparse. Recently, in [2] some results have been obtained on products of spin characters of $\tilde{S}_n$ which led to a classification of homogeneous spin products. Here, homogeneous products do occur for all triangular numbers $n$, but non-trivial irreducible products occur only for $n = 6$.

In this article, we consider mixed products of complex characters for the double covers $\tilde{S}_n$, i.e., products of a non-faithful character of $\tilde{S}_n$ (corresponding to a character of $S_n$) with a spin character. In this situation, there are some interesting homogeneous or almost homogeneous mixed products; finding such mixed products was greatly helped by the special maple packages SF and QF for dealing with symmetric functions by John Stembridge. Two families of homogeneous resp. almost homogeneous products are described; one for any composite number and the other one for triangular numbers. The irreducible mixed products are then classified; they occur for even numbers and triangular numbers satisfying a congruence condition.
2. Preliminaries.

We denote by $P(n)$ the set of partitions of $n$. For a partition $\lambda \in P(n)$, $l(\lambda)$ denotes its length, i.e., the number of (non-zero) parts of $\lambda$. The set of partitions of $n$ into odd parts only is denoted by $O(n)$, and the set of partitions of $n$ into distinct parts is denoted by $D(n)$. We write $D^+(n)$ resp. $D^-(n)$ for the sets of partitions $\lambda$ in $D(n)$ with $n - l(\lambda)$ even resp. odd; the partition $\lambda$ is then also called even resp. odd.

We write $S_n$ for the symmetric group on $n$ letters, and $\tilde{S}_n$ for one of its double covers; so $\tilde{S}_n$ is a non-split extension of $S_n$ by a central subgroup $\langle z \rangle$ of order 2. It is well-known that the representation theory of these double covers is ‘the same’ for all representation theoretical purposes. The spin characters of $\tilde{S}_n$ are those that do not have $z$ in their kernel. For an introduction to the properties of spin characters resp. for some results we will need in the sequel we refer to [5], [10], [11], [13]. Below we collect some of the necessary notation and some results from [13] that are crucial in later sections.

For $\lambda \in P(n)$, we write $[\lambda]$ for the corresponding irreducible character of $S_n$; this is identified with the corresponding character of $\tilde{S}_n$. The associate classes of spin characters of $\tilde{S}_n$ are labelled canonically by the partitions in $D(n)$. For each $\lambda \in D^+(n)$ there is a self-associate spin character $\langle \lambda \rangle = \text{sgn} \langle \lambda \rangle$, and for each $\lambda \in D^-(n)$ there is a pair of associate spin characters $\langle \lambda \rangle, \langle \lambda \rangle' = \text{sgn} \langle \lambda \rangle$. We write

$$\hat{\langle \lambda \rangle} = \begin{cases} \langle \lambda \rangle & \text{if } \lambda \in D^+(n) \\ \langle \lambda \rangle + \langle \lambda \rangle' & \text{if } \lambda \in D^-(n) \end{cases}$$

$$\varepsilon_\lambda = \begin{cases} 1 & \text{if } \lambda \in D^+(n) \\ \sqrt{2} & \text{if } \lambda \in D^-(n). \end{cases}$$

In [13], Stembridge introduces a projective analogue of the outer tensor product, called the reduced Clifford product, and proves a shifted analogue of the Littlewood-Richardson rule which we will need in the sequel. To state this, we first have to define some further combinatorial notions.

Let $A'$ be the ordered alphabet $\{1' < 1 < 2' < 2 < \ldots \}$. The letters $1', 2', \ldots$ are said to be marked, the others are unmarked. The notation $|a|$ refers to the unmarked version of a letter $a$ in $A'$. To a partition $\lambda \in D(n)$ we associate a shifted diagram

$$Y'(\lambda) = \{(i, j) \in \mathbb{N}^2 \mid 1 \leq i \leq l(\lambda), i \leq j \leq \lambda_i + i - 1 \}.$$ 

A shifted tableau $T$ of shape $\lambda$ is a map $T : Y'(\lambda) \rightarrow A'$ such that $T(i, j) \leq T(i + 1, j), T(i, j) \leq T(i, j + 1)$ for all $i, j$, and every $k \in \{1, 2, \ldots \}$ appears at most once in each column of $T$, and every $k' \in \{1', 2', \ldots \}$ appears at most once in each row of $T$. For $k \in \{1, 2, \ldots \}$, let $c_k$ be the number of boxes
(i,j) in Y'(λ) such that |T(i,j)| = k: Then we say that the tableau T has content (c_1, c_2, ...). Analogously, we define skew shifted diagrams and skew shifted tableaux of skew shape λ \ μ if μ is a partition with Y''(μ) ⊆ Y'(λ).

For a (possibly skew) shifted tableau S, we define its associated word w(S) = w_1w_2 ... by reading the rows of S from left to right and from bottom to top. By erasing the marks of S, we obtain the word |w|.

Given a word w = w_1w_2 ..., we define

\[ m_i(j) = \text{multiplicity of } i \text{ among } w_{n-j+1}, ..., w_n, \text{ for } 0 \leq j \leq n \]
\[ m_i(n+j) = m_i(n) + \text{multiplicity of } i' \text{ among } w_1, ..., w_j, \text{ for } 0 < j \leq n. \]

This function m_i corresponds to reading the rows of the tableau first from right to left and from top to bottom, counting the letter i on the way, and then reading from bottom to top and left to right, counting the letter i' on this way.

The word w satisfies the lattice property if, whenever \( m_i(j) = m_{i-1}(j), \) then

\[ w_{n-j} \neq i, i', \text{ if } 0 \leq j < n \]
\[ w_{j-n+1} \neq i-1, i', \text{ if } n \leq j < 2n. \]

For two partitions μ and ν we denote by μ ∪ ν the partition which has as its parts all the parts of μ and ν together.

**Theorem 2.1** ([13, 8.1 and 8.3]). Let μ ∈ D(k), ν ∈ D(n - k), λ ∈ D(n), and form the reduced Clifford product \( \langle μ \rangle ×_c \langle ν \rangle \). Then we have

\[ \left( \langle μ \rangle ×_c \langle ν \rangle \right) |^*_n (λ) = \frac{1}{ε_λε_με_ν} 2^{(l(μ)+l(ν)-l(λ))/2} g^λ_μν, \]

unless λ is odd and λ = μ ∪ ν. In that latter case, the multiplicity of \( \langle λ \rangle \) is 0 or 1, according to the choice of associates.

The coefficient \( g^λ_μν \) is the number of shifted tableaux S of shape λ \ μ and content ν such that the tableau word w = w(S) satisfies the lattice property and the leftmost i of |w| is unmarked in w for 1 ≤ i ≤ l(ν).

We will also use the following result from [13] on inner tensor products with the basic spin character \( \langle n \rangle \):

**Theorem 2.2** ([13, 9.3]). Let λ ∈ D(n), μ a partition of n. We have

\[ \langle \langle n \rangle \langle μ \rangle, \langle λ \rangle \rangle = \frac{1}{ε_λε_μ} 2^{(l(λ)-1)/2} g^λ_μ \]

unless λ = (n), n is even, and μ is a hook partition. In that case, the multiplicity of \( \langle λ \rangle \) is 0 or 1 according to choice of associates.

The coefficient \( g^λ_μ \) is the number of “shifted tableaux” S of unshifted shape μ and content λ such that the tableau word w = w(S) satisfies the lattice property and the leftmost i of |w| is unmarked in w for 1 ≤ i ≤ l(λ).

As an interesting consequence, this implies...
Corollary 2.3. Let $n \in \mathbb{N}$. Then
\[
\langle n \rangle \cdot \langle \hat{n} \rangle = [n] + \sum_{i=1}^{n-1} [n-i,1^{i}] .
\]

Finally, we collect some information about certain constituents in squares resp. in ‘almost’ squares.

Theorem 2.4 ([12], [14], [15]). Let $n \in \mathbb{N}$. Let $\lambda \in P(n)$ with $\lambda \neq (n)$. Define $a, b \in \mathbb{N}_0$ by
\[
[\lambda]^2 = [n] + a[n-2,2] + b[n-3,1^{3}] + \text{other constituents}.
\]
Denote by $h_i$ the number of hooks of $\lambda$ of length $i$, for $i \in \{1,2\}$. Let $h_{21}$ be the number of hooks of $\lambda$ of length 3 and arm length 1. Then we have:

(i) $a = h_2 + h_1(h_1 - 2)$.
In particular, $a > 0$ if $n \geq 4$.

(ii) $b = h_1(h_1 - 1)(h_1 - 3) + (h_1 - 1)(h_2 + 1) + h_{21}$.
In particular, $b > 0$, unless $\lambda$ is $(n-1,1)$ or $(n-1,1)'$.

Theorem 2.5 ([8, Theorem 4.3]). Let $n \in \mathbb{N}$, $n \geq 4$. Let $\mu \in D(n)$ with $\mu \neq (n)$ and $\mu \neq (k, k-1, \ldots, 2, 1)$. Let $s,t \in \mathbb{N}$ be defined by
\[
\langle \mu \rangle \cdot \langle \overline{\mu} \rangle = [n] + s[n-1,1] + t[n-2,2] + \text{other constituents}.
\]

Then:

(i) $t \geq 1$.
(ii) If $n \geq 5$, $\mu$ is even and $\mu$ is not of the form $(k+r, k-1+r, \ldots, 1+r)$ for some $r$, then $t \geq 2$.
(iii) If $\mu$ is not of the form $(k+r, k-1+r, \ldots, 1+r)$ for some $r$, then $s \geq 1$.

We have already considered the exceptional case $\mu = (n)$ in Corollary 2.3. Note that for even $n$ it is not clear which hook characters appear in the product $\langle \mu \rangle \cdot \langle \overline{\mu} \rangle$ and which appear in the product $\langle \overline{n} \rangle \cdot \langle \overline{n} \rangle$, except that each product contains one out of a pair of conjugate hook characters. Clearly, both products do not contain $[n-2,2]$ as a constituent, i.e., for the basic spin character we have $t = 0$ in the notation of the Theorem above.

The exceptional case of a staircase partition $\mu = (k, k-1, \ldots, 1)$ will be treated in the next section in Theorem 3.5.

3. Almost homogeneous mixed products.

We now want to study the case of mixed products, i.e., products of the form $\langle \mu \rangle \cdot \langle \nu \rangle$. Of course, if we know all the constituents in the case of products of spin characters then we also know all the coefficients in the case of mixed products, since
\[
((\mu)\langle \nu \rangle, [\lambda]) = (\langle \mu \rangle [\lambda], \langle \nu \rangle)
\]
and \( \langle \nu \rangle = \langle \nu \rangle' \) or \( \langle \nu \rangle'' \), depending on \( n - l(\nu) \mod 4 \). But it is not clear how to obtain a classification of the homogeneous mixed products from the results we know so far. In the following, we will describe some “combinatorially homogeneous” mixed products, and we classify the irreducible mixed products. We call a character of \( \tilde{S}_n \) “homogeneous” if it is of the form \( c\langle \lambda \rangle \) resp. “almost homogeneous” if it is of the form \( c\langle \lambda \rangle \) for some \( \lambda \in D(n) \) and \( c \in \mathbb{N} \).

We will need the following combinatorial result (see [2], Lemma 4.1 and its proof).

**Lemma 3.1** ([2]). Let \( k \in \mathbb{N} \). Let \( H(k) \) denote the product of the hook lengths in \( \nu = (k, k-1, \ldots, 2, 1) \), and let \( B(k) \) denote the product of the bar lengths in \( \nu \). Set \( n = \frac{1}{2}k(k+1) \). Then:

1. \( B(k) = 2^{n-k}H(k) \).
2. \( B(k+1) = B(k) \prod_{j=1}^{k+1} (k+j) \).

First we want to classify all homogeneous resp. almost homogeneous products with the basic spin character \( \langle n \rangle \).

**Theorem 3.2.** Let \( \mu \in P(n) \), \( \mu \neq (n),(1^n) \). Then the product \( \langle n \rangle \cdot [\mu] \) is almost homogeneous if and only if \( \mu \) is a rectangle.

Up to conjugation, we may assume in this case that \( \mu = (b^a) \) with \( 1 < a \leq b \), and then the product is

\[
\langle n \rangle \cdot [b^a] = \begin{cases} 
2^{a-3} \langle a+b-1, a+b-3, \ldots, b-a+1 \rangle, & \text{if } a \text{ is odd and } b \text{ is even} \\
2^{\frac{a+1}{2}} \langle a+b-1, a+b-3, \ldots, b-a+1 \rangle, & \text{else}.
\end{cases}
\]

**Proof.** First assume that \( \mu \) is not a rectangle. Let \( h_{i,i}, i = 1, \ldots, d = d(\mu) \), denote the principal hook lengths in \( \mu \). By Theorem 2.2 the product \( \langle n \rangle[\mu] \) always has a constituent \( \langle h_{11}, h_{22}, \ldots, h_{dd} \rangle \) as is illustrated by the following tableau for \( \mu = (7^3, 6^2) \):

\[
\begin{array}{cccccccc}
1' & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1' & 2' & 2 & 2 & 2 & 2 & 2 & 2 \\
1' & 2' & 3' & 3 & 3 & 3 & 3 & 3 \\
1' & 2' & 3' & 4' & 4 & 4 & 4 & 4 \\
1 & 2 & 3 & 4 & 5 & 5 & 5 & 5
\end{array}
\]

If \( \mu \) is not a rectangle, then take \( j \) maximal with \( \mu_j > \mu_{j+1} > 0 \). We can then replace the final two entries \( j'j \) in the \( j \)th column by the entries \( j' j+1 \) and still obtain a tableau of the type counted by the coefficients \( g_{\lambda\mu} \) occurring in Theorem 2.2, giving a constituent labelled by a partition different from
Thus if the product $\langle n \rangle \cdot [\mu]$ is almost homogeneous then $\mu$ has to be a rectangle.

So now we consider the case that $\mu$ is a rectangle, and we may assume that $\mu = (b^a)$ with $a \leq b$. In this situation, we have for the partition considered above:

$$\lambda = (h_{11}, h_{22}, \ldots, h_{dd}) = (a + b - 1, a + b - 3, \ldots, b - a + 1).$$

The multiplicity of the constituent $\langle a + b - 1, a + b - 3, \ldots, b - a + 1 \rangle$ in the product can be calculated by Theorem 2.2; it is easily seen that $g_{\lambda \mu} = 1$, and hence the multiplicity is

$$\frac{1}{\varepsilon_{\lambda \varepsilon(n)}} 2^{\frac{a-1}{2}} \begin{cases} 2^{\frac{a-3}{2}}, & \text{if } a \text{ is odd and } b \text{ is even} \\ 2^{\frac{a-1}{2}}, & \text{else}, \end{cases}$$

as is easily checked. In the first case, the character $\langle a + b - 1, a + b - 3, \ldots, b - a + 1 \rangle$ is not self-associate, and the associate character appears with the same multiplicity as $\mu$ is not a hook.

We now prove the statement of the Theorem by comparing degrees on both sides.

By the degree formulae for ordinary and spin characters we have for the left hand side:

$$\langle n \rangle [b^a](1) = 2^n \binom{n}{a} \frac{n!}{H(a, b)},$$

where we denote by $H(a, b)$ the product of the hook lengths in $(b^a)$. For the right hand side in the statement of the Theorem we obtain by the bar formula

$$2^n \binom{n}{a} \cdot 2^n \binom{n}{b} \frac{n!}{B(a, b)} = 2^n \binom{n}{a} \cdot \frac{n!}{B(a, b)},$$

where $B(a, b)$ denotes the product of the bar lengths in $(a + b - 1, a + b - 3, \ldots, b - a + 1)$. Hence we have to prove that for all $a \leq b$ we have $H(a, b) = B(a, b)$.

For this, we divide the Young diagram resp. the shifted diagram into three regions:

The middle region in both diagrams is of width $b - a$. It is easy to check that the hook lengths in the middle region of $(b^a)$ are exactly the same as the bar lengths in the middle region of $(a + b - 1, a + b - 3, \ldots, b - a + 1)$; let
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$H_m(a,b) = B_m(a,b)$ denote the product of the hook lengths resp. bar lengths in these middle regions. Let $H_l(a,b)$ and $H_r(a,b)$ denote the product of the hook lengths in the left resp. right region of the diagram for $(b^a)$; similarly, let $B_l(a,b)$, $B_r(a,b)$ denote the product of the bar lengths in the left resp. right region of the diagram for $(a + b - 1, a + b - 3, \ldots, b - a + 1)$. The bar lengths occurring in the left region of $(a + b - 1, a + b - 3, \ldots, b - a + 1)$ are exactly the same as the hook lengths in the left region of $(a + b - 1, a + b - 3, \ldots, b - a + 1)$ each multiplied by 2, so

$$B_l(a,b) = 2\binom{2}{2} H_l(a,b).$$

In the notation of Lemma 3.1 we have $H_r(a,b) = B(a)$ and $B_r(a,b) = H(a)$. Since by Lemma 3.1 we know that $B(a) = 2\binom{2}{2} H(a)$, we obtain

$$H(a,b) = H_l(a,b)H_m(a,b)H_r(a,b) = H_l(a,b)B_m(a,b) \cdot 2\binom{2}{2} B_r(a,b) = B_l(a,b)B_m(a,b)B_r(a,b) = B(a,b)$$

and thus the result is proved. □

Next we deal with the natural character $[n-1,1]$ and describe some almost homogeneous products with this character.

**Theorem 3.3.** Let $n$ be a triangular number, say $n = \binom{k+1}{2}$. Then

$$\langle k, k - 1, \ldots, 2, 1 \rangle \cdot [n - 1, 1] = \langle k + 1, k - 1, k - 2, \ldots, 3, 2 \rangle.$$

*Proof.* First we check that the character given on the right hand side in the statement above does indeed appear as a constituent:

$$\langle k, k - 1, \ldots, 2, 1 \rangle \cdot [n - 1, 1], \langle k + 1, k - 1, k - 2, \ldots, 3, 2 \rangle$$

$$= \langle k, k - 1, \ldots, 2, 1 \rangle \cdot \langle k + 1, k - 1, k - 2, \ldots, 3, 2 \rangle, [n - 1, 1]$$

$$= \langle k, k - 1, \ldots, 2, 1 \rangle \downarrow_{S_{n-1}}, \langle k + 1, k - 1, k - 2, \ldots, 3, 2 \rangle \downarrow_{S_{n-1}}$$

$$= 1$$

where the last equality follows from the spin branching theorem. Now to prove the assertion it suffices to check degrees on both sides.

Let again denote $B(k)$ the product of the bar lengths in $(k, k-1, \ldots, 2, 1)$, and let $B'(k)$ denote the product of the bar lengths in $(k + 1, k - 1, k - 2, \ldots, 2)$. Then by the bar formula we have to check whether the following equation holds:

$$2^{\binom{n-k}{2}} \binom{n!}{B(k)} (n-1) = 2^{\binom{n-k+2}{2}} \binom{n!}{B'(k)}$$

or equivalently,

$$(n-1)B'(k) = 2B(k).$$
We want to prove the claim by induction on \( k \), starting with \( k = 3 \), where the claim is easily checked. By Lemma 3.1

\[
B(k+1) = B(k) \prod_{j=k+1}^{2k+1} j.
\]

Let \( N = N(k-2) \) be the product of the bar lengths in \( (k-1, k-2, \ldots, 2) \). Then by considering the shifted diagrams one sees that

\[
B'(k) = N \cdot k(k+1) \prod_{j=k+3}^{2k} j
\]

\[
B'(k+1) = N \cdot (k-1)k \left( \prod_{j=k+2}^{2k-1} j \right) (k+1)(k+2) \prod_{i=k+4}^{2k+2} i.
\]

Hence

\[
B'(k+1) = B'(k) \frac{(k-1)(k+2)(2k+1)(2k+2)}{k+3} \prod_{j=k+2}^{2k-1} j.
\]

Since \( n = \binom{k+1}{2} \), we know by induction that

\[
(k^2 + k - 2)B'(k) = 4B(k),
\]

and we have to show that

\[
(k^2 + 3k)B'(k+1) = 4B(k+1).
\]

Using the relations given above this is a straightforward calculation.

Hence the assertion of the Theorem is proved.

\[\Box\]

**Theorem 3.4.** Let \( n \in \mathbb{N}, n \geq 3, \) and let \( \mu \in D(n) \). Then the product \( \langle \mu \rangle \cdot [n-1, 1] \) is irreducible if and only if \( n \) is a triangular number, say \( n = \binom{k+1}{2} \), with \( k \equiv 2 \) or \( 3 \) mod 4, and \( \mu = (k, k-1, \ldots, 2, 1) \). In this case,

\[
\langle k, k-1, \ldots, 2, 1 \rangle \cdot [n-1, 1] = \langle k+1, k-1, k-2, \ldots, 3, 2 \rangle.
\]

**Proof.** If \( \mu \) and \( k \) are as stated, then \( \mu \) is odd and \((k+1, k-1, k-2, \ldots, 3, 2)\) is even, and so by the previous Theorem the stated product is indeed irreducible.

Now assume that the product \( \langle \mu \rangle \cdot [n-1, 1] \) is irreducible. By the classification result for products with the basic spin character, we know that \( \mu \neq (n) \).

So assume now \( \mu \neq (k, k-1, \ldots, 1) \). Then by Theorem 2.5 resp. by Theorem 2.4 both \([n-2, 2]\) and \([n]\) are constituents of the product \( \langle \mu \rangle \cdot \langle \mu \rangle \) as well as of the square \([n-1, 1]^2\). Hence

\[
([n-1, 1] \cdot \langle \mu \rangle, [n-1, 1] \cdot \langle \mu \rangle) = ([\langle \mu \rangle \cdot \langle \mu \rangle], [n-1, 1] \cdot [n-1, 1]) \geq 0,
\]

so the product is not irreducible. \[\Box\]
For the classification of the irreducible mixed products, we will need some further information in the special case of staircase partitions.

**Theorem 3.5.** Let \(k \in \mathbb{N}, n = \binom{k + 1}{2} \). Define the coefficients \(a_1, a_2, a_3, b_2, b_3, c_3\) by

\[
\langle k, \ldots, 1 \rangle \cdot \langle k, \ldots, 1 \rangle = [n] + a_1[n - 1, 1] + a_2[n - 2, 2] + a_3[n - 3, 3] \\
+ b_2[n - 2, 1^2] + b_3[n - 3, 1^3] + c_3[n - 3, 2, 1] \\
+ \text{other constituents.}
\]

Then:

(i) \(a_1 = a_2 = c_3 = 0\).

(ii) \(b_2 = \begin{cases} 
1 & \text{if } k \equiv 0 \text{ or } 1 \mod 4 \\
0 & \text{if } k \equiv 2 \text{ or } 3 \mod 4.
\end{cases} \)

(iii) \(a_3 = b_3 = 1\).

**Proof.** Set \(\rho_k = \langle k, \ldots, 1 \rangle\) and \(\varphi_k = \langle k, \ldots, 1 \rangle\). Note that \(\rho_k \in D^+\) if and only if \(k \equiv 0\) or \(1 \mod 4\). Also, let \(\pi_\alpha = 1_{S_\alpha} \uparrow^{S_n} = 1_{S_\alpha} \uparrow^{S_n}\). So

\[
(\varphi_k \cdot \overline{\varphi_k}, \pi_\alpha) = (\varphi_k \downarrow_{S_\alpha} \varphi_k \downarrow_{S_\alpha}),
\]

and for computing the restriction we use the spin branching theorem resp. the shifted Littlewood-Richardson Rule provided by Theorem 2.1.

Since \(\varphi_k \downarrow_{S_{(n-1,1)}} = \langle k, \ldots, 2 \rangle\), \((\varphi_k \cdot \overline{\varphi_k}, \pi_{(n-1,1)}) = 1\). As \([n - 1, 1] = \pi_{(n-1,1)} - [n]\), this yields

\[a_1 = (\varphi_k \cdot \overline{\varphi_k}, [n - 1, 1]) = 0.\]

Next, \(\varphi_k \downarrow_{S_{(n-2,2)}}\) is the irreducible character \(\langle k, \ldots, 3, 1 \rangle \times_c \langle 2 \rangle\) (up to the choice of associates in the case \(k \equiv 0\) or \(1 \mod 4\)), hence \((\varphi_k \cdot \overline{\varphi_k}, \pi_{(n-2,2)}) = 1\). As \([n - 2, 2] = \pi_{(n-2,2)} - \pi_{(n-1,1)}\), this implies

\[a_2 = (\varphi_k \cdot \overline{\varphi_k}, [n - 2, 2]) = 0.\]

The restriction \(\varphi_k \downarrow_{S_{(n-3,3)}}\) has the two irreducible constituents \(\langle k, \ldots, 4, 2, 1 \rangle \times_c \langle 3 \rangle\) and \(\langle k, \ldots, 3 \rangle \times_c \langle 2, 1 \rangle\) (up to the choice of associates in the case \(k \equiv 2\) or \(3 \mod 4\)). Hence \((\varphi_k \cdot \overline{\varphi_k}, \pi_{(n-3,3)}) = 2\), and from the equation \([n - 3, 3] = \pi_{(n-3,3)} - \pi_{(n-2,2)}\) we now deduce

\[a_3 = (\varphi_k \cdot \overline{\varphi_k}, [n - 3, 3]) = 1.\]

Now \(\varphi_k \downarrow_{S_{(n-2,1^2)}} = \langle k, \ldots, 3, 1 \rangle\). Thus,

\[
(\varphi_k \cdot \overline{\varphi_k}, \pi_{(n-2,1^2)}) = \begin{cases} 
1 & \text{if } k \equiv 2 \text{ or } 3 \mod 4 \\
2 & \text{if } k \equiv 0 \text{ or } 1 \mod 4.
\end{cases}
\]
As \([n - 2, 1^2] = \pi_{(n-2,1^2)} - \pi_{(n-2,2)} - [n - 1, 1]\), we obtain
\[
b_2 = (\varphi_k \cdot \overline{\varphi_k}, [n - 2, 1^2]) = \begin{cases} 0 & \text{if } k \equiv 2 \text{ or } 3 \mod 4 \\
1 & \text{if } k \equiv 0 \text{ or } 1 \mod 4. \end{cases}
\]
For the restriction \(\varphi_k \mid \overline{\mathcal{S}}_{(n-3,2,1)}\) we obtain
\[
\langle k, \ldots, 4, 2, 1 \rangle \times_c \langle 2 \rangle + \langle k, \ldots, 4, 3 \rangle \times_c \langle 2 \rangle
\]
if \(k \equiv 0 \text{ or } 1 \mod 4\), and
\[
\langle k, \ldots, 4, 2, 1 \rangle \times_c \langle 2 \rangle + \langle k, \ldots, 4, 3 \rangle \times_c \langle 2 \rangle
\]
up to a choice of associates for the second summand, if \(k \equiv 2 \text{ or } 3 \mod 4\). Hence
\[
(\varphi_k \cdot \overline{\varphi_k}, \pi_{(n-3,2,1)}) = \begin{cases} 3 & \text{if } k \equiv 0 \text{ or } 1 \mod 4 \\
2 & \text{if } k \equiv 2 \text{ or } 3 \mod 4. \end{cases}
\]
As
\[
[n - 3, 2, 1] = \frac{1}{2} (\pi_{(n-3,2,1)} - \pi_{(n-3,3)} - [n - 2, 2] - [n - 1, 1] - [n - 2, 1^2]),
\]
we obtain
\[
c_3 = (\varphi_k \cdot \overline{\varphi_k}, [n - 3, 2, 1]) = 0.
\]
Finally, we have
\[
\varphi_k \mid \overline{\mathcal{S}}_{(n-3,1^3)} = \begin{cases} \langle k, \ldots, 3 \rangle + 2\langle k, \ldots, 4, 2, 1 \rangle & \text{if } k \equiv 0 \text{ or } 1 \mod 4 \\
\langle k, \ldots, 3 \rangle + \langle k, \ldots, 4, 2, 1 \rangle & \text{if } k \equiv 2 \text{ or } 3 \mod 4. \end{cases}
\]
Thus
\[
(\varphi_k \cdot \overline{\varphi_k}, \pi_{(n-3,1^3)}) = \begin{cases} 6 & \text{if } k \equiv 0 \text{ or } 1 \mod 4 \\
3 & \text{if } k \equiv 2 \text{ or } 3 \mod 4. \end{cases}
\]
As
\[
[n - 3, 1^3] = \pi_{(n-3,1^3)} - \pi_{(n-3,2,1)} - [n - 1, 1] - [n - 2, 2] - 2[n - 2, 1^2],
\]
we obtain
\[
b_3 = (\varphi_k \cdot \overline{\varphi_k}, [n - 3, 1^3]) = 1.
\]

\[\square\]

**Theorem 3.6.** Let \(\lambda \in P(n), \lambda \neq (n), (1^n)\). Let \(\mu \in D(n)\). Then \([\lambda] \cdot [\mu]\) is irreducible if and only if one of the following occurs:

(i) \(n = 2k, \lambda = (k, k)\) or \((2^k)\) and \(\mu = (n)\).

Here the products are
\[
[k, k] \cdot \langle n \rangle = [2^k] \cdot \langle n \rangle = \langle k + 1, k - 1 \rangle.
\]

(ii) \(n = \binom{k+1}{2}\) for some \(k \in \mathbb{N}\) with \(k \equiv 2 \text{ or } 3 \mod 4\), \(\lambda = (n - 1, 1)\) or \((2, 1^{n-2})\) and \(\mu = (k, k - 1, \ldots, 2, 1)\).

Here the products are
\[
[n - 1, 1] \cdot \langle k, \ldots, 1 \rangle = [2, 1^{n-2}] \cdot \langle k, \ldots, 1 \rangle = \langle k + 1, k - 1, k - 2, \ldots, 2 \rangle.
\]
Proof. In the cases (i) and (ii) described above, the product is irreducible by Theorem 3.2 resp. Theorem 3.4. In the following we may assume that $n \geq 4$.

Now assume that $|\lambda| \cdot \langle \mu \rangle$ is irreducible. Then

$$1 = ([\lambda] \cdot \langle \mu \rangle, [\lambda] \cdot \langle \mu \rangle) = ([\lambda]^2, \langle \mu \rangle \cdot \overline{\langle \mu \rangle}).$$

As both $[\lambda]^2$ and $\langle \mu \rangle \cdot \overline{\langle \mu \rangle}$ have $[n]$ as a constituent, it suffices to find a further common constituent in all situations not covered by (i) and (ii).

By Theorem 2.4, $[\lambda]^2$ always contains a constituent $[n-2,2]$, and by Theorem 2.5, also $\langle \mu \rangle \cdot \overline{\langle \mu \rangle}$ contains a constituent $[n-2,2]$, unless $\mu = (n)$ or $\mu = (k,k-1,\ldots,1)$. So we only have to consider mixed products $[\lambda] \cdot \langle \mu \rangle$ with $\mu$ of these two exceptional types.

For $\mu = (n)$, this was done in Theorem 3.2, giving the products described in case (i) as the only irreducible mixed products with the basic spin character.

So it remains to deal with the case of a staircase $\mu = (k,k-1,\ldots,1)$. By Theorem 2.4 and Theorem 3.5 we then find a common constituent $[n-3,1^2]$ in $[\lambda]^2$ and $\langle \mu \rangle \cdot \overline{\langle \mu \rangle}$, unless $\lambda$ is $(n-1,1)$ or $(n-1,1)'$. But in this latter situation, we can apply Theorem 3.3 which leads exactly to the irreducible mixed products given in (ii). \hfill $\square$

Acknowledgements. The financial support by the Deutsche Forschungsgemeinschaft (grants Be 923/6-1 and JAP-115/169/0) at different stages of the work for this article is gratefully acknowledged. Special thanks for the hospitality enjoyed at the University of Osaka where some part of the work for this article was done go to Professor Uno. The author is also grateful to John Stembridge for sharing his maple packages SF and QF, that helped finding homogeneous mixed products.

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Received September 9, 1999.

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CLOSED CONFORMAL VECTOR FIELDS AND LAGRANGIAN SUBMANIFOLDS IN COMPLEX SPACE FORMS

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We study a wide family of Lagrangian submanifolds in non-flat complex space forms that we will call pseudoumbilical because of their geometric properties. They are determined by admitting a closed and conformal vector field $X$ such that $X$ is a principal direction of the shape operator $A_{jX}$, being $J$ the complex structure of the ambient manifold. We emphasize the case $X = JH$, where $H$ is the mean curvature vector of the immersion, which are known as Lagrangian submanifolds with conformal Maslov form. In this family we offer different global characterizations of the Whitney spheres in the complex projective and hyperbolic spaces.

Let $M^n$ be a Kaehler manifold of complex dimension $n$. The Kaehler form $\Omega$ on $\overline{M}$ is given by $\Omega(v, w) = \langle v, Jw \rangle$, being $\langle , \rangle$ the metric and $J$ the complex structure on $\overline{M}$. An immersion $\phi : M \rightarrow \overline{M}$ of an $n$-dimensional manifold $M$ is called Lagrangian if $\phi^*\Omega \equiv 0$. This property involves only the symplectic structure of $\overline{M}$. In this family of Lagrangian submanifolds, one can study properties of the submanifold involving the Riemannian structure of $\overline{M}$. One must take into account the nice property of the second fundamental form $\sigma$ of these submanifolds which says that the trilinear form

$$\langle \sigma(v, w), Jz \rangle$$

is totally symmetric. Sometimes this property of the second fundamental form gives obstructions to the existence of examples satisfying classical Riemannian properties. So, if one considers like classical property the umbilicity, automatically our umbilical Lagrangian submanifold is totally geodesic and only appear trivial examples. Then two natural questions arise: What is the Lagrangian version of umbilicity? Which are the corresponding examples?

In [RU] it was proposed that the Lagrangian version of umbilicity was that the second fundamental form satisfies

$$(0.1) \quad \langle \sigma(v, w), Jz \rangle = \frac{n}{n + 2} \partial_{v, w, z} \langle v, w \rangle \langle H, Jz \rangle,$$
where the symbol $\partial_{v,w,z}$ means cyclic sum over $v, w, z$ and $H$ is the mean curvature vector of the submanifold. The corresponding classification was also made when the ambient space was complex Euclidean space $\mathbb{C}^n$. Besides the linear Lagrangian subspaces, the Whitney spheres ([RU], see Paragraph 4 for the definition) were the only examples, which can be considered like the Lagrangian version of the round hyperspheres of Euclidean space. For $n = 2$ this classification was made in [CU1].

In [CU2] when $n = 2$ and in [Ch1] in arbitrary dimension, the Lagrangian submanifolds satisfying (0.1) of the complex projective and hyperbolic spaces $\mathbb{CP}^n$ and $\mathbb{CH}^n$ were classified. Again, in the complex projective space only the Whitney spheres appeared, but in the complex hyperbolic space, besides the Whitney spheres, two new families of noncompact examples appeared, topologically equivalent to $\mathbb{S}^1 \times \mathbb{R}^{n-1}$ and $\mathbb{R}^n$, which can be considered like the Lagrangian version of the tubes over hyperplanes and the horospheres in the real hyperbolic space.

Following with the analogy between umbilical hypersurfaces of real space forms and our family of Lagrangian submanifolds of complex space forms, the umbilical hypersurfaces are the easiest examples of submanifolds with constant mean curvature. Our Lagrangian examples, except the totally geodesic ones, do not have parallel mean curvature vector, property which is usually taken as a version on higher codimension of the notion of constant mean curvature, but their mean curvature vectors $H$ satisfy that $JH$ are conformal fields on the submanifolds. So, we will take this property like the Lagrangian version of the concept of hypersurfaces of constant mean curvature. As the dual form of $JH$ is the well-known Maslov form, we will refer to these submanifolds as Lagrangian submanifolds with conformal Maslov form. In [RU] this family was studied when the ambient space is $\mathbb{C}^n$.

The Lagrangian submanifolds satisfying (0.1) also verify that $JH$ is a principal direction of $A_H$. Motivated by this fact, in this paper we study a wide family of Lagrangian submanifolds, defined by the property that the submanifold admits a closed and conformal vector field $X$ ($JH$ is always a closed field) such that $X$ is a principal direction of $A_{JX}$, without assuming that $JX$ is the mean curvature vector $H$. In this family a minor important degenerate case appears, which is completely studied and classified in Paragraph 3. Motivated by Proposition 1, the nondegenerate submanifolds of our family are called pseudoumbilical. A particular family of this kind of submanifolds, which was called H-umbilical by B.Y. Chen, was studied in [Ch2].

In Paragraphs 1 and 2 we study deeply pseudoumbilical Lagrangian submanifolds of $\mathbb{CP}^n$ and $\mathbb{CH}^n$. In Theorem 1, we describe the pseudoumbilical Lagrangian submanifolds (see Definition 1), showing that they have a similar behavior to the “umbilical” Lagrangian submanifolds. Theorem 1 allows
us to classify pseudoumbilical Lagrangian submanifolds of $\mathbb{C}P^n$ and $\mathbb{C}H^n$ (Corollary 1 and Theorem 2); they are described in terms of planar curves and Lagrangian (n-1)-submanifolds of the complex projective, hyperbolic or Euclidean spaces depending on the elliptic, hyperbolic or parabolic character of the submanifold.

In Paragraph 4, we use the above classification in order to study Lagrangian submanifolds with conformal Maslov form. Among the most important global results, we show the following Lagrangian version of a Hopf theorem:

The Whitney spheres are the only compact (nonminimal) Lagrangian submanifolds of $\mathbb{C}P^n$ and $\mathbb{C}H^n$ with conformal Maslov form and null first Betti number.

Another global result is established in terms of the Ricci curvature. In this context, we prove in Corollary 3 that the Whitney spheres are the only compact (nonminimal) Lagrangian submanifolds of $\mathbb{C}P^n$ with conformal Maslov form such that $\text{Ric}(JH) \geq (n-1)|H|^2$.

Finally, we prove that all the orientable compact Lagrangian submanifolds of $\mathbb{C}P^n$ and $\mathbb{C}H^n$ with nonparallel conformal Maslov form and first Betti number one are elliptic pseudoumbilical and then this allows describe them (Corollary 4).

1. Closed conformal fields and Lagrangian submanifolds.

The Lagrangian submanifolds we are going to consider in this paper will have a closed and conformal vector field. So, in this section, we will describe properties of this kind of vector fields, as well as properties of Lagrangian submanifolds admitting this kind of vector fields.

The following lemma summerizes some of the known results about Riemannian manifolds which admit closed and conformal vector fields (see [RU] and references there in).

**Lemma 1.** Let $(M, \langle \cdot, \cdot \rangle)$ be an $n$-dimensional Riemannian manifold endowed with a nontrivial vector field $X$ which is closed and conformal. Then:

(i) The set $Z(X)$ of the zeros of $X$ is a discrete set.

(ii) If $\text{div } X$ denotes the divergence of $X$, then

\[ \nabla_V X = \frac{\text{div } X}{n} V, \quad |X|^2 \nabla (\text{div } X) = -\frac{n \text{Ric}(X)}{n-1} X, \]

for any vector field $V$ on $M$, where $\text{Ric}$ denotes the Ricci curvature of $M$.

(iii) The curvature tensor $R$ of $M$ satisfies

\[ |X|^2 R(v, w) X = \frac{\text{Ric}(X)}{n-1} \{ \langle w, X \rangle v - \langle v, X \rangle w \}. \]
(iv) If \( M' = M - \mathcal{Z}(X) \), then
\[
p \in M' \mapsto D(p) = \{ v \in T_pM \mid \langle v, X \rangle = 0 \}
\]
defines an umbilical foliation on \((M', \langle \cdot, \cdot \rangle)\). In particular, \(|X|^2\) and \(\text{div} \, X\) are constant on the connected leaves of \(D\).

(v) \((M', g)\) with \(g = |X|^{-2} \langle \cdot, \cdot \rangle\), is locally isometric to \((I \times N, dt^2 \times g')\), where \(I\) is an open interval in \(\mathbb{R}\), \(\{t\} \times N\) is a leaf of the foliation \(D\) for any \(t \in \mathbb{R}\), and \(X = (\partial/\partial t, 0)\). Moreover, if \(\Delta^g\) is the Laplacian of \(g\) then
\[
\Delta^g \log |X| + \frac{\text{Ric}(X)}{n-1} = 0.
\]

In this paper, \(M^n(c)\) will denote a complete simply-connected complex space form with constant holomorphic sectional curvature \(c\), with \(c = 4, 0, -4\), i.e., complex Euclidean space \(\mathbb{C}^n\) if \(c = 0\), the complex projective space \(\mathbb{CP}^n\) if \(c = 4\) and the complex hyperbolic space \(\mathbb{CH}^n\) if \(c = -4\).

Let \(S^{2n+1} = \{ z \in \mathbb{C}^{n+1} : \langle z, z \rangle = 1 \}\) be the hypersphere of \(\mathbb{C}^{n+1}\) centered at the origin with radius 1, where \(\langle \cdot, \cdot \rangle\) denotes the inner product on \(\mathbb{C}^{n+1}\).
We consider the Hopf fibration \(\Pi : S^{2n+1} \to \mathbb{CP}^n\), which is a Riemannian submersion. We can identify \(\mathbb{C}^n\) with the open subset of \(\mathbb{CP}^n\) defined by
\[
\{ \Pi(z_1, \ldots, z_{n+1}) \in \mathbb{CP}^n : z_{n+1} \neq 0 \}.
\]
Then the Fubini-Study metric \(g\) is given on \(\mathbb{C}^n\) by
\[
g_p = \frac{1}{1 + |p|^2} \left\{ \langle \cdot, \cdot \rangle - \frac{\mathcal{R}(\alpha \otimes \overline{\alpha})}{1 + |p|^2} \right\}
\]
where \(\langle \cdot, \cdot \rangle\) is the Euclidean metric, \(\mathcal{R}\) denotes real part and \(\alpha\) is the complex 1-form on \(\mathbb{C}^n\) given by
\[
\alpha_p(v) = \langle v, p \rangle + i \langle v, Jp \rangle.
\]

On the other hand, if \(c = -4\), let \(\mathbb{H}^{2n+1}_i = \{ z \in \mathbb{C}^{n+1} : \langle z, z \rangle = -1 \}\) be the anti-De Sitter space where \(\langle \cdot, \cdot \rangle\) denotes the hermitian form
\[
\langle z, w \rangle = \sum_{i=1}^{n} z_i \overline{w}_i - z_{n+1} \overline{w}_{n+1},
\]
for \(z, w \in \mathbb{C}^{n+1}\). Then \(\langle z, w \rangle = \mathcal{R}(z, w)\) induces on \(\mathbb{H}^{2n+1}_i\) a Lorentzian metric of constant curvature \(-1\). If \(\Pi : \mathbb{H}^{2n+1}_i \to \mathbb{CH}^n\) denotes the Hopf fibration, it is well-known that \(\Pi\) is a Riemannian submersion. Then \(\mathbb{CH}^n\) can be identified with the unit ball
\[
\mathbb{B}^n = \left\{ (z_1, \ldots, z_n) \in \mathbb{C}^n : \sum_{i=1}^{n} |z_i|^2 < 1 \right\},
\]
endowed with the Bergmann metric

\[ g_p = \frac{1}{1 - |p|^2} \left\{ \langle \cdot, \cdot \rangle + \frac{\mathcal{R}(\alpha \otimes \bar{\alpha})}{1 - |p|^2} \right\}. \]

We recall that \( \mathbb{CP}^n \) has a smooth compactification \( \mathbb{CP}^n \cup S^{2n-1}(\infty) \), where \( S^{2n-1}(\infty) \) can be identified with asymptotic classes of geodesic rays in \( \mathbb{CP}^n \).

It is known that \( S^{2n-1}(\infty) \) can be also identified with \( S^{2n-1}(\infty) = \pi(N) \), being \( N = \{ z \in \mathbb{C}^{n+1} - \{0\} : (z, z) = 0 \} \) and \( \pi \) the projection given by the natural action of \( \mathbb{C}^* \) over \( N \). This identification is given by

\[ [\beta(s)] \mapsto \pi \left( \tilde{\beta}(s) + \frac{\tilde{\beta}'(s)}{|\tilde{\beta}'(s)|} \right) \]

being \( \beta(s), s \geq 0 \), a geodesic ray in \( \mathbb{CP}^n \) and \( \tilde{\beta}(s), s \geq 0 \), a geodesic ray in \( \mathbb{H}^{2n+1}_1 \) with \( \Pi(\tilde{\beta}(s)) = \beta(s) \).

Finally, in order to understand Theorem 1, it is convenient to remember that although in \( \mathbb{CP}^n \) and \( \mathbb{CH}^n \) there do not exist umbilical real hypersurfaces, M. Kon in [K] and S. Montiel in [M] proved that the geodesic spheres of \( \mathbb{CP}^n \) and the geodesic spheres, the tubes over complex hyperplanes and the horospheres of \( \mathbb{CH}^n \) are the only \( \eta \)-umbilic real hypersurfaces of \( \mathbb{CP}^n \) and \( \mathbb{CH}^n \). The horospheres of \( \mathbb{CH}^n \) with infinity point \( C \in S^{2n-1}(\infty) \) and radius \( \lambda > 0 \) are defined by

\[ \{ \Pi(z) \in \mathbb{CH}^n : |(z, \tilde{C})|^2 = \lambda^2 |\tilde{C}_{n+1}|^2 \} \]

where \( \tilde{C} = (\tilde{C}_1, \ldots, \tilde{C}_{n+1}) \) is a point in \( N \) such that \( \pi(\tilde{C}) = C \).

Let \( \phi \) be an isometric immersion of a Riemannian \( n \)-manifold \( M \) in \( \mathcal{C}^n(c) \). If the almost complex structure \( J \) of \( \mathcal{C}^n(c) \) carries each tangent space of \( M \) into its corresponding normal space, \( \phi \) is called Lagrangian. We denote the Levi-Civita connection of \( M \) and the connection on the normal bundle by \( \nabla \) and \( \nabla^\perp \), respectively. The second fundamental form will be denoted by \( \sigma \) and the shape operator by \( A_{\xi} \). If \( \phi \) is Lagrangian, the formulas of Gauss and Weingarten lead to

\[ \nabla^\perp_Y JZ = J\nabla_Y Z, \]
\[ \sigma(Y, Z) = JA_{\nabla_Y Z} = JA_{\nabla_Z Y}, \]

for tangent vector fields \( Y \) and \( Z \). These formulas imply that \( \langle \sigma(Y, Z), JW \rangle \) is totally symmetric, where \( \langle \cdot, \cdot \rangle \) denotes the metric in \( \mathcal{C}^n(c) \) and the induced one in \( M \) by \( \phi \). Using the Codazzi equation, \( \langle (\nabla \sigma)(Y, Z, W), JU \rangle \) is
also totally symmetric, where \( \nabla \sigma \) is the covariant derivative of the second fundamental form.

If \( \phi : M \rightarrow \mathbb{CP}^n \) is a Lagrangian immersion of a simply-connected manifold \( M \), then it is well-known that \( \phi \) has a horizontal lift (with respect to the Hopf fibration) to \( S^{2n+1} \) which is unique up to rotations on \( S^{2n+1} \). We will denote by \( \tilde{\phi} \) this horizontal lift. The horizontality means that \( \langle \tilde{\phi}_s(v), J\tilde{\phi} \rangle = 0 \) for any tangent vector \( v \) to \( M \), where \( J \) is the complex structure of \( \mathbb{C}^{n+1} \). We remark that only the Lagrangian immersions in \( \mathbb{CP}^n \) have (locally) horizontal lifts.

In a similar way, if \( \phi : M \rightarrow \mathbb{CH}^n \) is a Lagrangian immersion of a simply-connected manifold \( M \), then \( \phi \) has a horizontal lift to \( \mathbb{H}^{2n+1} \), that we will denote by \( \tilde{\phi} \). We also remark that only the Lagrangian immersions in \( \mathbb{CH}^n \) have (locally) horizontal lifts.

**Proposition 1.** Let \( \phi : M^n \rightarrow \overline{M}^n(c) \) be a Lagrangian isometric immersion of a connected Riemannian manifold \( M \) endowed with a closed and conformal field \( X \). Following the notation of Lemma 1, suppose that \( \sigma(X,X) = \rho JX \), for some function \( \rho \) defined on \( M' = M - Z(X) \). Then \( \rho \) is constant on the connected leaves of the foliation \( \mathcal{D} \) and we have only two possibilities:

- **(A)** At any point \( p \) of \( M \), \( A_{JX} \) has two constant eigenvalues \( b_1 \) and \( b_2 \) on \( \mathcal{D}(p) \), with constant multiplicities \( n_1 \) and \( n_2 \). In this case, \( \rho \) is constant and the field \( X \) is parallel.
- **(B)** At any point \( p \) of \( M' \), \( A_{JX} \) has only one eigenvalue \( b(p) \) on \( \mathcal{D}(p) \). In this case, \( b \) is constant on the connected leaves of the foliation \( \mathcal{D} \) and

\[
\frac{c}{4} |X|^2 + \left( \frac{\text{div} \ X}{n} \right)^2 + b^2 = \lambda \in \mathbb{R}.
\]

**Remark 1.** An immersion \( \phi \), under the assumptions of Proposition 1, admits a 1-parameter family of closed and conformal vector fields \( X_\mu = \mu X \) with \( \mu \) a nonnull real number, satisfying \( \sigma(X_\mu, X_\mu) = \mu \rho J(X_\mu) \). So, if (A) is satisfied, we always will consider one of the two parallel fields \( X \) such that \( |X| = 1 \). If condition (B) is satisfied, then \( \sigma(V, X_\mu) = \mu bJV \) for \( V \) orthogonal to \( X_\mu \). So

\[
\frac{c}{4} |X_\mu|^2 + \left( \frac{\text{div} \ X_\mu}{n} \right)^2 + (\mu b)^2 = \mu^2 \lambda.
\]

When \( \lambda \neq 0 \) we always will consider one of the two closed and conformal fields \( X \) such that the corresponding constant \( \lambda \) will be 1 or \(-1\) and we will refer to them to be **elliptic** or **hyperbolic** cases. The **parabolic** case will correspond to \( \lambda = 0 \).

Proposition 1 motivates the following definition.
Definition 1. A Lagrangian immersion $\phi : M^n \rightarrow \mathbb{M}^m(c)$ is said to be pseudoumbilical if the Riemannian manifold $M$ is endowed with a closed and conformal field $X$ (without zeros) such that $\sigma(X, X) = \rho JX$ for certain function $\rho$ on $M$ and condition (B) in Proposition 1 is satisfied.

Remark 2. When $c = 4$, a pseudoumbilical Lagrangian immersion is always elliptic. When $c = 0$, it can be either parabolic or elliptic. And when $c = -4$, it can be elliptic, parabolic or hyperbolic.

Proof of Proposition 1. We start proving that $\nabla \rho = hX$ for a certain function $h$, which means that $\rho$ is constant on the connected leaves of $\mathcal{D}$. For this purpose, derivating $\sigma(X, X) = \rho JX$ with respect to a tangent vector field $V$ and using Lemma 1, (ii) we get

$$\langle (\nabla \sigma)(V, X, X), JW \rangle + \frac{2 \text{div } X}{n} \langle \sigma(V, X), JW \rangle = \langle \nabla \rho, V \rangle \langle X, W \rangle + \rho \text{div } X \langle V, W \rangle,$$

for any field $W$. Using properties of $\phi$ and the Codazzi equation of $\phi$ we obtain that all the terms in the above equation are symmetric on $V$ and $W$, except possible $\langle \nabla \rho, V \rangle \langle X, W \rangle$. So this one must be also symmetric and this means that $\nabla \rho = hX$ for certain function $h$. In particular $\rho$ is constant on the leaves of $\mathcal{D}$.

On the other hand, as $X(p)$ is an eigenvector of $A_{JX}$, then $A_{JX}$ can be diagonalized on $\mathcal{D}(p)$. Now using Lemma 1, (iii) and the Gauss equation, the eigenvalues $\lambda's$ of $A_{JX}$ on $\mathcal{D}(p)$ satisfy the equation

$$\lambda^2 - \rho \lambda + \frac{\text{Ric}(X)}{n - 1} - \frac{c}{4} |X|^2 = 0. \quad (1.2)$$

In particular there are at most two. From Lemma 1, (v), $\text{Ric}(X)$ is constant on the connected leaves of $\mathcal{D}$. So the two possible eigenvalues of $A_{JX}$ on $\mathcal{D}(p)$ are also constant on the leaves of $\mathcal{D}$.

Let

$$M_0 = \{ p \in M' / A_{JX} \text{ has two different eigenvalues on } \mathcal{D}(p) \}.$$

Claim. $M_0$ is a closed subset of $M'$.

If these eigenvalues are denoted by $b_i, i = 1, 2$, and their multiplicities by $n_i, i = 1, 2$, then $n(H, JX) = \rho + n_1 b_1 + n_2 b_2$. Because $b_i$ are smooth functions on $M_0$, we have that $n_i, i = 1, 2$ are constant on each connected component of $M_0$. So,

$$\mathcal{D}_i(p) = \{ v \in \mathcal{D}(p) / A_{JX} v = b_i v \}, \ i = 1, 2,$$

define distributions on the connected components of $M_0$ such that $\mathcal{D} = \mathcal{D}_1 \oplus \mathcal{D}_2$. 

If \( V_i \) are vector fields on \( \mathcal{D}_i \), from Lemma 1, (iii) and the Gauss equation of \( \phi \), we have that

\[
\sigma(V_i, V_2) = 0. \tag{1.3}
\]

Also derivating \( \sigma(V_i, V_i), JX) = b_i \langle V_i, V_i \rangle \) with respect to \( V_j \), with \( j \neq i \), and using (1.3) and the fact that \( \nabla b_i = h_i X \) for certain functions \( h_i \), we obtain that

\[
\langle (\nabla \sigma)(V_j, V_i, V_i), JX \rangle = 0, \tag{1.4}
\]

for \( V_i \in \mathcal{D}_i, V_j \in \mathcal{D}_j \) and \( i \neq j \). But derivating with respect to \( V_i \), \( \langle \sigma(X, V_i), JV_j \rangle = 0 \) and using (1.3) and (1.4) we get

\[
(b_i - b_j) \langle \nabla V_i V_i, V_j \rangle = 0
\]

and then \( \langle \nabla V_i V_i, V_j \rangle = 0 \) when \( i \neq j \). So

\[
\nabla V_i V_i = - \frac{\text{div} X}{n} |X|^{-2} |V_i|^2 X + (\nabla V_i V_i)^i,
\]

where \( ^i \) means component on \( \mathcal{D}_i \). Finally, taking \( i \neq j \), derivating \( \sigma(V_i, V_j) = 0 \) with respect to \( V_i \) and using the above equation and (1.3) we get

\[
\langle (\nabla \sigma)(V_i, V_j, V_j), JV_j \rangle = b_j (\text{div} X/n) |X|^{-2} |V_i|^2 |V_j|^2
\]

for \( i \neq j \). Changing the roles of \( i \) and \( j \) and using the symmetry of \( \nabla \sigma \) we finally get

\[
(b_i - b_j) \text{div} X |V_i|^2 |V_j|^2 = 0.
\]

Since \( i \neq j \) we get that \( \text{div} X = 0 \) and then \( X \) is a parallel field on \( \mathcal{M}_0 \).

Now we see that \( b_i \) are constant on each connected component of \( \mathcal{M}_0 \). Taking in Equation (1.1) \( V = V_i \) and \( W = V_j \) with \( i \neq j \) we obtain

\[
\langle (\nabla \sigma)(V_i, X, X), JV_j \rangle = 0.
\]

Derivating \( \langle \sigma(X, V_i), JV_j \rangle = 0 \) with respect to \( X \) and using the above equation and (1.3) we obtain

\[
(b_i - b_j) \langle \nabla_X V_i, V_j \rangle = 0,
\]

and so \( \nabla_X V_i \) is a field on \( \mathcal{D}_i \). Now if we derivate \( \sigma(X, V_i) = b_i JV_i \) with respect to \( X \), and use the above information and the parallelism of \( X \) in (1.1), we obtain that \( X(b_i) = 0 \), and therefore \( b_i \) are constant on each component of \( \mathcal{M}_0 \). So, by continuity, \( A_{J_X} \) on \( \mathcal{D} \) has also two eigenvalues on the clousure of \( \mathcal{M}_0 \), and we get that \( \mathcal{M}_0 \) is a closed subset of \( \mathcal{M}' \). This proves the claim.

As \( \mathcal{M}' \) is also connected, there are only two possibilities: Either \( \mathcal{M}_0 = \mathcal{M}' \) or \( \mathcal{M}_0 = \emptyset \).

If \( \mathcal{M}_0 = \mathcal{M}' \), then the parallel field \( X \) has no zeros and \( \mathcal{M}' = \mathcal{M} \). As \( \rho = b_1 + b_2 \) then \( \rho \) is also constant and we prove (A).
If $M_0 = \emptyset$, then for any point $p$ of $M'$, $A_{JX}$ has only one eigenvalue $b(p)$ on $D(p)$, which is constant on the leaves of $D$.

Given any field $V$ on $D$, we have that $\sigma(X, V) = bJV$. Then derivating with respect to $X$ and using that $\nabla_XV$ is a field on $D$ we obtain

$$(\nabla \sigma)(X, X, V) + \frac{\text{div} X}{n} bJV = X(b)JV.$$  

But from (1.1) we have that

$$(\nabla \sigma)(V, X, X) = (\rho - 2b) \frac{\text{div} X}{n} JV,$$

which, joint to the above equation, gives

(1.5) \hspace{1cm} X(b) = (\rho - b) \frac{\text{div} X}{n}.

If $h : M' \rightarrow \mathbb{R}$ is the function defined by

$$h = \frac{c}{4} |X|^2 + \left( \frac{\text{div} X}{n} \right)^2 + b^2,$$

then using Lemma 1, (ii), (1.5), the fact that $b$ is constant on the leaves of $D$ and (1.2), we have

$$\nabla h = \frac{-2\text{div} X}{n|X|^2} \left( (-c/4)|X|^2 + \frac{\text{Ric}(X)}{n-1} + b^2 - \rho b \right) X = 0.$$  

So $h$ is a constant $\lambda$. This finishes the proof. \hfill $\square$

2. Pseudoumbilical Lagrangian submanifolds of complex space forms.

In the following result we characterize pseudoumbilical Lagrangian immersions in complex space forms (see Definition 1).

Theorem 1. Let $\phi : M^n \rightarrow \overline{M}^n(c)$ be a Lagrangian immersion of a connected manifold $M$ and $\exp : T\overline{M} \rightarrow \overline{M}$ the exponential map of $\overline{M}$.

(i) $\phi$ is elliptic pseudoumbilical if and only if there exist a point $C \in \overline{M}$, a vector field (without zeros) $X$ on $M$ and a nontrivial smooth function $h : M \rightarrow \mathbb{C}$ satisfying $\frac{c}{4}|X|^2 + |h|^2 = 1$, such that for any point outside the zeros of $h$, $p \in M - Z(h)$, the geodesic

$$\beta_p(s) = \exp(\phi(p), sf(p)h(p)X_p)$$
with

\[
f(p) = \begin{cases} 
1 & \text{when } c = 0 \\
\arccos \frac{|h|}{|hX|}(p) & \text{when } c = 4 \\
\cosh^{-1} \frac{|h|}{|hX|}(p) & \text{when } c = -4 
\end{cases}
\]

pass through the point \( C \) at \( s = 1 \). This point \( C \) of \( \overline{M} \) will be called the center of \( \phi \).

(ii) \( \phi : M \to \mathbb{CH}^n \) is hyperbolic pseudoumbilical if and only if there exist a complex hyperplane \( \mathbb{CH}^{n-1} \), a vector field \( X \) on \( M \) and a nontrivial smooth function \( h : M \to \mathbb{C} \) satisfying \( |h|^2 - |X|^2 = -1 \), such that for any point outside the zeros of \( h \), \( p \in M - Z(h) \), the geodesic

\[
\beta_p(s) = \exp(\phi(p), s f(p)h(p)X_p)
\]

with \( f(p) = \frac{\cosh^{-1}|X|}{|hX|}(p) \) cuts orthogonally to \( \mathbb{CH}^{n-1} \) at \( s = 1 \).

(iii) \( \phi : M \to \mathbb{CH}^n \) is parabolic pseudoumbilical if and only if there exist a vector field (without zeros) \( X \) on \( M \) and a smooth function \( h : M \to \mathbb{C} \) satisfying \( |h|^2 - |X|^2 = 0 \) such that the map

\[
p \in M \mapsto [\beta_p] \in S^{2n-1}(\infty),
\]

with \( \beta_p \) the geodesic ray \( \beta_p(s) = \exp(\phi(p), s h(p)X_p) \), \( s \geq 0 \), is a constant \( C \in S^{2n-1}(\infty) \) and the horosphere with infinity point \( C \), where \( \phi(p) \) lies in, has radius \( \delta|X_p| \), with \( \delta \) a positive real number.

Remark 3. If \( \phi : M \to \overline{M}^n(c) \) is an elliptic pseudoumbilical Lagrangian immersion and \( d \) is the distance on \( \overline{M}^n(c) \), then

\[
d(C, \phi) = \begin{cases} 
|X| & \text{when } c = 0 \\
\arccos |h| & \text{when } c = 4 \\
\cosh^{-1} |h| & \text{when } c = -4 
\end{cases}
\]

So, if \( N \) is the leaf of the foliation \( D \) passing through a point \( p \in M - Z(h) \), then \( \phi(N) \) lies on the geodesic sphere of \( \overline{M}^n(c) \) centered at \( C \) and radius \( d(C, \phi(p)) \).

When \( c = 0 \) or \( c = -4 \), \( Z(h) = \emptyset \). When \( c = 4 \), \( h \) can have zeros. By continuity, \( Z(h) \) is given by

\[
Z(h) = \{ p \in M : d(\phi(p), C) = \pi/2 \},
\]

i.e., \( \phi(Z(h)) \) is the intersection of \( \phi(M) \) with the cut locus of the point \( C \).
Also, if $\overline{M}^n = \mathbb{C}^n$, $\phi$ is given by $\phi = C + hX$ with $|h| = 1$. In this case, if $G : \mathbb{C}^n - \{C\} \rightarrow \mathbb{C}^n - \{C\}$ is the inversion centered at the point $C$, then $G \circ \phi$ is also an elliptic pseudoumbilical Lagrangian immersion with center $C$. The corresponding closed and conformal field is $X/|X|^2$.

**Remark 4.** If $\phi : M \rightarrow \mathbb{CH}^n$ is a hyperbolic pseudoumbilical Lagrangian immersion and $d$ is the distance on $\mathbb{CH}^n$, then

$$d(\mathbb{CH}^{n-1}, \phi) = \cosh^{-1} |X|.$$ 

So, if $N$ is the leaf of the foliation $\mathcal{D}$ passing through a point $p \in M$, then $d(\mathbb{CH}^{n-1}, \phi(N)) = \cosh^{-1} |X_p|$, which means that $\phi(N)$ lies on the tube over $\mathbb{CH}^{n-1}$ of radius $\cosh^{-1} |X_p|$. Also

$$Z(h) = \{p \in M : \phi(p) \in \mathbb{CH}^{n-1}\}.$$ 

Finally, if $\phi : M \rightarrow \mathbb{CH}^n$ is a parabolic pseudoumbilical Lagrangian immersion, and $N$ the leaf of the foliation $\mathcal{D}$ passing through a point $p \in M$, then $\phi(N)$ lies on the (2n-1)-dimensional horosphere of $\mathbb{CH}^n$ with infinity point $C$ and radius $\delta |X_p|$.

These $\eta$-umbilic hypersurfaces of $\mathbb{CH}^n$ carry a contact structure and the leaves of the foliation $\mathcal{D}$ of our pseudoumbilical Lagrangian submanifolds are integral submanifolds of maximal dimension of such structure (see [B] for details).

**Proof.** Let $\phi : M \rightarrow \overline{M}^n(c)$ a Lagrangian immersion, $h : M \rightarrow \mathbb{C}$ a nontrivial smooth function, $X$ a vector field without zeros on $M$ and $f : M - Z(h) \rightarrow \mathbb{R}$ a smooth function. For each $p \in M - Z(h)$, $\beta_p(s)$ will denote the geodesic $\exp(\phi(p), sf(p)h(p)X_p)$.

Given a vector $v$ tangent to $M$ at $p$ and a curve $\alpha : (-\epsilon, \epsilon) \rightarrow M - Z(h)$ with $\alpha(0) = p$ and $\alpha'(0) = v$, we consider a variation $G : (-\epsilon, \epsilon) \times \mathbb{R} \rightarrow \overline{M}$ of the geodesic $\beta_p$ given by $G(t, s) = \gamma_t(s)$, where $\gamma_t$ is the geodesic on $\overline{M}$ with $\gamma_t(0) = \phi(\alpha(t))$ and $\gamma_t'(0) = (fhX)(\alpha(t))$. Then $K(s) = \frac{\partial G}{\partial t}(0, s)$ is a Jacobi field along the geodesic $\beta_p$.

To determine $K(s)$ we need to control its initial conditions $K(0)$ and $K'(0)$. It is clear that $K(0) = \phi_*(v) \equiv v$. Also

$$K'(0) = \frac{\partial^2 G}{\partial s \partial t}(0, 0) = \frac{\partial}{\partial t} (\frac{\partial G}{\partial s} (t, 0) = d(fh)_p(v)X_p + (fh)(p)\nabla_v X,$$

where $\nabla$ is the Levi-Civita connection on $\overline{M}$. 

As \( \beta_p'(0) = (fhX)(p) \), we decompose \( K(0) \) and \( K'(0) \) in the following way

\[
K(0) = \mathcal{R}(h) \langle X_p, v \rangle \beta_p'(0) - \frac{\mathcal{I}(h)}{f|hX|^2} \langle X_p, v \rangle J\beta_p'(0) + v^+, \\
K'(0) = \langle \nabla \log f|hX|, v \rangle \beta_p'(0) \\
+ \left( \frac{\langle \sigma(X_p, X_p), Jv \rangle}{|X|^2} + \frac{\mathcal{I}(v(h)\bar{h})}{|h|^2} \right) J\beta_p'(0) + fh(\nabla_v X)^+, 
\]

where \( \perp \) means the component orthogonal to the complex plane spanned by \( X_p \). \( \mathcal{R} \) (resp. \( \mathcal{I} \)) denotes real (resp. imaginary) part, and \( \nabla \) is the gradient of the induced metric. Now, taking into account (2.1), our Jacobi field \( K \) along \( \beta_p \) is given by

\[
K(s) = (\nu s + \mu)\beta_p'(s) + K_1(s) + \hat{K}(s),
\]

being

\[
\nu = \langle \nabla \log f|hX|, v \rangle, \quad \mu = \frac{\mathcal{R}(h)\langle v, X \rangle}{f|hX|^2}
\]

and \( K_1(s) \) and \( \hat{K}(s) \) the Jacobi fields along \( \beta_p \) given by

\[
K_1(s) = \begin{cases} 
A_1(s) + sB_1(s) & \text{when } c = 0 \\
\cos(2\sqrt{\lambda}s)A_1(s) + \frac{\sin(2\sqrt{\lambda}s)}{2\sqrt{\lambda}}B_1(s) & \text{when } c = 4 \\
\cosh(2\sqrt{-\lambda}s)A_1(s) + \frac{\sinh(2\sqrt{-\lambda}s)}{2\sqrt{-\lambda}}B_1(s) & \text{when } c = -4,
\end{cases}
\]

\[
\hat{K}(s) = \begin{cases} 
\hat{A}(s) + s\hat{B}(s) & \text{when } c = 0 \\
\cos(\sqrt{\lambda}s)\hat{A}(s) + \frac{\sin(\sqrt{\lambda}s)}{\sqrt{\lambda}}\hat{B}(s) & \text{when } c = 4 \\
\cosh(\sqrt{-\lambda}s)\hat{A}(s) + \frac{\sinh(\sqrt{-\lambda}s)}{\sqrt{-\lambda}}\hat{B}(s) & \text{when } c = -4,
\end{cases}
\]

where \( \lambda = \frac{c^2}{4}(f|hX|^2)(p) \) and \( A_1(s), B_1(s), \hat{A}(s), \hat{B}(s) \) are, respectively, the parallel fields along \( \beta_p(s) \) with

\[
A_1(0) = -\frac{\mathcal{I}(h)}{f|hX|^2} \langle X_p, v \rangle J\beta_p'(0), \\
B_1(0) = \left( \frac{\langle \sigma(X_p, X_p), Jv \rangle}{|X|^2} + \frac{\mathcal{I}(v(h)\bar{h})}{|h|^2} \right) J\beta_p'(0), \\
\hat{A}(0) = v^+, \quad \hat{B}(0) = fh(\nabla_v X)^+.
\]
We remark that $K_1(s)$ is a Jacobi field colinear with $J\beta'_{p}(s)$ and $\hat{K}(s)$ is a Jacobi field orthogonal to $\beta'_{p}(s)$ and $J\beta'_{p}(s)$.

**Proof of (i).** First, we suppose that $\phi$ is elliptic pseudounmbilical. Then (see Definition 1 and Proposition 1) there exists a closed and conformal vector field $X$ (without zeros) such that $\sigma(X,X) = \rho JX$ and $\sigma(X,V) = bJV$ for any field $V$ orthogonal to $X$. Defining $h$ by

$$h = -\frac{\text{div}X}{n} + ib,$$

$X$ can be chosen (see Remark 1) in such a way that $|h|^2 + \frac{c}{4}|X|^2 = 1$. If $h \equiv 0$, then $X$ is a parallel vector field and hence $\text{Ric}(X) = 0$. As $b$ also vanishes, (1.2) says that $c = 0$, which contradicts the fact that $|h|^2 + \frac{c}{4}|X|^2 = 1$. So $h$ is nontrivial. In fact the set of points where $h$ does not vanish is dense in $M$.

Now we consider the function $F : M - \mathcal{Z}(h) \rightarrow \overline{M}$ given by

$$F(p) = \beta_p(1) = \exp(\phi(p), f(p)h(p)X_p).$$

We are going to compute $dF_p(v)$ for any $p \in M - \mathcal{Z}(h)$ and any $v \in T_pM$. It is clear that $dF_p(v) = K(1)$, being $\hat{K}(s)$ the Jacobi field associated to $v$ (see the beginning of the proof).

From the definition of $f(p)$ and the fact that $|h|^2 = 1 - \frac{c}{4}|X|^2$ we can directly get

$$\sin \sqrt{\lambda} = |X|, \quad \cos \sqrt{\lambda} = |h|, \quad \text{when } c = 4$$

$$\sinh \sqrt{-\lambda} = |X|, \quad \cosh \sqrt{-\lambda} = |h|, \quad \text{when } c = -4.$$  

From Lemma 1, Proposition 1, (1.2), (1.5) and the expression of $f$, we get that $\mu + \nu = 0$. Also, from Lemma 1, Proposition 1, (1.5) and an easy computation, it follows that

$$B_1(0) = b \left( \frac{1}{|X|^2} - \frac{c}{4|h|^2} \right) (X_p,v)J\beta'_{p}(0),$$

which allows to prove, using (2.2), that $K_1(1) = 0$.

Finally, from Lemma 1 we have that $\hat{B}(0) = -f|h|^2v \perp$, which implies using again (2.2) that $\hat{K}(1) = 0$. So we have got that

$$dF_p(v) = K(1) = 0,$$

for any tangent vector $v \in T_pM$ and any $p \in M - \mathcal{Z}(h)$.

If $c = 0$ or $c = -4$, $\mathcal{Z}(h) = \emptyset$ and then $F : M \rightarrow \overline{M}$ is a constant function $C$.

If $c = 4$, let $M_1$ and $M_2$ be two different connected components of $M - \mathcal{Z}(h)$ with $\overline{M}_1 \cap \overline{M}_2 \neq \emptyset$. Then $F$ is constant on each $M_i$, i.e., $F(M_i) = C_i$ for $i = 1, 2$. Let $p \in \mathcal{Z}(h) \cap \overline{M}_1 \cap \overline{M}_2$. From Lemma 1, (v) we can parameterize $M$ around $p$ like $(-\delta, \delta) \times \mathbb{R}^{n-1}$ such that $\phi(0, x_0) = p$. In this
neighborhood, \( h \) is a function of \( t \in (-\delta, \delta) \) (see Lemma 1 again), and so \( \{0\} \times N \subset \mathcal{Z}(h) \). So 
\[
\phi_0 : N \rightarrow \mathbb{CP}^n 
\]
defined by \( \phi_0(x) = \phi(0, x) \) is a Lagrangian immersion lying in the cut locus of \( C_1 \) and \( C_2 \). If \( C_1 \neq C_2 \), then the image of \( \phi_0 \) lies in the intersection of both, which is a linear \( (n - 2) \)-subspace of \( \mathbb{CP}^n \). This is impossible because \( \phi_0 \) is Lagrangian and \( N \) is \( (n - 1) \)-dimensional. So we have obtained that \( C_1 = C_2 = C \) and the necessary condition is proved.

Conversely, suppose that there exist a tangent vector field \( X \) (without zeros) and a nontrivial function \( h : M \rightarrow \mathbb{C} \) such that \( F(p) = \beta_p(1) = \exp(\phi(p), f(p)h(p)X_p) : M - \mathcal{Z}(h) \rightarrow M \) is a constant function. Then \( dF_p(v) = 0 \) for any \( p \in M - \mathcal{Z}(h) \) and for any \( v \in T_pM \). So the corresponding Jacobi field \( K(s) \) along the geodesic \( \beta_p \) satisfies \( K(1) = dF_p(v) = 0 \). Looking at the expression of \( K(s) \), this means that
\[
\mu + \nu = 0, \quad K_1(1) = 0 \quad \text{and} \quad \dot{K}(1) = 0.
\]
But \( \mu + \nu = 0 \) implies, taking into account the expression of \( f \), that
\[
(2.3) \quad \nabla|X|^2 = -2\mathcal{R}(h)X.
\]
Also, from \( \dot{K}(1) = 0 \) and using \( (2.2) \), we obtain that
\[
|h|^2v^\perp + h(\nabla_vX)^\perp = 0.
\]
Now by decomposing the above equation in tangential and normal components to \( \phi \) we get
\[
(2.4) \quad (\nabla_vX)^\perp = -\mathcal{R}(h)v^\perp, \quad \sigma(X_p,v)^\perp = I(h)Jv^\perp.
\]
Now \( (2.3) \) and the first part of Equation \( (2.4) \) say that
\[
\nabla_vX = -\mathcal{R}(h)v,
\]
for any \( v \in T_pM \), and for any \( p \in M - \mathcal{Z}(h) \). This means that \( X \) is a closed and conformal vector field on \( M - \mathcal{Z}(h) \) with \( \text{div}(X) = -n\mathcal{R}(h) \).

Also, taking \( v = X_p \) in the second part of Equation \( (2.4) \) we obtain that
\[
\sigma(X,X) = \rho JX \quad \text{for certain function } \rho.
\]
Finally, taking \( v \) orthogonal to \( X_p \) in the same Equation \( (2.4) \), we get \( \sigma(v,X_p) = I(h)Jv \). So (see Proposition 1) \( \phi : M - \mathcal{Z}(h) \rightarrow M \) is an elliptic pseudoumbilical Lagrangian immersion.

If \( c = 0, -4 \) the proof is finished. If \( c = 4, \phi(\mathcal{Z}(h)) \) is contained in the cut locus of the center \( C \), which is a linear \( (n - 1) \)-subspace of \( \mathbb{CP}^n \). As \( \phi \) is a Lagrangian immersion, \( \mathcal{Z}(h) \) has no interior points and \( M - \mathcal{Z}(h) \) is dense on \( M \). So the whole \( \phi \) is an elliptic pseudoumbilical immersion.

**Proof of (ii).** First, we suppose that \( \phi : M \rightarrow \mathbb{CP}^n \) is hyperbolic pseudoumbilical. Then, from Proposition 1 and Definition 1, there exists a closed
and conformal vector field \( X \) (without zeros) such that \( \sigma(X,X) = \rho JX \) and \( \sigma(X,V) = bJV \) for any \( V \) orthogonal to \( X \). Defining \( h \) by

\[
h = -\frac{\text{div} X}{n} + ib,
\]

\( X \) can be chosen (see Remark 1) in such a way that \(|h|^2 - |X|^2 = -1\). If \( h \equiv 0 \), then \( X \) is parallel and so \( \text{Ric}(X) = 0 \). As \( b = 0 \), (1.2) gives a contradiction. So \( h \) is nontrivial.

In this case, we consider \( F : M - Z(h) \rightarrow \mathbb{C}^{\mathbb{H}^n} \) given by

\[
F(p) = \beta_p(1) = \exp(\phi(p), f(p)h(p)X_p).
\]

Following the proof of (i), we can prove in this case that \( \mu + \nu = 0 \) and \( K_1(1) = 0 \) and hence

\[
dF_p(v) = \tilde{K}(1),
\]

which means that \( dF_p(v) \) is orthogonal to the complex plane spanned by \( \beta_p'(1) \). Also, using the expressions of \( \tilde{K}(s) \) and \( f \), it is easy to check that \( \tilde{K}'(1) = 0 \), and then \( \{\beta_p'(1) : p \in M - Z(h)\} \) spans a 1-complex dimensional parallel subbundle on \( \mathbb{C}^{\mathbb{H}^n} \) along \( M - Z(h) \). So, for each connected component \( C_i \) of \( M - Z(h) \) there exists a complex hyperplane \( \mathbb{C}^{\mathbb{H}^n}_{i-1} \) of \( \mathbb{C}^{\mathbb{H}^n} \) such that \( F(C_i) \subset \mathbb{C}^{\mathbb{H}^n}_{i-1} \). It is clear that for any \( p \in C_i \), \( \beta_p(s) \) cuts orthogonally to \( \mathbb{C}^{\mathbb{H}^n}_{i-1} \) at \( s = 1 \).

Following a similar reasoning as in (i), we can prove that \( M - Z(h) \) is connected and the proof of the necessary condition is finished.

Conversely, suppose that there exist a vector field \( X \) without zeros on \( M \), a nontrivial smooth function \( h : M \rightarrow \mathbb{C} \) satisfying \(-|X|^2 + |h|^2 = -1\) and a linear hyperplane \( \mathbb{C}^{\mathbb{H}^n}_{i-1} \) such that the geodesic \( \beta_p(s) = \exp(\phi(p), f|hx|(p)) \) cuts orthogonally to \( \mathbb{C}^{\mathbb{H}^n}_{i-1} \) at \( s = 1 \). We follow again the proof of (i), and then, if \( K \) is the corresponding Jacobi field along \( \beta_p \), \( dF_p(v) = K(1) \) is tangent to \( \mathbb{C}^{\mathbb{H}^n}_{i-1} \). Also, \( \beta_p'(1) \) spans the normal space to \( \mathbb{C}^{\mathbb{H}^n}_{i-1} \) and hence \( K'(1) \) is a normal vector to \( \mathbb{C}^{\mathbb{H}^n}_{i-1} \) at \( \beta_p(1) \). This means that \( K'(1) = 0 \). From these two facts, we obtain using the expression of \( \tilde{K}(s) \) that

\[
\mu + \nu = 0, \quad K_1(1) = 0, \quad \tilde{K}'(1) = 0.
\]

Using the above like in the proof of (i), we can prove that \( X \) is a closed and conformal vector field such that \( \sigma(X,X) = \rho X \) and \( \sigma(X,V) = JXV \) for any \( V \) orthogonal to \( X \). So \( \phi : M - Z(h) \rightarrow \mathbb{C}^{\mathbb{H}^n} \) is a hyperbolic pseudoumbilical Lagrangian immersion. By continuity, \( \phi(Z(h)) \subset \mathbb{C}^{\mathbb{H}^n}_{i-1} \), and since \( \phi \) is Lagrangian, \( Z(h) \) has no interior points, and \( M - Z(h) \) is dense in \( M \). This means that the whole \( \phi \) is hyperbolic pseudoumbilical.

**Proof of (iii).** We denote by \( F : M \rightarrow \mathbb{S}^{2n-1}(\infty) \) the map \( F(p) = [\beta_p] \). Let \( \phi \) be a local horizontal lift of \( \phi \) to \( \mathbb{H}^{2n+1}_{\mathbb{T}} \). Using the notation of Paragraph 1,
\[ \beta_p = \Pi(\widetilde{\beta}_p), \text{ with } \widetilde{\beta}_p(s) = \cosh(s|X_p|^2)\phi(p) + \sinh(s|X_p|^2)h(p)X_p/|X_p|^2, \]

where we have identified \( X \) and \( \phi \). From the identification \( S^{2n-1}(\infty) \equiv \pi(N) \), we have that \( F \equiv \pi \circ \phi, \) with \( \phi = \phi + hX/|X|^2 \).

Suppose now that \( \phi \) is parabolic pseudoumbilical. Then, from Proposition 1 and Definition 1 there exists a closed and conformal vector field \( X \) (without zeros) such that \( \sigma(X,X) = \rho JX \) and \( \sigma(X,V) = bJV \) for any \( V \) orthogonal to \( X \). We define

\[ h = - \text{div}X \]

Then Proposition 1 and Remark 1 say that we can take \( X \) in order to \( |h|^2 - |X|^2 = 0 \). For any tangent vector \( v \in T_pM \), using Lemma 1, (1.2) and (1.5), we obtain \( v(\phi) = \frac{h}{|X|^2} \langle v, X \rangle \). This means that \( dF_p(v) = 0 \) and from the connection of \( M \) this implies that \( F \) is a constant function \( C \in S^{2n-1}(\infty) \).

Following the notation of Paragraph 1, we can take \( \tilde{C} = \phi \). Using once more Lemma 1, (1.2) and (1.5), it is straightforward to prove that

\[ v \left( \frac{|(\phi, \tilde{C})|^2}{|X|^2|\tilde{C}|_{n+1}^2} \right) = 0. \]

This finishes the proof of the necessary condition.

Conversely, if \( v \) is any tangent vector to \( M \), then \( dF_p(v) = 0 \), which implies that

\[ v(\phi) = \frac{h}{|X|^2} \langle v, X \rangle \phi. \]

On the other hand, since \( \phi(p) \) lies in the horosphere with infinity point \( C \) and radius \( \delta|X_p| \), we deduce that \( v(|\phi|_{n+1}^2|X|^2) = 0 \). Using (2.5) in this equation we get \( \nabla|X|^2 = -2R(h,X) \). Using this information in (2.5) and following a similar reasoning like in the proof of (i), we arrive at \( \phi \) is parabolic pseudoumbilical.

Theorem 1 shows that elliptic pseudoumbilical Lagrangian immersions have a center and that they can be constructed using the exponential map of the ambient space. This fact will imply that they are invariant under certain transformations of the ambient space, what allows us construct them from the classification given by A. Ros and the third author for complex Euclidean space.

\textbf{Proposition 2.} (i) Let \( \phi : M^n \to \mathbb{C}^n \) be an immersion, \( \langle \cdot, \cdot \rangle \) the Euclidean metric on \( \mathbb{C}^n \) and \( g \) the Fubini-Study metric on \( \mathbb{C}^n \). Then \( \phi \) is
pseudoumbilical Lagrangian with center 0 in \((\mathbb{C}^n, g)\) if and only if \(\phi\) is ellipic pseudoumbilical Lagrangian with center 0 in \((\mathbb{C}^n, \langle \cdot, \cdot \rangle)\).

(ii) Let \(\phi : M^n \rightarrow \mathbb{B}^n \subset \mathbb{C}^n\) be an immersion and \(g\) the Bergmann metric on \(\mathbb{B}^n\). Then \(\phi\) is elliptic pseudoumbilical Lagrangian with center 0 in \((\mathbb{B}^n, g)\) if and only if \(\phi\) is elliptic pseudoumbilical Lagrangian with center 0 in \((\mathbb{B}^n, \langle \cdot, \cdot \rangle)\).

**Remark 5.** Proposition 2 is not true when the center of the immersion is a point \(C \in \mathbb{C}^n\) different from 0.

**Proof of (i).** From the definition of \(g\), if \(\Omega^g\) and \(\Omega\) denote the Kaehler two-forms on \((\mathbb{C}^n, g)\) and \((\mathbb{C}^n, \langle \cdot, \cdot \rangle)\) respectively, then

\[
\phi^* \Omega^g = \frac{1}{1 + |p|^2} \left\{ \phi^* \Omega - \frac{\phi^* (\Im(\alpha \wedge \bar{\alpha}))}{1 + |p|^2} \right\}.
\]

(2.6)

Also, the expressions of the exponential maps at \(0 \in \mathbb{C}^n\) with respect to the metrics \(\langle \cdot, \cdot \rangle\) and \(g\) are given respectively by

\[
\exp(0, v) = v, \quad \exp^g(0, v) = \cos(|v|) \frac{v}{|v|},
\]

and it is not difficult to see that the parallel transport along the geodesic \(\exp^g(0, tv)\) is (like in the Euclidean case) the identity.

First, we suppose that \(\phi\) is an elliptic pseudoumbilical Lagrangian immersion with respect to \((\mathbb{C}^n, \langle \cdot, \cdot \rangle)\). From Theorem 1 we have that

\[
\phi = hX,
\]

with \(|h|^2 = 1\).

Then, \((\phi^* \alpha)(v) = \bar{\alpha}(v, X_p)\). So from (2.6) we have that \(\phi^* \Omega^g = 0\), which means that \(\phi\) is a Lagrangian immersion in \((\mathbb{C}^n, g)\).

On the other hand, our immersion \(\phi = hX\) can be rewritten as

\[
\phi(p) = \exp^g(0, f(p)(\hat{h}\hat{X})(p)),
\]

with \(\hat{h} = \frac{h}{\sqrt{1 + |X|^2}}, \quad \hat{X} = \sqrt{1 + |X|^2}X\) and \(f(p)\) the corresponding function given in Theorem 1. Although \((\mathbb{C}^n, g)\) is not complete, it is easy to check that Theorem 1 can be used here in order to say that \(\phi\) is a pseudoumbilical Lagrangian immersion in \((\mathbb{C}^n, g)\) with \(\mathcal{Z}(h) = \emptyset\).

Conversely, we suppose that \(\phi\) is a pseudoumbilical Lagrangian immersion with respect to \((\mathbb{C}^n, g)\). From Theorem 1 and the fact that the cut locus of 0 in \(\mathbb{C}^n\) is empty we have that

\[
\phi = \exp^g(0, fhX),
\]

with \(g(X, X) + |h|^2 = 1, \quad f = \frac{\arccos|h|}{|h|\sqrt{g(X, X)}}\) and \(\mathcal{Z}(h) = \emptyset\). So using the expression of \(\exp^g(0, -)\) we have that

\[
\phi = hX.
\]
So \((\phi^* \alpha)(v) = \frac{\tilde{h}}{\sqrt{g(X,X)}} (v, X_p)\) and from (2.6) we have that \(\phi^* \Omega = 0\), which means that \(\phi\) is a Lagrangian immersion in \((\mathbb{C}^n, \langle \cdot, \cdot \rangle)\).

Now, our immersion \(\phi\) can be rewritten as \(\phi = \exp(0, \tilde{h} \tilde{X})\) with \(\tilde{h} = \frac{h}{|h|}\) and \(\tilde{X} = |h|X\). This means, using Theorem 1 again, that \(\phi\) is an elliptic pseudoumbilical Lagrangian immersion in \((\mathbb{C}^n, \langle \cdot, \cdot \rangle)\). This finishes the proof of (i).

To prove (ii) we follow the proof of (i) taking into account that in this case \(\exp^g(0, v) = \cosh(|v|) \sinh(|v|) \frac{v}{|v|}\), and the parallel transport along the geodesic \(\exp^g(0, tv)\) is the identity. \(\square\)

Proposition 2 allows us, using the results got by A. Ros and the third author for the case of complex Euclidean space, to classify the elliptic pseudoumbilical Lagrangian submanifolds of \(\mathbb{CP}^n\) and \(\mathbb{CH}^n\). In fact, we start recalling the classification for complex Euclidean space.

Theorem ([RU]). Let \(\phi : M^n \to \mathbb{C}^n\) be an elliptic pseudoumbilical Lagrangian immersion with center \(C\). Then locally \(M^n\) is a product \(I \times N\), where \(I\) is an interval of \(\mathbb{R}\) and \(N\) a simply-connected manifold, and \(\phi\) is given by
\[
\phi(t, x) - C = \alpha(t) \tilde{\psi}(x),
\]
where \(\tilde{\psi} : N \to S^{2n-1} \subset \mathbb{C}^n\) is a horizontal lift to \(S^{2n-1}\) of a Lagrangian immersion \(\psi : N \to \mathbb{CP}^{n-1}\) and \(\alpha : I \to \mathbb{C}^*\) a regular curve.

The proof of the statements given in the following families of examples is straightforward and so will be omitted.

Examples 1. Let \(\alpha : I \to \mathbb{CP}^1 - \{\Pi(0,1)\}\) be a regular curve, \(\psi : N \to \mathbb{CP}^{n-1}\) a Lagrangian immersion of an \((n-1)\)-dimensional simply-connected manifold \(N\), and \(\tilde{\alpha} = (\tilde{\alpha}_1, \tilde{\alpha}_2) : I \to S^3 \subset \mathbb{C}^2\) and \(\tilde{\psi} : N \to S^{2n-1} \subset \mathbb{C}^n\) horizontal lifts of \(\alpha\) and \(\psi\) respectively. Then,
\[
\alpha \ast \psi : I \times N \to \mathbb{CP}^n
\]
given by
\[
(\alpha \ast \psi)(t, x) = \Pi(\tilde{\alpha}_1(t) \tilde{\psi}(x); \tilde{\alpha}_2(t)),
\]
is a pseudoumbilical Lagrangian immersion with center \(C = \Pi(0,\ldots,0,1)\).

Remark 6. Looking at Remark 3, it is clear that this immersion \(\alpha \ast \psi\) verifies that
\[
Z(h) = \{(t, x) \in I \times N : \alpha(t) = \Pi(1,0)\}.
\]
Examples 2. Let \( \alpha : I \rightarrow \mathbb{C}^1 \setminus \{\Pi(0,1)\} \) be a regular curve, \( \psi : N \rightarrow \mathbb{C}^{n-1} \) a Lagrangian immersion of an \((n-1)\)-dimensional simply-connected manifold \( N \), and \( \tilde{\alpha} = (\tilde{\alpha}_1, \tilde{\alpha}_2) : I \rightarrow \mathbb{H}_3^{1} \subset \mathbb{C}^2 \) and \( \tilde{\psi} : N \rightarrow \mathbb{S}^{2n-1} \subset \mathbb{C}^n \) horizontal lifts of \( \alpha \) and \( \psi \) respectively. Then,

\[
\alpha \ast \psi : I \times N \rightarrow \mathbb{C}^n
\]
given by

\[
(\alpha \ast \psi)(t, x) = \Pi(\tilde{\alpha}_1(t)\tilde{\psi}(x); \tilde{\alpha}_2(t)),
\]
is an elliptic pseudoumbilical Lagrangian immersion with center \( C = \Pi(0, \ldots, 0, 1) \).

Remark 7. The immersions \( \alpha \ast \psi \) given in Examples 1 and 2 do not depend on the horizontal lift of the curve \( \alpha \) and, up to holomorphic isometries of the ambient space, do not depend on the horizontal lift of \( \psi \).

Corollary 1. Let \( \phi : M^n \rightarrow \mathbb{CP}^n \) (respectively \( \phi : M^n \rightarrow \mathbb{CH}^n \)) be an elliptic pseudoumbilical Lagrangian immersion. Then \( \phi \) is locally congruent to some of the immersions \( \alpha \ast \psi \) described in Examples 1 (respectively in Examples 2).

Proof. We start by proving the projective case. From Theorem 1, \( \phi \) has a center \( C \in \mathbb{CP}^n \), and there is no restriction if we take it as \( C = \Pi(0, \ldots, 0, 1) \). Let

\[
\mathbb{CP}^{n-1} = \{ \Pi(z_1, \ldots, z_{n+1}) \in \mathbb{CP}^n : z_{n+1} = 0 \}
\]
be the cut locus of the point \( C \). Then

\[
F : \mathbb{C}^n \rightarrow \mathbb{CP}^n - \mathbb{CP}^{n-1},
\]
given by

\[
F(p) = \Pi \left( \frac{1}{\sqrt{1 + |p|^2}} (p, 1) \right),
\]
is a diffeomorphism with \( F(0) = C \) and \( F^* \langle , \rangle = g \), where \( \langle , \rangle \) (respectively \( g \)) denotes the Fubini-Study metric on \( \mathbb{CP}^n \) (respectively on \( \mathbb{C}^n \)).

From Remark 2 and Proposition 2 it is easy to see that

\[
F^{-1} \circ \phi : M - B \rightarrow \mathbb{C}^n,
\]
with \( B = \{ p \in M : \phi(p) \in \mathbb{CP}^{n-1} \} \) is an elliptic pseudoumbilical Lagrangian immersion in \((\mathbb{C}^n, \langle , \rangle)\) with center 0, which means, using Theorem [RU], that locally

\[
F^{-1} \circ \phi = \beta \tilde{\psi}
\]
with $\beta : I \to \mathbb{C}^*$ a regular curve and $\tilde{\psi}$ the horizontal lift to $\mathbb{S}^{2n-1}$ of a Lagrangian immersion $\psi : N \to \mathbb{C}P^{n-1}$, being $N$ an $(n-1)$-dimensional simply-connected manifold. So, $\phi : M - B \to \mathbb{C}P^n - \mathbb{C}P^{n-1}$ is given by

$$\phi = \Pi \left( \frac{1}{\sqrt{1 + |\beta|^2}} (\beta \tilde{\psi}, 1) \right) = \Pi(\tilde{\alpha}_1 \tilde{\psi}, \tilde{\alpha}_2),$$

where

$$(\tilde{\alpha}_1, \tilde{\alpha}_2) = \frac{e^{i\theta}}{\sqrt{1 + |\beta|^2}} (\beta, 1),$$

being

$$\theta(t) = -\int_{t_0}^{t} \frac{\langle \beta', J\beta \rangle}{1 + |\beta|^2} \, dr$$

with $t_0$ a point of the interval $I$. From (2.7) it is easy to see that $\tilde{\alpha} = (\tilde{\alpha}_1, \tilde{\alpha}_2)$ is a horizontal curve in $\mathbb{S}^3$ and so it is the horizontal lift to $\mathbb{S}^3$ of the regular curve $\alpha = \Pi \circ \tilde{\alpha}$ in $\mathbb{C}P^1$. This finishes the proof of the projective case.

To prove the hyperbolic case, we proceed in a similar way taking into account that the center $C$ of $\phi$ can be taken as $C = \Pi(0, \ldots, 0, 1)$ and that $F : \mathbb{B}^n \to \mathbb{C}H^n$ given by

$$F(p) = \Pi \left( \frac{1}{\sqrt{1 - |p|^2}} (p, 1) \right)$$

is a diffeomorphism with $F(0) = C$ and $F^* (\cdot) = g$, being $\langle \cdot, \cdot \rangle$ (respectively $g$) the metric of $\mathbb{C}H^n$ (respectively the Bergmann metric on $\mathbb{B}^n$).

To finish the description of these submanifolds in $\mathbb{C}H^n$, it remains the hyperbolic and parabolic cases. As always, we start describing examples whose assertions will not be proved and will take again into account Remark 7.

**Examples 3.** Let $\alpha : I \to \mathbb{C}H^1$ be a regular curve, $\psi : N \to \mathbb{C}H^{n-1}$ a Lagrangian immersion of an $(n-1)$-dimensional simply-connected manifold $N$, and $\tilde{\alpha} = (\tilde{\alpha}_1, \tilde{\alpha}_2) : I \to \mathbb{H}_1^3 \subset \mathbb{C}^2$ and $\tilde{\psi} : N \to \mathbb{H}_1^{2n-1} \subset \mathbb{C}^n$ horizontal lifts of $\alpha$ and $\psi$ respectively. Then

$$\alpha * \psi : I \times N \to \mathbb{C}H^n$$

given by

$$(\alpha * \psi)(t, x) = \Pi(\tilde{\alpha}_1(t), \tilde{\alpha}_2(t) \tilde{\psi}(x)),$$

is a hyperbolic pseudoumbilical Lagrangian immersion.

To describe the parabolic pseudoumbilical Lagrangian examples in $\mathbb{C}H^n$, we need some previous remarks.
First, we will consider the null vectors $e_1 = \frac{1}{2}(0, \ldots, 0, 1, -1)$ and $e_2 = \frac{1}{2}(0, \ldots, 0, 1, 1)$, in such a way that $(e_1, e_2) = 1/2$ where $(,)$ is the Hermitian product in $\mathbb{C}^{n+1}$.

Second, if $\psi : N^{n-1} \rightarrow \mathbb{C}^{n-1}$ is a Lagrangian immersion of a simply-connected manifold $N$, then the complex 1-form $\alpha$ on $\mathbb{C}^{n-1}$ defined before as

$$\alpha_p(v) = \langle v, p \rangle + i \langle v, Jp \rangle$$

verifies that $\psi^* \alpha$ is a closed 1-form and so there exists a complex function $f^\psi : N \rightarrow \mathbb{C}$ such that $df^\psi = 2\psi^* \alpha$.

**Examples 4.** Let $\alpha : I \rightarrow \mathbb{C}^1$ be a regular curve, $\psi : N \rightarrow \mathbb{C}^{n-1}$ a Lagrangian immersion of an $(n-1)$-dimensional simply-connected manifold $N$, and $\tilde{\alpha} = \tilde{\alpha}_1 e_1 + \tilde{\alpha}_2 e_2 : I \rightarrow \mathbb{H}_1^1 \subset \mathbb{C}^2$ a horizontal lift of $\alpha$. Then

$$\alpha \ast \psi : I \times N \rightarrow \mathbb{C}^n$$

given by

$$(\alpha \ast \psi)(t, x) = \Pi(\tilde{\alpha}_1(t)\psi(x); \tilde{\alpha}_1(t)e_1 + (\tilde{\alpha}_2(t) - f^\psi(x)\tilde{\alpha}_1(t))e_2)$$

with $\Re(f^\psi) = |\psi|^2$, is a parabolic pseudoumbilical Lagrangian immersion.

**Theorem 2.** Let $\phi : M^n \rightarrow \mathbb{C}^n$ be a pseudoumbilical Lagrangian immersion. If $\phi$ is hyperbolic (respectively parabolic), then $\phi$ is locally congruent to some of the immersions described in Examples 3 (respectively in Examples 4).

**Proof.** We first consider a horizontal lift $\tilde{\phi} : U \rightarrow \mathbb{H}_1^{2n+1}$ of $\phi$, where $U$ can be identified (see Lemma 1, (v)) to $I \times N$, where $I$ is an open interval with $0 \in I$ and $N$ is a simply-connected $(n-1)$-manifold. In addition, following the same notation of the proof of Theorem 1, we can take $X = (\partial/\partial t, 0)$ and, by identifying $X$ and $\tilde{\phi}_s X$, the second fundamental form $\tilde{\sigma}$ of $\tilde{\phi}$ satisfies

$$\tilde{\sigma}(X, X) = \rho JX + |X|^2\tilde{\phi}, \quad \tilde{\sigma}(X, v) = bJv,$$

for any vector $v$ orthogonal to $X$.

Suppose now $\phi$ is hyperbolic pseudoumbilical. From Theorem 1(ii), up to holomorphic isometries, we can take the complex hyperplane $\mathbb{C}^{n-1}$ as

$$\mathbb{C}^{n-1} = \{ \Pi(z_1, \ldots, z_{n+1}) : z_1 = 0 \}.$$ 

The equation of a tube of radius $r$ over $\mathbb{C}^{n-1}$ is $|z_1| = \sinh r$ (cf. [M]). If $r = \cosh^{-1}|X_p|$ (see Remark 4), it becomes in

$$|\tilde{\phi}_1|^2 = |X|^2 - 1 = |h|^2.$$

Using Theorem 1 again, the geodesic

$$\beta_p(s) = \Pi \left( \cosh(s f|hX|(p))\tilde{\phi} + \sinh(s f|hX|(p)) \frac{(hX)(p)}{|hX|(p)} \right)$$
with \( f(p) = \frac{\cosh^{-1}|X_p|}{|h(p)|} \) cuts orthogonally to \( \mathbb{CH}^{n-1} \) at \( s = 1 \). In particular, 
\[ \beta_p(1) = \Pi\left(|X|\phi + \frac{hX}{|X|}\right) \in \mathbb{CH}^{n-1}, \] 
which means that the first component of \( \Upsilon = |X|^2\tilde{\phi} + hX \) is zero.

On the other hand, using (2.8) we obtain that \( v(hX + |h|^2\tilde{\phi}) = 0 \) and so \( \tilde{\alpha}_1 = -(hX + |h|^2\tilde{\phi}) \) is a function of \( t \), with \(|\tilde{\alpha}_1|^2 = |h|^2 \) and, from (2.8) again, satisfies \( \tilde{\alpha}'_1 = -\frac{|X|^2}{\tilde{\alpha}_1} \).

Since \(|X|^2 - |h|^2 = 1 \) and \( \langle \Upsilon, \Upsilon \rangle = -|X|^2 \), from above we can write 
\[ \tilde{\phi} = \tilde{\alpha}_1 + \Upsilon \] 
as 
\[ \tilde{\phi}(t, x) = (\tilde{\alpha}_1(t), \Upsilon(t, x)). \]

Using (2.8), we obtain that \( \Upsilon' = -\tilde{K}\Upsilon \), so that \( \alpha_x(t) = \Upsilon(t, x) = e^{-\int_0^t \tilde{n}(s) ds} \alpha_x(0) \) is a plane curve too. If we now put 
\[ \tilde{\psi}(x) = \frac{\alpha_x(0)}{\sqrt{-\langle \alpha_x(0), \alpha_x(0) \rangle}}, \]
we have that \( \langle \tilde{\psi}, \tilde{\psi} \rangle = -1 \). From the above definition we can write 
\[ \tilde{\phi}(t, x) = (\tilde{\alpha}_1(t), \tilde{\alpha}_2(t)\tilde{\psi}(x)), \]
with \( \tilde{\alpha}_2(t) = |X|(0)e^{-\int_0^t \tilde{n}(s) ds}. \)

Since \( \tilde{\phi} \) is horizontal, we deduce easily that \( \tilde{\psi} \) is a horizontal immersion in a certain \( \mathbb{H}_1^{n-1} \) and that \( \tilde{\alpha} = (\tilde{\alpha}_1, \tilde{\alpha}_2) \) is a horizontal curve in \( \mathbb{H}_1^3 \). Thus \( \phi : U \rightarrow \mathbb{CH}^n \) is the example \( \alpha * \psi \) of Examples 3 with \( \alpha = \Pi(\tilde{\alpha}) \) and \( \psi = \Pi(\tilde{\psi}) \).

In the parabolic case, from Theorem 1, (iii), we can take up to holomorphic isometries the infinity point \( C = \pi(C) \) with \( \tilde{C} = (0, \ldots, 0, 1, 1) \) and identify, following the notation of Paragraph 1, \( \tilde{C} \) with \( \phi = \tilde{\phi} + \frac{hX}{|X|^2} \).

Then (2.8) says that \( v(\varphi) = 0 \), for any vector \( v \) orthogonal to \( X \). In this way, \( \varphi = \varphi(t) \) is a null plane curve satisfying, from (2.8) again, \( X(\varphi) = h\varphi \).

We now put \( \tilde{\alpha}_1 = -|X|^2\varphi \) and it is easy to check that \( \tilde{\alpha}'_1 = -\tilde{K}\tilde{\alpha}_1 \).

Now we define 
\[ \psi = \frac{1}{\tilde{\alpha}_1}(\tilde{\phi} - \tilde{\alpha}_1 e_1 - 2(\tilde{\phi}, e_1)e_2), \]
where \( e_1 = \frac{1}{2}(0, \ldots, 0, 1, -1) \) and \( e_2 = \frac{1}{2}(0, \ldots, 0, 1, 1) \). It is clear that \( (\psi, e_1) = 0 \) and from the properties of \( \tilde{\alpha}_1 \) (or \( \varphi \)) and the choosing of \( e_1 \) and \( e_2 \) it is not difficult to get that \( (\psi, e_2) = 0 \) and \( X(\psi) = 0 \).

On the other hand, for any \( v \) orthogonal to \( X \), we have that \( (\tilde{\phi}_* v, e_2) = 0 \) and this implies 
\[ (\psi_* v, \psi_* w) = \frac{1}{|\tilde{\alpha}_1|^2}(\phi_* v, \phi_* w), \quad (\psi_* v, \psi) = -\frac{(\phi_* v, e_1)}{|\tilde{\alpha}_1|}. \]
Let us take $\tilde{\alpha}_2$ as a solution to $\tilde{\alpha}_2' + \bar{n}\tilde{\alpha}_2 = -e^{-\int \bar{n}/h}$ and define $f = \tilde{\alpha}_2 - 2(\tilde{\alpha}_2 e_1)$. We compute that $X(f) = 0$ and $v(f) = 2(\psi_1 v, \psi)$.

As a summary, we have shown that $\tilde{\phi} = \tilde{\alpha}_1 \psi + \tilde{\alpha}_1 e_1 + (\tilde{\alpha}_2 - f\tilde{\alpha}_1) e_2$ and thus $\phi : U \longrightarrow \mathbb{C}P^n$ is the example $\alpha * \psi$ of Examples 4 with $\alpha = \Pi(\tilde{\alpha})$. \qed

3. The degenerate case for nonflat complex space forms.

In this paragraph we will classify the family of Lagrangian submanifolds of $M^n(c)$ admitting a closed and conformal field $X$ with $\sigma(X, X) = \rho JX$ and satisfying condition (A) of Proposition 1 when $\overline{M}$ is a nonflat complex space form. We will refer to this as the degenerate case. In order to make self contained the paper, we start describing this family when $\overline{M}^n(c)$ is the complex Euclidean space $\mathbb{C}^n$; this description was obtained by A. Ros and the third author in [RU, Proposition 2].

**Proposition ([RU]).** Let $\phi : M^n \longrightarrow \mathbb{C}^n$ be Lagrangian immersion of a connected manifold $M$ endowed with a closed and conformal vector field $X$ (without zeros) such that $\sigma(X, X) = \rho JX$. If $\phi$ is degenerate (i.e., condition (A) in Proposition 1 is satisfied) then locally $M^n$ is a Riemannian product $M_1^{n_1} \times M_2^{n_2}$ with $n_1 \geq 2, n_2 \geq 1$, and $\phi$ is the product of two Lagrangian immersions $\phi_i : M_i^{n_i} \longrightarrow \mathbb{C}^n_i, i = 1, 2$, being $\phi_1$ a spherical immersion (i.e., the image of $\phi_1$ lies in a hypersphere of $\mathbb{C}^{n_1}$).

The assertions given in the following examples are not proved because they are straightforward.

**Examples 5.** Let $\phi_i : (N_i, g_i) \longrightarrow \mathbb{C}P^{n_i}, i = 1, 2$ be Lagrangian immersions of $n_i$-dimensional simply-connected manifolds $N_i$, $i = 1, 2$ and $\tilde{\phi}_i : N_i \longrightarrow S^{2n_i+1}$ horizontal lifts of $\phi_i, i = 1, 2$. Given a real number $\delta \in (0, \pi/4]$ and being $n = n_1 + n_2 + 1$,

$$\phi^\delta_{1,2} : \mathbb{R} \times N_1 \times N_2 \longrightarrow \mathbb{C}P^n$$

given by

$$\phi^\delta_{1,2}(t, p, q) = \Pi(\cos \delta e^{it\tan \delta \tilde{\phi}_1(p)}; \sin \delta e^{-it\cot \delta \tilde{\phi}_2(q)}),$$

is a Lagrangian immersion where $X = \left(\frac{2}{\sqrt{t}}, 0, 0\right)$ is a parallel field on $(\mathbb{R} \times N_1 \times N_2, dt^2 \times \cos^2 \delta g_1 \times \sin^2 \delta g_2)$ with $\sigma(X, X) = -2\cot 2\delta JX$. Moreover $\phi$ is degenerate with $b_1 = \tan \delta$ and $b_2 = -\cot \delta$.

**Examples 6.** Let $\phi_1 : (N_1, g_1) \longrightarrow \mathbb{C}P^{n_1}$ and $\phi_2 : (N_2, g_2) \longrightarrow \mathbb{C}P^{n_2}$ be Lagrangian immersions of $n_i$-dimensional simply-connected manifolds $N_i$, $i = 1, 2$ and $\tilde{\phi}_i : N_i \longrightarrow S^{2n_i+1}$ and $\tilde{\phi}_2 : N_2 \longrightarrow \mathbb{H}^{2n_2+1}$ horizontal lifts of $\phi_i, i = 1, 2$. Given a positive real number $\delta$ and being $n = n_1 + n_2 + 1$,

$$\phi^\delta_{1,2} : \mathbb{R} \times N_1 \times N_2 \longrightarrow \mathbb{C}P^n$$
given by
\[ \phi^\delta_{1,2}(t, p, q) = \Pi(\sinh \delta e^{it\cosh \phi^\delta_1(p)}; \cosh \delta e^{it\tanh \phi^\delta_2(q)}), \]
is a Lagrangian immersion where \( X = (\frac{\partial}{\partial t}, 0, 0) \) is a parallel field on \((\mathbb{R} \times N_1 \times N_2, dt^2 \times \sinh^2 \delta g_1 \times \cosh^2 \delta g_2)\) with \( \sigma(X, X) = 2 \coth 2\delta JX \). Moreover \( \phi \) is degenerate with \( b_1 = \coth \delta \) and \( b_2 = \tanh \delta \).

**Proposition 3.** Let \( \phi : M^n \rightarrow \mathbb{CP}^n \) (respectively \( \phi : M^n \rightarrow \mathbb{CH}^n \)) be a Lagrangian immersion of a connected manifold \( M \) endowed with a closed and conformal vector field \( X \) (without zeros) such that \( \sigma(X, X) = \rho JX \). If \( \phi \) is degenerate, then \( \phi \) is locally congruent to some of the immersions \( \phi^\delta_{1,2} \) described in Examples 5 (respectively in Examples 6).

**Proof.** We start with the projective case. From Proposition 1, \( X \) is parallel and \( \rho \) is constant. We assume that \( |X| = 1 \). So the two eigenvalues of \( A_{JX} \) on \( \mathcal{D} \) (see Proposition 1) satisfy
\[ b_1 + b_2 = \rho, \quad b_1 b_2 = -1. \]

We start taking the distributions \( \mathcal{D}_i, i = 1, 2 \), on \( M \) (see the proof of Proposition 1) defined by
\[ \mathcal{D}_i(x) = \{ v \in \mathcal{D}(x) : A_{JX} v = b_i v \}, \]
of dimensions \( n_1 \) and \( n_2 \) respectively. Using a similar reasoning as in the proof of Proposition 1, it is not difficult to prove that
\[ \nabla_Z V_i \in \mathcal{D}_i, \]
for any \( Z \) tangent to \( M \) and \( V_i \in \mathcal{D}_i \). So using also Lemma 1, (v), locally \( M \) is the Riemannian product \((I \times N_1 \times N_2, dt^2 \times g_1 \times g_2)\), where \( I \) is an interval of \( \mathbb{R} \), and \( N_i \) are simply-connected manifolds of dimension \( n_i \).

We consider a horizontal lift \( \tilde{\phi} : I \times N_1 \times N_2 \rightarrow \mathbb{S}^{2n+1} \subset \mathbb{C}^{n+1} \) of \( \phi \), and identify \( X \) and \( \tilde{\phi}_*X \). If \( h \) denotes the second fundamental form of \( \tilde{\phi} \) in \( \mathbb{C}^{n+1} \), we have
\[ h(X, X) = \rho JX - \tilde{\phi}, \quad h(X, v_i) = b_i Jv_i, \]
for any tangent vector \( v_i \) to \( N_i \).

We can suppose \( b_1 > 0 > b_2 \) and define \( \gamma_i : I \times N_1 \times N_2 \rightarrow \mathbb{C}^{n+1}, i = 1, 2 \), by
\[ \gamma_1 = (b_2 - b_1)(b_2 \tilde{\phi} + JX), \quad \gamma_2 = (b_1 - b_2)(b_1 \tilde{\phi} + JX). \]
First, it is easy to check that
\[ |\gamma_1|^2 = b_2(b_2 - b_1)^3, \quad |\gamma_2|^2 = b_1(b_1 - b_2)^3, \quad \langle \gamma_1, \gamma_2 \rangle = \langle \gamma_1, J\gamma_2 \rangle = 0. \]
Also, using (3.1) we have
\[ v_2(\gamma_1) = 0, \quad X(\gamma_1) = b_1 J\gamma_1, \quad v_1(\gamma_2) = 0, \quad X(\gamma_2) = b_2 J\gamma_2, \]
and
\[ v_1(\gamma_1) = b_2 \gamma_1, \quad v_2(\gamma_2) = b_1 \gamma_2. \]
for any $v_i$ tangent to $N_i$, $i = 1, 2$. So in particular, from (3.2), (3.3) and (3.4), we recuperate $\tilde{\phi}$ in terms of $\gamma_i$ by

$$\tilde{\phi}(t, p, q) = \frac{1}{(b_1 - b_2)^2}(e^{ib_1 t}\gamma_1(0, p), e^{ib_2 t}\gamma_2(0, q)), $$

for any $(t, p, q) \in I \times N_1 \times N_2$.

Now we define

$$\tilde{\phi}_1(p) = \frac{\gamma_1(0, p)}{\gamma_1(0, p)} = \frac{\gamma_1(0, p)}{(b_1 - b_2)\sqrt{b_2(b_2 - b_1)}}, \quad p \in N_1,$n

$$\tilde{\phi}_2(q) = \frac{\gamma_2(0, q)}{\gamma_2(0, q)} = \frac{\gamma_2(0, q)}{(b_1 - b_2)\sqrt{b_1(b_1 - b_2)}}, \quad q \in N_2.$$

So we can rewrite (3.5) as

$$\tilde{\phi}(t, p, q) = \left(\sqrt{\frac{b_2}{b_2 - b_1}}e^{ib_1 t}\tilde{\phi}_1(p), \sqrt{\frac{b_1}{b_1 - b_2}}e^{ib_2 t}\tilde{\phi}_2(q)\right).$$

Now the properties of the lift $\tilde{\phi}$ say that $\tilde{\phi}_i$ are horizontal immersions in the corresponding $S^{2n_i+1} \subset \mathbb{C}^{n_i+1}$, which means that $\tilde{\phi}_i$ are horizontal lifts of Lagrangian immersions $\phi_i = \Pi(\tilde{\phi}_i)$, $i = 1, 2$, in $\mathbb{C}^{n_i}$.

Because $b_1 = \tan \delta$ for some $\delta \in (0, \frac{\pi}{2})$, if $\delta \in (0, \frac{\pi}{4}]$, our immersion is locally congruent to some of the immersions $\phi^\delta_{1,2}$ described in Examples 5. If $\delta \in (\frac{\pi}{4}, \frac{\pi}{2})$, then our immersion is locally congruent to some of the immersions $\phi^\delta_{1,2}$ described in Examples 5.

We follow with the hyperbolic case. As the proof is quite similar to the above proof, we will omit some details. From Proposition 1, $X$ is parallel and $\rho$ is constant. We assume that $|X| = 1$. So the two eigenvalues of $A_{iX}$ on $\mathcal{D}$ (see Proposition 1) satisfied

$$b_1 + b_2 = \rho, \quad b_1b_2 = 1.$$

Reasoning as before, locally $M$ is the Riemannian product $(I \times N_1 \times N_2, dt^2 \times g_1 \times g_2)$, where $I$ is an interval of $\mathbb{R}$, and $N_i$ are simply-connected manifolds of dimension $n_i$.

We consider a horizontal lift $\tilde{\phi} : I \times N_1 \times N_2 \rightarrow \mathbb{H}_1^{2n+1} \subset \mathbb{C}^{n+1}$, and identify $X$ and $\tilde{\phi}_*X$. If $h$ denotes the second fundamental form of $\tilde{\phi}$ in $\mathbb{C}^{n+1}$ we have

$$h(X, X) = \rho JX + \tilde{\phi}, \quad h(X, v_i) = b_iJv_i,$$

for any tangent vector $v_i$ to $N_i$.

We can suppose $b_1 > b_2 > 0$. Otherwise $0 > b_1 > b_2$ and changing $X$ by $-X$ we will be in the first case. We define $\gamma_i : I \times N_1 \times N_2 \rightarrow \mathbb{C}^{n+1}, i = 1, 2$, by

$$\gamma_1 = (b_2 - b_1)(b_2\tilde{\phi} + JX), \quad \gamma_2 = (b_1 - b_2)(b_1\tilde{\phi} + JX).$$
Following the same reasoning as in the projective case, we get that \( \bar{\phi} \) can be written as

\[
\bar{\phi}(t, p, q) = \left( \sqrt{\frac{b_2}{b_1 - b_2}} e^{ib_1 t} \bar{\phi}_1(p), \sqrt{\frac{b_1}{b_1 - b_2}} e^{ib_2 t} \bar{\phi}_2(q) \right).
\]

Now the properties of the lift \( \bar{\phi} \) say that \( \bar{\phi}_i \) are horizontal immersions in the corresponding \( \mathbb{S}^{2n_1+1} \) and \( \mathbb{H}^{n_2+1}_1 \), and then \( \phi_1 = \Pi(\bar{\phi}_1) \) and \( \phi_2 = \Pi(\bar{\phi}_2) \) are Lagrangian immersions in \( \mathbb{C}P^{n_1} \) and \( \mathbb{C}H^{n_2} \) respectively, with horizontal lifts \( \bar{\phi}_1 \) and \( \bar{\phi}_2 \). As \( b_1 \in (1, \infty) \), let \( \delta \) be the positive real number with \( \coth\delta = b_1 \). Then our immersion is locally congruent to some of \( \phi^\delta_{1,2} \) described in Examples 6.

\[\square\]

4. Lagrangian submanifolds with conformal Maslov form in nonflat complex space forms.

In this section, we will deal with the case that \( X = JH \). Since \( JH \) is a closed vector field (because the ambient manifold is a complex space form), we must only assume that \( JH \) is a conformal vector field on the submanifold. It is well-known that, up to a constant, \( JH \) is the dual vector field of the Maslov form and so this kind of Lagrangian submanifolds will be referred from now on as Lagrangian submanifolds with conformal Maslov form. They were deeply studied by A. Ros and the third author in complex Euclidean space (cf. [RU]).

In the following, we will describe a special family of Lagrangian submanifolds with conformal Maslov form in nonflat complex space forms in terms of (i) minimal Lagrangian immersions in complex projective, complex hyperbolic and complex Euclidean spaces of less dimension and (ii) a two parameter family of curves of \( \mathbb{C}P^1 \) and \( \mathbb{C}H^1 \).

To introduce this last family, we start to consider the following 2-parameter (\( \lambda, \mu \neq 0 \)) family of o.d.e.s:

\[
(4.1) \quad u'^2 + \frac{c}{4} e^{2u} + \left( \frac{n\mu}{n+2} e^{2u} + \lambda e^{-nu} \right)^2 = \begin{cases} 
1, & \text{if } c = 4 \\
A \in \{1, 0, -1\}, & \text{if } c = -4 
\end{cases}
\]

We put \( h = -u' - i \left( \frac{n\mu}{n+2} e^{2u} + \lambda e^{-nu} \right) \) and for the solution to (4.1) satisfying \( u'(0) = 0 \), following the notation of Examples 1-4, we define the curve \( \alpha_{\lambda,\mu} \) in \( \mathbb{C}P^1 \) if \( c = 4 \), in \( \mathbb{C}H^1 \) in the other cases, by means of \( \alpha_{\lambda,\mu} = \Pi \circ \bar{\alpha}_{\lambda,\mu} \).
where

\[ \tilde{\alpha}_{\lambda,\mu}(t) = \left( e^{u(t)} e^{-\int_0^t \overline{\alpha}(s)ds}, |h(0)| e^{\frac{2}{\lambda} \int_0^t \frac{2u(s)}{h(s)} - ds} \right), \text{ if } c = 4 \text{ or } c = -4, A = 1, \]

\[ \tilde{\alpha}_{\lambda,\mu}(t) = \left( |h(0)| e^{-\int_0^t \frac{e^{2u(s)}}{h(s)} - ds}, e^{u(t)} e^{-\int_0^t \overline{\alpha}(s)ds} \right), \text{ if } c = -4, A = -1, \]

\[ \tilde{\alpha}_{\lambda,\mu}(t) = e^{-\int_0^t \overline{\alpha}(s)ds} \left( 2e^{2u(0)} e_1 - \left( e^{-2u(0)/2} + \int_0^t ds/h(s) \right) e_2 \right), \]

if \( c = -4, A = 0. \)

**Corollary 2.** Let \( \phi : M^n \longrightarrow \overline{M}^n(c) \) be a nonminimal Lagrangian immersion. Then \( \phi \) has conformal Maslov form and \( JH \) is a principal direction of \( A_H \) if and only if around each point where \( H \) does not vanish, \( \phi \) is congruent to:

(a) Some \( \alpha \ast \psi \) of Examples 1-4, where \( \psi \) is a minimal immersion in \( \mathbb{CP}^{n-1} \), \( \mathbb{CH}^{n-1} \) or \( \mathbb{C}^{n-1} \) and \( \alpha = \alpha_{\lambda,\mu} \) is the curve given in (4.2). In this case, \( |H| = \mu e^u. \)

(b) Some \( \phi_1^{\delta} \) of Examples 5 with \( \phi_i \) minimal immersions in \( \mathbb{CP}^{n_i}, i = 1, 2, \)

\( (n_1 + n_2 + 1 = n) \) and arbitrary \( \delta \in (0, \pi/4] \), or some \( \phi_1^{\delta} \) of Examples 6 with \( \phi_1 \) a minimal immersion in \( \mathbb{CP}^{n_1}, \phi_2 \) a minimal immersion in \( \mathbb{CH}^{n_2}, (n_1 + n_2 + 1 = n) \) and arbitrary \( \delta > 0. \)

**Remark 8.** In [Ch2], B.Y. Chen introduced the notion of Lagrangian H-umbilical submanifolds in Kaehler manifolds. This kind of submanifolds correspond to the simplest Lagrangian submanifolds satisfying the two following conditions: \( JH \) is an eigenvector of the shape operator \( A_H \) and the restriction of \( A_H \) to the orthogonal subspace to \( JH \) is proportional to the identity. He classified them in \( \mathbb{CP}^n \) and \( \mathbb{CH}^n \) (cf. [Ch2]) proving that, except in some exceptional cases, they are obtained from Legendre curves in \( S^3 \) or in \( \mathbb{E}^3_1 \) via warped products. In Corollary 2, they appear in (a) by considering \( \psi \) totally geodesic.

**Proof.** It is a consequence of Proposition 1, Corollary 1, Theorem 2, Proposition 3 and the following facts:

First, when we study the geometric properties of the examples \( \alpha \ast \psi \) we arrive at the following expressions for its mean curvature vector:

\[ nH^* = \frac{1}{|\alpha_1|^2} \left( (\rho + (n - 1)b)JX + (n - 1)(\tilde{\alpha}_1 H^*_\psi; 0) \right) \]

\[ \text{resp. } nH^* = \frac{1}{|\alpha_2|^2} \left( (\rho + (n - 1)b)JX + (n - 1)(\tilde{\alpha}_2 H^*_\psi) \right) \]
for Examples 1, 2, 4 (resp. for Examples 3), where
\[
\rho = \frac{|\tilde{\alpha}_1|}{|\tilde{\alpha}'|} \langle \tilde{\alpha}'', J \tilde{\alpha}' \rangle, \quad b = \frac{\langle \tilde{\alpha}_1', J \tilde{\alpha}_1 \rangle}{|\tilde{\alpha}_1||\tilde{\alpha}'|},
\]
(resp. \( \rho = \frac{|\tilde{\alpha}_2|}{|\tilde{\alpha}'|} \langle \tilde{\alpha}'', J \tilde{\alpha}' \rangle \), \( b = \frac{\langle \tilde{\alpha}_2', J \tilde{\alpha}_2 \rangle}{|\tilde{\alpha}_2||\tilde{\alpha}'|} \)).

So if we put \( X_\mu = \mu X = JH \) (where \( X \) is normalized according to Remark 1) and look at the foregoing expressions we deduce that \( H^*_\psi = 0 \) and hence \( \psi \) is a minimal immersion, and using that \( |X|^2 = |\tilde{\alpha}_1|^2 \) in Examples 1, 2, 4 (resp. \( |X|^2 = |\tilde{\alpha}_2|^2 \) in Examples 3), we obtain
\[
-\mu n|\tilde{\alpha}_1|^2 = \rho + (n - 1)b \quad \text{(resp. }-\mu n|\tilde{\alpha}_2|^2 = \rho + (n - 1)b) \]
for Examples 1, 2, 4 (resp. for Examples 3). From (1.5) and the above equations, we deduce
\[
b' + nb = -n\mu e^{2u}
\]
where \( e^{2u} = |X|^2 = |H|^2/\mu^2 \), and then
\[
b = -\left( \frac{n\mu}{n + 2} e^{2u} + \lambda e^{-nu} \right).
\]

Second, the expressions given for \( \tilde{\alpha} \) in Corollary 2 can be deduced following a constructive reasoning from the proof of Corollary 1 and Theorem 2 in all the cases.

Finally, a similar technique can be used to get the cases given in (b). \( \Box \)

Next we will consider the easiest examples provided in Corollary 2, taking \( \psi = \psi_0 \) totally geodesic and the curves \( \alpha_{0,\mu} \), i.e., putting \( \lambda = 0 \) in (4.1). For our purposes, it is enough to consider \( \mu > 0 \). We will state along this paragraph that these examples play the role of “umbilical” Lagrangian immersions in non flat complex space forms. If \( \lambda = 0 \), (4.1) becomes in
\[
(4.3) \quad u'^2 + \frac{c}{4} e^{2u} + \nu^2 e^{4u} = \begin{cases} 
1, & \text{if } c = 4 \\
A \in \{1, 0, -1\}, & \text{if } c = -4
\end{cases},
\]
with \( \nu = \frac{n\mu}{n+2} > 0 \), and we must now take into account that \( h = -u' - i\nu e^{2u} \) for the curves given in (4.2). The solution to (4.3) satisfying \( u'(0) = 0 \) is
given by
\[
e^{2u(t)} = \begin{cases} \frac{2}{\sqrt{1+4\nu^2} \cosh(2t)+c/4}, & \text{if } c = 4 \text{ or } c = -4, A = 1, \\ \frac{2}{1-\sqrt{1-4\nu^2} \cos(2t)}, & \text{if } c = -4, A = -1, \\ \frac{1}{\nu^2+t^2}, & \text{if } c = -4, A = 0. \end{cases}
\]

The only constant solution appears in the hyperbolic case \(c = -4, A = -1\), just when \(\nu = 1/2\). In this case \(e^{2u(t)} \equiv 2\) and the corresponding immersion has parallel mean curvature vector, just like the examples collected in Corollary 2, (b).

In the elliptic case, we define \(\theta = (1/2) \cosh^{-1} \sqrt{1+4\nu^2} > 0\) and abbreviate \(ch_\theta = \cosh \theta\) and \(sh_\theta = \sinh \theta\); now (4.3) and its solution are rewritten as
\[
u^2 + \frac{c}{4} e^{2u} + ch_\theta^2 sh_\theta^2 e^{4u} = 1
\]
and
\[
e^{2u(t)} = \frac{2}{ch_\theta \cosh(2t) + c/4}.
\]

From (4.2), after a long straightforward computation, we arrive at
\[
\bar{\alpha}_\theta(t) = \frac{1}{ch_\theta \cosh t + ish_\theta \sinh t}(1, sh_\theta \cosh t + ich_\theta \sinh t),
\]
\[
\bar{\alpha}_\theta(t) = \frac{1}{sh_\theta \cosh t + ich_\theta \sinh t}(1, ch_\theta \cosh t + ish_\theta \sinh t),
\]
for \(c = 4\) and \(c = -4\) respectively. By identifying \(\mathbb{R}^n - \{0\}\) with \(\mathbb{R} \times S^{n-1}\) via the conformal transformation \(w \mapsto (\log |w|, w/|w|)\), \(\alpha_\theta * \psi_0\) defines a Lagrangian immersion \(\phi_\theta : \mathbb{R}^n - \{0\} \to \mathbb{C}P^n\), and it is not complicated to check that \(\phi_\theta\) extends regularly to 0 and \(\infty\); hence via stereographic projection we obtain a family of Lagrangian immersions
\[(4.4) \quad \phi_\theta : \mathbb{S}^n \to \mathbb{C}P^n, \theta > 0,
\]
given by
\[
\phi_\theta(x_1, \ldots, x_n, x_{n+1}) = \left[ \left( \frac{x_1, \ldots, x_n}{ch_\theta + ish_\theta x_{n+1}} ; \frac{sh_\theta ch_\theta(1 + x_{n+1}^2) + ix_{n+1}}{ch_\theta^2 + sh_\theta^2 x_{n+1}^2} \right) \right]
\]
that we will call the Whitney spheres of \(\mathbb{C}P^n\). We notice that \(\phi_\theta\) are embeddings except in a double point and that if \(\theta \to 0\) it appears the totally geodesic immersion of \(\mathbb{S}^n\) in \(\mathbb{C}P^n\).
In a similar way we obtain the Whitney spheres of $\mathbb{CH}^n$,
\[(4.5) \quad \Phi_\theta : \mathbb{S}^n \rightarrow \mathbb{CH}^n, \theta > 0,\]
given by
\[
\Phi_\theta(x_1, \ldots, x_n, x_{n+1}) = \left[ \left( \frac{x_1, \ldots, x_n}{sh_\theta + ich_\theta x_{n+1}}, \frac{sh_\theta ch_\theta(1 + x_{n+1}^2) - ix_{n+1}}{sh_\theta^2 + ch_\theta^2 x_{n+1}^2} \right) \right],
\]
which are also embeddings except in a double point.

In the hyperbolic case, we define $\beta = (1/2) \cos^{-1}(\sqrt{1 - 4\nu^2}) \in (0, \pi/4]$ and abbreviate $c_\beta = \cos \beta$ and $s_\beta = \sin \beta$; now (4.3) and its solution are rewritten as
\[
u^2 - e^{2\nu} + s_\beta^2 c_\beta^2 e^{4\nu} = -1
\]
and
\[
e^{2\nu(t)} = \frac{2}{1 - c_\beta \cos 2t}.
\]
From (4.2), after a long straightforward computation, we arrive at
\[
\tilde{\alpha}_\beta(t) = \frac{1}{s_\beta \cos t + ic_\beta \sin t} (c_\beta \cos t - is_\beta \sin t, 1).
\]
If $\mathbb{RH}^{n-1} = \{ y = (y_1, \ldots, y_n) \in \mathbb{R}^n : y_1^2 + \ldots y_{n-1}^2 - y_n^2 = -1 \}$ denotes the $(n-1)$-dimensional real hyperbolic space, $\alpha_\beta * \psi_0$ defines a family of Lagrangian embeddings
\[(4.6) \quad \Psi_\beta : \mathbb{S}^1 \times \mathbb{RH}^{n-1} \rightarrow \mathbb{CH}^n, \beta \in (0, \pi/4],\]
given by
\[
\Psi_\beta(e^it, y) = \left[ \frac{1}{s_\beta \cos t + ic_\beta \sin t} (c_\beta \cos t - is_\beta \sin t; y) \right].
\]
We note that $\Psi_{\pi/4}(e^{2it}, \sqrt{2}e^{-it}y)$ is flat.

Finally, in the parabolic case, using (4.2) a straightforward computation leads to
\[
\tilde{\alpha}_\nu(t) = \frac{1}{\nu + it} \left( \frac{2}{\nu} e_1 - \nu \left( \frac{\nu^2 + t^2}{2} + i\nu t \right) e_2 \right).
\]
From Examples 4, $\alpha_\nu * \psi_0$ defines a one-parameter family of Lagrangian embeddings
\[(4.7) \quad \varphi_\nu : \mathbb{R}^n \equiv \mathbb{R} \times \mathbb{R}^{n-1} \rightarrow \mathbb{CH}^n, \nu > 0,\]
given by
\[
\varphi_\nu(t, x) = \left[ \frac{1}{\nu + it} \left( \frac{2}{\nu} x; \frac{2}{\nu} e_1 - \left( \frac{\nu(\nu^2 + t^2)}{2\nu} + 2|x|^2 \nu + i\nu^2 t \right) e_2 \right) \right].
\]
Theorem 3. Let $\phi : M^n \rightarrow \mathbb{C}P^n$ (respectively $\phi : M^n \rightarrow \mathbb{C}H^n$) be a Lagrangian immersion.

(i) The second fundamental form of $\phi$ is given by

\begin{equation}
\langle \sigma(v, w), Jz \rangle = \frac{n}{n+2} \partial_{v, w, z} \langle v, w \rangle \langle H, Jz \rangle
\end{equation}

for any tangent vectors $v$, $w$ and $z$, (where the symbol $\partial_{v, w, z}$ means cyclic sum over $v, w, z$) if and only if either $\phi$ is totally geodesic or $\phi(M)$ is an open set of some of the Lagrangian submanifolds (4.4) in $\mathbb{C}P^n$ (respectively of the Lagrangian submanifolds (4.4), (4.5), (4.6) or (4.7) in $\mathbb{C}H^n$).

(ii) [B.Y. Chen] The scalar curvature $\tau$ of $M$ satisfies

$$\tau \leq \frac{n^2(n-1)}{n+2} |H|^2 + n(n-1) \frac{c}{4}$$

and the equality holds if and only if $\phi$ is totally geodesic or $\phi$ is an open set of some of the Lagrangian submanifolds (4.4) in $\mathbb{C}P^n$ (respectively of the Lagrangian submanifolds (4.5), (4.6) or (4.7) in $\mathbb{C}H^n$).

Remark 9. In [CU2] when $n = 2$ and in [Ch1] in arbitrary dimension, the sharp inequality of (ii) between the squared mean curvature and the scalar curvature for a Lagrangian submanifold in a nonflat complex space form was established. By utilising Jacobi’s elliptic functions and warped products, B.Y. Chen introduced in [Ch1] three families of Lagrangian submanifolds and two exceptional ones characterized by satisfying the equality in (ii) besides the totally geodesic ones. Later, in [ChV], explicit expressions of these Lagrangian immersions were found in a different context from ours. All of them are exactly the Lagrangian submanifolds described in (4.4), (4.5), (4.6) and (4.7).

Proof. For the proof of (i), it is straightforward to verify that the totally geodesic immersions and the Lagrangian immersions (4.4)-(4.7) satisfy (4.8).

Conversely, exactly the same proof of Theorem 2 in [RU] works also here to get that $JH$ is a conformal vector field on $M$. If $H \equiv 0$ then (4.8) implies that $\phi$ is totally geodesic. If $H$ is nontrivial, Lemma 1, (i) says that the zeros of $H$ are isolated. Hence the set of points of $M$ where $H$ does not vanish, say $M'$, is a connected open dense subset of $M$. We will work in it. From (4.8) we first deduce that $\sigma(JH, JH) = \frac{3n}{n+2} |H|^2 H$. Then Corollary 2 says that $\phi$ is locally congruent to some of the examples described there. Second, (4.8) also gives us that $\sigma(v, JH) = -\frac{n}{n+2} |H|^2 Jv$ for $v$ orthogonal to $JH$ and this implies that necessarily that $\phi$ is locally congruent to some $\alpha \ast \psi$ of Corollary 2, (a) and, in addition, $\lambda = 0$ in (4.1). Using again (4.8) we obtain, for $v$ and $w$ orthogonal to $JH$, that $\sigma(v, w) = \frac{n}{n+2} \langle v, w \rangle H$ and this means that $\sigma^* \psi = 0$ when we study the second fundamental form of such an immersion $\alpha \ast \psi$. Thus $\psi$ is totally geodesic. As $M'$ is connected and
dense in $M$, a standard argument shows that $\phi(M)$ is an open set in some of the Lagrangian submanifolds (4.4)-(4.7).

Our proof of (ii) follows from (i) and a standard interpolation argument.

\[\square\]

**Theorem 4.** Let $\phi : M^n \to \mathbb{CP}^n$ (respectively $\phi : M^n \to \mathbb{CH}^n$) be a Lagrangian immersion of a compact manifold $M$ with conformal Maslov form. If the first Betti number of $M$ vanishes, then $\phi$ is either minimal (necessarily in $\mathbb{CP}^n$) or congruent to some of the Whitney spheres (4.4) in $\mathbb{CP}^n$ (respectively congruent to some of the Whitney spheres (4.5) in $\mathbb{CH}^n$).

**Proof.** Suppose (only in the projective case) that $H$ does not vanish identically. Since $JH$ is a closed vector field and the first Betti number of $M$ is zero, there exists a function $f$ on $M$ such that $JH = \nabla f$. So $H$ has zeros at the critical points of $f$. Hence $JH$ is a closed and conformal vector field with at least a zero. Under these conditions, exactly the same reasoning of Theorem 3 in [RU] proves that $M$ is conformally equivalent to a round sphere, the leaves of the umbilical foliation $D$ (see Lemma 1) are spheres and $Y = J\sigma(JH, JH) - (3n/(n + 2))|H|^2JH$ is a harmonic vector field. Since the first Betti number of $M$ is zero, we deduce that $JH$ is an eigenvector of $A_H$. If $N$ is one of the leaves of $D$, using Corollary 2 we have a minimal Lagrangian immersion $\psi$ necessarily in $\mathbb{CP}^{n-1}$ because in $\mathbb{C}^{n-1}$ and $\mathbb{CH}^{n-1}$ there are no compact minimal submanifolds. Then we can use the same proof in the above mentioned theorem of [RU] to conclude that $\psi$ is totally geodesic and that $\sigma$ is given as in (4.8) on the whole of $M$. The proof finishes thanks to Theorem 3. \[\square\]

Next, we obtain the following corollary using the same argument that in Corollary 5 of [RU], where the Whitney spheres are characterised in terms of the behaviour of the Ricci curvature.

**Corollary 3.** Let $\phi : M^n \to \mathbb{CP}^n$ be a Lagrangian nonminimal immersion of a compact manifold $M$ with conformal Maslov form. Then:

(i) $\text{Ric}(JH) \geq (n - 1)|H|^2$ if and only if $\phi$ is congruent to some of the Whitney spheres (4.4) in $\mathbb{CP}^n$;

(ii) $\text{Ric}(JH) \geq 0$ if and only if either $\phi$ has parallel mean curvature vector or $\phi$ is congruent to some of the Whitney spheres (4.4) in $\mathbb{CP}^n$.

To finish this section we state that if our Lagrangian submanifold has first Betti number equal to one then it belongs to our family.

**Corollary 4.** Let $\phi : M^n \to \mathbb{CP}^n$ (respectively $\phi : M^n \to \mathbb{CH}^n$) be a Lagrangian immersion of an orientable compact manifold $M$ with nonparallel conformal Maslov form such that the first Betti number of $M$ is one.
Then the universal covering of $\phi$ is congruent to some $\alpha * \psi$ as described in Corollary 2,(a).

Proof. From the proof of Theorem 4 we can deduce that $H$ has no zeros on $M$ and so the vector field $|H|^{-n}JH$ is well-defined on $M$. Using Lemma 1, one can check that it is a harmonic vector field on $M$ and the same happens for $Y = J\sigma(JH,JH) - (3n/(n+2))|H|^2JH$. By the hypothesis, there is a constant $a \in \mathbb{R}$ such that

$$\sigma(JH,JH) = \left(\frac{3n}{n+2}|H|^2 + a|H|^{-n}\right)H$$

and we can go to Corollary 2 to get the result. \qed

References


Received September 13, 1999 and revised March 31, 2000. This research was partially supported by a DGICYT grant No. PB97-0785.

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THE SUPPORT OF THE EQUILIBRIUM MEASURE FOR A
CLASS OF EXTERNAL FIELDS ON A FINITE INTERVAL

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We investigate the support of the equilibrium measure associated with a class of nonconvex, nonsmooth external fields on a finite interval. Such equilibrium measures play an important role in various branches of analysis. In this paper we obtain a sufficient condition which ensures that the support consists of at most two intervals. This is applied to external fields of the form \(-c\, \text{sign}(x)|x|^{\alpha}\) with \(c > 0\), \(\alpha \geq 1\) and \(x \in [-1, 1]\). If \(\alpha\) is an odd integer, these external fields are smooth, and for this case the support was studied before by Deift, Kriecherbauer and McLaughlin, and by Damelin and Kuijlaars.

1. Introduction.

In recent years, equilibrium measures with external fields have found an increasing number of applications in a variety of areas. We refer to [2, 3, 4, 5, 8, 10, 14, 15] for these relations, ranging from classical topics as weighted transfinite diameter and weighted Chebyshev polynomials, to more recent developments in weighted approximation, orthogonal polynomials, integrable systems, and random matrix theory.

In the present paper we consider equilibrium problems on the interval \([-1, 1]\). With a continuous function \(Q : [-1, 1] \to \mathbb{R}\), we associate the weighted energy of a measure \(\mu\) on \([-1, 1]\) as follows

\[
I_Q(\mu) = \int \int \log \frac{1}{|s-t|} d\mu(s)d\mu(t) + 2 \int Q(t)d\mu(t).
\]

The equilibrium measure in the presence of the external field \(Q\) is the unique probability measure \(\mu_Q\) on \([-1, 1]\) minimizing the weighted energy among all probability measures. Thus

\[
I_Q(\mu_Q) = \min\{I_Q(\mu) : \mu \in \mathcal{P}([-1, 1])\}
\]

where \(\mathcal{P}([-1, 1])\) denotes the class

\[
\mathcal{P}([-1, 1]) = \{\mu : \mu \text{ is a Borel probability measure on } [-1, 1]\}.
\]
The determination of the support of the equilibrium measure is a major step in obtaining the measure. As described by Deift [2, Chapter 6] the information that the support consists of $N$ disjoint closed intervals, allows one to set up a system of equations for the endpoints, from which the endpoints may be calculated. Knowing the endpoints, the equilibrium measure may be obtained from a Riemann-Hilbert problem or, equivalently, a singular integral equation.

There are two general useful facts about the equilibrium measure. The first one, due to Mhaskar and Saff [12], says that for a convex external field, the support is always one single interval. The other one, due to Deift, Kriecherbauer and McLaughlin [3], says that for a real analytic external field, the support always consists of a finite number of intervals. The actual determination of this number is a nontrivial problem. To illustrate the difficulties, Deift, Kriecherbauer and McLaughlin considered explicitly the families of monomial external fields $Q(x) = -cx^n$ with $c \neq 0$, $n \in \mathbb{N}$ and $x \in [-1, 1]$.

In the even case ($n = 2m$) the external field is convex if $c < 0$, and therefore the support is a single interval. For $c > 0$, the external field is concave, and the analysis becomes more involved. Independently from [3], this case was considered in [9], and it was shown that for every $c > 0$, there are at most three intervals in the support of the equilibrium measure. The same result was also found to be valid for the nonsmooth (i.e., not real analytic) external fields $Q(x) = -c|x|^\alpha$ with $\alpha \geq 1$ not necessarily an even integer.

In the odd case ($n = 2m + 1$) the external field is an odd function, and, by symmetry, we may restrict attention to $c > 0$. In this case the results of [3] were extended to the full range of parameters in [1]. For all $c$ and all odd integers $n$, it was shown that the support of the equilibrium measure consists of at most two intervals.

It is the aim of the present paper to study the nonsmooth analogues of $-cx^{2m+1}$ given by

$$Q_{\alpha,c}(x) := -c \text{sign}(x)|x|^\alpha = \begin{cases} c|x|^\alpha & \text{for } x \in [-1, 0], \\ -cx^\alpha & \text{for } x \in [0, 1], \end{cases}$$

with a real number $\alpha \geq 1$ and $c > 0$. The functions (1.3) are both non-convex and nonsmooth, and therefore it is of interest to develop methods to determine the nature of the support of the equilibrium measures associated with these external fields.

Our first theorem presents a sufficient condition which ensures that the support of the equilibrium measure is the union of at most two intervals. To formulate it, we use $C^{1+\varepsilon}([-1, 1])$ to denote the class of differentiable functions on $[-1, 1]$, whose derivative satisfies a Hölder condition for some
positive exponent. Thus $Q \in C^{1+\varepsilon}([-1, 1])$ if and only if

$$|Q'(x) - Q'(y)| \leq C|x - y|^{\varepsilon}, \quad x, y \in [-1, 1]$$

for some $\varepsilon > 0$ and some positive constant $C$ independent of $x$ and $y$.

**Theorem 1.1.** Let $Q \in C^{1+\varepsilon}([-1, 1])$. Suppose that there exists a number $a_1 \in [-1, 1]$ such that

(a) $Q$ is convex on $[-1, a_1]$, and
(b) for every $a \in [-1, a_1]$, there is $t_0 \in [a_1, 1]$, such that the function

$$t \mapsto \frac{1}{\pi} \int_a^1 \frac{Q'(s)}{s - t} \sqrt{(1 - s)(s - a)} ds$$

(1.4)

is nonincreasing on $(a_1, t_0)$ and nondecreasing on $(t_0, 1)$. The integral in (1.4) is a principal value integral.

Then $\text{supp } (\mu_Q)$ is the union of at most two intervals.

**Remark 1.2.** For the special case $a_1 = -1$, Theorem 1.1 was given already in [9, Theorem 2].

In our second main result we show that the conditions of Theorem 1.1 are satisfied for the external fields (1.3).

**Theorem 1.3.** For $\alpha \geq 1$ and $c > 0$, let $Q_{\alpha,c}$ be given by (1.3). Then for every $a \in [-1, 0]$, there exists $t_0 \in [0, 1)$ such that

$$\frac{1}{\pi} \int_a^1 \frac{Q'_{\alpha,c}(s)}{s - t} \sqrt{(1 - s)(s - a)} ds$$

(1.5)

decreases on $(0, t_0)$ and increases on $(t_0, 1)$. As a result, the support of $\mu_{Q_{\alpha,c}}$ consists of at most two intervals.

**Remark 1.4.** For $\alpha$ an odd integer, Theorem 1.3 was established in [1].

The proof for this special case differs from the one given here in several respects. For example, the function (1.5) is a polynomial in $t$ whenever $\alpha$ is an odd integer. The proof of the decreasing/increasing property of (1.5) was based in [1] on the calculation of the polynomial coefficients and the Descartes’ rule of signs for polynomials.

Another difference between [1] and the present paper is that in [1] the problem was viewed in terms of the parameter $c$. Quite complicated perturbation arguments were used to obtain from the decreasing/increasing property of (1.5) the conclusion that the support consists of at most two intervals. Here we use Theorem 1.1 and this simplifies the arguments considerably, also in the case where $\alpha$ is an odd integer.

**Remark 1.5.** To view the problem in terms of the parameter $c$ is quite natural, since there is a monotonicity with respect to $c$. To be precise, if $Q$ is fixed then the support $\text{supp } (\mu_{c,Q})$ is decreasing as $c$ increases, see
Using this, we can show the following behavior of the support depending on the parameter in case $\alpha > 1$. There exist three critical values $0 < c_1 < c_2 < c_3$ depending on $\alpha$ such that:

(a) For $0 < c \leq c_1$, the support $\text{supp}(\mu_{Q,\alpha,c})$ is equal to the full interval $[-1, 1]$.
(b) For $c_1 < c \leq c_2$, there exists $a \in (-1, 0)$ such that $\text{supp}(\mu_{Q,\alpha,c}) = [a, 1]$.
(c) For $c_2 < c < c_3$, there exist $a_1, b_1$ and $a_2$ such that $-1 < a_1 < b_1 < a_2 < 1, a_1 < 0$, and $\text{supp}(\mu_{Q,\alpha,c}) = [a_1, b_1] \cup [a_2, 1]$.
(d) For $c \geq c_3$, there exists $a \in (0, 1)$ such that $\text{supp}(\mu_{Q,\alpha,c}) = [a, 1]$.

See [1, Theorem 1.1] where this was shown for odd integers $\alpha \geq 3$.

Note that for $\alpha = 1$, the external field (1.3) is convex and the support of $\mu_{Q_1,c}$ is an interval containing 1 for every $c > 0$.

Acknowledgement. This work would not have been possible without grants of our institutions that have allowed for mutual visits. We are grateful for their support.

2. The Proof of Theorem 1.1.

In this section, we shall prove Theorem 1.1.

2.1. Preliminaries. Let $Q \in C^{1+\varepsilon}([-1, 1])$ be fixed. The equilibrium measure $\mu_Q$ is characterized by the Euler-Lagrange variational conditions associated with the extremal problem (1.2), which are

\begin{equation}
U^\mu(x) + Q(x) = F \quad \text{for } x \in \text{supp}(\mu),
\end{equation}

\begin{equation}
U^\mu(x) + Q(x) \geq F \quad \text{for } x \in [-1, 1],
\end{equation}

where $F$ is a constant and

\begin{equation}
U^\mu(x) = \int \log \frac{1}{|x - t|} \, d\mu(t)
\end{equation}

denotes the logarithmic potential of $\mu$, see [2, 14]. The equilibrium measure $\mu_Q$ is the only measure from $\mathcal{P}([-1, 1])$ satisfying (2.1) and (2.2) for some constant $F$.

If $\text{supp}(\mu_Q) = \Sigma$ and if $\mu_Q$ has a density $v$, then Equation (2.1) yields

\begin{equation}
\int_\Sigma \log |x - t| v(t) \, dt = Q(x) - F, \quad x \in \Sigma.
\end{equation}
Then there is a unique constant $F$, such that the integral equation (2.4) has a solution $v(t)$ satisfying

$$\int_{\Sigma} v(t) dt = 1.$$  \hspace{1cm} (2.5)

If $\Sigma$ consists of a finite number of nondegenerate closed intervals, then (2.4) may be differentiated for $x$ in the interior of $\Sigma$ (since $Q'$ is Hölder continuous) to give the singular integral equation

$$\int_{\Sigma} \frac{v(t)}{x-t} dt = Q'(x), \quad x \in \text{int} \Sigma.$$  \hspace{1cm} (2.6)

It is well-known, see [7, §42.3], that the general solution of (2.6) depends on $N$ parameters, where $N$ is the number of intervals in $\Sigma$. These parameters are uniquely determined by the normalization (2.5) and the conditions that the constant $F$ in (2.4) should be the same on each interval of $\Sigma$. We also recall that the solutions of (2.6) are Hölder continuous on the interior of $\Sigma$, and may become unbounded at endpoints of $\Sigma$, cf. [7, §5, §42.3].

If we do not know that $\Sigma$ is the support of $\mu_Q$, we can still consider the function $v(t)$ determined by Equations (2.4) and (2.5). Then in general the function $v(t)$ will not be nonnegative on $\Sigma$. Thus $v(t)$ is the density of a signed measure $\eta$ that depends on $\Sigma$:

$$d\eta(t) = d\eta_{\Sigma}(t) = v(t) dt.$$  

The signed measure $\eta_{\Sigma}$ satisfies

$$\text{supp} (\eta_{\Sigma}) \subset \Sigma, \quad \int d\eta_{\Sigma} = 1,$$  \hspace{1cm} (2.7)

and it minimizes the weighted energy $I_Q(\eta)$ amongst all signed measures satisfying (2.7).

For the special case $\Sigma = [a, 1]$, with $a \in [-1, 1)$, we have that

$$\frac{d\eta_{\Sigma}}{dt} = v(t) = \frac{1}{\pi \sqrt{(1-t)(t-a)}} \left[1 + G(t)\right], \quad a < t < 1,$$  \hspace{1cm} (2.8)

with

$$G(t) = \frac{1}{\pi} \int_{a}^{1} \frac{Q'(s)}{s-t} \sqrt{(1-s)(s-a)} ds,$$  \hspace{1cm} (2.9)

see [7, §42.3] or [16, §4.3]. Note that (2.9) is equal to the function from (1.4).

Next, we recall the notion of balayage of a measure. The balayage of a nonnegative measure $\nu$ with compact support and continuous potential onto a set $\Sigma$ of positive capacity, is the unique measure $\hat{\nu}$ such that $\text{supp} (\hat{\nu}) \subset \Sigma$, $\|\nu\| = \|\hat{\nu}\|$ and for some constant $c$,

$$U^{\hat{\nu}}(x) = U^{\nu}(x) + c, \quad \text{for quasi every } x \in \Sigma.$$  \hspace{1cm} (2.10)
Here ‘quasi every’ means with the possible exception of a set of capacity zero. We refer the reader to [11, 13, 14] for these and other notions from logarithmic potential theory. Instead of $\hat{\nu}$ we also write $Bal(\nu; \Sigma)$. For a signed measure $\nu$ with Jordan decomposition $\nu = \nu^+ - \nu^-$, the balayage of $\nu$ onto $\Sigma$ is

$$Bal(\nu; \Sigma) = Bal(\nu^+; \Sigma) - Bal(\nu^-; \Sigma)$$

provided the balayages of $\nu^+$ and $\nu^-$ exist.

From their defining properties it is then easy to see that the measures $\eta_{\Sigma}$ are related by balayage. That is, if $\Sigma_1 \subset \Sigma_2$, then

$$\eta_{\Sigma_1} = Bal(\eta_{\Sigma_2}; \Sigma_1). \tag{2.11}$$

The following result will be used in the proof of Theorem 1.1 below. We say that two sets $A$ and $B$ are quasi-equal, if $A \setminus B$ and $B \setminus A$ have capacity zero.

**Lemma 2.1.** Let $\Sigma$ and $\Sigma_n, n \in \mathbb{N}$, be closed subsets of $[-1,1]$ having positive capacity such that

$$\Sigma = \bigcap_n \bigcup_{k \geq n} \Sigma_k \tag{2.12}$$

and $\Sigma$ is quasi-equal to

$$\bigcup_n \bigcap_{k \geq n} \Sigma_k. \tag{2.13}$$

Then the following hold.

(a) For every finite measure $\nu$ with compact support and continuous potential, we have

$$\lim_{n \to \infty} Bal(\nu; \Sigma_n) = Bal(\nu; \Sigma)$$

with convergence in the sense of weak* convergence of measures on $[-1,1]$.

(b) If $\Sigma$ and $\Sigma_n, n \in \mathbb{N}$, are finite unions of closed intervals, then

$$\lim_{n \to \infty} \eta_{\Sigma_n} = \eta_{\Sigma}$$

in the sense of weak* convergence of signed measures.

**Proof.** (a) Let us write $\nu_n = Bal(\nu; \Sigma_n)$. Then by (2.10), we have for some constant $c_n$,

$$U^{\nu_n}(x) = U^{\nu}(x) + c_n \quad \text{for quasi every } x \in \Sigma_n.$$

By weak* compactness, we may assume that $(\nu_n)$ converges, say with weak* limit $\nu^*$. Then $\|\nu^*\| = \|\nu\|$ and because of (2.12) we have $\operatorname{supp}(\nu^*) \subset \Sigma$.

The lower envelope theorem [14] says that

$$U^{\nu^*}(x) = \liminf_{n \to \infty} U^{\nu_n}(x) \quad \text{for quasi every } x \in C.$$
Since $\Sigma$ is quasi-equal to (2.13) it then follows that
\[ U^{\nu^*}(x) = U^{\nu}(x) + \lim\inf_{n \to \infty} c_n \quad \text{for quasi every } x \in \Sigma. \]

Then $\lim\inf c_n$ is finite and it follows that $\nu^*$ is the balayage of $\nu$ onto $\Sigma$.

(b) Let $\eta_0 = \eta_{[-1,1]}$. The positive and negative parts of $\eta_0$ in the Jordan decomposition $\eta_0 = \eta_0^+ - \eta_0^-$ are compactly supported. They also have continuous potentials. Indeed, the function $G$ from (2.9) (with $a = -1$) is continuous, and therefore it is bounded on $[-1,1]$. Then it follows from the representation (2.8)–(2.9) for $\eta_0$, that both $\eta_0^+$ and $\eta_0^-$ are bounded above by a constant times the measure $1/(\pi\sqrt{1-t^2})dt$. This measure has a continuous potential — in fact its potential is constant on $[-1,1]$ — and therefore the potentials of $\eta_0^+$ and $\eta_0^-$ are continuous as well, see [6, Lemma 5.2]. Thus it follows from part (a) that
\[ \text{Bal}(\eta_0^+; \Sigma_n) \xrightarrow{\ast} \text{Bal}(\eta_0^+; \Sigma) \]
and
\[ \text{Bal}(\eta_0^-; \Sigma_n) \xrightarrow{\ast} \text{Bal}(\eta_0^-; \Sigma). \]

Then
\[ \text{Bal}(\eta_0; \Sigma_n) \xrightarrow{\ast} \text{Bal}(\eta_0; \Sigma). \]

Since $\eta_\Sigma$ is equal to the balayage of $\eta_0$ onto $\Sigma$, and similarly $\eta_{\Sigma_n}$ is the balayage of $\eta_0$ onto $\Sigma_n$, part (b) follows. \qed

### 2.2. A lemma on convexity

The convexity assumption (a) of Theorem 1.1 will be used via the following lemma.

**Lemma 2.2.** Let $Q \in C^{1+\varepsilon}([-1,1])$. Let $\Sigma \subset [-1,1]$ be a finite union of nondegenerate closed intervals. Let $\eta = \eta_\Sigma$ be the signed measure associated with $\Sigma$, as described in Section 2.1, and let $v$ be the density of $\eta$. Suppose that $[a,b] \subset \Sigma$ and that
\begin{itemize}
  \item[(a)] $Q$ is convex on $[a,b]$,
  \item[(b)] $v(a) \geq 0$, and $v(b) \geq 0$,
  \item[(c)] $v(t) \geq 0$ on $\Sigma \setminus [a,b]$.
\end{itemize}
Then $v(t) > 0$ for all $t \in (a,b)$.

**Remark 2.3.** The density $v$ is continuous on the interior of $\Sigma$, and may become unbounded ($\pm \infty$) at endpoints of $\Sigma$. The assumption (b) is also satisfied if $v(a) = +\infty$ in case $a$ is an endpoint, and similarly for $b$.

**Proof.** First, we reduce the problem to the case $\Sigma = [a,b]$. Write $\eta = \eta_1 + \eta_2$, where $\eta_1$ is the restriction of $\eta$ to $[a,b]$ and $\eta_2$ the restriction to $\Sigma \setminus [a,b]$. From (2.4) we get
\[ U^{\eta_1}(x) + Q_1(x) = F \quad \text{for } x \in [a,b], \]
where
\[ Q_1(x) = U_{\eta_2}(x) + Q(x), \quad x \in [a, b]. \]
The measure \( \eta_2 \) is nonnegative by assumption (c). Then it is easy to see from (2.3) that the logarithmic potential \( U_{\eta_2} \) is convex on \( [a, b] \). Thus \( Q_1 \) is convex on \( [a, b] \) because of assumption (a). The potential \( U_{\eta_2} \) is real analytic on the open interval \( (a, b) \), and therefore \( Q_1' \) satisfies a H"older condition on \( (a, b) \). At the endpoints \( a \) and \( b \), \( Q_1' \) could have a singularity of logarithmic type, but this will not affect the arguments that follow. In particular, the representation (2.14) below, remains valid, cf. [16, §4.3].

Therefore, we have reduced the proof of the lemma to the case when \( \Sigma = [a, b] \). Without loss of generality, we may also assume that \( [a, b] = [-1, 1] \). Then, as in (2.8), the density \( v \) is given by
\[ v(t) = \frac{1}{\pi \sqrt{1 - t^2}} [1 + G(t)] \]
and
\[ G(t) = \frac{1}{\pi} \int_{-1}^{1} \frac{Q'(s)}{s - t} \sqrt{1 - s^2} ds. \]
In the principal value integral we remove the singular part as follows
\[ G(t) = \frac{1}{\pi} \int_{-1}^{1} \frac{Q'(s) - Q'(t)}{s - t} \sqrt{1 - s^2} ds + \frac{Q'(t)}{\pi} \int_{-1}^{1} \frac{1}{s - t} \sqrt{1 - s^2} ds. \]
The remaining principal value integral we write as
\[ \int_{-1}^{1} \frac{1}{s - t} \sqrt{1 - s^2} ds = \int_{-1}^{1} \frac{(1 - s^2) - (1 - t^2)}{s - t} \sqrt{1 - s^2} ds = - \int_{-1}^{1} \frac{1}{s - t} \sqrt{1 - s^2} ds, \]
where we used the fact that
\[ \int_{-1}^{1} \frac{1}{s - t} \sqrt{1 - s^2} ds = 0. \]
Thus
\[ G(t) = \frac{1}{\pi} \int_{-1}^{1} \frac{Q'(s) - Q'(t)}{s - t} \sqrt{1 - s^2} ds - \frac{Q'(t)}{\pi} \int_{-1}^{1} \frac{s + t}{s - t} \sqrt{1 - s^2} ds. \]
Next, we have that
\[ \left( \frac{1 + t}{2} \right) G(1) + \left( \frac{1 - t}{2} \right) G(-1) = - \left( \frac{1 + t}{2} \right) \frac{1}{\pi} \int_{-1}^{1} Q'(s)(1 + s) \frac{ds}{\sqrt{1 - s^2}} \]
\[ + \left( \frac{1 - t}{2} \right) \frac{1}{\pi} \int_{-1}^{1} Q'(s)(1 - s) \frac{ds}{\sqrt{1 - s^2}}. \]
Combining the two integrals, and using
\[ - \left( \frac{1 + t}{2} \right) (1 + s) + \left( \frac{1 - t}{2} \right) (1 - s) = -(s + t), \]
we obtain

\[(2.16) \quad \left(\frac{1+t}{2}\right) G(1) + \left(\frac{1-t}{2}\right) G(-1) = -\frac{1}{\pi} \int_{-1}^{1} Q'(s) \frac{s + t}{\sqrt{1 - s^2}} ds.\]

From (2.15) and (2.16) we learn that

\[(2.17) \quad G(t) - \left(\frac{1+t}{2}\right) G(1) - \left(\frac{1-t}{2}\right) G(-1) = \frac{1}{\pi} \int_{-1}^{1} \frac{Q'(s) - Q'(t)}{s - t} \sqrt{1 - s^2} ds - \frac{1}{\pi} \int_{-1}^{1} (Q'(t) - Q'(s)) \frac{s + t}{\sqrt{1 - s^2}} ds\]

\[= \frac{1}{\pi} \int_{-1}^{1} \frac{Q'(t) - Q'(s)}{t - s} \sqrt{1 - s^2} ds - \frac{1}{\pi} \int_{-1}^{1} \frac{Q'(t) - Q'(s)}{t - s} t^2 - s^2 \sqrt{1 - s^2} ds\]

\[= \frac{1}{\pi} \int_{-1}^{1} \frac{Q'(t) - Q'(s)}{t - s} \left[\sqrt{1 - s^2} - \frac{t^2 - s^2}{\sqrt{1 - s^2}}\right] ds\]

\[= \frac{1}{\pi} \int_{-1}^{1} \frac{Q'(t) - Q'(s)}{t - s} \frac{1 - t^2}{\sqrt{1 - s^2}} ds.\]

The convexity of \(Q\) implies that

\[\frac{Q'(t) - Q'(s)}{t - s} \geq 0\]

for every \(s\) and \(t\) in \((-1, 1)\). Then for \(t \in (-1, 1)\), the integral (2.17) is nonnegative and this proves the inequality

\[(2.18) \quad G(t) \geq \left(\frac{1+t}{2}\right) G(1) + \left(\frac{1-t}{2}\right) G(-1), \quad -1 < t < 1.\]

Actually, we have strict inequality in (2.18), unless \(Q'\) is a constant. Indeed, if equality holds in (2.18) at a certain \(t \in (-1, 1)\), then it follows from (2.17) that

\[\frac{Q'(t) - Q'(s)}{t - s} = 0\]

for almost all \(s \in (-1, 1)\). Since \(Q'\) is continuous, this can only happen if \(Q'(s) = Q'(t)\) for all \(s\), and this means that \(Q'\) is constant.

Thus, if \(Q'\) is not a constant, we see that

\[(2.19) \quad G(t) > \left(\frac{1+t}{2}\right) G(1) + \left(\frac{1-t}{2}\right) G(-1), \quad -1 < t < 1,\]

and then it follows from assumption (b) and (2.14) that \(1 + G(1) \geq 0\) and \(1 + G(-1) \geq 0\). The right-hand side of (2.19) is a convex combination of \(G(1)\) and \(G(-1)\). Thus it follows from (2.19) that \(1 + G(t) > 0\) for all \(t \in (-1, 1)\). In view of (2.14), we then have \(v(t) > 0\) in case \(Q'\) is not a constant.
If $Q'$ is a constant, say $Q'(t) = k$, then we obtain from (2.15) that $G(t) = -kt$. Hence

$$v(t) = \frac{1 - kt}{\pi \sqrt{1 - t^2}}, \quad -1 < t < 1.$$  

Then from $v(-1) \geq 0$ and $v(1) \geq 0$, we get $|k| \leq 1$, and then clearly $v(t) > 0$ on $(-1, 1)$. This completes the proof of Lemma 2.2. □

2.3. Proof of Theorem 1.1.

Proof. We write $\mu = \mu_Q$. Let us first assume that $\text{supp } (\mu) \subset [a_1, 1]$. From the assumption (b) of Theorem 1.1 with $a = a_1$, we have that there exists $t_0 \in (a_1, 1)$, such that

$$\frac{1}{\pi} \int_{a_1}^{1} \frac{Q'(s)}{s-t} \sqrt{(1-s)(s-a_1)} ds$$

is nonincreasing on $(a_1, t_0)$ and nondecreasing on $(t_0, 1)$. As no points of $\text{supp } (\mu)$ lie to the left of $a_1$, we may apply [9, Theorem 2] on the restricted interval $[a_1, 1]$ and deduce that $\text{supp } (\mu)$ is either an interval containing $a_1$, or an interval containing 1, or the union of an interval containing $a_1$ with an interval containing 1. This proves the theorem in case the support of $\mu$ is contained in $[a_1, 1]$.

For the rest of the proof, we shall assume that $\text{supp } (\mu)$ is not contained in $[a_1, 1]$. Let

$$a := \min \{x : x \in \text{supp } (\mu)\}$$

so that $a < a_1$.

For every pair $(p, q)$ with $a < p \leq q \leq 1$, we let $v_{p,q}$ be the density of the signed measure $\eta_\Sigma$ with $\Sigma = [a, p] \cup [q, 1]$ if $q < 1$ and $\Sigma = [a, p]$ if $q = 1$. See Section 2.1 for the definition of $\eta_\Sigma$.

We introduce the set $Z$ consisting of all pairs $(p, q)$ satisfying the following four conditions:

(a) $a < p \leq q \leq 1$ and $q \geq a_1$.
(b) $\text{supp } (\mu) \subset [a, p] \cup [q, 1]$.
(c) If $q < 1$ then $\pi \sqrt{(1-t)(t-a)} v_{p,q}(t)$ is nondecreasing for $t \in (q, 1)$.
(d) If $p > a_1$ then $\pi \sqrt{(1-t)(t-a)} v_{p,q}(t)$ is nonincreasing for $t \in (a_1, p)$.

We observe first that $Z \neq \emptyset$. Indeed, from the assumption (b) of Theorem 1.1 it follows that there exists $t_0 \in [a_1, 1]$ such that

$$\frac{1}{\pi} \int_{a}^{1} \frac{Q'(s)}{s-t} \sqrt{(1-s)(s-a)} ds, \quad a < t < 1$$
is nonincreasing on \((a_1, t_0)\) and nondecreasing on \((t_0, 1)\). Since for \(a < t < 1\), we have
\[
\pi \sqrt{(1-t)(t-a)} v_{a,t_0}(t) = 1 + \frac{1}{\pi} \int_a^1 \frac{Q'(s)}{s-t} \sqrt{(1-s)(s-a)} ds,
\]
by (2.8) and (2.9), we see that properties \((c)\) and \((d)\) are satisfied for the pair \((t_0, t_0)\). Properties \((a)\) and \((b)\) are trivially satisfied, so that \((t_0, t_0)\) belongs to \(Z\). Hence \(Z\) is nonempty indeed.

Next, we want to show that \(Z\) is closed. To this end, we take \((p, q) \in \bar{Z}\) and we choose sequences \((p_n)\) and \((q_n)\) such that
\[
(p_n, q_n) \in Z, \quad p_n \to p, \quad q_n \to q.
\]
We verify that the properties \((a)-(d)\) hold for the pair \((p, q)\). Since \((p_n, q_n)\) belongs to \(Z\), we have by \((b)\) that \([a, p_n] \cup [q_n, 1]\) contains the support of \(\mu\) for every \(n\). It then follows that \([a, p] \cup [q, 1]\) contains \(\text{supp}(\mu)\). Thus \((b)\) holds. Since \(a \in \text{supp}(\mu)\) and \(\text{supp}(\mu)\) does not have isolated points, we find that \(p > a\). The other inequalities of \((a)\) are immediate. To establish \((c)\) and \((d)\), we first note that by Lemma 2.1 we have in the sense of weak* convergence of signed measures
\[
(2.21) \quad v_{p_n, q_n}(t) dt \overset{*}{\to} v_{p, q}(t) dt.
\]
Now suppose that \((c)\) does not hold. Then there exist \(t_1\) and \(t_2\) with \(q < t_1 < t_2 < 1\) such that
\[
\pi \sqrt{(1-t_1)(t_1-a)} v_{p,q}(t_1) > \pi \sqrt{(1-t_2)(t_2-a)} v_{p,q}(t_2).
\]
Since \(v\) is continuous, there exists \(\varepsilon > 0\) such that
\[
\pi \int_{t_1-\varepsilon}^{t_1+\varepsilon} \sqrt{(1-t)(t-a)} v_{p,q}(t) dt > \pi \int_{t_2-\varepsilon}^{t_2+\varepsilon} \sqrt{(1-t)(t-a)} v_{p,q}(t) dt.
\]
We may assume that \(\varepsilon\) is chosen sufficiently small so that \([t_1-\varepsilon, t_1+\varepsilon]\) and \([t_2-\varepsilon, t_2+\varepsilon]\) are disjoint intervals that are both contained in \((q, 1)\). From the weak* convergence \((2.21)\) it then easily follows that we must have for \(n\) large enough,
\[
\pi \int_{t_1-\varepsilon}^{t_1+\varepsilon} \sqrt{(1-t)(t-a)} v_{p_n, q_n}(t) dt > \pi \int_{t_2-\varepsilon}^{t_2+\varepsilon} \sqrt{(1-t)(t-a)} v_{p_n, q_n}(t) dt.
\]
For \(n\) large enough, we also have \(q_n < t_1 - \varepsilon\). Then we arrive at a contradiction, since \((c)\) holds for the pair \((p_n, q_n)\). Thus property \((c)\) holds for the pair \((p, q)\). In a similar way, it follows that \((d)\) holds. Therefore \(Z\) is a closed set.

Since \(Z\) is a closed nonempty subset of \([a, 1] \times [a_1, 1]\), we can find a pair in \(Z\) for which the difference \(q - p\) is maximal. Such a maximizer may not be unique (when we have finished the proof, we will see that it is), but we take any such pair and denote it by \((p^*, q^*)\). Let \(\Sigma = [a, p^*] \cup [q^*, 1]\) in case
that \( q^* < 1 \), and \( \Sigma = [a, p^*] \) in case \( q^* = 1 \). For brevity, we write \( v^* \) instead of \( v_{p^*, q^*} \). Our aim is to show that \( \operatorname{supp}(\mu) = \Sigma \). Having established that, it will follow from the uniqueness of \( \mu \) that \((p^*, q^*)\) is the only maximizer for the difference \( q - p \). We prove that \( \operatorname{supp}(\mu) = \Sigma \) by showing that \( v^* \) is positive on the interior of \( \Sigma \).

We consider several cases. First we assume \( q^* < 1 \) and we consider the interval \((q^*, 1)\). Suppose that \( v^* \) is nonpositive somewhere on \((q^*, 1)\). Then by property (c) it follows that there exists \( \varepsilon \in (0, 1 - q^*) \) such that \( v^* \) is nonpositive on \([q^*, q^* + \varepsilon]\). We claim that \((p^*, q^* + \varepsilon)\) satisfies the conditions (a)-(d). It is clear that (a) is satisfied. For (b), we recall from [9, Lemma 3] that

\[
\operatorname{supp}(\mu) \subset \{ x : v^*(x) > 0 \} \subset [a, p^*] \cup [q^* + \varepsilon, 1].
\]

To see (c) and (d), we note that \( v_{p^*, q^* + \varepsilon} \) is obtained from \( v^* \) by taking the balayage of \( v^* \) onto \([a, p^*] \cup [q^* + \varepsilon, 1]\). Since \( v^* \) is nonpositive on the gap \((p^*, q^* + \varepsilon)\), we see using [9, Lemma 4 (2)], that this process preserves the properties (c) and (d). Thus \((p^*, q^* + \varepsilon) \in Z\). However, this contradicts the maximality of \( q^* - p^* \). Thus our assumption that \( v^* \) is nonpositive somewhere in \((q^*, 1)\) is incorrect, and it follows that \( v^* \) is positive on the interval \((q^*, 1)\).

Now consider the case \( p^* > a_1 \). In a similar way as above it follows that \( v^* \) is positive on \((a_1, p^*)\). Because of property (d) and the continuity of \( v^* \), we find \( v^*(a_1) > 0 \). Since

\[
\operatorname{supp}(\mu) \subset \{ x : v^*(x) > 0 \},
\]

we also have \( v^*(a_1) \geq 0 \). Since \( Q \) is convex on \([a, a_1]\) and \( v^* \geq 0 \) outside \([a, a_1]\), it follows from Lemma 2.2 that \( v^* > 0 \) on \((a_1, a_1)\). So we have shown that \( v^* > 0 \) on the interval \((a_1, p^*)\) in case \( p^* > a_1 \).

What remains is the case \( p^* \leq a_1 \). If \( v^*(p^*) < 0 \), then \( v^* \) is negative on \([p^* - \varepsilon, p^*]\) for some \( \varepsilon > 0 \) with \( \varepsilon < p^* - a \). Then we may take the balayage of this negative part onto \([a, p^* - \varepsilon] \cup [q^*, 1]\) and it follows as above that the pair \((p^* - \varepsilon, q^*)\) belongs to \( Z \). This is a contradiction. Thus \( v^*(p^*) \geq 0 \). Since \( Q \) is convex on \([a, p^*]\) with \( v^*(a) \geq 0 \), \( v^*(p^*) \geq 0 \), and \( v^* \geq 0 \) outside of \([a, p^*]\), it follows again from Lemma 2.2 that \( v^* \) is positive on \((a, p^*)\).

Thus in both cases, we find that \( v^* > 0 \) on \((a, p^*)\). We also proved that \( v^* > 0 \) on \((q^*, 1)\) in case \( q^* < 1 \). Thus \( v^* \) is positive on the interior of \( \Sigma \). It follows that \( \operatorname{supp}(\mu) = \Sigma \). This completes the proof of Theorem 1.1, since \( \Sigma \) is the union of at most two intervals. \( \square \)

### 3. The Proof of Theorem 1.3.

**Proof.** We write \( Q = Q_{a,c} \). Clearly \( Q \) is convex on \([-1, 0]\). Let us set for \( a \in [-1, 0] \) and for \( t \in [0, 1] \),
Here the second integral $I_2$ is a principal value integral.

We have to prove that there exists $t_\alpha \in [0,1)$ so that $G_\alpha$ decreases in $(0,t_\alpha)$ and increases in $(t_\alpha,1)$ (if $t_\alpha = 0$ then the first condition is an empty one). We establish the following properties:

(i) $G_\alpha(0) \leq 0$;
(ii) $G_\alpha(1) > 0$;
(iii) For every $\alpha > 1$, there is $t_\alpha \in [0,1)$, such that $G'_\alpha(t) < 0$ on $(0,t_\alpha)$, $G'_\alpha(t) > 0$ on $(t_\alpha,1)$, and $G''_\alpha(t) \geq 0$ on $(t_\alpha,1)$.

Clearly, then (iii) implies the decreasing/increasing property of $G_\alpha$.

To show (i), we write

$$G_\alpha(0) = -\frac{1}{\pi} \left[ \int_0^1 s^{\alpha-2} \sqrt{(1-s)(s-a)} ds - \int_a^0 |s|^{\alpha-1} \sqrt{(1-s)(s-a)} ds \right]$$

and in the second integral we make the change of variables $s \mapsto as$, to find

$$G_\alpha(0) = -\frac{1}{\pi} \int_0^1 s^{\alpha-2} \sqrt{1-s} \left( \sqrt{s-a} - |a|^{\alpha-1} \frac{\sqrt{1-as}}{\sqrt{1-s}} \right) ds.$$ 

Since $\sqrt{s-a}$ is greater than or equal to $|a|^{\alpha-\frac{1}{2}} \sqrt{1-as}$ for $s$ in the interval $[0,1]$, we find that $G_\alpha(0) \leq 0$, as claimed in (i). Note that $G_\alpha(0) = 0$ if and only if $a = -1$.

Next, it is easy to see from (3.1) that

$$G_\alpha(1) = \frac{1}{\pi} \int_a^1 \frac{s^{\alpha-1} \sqrt{s-a}}{\sqrt{1-s}} ds > 0,$$

which establishes (ii) for all $\alpha \geq 1$.

We now prove (iii) by induction on $k = \lfloor \alpha \rfloor$, where $\lfloor \alpha \rfloor$ denotes the integer part of $\alpha$.

For $\alpha = 1$, we find by explicit calculation

$$G_1(t) = -\frac{1}{\pi} \int_a^1 \frac{\sqrt{(1-s)(s-a)}}{s-t} ds = t - \frac{1+a}{2}.$$
Then (iii) is satisfied with $t_1 = t_\alpha = 0$. Suppose now $1 < \alpha < 2$. Consider the analytic function

$$f(z) := z^{\alpha - 1} [(z - 1)(z - a)]^{1/2}$$

defined for $z \in \mathbb{C} \setminus (-\infty, 1]$, where that branch of the square root is chosen which is positive for real $z > 1$. Then the principal value integral $I_2$ may be written as

$$I_2 = -\frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{z - t} \, dz$$

with the contour $\gamma$ going from 0 to 1 on the upper side of the cut $(-\infty, 1]$ and back from 1 to 0 on the lower side.

We transform $\gamma$ into the contour $\Gamma_R$ going from 0 to $-R$ on the upper side of the cut, continuing along the big circle $C_R$ of radius $R$ going to $-R$ on the lower side of the cut, and then going from $-R$ to 0 on the lower side of the cut. We choose $R > 1$. See Figure 1 for $\gamma$ and $\Gamma_R$.

The contribution from the upper and lower sides of the cut comes from the imaginary part of $f$, which is

$$\Im f(x + i0) = \begin{cases} 
\sin(\alpha \pi)|x|^{\alpha - 1} \sqrt{(1 - x)(a - x)} & \text{for } x < a, \\
-\cos(\alpha \pi)|x|^{\alpha - 1} \sqrt{(1 - x)(x - a)} & \text{for } a < x < 0,
\end{cases}$$

**Figure 1.** The contours $\gamma$ and $\Gamma_R$. 
and $\Im f(x - i0) = -\Im f(x + i0)$. Therefore

$$I_2 = -\frac{1}{2\pi i} \int_{C_R} \frac{f(z)}{z - t} \, dz$$

$$+ \frac{\sin \alpha \pi}{\pi} \int_{-R}^{a} \frac{|x|^{\alpha-1} \sqrt{(1-x)(a-x)}}{x-t} \, dx$$

$$- \frac{\cos \alpha \pi}{\pi} \int_{a}^{0} \frac{|x|^{\alpha-1} \sqrt{(1-x)(x-a)}}{x-t} \, dx.$$ 

Thus we have shown that

$$(3.3) \quad G_\alpha(t) = -\frac{1}{2\pi i} \int_{C_R} \frac{f(z)}{z - t} \, dz$$

$$+ \frac{\sin \alpha \pi}{\pi} \int_{-R}^{a} \frac{|x|^{\alpha-1} \sqrt{(1-x)(a-x)}}{x-t} \, dx$$

$$- \left(1 + \cos \alpha \pi\right) \pi \int_{0}^{a} \frac{|x|^{\alpha-1} \sqrt{(1-x)(x-a)}}{(x-t)^3} \, dx.$$ 

From (3.3), we obtain for the second derivative

$$(3.4) \quad G''_\alpha(t) = -\frac{1}{\pi i} \int_{C_R} \frac{f(z)}{(z-t)^3} \, dz$$

$$+ \frac{2 \sin \alpha \pi}{\pi} \int_{-R}^{a} \frac{|x|^{\alpha-1} \sqrt{(1-x)(a-x)}}{(x-t)^3} \, dx$$

$$- \frac{2 (1 + \cos \alpha \pi)}{\pi} \int_{0}^{a} \frac{|x|^{\alpha-1} \sqrt{(1-x)(x-a)}}{(x-t)^3} \, dx.$$ 

We let $R \to \infty$ in (3.4). Then the integral over the circle $C_R$ tends to 0, since the integrand behaves like $z^{\alpha-3}$ as $|z| \to \infty$. Then we get the representation

$$(3.5) \quad G''_\alpha(t) = \frac{2 \sin \alpha \pi}{\pi} \int_{-\infty}^{a} \frac{|x|^{\alpha-1} \sqrt{(1-x)(a-x)}}{(x-t)^3} \, dx$$

$$- \frac{2 (1 + \cos \alpha \pi)}{\pi} \int_{0}^{a} \frac{|x|^{\alpha-1} \sqrt{(1-x)(x-a)}}{(x-t)^3} \, dx.$$ 

Note that the improper integral is convergent because $\alpha < 2$. Since $1 < \alpha < 2$, we have $\sin \alpha \pi < 0$. Also $(x-t)^3 < 0$ whenever $x < 0 < t$. Thus we conclude that

$$(3.6) \quad G''_\alpha(t) > 0, \quad \text{for } t \in (0, 1),$$

in case $1 < \alpha < 2$. Thus $G_\alpha$ is strictly convex on $(0, 1)$. Since $G_\alpha(0) < G_\alpha(1)$ by properties (i) and (ii) the property (iii) follows for $\alpha \in (1, 2)$. Thus we have established (iii) whenever $k = [\alpha] = 1$. 


Now let \( k \geq 2 \) and suppose that (iii) is true for all \( \alpha \) with \( [\alpha] = k - 1 \).
Let \( \alpha \in [k, k + 1) \). From (3.1) we obtain for \( 0 < t < 1 \),
\[
(3.7) \quad G_\alpha(t) = -\frac{1}{\pi} \int_a^1 \frac{|x|^{\alpha - 1} - t|x|^{\alpha - 2} + t|x|^{\alpha - 2}}{x - t} \sqrt{(1 - x)(x - a)} \, dx
= -\frac{1}{\pi} \int_a^1 \frac{|x|^{\alpha - 1} - t|x|^{\alpha - 2}}{x - t} \sqrt{(1 - x)(x - a)} \, dx + tG_{\alpha - 1}(t)
= F(t) + tG_{\alpha - 1}(t).
\]
We can write that
\[
F(t) := -\frac{1}{\pi} \int_a^1 \frac{|x - t|}{x - t} |x|^{\alpha - 2} \sqrt{(1 - x)(x - a)} \, dx
= \frac{1}{\pi} \int_a^0 \frac{x + t}{x - t} |x|^{\alpha - 2} \sqrt{(1 - x)(x - a)} \, dx
- \frac{1}{\pi} \int_0^1 |x|^{\alpha - 2} \sqrt{(1 - x)(x - a)} \, dx,
\]
from which we obtain
\[
(3.8) \quad F'(t) = \frac{1}{\pi} \int_a^0 \frac{2x}{(x - t)^2} |x|^{\alpha - 2} \sqrt{(1 - x)(x - a)} \, dx < 0, \quad 0 < t < 1
\]
and
\[
(3.9) \quad F''(t) = \frac{1}{\pi} \int_a^0 \frac{4x}{(x - t)^3} |x|^{\alpha - 2} \sqrt{(1 - x)(x - a)} \, dx > 0, \quad 0 < t < 1.
\]
Differentiating (3.7) we get
\[
(3.10) \quad G''_\alpha(t) = F''(t) + G_{\alpha - 1}(t) + tG''_{\alpha - 1}(t)
\]
and
\[
(3.11) \quad G''_\alpha(t) = F''(t) + 2G'_{\alpha - 1}(t) + tG''_{\alpha - 1}(t).
\]
By the inductive hypothesis, there exists \( t_\alpha - 1 \), such that \( G'_{\alpha - 1}(t) \) is negative on \((0, t_\alpha - 1)\) and positive on \((t_\alpha - 1, 1)\), as well as \( G''_{\alpha - 1}(t) \geq 0 \) on \((t_\alpha - 1, 1)\).
Suppose first that \( t_\alpha - 1 > 0 \). Since \( G_{\alpha}(0) \leq 0 \) and \( G'_{\alpha - 1}(t) < 0 \) on \((0, t_\alpha - 1)\), we have that \( G_{\alpha - 1}(t) \) is strictly decreasing on \((0, t_\alpha - 1)\), and therefore is negative there. This, together with (3.8) and (3.10), implies that \( G''_{\alpha}(t) < 0 \) on \((0, t_\alpha - 1)\). On the other hand from (3.9), (3.11) and the inductive hypothesis, we obtain that \( G'''_{\alpha}(t) > 0 \) on \([t_\alpha - 1, 1)\). This implies that \( G_{\alpha}(t) \) is strictly convex on \((t_\alpha - 1, 1)\). Since \( G_{\alpha} \) and \( G'_{\alpha} \) are negative on \((0, t_\alpha - 1)\), and \( G_{\alpha}(1) > 0 \), we see that there exists \( t_\alpha \in (t_\alpha - 1, 1)\), such that \( G''_{\alpha}(t) \) is negative on \((0, t_\alpha)\) and positive on \((t_\alpha, 1)\). It is clear also that \( G''_{\alpha}(t) > 0 \) on \((t_\alpha, 1)\). Thus property (iii) holds in case \( t_\alpha - 1 > 0 \).
If $t_{\alpha-1} = 0$, then we still use (3.9) and (3.11) to derive $G_{\alpha}''(t) > 0$ on $(0, 1)$, which implies that $G_{\alpha}$ is strictly convex on $[0, 1]$. Since $G_{\alpha}(0) < G_{\alpha}(1)$, the property (iii) follows as well.

The property (iii) is now established whenever $\lceil \alpha \rceil = k$. By induction we derive that it is true for every $k \geq 1$, that is it holds for every $\alpha \geq 1$. Thus there exists $t_0 \in [0, 1)$ such that (1.5) decreases on $(0, t_0)$ and increases on $(t_0, 1)$. Since $Q$ is convex on $[-1, 0]$, the conditions of Theorem 1.1 are satisfied with $a_1 = 0$. It follows from Theorem 1.1 that the support of the equilibrium measure consists of at most two intervals. □

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Received July 16, 1999 and revised October 11, 2000. The research of the first author began while visiting the third author at the Mathematics Department of the City University of Hong Kong and was supported partly by grants from the Centre for Applicable Analysis and Number Theory, Johannesburg, and Georgia Southern University. The work of the second author was supported by an Indiana University–Purdue University Fort Wayne Summer Research Grant. The work of the third author started while at the City University of Hong Kong. His research is supported in part by FWO research project G.0278.97, by a research grant of the Fund for Scientific Research – Flanders, and by a University of the Witwatersrand Research grant, jointly with S.B. Damelin.

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INVARnants of GENERIC IMMERSIONS

Tobias Ekholm

First order invariants of generic immersions of manifolds of dimension \( nm - 1 \) into manifolds of dimension \( n(m + 1) - 1 \), \( m, n > 1 \) are constructed using the geometry of self-intersections. The range of one of these invariants is related to Bernoulli numbers.

As by-products some geometrically defined invariants of regular homotopy are found.

1. Introduction.

An immersion of a smooth manifold \( M \) into a smooth manifold \( W \) is a smooth map with everywhere injective differential. Two immersions are regularly homotopic if they can be connected by a continuous 1-parameter family of immersions.

An immersion is generic if all its self-intersections are transversal. In the space \( \mathcal{F} \) of immersions \( M \to W \), generic immersions form an open dense subspace. Its complement is the discriminant hypersurface, \( \Sigma \subset \mathcal{F} \). Two generic immersions belong to the same path component of \( \mathcal{F} - \Sigma \) if they can be connected by a regular homotopy, which at each instance is a generic immersion. We shall consider the classification of generic immersions up to regular homotopy through generic immersions. It is similar to the classification of embeddings up to diffeotopy (knot theory). In both cases, all topological properties of equivalent maps are the same.

An invariant of generic immersions is a function on \( \mathcal{F} - \Sigma \) which is locally constant. The value of such a function along a path in \( \mathcal{F} \) jumps at intersections with \( \Sigma \). Invariants may be classified according to the complexity of their jumps. The most basic invariants in this classification are called first order invariants (see Section 2).

In [2], Arnold studies generic regular plane curves (i.e., generic immersions \( S^1 \to \mathbb{R}^2 \)). He finds three first order invariants \( J^+, J^-, \) and St. In [4], the author considers the case \( S^3 \to \mathbb{R}^5 \). Two first order invariants \( J \) and \( L \) are found. In both these cases, the only self-intersections of generic immersions are transversal double points and in generic 1-parameter families there appear isolated instances of self-tangencies and triple points. The values of the invariants \( J^\pm \) and \( J \) change at instances of self-tangency and remain
constant at instances of triple points. The invariants \(S_t\) and \(L\) change at triple points and remain constant at self-tangencies.

In this paper we shall consider high-dimensional analogs of these invariants. The most straightforward generalizations arise for immersions \(M^{2n-1} \to W^{3n-1}\), where self-tangencies and triple points are the only degeneracies in generic 1-parameter families. The above range of dimensions is included in a 2-parameter family, \(M^{nm-1} \to W^{n(m+1)-1}\), \(m, n > 1\), where generic immersions do not have \(k\)-fold self-intersection points if \(k > m\) and in generic 1-parameter families there appear isolated instances of \((m+1)\)-fold self-intersection. Under these circumstances, we find first order invariants of generic immersions. (For precise statements, see Theorem 1 and Theorem 3 in Section 2, where the main results of this paper are formulated.)

In particular, if \(n\) is even and \(M^{nm-1}\) is orientable and satisfies a certain homology condition (see Theorem 1 (b)) then there exists an integer-valued invariant \(L\) of generic immersions \(M^{nm-1} \to \mathbb{R}^{n(m+1)-1}\) which is an analog of Arnold’s \(S_t\): It changes under instances of \((m+1)\)-fold self-intersection, does not change under other degeneracies which appear in generic 1-parameter families, and is additive under connected sum. The value of \(L\) at a generic immersion \(f\) is the linking number of a copy of the set of \(m\)-fold self-intersection points of \(f\), shifted in a special way, with \(f(M)\) in \(\mathbb{R}^{n(m+1)-1}\). (See Definition 4.13.)

For codimension two immersions of odd-dimensional spheres into Euclidean space there appear restrictions on the possible values of \(L\) (see Theorem 2). This phenomenon is especially interesting in the case of immersions \(S^{4j-1} \to \mathbb{R}^{4j+1}\).

In general, it is not known if these restrictions are all the restrictions on the range of \(L\). However, in special cases they are. For example, for immersions \(S^5 \to \mathbb{R}^5\) the range of \(L\) is \(\mathbb{Z}\) (see [4] and also Remark 9.3) and for immersions \(S^{67} \to \mathbb{R}^{69}\) it is \(35\mathbb{Z}\) (see Section 9.3).

In [5], the author gives a complete classification of generic immersions \(M^k \to \mathbb{R}^{2k-r}\), \(r = 0, 1, 2, k \geq 2r + 4\) up to regular homotopy through generic immersions (under some conditions on the lower homotopy groups of \(M\)). Here, the class of a generic immersion is determined by its self-intersection with induced natural additional structures (e.g. spin structures). The existence of invariants such as \(L\) mentioned above implies that the corresponding classification in other dimensions is more involved (see Remark 9.2).

The invariants of generic immersions in Theorems 1 and 3 give rise to invariants of regular homotopy which take values in finite cyclic groups (Section 7). We construct examples showing that, depending on the source and target manifolds, these regular homotopy invariants may or may not be trivial (Section 9). It would be interesting to relate these invariants to invariants arising from the cobordism theory of immersions (see for example Eccles [3]).
2. Statements of the main results.

Before stating the main results we define the notions of invariants of orders zero and one. They are similar to knot invariants of finite type introduced by Vassiliev in [13].

As in the Introduction, let \( \mathcal{F} \) denote the space of immersions and let \( \Sigma \subset \mathcal{F} \) denote the discriminant hypersurface. The set \( \Sigma^1 \subset \Sigma \) of non-generic immersions which appear at isolated instances in generic 1-parameter families (see Lemmas 3.5 and 3.6 for descriptions of such immersions) is a smooth submanifold of \( \mathcal{F} \) of codimension one. (\( \mathcal{F} \) is an open subspace of the space of smooth maps, thus an infinite dimensional manifold and the notion of codimension in \( \mathcal{F} \) makes sense.)

If \( f_0 \in \Sigma^1 \) then there is a neighborhood \( U(f_0) \) of \( f_0 \) in \( \mathcal{F} \) cut in two parts by \( \Sigma^1 \). A coherent choice of a positive and negative part of \( U(f_0) \) for each \( f_0 \in \Sigma^1 \) is a coorientation of \( \Sigma \). (The coorientation of the discriminant hypersurface in the space of immersions is considered in Section 6.1.)

Let \( a \) be an invariant of generic immersions and let \( f_0 \in \Sigma^1 \). Define the jump \( \nabla a \) of \( a \) as

\[
\nabla a(f_0) = a(f_+ - a(f_-),
\]

where \( f_+ \) and \( f_- \) are generic immersions in the positive respectively negative part of \( U(f_0) \). Then \( \nabla a \) is a locally constant function on \( \Sigma^1 \).

An invariant \( a \) of generic immersions is a zero order invariant (or which is the same, an invariant of regular homotopy) if \( \nabla a \equiv 0 \).

Let \( \Sigma^2 \) be the set of all immersions which appear in generic 2-parameter families but can be avoided in generic 1-parameter families (see Lemma 3.7 for the case \( M^{nm-1} \rightarrow W^{n(m+1)-1} \) is considered in Section 6.1.)

An invariant \( a \) of generic immersions is a first order invariant if \( \nabla a(f_0) = \nabla a(f_1) \) for any immersions \( f_0, f_1 \in \Sigma^1 \) which can be joined by a path in \( \Sigma^1 \cup \Sigma^2 \) such that at intersections with \( \Sigma^2 \) its tangent vector is transversal to the tangent space of \( \Sigma^2 \).

**Theorem 1.** Let \( m > 1 \) and \( n > 1 \) be integers and let \( M^{nm-1} \) be a closed manifold.

(a) If \( m \) is even and \( H_{n-1}(M; \mathbb{Z}_2) = 0 = H_n(M; \mathbb{Z}_2) \) then there exists a unique (up to addition of zero order invariants) first order \( \mathbb{Z}_2 \)-valued invariant \( \Lambda \) of generic immersions \( M^{nm-1} \rightarrow \mathbb{R}^{n(m+1)-1} \) satisfying the following conditions: It jumps by 1 on the part of \( \Sigma^1 \) which consists of immersions with one \( (m + 1) \)-fold self-intersection point and does not jump on other parts of \( \Sigma^1 \).

(b) If \( n \) is even, \( M \) is orientable, and \( H_{n-1}(M; \mathbb{Z}) = 0 = H_n(M; \mathbb{Z}) \) then there exists a unique (up to addition of zero order invariants) first
order integer-valued invariant $L$ of generic immersions $M^{nm-1} \to \mathbb{R}^{n(m+1)-1}$ satisfying the following conditions: It jumps by $m+1$ on the part of $\Sigma^1$ which consists of immersions with one $(m+1)$-fold self-intersection point and does not jump on other parts of $\Sigma^1$.

Theorem 1 is proved in Section 6.4. The invariants $\Lambda$ and $L$ are defined in Definition 4.12 and Definition 4.13, respectively. If appropriately normalized, $\Lambda$ and $L$ are additive under connected summation of generic immersions (see Section 5.1).

**Theorem 2.** Let $f: S^{2m-1} \to \mathbb{R}^{2m+1}$ be a generic immersion.

(a) If $m = 4j + 1$ then $2j + 1$ divides $L(f)$.
(b) If $m = 4j + 3$ then $4j + 4$ divides $L(f)$.
(c) If $m = 2j$ then $p^r$ divides $L(f)$ for every prime $p$ and integer $r$ such that $p^{r+k}$ divides $2j + 1$ and $p^{k+1}$ does not divide $\mu_j$ for some integer $k$. Here $\mu_j$ is the denominator of $\frac{B_j}{4j}$, where $B_j$ is the $j$th Bernoulli number.

We prove Theorem 2 in Section 8.2.

**Theorem 3.** Let $m > 1$ and $n > 1$ be integers and let $M^{nm-1}$ be a closed manifold and let $W^{n(m+1)-1}$ be a manifold.

(a) If $n$ is odd then, for each integer $2 \leq r \leq m$ such that $m - r$ is even, there exist a unique (up to addition of zero order invariants) first order integer-valued invariant $J_r$ of generic immersions $M^{nm-1} \to W^{n(m+1)-1}$ satisfying the following conditions: It jumps by 2 on the part of $\Sigma^1$ which consists of immersions with one degenerate $r$-fold self-intersection point and does not jump on other parts of $\Sigma^1$.

(b) If $n = 2$ then there exists a unique (up to addition of zero order invariants) first order integer-valued invariant $J$, of generic immersions $M^{2m-1} \to W^{2m+1}$ satisfying the following conditions: It jumps by 1 on the part of $\Sigma^1$ which consists of immersions with one degenerate $m$-fold self-intersection point and does not jump on other parts of $\Sigma^1$.

Theorem 3 is proved in Section 6.3. If appropriately normalized, $J_r$ and $J$ are additive under connected summation of generic immersions (see Section 5.1). The value of $J_r$ on a generic immersion is the Euler characteristic of its resolved $r$-fold self-intersection manifold, the value of $J$ is the number of components in its $m$-fold self-intersection (which is a closed 1-dimensional manifold).
3. Generic immersions and generic regular deformations.

In this section we define generic immersions and describe the immersions corresponding to non-generic instances in generic 1- and 2-parameter families. We also describe the versal deformations of these non-generic immersions.

3.1. Generic immersions. Let \( M \) and \( W \) be smooth manifolds and let \( f: M \to W \) be an immersion. A point \( q \in W \) is a \( k \)-fold self-intersection point of \( f \) if \( f^{-1}(q) \) consists of exactly \( k \) points. Let \( \Gamma_k(f) \subset W \) denote the set of \( k \)-fold self-intersection points of \( f \) and let \( \tilde{\Gamma}_k(f) = f^{-1}(\Gamma_k(f)) \subset M \) denote its preimage. Note that \( f^*\tilde{\Gamma}_k(f) \to \Gamma_k(f) \) is a \( k \)-fold covering.

**Definition 3.1.** Let \( M^m \to W^{n(m+1)-1} \) be a generic immersion if

1. \( \Gamma_r(f) \) is empty for \( r > m \), and
2. if \( q = f(p_1) = \cdots = f(p_k) \in \Gamma_k(f) \) \((k \leq m)\) then for any \( i, 1 \leq i \leq k \)
   \[ df_{\tilde{T}_{p_i}M} + \bigcap_{s \neq i} df_{\tilde{T}_{p_s}M} = T_qW. \]

A standard application of the jet-transversality theorem shows that the set of generic immersions is open and dense in the space of all immersions \( \mathcal{F} \).

We remark that the set of \( k \)-fold self-intersection points \( \Gamma_k(f) \) of a generic immersion \( f: M^{nm} \to W^{n(m+1)-1} \) is a smooth submanifold of \( W \) of dimension \( n(m - k + 1) - 1 \). Moreover, the deepest self-intersection \( \Gamma_m(f) \) is a closed manifold.

**Lemma 3.2.** Below gives a local coordinate description of a generic immersion close to a \( k \)-fold self-intersection point. To state it we introduce some notation: Let \( x_i \in \mathbb{R}^{nm-1} \), we write \( x_i = (x_i^0, x_i^1, \ldots, x_i^{k-1}) \), where \( x_i^0 \in \mathbb{R}^{(n-m-1)-1} \) and \( x_i^r \in \mathbb{R}^m \), for \( 1 \leq r \leq k - 1 \). Similarly, we write \( y \in \mathbb{R}^{n(m+1)-1} \) as \( y = (y^0, y^1, \ldots, y^k) \), where \( y^0 \in \mathbb{R}^{n(m-k-1)-1} \) and \( y^r \in \mathbb{R}^n \), for \( 1 \leq r \leq k \).

**Lemma 3.2.** Let \( f: M^{nm} \to W^{n(m+1)-1} \) be a generic immersion and let \( q = f(p_1) = \cdots = f(p_k) \) be a \( k \)-fold self-intersection point of \( f \). Then there are coordinates \( y \in V \subset W^{n(m+1)-1} \) centered at \( q \) and coordinates \( x_i \) on \( U_i \subset M^{nm} \) centered at \( p_i \), \( 1 \leq i \leq k \) such that \( f \) is given by

\[
\begin{align*}
  f(x_1) &= (x_1^0, x_1^1, \ldots, x_1^{k-2}, x_1^{k-1}, 0), \\
  f(x_2) &= (x_2^0, x_2^1, \ldots, x_2^{k-2}, 0, x_2^{k-1}), \\
  & \vdots \\
  f(x_k) &= (x_k^0, 0, x_k^1, x_k^2, \ldots, x_k^{k-1}).
\end{align*}
\]

(That is, if \( y = f(x_i) \) then \( y^0(x_i) = x_i^0 \), \( y^r(x_i) = x_i^r \) for \( 1 \leq r \leq k - i \), \( y^{k-i+1}(x_i) = 0 \), and \( y^r(x_i) = x_i^{r-1} \) for \( k - i + 2 \leq r \leq k \).)
Proof. The proof is straightforward.

When the source and target of a generic immersion are oriented and the codimension is even then there are induced orientations on the self intersection manifolds:

**Proposition 3.3.** Let \( n = 2j > 1, m > 1, 2 \leq k \leq m, \) and let \( f: \mathcal{M}^{2j(m-1)} \rightarrow \mathcal{W}^{2j(m+1)-1} \) be a generic immersion. Orientations on \( \mathcal{M}^{2j(m-1)} \) and \( \mathcal{W}^{2j(m+1)-1} \) induce an orientation on \( \Gamma_k(f) \).

**Proof.** Let \( N \) denote the normal bundle of the immersion. The decomposition \( f^* \mathcal{W}^{2j(m+1)-1} = T \mathcal{M}^{2j(m-1)} \oplus N \) induces an orientation on \( N \). If \( q \in \Gamma_k(f), q = f(p_1) = \cdots = f(p_k) \) then \( T_q \mathcal{W}^{2j(m+1)-1} = T_q \Gamma_k(f) \oplus \bigoplus_{i=1}^k N_{p_i} \).

The orientation on \( N \) induces an orientation on \( \bigoplus_{i=1}^k N_{p_i} \). Since the dimension of the bundle \( N \) is \( 2j \) which is even, the orientation on the sum is independent on the ordering of the summands. Hence, the decomposition above induces a well-defined orientation on \( T \Gamma_k(f) \). \( \square \)

**3.2. The codimension one part of the discriminant hypersurface.**

Our next result describes (the 1-jets of) immersions in \( \Sigma^1 \) (see Section 2).

**Lemma 3.4.** If \( f_0: \mathcal{M}^{nm-1} \rightarrow \mathcal{W}^{n(m+1)-1} \) is an immersion in \( \Sigma^1 \) then \( g_1 \) and \( g_2 \) of Definition 3.1 holds, except at one \( k \)-fold \( (2 \leq k \leq m+1) \) self-intersection point \( q = f_0(p_1) = \cdots = f_0(p_k) \), where,

(a) if \( k = 2 \),

\[
\dim(df_0 T_{p_1} M + df_0 T_{p_2} M) = n(m + 1) - 2,
\]

or

(b) if \( 2 < k \leq m + 1 \), for \( i \neq l \)

\[
\dim(df_0 T_{p_i} M + \bigcap_{r \neq i, r \neq l} df_0 T_{p_r} M) = n(m + 1) - 1
\]

and

\[
\dim(df_0 T_{p_i} M + \bigcap_{r \neq i} df_0 T_{p_r} M) = n(m + 1) - 2.
\]

**Proof.** We have to show that degeneracies as above appears at isolated parameter values in generic 1-parameter families, and that further degeneracies (of the 1-jet) may be avoided. This follows from the jet-transversality theorem applied to maps \( \mathcal{M}^{nm-1} \times [-\delta, \delta] \rightarrow \mathcal{W}^{n(m+1)-1} \). \( \square \)

A \( k \)-fold self-intersection point \( q \) of an immersion \( f_0: \mathcal{M}^{nm-1} \rightarrow \mathcal{W}^{n(m+1)-1} \) where \( g_1 \) and \( g_2 \) of Definition 3.1 does not hold will be called a degenerate self-intersection point of \( f_0 \). If \( 2 \leq k \leq m \) we say that \( q \) is a \( k \)-fold self-tangency point of \( f_0 \) if \( k = m + 1 \) we say that \( q \) is a \((m+1)\)-fold self-intersection point of \( f_0 \).
Recall that a deformation $F$ of a map $f$ is called versal if any deformation of $f$ is equivalent (up to left-right action of diffeomorphisms) to one induced from $F$.

Let $f_0$ be an immersion in $\Sigma^1$. Then its versal deformation $f_t$ is a 1-parameter deformation. In other words, it is a path $\lambda(t) = f_t$ in $\mathcal{F}$ which intersects $\Sigma^1$ transversally at $f_0$ and thus, $f_t$ are generic immersions for small $t \neq 0$.

Next, we shall describe immersions $f_0 \in \Sigma^1$ in local coordinates close to their degenerate self intersection point. Self-tangency points and $(m + 1)$-fold self-intersection points are treated in Lemma 3.5 and Lemma 3.6, respectively.

To accomplish this we need a more detailed description of coordinates than that given in Lemma 3.2, we use coordinates as there with one more ingredient: We write (when necessary) $x^r_i \in \mathbb{R}^n$ and $y^r_i \in \mathbb{R}^n, r \geq 1$ as $x^r_i = (\xi^r_i, u^r_i)$ and $y^r_i = (\eta^r_i, v^r_i)$, where $\xi^r_i, \eta^r_i \in \mathbb{R}$ and $u^r_i, v^r_i \in \mathbb{R}^{n-1}$. (Greek letters for scalars and Roman for vectors.)

**Lemma 3.5.** Let $f_0: M^{nm-1} \rightarrow W^{n(m+1)-1}$ be an immersion in $\Sigma^1$ and let $q = f_0(p_1) = \cdots = f_0(p_k)$ be a point of degenerate $k$-fold self intersection $2 \leq k \leq m$. Then there are coordinates $y$ on $V \subset W$ centered at $q$ and coordinates $x_i$ on $U_i \subset M$ centered at $p_i, 1 \leq i \leq k$ such that in these coordinates the versal deformation $f_t, -\delta < t < \delta$ of $f_0$ is constant outside of $\cup_i U_i$ and in neighborhoods of $p_i \in U_i$ it is given by

(a) \[
\begin{align*}
    f_t(x_1) &= (x^0_1, x^1_1, \ldots, x^{k-2}_1, x^{k-1}_1, 0), \\
    f_t(x_2) &= (x^0_2, x^1_2, \ldots, x^{k-2}_2, 0, x^{k-1}_2), \\
    &\vdots \\
    f_t(x_{k-1}) &= (x^0_{k-1}, x^1_{k-1}, 0, x^2_{k-1}, \ldots, x^{k-1}_{k-1}).
\end{align*}
\]
(That is, if $y = f_t(x_i)$ then for $1 \leq i \leq k - 1$, $y^0(x_i) = x^0_i$, $y^1(x_i) = x^1_i$, $y^r(x_i) = x^r_i$ for $1 \leq r \leq k - i$, $y^{k-1}(x_i) = 0$, and $y^r(x_i) = x^{r-1}_i$ for $k - i + 2 \leq r \leq k$.)

(b) \[
\begin{align*}
    f_t(x_k) &= \left( x^0_k, (\xi^1_k, 0), (\xi^2_k, u^1_k), \ldots, (\xi^{k-1}_k, u^{k-2}_k), \\
    &\quad \left( -\xi^2_k - \cdots - \xi^{k-1}_k, u^{k-1}_k \right) \right) \\
    &\quad + \left( 0, (0, 0), (0, 0), \ldots, (0, 0), (Q(x^0_k, \xi^1_k) + t, 0) \right)
\end{align*}
\]
where $Q$ is a non-degenerate quadratic form in the $n(m - k + 1)$ variables $(x^0_k, \xi^1_k)$.
Proof. It is straightforward to see that we can find coordinates \( x_i \) on \( U_i, \)
\( i = 1, \ldots, k, \) so that up to first order of approximation \( f_0|U_i \) is given by the
expressions in (a) and the first term in (b) above.

We must consider also second order terms: Let \( N \) be a linear subspace in
the coordinates \( y \) transversal to \( T_0(U_1) \cap \cdots \cap T_0(U_{k-1}) \). Let \( \phi: f(U_k) \to \)
\( N \) be orthogonal projection into \( N \). Then \( \ker(\phi) = T_0(U_1) \cap \cdots \cap T_0(U_k) \)
and \( \coker(\phi) \) is 1-dimensional. The second derivative \( d^2 \phi: \ker(\phi) \to \)
\( \coker(\phi) \) must be a non-degenerate quadratic form, otherwise, we can avoid
\( f_0 \) in generic 1-parameter families. (This is a consequence of the jet transversality
theorem.) The Morse lemma then implies that, after possibly adjusting the coordinates
in \( U_k \) by adding quadratic expressions in \( (x_k^0, \xi_k^1) \to \xi_k^2, \)
\( j > 1, \) there exists coordinates for \( f_0 \) as stated.

Finally, we must prove that the deformation \( f_t \) as given above is versal.
A result of Mather [9] says that it is enough to prove that the deformation
is infinitesimally versal. This is straightforward. \( \square \)

In Lemma 3.6 below we will use coordinates as at generic \( m \)-fold self intersection points. That is, \( x = (x^0, x^1, \ldots, x^{m-1}) \in \mathbb{R}^{nm-1} \) and \( y = \)
\( (y^0, y^1, \ldots, y^m) \in \mathbb{R}^{n(m+1)-1} \), where \( x^0, y^0 \in \mathbb{R}^{n-1} \) and \( x_k, y_k \in \mathbb{R}^n \) for
\( k \neq 0. \)

**Lemma 3.6.** Let \( f_0: M^{nm-1} \to W^{n(m+1)-1} \) be an immersion in \( \Sigma^1 \) and let
\( q = f_0(p_1) = \cdots = f_0(p_{m+1}) \) be a point of \( (m+1) \)-fold self-intersection.
Then there are coordinates \( y \) on \( V \subset W \) centered at \( q \) and coordinates \( x_i \)
on \( U_i \subset M \) centered at \( p_i, 1 \leq i \leq m+1 \) such that in these coordinates the
versal deformation \( f_t, -\delta \leq t < \delta \) of \( f_0 \) is constant outside of \( \cup_i U_i \) and in
neighborhoods of \( p_i \in U_i \) it is given by

\[
\begin{align*}
(a) \quad f_t(x_1) &= (x_1^0, x_1^1, \ldots, x_1^{m-2}, x_1^{m-1}, 0), \\
f_t(x_2) &= (x_2^0, x_2^1, \ldots, x_2^{m-2}, 0, x_2^{m-1}), \\
&\vdots \\
f_t(x_m) &= (x_m^0, 0, x_m^1, x_m^2, \ldots, x_m^{m-1}).
\end{align*}
\]

(That is, if \( y = f_t(x_i) \) then for \( 1 \leq i \leq m, y^0(x_i) = x_i^0, y^r(x_i) = x_i^r \) for
\( 1 \leq r \leq m - i, y^{m-i+1}(x_i) = 0, \) and \( y^r(x_i) = x_i^{r-1} \) for \( m - i + 2 \leq r \leq m. \))

\[
(b) \quad f_t(x_k) = \left(0, (\xi_k^0, x_k^0), (\xi_k^2, u_k^1), \ldots, (\xi_k^{m-1}, u_k^{m-2}), \right.
\left.-\xi_k^1 - \cdots - \xi_k^{m-1} + t, u_k^{m-1}\right).
\]

**Proof.** The proof is similar to the proof of Lemma 3.5, but easier. \( \square \)
3.3. The codimension two part of the discriminant hypersurface.

Lemma 3.7. Let \( f_{0,0} : M^{nm-1} \to W^{n(m+1)-1} \) be an immersion in \( \Sigma^2 \). Then either (a) or (b) below holds.

(a) \( f_{0,0} \) has two distinct degenerate self-intersection points \( q_1 \) and \( q_2 \). Locally around \( q_i \), \( i = 1, 2 \), \( f_{0,0} \) is as in Lemma 3.5 or Lemma 3.6. The versal deformation of \( f_{0,0} \) is a product of the corresponding 1-parameter versal deformations.

(b) \( f_{0,0} \) has one degenerate \( k \)-fold \( 2 \leq k \leq m \) self-intersection point \( q \). There are coordinates \( y \) centered at \( q \) and coordinates \( x_i \) centered at \( p_i \), \( 1 \leq i \leq k \) such that in these coordinates the versal deformation \( f_{s,t} \), \( -\delta < t, s < \delta \) of \( f_{0,0} \) is constant outside of \( \cup_i U_i \) and in a neighborhood of \( p_i \in U_i \) it is given by

\[
\begin{align*}
\phi = f_{s,t}(x_1) &= (x_1^0, x_1^1, \ldots, x_1^{k-2}, x_1^{k-1}, 0), \\
\phi = f_{s,t}(x_2) &= (x_2^0, x_2^1, \ldots, x_2^{k-2}, 0, x_2^{k-1}), \\
\vdots
\end{align*}
\]

That is, if \( y = f_{s,t}(x_i) \) then for \( 1 \leq i \leq k-1 \), \( y^0(x_i) = x_i^0 \), \( y^1(x_i) = x_i^1 \), \( y^r(x_i) = x_i^r \) for \( 1 \leq r \leq k-i \), \( y^{k-i+1}(x_i) = 0 \), and \( y^r(x_i) = x_i^{r-1} \) for \( k-i+2 \leq r \leq k \).

(b) \( f_{s,t}(x_k) = (x_k^0, (\xi_k^1, 0), (\xi_k^2, u_k^1), \ldots, (\xi_k^{k-1}, u_k^{k-2}), (-\xi_k^2 - \cdots - \xi_k^{k-1}, u_k^{k-1})) + (0, (0, 0), (0, 0), \ldots, (0, 0), (Q(x_k^0) + \xi_k^1((\xi_k^2)^2 + s) + t, 0)) \)

where \( Q \) is a non-degenerate quadratic form in the \( n(m-k+1) - 1 \) variables \( x_k^0 \).

Proof. The jet-transversality theorem applied to maps of \( M^{2jm-1} \times [-\delta, \delta]^2 \) into \( W^{2j(m+1)-1} \) shows that immersions with points of \( k \)-fold self-intersection \( k \geq m+2 \), as well as immersions with points of \( k \)-fold self-intersection points \( 2 \leq k \leq m+1 \) at which the 1-jet has further degenerations than the 1-jets of Lemma 3.4 can be avoided in generic 2-parameter families.

Assume that \( f_{0,0} \) has a degenerate \( k \)-fold self-intersection point. If \( k = m+1 \) it is easy to see that \( f_{0,0} \) has the same local form as the map in Lemma 3.6. So, immersions with \((m+1)\)-fold self-intersection points appears along 1-parameter subfamilies in generic 2-parameter families. If \( k < m+1 \) we proceed as in the proof of Lemma 3.5 and construct the map \( \phi \). Applying the jet transversality theorem once again we see that if the rank of \( d^2\phi \) is smaller than \( n(m-k+1) - 1 \) then \( f_{0,0} \) can be avoided in generic 2-parameter families. If the rank of \( d^2\phi \) equals \( n(m-k+1) - 1 \) then the third derivative
of \( \phi \) in the direction of the null-space of \( d^2\phi \) must be non-zero otherwise \( f_{0,0} \) can be avoided in generic 2-parameter families.

With this information at hand we can find coordinates as claimed. Finally, as in the proof of Lemma 3.5, we must check that the deformation given is infinitesimally versal. This is straightforward.

\[ \xymatrix{ & \Sigma \ar[dl] \ar[dr] & \\
(\text{a}) & & (\text{b}) } \]

**Figure 1.** The discriminant intersected with a small generic 2-disk.

Pictures (a) and (b) in Figure 1 correspond to cases (a) and (b) in Lemma 3.7. The codimension two parts of the discriminant are represented by points. In (a) two branches of the discriminant, which consist of immersions with one degenerate \( r \)-fold and one degenerate \( k \)-fold self-intersection point respectively, intersect in \( \Sigma^2 \). In (b) the smooth points of the semi-cubical cusp represents immersions with one degenerate \( k \)-fold self-intersection point. The singular point represents \( \Sigma^2 \). If an immersion in \( \Sigma \) above the singular point (\( \Sigma^2 \)) is moved below it then the index of the quadratic form \( Q \) in local coordinates close to the degenerate self-intersection point (see Lemma 3.5) changes.

4. Definition of the invariants.

In this section we define the invariants \( J_r, J, L \) and \( \Lambda \). To this end, we describe resolutions of self-intersections (for the definitions of \( J_r \) and \( J \)) and we compute homology of image-complements of and of normal bundles (for the definitions of \( L \) and \( \Lambda \)).

4.1. Resolution of the self intersection. For generic immersions \( f: M^{nm-1} \to W^{n(m+1)-1} \) let \( \Gamma^j(f) = \Gamma_j(f) \cup \Gamma_{j+1}(f) \cup \cdots \cup \Gamma_m(f) \), for \( 2 \leq j \leq m \) and let \( \tilde{\Gamma}^j(f) = f^{-1}(\Gamma^j(f)) \). Resolving \( \Gamma^j(f) \) we obtain a smooth manifold \( \Delta^j(f) \):

**Lemma 4.1.** Let \( m > 1 \) and \( n > 1 \) and \( 2 \leq j \leq m \) be integers. Let \( f: M^{nm-1} \to W^{n(m+1)-1} \) be a generic immersion. Then there exists closed
\[(n(m-j+1)-1)\)-manifolds \(\tilde{\Delta}^j(f)\) and \(\Delta^j(f)\), unique up to diffeomorphisms, and immersions \(\sigma: \tilde{\Delta}^j(f) \to M\) and \(\tau: \Delta^j(f) \to W\) such that the diagram
\[
\begin{array}{ccc}
\tilde{\Delta}^j(f) & \xrightarrow{\sigma} & f^{-1}(\Gamma^j(f)) \subset M \\
p \downarrow & & \downarrow f \\
\Delta^j(f) & \xrightarrow{\tau} & \Gamma^j(f) \subset W
\end{array}
\]
commutes. The maps \(\sigma\) and \(\tau\) are surjective, have multiple points only along \(\tilde{\Gamma}^{j+1}(f)\) and \(\Gamma^{j+1}(f)\) respectively, and \(p\) is a \(j\)-fold cover.

**Proof.** This is immediate from Lemma 3.2: Close to a \(k\)-fold self intersection point \(\Gamma_k(f)\) looks like the intersection of \(k\) \((nm-1)\)-planes in general position in \(\mathbb{R}^{n(m+1)-1}\). \(\square\)

**4.2. Definition of the invariants \(J_r\) and \(J\).**

**Definition 4.2.** Let \(m>1\) and \(n>1\) be integers and assume that \(n\) is odd. For a generic immersion \(f: M^{nm-1} \to W^{n(m+1)-1}\) and an integer \(2 \leq r \leq m\) such that \(m-r\) is even, define
\[J_r(f) = \chi(\Delta^r(f)),\]
where \(\chi\) denotes Euler characteristic.

**Definition 4.3.** Let \(m>1\) be an integer. For a generic immersion \(f: M^{2m-1} \to W^{2m+1}\), define
\[J(f) = \text{The number of components of } \Gamma_m(f)\]

**Lemma 4.4.** The functions \(J_r\) and \(J\) are invariants of generic immersions.

**Proof.** Let \(f_t, 0 \leq t \leq 1\) be a regular homotopy through generic immersions. If \(F: M \times [0,1] \to W \times [0,1]\) is the immersion \(F(x,t) = (f_t(x),t)\) then \(\Gamma^j(F) \cong \Gamma^j(f_0) \times I\). It follows that \(\Delta^j(f_0)\) is diffeomorphic to \(\Delta^j(f_1)\). \(\square\)

**4.3. Homology of complements of images.**

**Lemma 4.5.** Let \(f: M^{nm-1} \to \mathbb{R}^{n(m+1)-1}\) be a generic immersion. Assume that \(M\) is closed. Then
\[(a) \quad H_{n-1}(\mathbb{R}^{n(m+1)-1} - f(M); \mathbb{Z}_2) \cong H^{nm-1}(M; \mathbb{Z}_2) \cong \mathbb{Z}_2,\]
and
\[(b) \quad \text{if } M \text{ is oriented then } \quad H_{n-1}(\mathbb{R}^{n(m+1)-1} - f(M); \mathbb{Z}) \cong H^{nm-1}(M; \mathbb{Z}) \cong \mathbb{Z}.\]
Proof. Alexander duality implies that
\[
H_{n-1}(\mathbb{R}^{n(m+1)-1} - f(M)) \cong H^{nm-1}(f(M)).
\]
By Lemma 3.2, we can choose triangulations of \(M\) and \(f(M)\) such that the map \(f: M \to f(M)\) induces an isomorphism of the associated cellular cochain complexes in dimensions \((nm - k), 1 \leq k \leq n\).

Remark 4.6. In case (a) of Lemma 4.5 we will use the unique \(\mathbb{Z}_2\)-orientation of \(M\) and the duality isomorphism to identify \(H_{n-1}(\mathbb{R}^{n(m+1)-1} - f(M); \mathbb{Z}_2)\) with \(\mathbb{Z}_2\). In case (b) of Lemma 4.5, a \(\mathbb{Z}\)-orientation of \(M\) determines an isomorphism \(H^{nm-1}(M; \mathbb{Z}) \to \mathbb{Z}\). We shall use this and the duality isomorphism to identify \(H_{n-1}(\mathbb{R}^{n(m+1)-1} - f(M); \mathbb{Z})\) with \(\mathbb{Z}\).

A small \((n - 1)\)-dimensional sphere going around a fiber in the normal bundle of \(f(M)\) generates these groups.

4.4. Homology of normal bundles of immersions. Consider an immersion \(f: M^{nm-1} \to W^{n(m+1)-1}\). Let \(N\) denote its normal bundle. Then \(N\) is a vector bundle of dimension \(n\) over \(M\). Choose a Riemannian metric on \(N\) and consider the associated bundle \(\partial N\) of unit vectors in \(N\). This is an \((n - 1)\)-sphere bundle over \(M\). Let \(\partial F \cong S^{n-1}\) denote a fiber of \(\partial N\).

Lemma 4.7. Let \(f: M^{nm-1} \to W^{n(m+1)-1}\) be an immersion. Let \(i: \partial F \to \partial N\) denote the inclusion of the fiber.

(a) If \(H_{n-1}(M; \mathbb{Z}_2) = 0 = H_n(M; \mathbb{Z}_2)\) then
\[
i_*: H_{n-1}(\partial F; \mathbb{Z}_2) \to H_{n-1}(\partial N; \mathbb{Z}_2),
\]
is an isomorphism.

(b) If \(M\) and \(W\) are oriented and \(H_{n-1}(M; \mathbb{Z}) = 0 = H_n(M; \mathbb{Z})\) then
\[
i_*: H_{n-1}(\partial F; \mathbb{Z}) \to H_{n-1}(\partial N; \mathbb{Z}),
\]
is an isomorphism.

Proof. This follows from the Leray-Serre spectral sequence.

Remark 4.8. In case (a) of Lemma 4.7 we use the isomorphism \(i_*\) and a canonical generator of \(H_{n-1}(\partial F; \mathbb{Z}_2)\) to identify \(H_{n-1}(\partial N; \mathbb{Z}_2)\) with \(\mathbb{Z}_2\). In case (b) of Lemma 4.7, orientations of \(M\) and \(W\) induce an orientation of the fiber sphere \(\partial F\). We use this orientation and \(i_*\) in Lemma 4.7 to identify \(H_{n-1}(\partial N; \mathbb{Z})\) with \(\mathbb{Z}\).

4.5. Shifting the \(m\)-fold self-intersection. Let \(f: M^{nm-1} \to W^{n(m+1)-1}\) be a generic immersion. Recall that the set of \(m\)-fold self-intersection points of \(f\) is a closed submanifold \(\Gamma_m(f)\) of \(W^{n(m+1)-1}\) of dimension \(n - 1\) and its preimage \(\tilde{\Gamma}_m(f)\) is a closed submanifold of the same dimension in \(M^{nm-1}\).

Lemma 4.9. If \(f: M^{nm-1} \to W^{n(m+1)-1}\) is an immersion then there exists a smooth section \(s: \tilde{\Gamma}_m(f) \to \partial N\), where \(\partial N\) is the unit sphere bundle of the normal bundle of \(f\).
Definition 4.12. Let $f: M^{nm-1} \to \mathbb{R}^{n(m+1)-1}$ be a generic immersion. Assume that $M$ satisfies

\[(CA) \quad H_{n-1}(M; \mathbb{Z}) = 0 = H_n(M; \mathbb{Z}).\]

Define $\Lambda(f) \in \mathbb{Z}$ as

$$\Lambda(f) = [\Gamma'_m(f, v, \epsilon)] - s_s[\Gamma_m(f)] \in \mathbb{Z},$$

where $\epsilon > 0$ is very small and $[\Gamma'_m(f, v, \epsilon)] \in H_{n-1}(\mathbb{R}^{n(m+1)-1} - f(M); \mathbb{Z}) \cong \mathbb{Z}$ (see Remark 4.6) and $s_s[\Gamma_m(f)] \in H_{n-1}(\partial N; \mathbb{Z}) \cong \mathbb{Z}$ (see Remark 4.8).

Definition 4.13. Let $f: M^{nm-1} \to \mathbb{R}^{n(m+1)-1}$ be a generic immersion. Assume that $n$ is even and that $M$ is oriented and satisfies

\[(CL) \quad H_{n-1}(M; \mathbb{Z}) = 0 = H_n(M; \mathbb{Z}).\]

Define $L(f) \in \mathbb{Z}$ as

$$L(f) = [\Gamma'_m(f, v, \epsilon)] - s_s[\Gamma_m(f)] \in \mathbb{Z},$$

where $\epsilon > 0$ is very small and $[\Gamma'_m(f, v, \epsilon)] \in H_{n-1}(\mathbb{R}^{2m(j+1)-1} - f(M); \mathbb{Z}) \cong \mathbb{Z}$ (see Remark 4.6) and $s_s[\Gamma_m(f)] \in H_{n-1}(\partial N; \mathbb{Z}) \cong \mathbb{Z}$ (see Remark 4.8).
Remark 4.14. If $\text{Tor}(H_{n-2}(M; \mathbb{Z}), \mathbb{Z}_2) = 0$ and $M$ satisfies condition (CL) in Definition 4.13 then by the universal coefficient theorem $M$ also satisfies condition (CA) in Definition 4.12. It follows from Remarks 4.6, 4.8 that in this case $\Lambda = L \mod 2$.

Lemma 4.15. $\Lambda$ and $L$ are well-defined. That is, they do neither depend on the choice of $s$, nor on the choice of $\epsilon > 0$.

Remark 4.16. We shall often drop the awkward notation $\Gamma'_m(f, v, \epsilon)$ and simply write $\Gamma'(f)$. This is justified by Lemma 4.15.

Proof. The independence of $\epsilon > 0$ follows immediately from Remark 4.11. We will therefore not write all $\epsilon$’s out in the sequel of this proof.

Let $s_0$ and $s_1$ be two homotopic sections of $\partial N | \tilde{\Gamma}_m(f)$. Let $v_0$ and $v_1$ be the corresponding normal vector fields along $\Gamma_m(f)$.

A homotopy $s_t$ between $s_0$ and $s_1$ induces a homotopy $v_t$ between $v_0$ and $v_1$ and hence between $\Gamma'_m(f, v_0)$ and $\Gamma'_m(f, v_1)$ in $\mathbb{R}^{n(m+1)-1} - f(M)$. This shows that $\Lambda$ and $L$ only depend on the homotopy class of $s$.

To see that $\Lambda$ and $L$ are independent of the vector field we introduce the notion of adding a local twist: Let $s: \tilde{\Gamma}_m(f) \to \partial N$ be a section and let $p \in \tilde{\Gamma}_m(f)$. Choose a neighborhood $U$ of $p$ in $\tilde{\Gamma}_m(f)$ and a trivialization $\partial N | U \cong U \times S^{n-1}$ such that $s(u) = (u, w)$, where $w$ is a point in $S^{n-1}$. Let $D^{n-1}$ be disk inside $U$ and let $\sigma: D \to S^{n-1}$ be a smooth map of degree $\pm 1$ such that $\sigma \equiv w$ in a neighborhood of $\partial D$. Let $s^{tw}$ be the section which is $s$ outside of $D$ and $\sigma$ in $D$. We say that $s^{tw}$ is the result of adding a local twist to $s$. (In the case when $M$ is oriented and $n$ is even, the local twist is said to be positive if the degree of $\sigma$ is $+1$ and negative if the degree of $\sigma$ is $-1$.)

Let $s^{tw}$ be the vector field obtained by adding a twist to $s$. Let $v^{tw}$ and $v$ be the vector fields along $\Gamma_m(f)$ obtained from $s^{tw}$ and $s$ respectively. Clearly,

$$s^{tw}_*[\Gamma_m(f)] = s_*[\Gamma_m(f)] \pm 1 \quad \text{and} \quad [\Gamma'_m(f, v^{tw})] = [\Gamma'_m(f, v)] \pm 1.$$ 

Hence, $\Lambda$ and $L$ are invariant under adding local twists.

A standard obstruction theory argument shows that if $s$ and $s'$ are two sections of $\partial N | \Gamma_m$ then by adding local twists to $s'$ we can obtain a section $s''$ which is homotopic to $s$. Hence, $\Lambda$ and $L$ are independent of the choice of section. \hfill $\square$

Lemma 4.17. $\Lambda$ and $L$ are invariants of generic immersions.

Proof. Let $f_t$, $0 \leq t \leq 1$ be a regular homotopy through generic immersions. Consider the induced map

$$F: M \times I \to \mathbb{R}^{n(m+1)-1} \times I, \quad F(x, t) = (f_t(x), t),$$

where

$$F_*[\Gamma_m(f)] = F_*(\Gamma_m(f)) \quad \text{and} \quad F_*(\Gamma'_m(f, v)) = \Gamma'_m(f, v),$$

hold. Hence, $\Lambda$ and $L$ are invariant under generic homotopies.
where \( I \) is the unit interval \([0, 1]\). Shifting \( \Gamma_m(F) \cong \Gamma_m(f_0) \times I \) off \( F(M \times I) \) using a suitable vector field in \( N[\Gamma_m(F)] \) it is easy to see that \( L(f_0) = L(f_1) \).

5. Additivity properties, connected sum, and reversing orientation.

In this section we study how our invariants behave under two natural operations on generic immersions: Connected sum and reversing orientation.

5.1. Connected summation of generic immersions. For (oriented) manifolds \( M \) and \( V \) of the same dimension, let \( M \sharp V \) denote the (oriented) connected sum of \( M \) and \( V \).

Let \( f : M^{nm-1} \to \R^{n(m+1)-1} \) and \( g : V^{nm-1} \to \R^{n(m+1)-1} \) be two generic immersions. We shall define the connected sum \( f \# g \) of these. It will be a generic immersion \( M \sharp V \to \R^{n(m+1)-1} \).

Let \((u,x)\), \( u \in \R \) and \( x \in \R^{n(m+1)-2} \) be coordinates on \( \R^{n(m+1)-1} \). Composing the immersions \( f \) and \( g \) with translations we may assume that \( f(M) \subseteq \{u \leq -1\} \) and \( g(V) \subseteq \{u \geq 1\} \). Choose a point \( p \in M \) and a point \( q \in V \) such that there is only one point in \( f^{-1}(f(p)) \) and in \( g^{-1}(g(q)) \). Pick an arc \( \alpha \) in \( \R^{n(m+1)-1} \) connecting \( f(p) \) to \( g(q) \) and such that \( \alpha \cap (f(M) \cup g(V)) = \{f(p), g(q)\} \). Moreover, assume that \( \alpha \) meets \( f(M) \) and \( g(V) \) transversally at its endpoints. Let \( N \) be the normal bundle of \( \alpha \). Pick a (oriented) basis of \( T_f(p) \cdot f(M) \) and an (anti-oriented) one of \( T_g(q) \cdot g(V) \). These give rise to \( nm - 1 \) vectors over \( \partial a \) in \( N \). Extend these vectors to \( nm - 1 \) independent normal vector fields along \( \alpha \). Using a suitable map of \( N \) into a tubular neighborhood of \( \alpha \) these vector fields give rise to an embedding \( \phi : \alpha \times D \to \R^{n(m+1)-1} \), where \( D \) denotes a disk of dimension \( nm - 1 \), such that \( \phi|f(p) \times D \) is an (orientation preserving) embedding into \( f(M) \) and \( \phi|g(q) \times D \) is an (orientation reversing) embedding into \( g(V) \). The tube \( \phi(\alpha \times \partial D) \) now joins \( f(M) - \phi(f(p) \times \text{int}(D)) \) to \( g(V) - \phi(g(q) \times \text{int}(D)) \). Smoothing the corners we get a generic immersion \( f \# g : M \sharp V \to \R^{n(m+1)-1} \).

**Lemma 5.1.** Let \( f : M^{nm-1} \to \R^{n(m+1)-1} \) and \( g : V^{nm-1} \to \R^{n(m+1)-1} \) be generic immersions. The connected sum \( f \# g \) is independent of both the choices of points \( f(p) \) and \( g(q) \) and the choice of the path \( \alpha \) used to connect them, up to regular homotopy through generic immersions.

**Proof.** This is straightforward. (Note that the preimages of self-intersections has codimension \( n > 1 \).) \( \square \)

**Lemma 5.2.** If \( f, f' : M^{nm-1} \to \R^{n(m+1)-1} \) and \( g, g' : V^{nm-1} \to \R^{n(m+1)-1} \), \( m, n > 1 \) are regularly homotopic through generic immersions then \( f \# g \) and \( f' \# g' \) are regularly homotopic through generic immersions.
Proof. This is straightforward.

**Proposition 5.3.** The invariants $J_r$ and $J$ are additive under connected summation.

Proof. $\Delta^j(f \ast g) = \Delta^j(f) \sqcup \Delta^j(g)$. □

Note that if $M^{nm-1}$ and $V^{nm-1}$ are manifolds which both satisfy condition (CA) in Definition 4.12 or condition (CL) in Definition 4.13 then so does $M \ast V$.

**Proposition 5.4.** Let $f : M^{nm-1} \to \mathbb{R}^{n(m+1)-1}$ and $g : V^{nm-1} \to \mathbb{R}^{n(m+1)-1}$ be generic immersions.

(a) If $M$ and $V$ both satisfy condition (CA) then

$$\Lambda(f \ast g) = \Lambda(f) + \Lambda(g).$$

(b) If $n$ is even and $M$ and $V$ are both oriented and both satisfy condition (CL) then

$$L(f \ast g) = L(f) + L(g).$$

Proof. Consider case (b). Choose $2j$-chains $D$ and $E$ in $\{u < -1\}$ and $\{u > 1\}$ bounding $\Gamma_m(f)$ and $\Gamma_m(g)$ respectively and disjoint from the arc $\alpha$, used in the construction of $f \ast g$. Then

$$L(f \ast g) = (D \sqcup E) \cdot f \ast g(M \ast V) = D \cdot f(M) + E \cdot g(V) = L(f) + L(g).$$

Case (a) is proved in exactly the same way. □

**5.2. Changing orientation.** The invariants $J_r$, $J$, and $\Lambda$ are clearly orientation independent. In contrast to this, the invariant $L$ is orientation sensitive.

To have $L$ defined, let $n = 2j$ and consider an oriented closed manifold $M^{2jm-1}$ which satisfies condition (CL).

**Proposition 5.5.** Assume that there exists an orientation reversing diffeomorphism $r : M \to M$. Let $f : M^{2jm-1} \to \mathbb{R}^{2j(m+1)-1}$ be a generic immersion. Then $f \circ r$ is a generic immersion and

$$L(f \circ r) = (-1)^{m+1}L(f).$$

Proof. Note that $\Gamma_m(f \circ r) = \Gamma_m(f) = \Gamma_m$. The orientation of $\Gamma_m$ is induced from the decomposition

$$T_q\mathbb{R}^{2j(m+1)-1} = T_q\Gamma_m(f) \oplus N_1 \oplus \cdots \oplus N_m.$$ 

The immersions $f$ and $f \circ r$ induces opposite orientations on each $N_i$ hence the orientations induced on $\Gamma_m$ agrees if $m$ is even and does not agree if $m$ is odd. Let $D$ be a $2j$-chain bounding $\Gamma_m$, with its orientation induced from $f$. If $m$ is even then

$$L(f \circ r) = D \cdot f(r(M)) = D \cdot -f(M) = -L(f).$$
If $m$ is odd then
\[ L(f \circ r) = -D \cdot f(r(M)) = -D \cdot f(M) = L(f). \]

Proposition 5.6. Let $R: \mathbb{R}^{2j(m+1)-1} \to \mathbb{R}^{2j(m+1)-1}$ be reflection in a hyperplane. Let $f: M^{2jm-1} \to \mathbb{R}^{2j(m+1)-1}$ be a generic immersion. Then $R \circ f$ is a generic immersion and
\[ L(R \circ f) = (-1)^m L(f). \]

Proof. Note that the oriented normal bundle of $Rf$ is $-RN$. So the correctly oriented $2j$-chain bounding $\Gamma_m(R \circ f)$ is $(-1)^{m+1}RD$. Thus,
\[ L(R \circ f) = (-1)^{m+1}(RD \cdot Rf(M)) = (-1)^m(D \cdot f(M)) = (-1)^m L(f). \]

6. Coorientations and proofs of Theorems 1 and 3.

In this section we prove Theorems 1 and 3. To do that we need a coorientation of the discriminant hypersurface in the space of immersions.

6.1. Coorienting the discriminant.

Remark 6.1. Let $a$ be an invariant of generic immersions. If $a$ is $\mathbb{Z}_2$-valued then $\nabla a$ (see Section 2) is well-defined without reference to any coorientation of $\Sigma$. Moreover, if $a$ is integer-valued and $\Delta$ is a union of path components of $\Sigma^1$ then the notion $\nabla a|\Delta \equiv 0$ is well-defined without reference to a coorientation of $\Sigma$.

We shall coorient the relevant parts (see Remark 6.1) of the discriminant hypersurface. That is, we shall find coorientations of the parts of the discriminant hypersurface in the space of generic immersions where our invariants have non-zero jumps. These coorientations will be continuous (see [6], Section 7). That is, the intersection number of any generic small loop in $\mathcal{F}$ and $\Sigma^1$ vanishes and the coorientation extends continuously over $\Sigma^2$.

Definition 6.2. Let $m > 1$ and $n > 1$. Assume that $n$ is odd. Let $2 \leq k \leq m$ and assume that $m - k$ is even. Let $f_0: M^{nm-1} \to W^{n(m+1)-1}$ be a generic immersion in $\Sigma^1$ with one degenerate $k$-fold self-intersection point. Let $f_\delta$ be a versal deformation of $f_0$. We say that $f_\delta$ is on the positive side of $\Sigma^1$ and $f_{-\delta}$ on the negative side if $\chi(\Delta^k(f_\delta)) > \chi(\Delta^k(f_{-\delta}))$.

Remark 6.3. Note that by Lemma 3.5 $\chi(\Delta^k(f_\delta))$ is obtained from $\chi(\Delta^k(f_{-\delta}))$ by a Morse modification. Since the dimension of $\chi(\Delta^k(f_\delta))$ is $n(m - k + 1) - 1$ is even the Euler characteristic changes under such modifications.
Definition 6.4. Let $m > 1$. Let $f_0 : M^{2m-1} \to W^{2m+1}$ be a generic immersion in $\Sigma^1$ with one degenerate $m$-fold self-intersection point. Let $f_\delta$ be a versal deformation of $f_0$. We say that $f_\delta$ is on the positive side of $\Sigma^1$ and $f_{-\delta}$ on the negative side if the number of components in $\Gamma_m(f_\delta)$ is larger than the number of components in $\Gamma_m(f_{-\delta})$.

Remark 6.5. Note that by Lemma 3.5 $\Gamma_m(f_\delta)$ is obtained from $\Gamma_m(f_{-\delta})$ by a Morse modification. Since these are 1-manifolds the number of components change.

The construction of the relevant coorientation for $L$ is more involved. It is based on a high-dimensional counterpart of the notion of over- and under-crossings in classical knot theory.

Let $f_0 : M^{2jm-1} \to W^{2j(m+1)-1}$ be an immersion of oriented manifolds. Assume that $f_0 \in \Sigma^1$ has an $(m+1)$-fold self-intersection point, $q = f_0(p_1) = \cdots = f_0(p_{m+1})$. Let $f_t$ be a versal deformation of $f_0$. Let $U_i$ be small neighborhoods of $p_i$ and let $S^i_t$ denote the oriented sheet $f_t(U_i)$. Note that $S^i_0 \cap \cdots \cap S^m_{m+1} = q$ and that $S^i_1 \cap \cdots \cap S^m_{m+1} = \emptyset$ if $t \neq 0$.

Let $D^i_1 = \cap_{j\neq i} S^j_1$. Let $w_i$ be a line transversal to $T S^0_i + T D^0_i$ at $q$. For small $t \neq 0$ both $S^i_t$ and $D^i_t$ intersects $w_i$. Orienting the line from $S^i_t$ to $D^i_t$ gives a local orientation $(D^i_t, S^i_t, \vec{w}_i)$ of $\mathbb{R}^{2j(m+1)-1}$. Comparing it with the standard orientation of $\mathbb{R}^{2j(m+1)-1}$ we get a sign $\sigma_t(i) = \text{Or}(D^i_t, S^i_t, \vec{w}_i)$, where $\text{Or}$ denotes the sign of the orientation. Note that, if we orient the line from $D^i_1$ to $S^i_1$ we get the opposite orientation $-\vec{w}_i$ of $w$ and

$$\text{Or}(D^i_1, S^i_1, -\vec{w}_i) = \text{Or}(S^i_1, D^i_1, -\vec{w}_i),$$

since $D^i_1$ and $S^i_1$ are both odd-dimensional. Note also that $\sigma_t(i) = -\sigma_{-t}(i)$ (see Lemma 3.6).

We next demonstrate that $\sigma_t(i) = \sigma_t(j)$ for all $i, j$: Let $N^i_t$ be the oriented normal bundle of $S^i_t$. Let $w$ be a vector from $D^i_t$ to $D^j_t$, transversal to both $T S^0_1 + T D^0_1$ and $T S^0_2 + T D^0_2$. Orient it from $D^i_1$ to $D^j_2$. Then

$$\sigma_t(1) = \text{Or}(S^1_1, D^1_1, \vec{w}),$$

and hence $TD^1_t + w$ gives a normal bundle $N^1_t$ and by the convention used to orient the normal bundle $\text{Or}(N^1_t) = \sigma_t(1) \text{Or}(D^1_t, \vec{w})$. In a similar way it follows that $\text{Or}(N^2_t) = \sigma_t(2) \text{Or}(D^2_t, -\vec{w})$.

Now, by definition

$$1 = \text{Or}(D^1_t, N^2_t, \ldots, N_m^t) = \sigma_t(2) \text{Or}(D^1_t, (D^2_t, -\vec{w}), N^3_t, \ldots, N^t_m)$$

$$= [\dim(D^1_t) \text{ is odd}] = -\sigma_t(2) \text{Or}(D^1_t, -\vec{w}, D^2_t, N^3_t, \ldots, N^t_m)$$

$$= \sigma_t(2)\sigma_t(1) \text{Or}(N^1_t, D^2_t, N^3_t, \ldots, N^t_m) = \sigma_t(1)\sigma_t(2).$$

Hence, $\sigma_t(1)\sigma_t(2) = 1$ as claimed.
Definition 6.6. Let \( f_0 : M^{2j(m+1)-1} \to W^{2j(m+1)-1} \) be an immersion of oriented manifolds. Assume that \( f_0 \) has an \((m+1)\)-fold self-intersection point. Let \( f_t \) be a versal deformation of \( f_0 \). We say that \( f_\delta \) is on the positive side of \( \Sigma^1 \) at \( f_0 \) if
\[
\sigma_\delta(1) = \cdots = \sigma_\delta(m+1) = +1.
\]

6.2. First order invariants. The following obvious lemma will be used below.

Lemma 6.7. Any first order invariant of generic immersions is uniquely (up to addition of zero order invariants) determined by its jump.

6.3. Proof of Theorem 3. We know from Lemma 4.4 that \( J \) and \( J_r \) are invariants of generic immersions. We must calculate their jumps.

We start in case (a): Assume that \( n \) is odd. Let \( f_t : M^{nm-1} \to \mathbb{R}^{n(m+1)-1}, t \in [-\delta, \delta] \) be a versal deformation of \( f_0 \in \Sigma^1 \) and fix \( r, 2 \leq r \leq m \) such that \( m-r \) is even.

If \( f_0 \) has a degenerate \( k \)-fold intersection point \( 2 \leq k \leq m+1 \) and \( k \neq r \) then \( \Delta^r(f_{-\delta}) \) is diffeomorphic to \( \Delta^r(f_\delta) \): If \( k < r \) then \( \Delta^r(f_{-\delta}) \) is not affected at all by the versal deformation. If \( k > r \) then the immersed submanifold \( \tau^{-1}(\Gamma_k(f_{-\delta})) \) of \( \Delta^r(f_{-\delta}) \) is changed by surgery (or is deformed by regular homotopy if \( k = m+1 \)) under the versal deformation. This does not affect the diffeomorphism class of \( \Delta^r(f_{-\delta}) \).

If \( f_0 \) has a degenerate \( r \)-fold self-intersection point then \( \Delta^r f_\delta \) is obtained from \( \Delta^r f_{-\delta} \) by a surgery (see Lemma 3.5). Since the dimension of \( \Delta^r(f_{-\delta}) \) is \( n(m-r+1)-1 \) which is even this changes the Euler characteristic by \( \pm 2 \).

According to our coorientation conventions \( \nabla J_r(f_0) = 2 \) if \( f_0 \) has a degenerate \( r \)-fold self-intersection point. Also, \( \nabla J_r(f_0) = 0 \) if \( f_0 \) has any other degeneracy.

By the same argument, the jump of \( J \) in case (b) is 1 at degenerate \( m \)-fold self intersection points and 0 at other degeneracies.

It is evident from Lemma 3.7 (see Figure 1) that \( J \) and \( J_r \) are first order invariants. The theorem now follows from Lemma 6.7.

6.4. Proof of Theorem 1. We start with (b): Let \( f_t : M^{2jm-1} \to \mathbb{R}^{2j(m+1)-1}, t \in [-\delta, \delta] \) be a generic one-parameter family intersecting \( \Sigma^1 \) in \( f_0 \). If the degenerate intersection point of \( f_0 \) is a \( k \)-fold intersection point with \( 2 \leq k \leq m-1 \) then clearly \( L(f_{-\delta}) = L(f_\delta) \), since the \( m \)-fold self-intersection is not affected under such deformations.

Assume that \( f_0 \) has a degenerate \( m \)-fold self-intersection point. Without loss of generality we may assume that \( f_t \) is a deformation of the form in Lemma 3.5 (we use coordinates as there):

Let \( F(x,t) = (f_t(x), t) \). Shifting \( ((Q(y^0, v^1) = t), 0, \ldots, 0, t) \) we obtain a \( 2j \)-chain bounded by \( \Gamma_m(f_\delta) \times -\delta - \Gamma_m(f_{-\delta}) \times -\delta \) in \( \mathbb{R}^{2jm+1-1} \times [-\delta, \delta] - F(M \times [-\delta, \delta]) \). It follows that \( L(f_\delta) = L(f_{-\delta}) \).
If $f_0$ has an $(m + 1)$-fold self-intersection then the discussion preceding Definition 6.6 shows that $L(f_0) = L(f_{\delta}) + (m + 1)$: At an $(m + 1)$-fold self intersection point $m + 1$ crossings are turned into crossings of opposite sign. Hence, $\nabla L(f_0) = m + 1$ if $f_0$ has an $(m + 1)$-fold self-intersection and $\nabla L(f_0) = 0$ if $f_0$ has any other degeneracy.

The calculation of $\nabla \Lambda$ in (a) is analogous. Let us just make a remark about the parity of $m$: At an $(m + 1)$-fold self-intersection point $m + 1$ crossings are changed. Hence, at instances of $(m + 1)$-fold self-intersection the invariant $\Lambda$ changes by $1 \in \mathbb{Z}_2$ if $m + 1$ is odd and does not change if $m + 1$ is even.

It is immediate from Lemma 3.7, see Figure 1 that $\Lambda$ and $L$ are first order invariants. The theorem now follows from Lemma 6.7. □

7. Invariants of regular homotopy.

A function of immersions $M \to W$ which is constant on path components of $\mathcal{F}$ will be called an invariant of regular homotopy. Our geometrically defined invariants of generic immersions give rise to torsion invariants of regular homotopy.

**Definition 7.1.** Let $n$ be odd. Let $f: M^{nm-1} \to W^{n(m+1)-1}$ be an immersion. Let $f'$ be a generic immersion regularly homotopic to $f$. For $2 \leq r \leq m$ such that $m - r$ is even, define $j_r(f) \in \mathbb{Z}_2$ as

$$j_r(f) = J_r(f') \mod 2.$$  

**Proposition 7.2.** The function $j_r$ is an invariant of regular homotopy.

**Proof.** Clearly it is enough to show that for any two regularly homotopic generic immersions $f_0$ and $f_1$, $j_r(f_0) = j_r(f_1)$. Let $f_i$ be a generic regular homotopy from $f_0$ to $f_1$. Then $f_i$ intersects $\Sigma$ transversally in a finite number of points in $\Sigma^1$. It follows from Theorem 3 that $j_r$ remains unchanged at such intersections. Hence, $j_r(f_0) = j_r(f_1)$. □

**Definition 7.3.** Let $n$ be even. Assume that $M^{nm-1}$ is a manifold which satisfy condition (CL). Let $f: M^{nm-1} \to \mathbb{R}^{n(m+1)-1}$ be an immersion. Let $f'$ be a generic immersion regularly homotopic to $f$. Define $l(f) \in \mathbb{Z}_{m+1}$ as

$$l(f) = L(f') \mod (m + 1).$$

**Proposition 7.4.** The function $l$ is an invariant of regular homotopy.

**Proof.** The proof is identical to the proof of Proposition 7.2. □

**Definition 7.5.** Let $m > 1$ be odd. Assume that $M^{nm-1}$ is a manifold which satisfy condition (CA). Let $f: M^{nm-1} \to \mathbb{R}^{n(m+1)-1}$ be an immersion. Let $f'$ be a generic immersion regularly homotopic to $f$. Define $\lambda(f) \in \mathbb{Z}_2$ as

$$\lambda(f) = \Lambda(f').$$
Proposition 7.6. The function $\lambda$ is an invariant of regular homotopy.

Proof. This follows from the proof of Theorem 1, where it is noted that when $m$ is odd $\Lambda$ remains constant when a regular homotopy intersects $\Sigma^1$. □

8. Sphere-immersions in codimension two.

In this section Theorem 2 will be proved. To do that we will first discuss the classifications of sphere-immersions and sphere-embeddings up to regular homotopy.

8.1. The Smale invariant. In Smale’s classical work [12] it is proved that there is a bijection between the set of regular homotopy classes $\text{Imm}(k,n)$ of immersions $S^k \rightarrow \mathbb{R}^{k+n}$ and the elements of the group $\pi_k(V_{k+n,k})$, the $k$th homotopy group of the Stiefel manifold of $k$-frames in $(k+n)$-space. If $f: S^k \rightarrow \mathbb{R}^{k+n}$ is an immersion we let $\Omega(f) \in \pi_k(V_{k+n,k})$ denote its Smale invariant. Via $\Omega$ we can view $\text{Imm}(k,n)$ as an Abelian group.

If the codimension is two the groups appearing in Smale’s classification are easily computed: The exact homotopy sequence of the fibration $SO(2) \hookrightarrow SO(2m+1) \rightarrow V_{2m+1,2m-1}$ implies that $\pi_{2m-1}(V_{2m+1,2m-1}) \cong \pi_{2m-1}(SO(2m+1))$. Bott-periodicity then gives:

$$\pi_{2m-1}(V_{2m+1,2m-1}) = \begin{cases} \mathbb{Z} & \text{if } m = 2j, \\ \mathbb{Z}_2 & \text{if } m = 4j + 1, \\ 0 & \text{if } m = 4j + 3. \end{cases}$$

Remark 8.1. It is possible to identify the group operations in $\text{Imm}(k,2)$ geometrically. Kervaire [8] proves that the Smale invariant $\Omega$ is additive under connected sum of immersions. This gives the geometric counterpart of addition. If $f: S^n \rightarrow \mathbb{R}^{n+2}$ is an immersion, $r: S^n \rightarrow S^n$ is an orientation reversing diffeomorphism, and $n \neq 2$ then $\Omega(f) = -\Omega(f \circ r)$. (See [4] for the case $S^3 \rightarrow \mathbb{R}^5$, the other cases are analogous.) This gives the geometric counterpart of the inverse operation in $\text{Imm}(n,2)$ for $n \neq 2$.

It is interesting to note that for immersions $f: S^2 \rightarrow \mathbb{R}^4$, $\Omega(f) = \Omega(f \circ r)$: The Smale invariant can in this case be computed as the algebraic number of self-intersection points. This number is clearly invariant under reversing orientation. (The same is true also for immersions $S^{2k} \rightarrow \mathbb{R}^{4k}$, $k \geq 1$.)

Lemma 8.2. The regular homotopy invariant $l$ induces a homomorphism $\text{Imm}(2m-1,2) \rightarrow \mathbb{Z}_{m+1}$.

Proof. Note that the dimensions are such that $L$ is defined and spheres certainly satisfy condition (CL). The invariant $L$ is additive under connected summation and changes sign if an immersion is composed on the left with
an orientation reversing diffeomorphism. Hence, \( \ell \) induces a homomorphism. \( \square \)

Let \( \text{Emb}(n, 2) \subset \text{Imm}(n, 2) \) be the set of regular homotopy classes which contain embeddings. By Remark 8.1 \( \text{Emb}(n, 2) \) is a subgroup of \( \text{Imm}(n, 2) \).

A result of Hughes and Melvin \([7]\) states that \( \text{Emb}(4j - 1, 2) \subset \text{Imm}(4j - 1, 2) \cong \mathbb{Z} \) is a subgroup of index \( \mu_j \), where \( \mu_j \) is the order of the image of the \( J \)-homomorphism, \( J : \pi_{4j-1}(SO(4j + 1)) \to \pi_{8j}(S^{4j+1}) \). The Adams conjecture, formulated in \([1]\) and proved in \([11]\) implies that \( B_j = \frac{\nu_j}{\mu_j} \), where \( B_j \) is the \( j \)th Bernoulli number and \( \nu_j \) and \( \mu_j \) are coprime integers.

8.2. Proof of Theorem 2. By Lemma 8.2, \( l : \text{Imm}(2m - 1, 2) \to \mathbb{Z}/(m + 1)\mathbb{Z} \) is a homomorphism and clearly, \( \text{Emb}(2m - 1, 2) \subset \ker(l) \).

In case (b) \( \text{Imm}(8j + 5, 2) = 0 \) and hence the image of \( \ell \) is zero, which proves that \( L(f) \) is always divisible by \( m + 1 = 4j + 4 \).

In case (a) \( \text{Imm}(8j + 1, 2) \cong \mathbb{Z}/2\mathbb{Z} \). Hence, for any immersion \( f, \ell(f \sharp f) = 0 \). Thus, \( L(f \sharp f) = 2L(f) \) is divisible by \( m + 1 = 4j + 2 \), which implies that \( L(f) \) is divisible by \( 2j + 1 \).

In case (c) \( \text{Emb}(4j - 1, 2) \cong \mu_j \mathbb{Z} \) as a subgroup of \( \text{Imm}(4j - 1, 2) \cong \mathbb{Z} \).

Hence, for any immersion \( f \), \( m + 1 = 2j + 1 \) divides \( L(f \sharp \cdots \sharp f) = \mu_j L(f) \).

Thus, if if \( p \) is a prime and \( r, k \) are integers such that \( p^{r+k} \) divides \( 2j + 1 \) and \( p^{k+1} \) does not divide \( \mu_j \) then \( p^r \) divides \( L(f) \). \( \square \)

9. Examples and problems.

In this section we construct examples of 1-parameter families of immersions which shows that all the first order invariants we have defined are non-trivial. We also discuss some problems in connection with the regular homotopy invariants defined in Section 7.

9.1. Examples. Using our local coordinate description of immersions with one degenerate self-intersection point we can construct examples showing that the invariants \( J, J_r, \Lambda \) and \( L \) are non-trivial. We start with \( \Lambda \) and \( L \):

Choose \( m \) standard spheres \( S_1, \ldots, S_m \) of dimension \( (nm - 1) \) intersecting in general position in \( \mathbb{R}^{n(m+1)-1} \) so that \( S_1 \cap \cdots \cap S_m \cong S^{n-1} \). Pick a point \( p \in S_1 \cap \cdots \cap S_m \) and let \( S_{m+1} \) be another standard \( (nm - 1) \)-sphere intersecting \( S_1 \cap \cdots \cap S_m \) at \( p \) so that in a neighborhood of \( p \) the embeddings are given by the expressions in Lemma 3.6 and so that \( p \) is the only degenerate intersection point. Let \( f_0 : S^{nm-1} \to \mathbb{R}^{n(m+1)-1} \) be the immersions which is the connected sum of \( S_1, \ldots, S_{m+1} \). Let \( f_t \) be a versal deformation of \( f \). Then, after possibly reversing the direction of the versal deformation we have \( \Lambda(f_t) = 1 \in \mathbb{Z}_2 \), if \( n \) is odd or \( L(f_t) = \pm(m + 1) \) if \( n \) is even.
In the latter case we would get the other sign of $L(f_\delta)$ if the orientation of $S_{m+1}$ in the construction above is reversed. To distinguish these two immersions denote one by $h^+$ and the other by $h^-$, so that $L(h^+) = m+1 = -L(h^-)$. Then it is an immediate consequence of Theorem 1 that:

Any generic regular homotopy from $h^+$ to $h^-$ has at least two instances of $(m+1)$-fold self-intersection.

Theorem 1 has many corollaries similar to the one just mentioned. To see that $L$ and $\Lambda$ are non-trivial for other source manifolds. We use connected sum with the immersions $S^{nm-1} \to \mathbb{R}^{n(m+1)-1}$ just constructed and Proposition 5.4.

The proofs that the invariants $J$ and $J_\tau$ are non-trivial are along the same lines: Construct a sphere-immersion into Euclidean space with one degenerate self-intersection point as in the local models in Lemma 3.5. Then embed this Euclidean space with immersed sphere into any target manifold and use connected sum together with Proposition 5.3.

**Remark 9.1.** Applying connected sum to the sphere-immersion constructed above in the case when it is an immersion $S^{2m-1} \to \mathbb{R}^{2m+1}$ shows that for any integer $k$ there exists a generic immersions $f: S^{2m-1} \to \mathbb{R}^{2m+1}$ such that $L(f) = (m+1)k$.

**Remark 9.2.** Let $f_0: M^{nm-1} \to \mathbb{R}^{n(m+1)-1}$ be an immersion in $\Sigma^1$ with an $(m+1)$-fold self-intersection point and let $f_t$, $-\delta < t < \delta$ be a versal deformation of $f_0$.

Let $\Gamma_\pm = \bigcup_{i=1}^m \Gamma_i(f_{\pm\delta})$ and $\tilde{\Gamma}_\pm = f_{\pm\delta}^{-1}(\Gamma_\pm)$. Then if $U_\pm$ are small regular neighborhoods of $\Gamma_\pm$ and $\tilde{U}_\pm = f_{\pm\delta}^{-1}(U_\pm)$ then there exist diffeomorphisms $\phi$ and $\psi$ such that the following diagram commutes

\[
\begin{array}{ccc}
(U_-, \Gamma_-) & \xrightarrow{\phi} & (U_+, \Gamma_+) \\
\downarrow f_{-\delta} & & \downarrow f_{+\delta} \\
(\tilde{U}_-, \tilde{\Gamma}_-) & \xrightarrow{\psi} & (\tilde{U}_+, \tilde{\Gamma}_+) 
\end{array}
\]

That is, the local properties of a generic immersion close to its self-intersection does not change at instances of $(m+1)$-fold self-intersection points in generic 1-parameter families.

Assume that $M$ fulfills the requirements of Theorem 1. Then $\Lambda$ or $L$ is defined and $\Lambda(f_{-\delta}) \neq \Lambda(f_\delta)$ or $L(f_{-\delta}) \neq L(f_\delta)$, respectively. This implies that $f_{-\delta}$ and $f_\delta$ are not regularly homotopic through generic immersions. Hence, knowledge of the local properties of a generic immersion $M^{nm-1} \to \mathbb{R}^{n(m+1)-1}$ close to its self-intersection is not enough to determine it up to regular homotopy through generic immersions.

**9.2. Problems.** Are the regular homotopy invariants $l$, $\lambda$ and $j_\tau$ non-trivial?
A negative answer to this question implies restrictions on self-intersection manifold. A positive answer gives non-trivial geometrically defined regular homotopy invariants.

9.3. Remarks on the invariant \( l \). Theorem 2 gives information on the possible range of \( l \) for sphere-immersions in codimension two. We consider the most interesting cases of immersions \( S^{4j-1} \to \mathbb{R}^{4j+1}, \ j \geq 1 \) (Theorem 2 (c)). In the first case \( S^3 \to \mathbb{R}^5 \), \( l \) is non-trivial. This was shown in [4]. Hence, for any integer \( b \) there are generic immersions \( f : S^3 \to \mathbb{R}^5 \) such that \( L(f) = b \).

Remark 9.3. The invariant \( l \) is called \( \lambda \) in [4] and is the mod 3 reduction of an integer-valued invariant called \( \text{lk} \). The definition of \( \text{lk} \) given in [4] differs slightly from the definition of \( L \) given here. There is an easy indirect way to see that, nonetheless, the two invariants are the same: Due to Theorem 1 and Theorem 2 in [4], \( L - \text{lk} \) is an invariant of regular homotopy which is 0 on embeddings and additive under connected summation. Assume that \( L(f) - L'(f) = a \) for some immersion. Then, since \( \mu_1 = 24 \), the connected sum of 24 copies of any immersion is regularly homotopic to an embedding. Hence, \( 24a = 0 \) and therefore \( a = 0 \). Thus, \( L \equiv \text{lk} \).

If \( 2j + 1 \) is prime then Theorem 2 does not impose any restrictions on \( l \) since, in this case, \( 2j + 1 \) divides \( \mu_j \) (see Milnor and Stasheff [10]).

On the other hand, if none of the prime factors of \( 2j + 1 \) divides \( \mu_j \), Theorem 2 implies that \( l \) is trivial and in such cases Remark 9.1 allows us to determine the range of \( L \) which is \((2j + 1)\mathbb{Z} \). The first two cases where this happens are \( S^57 \to \mathbb{R}^{59} \) and \( S^{407} \to \mathbb{R}^{109} \), where \( 2j + 1 \) equals 35 = \( 5 \cdot 7 \) and 55 = \( 5 \cdot 11 \) and \( \mu_j \) equals 24 = \( 2^3 \cdot 3 \) and 86184 = \( 2^3 \cdot 3^4 \cdot 7 \cdot 19 \), respectively.

9.4. Remarks on the invariant \( j_2 \). The first case in which to consider the invariant \( j_2 \) are immersions of a 5-manifold into an 8-manifold.

Theorem 7.30 in [5] shows that the self-intersection surface of a generic immersion \( S^5 \to \mathbb{R}^8 \) must have even Euler characteristic (even though it may be non-orientable). Thus, in this case \( j_2 \) is trivial. (Note that \( \pi_5(V_{5,5}) \cong \mathbb{Z}_2 \) so that there are two regular homotopy classes of immersions \( S^5 \to \mathbb{R}^8 \). Hence, it is non-trivial to see that \( l \) is trivial in this case.)

In contrast, the invariant \( j_2 \) is non-trivial for immersions \( S^5 \to \mathbb{R}P^8 \): Clearly, \( S^5 \) embeds in \( \mathbb{R}P^8 \). Thus, there is an immersion \( f \) with \( j_2(f) = 0 \).

Consider three hyperplanes \( H_1, H_2, H_3 \) in general position in \( \mathbb{R}P^8 \). Fix a point \( q \in \mathbb{R}P^8 \setminus (H_1 \cup H_2 \cup H_3) \). Note that \( X = \mathbb{R}P^8 - \{q\} \) is a non-orientable line bundle over \( H_i \), \( i = 1, 2, 3 \). Choose sections \( v_i \) of \( X \) over \( H_i \) such that \( \{v_i = 0\} \cong \mathbb{R}P^6 \) meets \( H_1 \cap H_2 \cap H_3 \) transversally in \( H_i \) for \( i = 1, 2, 3 \). Now, \( H_1 \cap H_2 \cap H_3 \cong \mathbb{R}P^5 \). Let \( g : \mathbb{R}P^5 \to \mathbb{R}P^8 \) be the corresponding embedding. Restricting \( v_i \) to \( \mathbb{R}P^5 \) we get three normal vector fields \( v_1, v_2, v_3 \) along \( \mathbb{R}P^5 \subset \mathbb{R}P^8 \) and \( \{v_1 = v_2 = v_3 = 0\} \cong \mathbb{R}P^2 \).
Let \( p: S^5 \to \mathbb{RP}^5 \) be the universal cover. Then \( h = g \circ p: S^5 \to \mathbb{RP}^8 \) is an immersion. Let \( K_i = h^{-1}\{v_i = 0\} \cong S^4 \). Choose Morse functions \( \phi_i: S^5 \to \mathbb{R} \) such that \( \{\phi_i = 0\} = K_i \). Let \( \epsilon > 0 \) be small. Then \( f: S^5 \to \mathbb{RP}^8 \), given by
\[
f(x) = h(x) + \epsilon \sum_{i=1}^{3} \phi_i(x)v_i(h(x)) \quad \text{for } x \in S^5,
\]
is a generic immersion with \( \Gamma_2(f) \cong \mathbb{RP}^2 \). Hence, \( j_2(f) = 1 \).

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Received May 24, 1999.

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We give a short proof of the theorem of Brown, Douglas and Fillmore that an essentially normal operator on a Hilbert space is of the form “normal plus compact” if and only if it has trivial index function. The proof is basically a modification of our short proof of Lin’s theorem on almost commuting self-adjoint matrices that takes into account the index.

Using similar methods we obtain new results, generalizing results of Lin, on approximating normal operators by ones with finite spectrum.

1. Introduction.

Let $H$ be an infinite-dimensional separable Hilbert space, let $\mathcal{K}$ denote the compact operators on $H$, and consider the short-exact sequence

$$0 \longrightarrow \mathcal{K} \longrightarrow B(H) \xrightarrow{\pi} Q(H) \longrightarrow 0$$

where $Q(H)$ is the Calkin algebra $B(H)/\mathcal{K}$. An operator $T \in B(H)$ is \textit{essentially normal} if $T^*T - TT^* \in \mathcal{K}$, or equivalently, if $\pi(T)$ is normal.

An operator $T \in B(H)$ is Fredholm if $\pi(T)$ is invertible in $Q(H)$, and its Fredholm index is denoted by $\text{index}(T)$. The essential spectrum $\text{sp}_{\text{ess}}(T)$ is the spectrum of $\pi(T)$. The \textit{index function} of $T$ is the map

$$\mathbb{C} \setminus \text{sp}_{\text{ess}}(T) \rightarrow \mathbb{Z}; \quad \lambda \mapsto \text{index}(T - \lambda \cdot 1).$$

The index function is invariant under compact perturbations. Hence we may define the index function of an invertible $S \in Q(H)$ to be that of any lift $T \in B(H)$ of $S$.

The index function is continuous and hence constant on each connected component of its domain. It vanishes on the unbounded connected component of its domain. We say that an operator has trivial index function if its index function is zero everywhere on its domain.

In Section 2 we give a new proof of the following:

\textbf{Theorem 1.1 ([BDF1, Cor. 11.2])}. An essentially normal operator on a Hilbert space is a compact perturbation of a normal operator if and only if it has trivial index function.
The theorem is a special case of the result by Brown-Douglas-Fillmore ([BDF1, Theorem 11.1]) that two essentially normal operators are unitarily equivalent modulo a compact perturbation if and only if they have the same essential spectrum and index function. In fact the general theorem is a fairly straightforward consequence of Theorem 1.1, see [BD, Theorem 5.8] and [D, Proposition 4.1].

Our proof of Theorem 1.1 has two steps. The first step is to show that an essentially normal operator with trivial index function is in the closure of the set of compact perturbations of normal operators on $H$. We prove this by showing that a normal element in the Calkin algebra can be approximated by normal elements with finite spectra if it has trivial index function (see also Lin [L2]). The methods used here follow closely the methods we used in [FR]. The main difference is that we here must keep track of the index (or $K_1$-class) of the invertible operators used in the various approximation steps. Step two is then to show that the set of compact perturbations of normal operators is norm closed. This step involves quasidiagonal essentially normal operators, and its proof is a straightforward consequence of Lin’s Theorem:

**Theorem 1.2** (H. Lin, [L2]). For every $\varepsilon > 0$ there is a $\delta > 0$ such that for every finite dimensional C*-algebra $A$ and every element $T \in A$ such that $\|T\| \leq 1$ and $\|T^* T - TT^*\| < \delta$

there is a normal element $N \in A$ such that $\|T - N\| < \varepsilon$.

See also the short proof of Lin’s theorem in [FR].

In Section 3 we consider the general problem of approximating normal elements of a C*-algebra by normal elements with finite spectra.

We show that a normal element $a$ of a unital C*-algebra $A$ of real rank zero can be approximated by normal elements with finite spectra if and only if all its translates, $a - \lambda \cdot 1$, $\lambda \in \mathbb{C}$, belong to the closure of $GL_0(A)$, the connected component of the identity in $GL(A)$ (Theorem 3.2). A key step towards this end is that a normal element $a$ in any unital C*-algebra $A$ can be approximated by normal elements $b \in A$, with 1-dimensional spectra and with $b - \lambda \cdot 1 \in GL_0(A)$ for all $\lambda$ not in the spectrum of $b$, if and only if all translates $a - \lambda \cdot 1$ belong to the closure of $GL_0(A)$. To complete the argument we use a theorem of Lin ([L3, Theorem 5.4]) that every normal element with 1-dimensional spectrum in a C*-algebra of real rank zero is the norm-limit of normal elements with finite spectra.

In Corollary 3.12 to Theorem 3.2 it is shown that if $A$ is a unital C*-algebra of real rank zero and stable rank one, then every normal element $a \in A$, satisfying $\pi(a) - \lambda \cdot 1 \in GL_0(A/I)$ for all proper ideals $I$ of $A$ and for all $\lambda \in \mathbb{C} \setminus \text{sp}(\pi(a))$, can be approximated by normal elements with finite spectra. This result should be compared with the theorem of Lin in [L4]
which says that in a simple C*-algebra $A$ of real rank zero and with property (IR) — a property, considered in [FR], which is weaker than stable rank one — every normal element $a$, with $a - \lambda \cdot 1 \in \text{GL}_0(A)$ for every $\lambda \in \mathbb{C} \setminus \text{sp}(a)$, is the norm-limit of normal operators with finite spectra.

The first operator theoretic proof of Theorem 1.1 is due to Berg and Davidson [BD]. In fact, their analysis gives a stronger quantitative version which, subject to a natural resolvent condition on the operator $T$, gives a bound on the norm of the compact perturbation in terms of the norm of the self-commutator $T^*T - TT^*$.

Also let us mention that Lin has generalized Theorem 1.1 to essentially normal elements of $M(A)/A$ for certain AF-algebras $A$ (see [L5]).

We thank Larry Brown for several suggestions that helped improve our results and our exposition.

2. Proof of Theorem 1.1.

The main part of the proof of Theorem 1.1 consists of showing the theorem below, which — as indicated — has already been proved by Huaxin Lin (noting that $Q(H)$ is purely infinite and simple). The proof presented here, we believe is shorter and more direct than Lin’s proof.

**Theorem 2.1** (cf. H. Lin, [L3, Theorem 4.4]). *Let $T$ be a normal element in $Q(H)$. Then $T$ is the norm limit of a sequence of normal elements in $Q(H)$ with finite spectra if and only if $T$ has trivial index function.*

The lemmas below serve to prove the "if"-part of the theorem.

**Lemma 2.2.** *Let $T$ be a normal element in $Q(H)$, let $\lambda \in \text{sp}(T)$ and let $\varepsilon > 0$. There exists a normal element $S \in Q(H)$ with $\|T - S\| \leq 2\varepsilon$, $\lambda \notin \text{sp}(S)$, $\text{index}(S - \lambda \cdot 1) = 0$, and

$$\text{sp}(S) \setminus B(\lambda, \varepsilon) = \text{sp}(T) \setminus B(\lambda, \varepsilon).$$

*Proof.* We may without loss of generality assume that $\lambda = 0$. Let $R \in B(H)$ be any lift of $T$, and let $R = V|R|$ be the polar decomposition of $R$. The operator $Vf_\varepsilon(|R|)$ and its adjoint, $V^*f_\varepsilon(|R^*|)$, have infinite-dimensional kernels (because $\pi(|R|) = |T| = \pi(|R^*|)$ is non-invertible). Let $Wf_\varepsilon(|R|)$ be the polar decomposition of $Vf_\varepsilon(|R|)$. The argument above shows that $1 - WW^*$ and $1 - W^*W$ are both infinite-dimensional projections. Hence $W$ extends to a unitary $U \in B(H)$ with $Vf_\varepsilon(|R|) = Uf_\varepsilon(|R|)$.

Notice that $\pi(Vf_\varepsilon(|R|)) = \pi(V)f_\varepsilon(|T|) = \pi(U)f_\varepsilon(|T|)$ is normal (because $\pi(V)$ commutes with $|T|$). This implies that $\pi(U)$ commutes with $f_\varepsilon(|T|)$,
and therefore $S = \pi(U)(f_{\varepsilon}(|T|) + \varepsilon \cdot 1)$ is normal. Clearly, $S$ is invertible, and $S$ lifts to the invertible operator $U(f_{\varepsilon}(|R|) + \varepsilon \cdot 1)$, and so $\text{index}(S) = 0$. The distance between $S$ and $T$ is estimated by

$$
\|T - S\| \leq \|\pi(V)|T| - \pi(V)f_{\varepsilon}(|T|)\| + \|\pi(U)f_{\varepsilon}(|T|) - \pi(U)(f_{\varepsilon}(|T|) + \varepsilon \cdot 1)\| \leq 2\varepsilon.
$$

Let $E \in Q(H)^{**}$ be the spectral projection for $|T|$ corresponding to the interval $[0, \varepsilon]$. Since $|S| = f_{\varepsilon}(|T|) + \varepsilon \cdot 1$, it follows that $E$ is the spectral projection of $|S|$ corresponding to $\{\varepsilon\}$ in $Q(H)^{**}$. Since $T$ and $S$ are normal, $E$ commutes with $T$ and $S$ (being a Borel function of $T$ and of $S$). We therefore have

$$
\text{sp}(T) \cup \{0\} = \text{sp}(TE) \cup \text{sp}(T(1 - E)), \quad \text{and}
$$

$$
\text{sp}(S) \cup \{0\} = \text{sp}(SE) \cup \text{sp}(S(1 - E)).
$$

To complete the argument, notice that $|T|(1 - E) = (f_{\varepsilon}(|T|) + \varepsilon \cdot 1)(1 - E) = |S|(1 - E)$, and hence $T(1 - E) = S(1 - E)$. Also, $\|TE\| = \|T|E|\| \leq \varepsilon$ and $\|SE\| = \|S|E|\| \leq \varepsilon$, which entails that $\text{sp}(TF)$ and $\text{sp}(SF)$ are both contained in $B(0, \varepsilon)$.

**Lemma 2.3.** Let $F$ be a finite subset of $\mathbb{C}$. The set of elements $S \in Q(H)$, satisfying $\text{sp}(S) \cap F = \emptyset$ and $\text{index}(S - \lambda \cdot 1) = 0$ for all $\lambda \in F$, is open.

**Proof.** The set in question is a finite intersection of open sets of the form $\text{GL}_0(Q(H)) + \lambda \cdot 1$. \hfill \Box

**Lemma 2.4.** Let $T \in Q(H)$ be normal with trivial index function, let $F$ be a finite subset of $\mathbb{C}$, and let $\varepsilon > 0$. Then there exists a normal element $S \in Q(H)$ such that $\|T - S\| \leq \varepsilon$, $\text{sp}(S) \cap F = \emptyset$, and

$$
\text{index}(S - \lambda \cdot 1) = 0,
$$

for all $\lambda \in F$.

**Proof.** For some $0 \leq k \leq n$ we can write $F = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ so that $F \cap \text{sp}(T) = \{\lambda_{k+1}, \lambda_{k+2}, \ldots, \lambda_n\}$. By assumption, $\text{index}(T - \lambda_i \cdot 1) = 0$ for each $j \leq k$.

Using Lemmas 2.2 and 2.3 we find successively normal elements $T_k = T, T_{k+1}, T_{k+2}, \ldots, T_n$ in $Q(H)$ satisfying

- $\|T_j - T_{j-1}\| \leq \varepsilon/(n - k)$,
- $\lambda_1, \lambda_2, \ldots, \lambda_j \notin \text{sp}(T_j)$,
- $\text{index}(T_j - \lambda_i \cdot 1) = 0$, for $i = 1, 2, \ldots, j$, and
- $\lambda_{j+1}, \lambda_{j+2}, \ldots, \lambda_n \in \text{sp}(T_j)$.

Finally, $S = T_n$ will be as desired. \hfill \Box
For each \( \varepsilon > 0 \) consider the \( \varepsilon \)-grid \( \Gamma_\varepsilon \) in \( \mathbb{C} \) defined by
\[
\Gamma_\varepsilon = \{ x + iy \in \mathbb{C} \mid x \in \varepsilon \mathbb{Z} \text{ or } y \in \varepsilon \mathbb{Z} \}.
\]

**Lemma 2.5.** Let \( T \in Q(H) \) be normal with trivial index function, and let \( \varepsilon > 0 \). Then there exists a normal element \( S \in Q(H) \) with trivial index function, \( \text{sp}(S) \subseteq \Gamma_\varepsilon \), and such that \( \|S - T\| \leq \varepsilon \).

**Proof.** Choose \( N \in \mathbb{N} \) such that \( N\varepsilon \geq \|T\| + \varepsilon/4 \). Put
\[
\Sigma_\varepsilon = \{ x + iy \in \mathbb{C} \mid x, y \in \varepsilon(\mathbb{Z} + \frac{1}{2}) \}, \quad X = \{ a + ib : |a| \leq N\varepsilon, \ |b| \leq N\varepsilon \}.
\]
Applying Lemma 2.4 to the finite set \( \Sigma_\varepsilon \cap X \) we get a normal operator \( S' \in Q(H) \) satisfying: \( \|S' - T\| \leq \varepsilon/4 \), \( \text{sp}(S') \cap (\Sigma_\varepsilon \cap X) = \emptyset \), and \( \text{index}(S' - \lambda \cdot 1) = 0 \) for all \( \lambda \in \Sigma_\varepsilon \cap X \). Note that \( \text{sp}(S') \subseteq X \setminus \Sigma_\varepsilon \) (= \( Y \)).

There is a continuous path \( t \mapsto f_t \), \( t \in [0,1] \), of continuous functions \( f_t : Y \to Y \) so that
- \( f_0(z) = z \) for all \( z \in Y \),
- \( f_1(Y) \subseteq \Gamma_\varepsilon \),
- \( f_t(z) = z \) for all \( z \in \Gamma_\varepsilon \cap X \) and for all \( t \),
- \( |f_t(z) - z| < (\sqrt{2}/2)\varepsilon \) for all \( z \in Y \).

Put \( S = f_1(S') \). Then \( S \) is normal, \( \text{sp}(S) \subseteq \Gamma_\varepsilon \cap X \), and \( \|T - S\| \leq \varepsilon/4 + (\sqrt{2}/2)\varepsilon \leq \varepsilon \).

If \( \lambda \in \mathbb{C} \setminus \text{sp}(S) \), then \( \lambda \) is in the same connected component of \( \mathbb{C} \setminus \text{sp}(S) \) as some \( \lambda' \in \Sigma_\varepsilon \cap X \), or \( \lambda \) is in the unbounded component of \( \mathbb{C} \setminus \text{sp}(S) \). In the latter case, \( \text{index}(S - \lambda \cdot 1) = 0 \), and in the former case,
\[
\text{index}(S - \lambda \cdot 1) = \text{index}(S - \lambda' \cdot 1) = \text{index}(f_1(S') - \lambda' \cdot 1) = \text{index}(f_0(S') - \lambda' \cdot 1) = \text{index}(S' - \lambda' \cdot 1) = 0.
\]
\( \square \)

The lemma below is a special case of the Alexander Duality theorem from topology. For a compact subset \( X \) of \( \mathbb{C} \) it says that \( \pi^1(X) \cong H^1(X; \mathbb{Z}) \cong H_0(\mathbb{C} \setminus X; \mathbb{Z}) \), and the latter is the free Abelian group generated by the bounded connected components of \( \mathbb{C} \setminus X \).

**Lemma 2.6.** Let \( X \) be a compact subset of \( \Gamma_\varepsilon \) for some \( \varepsilon > 0 \). Every continuous map \( f : X \to \mathbb{C} \setminus \{0\} \) is homotopic to a map of the form
\[
z \mapsto (z - \lambda_1)^{n_1}(z - \lambda_2)^{n_2} \cdots (z - \lambda_k)^{n_k},
\]
for some \( \lambda_i \in \mathbb{C} \setminus X \) and some \( n_i \in \mathbb{Z} \).

**Proof.** Choose \( n \in \mathbb{N} \) such that \( X \subseteq [-n\varepsilon, n\varepsilon]^2 \), and put \( Y = \Gamma_\varepsilon \cap [-n\varepsilon, n\varepsilon]^2 \). Then \( f \) extends to a continuous function \( g : Y \to \mathbb{C} \setminus \{0\} \). (Indeed, by considering only one line-segment of \( Y \) at a time, this follows from the elementary fact that if \( X_0 \) is a closed subset of the interval \( [0,1] \) and if
that FR

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Lemma 2.7 is a natural generalization of [FR, Lemma 2.3] and the proof requires only a simple modification using Lemma 2.6. For the reader’s convenience we include the entire proof.

Lemma 2.7. Let $T$ be a normal element in $Q(H)$ with trivial index function and with $\text{sp}(T) \subseteq \Gamma_{\varepsilon}$ for some $\varepsilon > 0$. Suppose that $I$ is a relatively open subset of $\text{sp}(T)$ which is homeomorphic to the open interval $(0, 1)$. Then for each $\lambda_0 \in I$ and for each $\delta > 0$ there is a normal element $S$ in $Q(H)$ with trivial index function such that $\|S - T\| \leq \delta$, and $\text{sp}(S) \subseteq \text{sp}(T) \setminus \{\lambda_0\}$.

Proof. Choose one point in each bounded connected component of $\mathbb{C} \setminus \text{sp}(T)$, and let $F$ be the finite set of all these points. If $S \in Q(H)$ satisfies $\text{sp}(S) \subseteq \text{sp}(T)$, then $S$ has trivial index function if $\text{index}(S - \lambda \cdot 1) = 0$ for all $\lambda \in F$. It follows from Lemma 2.3 that $S$ has these properties if $\|T - S\| \leq \delta_0$ for a sufficiently small $\delta_0 > 0$. We may assume that $\delta \leq \delta_0$, and the part of the statement regarding $S$ having trivial index function is then automatically taken care of.

Let $J$ be a relatively open subset of $I$ satisfying

\[ \lambda_0 \in J \subseteq \overline{J} \subseteq I, \quad \text{diam}(J) \leq \varepsilon. \]

Let $f_0: I \to \mathbb{T} \setminus \{-1\}$ be a homeomorphism, and extend $f_0$ to $f: \text{sp}(T) \to \mathbb{T}$ by setting $f(z) = 1$ for all $z \in \text{sp}(T) \setminus I$. Let $V$ be the unitary element $f(T)$ of $Q(H)$.

It follows from Lemma 2.6 that $f$ is homotopic, inside $C(\text{sp}(T), \mathbb{C} \setminus \{0\})$, to the function $g: \text{sp}(T) \to \mathbb{C} \setminus \{0\}$ given by

\[ g(z) = (z - \mu_1)^{n_1}(z - \mu_2)^{n_2} \cdots (z - \mu_k)^{n_k}, \]
for appropriate $\mu_i \in \mathbb{C} \setminus \text{sp}(T)$ and $n_i \in \mathbb{Z}$. Hence $f(T) \sim_h g(T)$ inside GL($Q(H)$) and
\[
\text{index}(f(T)) = \text{index}(g(T)) = \sum_{i=1}^{k} n_i \cdot \text{index}(T - \mu_i \cdot 1) = 0.
\]

This shows that $V = f(T)$ lifts to a unitary $U$ in $B(H)$. We may now proceed exactly as in [FR, Lemma 2.3] (see also [BDF1, Lemma 6.1]).

Let $E \in B(H)$ be the spectral projection for $U$ corresponding to the (relatively open) subset $f(J)$ of $\text{sp}(U)$, and put $F = \pi(E)$. Then $F$ is a projection in $Q(H)$. We show below that $F$ commutes with $T$, and that
\[
\text{sp}_{FQ(H)F}(TF) \subseteq J, \quad \text{sp}_{(1-F)Q(H)(1-F)}(T(1-F)) \subseteq \text{sp}(T) \setminus J.
\]

Once this has been established, we can choose any $\lambda_1 \in J \setminus \{\lambda_0\}$ and set
\[
S = \lambda_1 F + (1-F)T.
\]
Then $S$ is normal (because $T$ and $F$ commute),
\[
\text{sp}(T) \subseteq \{\lambda_1\} \cup \text{sp}(T) \setminus J \subseteq \text{sp}(T) \setminus \{\lambda_0\},
\]
and $\|S - T\| = \|TF - \lambda_1 F\| \leq \text{diam}(J) \leq \varepsilon$ as desired.

Suppose $\varphi: \text{sp}(T) \to \mathbb{C}$ is a continuous function which is zero on $\text{sp}(T) \setminus I$. Then
\[
\hat{\varphi}(z) = \begin{cases} 
(\varphi \circ (f|_I)^{-1})(z), & \text{if } z \in \mathbb{T} \setminus \{1\} \\
0, & \text{if } z = 1
\end{cases},
\]
defines a continuous function $\mathbb{T} \to \mathbb{C}$ satisfying $\varphi = \hat{\varphi} \circ f$, and hence $\varphi(T) = \hat{\varphi}(V)$. Since $E$ commutes with $U$, it follows that $F$ commutes with $V$ and hence with $\varphi(V) = \varphi(T)$.

If $\varphi$ in addition is constant equal to 1 on $J$, then $\hat{\varphi}$ is constant equal to 1 on $f(J)$, which implies that $\varphi(T)F = F\varphi(T) = F$. If $\varphi$ instead vanishes on $\text{sp}(T) \setminus J$, then $\hat{\varphi}$ vanishes on $\mathbb{T} \setminus f(J)$, whence $\varphi(T)F = F\varphi(T) = \varphi(T)$.

Let $h: \text{sp}(T) \to [0,1]$ be a continuous function with $h|_J = 1$ and $h|_{\text{sp}(T) \setminus I} = 0$. By the argument above, $h(T)F = Fh(T) = F$, and since the function $z \mapsto zh(z)$ vanishes on $\text{sp}(T) \setminus I$, we get
\[
TF = Th(T)F = FTh(T) = Fh(T)T = FT.
\]

To show (♠) it suffices to show that $\varphi(TF) = 0$ and $\psi(T(1-F)) = 0$ for every pair of continuous functions $\varphi, \psi: \text{sp}(T) \to \mathbb{C}$, where $\varphi$ vanishes on $J$ and $\psi$ vanishes on $\text{sp}(T) \setminus I$ (and where the continuous functions operate in the respective corner algebras). We may assume that $\varphi$ is equal to 1 on the set $\text{sp}(T) \setminus I$. From the argument in the previous paragraph we get
\[
\varphi(TF) = \varphi(T)F = F - (1 - \varphi(T))F = 0, \quad \psi(T(1-F)) = \psi(T)(1-F) = 0,
\]
as desired. \qed
Proof of Theorem 2.1. Assume that \( T_n \) is a sequence of normal elements of \( Q(H) \) so that \( T_n \) tends to \( T \) in norm, and \( T_n \) has finite spectrum for each \( n \). Let \( \lambda \in \mathbb{C} \setminus \text{sp}(T) \). Then \( \lambda \in \mathbb{C} \setminus \text{sp}(T_n) \) for all sufficiently large \( n \), and index\((T_n - \lambda 1) \) → index\((T - \lambda 1) \). But index\((T_n - \lambda 1) = 0 \) because \( \mathbb{C} \setminus \text{sp}(T_n) \) is connected, so index\((T - \lambda 1) = 0 \). This shows that \( T \) has trivial index function.

Assume now that \( T \) is a normal element of \( Q(H) \) with trivial index function, and let \( \varepsilon > 0 \). By Lemma 2.5 there is a normal element \( T' \in Q(H) \) with \( \text{sp}(T') \subseteq \Gamma_\varepsilon \), \( \|T - T'\| \leq \varepsilon \), and such that \( T' \) has trivial index function.

We show next that there exists a normal element \( T'' \in Q(H) \) such that

\[
\|T' - T''\| \leq \varepsilon, \quad \text{sp}(T'') \subseteq \Gamma_\varepsilon,
\]

and such that \( \text{sp}(T'') \) contains no entire line segment of \( \Gamma_\varepsilon \), where an entire line segment is a set of the form

\[
\{n + iy \mid m\varepsilon < y < (m + 1)\varepsilon\} \quad \text{or} \quad \{x + im \mid n\varepsilon < x < (n + 1)\varepsilon\},
\]

for some \( n, m \in \mathbb{Z} \).

Let \( \{I_1, I_2, \ldots, I_n\} \) be the set of entire line segments of \( \Gamma_\varepsilon \) that are contained in \( \text{sp}(T') \), and choose \( \lambda_j \in I_j \) for each \( j \). Apply Lemma 2.7 successively to obtain normal elements \( R_0 = T', R_1, R_2, \ldots, R_n \) in \( Q(H) \) such that

\[
\|R_{j+1} - R_j\| \leq \varepsilon/n, \quad \text{sp}(R_{j+1}) \subseteq \text{sp}(R_j) \setminus \{\lambda_{j+1}\},
\]

and each with trivial index function. The element \( T'' = R_n \) then has the desired property.

It now follows that \( \text{sp}(T'') \) can be partitioned into finitely many clopen sets \( C_1, C_2, \ldots, C_m \) each with diameter less than \( 2\varepsilon \). Choose \( \mu_i \in C_i \) for each \( i \), and let \( f : \text{sp}(T'') \rightarrow \{\mu_1, \mu_2, \ldots, \mu_m\} \) be the continuous function which maps \( C_i \) to \( \mu_i \). Then \( |f(z) - z| < 2\varepsilon \) for all \( z \in \text{sp}(T'') \). The element \( S = f(T'') \) is normal with \( \text{sp}(S) = \{\mu_1, \mu_2, \ldots, \mu_m\} \), and \( \|S - T\| \leq 2\varepsilon \), so that \( \|S - T\| \leq 4\varepsilon \).

Recall that an operator \( T \) is quasidiagonal if there is an increasing sequence \( \{E_n\}_{n=1}^\infty \) of finite rank projections converging strongly to \( 1 \) such that \( \lim_{n \to \infty} \|T E_n - E_n T\| = 0 \). The set of quasidiagonal operators is invariant under compact perturbations and is norm closed. That the set is norm closed can be seen by using the equivalent “local” definition of \( T \) being quasidiagonal, that for every finite rank projection \( E \in B(H) \) and for every \( \varepsilon > 0 \) there exists a finite rank projection \( F \in B(H) \) such that \( E \leq F \) and \( \|T F - F T\| \leq \varepsilon \).

Every normal element \( N \) with finite spectrum is quasidiagonal (write \( N = \sum_{j=1}^k \lambda_j P_j \) and put \( E_n = \sum_{j=1}^k f_n^{(j)} \), where \( \{f_n^{(j)}\}_{n=1}^\infty \) is an increasing sequence of finite rank projections converging strongly to \( P_j \)). Since every
normal operator in $B(H)$ can be approximated by normal operators with finite spectrum, every normal operator is in fact quasidiagonal.

The statements in the proposition below were consequences of Theorem 1.1 in [BDF1]. Here it is used as a step in our proof.

**Proposition 2.8** (cf. [BDF1, Cor. 11.4 and Cor. 11.12]). The set of compact perturbations of normal operators on a Hilbert space $H$ is equal to the set of quasidiagonal essentially normal operators. In particular, the set of compact perturbations of normal operators on $H$ is norm-closed.

**Proof.** Each compact perturbation of a normal operator is clearly essentially normal, and — by the remarks above — also quasidiagonal.

It is well-known that every quasidiagonal operator $T \in B(H)$ is block-diagonal plus compact, i.e., there exist $S \in B(H)$ and an increasing sequence $\{E_n\}_{n=1}^{\infty}$ of finite rank projections converging strongly to 1 such that $T - S \in K$ and $SE_n = E_nS$ for all $n$. (Indeed, choose $\{E_n\}_{n=1}^{\infty}$ such that $\sum_{n=1}^{\infty} \|E_n T - T E_n\| < \infty$. Put $E_0 = 0$, and put $S = \sum_{n=1}^{\infty} (E_n - E_{n-1}) T (E_n - E_{n-1})$. Then

$$T - S = \sum_{n=1}^{\infty} (E_n - E_{n-1}) T (1 - E_n) + \sum_{n=1}^{\infty} (E_n - E_{n-1}) T E_{n-1},$$

and the right-hand side is compact since the two terms are norm-convergent sums of compact operators.)

Assume now that $T$ is essentially normal and quasidiagonal. Notice that $S$, being a compact perturbation of $T$ is also essentially normal. Put

$$S_n = (E_n - E_{n-1}) T (E_n - E_{n-1}).$$

Since $S = \sum_{n=1}^{\infty} S_n$, and $SS^* - S^* S$ is compact, we have

$$\lim_{n \to \infty} \|S_n S_n^* - S_n^* S_n\| = 0.$$

Each $S_n$ lies in the finite dimensional C*-algebra $B(H_n)$, where $H_n = (E_n - E_{n-1})(H)$, and so Lin’s theorem (Theorem 1.2) says that there exist normal operators $R_n \in B(H_n)$ with $\lim_{n \to \infty} \|R_n - S_n\| = 0$. Put $R = \sum_{n=1}^{\infty} R_n$. Then $R$ is normal, and $R - S$ is compact. This proves that $T$ is a compact perturbation of a normal operator.

The set of essentially normal operators and the set of quasidiagonal operators are both closed. Hence so is their intersection. 

**Proof of Theorem 1.1.** A compact perturbation of a normal operator has trivial index function since this is the case for a normal operator and since the index function is invariant under compact perturbation.

Assume now that $T$ is essentially normal with trivial index function. Then $\pi(T) \in Q(H)$ is normal with trivial index function. Hence, by Theorem 2.1, there is a sequence $\{S_n\}_{n=1}^{\infty}$ of normal elements of $Q(H)$ with finite spectra such that $S_n \to \pi(T)$ in norm. Lift $S_n$ to $T_n \in B(H)$ such that $T_n \to T$ in
norm. Every normal operator in $Q(H)$ with finite spectrum has a lift to a normal operator in $B(H)$. Hence we can find normal operators $R_n \in B(H)$ with $\pi(R_n) = S_n = \pi(T_n)$. It follows that each $T_n$ is a compact perturbation of a normal operator. By Proposition 2.8 this shows that $T$ itself is a compact perturbation of a normal operator.

**Corollary 2.9.** An essentially normal operator on a Hilbert space has trivial index function if and only if it is quasidiagonal.

**Proof.** Combine Theorem 1.1 with Proposition 2.8.

3. **Approximating normal elements with normal elements with finite spectra.**

In this section we prove various generalizations of Theorem 2.1 using more or less the same methods as in Section 2.

We first consider an obstruction — analogous to the index-obstruction in Theorem 2.1 — for a normal element of a C*-algebra to be a norm-limit of normal elements with finite spectra. The natural generalization of the index function of an element of $Q(H)$ to an element $a$ of a unital C*-algebra $A$ is the map

$$\mathbb{C} \setminus \text{sp}(a) \to K_1(A); \quad \lambda \mapsto [a - \lambda \cdot 1]_1.$$

Proposition 3.1 below shows that we must also take into account the index function of $a$ in every quotient of $A$. For each proper ideal $I$ of $A$ (proper ideal meaning $I \neq A$) we must consider the maps

$$\mathbb{C} \setminus \text{sp}(\pi_I(a)) \to K_1(A/I); \quad \lambda \mapsto [\pi_I(a) - \lambda \cdot 1]_1,$$

where $\pi_I$ denotes the quotient mapping $A \to A/I$. This additional obstruction was hidden in the case of the simple C*-algebra $Q(H)$.

**Proposition 3.1.** Let $A$ be a unital C*-algebra and let $a \in A$ be a normal element. If $a$ is the norm limit of normal elements in $A$ with finite spectra, then

$$(\diamond) \quad \pi_I(a) - \lambda \cdot 1 \in \text{GL}_0(A/I)$$

for every proper ideal $I$ of $A$ and every $\lambda \in \mathbb{C} \setminus \text{sp}(\pi_I(a))$.

Lin showed in [L3] that the natural map $\text{GL}(A)/\text{GL}_0(A) \to K_1(A)$ is injective if $\text{RR}(A) = 0$. Since the property real rank zero passes to quotients, this shows that $(\diamond)$ could be replaced by

$$(\heartsuit) \quad [\pi_I(a) - \lambda \cdot 1]_1 = 0 \text{ in } K_1(A/I)$$

if $\text{RR}(A) = 0$. 

**\[\square\]**
Proof. Suppose that \( \pi_I(a) - \lambda \cdot 1 \notin \text{GL}_0(A/I) \) for some proper ideal \( I \) of \( A \) and some \( \lambda \in \mathbb{C} \setminus \text{sp}(\pi_I(a)) \). Then \( a \) belongs to the open set
\[
\{ b \in A \mid \pi_I(b) - \lambda \cdot 1 \notin \text{GL}(A/I) \setminus \text{GL}_0(A/I) \}.
\]
This set can contain no element with finite spectrum, because if \( b \) was such an element, \( \pi_I(b) \in A/I \) would have finite spectrum and then either \( \pi_I(b) - \lambda \cdot 1 \) is not invertible or belongs to \( \text{GL}_0(A/I) \) since \( \mathbb{C} \setminus \text{sp}(\pi_I(b)) \) is connected. \( \square \)

Next, we investigate to what extent the converse of Proposition 3.1 holds. Recall the definition of the \( \epsilon \)-grid \( \Gamma_\epsilon \) given above Lemma 2.5.

**Theorem 3.2.** Let \( A \) be a unital \( C^* \)-algebra and let \( a \) be a normal element in \( A \). The following conditions are equivalent:

(i) \( a - \lambda \cdot 1 \) lies in the closure of \( \text{GL}_0(A) \) for every \( \lambda \in \mathbb{C} \),

(ii) for every \( \epsilon > 0 \) there exists a normal element \( b \in A \) such that

\[
\text{sp}(b) \subseteq \Gamma_\epsilon, \quad \|a - b\| \leq 2\epsilon, \quad b - \lambda \cdot 1 \in \text{GL}_0(A)
\]

for all \( \lambda \in \mathbb{C} \setminus \text{sp}(b) \).

If the real rank of \( A \) is zero, then (i) and (ii) are equivalent to

(iii) for every \( \epsilon > 0 \) there exists a normal element \( b \in A \) with finite spectrum and with \( \|a - b\| \leq \epsilon \).

The implication (ii) \( \Rightarrow \) (iii) of Theorem 3.2 is contained in a theorem of H. Lin, [L3, Theorem 5.4]. It also follows from [ELP, Theorem 3.1] after realizing that the conditions on \( a \) in (i) and (ii) imply that the map \( K_1(C^*(a, 1)) \rightarrow K_1(A) \), induced by the inclusion map, is zero. The implication (ii) \( \Rightarrow \) (i) follows easily from the fact that \( \mathbb{C} \setminus \Gamma_\epsilon \) is dense in \( \mathbb{C} \). The implication (iii) \( \Rightarrow \) (i) is also easy — see also the proof of Proposition 3.1 above.

One could alternatively prove (ii) \( \Rightarrow \) (iii) by mimicking the proof of Lemma 2.7. One would for this approach need Lin’s result, [L1], that if \( A \) is a unital \( C^* \)-algebra of real rank zero, then every unitary \( u \in \mathcal{U}_0(A) \) can be approximated by unitaries with finite spectra. To follow the proof of Lemma 2.7 we would need actual spectral projections for \( u \). They will in general not be available. Instead we can find projections, that approximately commute with \( u \) and that approximately divide the spectrum of \( u \) into two disjoint subsets. With some care, one can complete the proof of Lemma 2.7 in this fashion.

The proof of (i) \( \Rightarrow \) (ii) is contained in the three lemmas below:

**Lemma 3.3** (cf. [Rø1, Theorem 2.2]). Let \( A \) be a unital \( C^* \)-algebra and let \( a \) be an element in the closure of \( \text{GL}_0(A) \). Let \( a = v|a| \) be the polar decomposition of \( a \), with \( v \) a partial isometry in \( A^{**} \). For each continuous function \( f: [0, \epsilon] \rightarrow [0, \epsilon] \), such that \( f(0) \equiv 0 \) for some \( \epsilon > 0 \), there exists a unitary \( u \in \mathcal{U}_0(A) \) such that \( uf(|a|) = uf(|a|) \).
Lemma 3.4. Let \( f : \mathbb{R}^+ \to \mathbb{R}^+ \) be a normal element in \( A \), and let \( a = v|a| \) be a polar decomposition of \( a \), where \( v \in A^{**} \) is a partial isometry and \( |a| = (a^*a)^{1/2} \).

(i) If \( f : \mathbb{R}^+ \to \mathbb{R}^+ \) is continuous with \( f(0) = 0 \), then \( \hat{v}f(|a|) = \hat{f}(a) \).

(ii) If \( v f_\varepsilon(|a|) \) is continuous with \( f(0) = 0 \), then \( v f_\varepsilon(|a|) = \hat{f}(a) \).

Proof. (i). This follows easily by approximating \( f \) with functions \( f_0 \) of the form \( f_0(r) = rh(r) \), where \( h : \mathbb{R}^+ \to \mathbb{R}^+ \) is continuous (not necessarily with \( h(0) = 0 \)).

(ii). That \( b \) is normal, and \( \|a - b\| \leq 2\varepsilon \) can be seen as in the proof of Lemma 2.2. Notice that \( |b| = f_\varepsilon(|a|) + \varepsilon \cdot 1 \). This shows that \( |b| \) — and hence \( b \) — are invertible, and that the spectrum of \( b \) does not intersect the open ball with center 0 and radius \( \varepsilon \).
We shall apply the Borel function calculus inside the von Neumann algebra $A^{**}$. Denoting the indicator function of the (Borel) set $E$ by $1_E$, set $e_\varepsilon = 1_{[0,\varepsilon]}(|a|) = 1_{[0,\varepsilon]}(|b|) = 1_{\{\varepsilon\}}(|b|)$. Put $\varphi(t) = t 1_{(\varepsilon, \infty)}(t) = (f_\varepsilon(t) + \varepsilon) 1_{(\varepsilon, \infty)}(t)$ for $t \in \mathbb{R}^+$. 

$$|a|(1 - e_\varepsilon) = \varphi(|a|) = |b|(1 - e_\varepsilon).$$

Since $e_\varepsilon$ commutes with $|b|$, 

$$a(1 - e_\varepsilon) = v|a|(1 - e_\varepsilon) = v(1 - e_\varepsilon)|b| = u(1 - e_\varepsilon)|b| = b(1 - e_\varepsilon).$$

Now using that $e_\varepsilon$ commutes with $a$ and with $b$, we get 

$$g(a)(1 - e_\varepsilon) = g(a(1 - e_\varepsilon)) = g(b(1 - e_\varepsilon)) = g(b)(1 - e_\varepsilon)$$

for every continuous function $g: \mathbb{C} \to \mathbb{C}$.

Assume that $g$ is constant on $B(0, \varepsilon)$. Put $\psi(z) = g(z)1_{[0,\varepsilon]}(|z|) = g(0)1_{[0,\varepsilon]}(|z|)$. Then 

$$g(a)e_\varepsilon = \psi(a) = g(0)e_\varepsilon = \psi(b) = g(b)e_\varepsilon.$$ 

In conclusion, we have shown that $g(a) = g(b)$. The claim about the spectra follows from the previous statement.

We now show an analogue of Lemma 2.4.

**Lemma 3.5.** Let $A$ be a unital $C^*$-algebra, let $a$ be a normal element in $A$, and let $F$ be a finite subset of $\mathbb{C}$. If $a - \lambda \cdot 1$ lies in the closure of $\text{GL}_0(A)$ for all $\lambda \in F$, then for every $\varepsilon > 0$ there exists a normal element $b$ in $A$ with $

\|a - b\| \leq \varepsilon$ and $b - \lambda \cdot 1 \in \text{GL}_0(A)$ for every $\lambda \in F$.

**Proof.** Write $F = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$. We find successively normal elements $a_0 = a$, $a_1, a_2, \ldots, a_n$ in $A$ satisfying

- $\|a_{j+1} - a_j\| \leq \varepsilon/n$,
- $a_j - \lambda_i \cdot 1 \in \text{GL}_0(A)$ for $i = 1, 2, \ldots, j$,
- $a_j - \lambda_i \cdot 1$ lie in the closure of $\text{GL}_0(A)$ for $i = j + 1, j + 2, \ldots, n$.

The element $b = a_n$ will then be as desired.

Assume $a_{j-1}$ has been found and that $1 \leq j \leq n$. Choose $\delta > 0$ such that

- $\delta < \varepsilon/2n$,
- for every $c \in A$ with $\|c - a_{j-1}\| \leq \delta$, we have $c - \lambda_i \cdot 1 \in \text{GL}_0(A)$ for $i = 1, 2, \ldots, j - 1$.
- $|\lambda_i - \lambda_j| > \delta$ for $i = j + 1, j + 2, \ldots, n$,

Write $a_{j-1} - \lambda_j \cdot 1 = v|a_{j-1} - \lambda_j \cdot 1|$, with $v$ a partial isometry in $A^{**}$, and use Lemma 3.3 to find a unitary $u \in \mathcal{U}_0(A)$ so that $vf_\delta(|a_{j-1} - \lambda_j \cdot 1|) = uf_\delta(|a_{j-1} - \lambda_j \cdot 1|)$. Put 

$$a_j = u(f_\delta(|a_{j-1} - \lambda_j \cdot 1|) + \delta \cdot 1) + \lambda_j \cdot 1.$$
It then follows from Lemma 3.4 (ii) that \(a_j\) is normal, that \(a_j - \lambda_i 1 \in \text{GL}_0(A)\), and that \(\|a_j - a_{j-1}\| \leq 2\delta < \varepsilon/n\). By the choice of \(\delta\) this implies that \(a_j - \lambda_i 1 \in \text{GL}_0(A)\) for \(i = 1,2,\ldots,j\).

Let \(i \in \{j + 1, j + 2, \ldots, n\}\). We show that \(a_j - \lambda_i 1\) is in the closure of \(\text{GL}_0(A)\). By the choice of \(\delta\), we have that \(\lambda_i \notin B(\lambda_j, \delta)\). We can therefore find continuous functions \(f, g: \mathbb{C} \to \mathbb{C}\) such that

- \(f(z)g(z) = z - \lambda_i\) for all \(z \in \mathbb{C}\),
- \(f|_{B(\lambda_j, \delta)}\) is constant,
- \(g(z) = \exp(h(z))\) for some continuous function \(h: \mathbb{C} \to \mathbb{C}\).

The property of \(g\) entails that \(g(b) \in \text{GL}_0(A)\) for every normal element \(b \in A\). From Lemma 3.4 (ii) we conclude that \(f(a_{j-1}) = f(a_j)\). Hence

\[a_j - \lambda_i 1 = f(a_j)g(a_j) = f(a_{j-1})g(a_j) = (a_{j-1} - \lambda_i 1)g(a_{j-1})^{-1}g(a_j)\]

By assumption, \(a_{j-1} - \lambda_i 1\) lies in the closure of \(\text{GL}_0(A)\), and this shows that \(a_j - \lambda_i 1\) lies in the closure of \(\text{GL}_0(A)\).

Proof of (i) \(\Rightarrow\) (ii) in Theorem 3.2. Copy the proof of Lemma 2.5 using Lemma 3.5 instead of Lemma 2.4.

Recall from [FR] that a unital C*-algebra \(A\) is said to have property (IN) if every normal element belongs to the closure of \(\text{GL}(A)\). A non-unital C*-algebra \(A\) has property (IN) if its unitization \(\tilde{A}\) has property (IN).

**Definition 3.6.** We say that a unital C*-algebra \(A\) has property (IN\(_0\)) if every normal element that has 0 as an interior point of its spectrum lies in the closure of \(\text{GL}_0(A)\).

A non-unital C*-algebra \(A\) is said to have property (IN\(_0\)) if \(\tilde{A}\) has property (IN\(_0\)).

It is clear that property (IN\(_0\)) implies property (IN). Property (IN) does not imply (IN\(_0\)), not even for C*-algebras of real rank zero and stable rank one as Example 3.7 below shows.

Examples of C*-algebras satisfying (IN\(_0\)) are given in Proposition 3.8. The reader may prefer to consider the slightly more restrictive condition (IN\(_{00}\)) of a unital C*-algebra \(A\), defined by requiring all normal non-invertible elements of \(A\) to belong to the closure of \(\text{GL}_0(A)\). Trivially, (IN\(_{00}\)) implies (IN\(_0\)), but Example 3.9 gives a C*-algebra of real rank zero for which the reverse implication does not hold.

**Example 3.7.** Property (IN) does not imply property (IN\(_0\)).

Let \(B_1, B_2\) be two unital C*-algebras of stable rank one, real rank zero, so that the unitary group of \(B_1\) is disconnected, and \((B_1, \text{and}) B_2\) are non-scattered. (One could for example take \(B_1 = B_2\) to be an irrational rotation C*-algebra.) Let \(A\) be the C*-algebra \(B_1 \oplus B_2\). Then \(A\) is unital, \(\text{sr}(A) = 1\) and \(\text{RR}(A) = 0\).
Choose normal elements $b_1 \in B_1$ and $b_2 \in B_2$ with

$$\text{sp}(b_1) = \{ z \in \mathbb{C} : 1/2 \leq |z| \leq 1 \}, \quad \text{sp}(b_2) = \{ z \in \mathbb{C} : |z| \leq 1/2 \},$$

and with $b_1 \notin \text{GL}_0(B_1)$. Set $a = (b_1, b_2) \in A$. Then $\text{sp}(a) = \mathbb{D}$, the closed unit disc in the complex plane. Hence $a - \lambda b \in \text{GL}_0(A)$ for every $\lambda \in \mathbb{C}\setminus\text{sp}(a)$ (because $\mathbb{C}\setminus\text{sp}(a)$ is connected). But $a$ does not belong to the closure of $\text{GL}_0(A)$, since $b_1$ does not belong to the closure of $\text{GL}_0(B_1)$.

Hence $A$ does not have property (IN$_0$), but $\text{sr}(A) = 1$ (so $A$ has property (IN)), and $\text{RR}(A) = 0$.

**Proposition 3.8.** Every simple unital C*-algebra, which has stable rank one or is purely infinite, has property (IN$_0$).

**Proof.** The two classes of C*-algebras in the proposition have in common that they have property (IN) (see [Ro2, Theorem 4.4]), and if $B$ is a non-zero hereditary sub-C*-algebra of $A$, then the natural map $U(\overline{B}) \to U(A)/U_0(A)$ is surjective (see [Ri, Theorem 10.10] and [C, Theorem 1.9]).

Let $a$ be a normal non-invertible element of $A$, and let $\varepsilon > 0$. Let $a = |a|v|a|$ be the polar decomposition for $a$, with $v \in A^{**}$, and recall from [Ro1, Theorem 2.1] that there exists a unitary $u \in A$ so that $vf_\varepsilon(|a|) = uf_\varepsilon(|a|)$. Let $g: \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous function that vanishes on $[\varepsilon, \infty)$, and with $g(0) = 1$. Then $g(|a|) \neq 0$ because $a$ is non-invertible, and we can therefore find a unitary $w \in g(|a|)Ag(|a|) + 1$ such that $uw \sim_h 1$. Set $b = uw(f_\varepsilon(|a|) + \varepsilon \cdot 1)$. Then $b \in \text{GL}_0(A)$ (we are not claiming here that $b$ is normal), and

$$\|a - b\| \leq \|a - uf_\varepsilon(|a|) + \varepsilon \cdot 1\| + \|w(f_\varepsilon(|a|) + \varepsilon \cdot 1) - (f_\varepsilon(|a|) + \varepsilon \cdot 1)\| \leq 4\varepsilon,$$

where we have used that $wf_\varepsilon(|a|) = f_\varepsilon(|a|)$.

This argument shows that $A$ actually has property (IN$_{00}$). \hfill $\Box$

Villadsen has found an example of a simple C*-algebra with stable rank 2, showing that a specific normal element of the constructed C*-algebra cannot be approximated by invertible elements (see [V]). This example is therefore a simple, stably finite C*-algebra that does not have property (IN), and hence neither (IN$_0$) nor (IN$_{00}$).

**Example 3.9.** Property (IN$_0$) does not imply property (IN$_{00}$).

Let $B$ be any real rank zero, unital C*-algebra that has property (IN$_0$) and non-connected group of unitary elements, and set $A = B \oplus \mathbb{C}$. (Here $B$ could be an irrational rotation C*-algebra, cf. Proposition 3.8.) If zero is an interior point of the spectrum of $a = (b, \lambda) \in A$, then zero is an interior point of the spectrum of $b$. Hence $b$ lies in the closure of $\text{GL}_0(B)$, and from this we get that $a$ lies in the closure of $\text{GL}_0(A)$.

The C*-algebra $A$ does not have property (IN$_{00}$). Indeed, if $u \in U(B) \setminus U_0(B)$, and if $a = (u, 0) \in A$, then $a$ is normal and non-invertible, but $a$ is not in the closure of $\text{GL}_0(A)$. 

Corollary 3.10. Let $A$ be a unital $C^*$-algebra. The following two conditions are equivalent:

(i) Every normal element $a \in A$, satisfying

\[ a - \lambda \cdot 1 \in \text{GL}_0(A) \]

for all $\lambda \in \mathbb{C} \setminus \text{sp}(a)$, is the norm-limit of normal elements in $A$ with finite spectrum.

(ii) $A$ has real rank zero and property (IN$_0$).

Proof. (i) $\Rightarrow$ (ii). Assume that (i) holds. Every self-adjoint element $a$ in $A$ is normal and satisfies $a - \lambda 1 \in \text{GL}_0(A)$ for all $\lambda \in \mathbb{C} \setminus \text{sp}(a)$ (because $\mathbb{C} \setminus \text{sp}(a)$ is connected). Hence $a$ can be approximated within any given tolerance by a normal element $b$ with finite spectrum. It is easily seen, that $(b + b^*)/2$ is a self-adjoint element with finite spectrum whose distance to $a$ is at most $\|a - b\|$. This shows that $A$ has real rank zero.

We proceed to show that $A$ has property (IN$_0$). Let $a$ be a normal element in $A$, and assume that there exists an $r > 0$ such that $B(0, r) \subseteq \text{sp}(a)$. Let $a = va$ be the polar decomposition of $a$, with $v$ a partial isometry in $A^{**}$.

Consider the continuous functions $f : \mathbb{R}_+ \to \mathbb{R}^+$, $g : \mathbb{R}_+ \to \mathbb{R}^+$, given by $f(t) = \min\{r^{-1}t, 1\}$ and $g(t) = \max\{r, t\}$, and $h = f : \mathbb{C} \to \mathbb{C}$ (see above Lemma 3.4). Notice that $f(t)g(t) = t$, and that $vf(|a|) = h(a)$ (by Lemma 3.4 (i)).

The element $h(a)$ is therefore normal and $\text{sp}(h(a)) = h(\text{sp}(a)) = \mathbb{D}$ (the closed unit disc in the complex plane). Consequently, $h(a) - \lambda \cdot 1 \in \text{GL}_0(A)$ for all $\lambda \in \mathbb{C} \setminus \text{sp}(h(a))$ (because $\mathbb{C} \setminus \text{sp}(h(a))$ is connected). Assuming (i), we conclude that $h(a)$ can be approximated by normal elements with finite spectra, and therefore $h(a)$ lies in the closure of $\text{GL}_0(A)$ (cf. the proof of Proposition 3.1). Since $a = vf(|a|)g(|a|) = h(a)g(|a|)$, and since $g(|a|) \in \text{GL}_0(A)$, we conclude that $a$ lies in the closure of $\text{GL}_0(A)$. It has now been proved that $A$ has property (IN$_0$).

(ii) $\Rightarrow$ (i). By Theorem 3.2 it suffices to show that $a - \lambda \cdot 1$ lies in the closure of $\text{GL}_0(A)$ for all $\lambda \in \mathbb{C}$. This is the case by assumption on $a$ if $\lambda \notin \text{sp}(a)$. By continuity, $a - \lambda \cdot 1$ lies in the closure of $\text{GL}_0(A)$ for all $\lambda$ in the closure of $\mathbb{C} \setminus \text{sp}(a)$. The remaining points, $\lambda$, are the interior points of the spectrum of $a$, and there $a - \lambda \cdot 1$ is in the closure of $\text{GL}_0(A)$ by the assumption that $A$ has property (IN$_0$). \hfill $\Box$

Corollary 3.11. Let $A$ be a unital $C^*$-algebra. Assume that $\text{RR}(A) = 0$, that $A$ has property (IN), and that

(NT) for every hereditary sub-$C^*$-algebra $B$ of $A$, the map

\[ \mathcal{U}(\tilde{B}) \to \mathcal{U}(\tilde{I})/\mathcal{U}_0(\tilde{I}), \]

where $I$ is the ideal of $A$ generated by $B$, is surjective.
Then every normal element \( a \in A \), satisfying
\[
\pi_I(a) - \lambda \cdot 1 \in \text{GL}_0(A/I)
\]
for every proper ideal \( I \) of \( A \), and for every \( \lambda \in \mathbb{C} \setminus \text{sp}(\pi_I(a)) \), is a norm limit of normal elements in \( A \) with finite spectra.

Proof. By Theorem 3.2 it suffices to show that every normal element \( a \in A \), satisfying
\[
\pi_I(a) \in \text{GL}(A/I) \Rightarrow \pi_I(a) \in \text{GL}_0(A/I),
\]
lies in the closure of \( \text{GL}_0(A) \). Let \( \varepsilon > 0 \). Write \( a = v|a| \) with \( v \) a partial isometry in \( A^\ast \ast \). By the assumption that \( A \) has property (IN), and by Lemma 3.3, there is a unitary \( u \in A \) so that \( v f_\varepsilon(|a|) = u f_\varepsilon(|a|) \). We wish to replace \( u \) by another unitary that belongs to \( U_0(A) \). Since \( A \) is of real rank zero, we can find a projection \( p \) in the hereditary subalgebra of \( A \) generated by \( f_\varepsilon(|a|) \) so that \( \| (1 - p) f_\varepsilon(|a|) \| \leq \varepsilon \). We show that there is a unitary \( v \in (1 - p)A(1 - p) \) such that \( u^* \sim_h p + v \) in \( U(A) \). This will imply that \( b = u(p + v^*)(f_\varepsilon(|a|) + \varepsilon \cdot 1) \in \text{GL}_0(A) \), and
\[
\| a - b \| \leq \| a - u(f_\varepsilon(|a|) + \varepsilon \cdot 1) \|
\]
\[
+ \| (p + v^*)(f_\varepsilon(|a|) + \varepsilon \cdot 1) - (f_\varepsilon(|a|) + \varepsilon \cdot 1) \|
\]
\[
\leq 2\varepsilon + \| (v^* - (1 - p))(f_\varepsilon(|a|) + \varepsilon \cdot 1) \| \leq 6\varepsilon.
\]

Let \( I \) be the closed two-sided ideal in \( A \) generated by \( 1 - p \). If \( I = A \), then \( (1 - p)A(1 - p) \) contains a unitary \( v \) with \( u^* \sim_h p + v \) in \( U(A) \) by the assumption (NT). Assume that \( I \neq A \). Because \( p \) lies in the hereditary subalgebra generated by \( f_\varepsilon(|a|) \), we have \( p|a|p \geq \varepsilon p \); this entails \( \pi_I(|a|) \geq \varepsilon 1 \), and so \( \pi_I(a) \in \text{GL}(A/I) \), which by the assumption on \( a \) implies \( \pi_I(a) \in \text{GL}_0(A/I) \). It follows that
\[
\pi_I(u) \sim_h \pi_I(u f_\varepsilon(|a|)) \sim_h \pi_I(a) \sim_h 1
\]
in \( \text{GL}(A/I) \). We can therefore find \( w \in U_0(A) \) with \( \pi_I(w) = \pi_I(u) \). Since \( w^* u \in U(I) \) it follows from the assumption (NT) that there exists a unitary \( v \) in \( (1 - p)A(1 - p) \) with \( (w^* u)^* \sim_h p + v \). This completes the proof, because \( w^* u \sim_h u \). \( \square \)

Every C*-algebra of stable rank one has property (IN) (for trivial reasons), and also property (NT) (for less trivial reasons). For the latter one can use [Ri, Theorem 10.10]. With these observations we get the following corollary to Corollary 3.11:

Corollary 3.12. Let \( A \) be a unital C*-algebra of real rank zero and stable rank one. Then every normal element \( a \in A \), satisfying
\[
\pi_I(a) - \lambda \cdot 1 \in \text{GL}_0(A/I)
\]
for every proper ideal \( I \) of \( A \) and every \( \lambda \in \mathbb{C} \setminus \text{sp}(\pi_I(a)) \), is a norm limit of normal elements in \( A \) with finite spectra.
Corollary 3.12 can for example be applied to the real rank zero \( \mathcal{A} \)-algebras classified by George Elliott in [E]. (\( \mathcal{A} \)-algebras is the class of \( \mathcal{C}^* \)-algebras obtained from the \( \mathcal{C}^* \)-algebra \( C(\mathbb{T}) \) with the operations of tensoring by \( M_n(\mathbb{C}) \), taking direct sums, and taking inductive limits.)

It would be interesting to know if one can replace the assumption in Corollary 3.12 that \( \text{sr}(A) = 1 \) with the weaker assumption that \( A \) has property \((\text{IR})\) (cf. [FR, 3.1] and the main theorem of [L4]). By Corollary 3.11 that would be the case if property \((\text{IR})\) (together with \( \text{RR}(A) = 0 \)) implies property \((\text{NT})\). This is known to be true for simple \( \mathcal{C}^* \)-algebras, because a simple unital \( \mathcal{C}^* \)-algebra \( A \) has property \((\text{IR})\) if and only if either \( \text{sr}(A) = 1 \) or \( A \) is purely infinite.

**Example 3.13.** Of the three sufficient conditions in Corollary 3.11, the real rank zero condition is clearly also necessary (cf. the proof of Corollary 3.10). It is possible that the condition \((\text{NT})\) always holds for real rank zero \( \mathcal{C}^* \)-algebras. Larry Brown has informed us that examples of \( \mathcal{C}^* \)-algebras of real rank zero, which do not have property \((\text{IN})\), exist. Condition \((\text{NT})\) is necessary in Corollary 3.11, at least when \( A \) is simple, as the following example shows:

Assume that \( A \) is a simple unital \( \mathcal{C}^* \)-algebra of real rank zero where property \((\text{NT})\) of Corollary 3.11 does not hold — if such an example exists. Since every hereditary sub-\( \mathcal{C}^* \)-algebra of \( A \) is the inductive limit of corner algebras \( pAp \), where \( p \) is a projection in \( A \), there is a unitary \( u \in A \) and a projection \( p \in A \) with the property that there is no unitary \( v \in pAp \) satisfying \( u \sim_h v + (1 - p) \). Since \( A \) is of real rank zero, and since \( \text{sp}(u) = \mathbb{T} \), there is a non-zero projection \( q \in A \) such that \( \|quq - q\| \leq 1/2 \). Then \((1 - q)u(1 - q)\) is invertible in \((1 - q)A(1 - q)\), and \( z = q + (1 - q)u(1 - q) \) is homotopic to \( u \) in \( \text{GL}_0(A) \). Put \( u_0 = |z|^{-1} \). Then \( u_0 \in \mathcal{U}(A) \), \( u \sim_h u_0 \) and \( qu_0 = u_0q = q \).

Let \( x \) be a non-zero element in \( qAp \), and let \( e \) be a non-zero projection in \( x\mathcal{A}x^* \). Then \( e \leq q \) and \( e \lesssim p \). It follows that \( eu_0e = e \) and that there is no unitary \( v \in eAe \) such that \( u_0 \sim_h v + (1 - e) \). The corner algebra \( eAe \) is non-scattered (because \( A \) must be infinite-dimensional), and we can therefore find a normal element \( c \in eAe \) with \( \text{sp}(c) = \mathbb{D} \).

Put \( a = e + (1 - e)u_0(1 - e) \). Then \( a \) is normal and \( \text{sp}(a) = \mathbb{D} \), and so \( a - \lambda 1 \in \text{GL}_0(A) \) for every \( \lambda \in \mathbb{C} \setminus \text{sp}(a) \). We claim that \( a \) is not in the closure of \( \text{GL}_0(A) \), and this will show that \( a \) cannot be approximated by normal elements with finite spectra, cf. Theorem 3.2. Indeed, assume that \( b \in \text{GL}_0(A) \) and that \( \|a - b\| < 1 \). Then

\[
\|(1 - e)u_0(1 - e) - (1 - e)b(1 - e)\| < 1,
\]

and \((1 - e)u_0(1 - e)\) is unitary in \((1 - e)A(1 - e)\). Hence \((1 - e)b(1 - e)\) is invertible in \((1 - e)A(1 - e)\) with an inverse we denote by \( r \). As in a standard
\[2 \times 2\text{ matrix trick,}
\]
\[\begin{array}{c}
(1 - e)b(1 - e) + (ebe - ebrbe) = (1 - ebr(1 - e))b(1 - (1 - e)rbe)
\end{array}\]
\[\sim_h b\]

in \(\text{GL}(A)\). Hence \(d = ebe - ebrbe \in \text{GL}(eAe)\) and so \(d = v^*|d|\) for some unitary \(v \in eAe\). By (♠) we also have
\[\begin{array}{c}
(1 - e) + v \sim_h (1 - e) + d^{-1} \sim_h (1 - e)b(1 - e) + e
\end{array}\]
\[\sim_h (1 - e)u_0(1 - e) + e = u_0\]
in contradiction with the stipulated properties of \(u_0\) and \(e\).

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Received November 1, 1998. The first author was supported by The Danish Research Academy.

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PARAMETRIZATION OF THE IMAGE OF NORMALIZED
INTERTWINING OPERATORS

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In studying residual automorphic representations, we need to parametrize the image of normalized local intertwining operators. This has been done by Moeglin in the case of the residual spectrum attached to the trivial character of the torus for split classical groups. In this paper, we extend her result to non-trivial characters of the torus. To do this, we use Roche’s Hecke algebra isomorphisms and Barbasch-Moy’s graded algebra isomorphisms to reduce to the case of the trivial character. Along the way, we need to show that Roche’s Hecke algebra isomorphisms are compatible with induction in stages, construct a generalized Iwahori-Matsumoto involution, and show that the images of intertwining operators behave well with respect to the Hecke algebra and graded algebra isomorphisms. We note that this also gives a parameterization of the square-integrable and tempered representations supported on the Borel subgroup.

0. Introduction.

Let \( G = G(n) \) be a split classical group \( Sp(2n, F), SO(2n + 1, F), \) or \( O(2n, F) \) over a \( p \)-adic field \( F \) with odd residual characteristic (this condition comes from \([R]\)) and \( T \) be a maximal torus. Throughout this paper, we will drop \( F \) in the notation. Let \( \chi \) be a unitary character of \( T \) and \( p = (a_1, b_1, \ldots, a_s, b_s, a_{s+1}) \) be a chain (\( a_{s+1} \) is only for the cases \( Sp(2n, F) \) and \( SO(2n + 1, F) \); see Definition 3.1). Then \( p \) gives rise to a character

\[
\lambda_p = \left( \frac{a_1 - 1}{2}, \frac{a_1 - 3}{2}, \ldots, \frac{b_1 - 1}{2}, \ldots, \frac{a_s - 1}{2}, \frac{a_s - 3}{2}, \ldots, \frac{b_s - 1}{2}, \frac{a_{s+1} - 1}{2}, \frac{a_{s+1} - 3}{2}, \ldots, 1 \right).
\]

If we ignore the ordering, \( p \) gives rise to a unipotent orbit \( O \) in the dual group \( G^* = SO(2n + 1, \mathbb{C}), Sp(2n, \mathbb{C}), \) or \( O(2n, \mathbb{C}) \). To \( p \), we attach a Weyl group element \( w_p \). In this paper we parametrize the image of the local normalized intertwining operator \( R(w_p, \lambda_p, \chi) I (\lambda_p, \chi) \). We need this result in
the calculation of the residual spectrum [Ki3]. In addition to the application to the residual spectrum, our result has independent interest in that it parametrizes the square integrable representations which have support on Borel subgroups, via the generalized Iwahori-Matsumoto involution; actually it parametrizes tempered representations which have support on Borel subgroups (see Remark 3.2.4).

In a remarkable paper [M1], Mœglin solved the problem in the case when $\chi$ is trivial and $p$ satisfies a certain condition, that is, $p \in P(O)$ (see Section 2). She showed that $R(w_p, \lambda_p)I(\lambda_p)$ is semi-simple and its summands are parametrized by certain characters $\eta$ of $A(O)$, where $A(O)$ is a finite abelian group generated by the order two elements $\sigma(a_1), \sigma(b_1), \ldots, \sigma(a_s), \sigma(b_s)$, $\sigma(a_{s+1})$ (we take only distinct ones). If $O$ is a distinguished unipotent orbit (i.e., $a_i$'s, $b_j$'s are all distinct), then the characters are those which satisfy $\eta(\sigma(a_i)) = \eta(\sigma(b_i))$, $i = 1, \ldots, s$ and $\eta(\sigma(a_{s+1})) = 1$. We denote by $A(p)$ the set of such characters. Let Unip($p$) be the set of direct summands of $R(w_p, \lambda_p)I(\lambda_p)$. To a unipotent orbit $O$, she considered a certain set of ordered partitions $P(O)$ so that each chain $p' \in P(O)$ gives rise to a certain character $\lambda_{p'}$ which is a conjugate of $\lambda_p$. Let Unip($O$) be the union of all Unip($p$) as $p$ runs through $P(O)$. She showed that Unip($O$) is the set of irreducible constituents of the principal series $I(\lambda_p)$ whose Iwahori-Matsumoto involution is tempered. Then Unip($O$) is parametrized by Springer ($O$), which is the union of $A(p)$ as $p$ runs through $P(O)$; recall that the Springer correspondence is an injective map from the characters of $W$, the Weyl group, into the set of $(O, \eta)$, where $O$ is a unipotent orbit of $G^*$ and $\eta$ is a character of $A(O)$. Then Springer ($O$) is the set of characters of $A(O)$ which are in the image of the Springer correspondence. Thus Mœglin showed that Unip($O$) is the set of the local components of the residual spectrum attached to the trivial character of the maximal torus.

The basic approach of this paper is to reduce the problem to the trivial character case. However, there are a number of non-trivial obstacles, which we now describe.

First, the basic mechanism we use to reduce from the ramified case to the unramified case is the Hecke algebra isomorphisms of Roche [R]. The first basic problem we must deal with is that the representations we are interested in are not, in general, induced off the Borel, but rather are degenerate principal series. So, in order to implement our approach, we must establish that these isomorphisms behave well with respect to induction in stages. Since such results may be of broader use, we do this in the generality of [R], not just for the particular classical groups we deal with in the rest of this paper. In addition, we need to generalize the Iwahori-Matsumoto involution. (While this has been done in general in [Au1], [Au2], [Sc-St], these results are done in the Grothendieck group setting; we need to know
that composition series are respected as well.) Such a generalized Iwahori-Matsumoto involution can easily be defined using Roche’s isomorphism. But in order to verify some of the properties we need, we have to establish how it behaves with respect to induction in stages. Again, this comes down to showing that Roche’s isomorphisms respect induction in stages. These issues are addressed in Section 1.

Second, we need to deal with non-trivial unramified quadratic characters. Mœglin depended on Barbasch-Moy’s results \[\text{[B-Mo1]}\] which use Kazhdan-Lusztig’s parametrization of unramified representations and the Iwahori-Matsumoto involution. However, the technique cannot be extended to non-trivial unramified quadratic characters. In a subsequent work, Barbasch-Moy \[\text{[B-Mo2]}\] extended their results to non-trivial unramified quadratic characters, using graded Hecke algebras. We use their graded Hecke algebra isomorphisms to reduce to the trivial character case. See Section 3.1. Unfortunately, Barbasch-Moy’s results \[\text{[B-Mo2]}\] are stated for connected groups and we need them for the disconnected group \(O(2n)\). We have stated this as Assumption 3.1.1. We have no doubt that it is true. However, we were not able to verify it. Therefore, our results are complete only for odd orthogonal groups.

Third, we need to remove a restriction on \(p\) (see Remark 3.2) for an arbitrary chain when \(\chi\) is trivial. Note that an arbitrary chain does not belong to \(P(O)\) in general. But it always comes from the global consideration when \(\chi = \chi(\mu_1, \ldots, \mu_1, \ldots, \mu_k, \ldots, \mu_k, 1, \ldots, 1)\), where \(\mu_i\)’s are non-trivial quadratic grössencharacters such that \(\mu_{i\nu} = 1\). (See Remark 3.3.) Mœglin’s argument by induction shows that \(R(w_p, \lambda_p, 1)I(\lambda_p)\) is still semi-simple in the general case and we denote the set of direct summands still by \(\text{Unip}(p)\). However, Mœglin’s argument does not work in this case, since the normalized local intertwining operators could vanish. For this, we use the global method. By considering the iterated residue of the pseudo-Eisenstein series as in Mœglin \[\text{[M1]}\], we can show that \(\text{Unip}(p)\) is contained in \(\text{Unip}(O)\), where \(O\) is the unipotent orbit obtained by ignoring the ordering in \(p\). Recall that \(\text{Unip}(O)\) is the union of \(\text{Unip}(p)\) as \(p\) runs through \(P(O)\) and this shows that by considering arbitrary chains, we do not get a new component.

Fourth, we need to show that Hecke algebra isomorphisms of \[\text{[R]}\] and the graded Hecke algebra isomorphisms of \[\text{[B-Mo2]}\] commute with the intertwining operators. That reduces us to the case of the trivial character. More precisely, let \(p = (a_1, b_1, \ldots, a_s, b_s, a_{s+1})\) and \(\chi = \chi(\mu_1, \ldots, \mu_1, \ldots, \mu_k, 1, \ldots, 1)\), \(r_0 + \cdots + r_k = n\), \(r_1 \geq \cdots \geq r_k\), \(\mu_i\)’s are distinct quadratic characters. Here \(k \leq 3\). (Recall that we are dealing with a \(p\)-adic field
with odd residual characteristic and hence there are only three non-trivial distinct quadratic characters.) Set \( \mu_0 = 1 \). Let \( G' = G'_1 \times \cdots \times G'_k \times G'_0 \), where, for \( i = 1, \ldots, k \),

\[
G'_i = G(r_0), \quad G'_i = \begin{cases} O(2r_i), & \text{if } G = Sp(2n), O(2n) \\
SO(2r_i + 1), & \text{if } G = SO(2n + 1).
\end{cases}
\]

Here we note that \( G' \) is an endoscopic group of \( G \). Then by the Hecke algebra isomorphisms of \([R]\) and the graded algebra isomorphisms of \([B-Mo2]\), we get an equivalence of categories

\[
\mathcal{R}(G, \tau(\lambda_p, \chi)) \simeq \mathcal{R}(G', \tau'(\lambda_p, 1)),
\]

where \( \tau(\lambda_p, \chi) \) is the infinitesimal character associated to subquotients of \( I(\lambda_p, \chi) \) and \( \mathcal{R}(G, \tau(\lambda_p, \chi)) \) is the category of smooth finite-length representations of \( G \) having infinitesimal character \( \tau(\lambda_p, \chi) \). We note that \( \lambda_p \) may be viewed as a character of the maximal split torus of \( G' \) in the obvious way: If we write \( \lambda_p = \lambda_{p_1} \times \cdots \times \lambda_{p_k} \times \lambda_p \) (corresponding to the decomposition \( r_1 + \cdots + r_k + r_0 = n \) above), then \( \lambda_p \) may be viewed as a character of the maximal split torus of \( G'_i \). We show in Section 3.2 that under the category equivalence, the image \( R(w_p, \lambda_p, \chi)I(\lambda_p, \chi) \) corresponds to the image \( R(w_{p_1}, \lambda_{p_1}, 1)I(\lambda_{p_1}) \otimes \cdots \otimes R(w_{p_k}, \lambda_{p_k}, 1)I(\lambda_{p_k}) \otimes R(w_p, \lambda_p, 1)I(\lambda_p) \). This reduces us to the case when \( \chi = 1 \). Thus, by matching of the images of intertwining operators, we see that \( R(w_p, \lambda_p, \chi)I(\lambda_p, \chi) \) is semi-simple. Let \( \text{Unip}(\mathfrak{p}, \chi) \) be the set of the direct summands. Then under the category equivalence, \( \text{Unip}(\mathfrak{p}, \chi) \) is contained in the set \( \text{Unip}(O_1) \times \cdots \times \text{Unip}(O_k) \times \text{Unip}(O_0) \), where \( O_i \) are certain unipotent orbits obtained by ignoring the ordering in \( \mathfrak{p} \). In particular, the generalized Iwahori-Matsumoto involution of the elements in \( \text{Unip}(\mathfrak{p}, \chi) \) is tempered.

We let \( \chi = 1 \) and parametrize \( \text{Unip}(\mathfrak{p}) \). To each \((a_i, b_i)\), we can attach a Weyl group element \( \sigma(a_i, b_i) \). Then \( \sigma(a_i, b_i) \) defines a normalized operator \( R(\sigma(a_i, b_i)) \). It defines a homomorphism from the group \( \{id, \sigma(a_i, b_i)\} \) into the group of the intertwining operators of \( R(w_p, \lambda_p)I(\lambda_p) \). This means the following: For \( X \in \text{Unip}(\mathfrak{p}) \), let \( R(\sigma(a_i, b_i))X = \eta^p_X(\sigma(a_i, b_i))X \). Then \( \eta^p_X \) defines a character of \( A(O) \) such that \( \eta^p_X(\sigma(a_i)) = \eta^p_X(\sigma(b_i)) \). Since \( \text{Unip}(\mathfrak{p}) \subset \text{Unip}(O), \eta^p_X \in \text{Springer}(O) \). Therefore we have:

**Theorem 0.1** (Theorem 3.4.2). \( \text{Unip}(\mathfrak{p}) \) is parametrized by

\[
C(\mathfrak{p}) = \{\eta \in \text{Springer}(O) : \eta(\sigma(a_i)) = \eta(\sigma(b_i)) \text{ for } i = 1, \ldots, s, \text{ and } \eta(\sigma(a_{s+1})) = 1\}.
\]

Note that if \( \mathfrak{p} \) satisfies the condition (3.1) (that is, in Mœglin’s situation), then \( C(\mathfrak{p}) = \hat{A}(\mathfrak{p}) \) ([M1, Proposition 1.3.3]). In order to apply this theorem to the residual spectrum calculation, let \( O_1, O_2 \) be two distinguished unipotent orbits in \( G_1, G_2 \), resp. (If \( G = Sp(2n) \), then \( G_1^* = O(2r_1, \mathbb{C}) \))
and $G_{2}^{*} = SO(2r_{0} + 1, \mathbb{C})$. If $G = SO(2n + 1)$, then $G_{2}^{*} = Sp(2r_{1}, \mathbb{C})$ and $G_{2}^{*} = Sp(2r_{0}, \mathbb{C})$. If $G = O(2n)$, then $G_{2}^{*} = O(2r_{1}, \mathbb{C})$ and $G_{2}^{*} = O(2r_{0}, \mathbb{C})$.

Then we get a unipotent orbit $O$ in $G^{*}$ by combining $O_{1}$ and $O_{2}$. Further we have canonical embedding $A(O_{1}) \subset A(O)$. For $p_{i} \in P(O_{i})$, $i = 1, 2$, we get a chain $p_{1} \times p_{2}$ by shuffling the segments in $p_{1}$ and $p_{2}$ so that it satisfies (3.1), and thus we get $Unip(p_{1} \times p_{2})$. Let $Unip(O_{1}, O_{2})$ be the union of $Unip(p_{1} \times p_{2})$ as $p_{i}$ runs through $P(O_{i})$ for $i = 1, 2$. It is a subset of $Unip(O)$. Then we have:

**Theorem 0.2** (Theorem 3.4.3). $Unip(O_{1}, O_{2})$ is parametrized by

$$C(O_{1}, O_{2}) = \{ \eta \in Springer(O) : \eta|_{A(O_{1})} \in Springer(O_{1}), \eta|_{A(O_{2})} \in Springer(O_{2}) \}.$$ 

This can be easily generalized to an arbitrary character. Let us state the result on the local components of the residual spectrum attached to an arbitrary character in order to apply it to [Ki3]. Let $\chi = \chi_{\{\mu_{1}, \ldots, \mu_{1}, \ldots, r_{1}\}}{\mu_{k}, \ldots, \mu_{k}, 1, \ldots, 1}, r_{0} + \cdots + r_{k} = n, r_{1} \geq \cdots \geq r_{k} \geq 2, \mu_{i}$’s are distinct non-trivial quadratic grössencharacters. Let $O_{i}$ be a distinguished unipotent orbit in $G_{s}^{*}$ for $i = 0, 1, \ldots, k$, where, for $i = 1, \ldots, k$,

$$G_{i}^{*} = \begin{cases} 
O(2r_{i}, \mathbb{C}), & \text{if } G = Sp(2n), O(2n) \\
Sp(2r_{i}, \mathbb{C}), & \text{if } G = SO(2n + 1), 
\end{cases}$$

$$G_{0}^{*} = \begin{cases} 
SO(2r_{0} + 1, \mathbb{C}), & \text{if } G = Sp(2n) \\
Sp(2r_{0}, \mathbb{C}), & \text{if } G = SO(2n + 1) \\
O(2r_{0}, \mathbb{C}), & \text{if } G = O(2n). 
\end{cases}$$

Let $p_{i} \in P(O_{i})$ for $i = 0, \ldots, k$ and $p = p_{1} \times \cdots \times p_{k} \times p_{0}$. Then we can shuffle the segments in $p$ so that it satisfies the condition (3.1). We still call it $p$.

For a non-archimedean place $v$, let $Unip(O_{1}, \ldots, O_{k}, O_{0}, \chi_{v})$ be the set of union of $Unip(p_{1}, \ldots, p_{k}, p_{0}, \chi_{v})$ as $p_{i}$ runs through $P(O_{i})$ for $i = 0, \ldots, k$.

**Theorem 0.3** (Theorem 3.4.10). $\Pi_{res,v} = Unip(O_{1}, \ldots, O_{k}, O_{0}, \chi_{v})$ is parametrized by

$$C(O_{1}, \ldots, O_{k}, O_{0}, \chi_{v}) = [Springer(O_{1}) \times \cdots \times Springer(O_{k}) \times Springer(O_{0})],$$

where $[ ]$ is defined as follows: If $\mu_{1v} = \mu_{2v} \neq \mu_{iv}$ for $i = 0, 3, \ldots, k$, then we replace $Springer(O_{1}) \times Springer(O_{2})$ by

$$C(O_{1}, O_{2}, \mu_{1v}) = \{ \eta \in Springer(O) : \eta|_{A(O_{i})} \in Springer(O_{i}), \text{ for } i = 1, 2 \},$$
where \( O \) is the unipotent orbit of \( G_{12}^* \) obtained by combining \( O_1, O_2 \), where
\[
G_{12}^* = \begin{cases} 
O(2(r_1 + r_2), \mathbb{C}), & \text{if } G = Sp(2n), O(2n) \\
Sp(2(r_1 + r_2), \mathbb{C}), & \text{if } G = SO(2n + 1).
\end{cases}
\]

**Acknowledgments.** We would like to thank Prof. Shahidi for his guidance and encouragement throughout this paper. The second author would like to thank Prof. Mœglin for patiently answering many of his questions [M6] and for many correspondences. We would also like to thank Alan Roche for many useful conversations on Hecke algebra isomorphisms. Finally, we wish to thank Professors A. Moy, M. Reeder, A. Silberger, M. Tadić, and D. Vogan for many helpful correspondences.

1. **Hecke algebra isomorphisms and the generalized Iwahori-Matsumoto involution.**

In order to reduce from the case of ramified characters to the case of unramified characters, we use the Hecke algebra isomorphisms of Roche [R]. We also use these isomorphisms to construct a generalized Iwahori-Matsumoto involution. (The duality results of Aubert and Schneider-Stuhler are in the Grothendieck group setting; we need to deal with the composition series here.) We begin this section by reviewing the Iwahori-Matsumoto involution and the Hecke algebra isomorphisms of Roche. We then show that these isomorphisms behave well with respect to induction in stages. This will also allow us to verify certain properties of the generalized Iwahori-Matsumoto involution.

For this section, we work in a more general setting. Let \( G \) denote the \( F \)-rational points of a split connected reductive group defined over \( F \). In order to apply Roche’s results, we also assume the residue characteristic of \( F \) satisfies the conditions in [R]. For our applications to \( Sp(2n), SO(2n + 1) \), this requires odd residue characteristic. (We note that [Go], or more generally [Mr], gives similar Hecke algebra isomorphisms which could be used to extend the results of this section to cover characters of level 0 without the constraints on the residue characteristic.) Fix a set of positive roots \( \Phi^+ \) and a subset of simple roots \( \Pi \). Let
\[
\begin{align*}
B &= \text{Borel subgroup of } G \\
I &= \text{Iwahori subgroup of } G \\
K &= \mathcal{O} - \text{points in } G \text{ (a maximal compact subgroup)} \\
W &= \text{Weyl group of } G \\
W &= \text{affine Weyl group of } G \\
\ell(\cdot) &= \text{length function on } W \\
T &= \text{maximal split torus in } B
\end{align*}
\]
\( \delta(\cdot) = \) modular function on \( B \).

We may view \( \mathbf{W} \subset W \) by identifying \( \mathbf{W} \) with \( N(O)/A(O) \), where \( N = \text{Norm}_G(T) \). We use \( B_G, I_G, \) etc., if there is more than one group around and confusion is possible. Set

\[ \mathcal{H}(G) = C_c^\infty(G). \]

It is an algebra under convolution. If \( (\pi, G, V) \) is a smooth representation, we define \( \pi \) on \( \mathcal{H}(G) \) by

\[ \pi(h)v = \int_G h(g)\pi(g)vdg \]

for all \( h \in \mathcal{H}(G) \), \( v \in V \).

We begin by reviewing the Iwahori-Matsumoto involution. First, let us normalize Haar measure so that \( |I| = 1 \). Set \( 1_I = \text{char}\, I \).

Let

\[ \mathcal{H}(G, 1_I) = 1_I \cdot \mathcal{H}(G) \cdot 1_I = \{ f \in \mathcal{H}(G) | f(i_1 g i_2) = f(g) \text{ for all } i_1, i_2 \in I, g \in G \}. \]

The Iwahori-Matsumoto involution is an involution of \( \mathcal{H}(G, 1_I) \). In order to describe the Iwahori-Matsumoto involution, we must first discuss the structure of \( \mathcal{H}(G, 1_I) \). The following description is due to Bernstein-Zelevinsky (cf. \[ Lu2 \]); the classical description is due to Iwahori-Matsumoto \[ I-M \]. As a vector space, \( \mathcal{H}(G, 1_I) = \mathcal{H}(K, 1_I) \otimes \Theta \). Now, \( \mathcal{H}(K, 1_I) \) has \( \{ T_w \} \) as a basis, where \( T_w \) denotes the characteristic function of \( IwI \). Further, the multiplication is governed by

\[ T^2_w = (q - 1)T_w + q \text{ for } s \text{ simple} \]

\[ T_{w_1} T_{w_2} = T_{w_1 w_2} \text{ if } \ell(w_1) + \ell(w_2) = \ell(w_1 w_2). \]

\( \Theta \) is an abelian subalgebra of \( \mathcal{H}(G, 1_I) \) with basis \( \{ \theta_t | t \in T/T \cap K \} \). For \( t \in T \), choose \( t_1, t_2 \in T^- = \{ t \in T | |\alpha(t)| \leq 1 \text{ for all simple roots } \alpha \} \) such that \( t = t_1 t_2^{-1} \). Then, \( \theta_t = \delta^{\check{\alpha}}(t)T_{t_1} T_{t_2}^{-1} \). The multiplication between \( \mathcal{H}(K, 1_I) \) and \( \Theta \) is governed by the following: If \( s = s_\alpha, \alpha \) a simple root,

\[ \theta_t T_s = T_s \theta_{sts} + (q - 1) \frac{\theta_t - \theta_{sts}}{1 - \theta_\check{\alpha}(w^{-1})}, \]

where \( \check{\alpha} \) denotes the coroot associated to \( \alpha \). The Iwahori-Matsumoto involution \( j : \mathcal{H}(G, 1_I) \rightarrow \mathcal{H}(G, 1_I) \) is defined by

\[ j : T_w \mapsto (-q)^{\ell(w)}(T_{w^{-1}})^{-1} \]

\[ j : \theta_t \mapsto \theta_t^{-1} \]

for \( T_w \in \mathcal{H}(K, 1_I), \theta_t \in \Theta \).
We now discuss Roche’s results \([R]\). The applications to the representation theory of \(G\) will be discussed later; for now we focus on the results dealing with the structure of Hecke algebras. Fix a character \(\chi : T \cap K \longrightarrow \mathbb{C}^\times\). To the character \(\chi\), he associates an open compact subgroup \(J\) and a character \(\rho : J \longrightarrow \mathbb{C}^\times\) with \(\rho|_{T \cap K} = \chi\). The pair \((J, \rho)\) is a type in the sense of \([\text{Bu-K}]\). Let \(e_\rho\) be defined by

\[
e_\rho(g) = \begin{cases} \frac{1}{|J|}\rho^{-1}(g), & \text{if } g \in J \\ 0, & \text{if not.} \end{cases}
\]

Then, \(e_\rho^2 = e_\rho\). Let

\[
\mathcal{H}(G, \rho) = e_\rho \cdot \mathcal{H}(G) \cdot e_\rho
= \{ f \in \mathcal{H}(G) | f(j_1 gj_2) = \rho^{-1}(j_1 j_2)f(g) \text{ for all } j_1, j_2 \in J, g \in G \}.
\]

Roche constructs a split connected reductive group \(H\) and a finite abelian group \(C_\chi\), which acts on \(H\), such that

\[
\mathcal{H}(G, \rho) \cong \mathcal{H}(H, 1_1) \otimes \mathbb{C}[C_\chi].
\]

The notation \(\otimes\) is used to indicate that multiplication is governed by

\[
(T_{w_1} \otimes c_1) \cdot (T_{w_2} \otimes c_2) = T_{w_1 w_2 c_1(c_2)} \otimes c_1 c_2
\]

for \(w_1, w_2 \in H\), \(c_1, c_2 \in C_\chi\) (cf. Section 8 \([R]\)). He also constructs a disconnected group \(\tilde{H}\), with \(H\) the connected component of the identity in \(\tilde{H}\), which has \(\mathcal{H}(H, 1_1) \otimes \mathbb{C}[C_\chi] \cong \mathcal{H}(\tilde{H}, 1_{1_\mu})\). The following example will be of interest in \(\S 3.2\).

**Example 1.1.** Suppose \(G = SO(2(r_0 + r_1) + 1, F)\) or \(Sp(2(r_0 + r_1), F)\), \(F\) of odd residual characteristic. Let \(\mu\) denote a ramified quadratic character of \(F^\times\) and set \(\chi = \chi(\mu, \ldots, \mu, 1, \ldots, 1)\). (More precisely, it is \(\chi|_{\text{Tr}(\mathcal{O})}\) that is needed in Roche’s construction.) Then \(J_\chi = I\), and we have the following:

1. If \(G = SO(2(r_0 + r_1) + 1)\), then \(\mathcal{H}(G, \rho_\chi) \cong \mathcal{H}(\tilde{H}, 1)\), with \(\tilde{H} = SO(2r_1 + 1) \times SO(2r_0 + 1) = H'_1 \times H'_0\).
2. If \(G = Sp(2(r_0 + r_1))\), then \(\mathcal{H}(G, \rho_\chi) \cong \mathcal{H}(\tilde{H}, 1)\), with \(\tilde{H} = O(2r_1) \times Sp(2r_0) = H'_1 \times H'_0\).

Further, if \(\lambda\) is an unramified character of \(T\), write \(\lambda = \lambda_1 \times \lambda_0\) (with \(\lambda_1\) the character of \((F^\times)^{r_1}\) consisting of the first \(r_1\) terms of \(\lambda\) and \(\lambda_0\) the character of \((F^\times)^{r_0}\) consisting of the last \(r_0\) terms). Then, under the above Hecke algebra isomorphisms, \(\text{Ind}_B^G(\lambda \chi)\) is identified with \((\text{Ind}_{B_1}^{H'_1}(\lambda_1)) \otimes (\text{Ind}_{B_0}^{H'_0}(\lambda_0))\).

**Remark 1.1.** Because of the technical difficulties involved in dealing with disconnected groups, we will not pursue \(O(2n)\) in detail in this paper. However, in order that we may at least indicate what is expected in that case,
we briefly discuss how to extend Roche’s isomorphisms to get the analogue for $O(2n)$ of the example above.

Let us continue to assume $F$ of odd residual characteristic. Let $G = O(2n)$, $G = SO(2n)$, and $\mu \times \cdots \times \mu \times 1 \times \cdots \times 1$, where $r_0 + r_1 = n$. Let $\tilde{H} = O(2r_1) \times O(2r_0)$ and $\tilde{H} = S(O(2r_1) \times O(2r_0))$. By Roche’s results, there is a support-preserving isomorphism of Hecke algebras

$$\tilde{\Psi} : \mathcal{H}(G, \rho) \longrightarrow \mathcal{H}(\tilde{H}, 1).$$

We will use $\tilde{\Psi}$ to construct a support-preserving isomorphism of Hecke algebras

$$\Psi : \mathcal{H}(G, \rho) \longrightarrow \mathcal{H}(\tilde{H}, 1).$$

To this end, let $i_G : \mathcal{H}(G, \rho) \longrightarrow \mathcal{H}(G, \rho)$ and $i_H : \mathcal{H}(\tilde{H}) \longrightarrow \mathcal{H}(\tilde{H}, 1)$ denote the obvious embeddings. If we let $c_n$ denote the $n$th sign change (an element of $O(2n) \setminus SO(2n)$) and $C = \langle 1, c_n \rangle$, then we have $\mathcal{H}(G, \rho) \cong \mathcal{H}(G, \rho) \otimes \mathbb{C}[C]$. Similarly, if we let $H = SO(2r_1) \times SO(2r_0)$ and $C' = \langle 1, c_n, c_n, c_r, c_n \rangle$, $C'' = \langle 1, c_r, c_n \rangle$, then $\mathcal{H}(\tilde{H}, 1) \cong \mathcal{H}(H, 1) \otimes \mathbb{C}[C']$, $\mathcal{H}(\tilde{H}, 1) \cong \mathcal{H}(H, 1) \otimes \mathbb{C}[C'']$. We may then define $\tilde{\Psi}$ by

$$\Psi : h \otimes 1 \longmapsto i_H \circ \tilde{\Psi} \circ i_G^{-1}(h) \otimes 1$$

$$h \otimes c_n \longmapsto i_H \circ \tilde{\Psi} \circ i_G^{-1}(h) \otimes e_{c_n}'.$$

The definition of $\Psi$ ensures that it is a support-preserving linear isomorphism. We need to check that it also respects multiplication. For this, it is enough to show that $c_n(i_H \circ \tilde{\Psi} \circ i_G^{-1}(T_w)) = i_H \circ \tilde{\Psi} \circ i_G^{-1}(T_{c_n(w)})$. Since $\tilde{\Psi}$ is a support-preserving isomorphism, we may write $\tilde{\Psi}(T_w) = a_w T_w$. Then,

$$c_n(i_H \circ \tilde{\Psi} \circ i_G^{-1}(T_w)) = a_w T_{c_n(w)}'$$

and

$$i_H \circ \tilde{\Psi} \circ i_G^{-1}(T_{c_n(w)}) = a_{c_n(w)} T_{c_n(w)}'.$$

Therefore, it is enough to show that $a_w = a_{c_n(w)}$. For this, it is enough to show that $a_w = a_{c_n(w)}$ for $w \in \tilde{W}_\chi = \{ w \in \tilde{W}_\chi \mid w \cdot \chi = \chi \}$ and $w = y \in Y = \text{Hom}(G_m, T)$. Now, since $c_n \cdot \Pi = \Pi$ and $c_n \cdot \Pi_\chi = \Pi_\chi$ (cf. [R, p. 393]), we have $\ell(c_n(w)) = \ell(w)$ and $\ell_\chi(c_n(w)) = \ell_\chi(w)$ for $w \in \tilde{W}_\chi$, where $\ell_\chi = \ell_H$ denotes length taken with respect to $\Pi_\chi$. Since $\tilde{\Psi} : q^{-\frac{1}{2}\ell(w)} T_w \longmapsto q^{-\frac{1}{2}\ell_\chi(w)} T_{c_n(w)}$, this tells us that $a_w = a_{c_n(w)}$ for $w \in \tilde{W}_\chi$. This also tells us that $\delta \circ c_n = \delta$ and $\delta_H \circ c_n = \delta_H$. Therefore, $q^{\ell(y)} = q^{\ell(c_n(y))}$ for $y \in Y^+ = \{ y \in Y \mid \langle y, \alpha \rangle \geq 0 \text{ for all } \alpha \in \Phi^+ \}$ (cf. [Ca, Lemma 1.5.1]), and similarly for $\ell_\chi = \ell_H$. Since $\tilde{\Psi} : \delta^\frac{1}{2}(y) T_y \longmapsto \delta^\frac{1}{2}_H(y) T_y$ (cf. proof of Lemma 9.3 [R]), it follows that $a_y = a_{c_n(y)}$ for $y \in Y^+$, as needed.
The following results describe what happens when $\chi$ is replaced by $\chi^{-1}$. They will be needed below.

**Lemma 1.1.** In Roche’s construction, if $(J, \rho)$ is the type associated to $\chi$, then the type associated to $\chi^{-1}$ is $(J, \rho^{-1})$.

**Proof.** Write $(J_\chi, \rho_\chi)$, $(J_{\chi^{-1}}, \rho_{\chi^{-1}})$ for the types associated to $\chi, \chi^{-1}$, respectively. The functions $f_\chi$ and $f_{\chi^{-1}}$ from Definition 3.3 [R] must be the same. Therefore, $J_\chi = J_{f_\chi}$ and $J_{\chi^{-1}} = J_{f_{\chi^{-1}}}$ must be the same. It is then immediate from the definitions that $\rho_{\chi^{-1}} = (\rho_\chi)^{-1}$ (cf. [R, Proposition 3.6]).

**Corollary 1.2.** $\mathcal{H}(G, \rho) \cong \mathcal{H}(G, \rho^{-1})$ via a support-preserving isomorphism. (The isomorphism may also be taken to be $\ast$-preserving; cf. Section 6 [R].)

**Proof.** Observe that in the notation of [R], we have $\Phi_\chi = \Phi_{\chi^{-1}}$, $\Phi_{\chi, af} = \Phi_{\chi^{-1}, af}$ (cf. [R, Definition 6.1]) and $W_\chi = W_{\chi^{-1}}$ (cf. [R, Theorem 4.14]). Thus, $C_\chi = C_{\chi^{-1}}$, and therefore $\Pi_{\chi, af} = \Pi_{\chi^{-1}, af}$ (cf. [R, Section 6]). It then follows that $S_\chi^0 = S_{\chi^{-1}}^0$ and $\Omega_\chi = \Omega_{\chi^{-1}}$ (also [R, Section 6]). The corollary is now an immediate consequence of [R, Theorem 6.3].

We now define an involution $j$ on $\mathcal{H}(H, 1_I) \hat{\otimes} \mathbb{C}[C_\chi]$. For $h \in \mathcal{H}(H, 1_I)$, $j(h)$ is defined as above. For $\mathbb{C}[C_\chi]$, fix a character $\sigma : C_\chi \rightarrow \mathbb{C}^\times$ with $\sigma^2 = 1$. Then, for $c \in C_\chi$, set $j(e_c) = \sigma(c)e_c$. This extends to an involution of $\mathbb{C}[C_\chi]$ (trivial if $\sigma = 1$). We extend $j$ to $\mathcal{H}(H, 1_I) \hat{\otimes} \mathbb{C}[C_\chi]$ bilinearly.

**Proposition 1.3.** $j$ is an involution of $\mathcal{H}(H, 1_I) \hat{\otimes} \mathbb{C}[C_\chi]$.

**Proof.** We need to check that $j$ respects multiplication. For $h_1, h_2 \in \mathcal{H}(H, 1_I)$ and $c_1, c_2 \in C_\chi$, we have

$$j(h_1 \otimes e_{c_1}) \cdot j(h_2 \otimes e_{c_2}) = \sigma(c_1c_2)[j(h_1)c_1(j(h_2)) \otimes e_{c_1c_2}]$$

and

$$j((h_1 \otimes e_{c_1}) \cdot (h_2 \otimes e_{c_2})) = \sigma(c_1c_2)[j(h_1)c_1(j(h_2)) \otimes e_{c_1c_2}].$$

Thus, it suffices to show $j(c(h)) = c(j(h))$ for $h \in \mathcal{H}(H, 1_I)$, $c \in C_\chi$. Since the action of $C_\chi$ on $H$ preserves the set of simple roots of $H$, hence $\delta_H$ and $\ell_H$, we see that

$$j(c(T_w)) = (-q)^{\ell_H(w)}(T_{c(w^{-1})})^{-1} = c(j(T_w))$$

for all $w \in \overline{W}_H$. Also, since $c(\theta_t) = \theta_{c(t)}$, we get

$$j(c(\theta_t)) = \theta_{c(t)}^{-1} = c(j(\theta_t)),$$

as needed. \qed
We now consider how these restrict to Levi subgroups of standard parabolics. First, if $M$ is a standard Levi of $G$, we have support-preserving isomorphisms
\[
\Psi_G : \mathcal{H}(G, \rho) \longrightarrow \mathcal{H}(H, 1_{I_H}) \otimes \mathbb{C}[C_\chi] = \mathcal{H}(\hat{H}, 1_{I_{\hat{H}}})
\]
\[
\Psi_M : \mathcal{H}(M, \rho_M) \longrightarrow \mathcal{H}(L, 1_{I_L}) \otimes \mathbb{C}[D_\chi] = \mathcal{H}(\tilde{L}, 1_{I_{\tilde{L}}})
\]
constructed by Roche. In general, these are not unique. Similarly, let $j_H$, resp. $j_L$, denote the Iwahori-Matsumoto involutions for $\mathcal{H}(H, 1_I) \otimes \mathbb{C}[C_\chi]$, resp. $\mathcal{H}(L, 1_I) \otimes \mathbb{C}[D_\chi]$ constructed above. We note the following:

**Lemma 1.4.** We have $D_\chi \subset C_\chi$ and may take $L$ to be a standard Levi subgroup of $H$.

**Proof.** First, we check that $D_\chi \subset C_\chi$. In fact, we claim that $D_\chi = C_\chi \cap \overline{W}_M$. Recall that
\[
D_\chi = \{ w \in \overline{W}_{M,\chi} \mid w \cdot \Phi_{M,\chi}^+ = \Phi_{M,\chi}^+ \}.
\]
Now, it is immediate from the definitions that $\overline{W}_{M,\chi} = \overline{W}_M \cap \overline{W}_\chi$ and $\Phi_{M,\chi}^+ = \Phi_M \cap \Phi^+$. Therefore, if $w \in C_\chi \cap \overline{W}_M$, we have $w \in \overline{W}_{M,\chi}$. Further, $w \cdot \Phi_\chi^+ = \Phi_\chi^+$ (since $w \in C_\chi$) and $w \cdot \Phi_M = \Phi_M$ (since $w \in \overline{W}_M$), implying $w \cdot \Phi_{M,\chi}^+ = \Phi_{M,\chi}^+$. Thus, $C_\chi \cap \overline{W}_M \subset D_\chi$. On the other hand, suppose $d \in D_\chi$. It is automatic that $d \in \overline{W}_M$. To show $d \in C_\chi \cap \overline{W}_M$, it remains to check that $d \cdot \Phi_\chi^+ = \Phi_\chi^+$. From the definition of $D_\chi$, we have $d \cdot \Phi_{M,\chi}^+ = \Phi_{M,\chi}^+$. Suppose $\alpha \in \Phi_\chi^+ \setminus \Phi_{M,\chi}^+$. Since $\Phi_{M,\chi}^+ = \Phi_M \cap \Phi_\chi^+$, this forces $\alpha \in \Phi_M \setminus \Phi_{M,\chi}^+$. As $\overline{W}_M \cdot (\Phi_\chi^+ \setminus \Phi_{M,\chi}^+) = \Phi_\chi^+ \setminus \Phi_{M,\chi}^+$, we have $d \cdot \alpha \in \Phi_M \setminus \Phi_{M,\chi}^+$. Thus, $d \cdot \alpha \in (\Phi_\chi^+ \setminus \Phi_{M,\chi}^+) \cap \Phi_\chi \subset \Phi_\chi^+$, as needed. This tells us $D_\chi \subset C_\chi \cap \overline{W}_M$.

Thus, $D_\chi = C_\chi \cap \overline{W}_M$, as claimed.

Now, we identify $L$ as a standard Levi subgroup of $H$. First, we claim that if $\Pi_{M,\chi}$ denotes the simple roots in $\Phi_{M,\chi}$, then $\Pi_{M,\chi} = \Pi_\chi \cap \Phi_{M,\chi}$. Certainly, if $\alpha \in \Pi_\chi \cap \Phi_{M,\chi}$, then $\alpha \in \Pi_{M,\chi}$ (minimal with respect to $\Phi_\chi^+$ implies minimal with respect to $\Phi_{M,\chi}^+ = \Phi_M \cap \Phi_\chi^+$). In the other direction, suppose $\alpha \in \Pi_{M,\chi}$ but $\alpha \not\in \Pi_\chi \cap \Phi_{M,\chi}$. Since $\Pi_{M,\chi} \subset \Phi_{M,\chi}$, this requires $\alpha \not\in \Pi_\chi$. Therefore, we can write $\alpha = \beta + \gamma$ with $\beta, \gamma \in \Phi_\chi^+$. Write $\alpha = \sum b_i \alpha_i$, $\beta = \sum b_i \alpha_i$, $\gamma = \sum c_i \alpha_i$, where $\alpha_i$ runs over the simple roots $\Pi$ of $G$. Then, $a_i = b_i + c_i$ for all $i$. Further, $a_i, b_i, c_i \geq 0$ (since $\alpha, \beta, \gamma \in \Phi_\chi^+ \cup \Phi^+$). Now, $\alpha \in \Phi_M$ implies $a_i = 0$ for $\alpha_i \not\in \Pi_M$. Therefore, $b_i = c_i = 0$ for $\alpha_i \not\in \Pi_M$. Thus, $\beta, \gamma \in \Phi_M$. In particular, $\beta, \gamma \in \Phi_\chi^+ \cap \Phi_M = \Phi_{M,\chi}^+$. However, the decomposition $\alpha = \beta + \gamma$ with $\beta, \gamma \in \Phi_{M,\chi}^+$ contradicts the simplicity of $\alpha$ in $\Phi_{M,\chi}^+$. Thus, $\Pi_{M,\chi} = \Pi_\chi \cap \Phi_{M,\chi}$, as claimed. We now let $L$ be the Levi factor of the standard parabolic subgroup of $H$ associated to the subset of simple roots $\Pi_{M,\chi} \subset \Pi_\chi = \Pi_H$. Then, $L$ has the right root data to appear in the isomorphism $\mathcal{H}(M, \rho_M) \cong \mathcal{H}(L, 1_{I_L}) \otimes \mathbb{C}[D_\chi]$. 

\[\square\]
The next step is to identify $\mathcal{H}(M, \rho_M)$ as a subalgebra of $\mathcal{H}(G, \rho)$. Let $P = MU$ be the standard parabolic subgroup with Levi factor $M$; $U$ the unipotent radical opposite $U$. Set

$$J_\ell = J \cap \overline{U}, \quad J_M = J \cap M, \quad J_u = J \cap U,$$

which gives $J = J_\ell J_M J_u$. Let

$$\mathcal{I}_M^+ = \{ w \in W_M | wJ_u w^{-1} \subset J_u, \ w^{-1}J_\ell w \subset J_\ell \}.$$

We set

$$\mathcal{I}_{M, \chi}^+ = \mathcal{I}_M^+ \cap W_{M, \chi}.$$

If we let $\mathcal{H}^+(M, \rho_M)$ denote the space of functions in $\mathcal{H}(M, \rho_M)$ with support contained in $J_M \mathcal{I}_{M, \chi}^+ J_M$, then there is a support-preserving embedding (of $\mathbb{C}$-algebras with 1)

$$\mathcal{H}^+(M, \rho_M) \hookrightarrow \mathcal{H}(G, \rho)$$

([Bu-K, Corollary 6.12]). This may be extended uniquely to an embedding

$$\mathcal{H}(M, \rho_M) \hookrightarrow \mathcal{H}(G, \rho)$$

([Bu-K, Theorem 7.2], noting that the existence of $\zeta$ as in the hypotheses of Theorem 7.2 [Bu-K] follows immediately from the support-preserving isomorphism of Roche and the fact that elements of $\mathcal{H}(\tilde{H}, 1_{1_M})$ supported on a single double-coset are invertible). We fix such embeddings $\mathcal{H}(M, \rho_M) \hookrightarrow \mathcal{H}(G, \rho)$ and $\mathcal{H}(\tilde{L}, 1_{1_L}) \hookrightarrow \mathcal{H}(\tilde{H}, 1_{1_H})$.

Now, we note that $W_{M, \chi} = \overline{W}_{M, \chi} \cdot Y$, where $Y = \text{Hom}(G_m, T)$ (which may be viewed as a set of representatives for $T/T(O)$). If we let

$$Y_{M, \chi}^+ = \mathcal{I}_{M, \chi}^+ \cap Y,$$

we have the following:

**Lemma 1.5.**

$$\mathcal{I}_{M, \chi}^+ = \overline{W}_{M, \chi} \cdot Y_{M, \chi}^+.$$

**Proof.** First, we check that $\overline{W}_{M, \chi} \cdot Y_{M, \chi}^+ \subset \mathcal{I}_{M, \chi}^+$. By definition, $Y_{M, \chi}^+ \subset \mathcal{I}_{M, \chi}^+$. Thus, we need only check that $\overline{W}_{M, \chi} \subset \mathcal{I}_{M, \chi}^+$. Recall that Roche defines

$$f_\chi(\alpha) = \begin{cases} [c_\alpha/2], & \text{for } \alpha \in \Phi^+, \\ [(c_\alpha + 1)/2], & \text{for } \alpha \in \Phi^- , \end{cases}$$

where $c_\alpha = \text{cond}(\chi \circ \hat{\alpha})$ (where cond$(\lambda)$ is defined to be the lowest positive integer such that $1 + p^n \subset \ker(\lambda)$). Then,

$$J_\ell = \prod_{\alpha \in \Phi^- \setminus \Phi_M^-} U_{\alpha, f_\chi(\alpha)}, \quad J_M = \prod_{\alpha \in \Phi^+ \setminus \Phi_M^+} U_{\alpha, f_\chi(\alpha)}.$$
\[ J_M = \left( \prod_{\alpha \in \Phi_M^+} U_{\alpha,f_{\chi}(\alpha)} \right) \cdot A(\mathcal{O}) \cdot \left( \prod_{\alpha \in \Phi_M^+} U_{\alpha,f_{\chi}(\alpha)} \right). \]

Here, if \( \alpha \in \Phi^+ \),
\[ U_{\alpha,k} = \phi_\alpha \begin{pmatrix} 1 & p^k \\ 0 & 1 \end{pmatrix}, \quad U_{-\alpha,k} = \phi_\alpha \begin{pmatrix} 1 & 0 \\ p^k & 1 \end{pmatrix}, \]
with \( \phi_\alpha : SL_2 \rightarrow G \) as usual. Now, suppose \( w \in \overline{W}_M \) and \( \alpha \in \Phi^+ \). Then,
\[ w \cdot U_{\alpha,f_{\chi}(\alpha)} = \phi_{w\cdot\alpha} \begin{pmatrix} 1 & p^{f_{\chi}(\alpha)} \\ 0 & 1 \end{pmatrix}, \]
suitably interpreted if \( w \cdot \alpha < 0 \). To show that \( wJ_uw^{-1} \subset J_u \) for \( w \in \overline{W}_{M,X} \), it is (more than) enough to show that if \( \alpha \in \Phi^+ \setminus \Phi_M^+ \), then \( w \cdot \alpha \in \Phi^+ \) and \( f_{\chi}(w \cdot \alpha) = f_{\chi}(\alpha) \). Of course, \( (1) \) follows immediately from \( w \cdot (\Phi^+ \setminus \Phi_M^+) = \Phi^+ \setminus \Phi_M^+ \) (which holds for any \( w \in \overline{W}_M \)). \( (2) \) follows from \( w \cdot \chi = \chi; (w^{-1} \cdot \chi) \circ \check{\alpha} = \chi \circ \check{\alpha} \) implies \( c_{w^{-1}} = c_\alpha \), so that \( f_{\chi}(w \cdot \alpha) = f_{\chi}(\alpha) \), as needed. The same argument may be used to show \( w^{-1}J_{\ell}w \subset J_{\ell} \). Thus, we get \( \overline{W}_{M,X} \subset \mathcal{I}_{M,X}^+ \), as needed. In fact, this shows more: for \( w \in \overline{W}_{M,X} \), we have \( wJ_uw^{-1} = J_u \) and \( w^{-1}J_{\ell}w = J_{\ell} \).

The containment \( \mathcal{I}_{M,X}^+ \subset \overline{W}_{M,X} \cdot Y_{M,X}^+ \) is now easy. Take
\[ x \in \mathcal{I}_{M,X}^+ \subset W_{M,X} = \overline{W}_{M,X} \cdot Y, \]
and write \( x = wy, \ w \in \overline{W}_{M,X}, \ y \in Y \). Then, it suffices to show that \( y \in Y_{M,X}^+ \). Calculate:
\[
(wy)J_u(wy)^{-1} \subset J_u \implies yJ_uy^{-1} \subset w^{-1}J_uw = J_u
\]
since \( w \in \overline{W}_{M,X} \subset \mathcal{I}_{M,X}^+ \). Similarly, \( y^{-1}J_{\ell}y \subset J_{\ell} \). Thus, \( y \in Y_{M,X}^+ \) as needed.

Now, we can prove the following:

**Proposition 1.6.** (1) We may take \( \Psi_M = \Psi_G|_{\mathcal{H}(M,\rho_M)} \) to get a support-preserving isomorphism
\[ \Psi_M : \mathcal{H}(M,\rho_M) \rightarrow \mathcal{H}(L,1_{I_L})\hat{\otimes}C[D_{\chi}]. \]
(2) If we let \( j_H \), resp. \( j_L \), denote the Iwahori-Matsumoto involutions for \( \mathcal{H}(H,1_I)\hat{\otimes}C[C_{\chi}] \), resp. \( \mathcal{H}(L,1_{I_L})\hat{\otimes}C[D_{\chi}] \), then
\[ j_L = j_H|_{\mathcal{H}(L,1_{I_L})\hat{\otimes}C[D_{\chi}]} \]

**Proof.** We first say a few words about what it means for \( \Psi_G \) to be support-preserving. By construction, \( Y \) is associated to both \( G \) and \( H \); we use \( Y_G, Y_H \) when we wish to distinguish the context. Note that \( W_X = Y \times \overline{W}_X \). Also, since \( \Phi_H = \Phi_X \), we can identify \( \overline{W}_H^0 \) and \( \overline{W}_X^0 \). This gives rise to
an identification of $\overline{W}_\chi = \overline{W}_\chi^0 \times C_\chi$ with $\overline{W}_H^0 \times C_\chi$. Thus, we have an identification
\[
\psi_G : W_\chi = Y \times \overline{W}_\chi \leftrightarrow Y \times (\overline{W}_H^0 \times C_\chi) = W_H \times C_\chi.
\]
Then, $\Psi_G : \mathcal{H}(G, \rho) \longrightarrow \mathcal{H}(H, 1_T) \otimes \mathbb{C}[C_\chi]$ is support-preserving means that if $T \in \mathcal{H}(G, \rho)$ is supported on $J w J$, then $\Psi_G(T)$ is supported on $I_H \psi_G(w) I_H$.

Now, to make matters precise, let $t_M : \mathcal{H}(M, \rho_M) \hookrightarrow \mathcal{H}(G, \rho)$ and $t_\tilde{L} : \mathcal{H}(\tilde{L}, 1_{I_L}) \hookrightarrow \mathcal{H}(\tilde{H}, 1_{I_H})$ denote the fixed embeddings. To verify (1), we need to show that
\[
\Psi_G(t_M(\mathcal{H}(M, \rho_M))) = t_\tilde{L}(\mathcal{H}(\tilde{L}, 1_{I_L})).
\]
To this end, we first show that
\[
\Psi_G(t_M(\mathcal{H}^+(M, \rho_M))) \subset t_\tilde{L}(\mathcal{H}^+(\tilde{L}, 1_{I_L})).
\]
Since the restriction of $t_M$ to $\mathcal{H}^+(M, \rho_M)$ is support-preserving, and similarly for $t_\tilde{L}$, it is enough to show that
\[
\psi_G(I_{M,\chi}^+) \subset I_{\tilde{L}}^+ = \overline{W}_L \cdot D_\chi \cdot Y_L^+.
\]
By Lemma 1.5 and the fact that $\psi_G(\overline{W}_{M,\chi}) = \overline{W}_L \cdot D_\chi$, we are reduced to checking that $\psi_G(Y_{M,\chi}^+) \subset Y_L^+$. Now, if $y \in Y$ and $\alpha \in \Phi$, then $y U_{\alpha, k} y^{-1} = U_{\alpha, t(k)} y$. Therefore, $y \in Y_{M,\chi}^+$ if and only if $\alpha(y) \geq 0$ for all $\alpha \in \Phi^+ \setminus \Phi_M^+$ (noting that the corresponding condition for negative roots also reduces to this). On the other hand, $y \in Y_L^+$ if and only if $\alpha(y) \geq 0$ for all $\alpha \in \Phi_H^+ \setminus \Phi_L^+ = \Phi_X^+ \setminus \Phi_M^+ \subset \Phi^+ \setminus \Phi_M^+$, so we see that $\psi_G(Y_{M,\chi}^+) \subset Y_L^+$, as needed.

We now proceed to show that
\[
\Psi_G(t_M(\mathcal{H}(M, \rho_M))) = t_\tilde{L}(\mathcal{H}(\tilde{L}, 1_{I_L})).
\]
First, choose $\zeta$ and $\phi_m$ as in Proposition 7.1 [Bu-K]. Let $y_0 \in Y_{M,\chi}^+$ be a representative for $\zeta$. Then, $(\phi_m)^{-1} = \phi_m \in \mathcal{H}(M, \rho_M)$. We note that $\mathcal{H}^+(M, \rho_M)$ and $\phi^{-1}$ are sufficient to generate $\mathcal{H}(M, \rho_M)$. We already have $\Psi_G(t_M(\mathcal{H}^+(M, \rho_M))) \subset t_\tilde{L}(\mathcal{H}^+(\tilde{L}, 1_{I_L}))$. Since $y_0 \in I_{M,\chi}^+$, we have $\Psi_G(t_M(\phi_1)) \subset t_\tilde{L}(\mathcal{H}^+(\tilde{L}, 1_{I_L}))$. Further, since $t_M|_{\mathcal{H}^+(M, \rho_M)}$ and $\Psi_G$ are support-preserving, $\Psi_G(t_M(\phi_1))$ is supported on the double-coset $I_H y_0 I_H$. Therefore, $\Psi_G(t_M(\phi_1)) = t_\tilde{L}(\phi_1')$, where $\phi_1' \in \mathcal{H}(\tilde{L}, 1_{I_L})$ is supported on the double-coset $I_L y_0 I_L$. As a consequence, $\phi_1'$ is invertible and $\Psi_G(t_M(\phi_1^{-1})) = t_\tilde{L}(\phi_1'^{-1})$. Since $\mathcal{H}^+(M, \rho_M)$ and $\phi_1^{-1}$ are enough to generate $\mathcal{H}(M, \rho_M)$, we have $\Psi_G(t_M(\mathcal{H}(M, \rho_M))) \subset t_\tilde{L}(\mathcal{H}(\tilde{L}, 1_{I_L}))$. On the other hand, we know that $\Psi_G(t_M(\mathcal{H}(M, \rho_M)))$ contains the functions $t_{\tilde{L}}(T_w) \ y \in \psi_G(\overline{W}_{M,\chi}) = \overline{W}_L D_\chi \{ t_{\tilde{L}}(T_y) \ y \in Y_{M,\chi}^+ \}$, and $t_\tilde{L}(T_{y_0}^{-1})$. As these are enough to generate $t_\tilde{L}(\mathcal{H}(\tilde{L}, 1_{I_L}))$, we see that
\[
\Psi_G(t_M(\mathcal{H}(M, \rho_M))) = t_\tilde{L}(\mathcal{H}(\tilde{L}, 1_{I_L})).
\]
as needed.

We now know that \( t_M^{-1} \circ \Psi_G \circ t_M \) is an isomorphism from \( \mathcal{H}(M, \rho_M) \) to \( \mathcal{H}(\tilde{L}, 1_{I_L}) \). We must show that it is support-preserving. To this end, observe that since \( \zeta \in Z(M) \), we have \( \alpha(y_0) = 0 \) for all \( \alpha \in \Phi_M \). Therefore, viewing \( y_0 \) as an element of \( Y_H \), we have \( \alpha(y_0) = 0 \) for all \( \alpha \in \Phi_L = \Phi_{M,X} \).

In particular, \( y_0 U_{\alpha,k} y_0^{-1} = U_{\alpha,k + \alpha(y_0)} = U_{\alpha,k} \) for all \( \alpha \in \Phi_L \) allows us to conclude that \( I_L y_0 = y_0 I_L \). Therefore, in \( \mathcal{H}(\tilde{L}, 1_{I_L}) \), we have that \( T_{y_0} T_w \) is supported on \( I_L y_0 w I_L \) for any \( w \in W_L D_\chi \). In particular, \( T_{y_0}^{-1} \) is supported on \( I_L y_0^{-1} I_L \). Thus, we have that \( t_M^{-1} \circ \Psi_G \circ t_M \) is support-preserving on \( \mathcal{H}^+(M, \rho_M) \) and \( \phi^{-1} = \phi_{-1} \). By writing \( x \in W_{M,X} \approx y_0^{-m} w \), with \( w \in W_{M,X} \) ([Bu-K, Lemma 6.14]), one can then check that \( t_M^{-1} \circ \Psi_G \circ t_M(T_x) \) is supported on \( I_L y_0^{-m} \psi_G(w) I_L \), as needed.

It is now straightforward to check the second claim. This follows immediately from the following observations:

1. Since \( L \) is a standard Levi of \( H \), the Iwahori-Matsumoto involution on \( \mathcal{H}(H, 1_I) \) restricts to give the Iwahori-Matsumoto involution on \( \mathcal{H}(L, 1_{I_L}) \).
2. Since \( D_\chi \subset C_\chi \), we may choose \( \sigma|_{D_\chi} \) for the character of order \( \leq 2 \) of \( D_\chi \).

This finishes the proof of the proposition. \( \square \)

Suppose \( \mathcal{H}_1, \mathcal{H}_2 \) are Hecke algebras and \( \phi : \mathcal{H}_1 \longrightarrow \mathcal{H}_2 \) is an isomorphism. Suppose \( \pi \) is a representation of \( \mathcal{H}_1 \) with space \( \mathcal{V} \). Let \( \phi(\pi) \) denote the corresponding representation of \( \mathcal{H}_2 \) (i.e., \( \mathcal{H}_2 \) acting on \( \mathcal{V} \) by \( \pi \circ \phi^{-1} \)). We define induction as in [Bu-K]: If \( \mathcal{L}_1 \subset \mathcal{H}_1 \) is a subalgebra and \( \tau \) is a representation of \( \mathcal{L}_1 \), then \( \text{Ind}^{\mathcal{H}_1}_{\mathcal{L}_1} \tau \) has space \( \text{Hom}_{\mathcal{L}_1}(\mathcal{H}_1, \mathcal{V}_\tau) \) (where \( \mathcal{H}_1 \) is an \( \mathcal{L}_1 \)-module by left multiplication). The action is right translation. We claim that \( \phi \) respects induction. In particular, if \( \mathcal{L}_2 = \phi(\mathcal{L}_1) \), we have the following:

\textbf{Lemma 1.7.}

\[ \phi(\text{Ind}^{\mathcal{H}_1}_{\mathcal{L}_1} \tau) \cong \text{Ind}^{\mathcal{H}_2}_{\mathcal{L}_2} \phi(\tau). \]

\textit{Proof.} Let \( X \in \text{Hom}_{\mathcal{L}_1}(\mathcal{H}_1, \mathcal{V}_\tau) \). Then, it is straightforward to check that \( X \circ \phi^{-1} \in \text{Hom}_{\mathcal{L}_2}(\mathcal{H}_2, \mathcal{V}_{\phi(\tau)}) \) (noting that as a vector space, \( \mathcal{V}_{\phi(\tau)} = \mathcal{V}_\tau \); it is written as \( \mathcal{V}_{\phi(\tau)} \) to indicate the representation). In particular, \( \mathcal{E} : X \longrightarrow X \circ \phi^{-1} \) is a bijective linear map from \( \text{Hom}_{\mathcal{L}_1}(\mathcal{H}_1, \mathcal{V}_\tau) \) to \( \text{Hom}_{\mathcal{L}_2}(\mathcal{H}_2, \mathcal{V}_{\phi(\tau)}) \).

We claim that \( \mathcal{E} : \text{Hom}_{\mathcal{L}_1}(\mathcal{H}_1, \mathcal{V}_\tau) \longrightarrow \text{Hom}_{\mathcal{L}_2}(\mathcal{H}_2, \mathcal{V}_{\phi(\tau)}) \) gives the desired equivalence. In particular, if \( R_1 \) (resp. \( R_2 \)) denotes right translation on \( \text{Hom}_{\mathcal{L}_1}(\mathcal{H}_1, \mathcal{V}_\tau) \) (resp. \( \text{Hom}_{\mathcal{L}_2}(\mathcal{H}_2, \mathcal{V}_{\phi(\tau)}) \)), this requires

\[ \mathcal{E} R_1(\phi^{-1}(h_2)) X = R_2(h_2) \mathcal{E} X. \]
for all $X \in \text{Hom}_{L_1}(H_1, V_\tau)$, $h_2 \in H_2$. Now, it is easy to check that

$$[\mathcal{E} R_1(\phi^{-1}(h_2))X](h'_2) = X(\phi^{-1}(h'_2 h_2)) = [R_2(h_2) \mathcal{E} X](h'_2).$$

The desired equivalence follows. \hfill \Box

We now discuss the implications for the representation theory of $G$. First, if $(\pi, G, V)$ is a representation of $G$, let $(\pi, H(G, \rho), V^\rho)$ denote the corresponding Hecke algebra representation, where $V^\rho = \pi(e_\rho) V$ (the action is inherited from $H(G)$ acting on $V$). Now, fix a character $\tilde{\chi}$ extending $\chi$ to $T$. Let $\mathcal{R}_\chi(G)$ denote the category of smooth representations having the property that every irreducible subquotient of $\pi$ has supercuspidal support contained in $\{ \lambda(w \cdot \tilde{\chi}) | w \in W, \lambda \text{ an unramified character of } T \}$. As mentioned earlier, $(J, \chi)$ is a type. This means, among other things, that the map $(\pi, G, V) \mapsto (\pi, H(G, \rho), V^\rho)$ gives an equivalence of categories between $\mathcal{R}_\chi(G)$ and the category of $H(G, \rho)$ modules. This extends the results of Borel [Bo] and Casselman [Ca] on unramified principal series.

We now discuss the Iwahori-Matsumoto involution on representations. Let

$$\Psi_{G, \chi} : H(G, \rho) \longrightarrow H(H, 1_I) \hat{\otimes} \mathbb{C}[C_\chi]$$

$$\Psi_{G, \chi^{-1}} : H(G, \rho^{-1}) \longrightarrow H(H, 1_I) \hat{\otimes} \mathbb{C}[C_{\chi^{-1}}]$$

denote the isomorphisms above. Since $C_\chi = C_{\chi^{-1}}$ (which follows easily from the definition; cf. [R, Section 8]), the Hecke algebras on the right-hand side are identical. Therefore, we have an isomorphism

$$\phi_j : \Psi_{G, \chi^{-1}}^{-1} \circ j \circ \Psi_{G, \chi} : H(G, \rho) \longrightarrow H(G, \rho^{-1}).$$

If $(\gamma, V_\gamma)$ is a representation of $H(G, \rho)$, we let $\phi_j(\gamma)$ denote the representation of $H(G, \rho^{-1})$ associated to $\gamma$ by the Hecke algebra isomorphism above. Now, if $M$ is a standard Levi of $G$, let

$$E_{G, \chi} : R_\chi(G) \longrightarrow H(G, \rho) \mod$$

$$E_{M, \chi} : R_\chi(M) \longrightarrow H(M, \rho_M) \mod$$

denote the functors giving the equivalence of categories. If $\pi \in \mathcal{R}_\chi(G)$, we define the Iwahori-Matsumoto involution of $\pi$ by

$$j(\pi) = E_{G, \chi^{-1}}^{-1} \circ \phi_j \circ E_{G, \chi}(\pi).$$

Similarly, if $\tau \in \mathcal{R}_\chi(M)$, then $j(\tau) = E_{M, \chi^{-1}}^{-1} \circ \phi_j \circ E_{M, \chi}(\tau)$ (noting that $\phi_{j_M} = \phi_j |_{H(L,L_L) \hat{\otimes} \mathbb{C}[D_\chi]}$ from above).

Theorem 1.8.

$$j(\text{Ind}_P^G \tau) \cong \text{Ind}_P^G (j(\tau)).$$
Proof. First, by Corollary 8.4 of [Bu-K],
\[
E_{G,\chi}(\text{Ind}_{P}^{G}\tau) \cong \text{Ind}_{\mathcal{H}(M,\rho)}^{\mathcal{H}(G,\rho)}(E_{M,\chi}(\tau)).
\]

Therefore,
\[
j(\text{Ind}_{P}^{G}\tau) \cong E_{G,\chi}^{-1} \circ \phi_{j} \circ E_{G,\chi}(\text{Ind}_{P}^{G}\tau)
\cong E_{G,\chi}^{-1} \circ \phi_{j} \left( \text{Ind}_{\mathcal{H}(M,\rho)}^{\mathcal{H}(G,\rho)}(E_{M,\chi}(\tau)) \right)
\cong E_{G,\chi}^{-1} \left( \text{Ind}_{\mathcal{H}(M,\rho)}^{\mathcal{H}(G,\rho^{-1})} \phi_{j}(E_{M,\chi}(\tau)) \right),
\]
by Lemma 1.7 above. By definition, \(j(\tau) = E_{M,\chi}^{-1} \circ \phi_{j} \circ E_{M,\chi}(\tau)\), or \(E_{M,\chi}^{-1}(j(\tau)) = \phi_{j}(E_{M,\chi}(\tau))\). Thus,
\[
j(\text{Ind}_{P}^{G}\tau) \cong E_{G,\chi}^{-1} \left( \text{Ind}_{\mathcal{H}(M,\rho^{-1})}^{\mathcal{H}(G,\rho^{-1})} \phi_{j}(E_{M,\chi}(\tau)) \right)
\cong \text{Ind}_{P}^{G}(j(\tau)),
\]
again, by Corollary 8.4 of [Bu-K]. \(\square\)

We close by discussing a special case. First, let \(G = GL(k)\) and \(\mu\) a unitary character of \(F^\times\). Consider the character
\[
\tau = | \cdot |^{\alpha + \frac{k+1}{2}} \mu \otimes | \cdot |^{\alpha + \frac{k+1}{2} + 1} \mu \otimes \cdots \otimes | \cdot |^{\alpha + \frac{k+1}{2} \mu}
\]
of \(T\). By definition,
\[
j(\tau) = | \cdot |^{-\alpha + \frac{k-1}{2}} \mu^{-1} \otimes | \cdot |^{-\alpha + \frac{k-1}{2} - 1} \mu^{-1} \otimes \cdots \otimes | \cdot |^{-\alpha + \frac{k-1}{2} \mu^{-1}}.
\]
Since \((\mu \circ \det_{k})|\det_{k}|^{\alpha}\) is the unique irreducible subrepresentation of \(\text{Ind}_{P}^{G}\tau\), we see that \(j((\mu \circ \det_{k})|\det_{k}|^{\alpha})\) must be the unique irreducible subrepresentation of \(\text{Ind}_{B}^{G}(j(\tau))\), i.e.,
\[
j((\mu \circ \det_{k})|\det_{k}|^{\alpha}) = (\mu^{-1} \circ \det_{k})|\det_{k}|^{-\alpha} \otimes St_{k}.
\]

With this in hand, we may easily obtain the following:

\textbf{Corollary 1.9.} Let \(G = SO(2n + 1)\) (resp., \(G = Sp(2n)\)), \(P = MU\) a standard parabolic subgroup with \(M \cong GL(k) \times SO(2(n-k)+1)\) (resp., \(M \cong GL(k) \times Sp(2(n-k))\)), \(\mu\) a unitary character of \(F^\times\), and \(\pi\) a representation of \(SO(2(n-k)+1)\) (resp., \(Sp(2(n-k))\)). Then,
\[
j(\text{Ind}_{P}^{G}((\mu \circ \det_{k})|\det_{k}|^{\alpha} \otimes \pi)) \cong \text{Ind}_{P}^{G}((\mu^{-1} \circ \det_{k})|\det_{k}|^{-1} \otimes j(\pi)).
\]
2. The trivial character case; summary of Mœglin’s result.

We recall Mœglin’s results [M1]. All theorems in this section are due to Mœglin. Let $G(n) = Sp(2n), SO(2n+1), O(2n)$. Then the dual group is $G^*(n) = O(2n+1, \mathbb{C}), Sp(2n, \mathbb{C}), O(2n, \mathbb{C})$. Given a unipotent orbit $O \in O(2n+1, \mathbb{C}), Sp(2n, \mathbb{C})$, or $O(2n, \mathbb{C})$, Mœglin formed a set $P(O)$ of ordered partitions as follows:

- $p = (p_1, \ldots, p_r; q_1, \ldots, q_s) \in P(O)$ if and only if:
  1. $(p_1, p_1, \ldots, p_r; q_1, \ldots, q_s)$ is $O$ if we ignore the ordering.
  2. $q_i$ are distinct and odd integers in the case of $O_{2n+1}(\mathbb{C})$ and $O_{2n}(\mathbb{C})$, even integers in the case of $Sp_{2n}(\mathbb{C})$.
  3. For all $1 \leq j \leq \left\lfloor \frac{s+1}{2} \right\rfloor$, $q_{2j-1} > q_{2j}$ and there does not exist $1 \leq k \leq \left\lfloor \frac{s+1}{2} \right\rfloor$ such that $q_{2j-1} > q_{2k-1} > q_{2j} > q_{2k}$.
  4. If there exists a $1 \leq k \leq s$ such that $q_{2j-1} > q_k > q_{2j}$, then $k < 2j - 1$.

We set $q_{s+1} = 0$ if $s$ is odd. We can put an equivalence relation on $P(O)$ as follows: For $p = (p_1, \ldots, p_r; q_1, \ldots, q_s), p' = (p'_1, \ldots, p'_r; q'_1, \ldots, q'_s) \in P(O)$, $p \simeq p'$ if and only if for all $1 \leq i \leq \left\lfloor \frac{s+1}{2} \right\rfloor$, there exists $1 \leq j \leq \left\lfloor \frac{s+1}{2} \right\rfloor$ such that $q_{2j-1} = q'_{2j-1}, q_{2i} = q'_{2i}$. We note that $\{p_1, \ldots, p_r\} = \{p'_1, \ldots, p'_r\}$ as sets.

**Remark 2.1.** For a distinguished unipotent orbit, we have $r = 0$. In that case, we write $p = (q_1, \ldots, q_s)$.

**Example 2.1.** For a unipotent orbit of the form $(5, 3, 1)$ in $O_9(\mathbb{C})$, there are two non-equivalent elements in $P(O)$, namely, $(5,3,1)$ and $(3,1,5)$.

**Remark 2.2.** The conditions (3), (4) in $P(O)$ are such that the condition in [M1, Cor 0.10.2] is satisfied, and hence implies non-vanishing of normalized intertwining operators. We use this in Proposition 2.5. In that case, $x = \frac{q_1-1}{2}, \frac{q_2-3}{2}, \ldots, \frac{q_{s+1}}{2}$ and $X$ is a sub-module of

$$\text{Ind}_M 1 \times \left| \begin{array}{c} \frac{q_2-q_4}{4} \\ \vdots \\ \frac{q_{2i-1}+q_{2i}}{4} \\ \vdots \\ \frac{q_{2i-1}+q_{2i}}{4} \\ \vdots \\ \frac{q_{2s+1}-q_{2s}}{4} \end{array} \right|,$$

where $M = GL(q_2) \times GL(\frac{q_3+q_4}{2}) \times \cdots \times GL\left(\frac{q_{2s+1}+q_{2s}}{2}\right)$.

For $p = (p_1, \ldots, p_r; q_1, \ldots, q_s)$, we set, for $2 \leq i \leq r$, $p'_i = p_1 + \cdots + p_{i-1}$ and $p'_1 = 0$ and for $1 \leq i \leq \left\lfloor \frac{s+1}{2} \right\rfloor$,

$$T^d_i = \sum_{j=1}^{r} p_j + \sum_{1 \leq l < i} \frac{q_{2l-1} + q_{2l}}{2},$$

$$T^f_i = \sum_{j=1}^{r} p_j + \sum_{1 \leq l \leq i} \frac{q_{2l-1} + q_{2l}}{2}.$$
We recall the definition of \( \lambda_p \) and \( w_p \): \( \lambda_p = (\lambda_{p,1}, \ldots, \lambda_{p,n}) \), where
\[
\lambda_{p,p'_i+t} = \frac{p_i + 1}{2} - t, \text{ for } 1 \leq i \leq r \text{ and } 1 \leq t \leq p_i,
\]
\[
\lambda_{p,T^d_k+t} = \frac{q_{2k-1} + 1}{2} - t, \text{ for } 1 \leq k \leq \left\lfloor \frac{s+1}{2} \right\rfloor \text{ and } 1 \leq t \leq \frac{q_{2k_1} + q_{2k}}{2},
\]
\( w_p \) is an element of the Weyl group given by:
\[
w_p(p'_i + t) = p'_{i+1} - t + 1, \text{ for } 1 \leq i \leq r \text{ and } 1 \leq t \leq p_i
\]
\[
w_p(t) = -t, \text{ for } 1 \leq k \leq \left\lfloor \frac{s+1}{2} \right\rfloor \text{ and } T^d_k < t \leq T^d_k + \frac{q_{2k_1} - q_{2k}}{2},
\]
\[
w_p \left( T^d_k + \frac{q_{2k-1} - q_{2k}}{2} + t \right) = T^f_k - t + 1,
\]
for \( 1 \leq k \leq \left\lfloor \frac{s+1}{2} \right\rfloor \) and \( 1 \leq t \leq q_{2k} \).

**Remark 2.3.** All \( \lambda_p \) are conjugates and \( w_p^2 = 1 \). Let \( \lambda_O \) be the conjugate of \( \lambda_p \) which is in the closure of the positive Weyl chamber.

We also define \( \sigma_p_i \) for \( 1 \leq i \leq r \) and \( \sigma_k \) for \( 1 \leq k \leq \left\lfloor \frac{s}{2} \right\rfloor \) and let \( \text{Stab}(\lambda_p, \uparrow p) \) be the subgroup of \( \text{Stab}(\lambda_p) \) generated by these elements:
\[
\sigma_{p_i}(j) = j, \text{ if } j \notin [p'_i + 1, p'_{i+1}],
\]
\[
\sigma_{p_i}(p'_i + t) = -(p'_{i+1} - t + 1), \text{ if } t \in [1, p_i],
\]
\[
\sigma_k(j) = j, \text{ if } j \notin \left[ T^d_k + \frac{q_{2k-1} - q_{2k}}{2} + 1, T^f_k \right],
\]
\[
\sigma_k \left( T^d_k + \frac{q_{2k-1} - q_{2k}}{2} + t \right) = -(T^f_k - t + 1), \text{ if } t \in [1, q_{2k}].
\]

Let \( A(O) \) be a finite abelian group generated by the order two elements \( \sigma(p_1), \ldots, \sigma(p_r), \sigma(q_1), \ldots, \sigma(q_s) \). (We take only the distinct ones.) Let \( \overline{A}(p) = A(O)/K_p \), where \( K_p \) is generated by \( \sigma(q_{2k-1})\sigma(q_{2k})^{-1} \) for all \( 1 \leq i \leq \left\lfloor \frac{s+1}{2} \right\rfloor \). We set \( \sigma(q_{s+1}) = 1 \) if \( s \) is odd.

**Lemma 2.1.**
1. \( |\overline{A}(p)| = 2^{|\frac{s}{2}|} \).
2. \( \overline{A}(p) \) is isomorphic to the quotient of \( \text{Stab}(\lambda_p, \uparrow p) \) by the subgroup generated by \( \sigma(p_i)\sigma(p_j)^{-1} \) for \( p_i = p_j \) and \( \sigma(p_i)\sigma_k^{-1} \) for \( p_i = q_{2k-1} \) or \( p_i = q_{2k} \). The homomorphism \( \text{Stab}(\lambda_p, \uparrow p) \mapsto \overline{A}(p) \) is given by \( \sigma(p_i) \mapsto \sigma(p_i)K_p \) for \( i = 1, \ldots, r \) and \( \sigma_k \mapsto \sigma(q_{2k-1})K_p \).

Let Springer \( (O) \) is the set of characters of \( A(O) \) which is in the image of the Springer correspondence. We recall that the Springer correspondence is a one to one map from the set of characters of \( W \), the Weyl group of \( G^* \) into the set of pairs \( (O, \eta) \), where \( O \) is a unipotent orbit in \( G^* \) and \( \eta \) is a character of \( A(O) \). Then:
\textbf{Theorem 2.2.} Springer (O) \simeq \bigcup_{p \in P(O)} \hat{A}(p), where \( \hat{A}(p) \) is the character group of \( \hat{A}(p) \).

\textbf{2.1. Local theory.} Let \( I(\lambda_p) = \text{Ind}_B^G \exp(\langle \lambda_p, H_B \rangle) \) (normalized induction). The normalized intertwining operator \( R(w_p, \lambda) \) is not holomorphic at \( \lambda_p \) in general. Mœglin defined \( R(w_p, \lambda_p) \) as composition of several operators. (See [M1] or Section 3) Then we have:

\textbf{Theorem 2.3.}  
1. \( R(w_p, \lambda_p)I(\lambda_p) \) is a direct sum of \(|\hat{A}(p)| \) irreducible representations with multiplicity 1. Let \( \text{Unip}(p) \) be the set of the irreducible direct summands and \( \text{Unip}(O) = \bigcup_{p \in P(O)} \text{Unip}(p) \). Then the Iwahori-Matsumoto involution of elements in \( \text{Unip}(O) \) is tempered.
2. \( \text{Unip}(O) \) is exactly the set of irreducible representations of \( G(n) \) whose infinitesimal character is \( \lambda_\mathcal{O} \) and whose Iwahori-Matsumoto involution is tempered.
3. If \( r = 0 \), i.e., \( O \) is a distinguished unipotent orbit, then the Iwahori-Matsumoto involution of elements in \( \text{Unip}(O) \) is square integrable.

Mœglin proves Theorem 2.3 by induction: Let \( p = (p_1, \ldots, p_r; q_1, \ldots, q_s) \). Suppose \( r \neq 0 \). Set \( p' = (p_2, \ldots, p_r; q_1, \ldots, q_s) \). We have

\[ R(w_p, \lambda_p)I(\lambda_p) = \text{Ind}_{GL(p_1) \times G(n-p_1)} GL(1) \times R(w_{p'}, \lambda_{p'})I(\lambda_{p'}). \]

We denote by \( j \) the Iwahori-Matsumoto involution. Then

\[ jR(w_p, \lambda_p)I(\lambda_p) = \text{Ind}_{GL(p_1) \times G(n-p_1)} St(p_1) \times jR(w_{p'}, \lambda_{p'})I(\lambda_{p'}), \]

where \( St(p_1) \) is the Steinberg representation of \( GL(p_1) \). By induction, \( jR(w_{p'}, \lambda_{p'})I(\lambda_{p'}) \) is a semi-simple tempered representation of length \(|\hat{A}(p')|\). Let \( X' \in \text{Unip}(p') \).

\textbf{Proposition 2.4.} The induced representation

\[ \text{Ind}_{GL(p_1) \times G(n-p_1)} St(p_1) \times jX', \]

i.e., \( \text{Ind}_{GL(p_1) \times G(n-p_1)} GL(1) \times X' \) is irreducible if and only if \( p_1 = p_j \) or \( q_k \) for some \( j = 2, \ldots, r \) or \( k = 1, \ldots, s \). If it is reducible, then it is a sum of two irreducible representations.

For \( x \) small, the normalized intertwining operator associated to \( \sigma_{p_1} \) is

\[ \text{Ind}_{GL(1) \times \cdots \times GL(1) \times G(n-p_1)} | \frac{E_{11}^{-1} + x}{2} \times \cdots \times | \frac{E_{11}^{-1} + x}{2} \times X' \]

\[ \longrightarrow \text{Ind}_{GL(1) \times \cdots \times GL(1) \times G(n-p_1)} | \frac{E_{11}^{-1} - x}{2} \times \cdots \times | \frac{E_{11}^{-1} - x}{2} \times X'. \]

This operator is a product of the operator

\[ \text{Ind}_{GL(1) \times \cdots \times GL(1) \times G(n-p_1)} | \frac{E_{11}^{-1} + x}{2} \times \cdots \times | \frac{E_{11}^{-1} + x}{2} \times X' \]

\[ \longrightarrow \text{Ind}_{GL(p_1) \times G(n-p_1)} \det |x| \times X', \]
and the operator, \( R_{X'}(\sigma_{p_1}, x) \):

\[
\text{Ind}_{GL(p_1) \times G(n-p_1)} \left| \det \right|^{2} \times X' \mapsto \text{Ind}_{GL(p_1) \times G(n-p_1)} \left| \det \right|^{-x} \times X'.
\]

**Proposition 2.5.** \( R_{X'}(\sigma_{p_1}, x) \) is holomorphic at \( x = 0 \). Let \( R_{X'}(\sigma_{p_1}, 0) = R_{X'}(\sigma_{p_1}) \). Then \( R_{X'}(\sigma_{p_1})^2 = \text{id} \) and \( R_{X'}(\sigma_{p_1}) \) is the identity if and only if \( \text{Ind}_{GL(p_1) \times G(n-p_1)} 1 \times X' \) is irreducible. Let \( R(\sigma_{p_1}) \) be the sum of \( R_{X'}(\sigma_{p_1}) \) as \( X' \) runs through \( \text{Unip}(p') \). Then it defines an intertwining operator for \( R(w_p, \lambda_p)I(\lambda_p) \).

In a similar way, we can define \( R(\sigma_{p_1}) \).

Suppose \( r = 0 \). From [M1, 676] or Section 3, we know that

\[
R(w_p, \lambda_p)I(\lambda_p) \subset \text{Ind}_{GL(\frac{q_1+q_2}{2}) \times G(n-T_2)} \left| \det \right|^{-\frac{q_1-q_2}{4}} \times R(w_p, \lambda_{p_{\geq 3}})I(\lambda_{p_{\geq 3}}).
\]

We use induction: Let \( Y \in \text{Unip}(p_{\geq 3}) \) and consider

\[
W_Y = \text{Ind}_{GL(\frac{q_1+q_2}{2}) \times G(n-T_2)} \left| \det \right|^{\frac{q_1-q_2}{4}} \times Y.
\]

Let \( W_Y^* = \text{Ind}_{GL(\frac{q_1+q_2}{2}) \times G(n-T_2)} \left| \det \right|^{-\frac{q_1-q_2}{4}} \times Y \). Then we have

\[
jW_Y^* = \text{Ind}_{GL(\frac{q_1+q_2}{2}) \times G(n-T_2)} St \left( \frac{q_1 + q_2}{2} \right) \otimes \left| \det \right|^{\frac{q_1-q_2}{4}} \times jY.
\]

**Proposition 2.6.** \( jW_Y^* \) is reducible and its subrepresentations are tempered.

Consider the following commutative diagram:

\[
\begin{array}{ccc}
Z_Y = \text{Ind}_{GL(1) \times \cdots \times GL(1) \times GL(q_2) \times G(n-\frac{q_1+q_2}{2})} \left| \frac{q_1-1}{2} \right| \times \left| \frac{q_2-3}{2} \right| \times \cdots \times \left| \frac{q_2+1}{2} \right| \times 1 \times Y & \xrightarrow{R_Y^*} & W_{Y}^* \\
\downarrow R_Y^* & & \downarrow \\
Z_Y^* = \text{Ind}_{GL(1) \times \cdots \times GL(1) \times GL(q_1) \times G(n-\frac{q_1+q_2}{2})} \left| \frac{q_1-1}{2} \right| \times \left| \frac{q_1-3}{2} \right| \times \cdots \times \left| \frac{q_1+1}{2} \right| \times 1 \times Y & \xleftarrow{R_Y^*} & W_{Y} \\
\end{array}
\]

where \( R_Y^* \) is the normalized intertwining operator. Its image is \( R(w_p, \lambda_p)I(\lambda_p) \cap Z_Y^* \). By Proposition 2.4, \( \text{Ind}_{GL(q_2) \times G(n-\frac{q_1+q_2}{2})} 1 \times Y \) is semi-simple with length 2.

**Proposition 2.7.** The image of \( R_Y^* \) is semi-simple with length \( \leq 2 \).

**Proof.** Let \( X \) be a subrepresentation of \( Z_Y^* \). Let \( \xi = \left| \frac{q_1-1}{2} \right| \times \left| \frac{q_1-3}{2} \right| \times \cdots \times \left| \frac{q_2+1}{2} \right| \). Consider the subrepresentation of the Jacquet module of \( X \) with respect to \( M = \left( GL(1) \times \cdots \times GL(1) \times G(n-\frac{q_1+q_2}{2}) \right) \) such that the \( \frac{q_1-q_2}{2} \)...
copies of $GL(1)$ acts, after semi-simplification, according to $\xi$, i.e., the space of generalized weight $\xi$. We denote the space by Jac$_{\xi} X$. Then we have

$$Jac_{\xi} X \subset Jac_{\xi} Z^*_Y = \xi \times \text{Ind}_{GL(q_2) \times G(n - \frac{q_1 + q_2}{2})}^Y 1 \times Y.$$ 

Therefore our assertion follows. \hfill \Box

Let $\text{Ind}_{GL(q_2) \times G(n - \frac{q_1 + q_2}{2})}^Y 1 \times Y = X_1 + X_2$. Then by [M1, Cor. 0.10.2], the normalized intertwining operators

$$\text{Ind}_{GL(1) \times \cdots \times GL(1) \times G(n - \frac{q_1 - q_2}{2})}^{|-\frac{q_1 - 1}{2}} \times | \frac{q_1 - 3}{2} \times \cdots \times | \frac{q_2 + 1}{2} \times X_i \rightarrow \text{Ind}_{GL(1) \times \cdots \times GL(1) \times G(n - \frac{q_1 - q_2}{2})}^{|-\frac{q_1 - 1}{2}} \times | \frac{q_1 - 3}{2} \times \cdots \times | \frac{q_2 + 1}{2} \times X_i$$

are non-vanishing. Therefore:

**Proposition 2.8.** The image of $R_1'$ is a sum of two irreducible representations.

**Proposition 2.9.** Let $R_Y(\sigma_1, x)$ be the operator:

$$\text{Ind}_{GL(q_2) \times G(n - \frac{q_1 + q_2}{2})}^{|\det | x \times Y \rightarrow \text{Ind}_{GL(q_2) \times G(n - \frac{q_1 + q_2}{2})}^{|\det - x \times Y.}$$

$R_Y(\sigma_1, x)$ is holomorphic at $x = 0$ and let $R_Y(\sigma_1, 0) = R_Y(\sigma_1)$. Then $R_Y(\sigma_1)^2 = \text{id}$ and $R_Y(\sigma_1)$ is not the identity since $\text{Ind}_{GL(q_2) \times G(n - \frac{q_1 + q_2}{2})}^1 \times Y$ is reducible. Let $R(\sigma_1)$ be the sum of $R_Y(\sigma_1)$ as $Y$ runs through $\text{Unip}(p_{\geq 2})$. Then it defines an intertwining operator for $R(w_p, \lambda_p) I(\lambda_p)$.

In the same way, we can define $R(\sigma_k)$. For each $\sigma \in \text{Stab}(\lambda_p, \uparrow p)$, we define $R(\sigma)$ in a canonical way ([M1, 686]): If $\sigma = \sigma_{p_1} \cdots \sigma_{p_1} \sigma_{p_1} \cdots \sigma_k$, then $R(\sigma) = R(\sigma_{p_1}) \cdots R(\sigma_{p_1}) R(\sigma_{p_1}) \cdots R(\sigma_{p_1})$. We note that $R(\sigma w_p, \lambda_p) = R(\sigma) R(w_p, \lambda_p)$ for $\sigma \in \text{Stab}(\lambda_p, \uparrow p)$.

**Theorem 2.10.**

1. $\sigma \mapsto R(\sigma)$ is a homomorphism of the group $\text{Stab}(\lambda_p, \uparrow p)$ into the group of the intertwining operators of $R(w_p, \lambda_p) I(\lambda_p)$.
2. For $X \in \text{Unip}(p)$, let $R(\sigma) X = \eta_X^{\lambda_p}(\sigma) X$. Then $\eta_X^{\lambda_p}$ defines a character of $\text{Stab}(\lambda_p, \uparrow p)$.
3. If $p_i = p_j$, then $\eta_X^{\lambda_p}(\sigma(p_i)) = \eta_X^{\lambda_p}(\sigma(p_j))$. If $p_i = q_{p_1} - 1$ or $p_i = q_{p_2}$, then $\eta_X^{\lambda_p}(\sigma(p_i)) = \eta_X^{\lambda_p}(\sigma(p_j))$. Recall in Proposition 2.4 that these happen precisely when $\text{Ind}_{GL(p_i) \times G(m) 1 \times X'}$ is irreducible for an appropriate $m$. Therefore by Lemma 2.1, $\eta_X^{\lambda_p}$ defines a character of $\hat{\text{A}}(p)$.
4. By Proposition 2.5 and 2.9, passing to quotient, $X \mapsto \eta_X^{\lambda_p}$ gives rise to an isomorphism $\text{Unip}(p) \simeq \hat{\text{A}}(p)$ which is extended canonically to $\text{Unip}(O) \simeq \text{Springer}(O)$. 


2.2. Global Theory. Let $\sigma \mapsto R(\sigma)$ be an isomorphism of the group $\text{Stab}(\lambda_{\p}, \uparrow \p)$ onto the group of the intertwining operators of $R(w_{\p}, \lambda_{\p})I(\lambda_{\p})$.

We think of $\eta \in \widehat{\Lambda}(\p)$ as an element of $\widehat{\Lambda}(O)$ such that $\eta|_{K_{\p}} = 1$.

**Corollary 2.11.**

1. $\widehat{\Lambda}(\p)$ is isomorphic to $\text{Stab}(\lambda_{\p}, \uparrow \p)$.
2. $\sigma \mapsto R(\sigma)$ is an isomorphism of the group $\text{Stab}(\lambda_{\p}, \uparrow \p)$ onto the group of the intertwining operators of $R(w_{\p}, \lambda_{\p})I(\lambda_{\p})$.

**Definition 2.2.1.** For $\p = (p_1, \ldots, p_r; q_1, q_2, \ldots, q_s) \in P(O)$, let $S_{\p}$ be the set of positive roots defined as follows:

\[
\begin{align*}
t & = e_{j} - e_{j+1}, & & \text{if } \sum_{i=1}^{k} p_i < j < \sum_{i=1}^{k+1} p_i, \\
& = e_{j} - e_{j+1}, & & \text{for } T_i^d < j \leq T_i^f - 1 \text{ and } \frac{e_{T_i^d + 2q_i - 2q_i + 1} + e_{T_i^f}}{2}, & & \text{where } 1 \leq i \leq \left[ \frac{s}{2} \right], \\
& = e_{j} - e_{j+1}, & & \text{for } T_i^d < j < n, \\
& = 2e_n, & & \text{if } G = Sp(2n), \text{ is odd and } q_s > 1; \\
& = e_n, & & \text{if } G = SO(2n + 1), \text{ is odd.}
\end{align*}
\]

We note that $S_{\p} \subset \{ \alpha > 0 | w_{\p} \alpha < 0, \langle \lambda_{\p}, \alpha^\vee \rangle = 1 \}$ and $S_{\p}$ has exactly $n - r$ elements. We will take the iterated residue of the Eisenstein series along the $n$ singular hyperplanes $\langle \lambda_{\p}, \alpha^\vee \rangle = 1$ for $\alpha \in S_{\p}$.

**Definition 2.2.2.** Let $V(\p)$ (resp. $V'(\p)$) be the set of elements of the form $\lambda_{\p} + \eta$, where $\eta$ is a character of $M_{\p}(\mathbb{A})$ (resp. $M_{\p}'(\mathbb{A})$). Note that if $r = 0$, $V(\p) = \{ \lambda_{\p} \}$. We note that $V'(\p)$ is the intersection of the singular hyperplanes $\langle \lambda, \alpha^\vee \rangle = 1$, where $\alpha \in \{ e_j - e_{j+1} \text{ for } \sum_{i=1}^{k} p_i < j < \sum_{i=1}^{k+1} p_i, T_i^d < j \leq T_i^f - 1, i = 1, \ldots, [\frac{s}{2}] \text{ and } T_j^d < j < n \}$.
We denote the element in $V'(p)$ as

$$\lambda_p(x_1, \ldots, x_r, z_1, \ldots, z_{\left\lfloor \frac{s+1}{2} \right\rfloor}) = \lambda_p + \sum_{p_1} \frac{g_{1}^{i}}{f_{2}^{j}} + \sum_{p_2} \frac{q_{1}^{i}}{f_{2}^{j}} + \sum_{p_3} \frac{r_{1}^{i}}{f_{2}^{j}}.$$

**Definition 2.2.3.** For $1 \leq k \leq \left\lfloor \frac{s+1}{2} \right\rfloor$, we define

$$V'_k(p) = \{ \lambda_p(x_1, \ldots, x_r, z_1, \ldots, z_{\left\lfloor \frac{s+1}{2} \right\rfloor}) \in V'(p),$$

such that $z_i = 0$ for all $i > k$.

In particular, $V'_0(p) = V(p)$ and $V'_{\left\lfloor \frac{s+1}{2} \right\rfloor} = V'(p)$.

**Definition 2.2.4.** We define $W(\uparrow, p)$ to be the set of the Weyl group elements which send the positive roots of $M'_p$ to the positive roots of $M'_p$.

Let

$$d(p, \lambda) = \prod_{\alpha \in S_p} (\langle \lambda_p, \alpha^\vee \rangle - 1).$$

Let $\text{Unip}$ be the submodule of $\otimes'_v R_v(w_p, \lambda_p)I_v(\lambda_p)$ which is the sum of irreducible subrepresentations of type $\otimes'_v X_v$, where $X_v \in \text{Unip}(p)$ for all $v$ and there does not exist $p' > p$ and $X_v \in \text{Unip}(p')$ for all $v$.

Let $\text{proj}_{[p]}$ be the $G(\mathbb{A})$-projection $\otimes'_v R_v(w_p, \lambda_p)I_v(\lambda_p) \hookrightarrow \text{Unip}$. For $\phi \in PW$, the set of Paley-Wiener type functions, let

$$l_p(\phi, \lambda) = \sum_{w \in W} r(w, -\lambda)R(w_p w^{-1}, w \lambda)\phi(w \lambda).$$

Then we have:

1. $r(w_p, \lambda)d(p, \lambda)$ is holomorphic at $\lambda = \lambda_p$ and its value is non-zero.
2. The poles of $l_p(\phi, \lambda)$ in a neighborhood of $\lambda_p$ are contained in the local intertwining operators.
3. $r(w, -\lambda)$ is identically zero on $V'(p)$ if $w \notin W(\uparrow, p)$. So the restriction of $l_p(\phi, \lambda)$ to $V'(p)$ is given by

$$l_p(\phi, \lambda) = \sum_{w \in W(\uparrow, p)} r(w, -\lambda)R(w_p w^{-1}, w \lambda)\phi(w \lambda).$$

4. $l_p(\phi, \lambda)V(p)$ can be defined inductively by restricting it to $V'_k(p)$ from $k = \left\lfloor \frac{s+1}{2} \right\rfloor - 1$ to $k = 0$.
5. $l_p(\phi, \lambda)_V(p)$ is holomorphic at $\lambda_p$ and $l_p(\phi, \lambda_p) \in \otimes'_v R_v(w_p, \lambda_p)I_v(\lambda_p)$.

This depends only on $\phi$ and the equivalence class of $p$. Let $l_{[p]}(\phi, \lambda_p) = \text{proj}_{[p]}(\phi, \lambda_p)$. 
(6) Let $\langle , \rangle$ be the inner product in $L^2(G(F)\backslash \mathbb{A}_F)$. Then

$$\langle \theta_{\phi'}, \theta_{\phi} \rangle = \lim_{T \to \infty} \sum_{\phi \in \mathcal{P}(O)} \sum_{p \in \mathcal{P}(O)} c_p \int_{\mathcal{V}(p), \text{Re} \lambda = \lambda_p, ||\text{Im} \lambda|| \leq T} \langle \nu'_p(\phi'), \lambda \rangle, \nu_p(\phi, \lambda) \rangle_{\mathcal{V}(p)}(r(w_p, \lambda)d(p, \lambda))_{\mathcal{V}(p)},$$

where $O$ runs through the unipotent orbits in $G^*(n)$ and $p$ runs through the set of representatives in each equivalence classes in $\mathcal{P}(O)$.

(7) For $\phi \in \mathcal{P}W$, suppose $\nu_p(\phi, \lambda_p)$ generates an irreducible representation. Then for all $v$ finite places, let $X_v$ be the local representation of $G_v$ generated by $\nu_p(\phi, \lambda_p)$. Then $X_v \in \text{Unip}(p)$ and $\prod v \eta_{X_v} = 1$.

(8) Conversely, suppose $p = (p_1, \ldots, p_{r}; q_1, \ldots, q_s) \in \mathcal{P}(O)$ such that $p_i \neq p_j$ for $i, j \in [1, r]$ and $\pi = \otimes'_v X_v$ is an irreducible automorphic representation which satisfies the following: (a) $X_v \in \text{Unip}(p)$ for all $v$, (b) $X_v$ is spherical almost everywhere and at archimedean places, and (c) $\prod v \eta_{X_v} = 1$. Then there exists $\phi \in \mathcal{P}W$ such that the representation generated by $\nu_p(\phi, \lambda_p)$ is isomorphic to $\pi$.

(9) In fact, for an appropriate $\phi \in \mathcal{P}W$,

$$\nu_p(\phi, \lambda_p) = \sum_{\tau \in \text{Stab}(\lambda_p, 1)} R(\tau^{-1})R(w_p, \lambda_p)\phi(\lambda_p).$$

### 3. Arbitrary character case.

By conjugation, we can assume $\chi = \chi(\mu_1, \ldots, \mu_1, \ldots, \mu_k, \ldots, \mu_k, 1, \ldots, 1)$, $r_0 + \cdots + r_k = n$, $r_1 \geq \cdots \geq r_k$, $\mu_i$'s are distinct local quadratic characters. Here $k \leq 3$ (Recall that we are dealing with a $p$-adic field with odd residual characteristic and hence there are only three non-trivial distinct quadratic characters.) Set $\mu_0 = 1$. We use the following notation throughout this section:

1. If $G = \text{Sp}(2n)$, $G' = G'_1 \times \cdots \times G'_k \times G'_0$, where $G'_i = O(2r_i)$ for $i = 1, \ldots, k$, $G'_0 = \text{Sp}(2r_0)$. Also we denote $G^*_i = O(2r_i, \mathbb{C})$ for $i = 1, \ldots, k$, $G^*_0 = O(2r_0 + 1, \mathbb{C})$.

2. If $G = \text{SO}(2n + 1)$, $G' = G'_1 \times \cdots \times G'_k \times G'_0$, where $G'_i = \text{SO}(2r_i + 1)$ for $i = 1, \ldots, k$, $G'_0 = \text{SO}(2r_0 + 1)$. Also we denote $G^*_i = \text{Sp}(2r_i, \mathbb{C})$ for $i = 1, \ldots, k$, $G^*_0 = \text{Sp}(2r_0, \mathbb{C})$.

3. If $G = \text{O}(2n)$, $G' = G'_1 \times \cdots \times G'_k \times G'_0$, where $G'_i = \text{O}(2r_i)$ for $i = 1, \ldots, k$, $G'_0 = \text{O}(2r_0)$. Also we denote $G^*_i = \text{O}(2r_i, \mathbb{C})$ for $i = 1, \ldots, k$, $G^*_0 = \text{O}(2r_0, \mathbb{C})$.

We need to first generalize Mœglin’s results to an arbitrary chain.
**Definition 3.1.** By a segment attached to \((a,b)\), we mean a descending sequence of characters
\[
\lambda_{(a,b)} = |\lambda_1| \times |\lambda_2| \times \cdots \times |\lambda_n|,
\]
where \(\lambda_t = \frac{a+1}{2} - t, n = \frac{a+b}{2}\). We sometimes write it as \(\lambda_{(a,b)} = (\lambda_1, \ldots, \lambda_n)\).

We put the convention that \(a > b > 0\) if \((a,b)\) are odd in the case of \(Sp(2n)\) and \(O(2n)\), even in the case of \(SO(2n+1)\). To \((a)\), \(a \geq 3\), we attach a segment
\[
\lambda_{(a)} = |\frac{a-1}{2} - \frac{a-3}{2} \times \cdots \times |\frac{1}{2}|.
\]
We write it as \(\lambda_{(a)} = (\frac{a-1}{2}, \frac{a-3}{2}, \ldots, 1)\). By a chain we mean an ordered union of segments. We put a convention that the segment attached to \((a)\) appears at the end. It can appear only in the case of \(Sp(2n)\) and \(SO(2n+1)\).

A chain is denoted by \(p = (a_1, b_1, a_2, b_2, \ldots, a_s, b_s, a_{s+1})\), where \((a_i, b_i)\) is a segment and \(\lambda_p = \lambda_{(a_1, b_1)} \cdot \lambda_{(a_2, b_2)} \lambda_{(a_3, b_3)} \cdots \lambda_{(a_{s+1}, b_{s+1})}\) with an obvious meaning.

**Remark 3.1.** We note that \(\lambda_{(a,b)}\) is the intersection of the \(n\) singular hyperplanes \(e_1 - e_2 = 1, e_2 - e_3 = 1, \ldots, e_{n-1} - e_n = 1, e_{a+b+1} = 1\).

**Remark 3.2.** Suppose we ignore the ordering in \(p\). Then it corresponds to a unipotent orbit \(O\) in \(G^*(n)\). When \(O\) is a distinguished orbit, Mœglin's case is that \(a_i, b_i\) are all distinct and satisfy two additional conditions:

1. there does not exist \(1 \leq k \leq s\) such that \(a_i > a_k > b_i > b_k\).
2. if there exists \(a_k\) or \(b_k\) such that \(a_j > a_k > b_j\) or \(a_j > b_k > b_j\), then \(k < j\).

Let \(p = (a_1, b_1, \ldots, a_s, b_s, a_{s+1})\). If \(\{a_i, b_i\} \cap \{a_i+1, b_{i+1}\} \neq \emptyset\), then we could permute \((a_i, b_i)\) and \((a_{i+1}, b_{i+1})\) by [M1, Proposition 0.9.1]. But if \(\{a_i, b_i\} \cap \{a_{i+1}, b_{i+1}\} = \emptyset\), then we cannot permute \(\{a_i, b_i\}\) and \(\{a_{i+1}, b_{i+1}\}\) in general. From now on we assume that in a chain \(p\),
\[
(3.1) \quad \text{if } \{a_i, b_i\} \cap \{a_j, b_j\} = \emptyset \text{ for } i < j, \text{ it does not happen that } a_i > a_j > b_i > b_j,
\]
in order to use [M1, Cor. 0.10.2] on non-vanishing of normalized intertwining operators. We note that the condition \((3.1)\) is just a rephrasing of the condition \((2)\) in Remark 3.2.

**Remark 3.3.** Let \(O\) be a unipotent orbit obtained by ignoring the ordering in a chain \(p = (a_1, b_1, \ldots, a_s, b_s, a_{s+1})\). Suppose either \(O\) is not distinguished or \(p\) does not satisfy condition \((1)\) in Remark 3.2. Then \(p\) can be written as \(p = p_1 \times \cdots p_k \times p_0\), where \(p_i \in P(O_i)\) and \(O_i\) is a distinguished unipotent orbit in \(G_i^*\) for \(i = 0, 1, \ldots, k\). It means that it comes from global consideration when \(\chi = \chi(\mu_1, \ldots, \mu_1, \ldots, \mu_k, \ldots, \mu_k, 1, \ldots, 1)\), where \(\mu_i, i = 1, \ldots, k\) are non-trivial quadratic grössencharacters such that \(\mu_{iv} = 1\) for \(i = 1, \ldots, k\) for a given non-archimedean place \(v\).
For a segment \((a, b)\), we define the Weyl group elements \(w_{(a,b)}\), \(\sigma_{(a,b)}\) as follows (see [M1, p. 660]):

\[
\begin{align*}
    w_{(a,b)}(t) &= -t, \quad \text{if } 1 \leq t \leq \frac{a-b}{2}, \\
    w_{(a,b)}\left( t + \frac{a-b}{2} \right) &= \frac{a+b}{2} + 1 - t, \quad \text{if } 1 \leq t \leq b, \\
    \sigma_{(a,b)}(t) &= t, \quad \text{if } 1 \leq t \leq \frac{a-b}{2}, \\
    \sigma_{(a,b)}\left( t + \frac{a-b}{2} \right) &= -\left( \frac{a+b}{2} + 1 - t \right), \quad \text{if } 1 \leq t \leq b.
\end{align*}
\]

We note that

1. \(\sigma_{(a,b)}w_{(a,b)} = w_{(a,b)}\sigma_{(a,b)}\) is the longest element \(w_0\) of the Weyl group of \(Sp_{2n}\), \(n = \frac{a+b}{2}\), i.e., \(w_0 = c_1 c_2 \cdots c_n\), where \(c_i\)'s are sign changes.
2. \(\sigma_{(a,b)}\lambda_{(a,b)} = \lambda_{(a,b)}\).
3. \(l(\sigma_{(a,b)}w_{(a,b)}) = l(w_{(a,b)}) + l(\sigma_{(a,b)})\).
4. If \(q = 1\), \(\sigma_{(a,1)} = c_n\) and \(w_{(a,1)} = c_1 \cdots c_{n-1}\).

For a segment attached to \((a)\), we define \(w_{(a)} = c_1 c_2 \cdots c_{a-1}\) and \(\sigma_{(a)} = 1\).

\(\lambda_{(a)}\) is the intersection of the \(\frac{a-1}{2}\) singular hyperplanes \(c_1 - e_2 = 1, e_2 - e_3 = 1, \ldots, e_{a-3} - e_{a-2} = 1\).

For a chain \(\mathbf{p} = (a_1, b_1, \ldots, a_s, b_s, a_{s+1})\), we define \(w_\mathbf{p}\) as \(w_\mathbf{p} = w_{(a_1,b_1)} \cdots w_{(a_s,b_s)}w_{(a_{s+1})}\) with an obvious meaning and \(Stab(\lambda_\mathbf{p}, \mathbf{p})\) as the group generated by \(\sigma_{(a_i,b_i)}\) for \(i = 1, \ldots, s\).

In order to apply induction, we define \(\mathbf{p}_{\geq i} = (a_i, b_i, \ldots, a_s, b_s, a_{s+1})\). Let, for \(1 \leq i \leq s\), \(T^d_i = \sum_{k=1}^{i-1} \frac{a_k + b_k}{2}\) and \(T^f_i = \sum_{k=1}^{i} \frac{a_k + b_k}{2}\).

**Definition 3.2.** For a chain \(\mathbf{p} = (a_1, b_1, \ldots, a_s, b_s, a_{s+1})\), we define a Levi subgroup \(M'_\mathbf{p} = GL(\frac{a_1 + b_1}{2}) \times \cdots \times GL(\frac{a_s + b_s}{2}) \times GL(\frac{a_{s+1}}{2})\) and degenerate principal series

\[
\begin{align*}
    \bar{I}(\lambda_\mathbf{p}, \chi) &= \text{Ind}_{M'_\mathbf{p}} \bar{\chi} \otimes |\det| \frac{a_{k-1}}{4} \times \cdots \times |\det| \frac{a_{k-1}}{4} \times |\det| \frac{1}{2} \frac{a_{k+1}}{2} |, \\
    \tilde{I}(\lambda_\mathbf{p}, \chi) &= \text{Ind}_{M'_\mathbf{p}} \bar{\chi} \otimes |\det| - \frac{a_{k-1}}{4} \times \cdots \times |\det| - \frac{a_{k-1}}{4} \times |\det| - \frac{1}{2} \frac{a_{k+1}}{2} |, \\
\end{align*}
\]

where \(\bar{\chi}\) is the character of \(M'_\mathbf{p}\) induced by \(\chi\).

If we set \(w'_\mathbf{p}\) to be the longest Weyl group element of \(M'_\mathbf{p}\), then \(\bar{I}(\lambda_\mathbf{p}, \chi)\) is the image of the normalized intertwining operator \(R(w'_\mathbf{p}, \lambda_\mathbf{p}, \chi)\).

The normalized intertwining operator \(R(w'_\mathbf{p}, \lambda_\mathbf{p}, \chi)\) is not holomorphic at \(\lambda_\mathbf{p}\). In order to define \(R(w_\mathbf{p}, \lambda_\mathbf{p}, \chi)\), we need:

**Proposition 3.1.** For each segment \((a_i, b_i)\), \(R(w_{(a_i,b_i)}, \lambda_{(a_i,b_i)} \cdots \lambda_{(a_i,b_i)}, \chi)\) defines a holomorphic intertwining operator from

\[
\text{Ind}_{GL(1) \times \cdots \times GL(1) \times GL(n-T^d_i)} \lambda_{(a_1,b_1)} \cdots \lambda_{(a_1,b_1)} \times \tilde{I}(\lambda_{\mathbf{p}_{\leq i+1}}, \chi),
\]
into

\[ \text{Ind}_{GL(1) \times \cdots \times GL(n-T_f^f)} \chi(a_1,b_1) \lambda(a_1,b_1) \cdots \chi(a_{i-1},b_{i-1}) \lambda(a_{i-1},b_{i-1}) \]
\[ \times w(a_i,b_i) \chi(a_i,b_i) \lambda(a_i,b_i) \times \tilde{I}(-\lambda_{p \geq i+1}, \chi). \]

Its image is included in

\[ \text{Ind}_{GL(1) \times \cdots \times GL(n-T_{f+1}^f)} \chi(a_1,b_1) \lambda(a_1,b_1) \cdots \chi(a_{i-1},b_{i-1}) \lambda(a_{i-1},b_{i-1}) \times \tilde{I}(-\lambda_{p \geq i+1}, \chi). \]

Proof. The argument is like that in [M1, 0.13]; the introduction of quadratic characters does not create any new complications. □

We define the normalized intertwining operator \( R(w_p, \lambda_p, \chi) \) as the composition of the above operators. Then

\[ R(w_p, \lambda_p, \chi) \subset \tilde{I}(-\lambda_p, \chi) \]
\[ R(w_p, \lambda_p, \chi) \subset \text{Ind}_{GL(n-T_f^f) \times GL(n-T_f^f)} |\det|^{\frac{a_1-b_1}{4}} \]
\[ \times R(w_{p \geq 2}, \lambda_{p \geq 2}, \chi)I(\lambda_{p \geq 2}, \chi). \]

Here \( \chi \) in \( R(w_{p \geq 2}, \lambda_{p \geq 2}, \chi)I(\lambda_{p \geq 2}, \chi) \) should be interpreted appropriately.

Lemma 3.2. The normalized intertwining operator \( R(w_p, \lambda_p, \chi) \) does not vanish identically.

Proof. Let \( \lambda_O \) be the conjugate of \( \lambda_p \) which is in the closure of the positive Weyl chamber. Let \( w_1 \) be a Weyl group element such that \( \lambda_p = w_1 \lambda_O \).

Consider the following commutative diagram.

\[ I(\lambda_O, w_1^{-1} \chi) \xrightarrow{R(w_O, \lambda_O, w_1^{-1} \chi)} I(-\lambda_O, w_1^{-1} \chi) \]
\[ \downarrow R(w_1, \lambda_O, w_1^{-1} \chi) \quad \uparrow R(w_1^{-1}, -\lambda_p, \chi) \]
\[ I(\lambda_p, \chi) \xrightarrow{R(w_p, \lambda_p, \chi)} I(-\lambda_p, \chi). \]

Here \( R(w_O, \lambda_O, w_1^{-1} \chi) \) is the intertwining operator on \( \text{Ind}_{P}^G \lambda_O \otimes \text{Ind}_{P}^M w_1^{-1} \chi \), where \( P = MN \) is the parabolic subgroup such that \( \lambda_O \) is in the positive Weyl chamber with respect to \( P \). Then it is non-vanishing. Note that all the normalized intertwining operators are holomorphic. Therefore, \( R(w_p, \lambda_p, \chi) \) is non-vanishing. □

We first reduce to the case \( \chi = 1 \).
3.1. Review of the results of Barbasch-Moy [B-Mo2]. We now consider the case when $\chi$ is unramified. The following discussion is based on [B-Mo2].

Let $G = SO(2(n+1))$, $\theta = |a_1^2 \mu \times \cdots \times |a_{n_0}^n \mu \times |b_1^1 \times \cdots \times |b_{n_0}^n$, a character of $T \subset G$ ($n_0 + n_1 = n$), where $\mu$ is the non-trivial unramified quadratic character and $a_1, \ldots, a_{n_0}, b_1, \ldots, b_{n_0} \in \mathbb{R}$. Let $H' = H'_1 \times H'_0$, with $H'_1 = SO(2n_1 + 1), H'_0 = SO(2n_0 + 1)$. Set $\theta' = \theta'_1 \times \theta'_0$ with $\theta'_1 = |a_1^1 \times \cdots \times |b_{n_0}^1$, $\theta'_0 = |b_1^1 \times \cdots \times |b_{n_0}^n$ characters of $T'_1 \subset H'_1$, $T'_0 \subset H'_0$. If $\tau$ (resp. $\tau'$, $\tau'_1$, $\tau'_0$) denotes the infinitesimal character associated to subquotients of $\text{Ind}^G_B \theta$ (resp. $\text{Ind}^{H'}_B \theta'$, $\text{Ind}^{H'_1}_B \theta'_1$, $\text{Ind}^{H'_0}_B \theta'_0$), then the results of [B-Mo2] tell us that there is an equivalence of categories

$$\mathcal{R}(G, \tau) \simeq \mathcal{R}(H', \tau') \simeq \mathcal{R}(H'_1 \times H'_0, \tau'_1 \times \tau'_0),$$

where $\mathcal{R}(G, \tau)$ denotes the category of smooth finite-length representations of $G$ with infinitesimal character $\tau$.

Suppose $G = Sp(2n)$. Then the same discussion as above applies, except that in this case $H'_1 = O(2n_1)$ and $H'_0 = Sp(2n_0)$.

Ultimately, we are going to apply the results of [B-Mo2] to those of $[R]$ to deal with representations of $Sp(2n)$ and $SO(2n+1)$. Therefore, we also need to discuss [B-Mo2] for $O(2n)$ (cf. Example 1.1). Unfortunately, the results of [B-Mo2] do not apply to disconnected groups, nor does there appear to be any obvious way to extend the results of [B-Mo2] from $SO(2n)$ to $O(2n)$. Thus, we are forced to assume the results of [B-Mo2] hold for $O(2n)$ as well (cf. Assumption 3.1.1 below). The above discussion then applies to $G = O(2n)$ as well. In this case, $H'_1 = O(2n_1)$ and $H'_0 = O(2n_0)$.

**Assumption 3.1.1.** The results in Sections 1-6 of [B-Mo2] hold for $O(2n)$.

Next, we note that this equivalence respects induction, in a suitable sense. Let $G$ be one of the groups above ($SO(2n+1), Sp(2n), O(2n)$) and $M$ a standard Levi subgroup of $G$. Let $M', H'$ be the groups corresponding to $M, G$ under [B-Mo2]. Let $\tau_M, \tau'_M$ be infinitesimal characters for $M, M'$ which correspond under [B-Mo2], $\tau, \tau'$ the infinitesimal characters for $G, H'$ obtained by induction. Then by [B-Mo2, Theorem 6.2], the following diagram commutes:

$$\begin{array}{ccc}
\mathcal{R}(G, \tau) & \longrightarrow & \mathcal{R}(H', \tau') \\
\downarrow \otimes\text{-Ind} & & \downarrow \otimes\text{-Ind} \\
\mathcal{R}(M, \tau_M) & \longrightarrow & \mathcal{R}(M', \tau'_M)
\end{array}$$

where $\otimes\text{-Ind}$ denotes induction defined via tensor product at the Hecke algebra level.

**Example 3.1.1.** Let $\alpha_1, \ldots, \alpha_k \in \mathbb{R}$ and

$$\pi = \text{Ind}^G_B(|\det_{m_1}|^{\alpha_1} \mu \circ \det_{m_1} \times \cdots \times \det_{m_k}^{\alpha_k})$$

with $\mu$ the non-trivial unramified quadratic character of $\mathbb{R}$ and $\alpha_1, \ldots, \alpha_k \in \mathbb{R}$. Then $\text{Ind}^G_B|\det_{m_1}|^{\alpha_1} \mu \circ \det_{m_1} \times \cdots \times \det_{m_k}^{\alpha_k}$ is a smooth representation of $G$ with infinitesimal character $\pi$. Induction on the Hecke algebra level gives the following diagram,

$$\begin{array}{ccc}
\mathcal{R}(G, \pi) & \longrightarrow & \mathcal{R}(H', \pi') \\
\downarrow \otimes\text{-Ind} & & \downarrow \otimes\text{-Ind} \\
\mathcal{R}(M, \pi_M) & \longrightarrow & \mathcal{R}(M', \pi'_M)
\end{array}$$

where $\otimes\text{-Ind}$ denotes induction defined via tensor product at the Hecke algebra level.
and subquotients of the induced representation $I$. Under the Barbasch-Moy correspondence between subquotients of the induced representation $\pi \circ \mu$, we get an equivalence of categories, which gives rise to a correspondence $\pi' \in \mathcal{H}'$. Let $\lambda, \pi, \mu \in \mathcal{M}$ be irreducible representations of $G$. The correspondence behaves well with respect to representations of $G$.

3.2. Matching of images of intertwining operators under Hecke algebra isomorphisms. Let $\chi = \chi(\mu_1, \ldots, \mu_k, 1, \ldots, 1)$, where $r_0 + \cdots + r_k = n$, $r_1 \geq \cdots \geq r_k$, and the $\mu_i$'s are distinct local quadratic characters. Let $\mathcal{G}' = \mathcal{G}'_1 \times \cdots \times \mathcal{G}'_k \times \mathcal{G}'_0$. Combining the Hecke algebra isomorphisms of Roche and the graded algebra isomorphisms of Barbasch-Moy, we get an equivalence of categories, which gives rise to a correspondence between subquotients of the induced representation $I^G(\lambda, \chi)$ and subquotients of the induced representation $I^G(\lambda, 1)$ (cf. Example 1.1 and previous section). Note that if we write $\lambda = \lambda_1 \times \cdots \times \lambda_k \times \lambda_0$ with $\lambda_1 = \cdots = \lambda_r = a_1, \ldots, a_r \in \mathbb{R}$, etc. we have $I^G(\lambda, 1) \cong I^G_1(\lambda_1, 1) \times \cdots \times I^G_k(\lambda_k, 1) \times I^G_0(\lambda_0, 1)$. The correspondence preserves temperedness, square-integrability, etc. The correspondence behaves well with respect to intertwining operators.
Proposition 3.2.1. The above correspondence behaves well with respect to intertwinning operators, i.e., \( R(w_p, \lambda_p, \chi)I(\lambda_p, \chi) \) corresponds to

\[
R(w_{p_1}, \lambda_{p_1})I_{G_1}^I(\lambda_{p_1}) \times \cdots \times R(w_{p_k}, \lambda_{p_k})I_{G_k}^I(\lambda_{p_k}) \times R(w_{p_0}, \lambda_{p_0})I_{G_0}^I(\lambda_{p_0}),
\]

where \( w_p = w_{p_1} \times \cdots \times w_{p_k} \times w_{p_0} \) and \( \lambda_p = \lambda_{p_1} \times \cdots \times \lambda_{p_k} \times \lambda_{p_0} \).

We prove this in this section. The arguments used in this section are based largely on [Ca], [Ca2], [Re2] (with their presentation also influenced by [Re1]).

Since we are assuming \( F \) has odd residual characteristic here, we have that \( F \) admits three non-trivial quadratic characters. Let \( \mu, \mu_{nr}, \) denote a ramified, resp. unramified, non-trivial quadratic character (so that \( \mu, \mu_{nr}, \mu_{nr} \) are the three non-trivial quadratic characters). If we fix a uniformizer \( \varpi \), we may assume \( \mu \) is the ramified quadratic character satisfying \( \mu(\varpi) = 1 \). For convenience, assume \( \chi \) (cf. Section 1) has the form

\[
\chi = \mu \times \cdots \times \mu \times 1 \times \cdots \times 1.
\]

(As in Example 1.1, it is actually \( \chi|_{T(\mathcal{O})} \) that is needed in Roche’s construction.) Then we have \( J_{\chi} = I = \text{Iwahori subgroup} \)

\[
\mathcal{H}(G, \rho) \simeq \mathcal{H}((\bar{H}, 1_{I_{H}})).
\]

Here \( \rho = \rho_{\chi} \) as in Section 1. It will be convenient to write \( H' \) (resp., \( I', B' \)) for \( \bar{H} \) (resp., \( I_{H}, B_{H} \)). Recall the decomposition \( H' = H'_1 \times H'_0 \) from Example 1.1.

Now, \( \mathcal{H}(G, \rho) \) has linear basis \( \{ T_w \}_{w \in W_{\chi}} \), where \( T_w \) is supported on \( IwI \). If we identify \( W_{\chi} \) with representatives in \( G \) chosen as in the proof of Lemma 9.3 [R] (which in turn is based on [Mr]), we can normalize \( T_w \) so that it is 1 at \( w \). Also, observe that \( \overline{W}_{\chi} = \overline{W} \), which may be identified with \( \overline{W}(H'_1) \times \overline{W}(H'_0) \). We let \( w_0 \) denote the longest element of \( W \), and note that \( w_0 \in \overline{W}_{\chi} \).

If \( \pi = \text{Ind}^G_B(\lambda \chi) \) with \( \lambda \) unramified, then \( V_{\pi, \chi}^I \) has as basis \( \{ f_w \}_{w \in W_{\chi}} \), where

\[
f_w(g) = \begin{cases} 
\delta^\frac{1}{2}(t)\lambda(t)\chi(t)i, & \text{if } g = (tu)wi \in BwI \\
0, & \text{if not.}
\end{cases}
\]

Similarly, if \( \pi' = \text{Ind}^G_{B'} \lambda \), then \( V_{\pi', \chi}^I \) has basis \( \{ f'_w \}_{w \in \overline{W'}} \), where

\[
f'_w(g') = \begin{cases} 
\delta'^\frac{1}{2}(t')\lambda(t'), & \text{if } g' = (t'u')wi' \in B'wI' \\
0, & \text{if not.}
\end{cases}
\]

If we let \( T'_w, w \in W' \), denote the characteristic function of \( I'wI' \), we have the following:
Lemma 3.2.2. Let $s \in W$ be the reflection associated to the $\chi$-simple root $\alpha \in \Pi_\chi$. For $w \in W$, we have

$$\pi'(T_s)f_w' = \begin{cases} f_w \rho \alpha > 0, \\ qf_w + (q - 1)f_w' \text{ if } w\alpha < 0. \end{cases} \]$$

Further, for $c \in C_\chi$,

$$\pi'(T_c)f_w' = f_w'^{w_{wc} - 1}. \]

Proof. Since it is well-known how to do such calculations (and straightforward), we omit the details. \[\square\]

Corollary 3.2.3. Let $\mathcal{H}(K, \rho) \subset \mathcal{H}(G, \rho)$ denote the subalgebra consisting of functions supported on $K$. Then, $f_{w_0}$ generates $V^{I, \chi}$ under the action of $\mathcal{H}(K, \rho)$.\]

Proof. Observe that the corresponding result for $\pi'$ is straightforward: From the preceding lemma, $f_{w_1}' = \pi'(T_{w_1}')f_{w_1}'$. Therefore, $f_1' = \pi'(T_{w_1}' = 1)f_{w_0}'$. Thus, $f_{w_1}' = \pi'(T_{w_1}' = 1)f_{w_0}'$. To extend this to cover $\pi$, observe that $f_{w_1}' = T_{w_1}'f_{w_1}'$ and $f_{w_1}' = T_{w_1}'$. Therefore, for $w_1, w_2 \in W$, if $T_{w_2} = T_{w_1}' = \sum c_w T_{w_1}'$, then $\pi'(T_{w_1}')f_{w_1}' = \sum c_w f_{w_1}'$, and similarly for $\pi$. Suppose $\Psi : T_w \mapsto a_w T_w$. If $T_{w_1}' = \sum c_w T_{w_1}$, one can then conclude that $\pi(T_{w_1})f_{w_1} = \sum c_w a_w a_w a_w f_{w_1}$. The corollary follows. \[\square\]

If $B = TU$ is the Levi factorization of $B$, we use $\pi_U$ to denote the (non-normalized) Jacquet module of $\pi$ with respect to $U$. It has

space: $(V_\pi)_U = V_\pi / V_\pi(U),$

where $V_\pi(U) = \operatorname{span}\{\pi(u)v - v | u \in U, v \in V_\pi\},$

action: $\pi_U(t)(v + V_\pi(U)) = \pi(t)v + V_\pi(U).$

We have the following:

Lemma 3.2.4. The restriction of the quotient map $V_\pi \rightarrow (V_\pi)_U$ to $V^{I, \rho}_\pi$ gives rise to an isomorphism $V^{I, \rho}_\pi \cong (V^{I, \rho}_\pi)_U$ as vector spaces.\]

Proof. The proof is essentially the same as that done by Casselman for unramified principal series, so we just give a sketch here.

First, one checks that if $v \in V^{U_1 U_0, \chi}_\pi$, then $v - \pi(e_\rho)v \in V_\pi(U)$ (note that $e_\rho = T_1$). Consequently, $(V^{I, \rho}_\pi)_U = (V^{U_1 U_0, \chi}_\pi)_U$. Next, we choose a finite-dimensional subspace $X \subset V^{U_1 U_0, \chi}_\pi$ which maps onto $(V^{U_1 U_0, \chi}_\pi)_U$. If we take a compact subgroup $U^\pi_- \subset U^\pi$ which acts trivially on $X$ and $t \in T$ such that $t^{-1}U_1^- t \subset U_1^\pi$, we get $\pi(t)X \subset V^{U_1 U_0, \chi}_\pi$. Therefore, $(V^{I, \rho}_\pi)_U = \pi_U(t)X_U \subset (V^{I, \rho}_\pi)_U = (V^{I, \rho}_\pi)_U$, giving surjectivity. Injectivity then follows from a comparison of dimensions. \[\square\]
Lemma 3.2.5. For \( t \in T^- \), \( \pi(T_t)f_{w_0} = \delta^{-\frac{1}{2}}(t)(w_0\chi)(t)f_{w_0} \). (This result also holds when \( G = O(2n) \) and \( w_0 = c_1c_2 \cdots c_n \).)

Proof. Recall that we have a \( B \)-stable filtration \( V_{\pi} = \bigcup_{w \in W} (V_{\pi})_w \), where

\[
(V_{\pi})_w = \left\{ f \in V_{\pi} \mid \text{supp } f \subset \bigcup_{x \geq w} BxB \right\},
\]

(where \( \geq \) denotes the Bruhat order). This gives rise to a \( T \)-filtration \( (V_{\pi})_U = \bigcup_{w \in W} ((V_{\pi})_w)_U \). By Lemma 6.3.5 [Ca] (also, see the proof of Lemma 2.12 [B-Z]), \( T \) acts on \( ((V_{\pi})_w)_U \) by the character \( \delta^{\frac{1}{2}}(w_0\chi) \). Since \( Bw_0I \subset Bw_0B \), we see that \( f_{w_0} \) is a basis for \( V^{T^-}_\pi \cap (V_{\pi})_{w_0} \). Thus, by Lemma 3.2.4, \( f_{w_0} + V_{\pi}(U) \) is a basis for \( ((V_{\pi})_{w_0})_U \). In particular,

\[
\pi(t)(f_{w_0} + V_{\pi}(U)) = \delta^{\frac{1}{2}}(t)(w_0\chi)(t)f_{w_0} + V_{\pi}(U).
\]

Next, the same basic argument used in [Ca2, Proposition 2.5] tells us that \( |HI|^{-1}\pi(T_t)f_{w_0} \) and \( \pi(t)f_{w_0} \) have the same image in \( (V_{\pi})_U \). Since \( |HI| = \delta^{-1}(t) \), (Ca, Lemma 1.5.1), we see that

\[
\pi(T_t)f_{w_0} + V_{\pi}(U) = \delta^{-1}(t)\pi(t)f_{w_0} + V_{\pi}(U)
\]

\[
= \delta^{-\frac{1}{2}}(t)(w_0\chi)(t)f_{w_0} + V_{\pi}(U).
\]

Finally, if one writes \( \pi(T_t)f_{w_0} = \sum_{w \in W} c_w f_w \), taking Jacquet modules gives \( \pi(T_t)f_{w_0} + V_{\pi}(U) = \sum_{w \in W} c_w f_w + V_{\pi}(U) \). It is now immediate from the preceding lemma and the equality above that \( c_w = \delta^{-\frac{1}{2}}(t)(w_0\chi)(t)f_{w_0} \) and \( c_w = 0 \) for \( w \neq w_0 \).

For \( G = O(2n) \), let \( \lambda, \chi \) be as above and \( w_0 = c_1c_2 \cdots c_n \). Let \( \overline{G} = SO(2n) \) and \( \overline{w}_0 \in \overline{W}(\overline{G}) \) of maximal length. The preceding argument then tells us that for \( \overline{\pi} = \text{Ind}_{\overline{G}}^{\overline{G}}(\overline{\lambda}) \), \( \overline{\pi}(T_{t})f_{\overline{w}_0} = \delta^{-\frac{1}{2}}(t)(\overline{w}_0\chi)(t)f_{\overline{w}_0} \) when \( t \in T^- \). If \( \overline{w}_0 = w_0 \) (\( n \) even), the \( (2n) \) result is immediate. If \( \overline{w}_0 = w_0c_n \) (\( n \) odd), it follows easily from the fact that \( \pi(e_{c_n})f_{\overline{w}_0} = f_{w_0} \) and \( \delta \circ c_n = \delta \). \( \square \)

For \( \lambda, \chi \) as above, let \( \pi = \text{Ind}_{B}^{G}(\lambda \chi) \) and \( \pi' = \text{Ind}_{B'}^{G'}(\lambda) \). If \( \lambda = |x_1 \times \cdots \times |x_{n+1} \times |y_1 \cdots \chi| |y_{n_{\chi}} \), we set \( \lambda_1 = |x_1 \times \cdots \times |x_{n+1} \times |y_{n_{\chi}} \), and \( \lambda_0 = |y_1 \times \cdots \times |y_{n_{\chi}} \) (so that \( \lambda = \lambda_1 \times \lambda_0 \)). We note that \( \pi' \simeq \pi'_1 \otimes \pi'_0 \), where \( \pi'_1 = \text{Ind}_{H'_1}^{H_1} \lambda_1 \) and \( \pi'_0 = \text{Ind}_{B'_0}^{H_0} \lambda_0 \). We define a map \( \mathcal{M} : V_{\pi}^{1/\chi} \rightarrow V'_{\pi'} \) as follows: Let \( \mathcal{M} : f_{w_0} \mapsto f'_{w_0} \). (Note that under the identification \( \overline{W}_\chi = \overline{W}' = \overline{W}(H'_1) \times \overline{W}(H'_2) \), \( w_0 \) corresponds to \( w_{0,1}w_{0,0} \), with \( w_{0,3} \in \overline{W}(H'_2) \) consisting of all sign changes. Under the identification \( \pi' \simeq \pi'_1 \otimes \pi'_0 \), we have \( f'_{w_0} \mapsto f'_{w_{0,1}} \otimes f'_{w_{0,0}} \).)

Then by Corollary 3.2.3, we can extend \( \mathcal{M} \) to get a linear isomorphism satisfying \( \pi'(\Psi(h))(\mathcal{M}f) = \mathcal{M}(\pi(h)f) \) for all \( h \in H(K, \rho) \), \( f \in V_{\pi}^{1/\chi} \). We claim that the equivalence of categories \( \mathcal{R}_\chi(G) \simeq \mathcal{R}_1(H') \) comes from the map \( \mathcal{M} \). More precisely, we have the following:
Proposition 3.2.6. The pair \((\mathcal{H}(G, \rho), \pi)\) is equivalent to \((\mathcal{H}(H',1), \pi')\) under \((\Psi, \mathfrak{M})\). In particular,
\[
\pi'(\Psi(h))(\mathfrak{M}f) = \mathfrak{M}(\pi(h)f)
\]
for all \(h \in \mathcal{H}(G, \rho)\), \(f \in V^L_x\).

Proof. By definition, \(\pi'(\Psi(h))(\mathfrak{M}f) = \mathfrak{M}(\pi(h)f)\) holds for \(h \in \mathcal{H}(K, \rho)\). Take \(y \in Y^+\). Then \(\Psi : \delta^\frac{1}{2}(y)T_y \rightarrow \delta^\frac{1}{2}(y)T'_y\) (cf. proof of Lemma 9.3 [R]).

We note that by definition, \(T_y = T_{y(\tilde{w})}\). Since the extension of \(\chi\) from \(T(\mathcal{O})\) to \(T\) satisfies \(\chi(y(\tilde{w})) = 1\) for all \(y \in Y\), it follows from Lemma 3.2.5 that \(\pi'(\Psi(T_y))(\mathfrak{M}f_{w_0}) = \mathfrak{M}(\pi(T_y)f_{w_0})\) for \(y \in Y^+\). Therefore, \(\pi'(\Psi(T_y))(\mathfrak{M}f) = \mathfrak{M}(\pi(T_y)f)\) for all \(y \in Y\), \(f \in V^L_x\). The proposition follows. \(\square\)

We now give a technical lemma which we will need below. In the lemma, we use \(l_G\) to denote length for \(\mathcal{W}(G)\), \(l_H\) for length for \(\mathcal{W}(H)\).

Lemma 3.2.7. Let \(G, \chi\) be as above, \(M\) the Levi factor of a standard parabolic subgroup of \(G\). Then, there exists a set \(\mathcal{W}^M_x \subset \mathcal{W}_x\) such that the following all hold:

1. \(\mathcal{W}^M_x\) is a set of representatives for \(\mathcal{W}_x(M) \setminus \mathcal{W}_x\).
2. For \(x \in \mathcal{W}_x(M)\), \(w \in \mathcal{W}^M_x\), we have \(l_G(xw) = l_G(x) + l_G(w)\).
3. For \(x \in \mathcal{W}_x(M)\), \(w \in \mathcal{W}^M_x\), we have \(l_H(xw) = l_H(x) + l_H(w)\).

Proof. For explicitness, let \(G = \text{Sp}(2n)\). Write \(\chi = \mu \times \cdots \times \mu \times 1 \times \cdots \times 1\) and \(M \simeq \text{GL}(m_1) \times \cdots \times \text{GL}(m_l) \times \text{Sp}(2m_0)\), with \(n_0 + n_1 = m_0 + m_1 + \cdots + m_l = n\). Observe that \(H = H \times C_x\) with \(C_x = \{1, c_{n_1}\}\), where \(c_{n_1}\) denotes the \(n_1\)th sign change. Let \(\tilde{L}\) be the subgroup of \(\tilde{H}\) corresponding to \(M\) (cf. Lemma 1.4). We define

\[
\mathcal{W}^L_x(H) = \{w \in \mathcal{W}(H) | w^{-1} \alpha > 0 \text{ for all } \alpha \in \Phi^+_L\}
\]

\[
\mathcal{W}^L_x(\tilde{H}) = \{w \in \mathcal{W}(\tilde{H}) | w^{-1} \alpha > 0 \text{ for all } \alpha \in \Phi^+_L\}.
\]

It is known that if \(x \in \mathcal{W}(L)\), \(w \in \mathcal{W}^L_x(H)\), then \(l_H(xw) = l_H(x) + l_H(w)\); since \(l_H(c_{n_1}) = 0\), this result clearly extends to \(w \in \mathcal{W}^L_x(\tilde{H})\). We consider two cases.

Case 1: \(n_1 \leq m_1 + \cdots + m_l\).

In this case, \(\tilde{L} = L\) is connected. Set \(\mathcal{W}^M_x = \mathcal{W}^L_x(\tilde{H})\). Since \(\mathcal{W}_x(M) = \mathcal{W}(L)\), property (3) is immediate. The first property follows easily from the fact that \(\mathcal{W}^L_x(H)\) is a set of representatives for \(\mathcal{W}(L) \setminus \mathcal{W}(H)\) and \(\mathcal{W}^L_x(\tilde{H}) = \mathcal{W}^L_x(H) \cup \mathcal{W}^L_x(H)c_{n_1}\).
Finally, for the second property, one can directly check that there is a standard Levi $M'$ of $G$ such that $\Phi^+_{M,\chi} = \Phi^+_L = \Phi_{M'}^+$ (if $n_0 = m_1 + \cdots + m_i$ for some $i$, $M' = M$; otherwise, $M' < M$). Then $\mathcal{W}_\chi(M) = \mathcal{W}(M')$ and $\mathcal{W}_\chi^T = \mathcal{W}_{TL}(\tilde{H}) \subset \mathcal{W}_{TM}'$. The result follows.

Case 2: $n_1 > m_1 + \cdots + m_i$.

In this case, $\tilde{L} = L \times C_\chi$ is disconnected. Set $\mathcal{W}_\chi^T = \mathcal{W}_\chi^T \cap \mathcal{W}_\chi^T = \{w \in \mathcal{W}_\chi^T | w^{-1}\alpha > 0 \text{ for all } \alpha \in \Phi^+_M\}$. As a first step, we check that $\mathcal{W}_{TL}(\tilde{H}) = \mathcal{W}_\chi^T \cup c_n \bar{\mathcal{W}}_\chi^T$. It follows easily from the definitions that $\mathcal{W}_\chi^T \cup c_n \bar{\mathcal{W}}_\chi^T \subset \mathcal{W}_{TL}(\tilde{H})$. For the reverse containment, observe that $w \in \mathcal{W}_{TL}(\tilde{H})$ has $w \in \mathcal{W}_\chi$. Further, $w^{-1}\Phi^+_{M,\chi} = w^{-1}\Phi^+_L \subset \Phi^+_H = \Phi^+_\chi$.

Now, the only simple root in $\Pi_M$ which is not in $\Phi^+_{M,\chi}$ is

$$\alpha_{n_1} = \begin{cases} e_{n_1} - e_{n_1 + 1}, & \text{if } n_1 < n \\ 2e_n, & \text{if } n_1 = n. \end{cases}$$

There are two possibilities: $w^{-1}\alpha_{n_1} > 0$ or $w^{-1}\alpha_{n_1} < 0$. If $w^{-1}\alpha_{n_1} > 0$, then $w^{-1}\Pi_M \subset \Phi^+$, implying $w \in \mathcal{W}_{TM}$, as needed. If $w^{-1}\alpha_{n_1} < 0$, we claim $w^{-1}c_n\alpha_{n_1} > 0$. (This is clear for $n_1 = n$. Suppose $n_1 < n$. Observe that $w \in \mathcal{W}_\chi^T$ implies $we_n = \pm e_j$ with $j \leq n_1$ and $we_{n_1 + 1} = \pm e_k$ with $k > n_1$. Further, since $w \in \mathcal{W}_{TL}$, $w(2e_{n_1 + 1}) > 0$, so $we_{n_1 + 1} = e_k$. The claim follows.) Also $c_n(\Pi_M - \{\alpha_{n_1}\}) \subset c_n\Phi^+_{M,\chi} \subset \Phi^+_{M,\chi}$ (noting that $c_n \in \bar{\mathcal{W}}(\tilde{L}) = \bar{\mathcal{W}}(M)$). Thus, $w^{-1}c_n\Pi_M \subset \Phi^+$, implying $c_nw \in \mathcal{W}_{TM}$, as needed. Therefore, $\mathcal{W}_{TL}(\tilde{H}) = \mathcal{W}_\chi^T \cup c_n \bar{\mathcal{W}}_\chi^T$.

Since $|\mathcal{W}_\chi^T| = |\mathcal{W}(\tilde{H})|/|\mathcal{W}(\tilde{L})|$, the first property is equivalent to $\mathcal{W}(\tilde{L})\mathcal{W}_\chi^T = \mathcal{W}(\tilde{H})$. We calculate:

$$\mathcal{W}(\tilde{L})\mathcal{W}_\chi^T = (\mathcal{W}(\tilde{L}) \cup \mathcal{W}(\tilde{L})c_n)\mathcal{W}_\chi^T = \mathcal{W}(\tilde{L})(\mathcal{W}_{TM} \cup c_n \bar{\mathcal{W}}_\chi^T) = \mathcal{W}(\tilde{L})\mathcal{W}_{TL}(\tilde{H}).$$

That $\mathcal{W}(\tilde{L})\mathcal{W}_{TL}(\tilde{H}) = \mathcal{W}(\tilde{H})$ follows easily from $\mathcal{W}(\tilde{L})\mathcal{W}_{TL}(\tilde{H}) = \mathcal{W}(\tilde{H})$.

The second property follows immediately from $\mathcal{W}_\chi(M) \subset \mathcal{W}(M)$ and $\mathcal{W}_\chi^T \subset \mathcal{W}_{TL}$.

To check the third property, write $x = x_Hc_x, w = c_wHw_H \in C_\chi$ and $x_H, w_H \in \mathcal{W}(\tilde{H})$. Then since $l_{\tilde{H}}(c_n) = 0$, we have $l_{\tilde{H}}(x) = l_{\tilde{H}}(x_H), l_{\tilde{H}}(w) = l_{\tilde{H}}(w_H)$. Set $x'_H = (c_xc_w)^{-1}x_H(c_xc_w)$. Then $l_{\tilde{H}}(x'_H) = l_{\tilde{H}}(x)$. Since $x_H \in \mathcal{W}(\tilde{L})$ and $w_H \in \mathcal{W}_{TL}(\tilde{H})$, we have $l_{\tilde{H}}(x'_H) + l_{\tilde{H}}(w_H) = l_{\tilde{H}}(x'_Hw_H) = l_{\tilde{H}}(c_xc_wx'_Hw_H) = l_{\tilde{H}}(xw)$. The result follows.
The case of $G = SO(2n + 1)$ is much easier; since $H = \tilde{H}$ is connected, the argument is just that of Case 1 above.

**Lemma 3.2.8.** Suppose $M$ is a standard Levi for $G$, $\tilde{L}$ the corresponding subgroup of $\tilde{H}$. If $\nu \hookrightarrow \text{Ind}_{BM}^M \chi$ has space $V^{I \rho \lambda}_{\nu} \subset V^{I \rho \lambda}_{\text{Ind}_{BM}^M \chi}$, let $V^{I \nu}_{\nu'} \subset V^{I \nu}_{\text{Ind}_{BM}^M \chi}$ be its image under $\mathfrak{M}_M$ (with $\nu'$ denoting the restriction of $\text{Ind}_{BM}^M \chi$ to $V^{I \nu}_{\nu'}$). Then the image of $V^{I \rho}_{\text{Ind}_{BM}^M \nu} \subset V^{I \rho}_{\text{Ind}_{BM}^M \chi}$ under $\mathfrak{M}$ is $V^{I \rho}_{\nu'} \subset V^{I \rho}_{\nu'}$.

**Remark 3.2.1.** Certainly, the restriction of $\pi'$ to $\mathfrak{M}V^{I \rho}_{\text{Ind}_{BM}^M \nu}$ is equivalent to $\text{Ind}_{BM}^H \nu'$. But for our purposes, it is necessary to know that the subspaces actually match up.

**Proof of Lemma 3.2.8.** First we need to identify the subspaces for $\text{Ind}_{BM}^L \nu$ and $\text{Ind}_{BM}^H \nu'$. Suppose \{\$f^L_i = \{ \sum_{j \in \chi(M)} b^i_{j, x} f^\nu_{xw} \}_{i = 1, \ldots, k} \subset V^{I \nu}_{\nu'} \}$ and \{\$f^H_i = \{ \sum_{j \in \lambda(M)} c^i_{j, x} f^\nu_{xw} \}_{i = 1, \ldots, k} \subset V^{I \nu}_{\nu'} \}$ are bases for $V^{I \rho}_{\nu'} \subset V^{I \rho}_{\text{Ind}_{BM}^M \chi}$. Let $f_i = \sum_{j \in \chi(M)} b^i_{j, x} f^\nu_{xw} \in V^{I \rho}_{\nu'}$. Then a basis for $V^{I \rho}_{\text{Ind}_{BM}^M \chi}$ is

$$\{ \pi(T^{w-1}) f_i \}_{i = 1, \ldots, k} = \left\{ \sum_{w \in \mathcal{W}^{TM}_x} b^i_{j, x} f^\nu_{xw} \right\}_{i = 1, \ldots, k},$$

with $\mathcal{W}^{TM}_x$ as in the preceding lemma. The proof of this is essentially the same as that of [Ja1, Lemma 2.1.4] (noting that $\mathcal{W}^{TM}_x$ is a set of representatives for $\mathcal{W}(M) \backslash \mathcal{W}(G)$). Similarly, if \{\$f^H_i = \{ \sum_{j \in \lambda(M)} c^i_{j, x} f^\nu_{xw} \}_{i = 1, \ldots, j} \subset V^{I \nu}_{\nu'} \}$ is a basis for $V^{I \rho}_{\nu'} \subset V^{I \rho}_{\text{Ind}_{BM}^M \chi}$, then $\text{Ind}_{BM}^H \nu'$ has basis

$$\{ \pi'(T^{w-1}) f_i' \}_{i = 1, \ldots, j} = \left\{ \sum_{w \in \mathcal{W}^{TL}_x} c^i_{j, x} f^\nu_{xw} \right\}_{i = 1, \ldots, j}.$$ 

Here we are taking $\mathcal{W}^{TL}_x = \mathcal{W}^{TM}_x$. (By the preceding lemma, $\mathcal{W}^{TM}_x$ has the properties we need; we change notation only for appearance’s sake.)

We now check how subspaces match up. First observe that $\mathfrak{M}(f_w \mapsto \pi'(\Psi(T^{w-1})\Psi(T^{-1}_{w_0})))f^\nu_{w_0}$. Therefore, if $\Psi : T_w \mapsto a_w T_w'$, we see that $\mathfrak{M}(f_w = m(w) f^\nu_{w_0})$, where $m(w) = a_{w^{-1}} a_{w_0}^{-1}$. Similarly, using Proposition 1.6, we get $\mathfrak{M}_M f^M_w = m_M(w) f^\nu_{w_0}$ with $m_M(w) = a_{w^{-1}} a_{w_0}^{-1}$. Similarly, using Proposition 1.6, we get $\mathfrak{M}_M f^M_w = m_M(w) f^\nu_{w_0}$ with $m_M(w) = a_{w^{-1}} a_{w_0}^{-1}$ (where $M$ is the longest element...
of \( W_\chi(M) \). If \( \{ f^M_i \}_{i=1,...,k} \) as above is a basis for \( V'_{\nu}^{I_M,\rho_M} \), then \( V'_{\nu}^{I_\nu,\rho_{\nu}} \) has basis \( \left\{ \sum_{x \in W_\chi(M)} b^L_x f_x w \right\}_{i=1,...,k} \). Therefore, \( \mathfrak{M} V'_{\nu}^{I_\nu,\rho_{\nu}} \) has basis (using \( \mathfrak{W}_\chi(M) = \mathfrak{W}^L (\mathfrak{L}), \mathfrak{W}_\chi^T = \mathfrak{W}^T (\mathfrak{L}) \))

\[
\left\{ \sum_{x \in \mathfrak{W}(\mathfrak{L})} b^L_x \pi'(\Psi(T_{w-1,x-1}) \Psi(T^{-1}_{w_0-1})) f_x w \right\}_{i=1,...,k, w \in \mathfrak{W}_\chi^T}
\]

On the other hand, \( V'_{\nu}^{I_\nu,\rho_{\nu}} \) has basis

\[
\{ \mathfrak{M} f^M_i \}_{i=1,...,k} = \left\{ \sum_{x \in \mathfrak{W}(\mathfrak{L})} b^L_x m_M(x) f^L_x \right\}_{i=1,...,k}
\]

Therefore, \( V'_{\nu}^{I_\nu,\rho_{\nu}} \) has basis

\[
\left\{ \sum_{x \in \mathfrak{W}(\mathfrak{L})} b^L_x m_M(x) f^L_x \right\}_{i=1,...,k, w \in \mathfrak{W}_\chi^T}
\]

Finally, by Lemma 3.2.7, for \( x \in \mathfrak{W}_\chi(M) \) and \( w \in \mathfrak{W}_\chi^T \), we have \( T_{w-1,x-1} = T_{w-1} T_{x-1} \) and \( T'_{w-1,x-1} = T'_{w-1} T'_{x-1} \). From this, one sees that \( a_{w-1,x-1} = a_{w-1} a_{x-1} \). Therefore, \( m(xw) = \left( a_{w-1}^{-1} a_{w_0-1}^{-1} \right) m_M(x) \). The conclusion follows.

Let \( R(w_p, \lambda_p, \chi) \) denote the normalized standard intertwining operator defined earlier. Since \( w_p \in \mathfrak{W}_\chi \), we may identify \( w_p \in \mathfrak{W}' \) with \( w_{p_1} \cdot w_{p_0} \in \mathfrak{W}(I'_{\chi}) \times \mathfrak{W}(I'_{\nu}) \) \( (p_1, p_0) \) are ordered partitions for \( H'_{\chi}, H'_{\nu} \). If we use \( H' = H'_{\chi} \times H'_{\nu} \) to identify the degenerate principal series \( I_{H'}(\lambda_p, 1) \) with \( I_{H'_{\chi}}(\lambda_{p_1}, 1) \otimes I_{H'_{\nu}}(\lambda_{p_0}, 1) \), we have

\[
R'(w_p, \lambda_p, 1) = R'_1(w_{p_1}, \lambda_{p_1}, 1) \otimes R'_0(w_{p_0}, \lambda_{p_0}, 1),
\]

for the corresponding intertwining operators.

**Proposition 3.2.9.** \( R(w_p, \lambda_p, \chi) \) is a non-zero multiple of

\[
\mathfrak{M}^{-1} \circ R'_1(w_{p_1}, \lambda_{p_1}, 1) \otimes R'_0(w_{p_0}, \lambda_{p_0}, 1) \circ \mathfrak{M}.
\]

**Proof.** We argue as in [Re2]. Let \( x = (x_1, \ldots, x_s) \), \( y = (y_1, \ldots, y_{t+1}) \) and set

\[
\lambda_{p_1} + x = \det \left| \frac{a_{i-1} b_{i}}{4} + x_1 \right| \cdots \left| \det \frac{a_{s-1} b_{s}}{4} + x_s \right|
\]
\[ \lambda_{p_0} + y = |\det \begin{vmatrix} 2 \lambda_{p_0} + 1 & |x \cdot \cdots \cdot |x \cdot \cdots \cdot y_1 + 1 \end{vmatrix} \] 

and \( \lambda_{p_0} + (x, y) = (\lambda_{p_0} + x) \times (\lambda_{p_0} + y) \). Since \( I_1(\lambda_{p_0} + (x, y), \chi) \) and \( I_{H_1}^0(\lambda_{p_0} + x, 1) \) are irreducible representations for \( (x, y) \neq (0, 0) \) near zero, Schur’s lemma for \( p \)-adic groups tells us 

\[ (*) \quad R(w_p, h_\lambda + (x, y), \chi) = c(x, y) \mathcal{M}^{-1} \circ R_1'(w_{p_1}, \lambda_{p_1} + x, 1) \otimes R_0'(w_{p_0}, \lambda_{p_0} + y, 1) \circ \mathcal{M} \]

for such \( (x, y) \) \( (c(x, y) \) scalar). Now, observe that \( \mathcal{M} \) is independent of \( (x, y) \). By Proposition 3.1, Lemma 3.2 and [M1], \( R(w_p, h_\lambda + (x, y), \chi) \) and \( \mathcal{M}^{-1} \circ R_1'(w_{p_1}, \lambda_{p_1} + x, 1) \otimes R_0'(w_{p_0}, \lambda_{p_0} + y, 1) \circ \mathcal{M} \) are both holomorphic and non-zero at \( (x, y) = (0, 0) \). By analytic continuation, \( (*) \) holds at \( (x, y) = (0, 0) \). Further, we see that \( c(x, y) \) must be holomorphic and non-vanishing at \( (x, y) = (0, 0) \). The proposition follows. \( \square \)

The results above begin the process of separating the effects of the four characters of order \( 1 \) \( \mu_{nr} \) separately from \( \mu, \mu_{nr} \) (with \( \mu, \mu_{nr} \) associated to \( 1, \mu_{nr} \) for \( H_1' \)). To finish this process, we need to separate the effects of \( 1 \) and \( \mu_{nr} \) for both symplectic and orthogonal groups. To do this, we use an argument similar to that above, only with [B-Mo2] playing the role that [R] played above. We give the argument for the symplectic case; the orthogonal case is similar.

Let \( \chi = \left( \mu_{nr_1}, \ldots, \mu_{nr_n}, 1, \ldots, 1 \right) \) and \( \lambda = |x_1| \times \cdots \times |x_n| \times |y_1| \times \cdots \times |y_n| \)

with \( x_1, \ldots, x_n, y_1, \ldots, y_n \in \mathbb{R} \). Again, set \( H' = H_1' \times H_0' \). Let \( \pi = \text{Ind}^G_B(\lambda \chi) \) and \( \pi' = \text{Ind}^{H'}_{H_1'}(\lambda) = (\text{Ind}^H_{H_1'} \lambda_1) \otimes (\text{Ind}^H_{H_0'} \lambda_0) = \pi_1' \otimes \pi_0' \) (with \( \lambda_1 = |x_1| \times \cdots \times |x_n| \) and \( \lambda_0 = |y_1| \times \cdots \times |y_n| \)). We let \( \tau, \tau', \tau_1', \tau_0' \) denote the infinitesimal characters of \( \pi, \pi', \tau_1', \tau_0' \), resp.

We define a map \( \mathcal{M}_{\pi} : V_{\pi}^I \rightarrow V_{\pi'}^I \) (similar to the situation above, \( \mathcal{M}_{\pi} \) is not independent of \( \lambda \)). Let \( \mathcal{M}_{\tau} : f_{u_0} \mapsto f'_{u_0} \). If \( q, q' \) denote the quotient maps \( q : \mathcal{H}(G) \rightarrow \mathcal{H}(G) ', q' : \mathcal{H}(H') \rightarrow \mathcal{H}_{\tau'}(H') \) (cf. [B-Mo2], p. 619), then we certainly have \( f_{u_0}, f'_{u_0} \) generating \( V_{\pi}, V_{\pi'} \), under the action of \( \pi(q(\mathcal{H}(K))), \pi'(q'(\mathcal{H}(K'))) \). Thus, as above, we may extend \( \mathcal{M}_{\tau} \) to get a linear isomorphism satisfying \( \pi'(\Psi_{\tau}(h))(\mathcal{M}_{\tau} f) = \mathcal{M}_{\tau}(\pi(h) f) \) for all \( h \in q(\mathcal{H}(K)), f \in V_{\pi}^I \). Here \( \Psi : \mathcal{H}(G) \rightarrow \mathcal{H}(H') \) is the isomorphism of quotient algebras obtained by composing the isomorphisms in [B-Mo2].

With notation as above, we have the following:

**Lemma 3.2.10.** \( \mathcal{M}_{\tau} \) has the following properties:

1. \( \pi'(\Psi_{\tau}(h))(\mathcal{M}_{\tau} f) = \mathcal{M}_{\tau}(\pi(h) f) \) for all \( h \in \mathcal{H}(G), f \in V_{\pi}^I \).
2. \( \mathcal{M}_{\tau} \) is real analytic in \( x_1, \ldots, x_{n_1}, y_1, \ldots, y_{n_0} \).
Proof. The proof of (1) is similar to Proposition 3.2.6 above. For \( h \in q(\mathcal{H}(K)) \), it holds by definition. For \( t, t' \) corresponding to the same element of \( \mathcal{Y} \), we have that \( \Psi_\tau : \delta^1(t)q(T_i) \mapsto \chi(t)\delta^2(t')q'(T_{i}') \) (a consequence of [B-Mo2, (4.4)] and the construction of \( \Psi_\tau \)). With this observation, the rest of (1) follows as in the proof of Proposition 3.2.6.

For (2), if \( w \in \overline{W}_\chi = \overline{W}(H') \), we have
\[
\mathfrak{M}_\tau(f_w) = \pi'(\Psi_\tau(q(T_w)))f_{w_0}.
\]

It is sufficient to show \( \Psi_\tau(q(T_w)) \) is analytic, which follows from [B-Mo2, Theorem 4.3].

Let \( R(w_\lambda, \lambda_\chi, \chi) \) be the normalized standard intertwining operator defined earlier. As before, we can write \( w_\lambda = w_{\lambda_1}w_{\lambda_0} \in \overline{W}(H'_1) \times \overline{W}(H'_0) \), and \( \lambda_i \) is an ordered partition for \( H'_i \). We decompose \( \lambda_\chi = \lambda_{\lambda_1} \times \lambda_{\lambda_2} \times \lambda_{\lambda_0} \). Let \( R'(w_\lambda, \lambda_\chi, 1) = R'(w_{\lambda_1}, \lambda_{\lambda_2}, 1) \otimes R'(w_{\lambda_0}, \lambda_{\lambda_0}, 1) \) be the corresponding intertwining operator for \( H' \).

**Proposition 3.2.11.** \( R(w_\lambda, \lambda_\chi, \chi) = \mathfrak{M}_\tau \circ R'(w_{\lambda_1}, \lambda_{\lambda_2}, 1) \otimes R'(w_{\lambda_0}, \lambda_{\lambda_0}, 1) \circ \mathfrak{M}_\tau^{-1} \).

Proof. We again argue as in [Re2]. With notation as in the proof of Proposition 3.2.9, we again have that if \( (x, y) \neq 0 \) near 0,
\[
(\ast) \quad R(w_\lambda, \lambda_\chi + (x, y), \chi) = \mathfrak{M}_{\tau(x,y)} \circ R'(w_{\lambda_1} + x, 1) \otimes R'(w_{\lambda_0}, y, 1) \circ \mathfrak{M}_\tau^{-1},
\]
where \( \tau(x, y) \) is the infinitesimal character associated to \( \text{Ind}^G_B(\lambda_\chi + (x, y)) \). In this case, there is no need to introduce a scalar \( c(x, y) \)–the action of (\( \ast \)) on \( K \)-fixed vectors tells us we actually have equality. By Proposition 3.1 and the preceding lemma, both sides of (\( \ast \)) are analytic in \( (x, y) \). Therefore by analytic continuation, (\( \ast \)) holds at (0,0), as needed.

**Remark 3.2.2.** Hecke algebras are not available for archimedean places. Also Roche’s results are not available for the place \( v, v|2 \).

**Remark 3.2.3.** Above, we have used the results of Barbasch-Moy and Roche to identify \( \text{Ind}^G_B \lambda_\chi \) with \( \text{Ind}^G_{B_1^r} \lambda_1 \otimes \text{Ind}^G_{B_2^r} \lambda_2 \otimes \text{Ind}^G_{B_3^r} \lambda_3 \otimes \text{Ind}^G_{B_0^r} \lambda_0, \)
where \( \chi = (\mu_1, \mu_2, \mu_3, \mu_4, \mu_5, \mu_6, \mu_7, \mu_8, \mu_9, \mu_{10}, \mu_{11}, \mu_{12}) \) and \( \lambda = \lambda_1 \times \lambda_2 \times \lambda_3 \times \lambda_0 \). The same isomorphisms allow us to identify \( \text{Ind}^G_{B_1^r} \lambda_1 \) with \( \text{Ind}^G_{B_1} \lambda_1 \chi_1 \), where \( \chi_1 = (\mu_1, \mu_2, \mu_3, \mu_4, \mu_5, \mu_6, \mu_7, \mu_8, \mu_9, \mu_{10}, \mu_{11}, \mu_{12}) \) and \( G_1 = G_1(r_1) \) and similarly for \( G_2^r, G_3^r \).

Thus, we may also identify \( \text{Ind}^G_B \lambda_\chi \) with \( \text{Ind}^G_{B_1} \lambda_1 \chi_1 \otimes \text{Ind}^G_{B_2} \lambda_2 \chi_2 \otimes \text{Ind}^G_{B_3} \lambda_3 \chi_3 \otimes \text{Ind}^G_{B_0} \lambda_0 \) (with \( G_0 = G_0^r \)). This correspondence is done in
general in [Ja3]. However, there are two basic obstacles to using [Ja3] here. The first is that, as with [Au1], [Au2], [Sc-St], the results in [Ja3] are done in the Grothendieck group setting, hence do not deal with composition series. The second is that we deal with Ind\(_{\mathcal{B}_1}^G\) \(\lambda_1\), etc., by using Mœglin’s results. To work with Ind\(_{\mathcal{B}_1}^G\) \(\lambda_1\chi_1\), etc., we would have to establish the corresponding results ourselves.

To reduce our problem to that covered by Mœglin’s results, we will also need the following proposition. (We continue to use \(j\) for the Iwahori-Matsumoto involution.)

**Proposition 3.2.12.** Suppose \(\pi\) and \(\pi' = \pi'_1 \times \pi'_2 \times \pi'_3 \times \pi'_0\) are corresponding irreducible representations. Then, \(j(\pi)\) is tempered (resp., square-integrable) if and only if \(j(\pi'_1), j(\pi'_2), j(\pi'_3), j(\pi'_0)\) are all tempered (resp., square-integrable).

**Proof.** Note that \(j(\pi')\) is tempered (resp., square-integrable) if and only if \(j(\pi'_1), j(\pi'_2), j(\pi'_3), j(\pi'_0)\) are all tempered (resp., square-integrable).

Let us write \(\theta \in \text{Jac}(\pi)\) if \(\theta\) appears in the normalized Jacquet module of \(\pi\) (with respect to the Borel subgroup) with multiplicity at least one. Observe that by the abelianness of \(T\) and Frobenius reciprocity, we have \(\theta \in \text{Jac}(\pi)\) if and only if \(\pi \hookrightarrow \text{Ind}_B^G\theta\). Therefore, by Theorem 1.8 and Frobenius reciprocity, we see that \(\theta \in \text{Jac}(\pi)\) if and only if \(\theta^{-1} \in \text{Jac}(j(\pi))\).

Similarly, \(\theta_i \in \text{Jac}(\pi'_i)\) if and only if \(\theta_i^{-1} \in \text{Jac}(j(\pi'_i))\).

For notational convenience, let \(\mu^{(k)} = \mu \times \cdots \times \mu\). Let \(\mu_1 = \mu, \mu_2 = \mu \mu_{nr}, \mu_3 = \mu_{nr}, \mu_0 = 1\), as above. We claim that \(\lambda_1 \mu_1^{(r_1)} \times \lambda_2 \mu_2^{(r_2)} \times \lambda_3 \mu_3^{(r_3)} \times \lambda_0 \mu_0^{(r_0)} \in \text{Jac}(\pi)\) if and only if \(\lambda_i \in \text{Jac}(\pi_i')\) for \(i = 0, 1, 2, 3\). First, let \(G'' = G/\gamma_3 + r_0)\) and

\[
G''_1 = \begin{cases} O(2(r_1 + r_2)), & \text{if } G = Sp(2n), O(2n) \\ SO(2(r_1 + r_2) + 1), & \text{if } G = SO(2n + 1). \end{cases}
\]

Suppose that \(\pi\) corresponds to \(\pi''_{1} \times \pi''_{0}\) under Roche’s isomorphism. We then argue as follows: \((\lambda_1 \mu_1^{(r_1)} \times \lambda_2 \mu_2^{(r_2)} \times (\lambda_3 \mu_3^{(r_3)} \times \lambda_0 \mu_0^{(r_0)}) \in \text{Jac}(\pi)\) if and only if \(\pi \hookrightarrow \text{Ind}_B^G((\lambda_1 \mu_1^{(r_1)} \times \lambda_2 \mu_2^{(r_2)} \times (\lambda_3 \mu_3^{(r_3)} \times \lambda_0 \mu_0^{(r_0)})\)) if and only if \(\pi''_{1} \hookrightarrow \text{Ind}_{B_1}^{G''}((\lambda_1 \times \lambda_2 \mu_{nr}^{(r_2)})\) and \(\text{Ind}_{B}^{G''}((\lambda_3 \mu_{nr}^{(r_3)} \times \lambda_0)\) (cf. [R, Theorem 9.5]) if and only if \(\lambda_1 \times \lambda_2 \mu_{nr}^{(r_2)} \in \text{Jac}(\pi''_{1})\) and \(\lambda_3 \mu_{nr}^{(r_3)} \times \lambda_0 \in \text{Jac}(\pi''_{0})\). We use the same basic argument in conjunction with the results of Barbasch-Moy, making a few minor modifications to cover induction via tensor product. We argue as follows for \(\pi''_{1}: \lambda_1 \times \lambda_2 \mu_{nr}^{(r_2)} \in \text{Jac}(\pi''_{1})\) if and only if \(\pi''_{1} \hookrightarrow \text{Ind}_{B_1}^{G''}((\lambda_1 \times \lambda_2 \mu_{nr}^{(r_2)})\) if and only if \(\pi''_{1} \hookrightarrow \otimes_1 \text{Ind}_{B_1}^{G''}((\lambda_1^{-1} \times \lambda_2 \mu_{nr}^{(r_2)})\) if and only
if \( \pi'_1 \hookrightarrow \otimes\text{-}\text{Ind}_{B_1'}^{G'_1}(\lambda_1)^{-1} \) and \( \pi'_2 \hookrightarrow \otimes\text{-}\text{Ind}_{B_2'}^{G'_2}(\lambda_2)^{-1} \) (cf. [B-Mo2, Theorem 6.2]) if and only if \( \lambda_1 \in \text{Jac}(\pi'_1) \) and \( \lambda_2 \in \text{Jac}(\pi'_2) \). The argument for \( \pi''_0 \) is similar. This verifies our claim.

Next we claim that any \( \theta \in \text{Jac}(\pi) \) has the form \( \text{sh}(\lambda_1\mu_1^{(r_1)}, \lambda_2\mu_2^{(r_2)}, \lambda_3\mu_3^{(r_3)}, \lambda_0\mu_0^{(r_0)}) \) for some \( \lambda_1, \lambda_2, \lambda_3, \lambda_0 \) with \( \lambda_i \in \text{Jac}(\pi_i') \), \( i = 0, 1, 2, 3 \). Here, sh denotes a shuffle, used in the usual sense (e.g., see [Ja3, Definition 3.1]). This claim follows immediately from the discussion above and [Ja3, Lemma 5.4].

Since \( \theta \in \text{Jac}(\pi) \) if and only if \( \theta^{-1} \in \text{Jac}(j(\pi)) \), the inequalities required by the Casselman criteria for \( j(\pi) \) to be tempered (resp., square-integrable) have the same form as those in [M1, Remarque 1.3.5] (\( \pi'_1, \pi'_2, \pi'_3, \pi'_0 \) are already covered by [M1, Remarque 1.3.5]). Further, observe that \( \lambda_1, \lambda_2, \lambda_3, \lambda_0 \) each satisfy the inequalities of [M1, Remarque 1.3.5] if and only if \( \text{sh}(\lambda_1\mu_1^{(r_1)}, \lambda_2\mu_2^{(r_2)}, \lambda_3\mu_3^{(r_3)}, \lambda_0\mu_0^{(r_0)}) \) satisfies the inequalities of [M1, Remarque 1.3.5] for every shuffle sh. The proof of this is straightforward; essentially identical to that of [Ja3, Corollary 8.3]. Thus, \( j(\pi) \) is tempered (resp., square-integrable) if and only if \( j(\pi'_1), j(\pi'_2), j(\pi'_3), j(\pi'_0) \) are all tempered (resp., square-integrable), as needed.

\[ \square \]

**Remark 3.2.4.** We may also use the above result to classify the square-integrable (resp., tempered) representations of \( \text{Sp}(2n, F), \text{SO}(2n+1, F) \) supported on the Borel subgroup (at least for \( F \) having odd residual characteristic). By [Ta3, Theorem 6.2], such a square-integrable (resp., tempered) representation has cuspidal support contained in \( \{|^a\mu\}_{\alpha \in \mathbb{R}, \mu^2 = 1} \). Now, to classify such representations, it suffices to classify the representations \( \pi \) with cuspidal support in \( \{|^a\mu\}_{\alpha \in \mathbb{R}, \mu^2 = 1} \) such that \( j(\pi) \) is square-integrable (resp., tempered). By the preceding proposition, it suffices to classify the corresponding representations \( \pi'_1, \pi'_2, \pi'_3, \pi'_0 \) of \( G'_1, G'_2, G'_3, G'_0 \). This is done in [M1].

### 3.3. The definition of \( \text{Unip}(\mathfrak{p}, \chi) \)

**Proposition 3.3.1.**

1. \( \text{R}(\mathfrak{w}_p, \lambda_p, \chi)I(\lambda_p, \chi) \) is semi-simple and the generalized Iwahori-Matsuo moto involution of its direct summands is tempered. Let \( \text{Unip}(\mathfrak{p}, \chi) \) be the set of direct summands of \( \text{R}(\mathfrak{w}_p, \lambda_p, \chi)I(\lambda_p, \chi) \).

2. Under the Hecke algebra isomorphism, \( \text{Unip}(\mathfrak{p}, \chi) \) corresponds to a subset of \( \text{Unip}(O_{1}) \times \cdots \times \text{Unip}(O_{k}) \times \text{Unip}(O_{0}) \).

**Proof.** By Proposition 3.2.1, it is enough to prove for \( \chi = 1 \). For simplicity, we denote \( \text{Unip}(\mathfrak{p}, 1) = \text{Unip}(\mathfrak{p}) \). Let \( O \) be the unipotent orbit obtained from \( \mathfrak{p} \) by ignoring the ordering: We will prove \( \text{Unip}(\mathfrak{p}) \subset \text{Unip}(O) \), where \( O \) is a unipotent orbit obtained from \( \mathfrak{p} \) by ignoring the ordering.

If \( \mathfrak{p} = (a_1, b_1, \ldots, a_s, b_s, a_{s+1}) \) satisfies the two conditions in Remark 3.2, then we are in Moeblin’s situation. So it is clear by Moeblin’s result.
Otherwise, by Remark 3.3, such chain can be written as $p = p_1 \times \cdots \times p_k \times p_0$, where $p_i \in P(O_i)$ and $O_i$ is a distinguished unipotent orbit in $G_i^*$ for $i = 0, 1, \ldots, k$. Let $\mu_i, i = 1, \ldots, k$ be non-trivial quadratic grössencharacters such that $\mu_{i0} = 1$ for $i = 1, \ldots, k$ for a given non-archimedean place $v$. Let $\chi = \chi(\mu_1, \ldots, \mu_1, \ldots, \mu_k, \ldots, \mu_k, 1, \ldots, 1)$.

Consider the pseudo-Eisenstein series attached to $\chi$ from [Ki3]:

$$l_p(\phi, \lambda, \chi) = \sum_{i=0}^{k} \sum_{w_i \in W_i} r(w_i, -\lambda_i, \Phi_i) R(w_p w_k^{-1} \cdots w_0^{-1}, w_0 \cdots w_k \lambda, \chi)$$

where $\Phi_i$’s are given by:

$$\Phi_1 = \{e_i \pm e_j, \ 1 \leq i < j \leq r_1\},$$

$$\Phi_2 = \{e_{r_1+i} \pm e_{r_1+j}, \ 1 \leq i < j \leq r_2\},$$

$$\vdots$$

$$\Phi_k = \{e_{r_1+\cdots+r_k-i} \pm e_{r_1+\cdots+r_k+j}, \ 1 \leq i < j \leq r_k\},$$

$$\Phi_0 = \{e_{r_1+\cdots+r_k+i} \pm e_{r_1+\cdots+r_k+j}, \ 1 \leq i < j \leq r_0, \ 2e_{r_1+\cdots+r_k+i}, \ i = 1, \ldots, r_0\},$$

$$\Phi_D = \Phi^+ - \bigcup_{i=0}^{k} \Phi_i.$$

We note that the above is for $G = Sp(2n)$. If $G = SO(2n + 1)$, we need to add, to $\Phi_i$, $e_{r_1+\cdots+r_i+1+j}, \ j = 1, \ldots, r_i$, for $i = 1, \ldots, k$ and in $\Phi_0$, $2e_{r_1+\cdots+r_k+i}$ should be $e_{r_1+\cdots+r_k+i}$. If $G = O(2n)$, then $\Phi_0$ does not have the roots $2e_{r_1+\cdots+r_k+i}, \ i = 1, \ldots, r_0$.

Also $D$ is the set of distinguished coset representatives for $\theta = \Delta - \{e_{r_1+1}, e_{r_1+r_2+1}, \ldots, e_{r_1+\cdots+r_k+1}\} \subset \Delta = \{e_1 - e_2, \ldots, e_{n-1} - e_n\}$ and $W_i$ is the Weyl group of $G_i^*$ for $i = 0, 1, \ldots, k$. Let $\lambda = \lambda_1 + \cdots + \lambda_k + \lambda_0$, where $\lambda_i = a_{r_1+\cdots+r_i+1} + \cdots + a_{r_1+\cdots+r_i+r_i+1}$ for $i = 1, \ldots, k$ and $\lambda_0 = a_{r_1+\cdots+r_k+1} + \cdots + a_n e_n$.

Now we substitute $\chi = 1$ and we show that $l_p(\phi, \lambda_p, \chi = 1)$ is well-defined. Mœglin showed that $r(w_i, -\lambda_i, \Phi_i)$ is identically zero on $V'(p_i)$ if $w_i \notin W(\uparrow, p_i)$. Since the local intertwining operators $R(w_p, \lambda_p)$ are well-defined by Proposition 3.1, the only thing we need to show is that $r(dw_1 \cdots w_k w_0, -\lambda, \Phi_D)$ is holomorphic at $\lambda_p$ for $w_i \in W(\uparrow, p_i)$ even if $\chi = 1$. Recall that for non-trivial $\chi$, $\chi \circ \alpha$ is non-trivial for $\alpha \in \Phi_D$ and so it is holomorphic.
For this, we need to show that if for \( w_l \in W(\uparrow, p_l), w_l(e_{r_1+\ldots+r_{l-1}+i}) < 0 \) for some \( i = 1, \ldots, r_1 \), then \( w_l(e_{r_1+\ldots+r_{l-1}+i}) = -e_{r_1+\ldots+r_{l-1}+i} \). Then the poles and zeros in \( r(\alpha w_1 \cdots w_k w_0, -\lambda, \Phi_D) \) cancel each other. We show it for \( w_l \in W(\uparrow, p_l) \). The remaining cases are similar. Suppose \( w_l(e_i) = -e_j \) for \( 1 \leq i, j \leq r_1 \). Recall the definition of \( W(\uparrow, p_l) \). It is the set of the Weyl group elements of \( G_+ \) which send \( \{ e_i \} \) \( 1 \leq i \leq r_1 \) to itself. Then consider \( w_l(e_p - e_i) = w_l(e_p) + e_j \). So \( w_l(e_p) = -e_q \), where \( q > j \). Consider \( w_l(e_i - e_p) = -e_j + w_l(e_p) \). Then \( w_l(e_p) = e_q \), where \( q < j \). Therefore we can see that \( w_l(e_p) = -e_{r_1+1-p} \).

Since \( t_p(\phi, \lambda_p, \chi = 1) \) is well-defined, by Mœglin’s inner product formula (Section 2.2), it belongs to the residual spectrum \( L^2(G(F) \backslash G(\mathbb{A}))/\mathbb{T}(1) \). Its local components are precisely the image of intertwining operator \( R(\alpha_p, \lambda_p, \chi = 1)I(\lambda_p, \chi_v = 1) \). Even though Proposition 2.8 is no longer true since the normalized intertwining operators could vanish, we can show that the image of intertwining operator \( R(\alpha_p, \lambda_p, \chi_v = 1)I(\lambda_p, \chi_v = 1) \) is semi-simple in the same way as in Proposition 2.7. Then by Mœglin [M1, p. 734], it is included in \( \text{Unip}(O) \), where \( O \) is the unipotent orbit obtained by ignoring the ordering in \( p \).

### 3.4. Parametrization of \( \text{Unip}(p, \chi) \)

Next we parametrize \( \text{Unip}(p, \chi) \).

#### 3.4.1. The case \( \chi = 1 \)

Because of Proposition 3.3.1, we can still define \( R(\sigma(a_i, b_i)) \) by Proposition 2.9 and the following still holds.

**Proposition 3.4.1.** \( \sigma(a_i, b_i) \mapsto R(\sigma(a_i, b_i)) \) is a homomorphism of the group \( \{ id, \sigma(a_i, b_i) \} \) into the group of the intertwining operators of \( R(\alpha_p, \lambda_p)I(\lambda_p) \).

This means the following: For \( X \in \text{Unip}(p) \), let \( R(\sigma(a_i, b_i))X = \eta^p_X(\sigma(a_i, b_i))X \). Then \( \eta^p_X \) defines a character of \( A(O) \) such that \( \eta^p_X(\sigma(a_i)) = \eta^p_X(\sigma(b_i)) \).

Since \( \text{Unip}(p) \subset \text{Unip}(O) \), \( \eta^p_X \in \text{Springer}(O) \). Therefore we have:

**Theorem 3.4.2.** \( \text{Unip}(p) \) is parametrized by

\[
C(p) = \{ \eta \in \text{Springer}(O) : \eta(\sigma(a_i)) = \eta(\sigma(b_i)), \quad i = 1, \ldots, s, \eta(\sigma(a_{s+1})) = 1 \}.
\]

**Example 3.4.1.** Let \( G = Sp(8) \) and \( p = (5, 1, 3) \). Then \( \lambda_p = (2, 1, 0, 1) \) and \( \alpha_p = c_1c_2c_4. O = (5, 3, 1) \) and Springer \( (O) \) has 3 elements, namely, \( \eta \in \text{Springer}(O) \) if and only if \( \eta(\sigma(5)) = \eta(\sigma(3)) \), \( \eta(\sigma(1)) = 1 \) or \( \eta(\sigma(3)) = \eta(\sigma(1)) \), \( \eta(\sigma(5)) = 1 \). Therefore, \( \{ \eta \in \text{Springer}(O) : \eta(\sigma(5)) = \eta(\sigma(1)), \eta(\sigma(3)) = 1 \} \) has only the trivial character.

In order to apply the above theorem to the global situation, let \( O_1, O_2 \) be two distinguished unipotent orbits in \( G_1^* \) and \( G_2^* \), resp. (If \( G = Sp(2n) \), then \( G_1^* = O(2r_1, \mathbb{C}) \) and \( G_2^* = O(2r_2 + 1, \mathbb{C}) \). If \( G = SO(2n + 1) \), then
Let $G_1^* = Sp(2r_1, \mathbb{C})$ and $G_2^* = Sp(2r_2, \mathbb{C})$. If $G = O(2n)$, then $G_1^* = O(2r_1, \mathbb{C})$ and $G_2^* = O(2r_2, \mathbb{C})$. Then we get a unipotent orbit $O$ in $G_1^*$ by combining $O_1$ and $O_2$. Further we have canonical embedding $\tilde{A}(O_i) \subset A(O)$.

For $p_i \in P(O_i)$, $i = 1, 2$, we get a chain $p_1 \times p_2$ by shuffling the segments in $p_1$ and $p_2$ so that it satisfies (3.1), and thus we get $\text{Unip}(p_1 \times p_2)$. Let $\text{Unip}(O_1, O_2)$ be the union of $\text{Unip}(p_1 \times p_2)$ as $p_i$ runs through $P(O_i)$ for $i = 1, 2$. It is a subset of $\text{Unip}(O)$. Then we have:

**Theorem 3.4.3.** $\text{Unip}(O_1, O_2)$ is parametrized by

$$C(O_1, O_2) = \{ \eta \in \text{Springer}(O) : \eta |_{A(O_1)} \in \text{Springer}(O_1), \eta |_{A(O_2)} \in \text{Springer}(O_2) \}.$$  

This can be generalized easily. Let $O_i$ be distinguished unipotent orbits in $G_1^*$ for $i = 0, 1, \ldots, k$. Let $O$ be the unipotent orbit of $G^*$, obtained by combining $O_i$’s. Then we can define $\text{Unip}(O_1, \ldots, O_k, O_0)$ and it is a subset of $\text{Unip}(O)$ and:

**Theorem 3.4.4.** $\text{Unip}(O_1, \ldots, O_k, O_0)$ is parametrized by

$$C(O_1, \ldots, O_k, O_0) = \{ \eta \in \text{Springer}(O) : \eta |_{A(O_i)} \in \text{Springer}(O_i), \text{for } i = 0, \ldots, k \}.$$  

**Corollary 3.4.5.** Let $G = Sp(2n)$ and let $O_1 = (q_1, \ldots, q_{s-1}, 1)$, $s$ even, be a distinguished unipotent orbit of $O(2n, \mathbb{C})$ and $O_0 = (1)$. Then $\text{Unip}(O_1, O_0)$ is parametrized by $\text{Springer}(O')$, where $O' = (q_1, \ldots, q_{s-1})$.

### 3.4.2. The case $\chi = \chi(\mu, \ldots, \mu)$, $\mu$ non-trivial quadratic.

Let $p = (a_1, b_1, \ldots, a_s, b_s)$. In this case the above theorems for $\chi = 1$ case hold. We need to use the generalized Iwahori-Matsumoto involution in Section 1. We can define $R(\sigma(a_i, b_i), \mu)$ in the similar way as in Proposition 2.9.

Let $\text{Unip}(p, \mu)$ be the set of components of $R(wp, \lambda_p, \chi)I(\lambda_p, \chi)$. Let $O$ be a unipotent orbit obtained from $p$ by ignoring the ordering. Then:

**Theorem 3.4.6.** $\text{Unip}(p, \mu)$ is parametrized by

$$C(p) = \{ \eta \in \text{Springer}(O) : \eta(\sigma(a_i)) = \eta(\sigma(b_i)), i = 1, \ldots, s \}.$$  

Let $O_i$ be distinguished unipotent orbits in $G_1^*$ for $i = 1, \ldots, k$. Let $O$ be the unipotent orbit of $G^*$, obtained by combining $O_i$’s. Then we can define $\text{Unip}(O_1, \ldots, O_k, \mu)$ and:

**Theorem 3.4.7.** $\text{Unip}(O_1, \ldots, O_k, \mu)$ is parametrized by

$$C(O_1, \ldots, O_k, \mu) = \{ \eta \in \text{Springer}(O) : \eta |_{A(O_i)} \in \text{Springer}(O_i), \text{for } i = 1, \ldots, k \}.$$
3.4.3. The general case. It is enough to consider the case, $\chi = \chi(\mu_1, \ldots, \mu_k, 1, \ldots, 1)$, where $\mu$ is a non-trivial quadratic character. Let $p_i$, $i = 1, 2$, be chains in $G_1^\ast$, $G_2^\ast$, resp, from which we get unipotent orbits $O_1$, $O_2$, by ignoring the ordering in $p_i$’s. (If $G = Sp(2n)$, then $G_1^\ast = O(2r_1, \mathbb{C})$ and $G_2^\ast = O(2r_0 + 1, \mathbb{C})$. If $G = SO(2n + 1)$, then $G_1^\ast = Sp(2r_1, \mathbb{C})$ and $G_2^\ast = Sp(2r_0, \mathbb{C})$. If $G = O(2n)$, then $G_1^\ast = O(2r_1, \mathbb{C})$ and $G_2^\ast = O(2r_0, \mathbb{C})$.)

Let $p = p_1 \times p_2$.

We obtain $R(\sigma_i)$’s and $R(\sigma_j, \mu)$’s. Let $\text{Unip}(p, \chi)$ be the set of components of $R(w_p, \lambda_p, \chi)I(\lambda_p, \chi_v)$. Then:

**Theorem 3.4.8.** $\text{Unip}(p, \chi)$ is parametrized by $C(p_1) \times C(p_2)$.

Let $\chi = \chi(\mu_1, \ldots, \mu_k, 1, \ldots, 1)$, $r_0 + \cdots + r_k = n$, $r_1 \geq \cdots \geq r_k$, $\mu_i$’s are distinct non-trivial quadratic characters. Set $\mu_0 = 1$. Let $p_i$, $i = 0, 1, \ldots, k$, be chains in $G_i^\ast$, from which we get unipotent orbits $O_1, \ldots, O_k$, $O_0$, by ignoring the ordering in $p_i$’s. Let $p = p_1 \times \cdots p_k \times p_0$.

Then:

**Theorem 3.4.9.** $\text{Unip}(p, \chi)$ is parametrized by $C(p_1) \times \cdots \times C(p_k) \times C(p_0)$.

In order to apply the above theorem to the global situation, let $\chi = \chi(\mu_1, \ldots, \mu_k, 1, \ldots, 1)$, $r_0 + \cdots + r_k = n$, $r_1 \geq \cdots \geq r_k \geq 2$, $\mu_i$’s are distinct non-trivial quadratic gr"oszencharacters. Let $O_i$ be a distinguished unipotent orbit in $G_i^\ast$ for $i = 0, 1, \ldots, k$. Let $p_i \in P(O_i)$ for $i = 0, \ldots, k$ and $p = p_1 \times \cdots p_k \times p_0$. Then we can shuffle the segments in $p$ so that it satisfies the condition (3.1). We still call it $\rho$. For a non-archimedean place $v$, let $\text{Unip}(O_1, \ldots, O_k, O_0, \chi_v)$ be the set of union of $\text{Unip}(p_1, \ldots, p_k, p_0, \chi_v)$ as $p_v$ runs through $P(O_i)$ for $i = 0, \ldots, k$.

**Theorem 3.4.10.** $\Pi_{\text{res}, v} = \text{Unip}(O_1, \ldots, O_k, O_0, \chi_v)$ is parametrized by

\begin{equation}
C(O_1, \ldots, O_k, O_0, \chi_v) = [\text{Springer}(O_1) \times \cdots \times \text{Springer}(O_k) \times \text{Springer}(O_0)],
\end{equation}

where $[ ]$ is defined as follows: If $\mu_{1v} = \mu_{2v} \neq \mu_{iv}$ for $i = 0, 3, \ldots, k$, then we replace $\text{Springer}(O_1) \times \text{Springer}(O_2)$ by

$C(O_1, O_2, \mu_{1v}) = \{\eta \in \text{Springer}(O) : \eta|_{\text{A}(O_i)} \in \text{Springer}(O_i), \text{ for } i = 1, 2\}$.

where $O$ is the unipotent orbit of $G_{12}^\ast$ obtained by combining $O_1, O_2$, where

$G_{12}^\ast = \begin{cases}
O(2(r_1 + r_2), \mathbb{C}), & \text{if } G = Sp(2n), O(2n) \\
Sp(2(r_1 + r_2), \mathbb{C}), & \text{if } G = SO(2n + 1).
\end{cases}$
Example 3.4.2. Let $G = Sp(28)$ and $\chi = \chi(\mu, \ldots, \mu, 1, \ldots, 1)$, $\mu$ is a quadratic grössencharacter. Let $O_1 = (9, 7, 3, 1)$ be a unipotent orbit in $O_{20}(\mathbb{C})$ and $O_2 = (5, 3, 1)$ a unipotent orbit in $O_9(\mathbb{C})$. Then for a non-archimedean place $v$, if $\mu_v \neq 1$, then $\Pi_{res_v}$ is parametrized by Springer $(O_1) \times$ Springer $(O_2)$. It has 18 elements. Let $\mu_v = 1$. Let $O = (9, 7, 5, 3, 3, 1)$. Then $A(O)$ is an abelian group generated by order 2 elements $\sigma(1), \sigma(3), \sigma(5), \sigma(7), \sigma(9)$. Consequently

$$
\text{Springer} (O) = \{ \eta \in \widehat{A(O)} : \eta(\sigma(9)) = \eta(\sigma(7)), \eta(\sigma(5)) = 1 \\
or \eta(\sigma(7)) = \eta(\sigma(5)), \eta(\sigma(9)) = 1 \}.
$$

Then $C(O_1, O_2) = \{ \eta \in \text{Springer} (O) : \eta|_{A(O_i)} \in \text{Springer} (O_i), i = 1, 2 \}$. $C(O_1, O_2)$ has 5 elements.

| $\sigma(9)$ | 1 | -1 | 1 | 1 | -1 |
| $\sigma(7)$ | 1 | -1 | -1 | 1 | -1 |
| $\sigma(5)$ | 1 | 1 | -1 | 1 | 1 |
| $\sigma(3)$ | 1 | 1 | -1 | -1 | -1 |
| $\sigma(1)$ | 1 | 1 | 1 | -1 | -1 |

Remark 3.4.1. Let $\lambda_O$ be the conjugate of $\lambda_p$ which is the closure of the positive Weyl chamber as in Lemma 3.2. Then by inducing in stages, $I(\lambda_O, w_1^{-1} \chi_v) = \text{Ind}_P^G \lambda_O \otimes \text{Ind}_B^M w_1^{-1} \chi_v$, where $P = MN$ is the parabolic subgroup such that $\lambda_O$ is in the positive Weyl chamber with respect to $P$. Then we can consider the Knapp-Stein $R$-group of $\text{Ind}_B^M w_1^{-1} \chi_v$. It is a subset of (3.2). In fact, the Knapp-Stein $R$-group is spanned by the order 2 elements $c_{r_1 + \ldots + r_t}$ for $\mu_v \neq 1$. Therefore, we can think of (3.2) as a generalization of the Knapp-Stein $R$-group.

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[M6], Letters.


Received June 15, 1999. The second author was partially supported by NSF grant DMS9610387.

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RESIDUAL SPECTRUM OF SPLIT CLASSICAL GROUPS;
CONTRIBUTION FROM BOREL SUBGROUPS

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We completely determine the residual automorphic representations coming from the torus of odd orthogonal groups. Under certain technical assumption on disconnected groups, our results are valid for symplectic groups and even orthogonal groups. Moeglin has determined the residual spectrum attached to the trivial character of the torus for split classical groups. However, non-trivial characters present some non-trivial difficulties, both local and global. Local problems are resolved in a companion paper with C. Jantzen, where many other useful results are obtained.

Introduction.

Let $F$ be a number field and $\mathbb{A}$ its ring of adeles. Let $G$ be a reductive group defined over $F$. For simplicity we assume that the center of $G$ is anisotropic over $F$. A central problem in the theory of automorphic forms is to decompose the right regular representation of $G(\mathbb{A})$ acting on the Hilbert space $L^2(G(F)\backslash G(\mathbb{A}))$. It has continuous spectrum and discrete spectrum.

$$L^2(G(F)\backslash G(\mathbb{A})) = L^2_{\text{dis}}(G(F)\backslash G(\mathbb{A})) \oplus L^2_{\text{cont}}(G(F)\backslash G(\mathbb{A})).$$

We are mainly interested in the discrete spectrum. Main contributions have been made by Langlands and Arthur. First of all Langlands described, using his theory of Eisenstein series, an orthogonal decomposition of this space of the form:

$$L^2_{\text{dis}}(G(F)\backslash G(\mathbb{A})) = \bigoplus_{(M,\pi)} L^2_{\text{dis}}(G(F)\backslash G(\mathbb{A}))(M,\pi),$$

where $(M, \pi)$ is a Levi subgroup with a cuspidal automorphic representation $\pi$ taken modulo conjugacy (here we normalize $\pi$ so that the action of the maximal split torus in the center of $G$ at the archimedean places is trivial) and $L^2_{\text{dis}}(G(F)\backslash G(\mathbb{A}))(M,\pi)$ is a space of residues of Eisenstein series associated to $(M, \pi)$. Here we note that the subspace

$$\bigoplus_{(G,\pi)} L^2_{\text{dis}}(G(F)\backslash G(\mathbb{A}))(G,\pi),$$

is the space of cuspidal representations $L^2_{\text{cusp}}(G(F)\backslash G(\mathbb{A}))$. Its orthogonal complement in $L^2_{\text{dis}}(G(F)\backslash G(\mathbb{A}))$ is called the residual spectrum and we
denote it by $L_{\text{res}}^2 (G(F) \backslash G(\mathbb{A}))$. Therefore we have an orthogonal decomposition

$$L_{\text{dis}}^2 (G(F) \backslash G(\mathbb{A})) = L_{\text{cusp}}^2 (G(F) \backslash G(\mathbb{A})) \oplus L_{\text{res}}^2 (G(F) \backslash G(\mathbb{A})).$$

Arthur described, motivated by his trace formula, described a conjectural decomposition of this space as follows:

$$L_{\text{dis}}^2 (G(F) \backslash G(\mathbb{A})) = \bigoplus \psi L_{\text{cusp}}^2 (G(F) \backslash G(\mathbb{A})), \psi,$$

where $\psi$ runs, modulo conjugacy, through the set of morphism $\psi : L_F \times SL_2(\mathbb{C}) \rightarrow G^*$, where $L_F$ is the conjectural tannakian group, $G^*$ is the Langlands’ $L$-group and $\psi$ satisfies certain conditions; in particular, $\psi$ restricted to $SL_2(\mathbb{C})$ is algebraic, $\psi$ restricted to $L_F$ parametrizes a cuspidal tempered representation of a Levi subgroup and the image of $\psi$ is not included in proper Levi subgroups. The space $L_{\text{cusp}}^2 (G(F) \backslash G(\mathbb{A}))$ is defined by local data (3.3).

Let $\Pi_{\text{res}, \psi}$ be the set of the local components of the residual spectrum $L_{\text{dis}}^2 (G(F) \backslash G(\mathbb{A}))(M, \pi)$. It is of great importance to prove that $\Pi_{\text{res}, \psi} \subset \Pi_{\psi, \psi}$ for some $\psi, \psi$ and the multiplicity formula (3.1) holds: In other words, the Arthur parameters parametrize the residual spectrum. For this, it is necessary to construct a set of characters $C_{\text{res}, \psi} \subset \hat{C}_{\psi, \psi}$ attached to $\Pi_{\text{res}, \psi}$.

We should note that Arthur’s trace formula will not separate the cuspidal part from the residual one and therefore one needs to study the residual spectrum by computing the residues of Eisenstein series. It is noted that the first exotic example of the residual spectrum of the split group $G_2$, which was discovered by Langlands and further studied later by Moeglin and Waldspurger, has been of significance to Arthur in formulating his conjectures.

There are two problems in calculating the residual spectrum. The first problem is global. It is concerned with calculating the residues of the Eisenstein series via the constant term of the Eisenstein series. Computing residues of Eisenstein series is a very difficult problem and requires both the knowledge of the poles of corresponding automorphic $L$-functions, as well as handling very difficult computational combinatorics. These computations are already highly non-trivial even for rank 2 split groups. For example, for the split group $G_2$, the Eisenstein series, built out of the maximal parabolic subgroup attached to the long simple root, contains the third symmetric power $L$-function of $GL_2$. The precise location of the poles of the third symmetric power $L$-function was resolved only in the recent work of [Ki-Sh2] (see the introduction of the paper for the long history). The Eisenstein series built out of the minimal parabolic subgroups do not present a problem with regards to poles of automorphic $L$-functions since the $L$-functions are Hecke $L$-functions. However, in this case, cancelation of poles of the normalized intertwining operators presents a problem. Even for $Sp_4$ [Ki1], one has to use a non-trivial fact on the subtle analysis of the normalized intertwining
operators for $SL_2$ due to Labesse-Langlands [L-La]. The second problem is local. It is concerned with calculating the image of the normalized local intertwining operators of the generalized principal series. While the unitary principal series are well understood thanks to the Knapp-Stein $R$-group theory, the generalized principal series are not. Moeglin’s result [M1] on the unramified principal series of the split classical groups seems to be most satisfactory so far.

At this moment, the most satisfying result about the residual spectrum is due to Moeglin-Waldspurger [M-W2] who completely determined the residual spectrum for $GL_n$ and Arthur [A2] gave the Arthur parameters for them. Their result is that the residual spectrum for $GL_n$ is parametrized by the cuspidal representations of $GL_m$ and the principal unipotent orbit of $GL_{m}(\mathbb{C})$, where $m|n$. As we observed, Arthur’s parameters require the introduction of the hypothetical group $L_F$. However, there are parts of it that can be parametrized by our existing knowledge. This is the case in particular, if the Eisenstein series is built out of minimal parabolics. In this case, $L_F$ can be replaced by the global Weil group $W_F$. Therefore, we will restrict ourselves to the study of $L^2_{\text{dis}}(G(F)\backslash G(A))_{(T,\chi)}$, where $T$ is a maximal split torus and $\chi$ is a unitary character of $T$.

When $G$ is a split classical group and $\chi$ is trivial, Moeglin [M1] completely solved the problem with the restriction that archimedean components are spherical. In this case the Arthur parameter is given by

$$\psi : W_F \times SL_2(\mathbb{C}) \mapsto G^*,$$

where $\psi|_{W_F} = 1$ and $\psi|_{SL_2(\mathbb{C})}$ is determined by a distinguished unipotent orbit of $G^*$. Recall that the Springer correspondence is an injective map from the set of irreducible characters of $W$, the Weyl group, into the set of pairs $(O, \eta)$, where $O$ is a unipotent orbit and $\eta$ is an irreducible character of $A(u) = C(u)/C(u)^0$, $u \in O$, where $C(u)$ is the centralizer of $u$ in $G^*$. Let Springer $(O)$ be the set of characters of $A(u)$ which are in the image of the Springer correspondence. Then Moeglin’s result is that the local components of the residual spectrum are parametrized by the distinguished unipotent orbits $O$ of $G^*$ and Springer $(O)$. Let $\text{Unip}(O)$ be the set of the local components of the residual spectrum. Then there is a pairing between $\text{Unip}(O)$ and Springer $(O)$ which gives a desired Arthur’s multiplicity formula (3.2).

Moeglin [M4] also constructed the local representations associated to the remaining characters of $A(u)$ which are NOT in Springer $(O)$. These should be local components of cuspidal representations. She [M5] obtained a partial result on how to determine when $\pi \in \Pi_\psi$ is a cuspidal representation. It would be an important but a difficult problem. We note that Lusztig [Lu]
has a theory of the generalized Springer correspondence which gives the remaining characters of $A(u)$.

In this paper, we will extend Moeglin’s result when $\chi$ is an arbitrary unitary character of $T$. For simplicity, we will describe the result for $G = Sp_{2n}$. Because the result in [Ja-Ki] for $Sp_{2n}, O_{2n}$ is not complete, our result for $Sp_{2n}$ is not complete. This is due to the fact that Barbasch-Moy’s result [B-Mo2] is not available for disconnected groups like $O_{2n}$. However, we obtain a complete result for $SO_{2n+1}$. Nevertheless, the symplectic group case will illustrate the technique and the heart of the matter.

As in [M1], we use $G^* = O_{2n+1}(C)$ to denote its dual group. Let $\mu_1, \ldots, \mu_k$ be $k$ distinct non-trivial quadratic grossecharacters of $F$. Fix integers $r_1 \geq \cdots \geq r_k \geq 2$ and $r_0$ so that $r_0 + r_1 + \cdots + r_k = n$. Then $\chi = \chi(\mu_1, \ldots, \mu_k)$, $\ell_{r_0, \ldots, r_k}$, which decomposes according to eigenvalues $\mu_1, \ldots, \mu_k$, and $1$, with multiplicities $2r_1, \ldots, 2r_k$, and $2r_0 + 1$, respectively. Write $\mathbb{C}^{2n+1} = V_0 + V_1 + \cdots + V_k$, where each $V_i, \dim V_i = 2r_i$, is the eigenspace attached to eigenvalue $\mu_i$. $1 \leq i \leq k$, and $V_0$ is the trivial eigenspace of dimension $2r_0 + 1$. In this way we get an embedding of

$$\prod_{i=0}^{k} O(V_i) \subset O_{2n+1}(\mathbb{C}).$$

For each $i$, $0 \leq i \leq k$, let $O_i$ be the distinguished unipotent orbit of $O(V_i)$. Then Arthur parameter of interest to us is a homomorphism

$$\psi : W_F \times SL_2(\mathbb{C}) \longrightarrow \prod_{i=0}^{k} O(V_i) \subset O_{2n+1}(\mathbb{C}),$$

satisfying,

1. $\psi|_{W_F : w \mapsto 1 \times \mu_1(w) \times \cdots \times \mu_k(w) \in \{\pm 1\} \times \{\pm 1\} \times \cdots \times \{\pm 1\}$, where $\{\pm 1\}$ is the center of $O(V_i)$ for $i = 0, \ldots, k$.
2. By Jacobson-Morozov theorem, $\psi|_{SL_2(\mathbb{C})}$ defines the unipotent orbit $\prod_{i=0}^{k} O_i$ of $G^*$.

The unipotent orbits $O_i$ determine, through the Jacobson–Morozov’s theorem, certain conjugacy class of unramified characters of $T$. Let $\lambda_0 = \lambda_{1,0} + \cdots + \lambda_{k,0} + \lambda_{0,0}$ be the one in the positive Weyl chamber. To $\psi$, Arthur associates a Langlands’ parameter $\phi_\psi$:

$$\phi_\psi : W_F \longrightarrow O_{2n+1}(\mathbb{C}),$$
where $\phi_{\psi}(w) = \psi \left( w, \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{-1}{2} \end{pmatrix} w \right)$. It is given by $\phi_{\psi} = \chi \otimes \exp(\lambda_0, H_B(\cdot))$. Its non-tempered part is $\phi_{\psi}^+ = \exp(\lambda_0, H_B(\cdot))$.

In [Ki-Sh], we studied the special case when $O_i$ is the unipotent orbit of the form $(2r_i - 1, 1)$ for $i = 1, \ldots, k$ and $O_0$ is of the form $(2r_0 + 1)$. This is the opposite of the trivial character case. In this case, we showed that the local components of the residual spectrum in $\Pi_{\phi_{\psi}}$ are parametrized by the Knapp-Stein $R$-group of the unitary principal series $I_v = \text{Ind}_{B_0}^M \chi_v$, where $M$ is the Levi subgroup whose $L$-group is $M^* = \text{Cent}(im\phi_{\psi}^+, G^*)$. We note that $M$ is of the form $GL_{n_1} \times \cdots \times GL_{n_k} \times Sp_{2k}$. Keys-Shahidi pairing [Ke-Sh] gives a desired multiplicity formula (3.2). We note that the method of [Ki-Sh] can only prove “if” part in (3.2). We need the inner product of pseudo-Eisenstein series to prove “if and only if.”

In general case, the Knapp-Stein $R$-groups are not enough. Our result is that the local component of the residual spectrum $C_{\text{res}, v}$ is parametrized by the unipotent orbits $O_i$, $i = 0, \ldots, k$ and $[\text{Springer}(O_0) \times \cdots \times \text{Springer}(O_k)]$, where $[\cdot]$ is defined as follows (see Theorem 6.2.2): If $\mu_{1v} = \mu_{2v} \neq \mu_{iv}$ for $i = 3, \ldots, k$, then we replace $\text{Springer}(O_1) \times \text{Springer}(O_2)$ by

$$C(O_1, O_2, \mu_{1v}) = \{ \eta \in \text{Springer}(O) : \eta|_{A(O_i)} \in \text{Springer}(O_i), \text{ for } i = 1, 2 \},$$

where $O$ is a unipotent orbit of $Sp(2(r_1 + r_2), \mathbb{C})$ by combining $O_1, O_2$. In Section 6, we show that the calculation of the residues of the Eisenstein series is reduced to that of the Eisenstein series of $O_{2r_1} \times \cdots \times O_{2r_k} \times Sp_{2r_0}$ attached to the trivial character. There remains a local problem of calculating the image of the normalized local intertwining operators. This has been taken care of by [Ja-Ki], under some assumptions that Barbasch-Moy’s result [B-Mo2] holds for disconnected group $O_{2n}$. Therefore for symplectic groups, the result is not complete. We review the result in Section 6.

In Section 3, we review quadratic unipotent Arthur parameters and Moeglin’s reformulation of Arthur’s conjecture on the multiplicity formula. In Section 4, we review Eisenstein series and pseudo-Eisenstein series in the sense of [M-W1] attached to a character of a Borel subgroup. In Section 7, we show how automorphic representations obtained by Kudla-Rallis [Ku-Ra] by considering degenerate principal series, are fit as a special case. In Section 8, we explain how the result in [Ki-Sh] is fit as a special case.

Acknowledgments. We would like to thank Prof. Shahidi. Without his guidance and encouragement this paper could not have been finished. We would like to thank Prof. Moeglin for patiently answering many of our questions [M5] and for many correspondences. We also thank Dr. Jantzen for many discussions. Without his help on local questions [Ja-Ki], this paper could not have been finished.
1. Preliminaries.

Let $F$ be a field and let $G = SO_{2n+1}$, $Sp_{2n}$ or $SO_{2n}$ over $F$. Let $J_n$ be the $n \times n$ matrix given by

$$J_n = \begin{pmatrix} 1 & & & 1 \\ & 1 & & \\ & & \ddots & \\ 1 & & & 1 \end{pmatrix}.$$

Let $J'_{2n} = \begin{pmatrix} J_n & -J_n \\ -J_n & J_n \end{pmatrix}$. Then

$$Sp(2n) = \{ g \in GL(2n) \mid {}^t g J'_{2n} g = J'_{2n} \},$$

and

$$SO(n) = \{ g \in GL(n) \mid {}^t g J_n g = J_n; \det(g) = 1 \}.$$

In each case we let $T$ be the maximal split torus consisting of diagonal matrices in $G$. Then

$$T(F) = \left\{ t(l_1, \ldots, l_n) = \begin{pmatrix} l_1 \\ l_2 \\ \vdots \\ l_n \\ l_{n-1} \\ \vdots \\ l_2 \\ l_1 \end{pmatrix} \middle| l_i \in F^* \right\},$$

if $G = Sp(2n)$ or $SO(2n)$, and

$$T(F) = \left\{ t(l_1, \ldots, l_n) = \begin{pmatrix} l_1 \\ l_2 \\ \vdots \\ l_n \\ 1 \\ \vdots \\ l_2 \\ l_1 \end{pmatrix} \middle| l_i \in F^* \right\},$$

if $G = SO(2n + 1)$.

Let $\Phi(G, T)$ be the roots of $G$ with respect to $T$. We choose the ordering on the roots so that the Borel subgroup $B$ is the subgroup of upper triangular
matrices in $G$. Let $\Delta$ be the simple roots in $\Phi(G,T)$ given by $\Delta = \{\alpha_j\}_{j=1}^n$, with $\alpha_j = e_j - e_{j+1}$ for $1 \leq j \leq n - 1$, and

$$\alpha_n = \begin{cases} 
  e_n & G = SO(2n+1), \\
  2e_n & G = Sp(2n), \\
  e_{n-1} + e_n & G = SO(2n).
\end{cases}$$

We let $\langle \cdot, \cdot \rangle$ be the standard Euclidean inner product on $\Phi(G,T)$. If $G$ is a root system of type $B_n$, $C_n$, or $D_n$, then we denote by $G(\Phi)$ the split group with root system $\Phi$.

For $G = SO(2n+1)$ or $Sp(2n)$, the Weyl group $W(G/T) \simeq S_n \ltimes \mathbb{Z}_2^n$. $S_n$ acts by permutations on the $\lambda_i$, $i = 1, \ldots, n$. We will use standard cycle notation for the elements of $S_n$. Thus $(ij)$ interchanges $\lambda_i$ and $\lambda_j$. If $c_i$ is the non-trivial element in the $i$-th copy of $\mathbb{Z}_2$ then $c_i$ takes $\lambda_i$ to $\lambda_i^{-1}$. The element $c_i$ is called a sign change because its action on $\Phi(G,T)$ takes $e_i$ to $-e_i$. For $G = SO(2n)$, the Weyl group is given by $W(G/T) \simeq S_n \ltimes \mathbb{Z}_2^{n-1}$. $S_n$ acts by permutations on the $\lambda_i$, and $\mathbb{Z}_2^{n-1}$ acts by even numbers of sign changes. The requirement that the number of sign changes be even comes from the determinant condition in $SO(2n)$. Note that the sign change $c_i$ is an element of $O(2n)$ and normalizes $T(F)$. Each $c_i$ acts on $SO(2n)$ by conjugation, and $c_n$ induces the non-trivial graph automorphism on the Dynkin diagram of $\Phi(G,T)$.

2. Unipotent orbits of classical groups over $\mathbb{C}$.

Theory of Jordan normal forms implies that a unipotent matrix in $GL_N$ is conjugate to $J(p_1) \oplus J(p_2) \oplus \cdots \oplus J(p_s)$, $p_1 \geq p_2 \geq \cdots \geq p_s$, $p_1 + p_2 + \cdots + p_s = N$, where $J(p)$ is the $p \times p$ Jordan matrix with entries 1 just above the diagonal and the diagonal and zero everywhere else. Therefore unipotent classes in $GL_N$ are in 1 to 1 correspondence with partitions $\lambda$ of $N$. We use the following standard notation for $\lambda$: $\lambda = (1^{r_1}, 2^{r_2}, 3^{r_3}, \ldots)$, where $r_j$ is the number of $p_i$ equal to $j$.

Let $G$ be a classical group, of type $B_n$ ($O_{2n+1}(\mathbb{C})$), $C_n$ ($Sp_{2n}(\mathbb{C})$) or $D_n$ ($O_{2n}(\mathbb{C})$). We start with the following facts:

1. $X, X' \in G$ are conjugate in $G$ if and only if they are conjugate in $GL_N$, $N = 2n + 1$ or $2n$.

2. Let $X \in GL_N$ be unipotent. Then $X$ is conjugate to an element of $G$ if and only if $r_i$ is even for even $i$ in the orthogonal case and for odd $i$ in the symplectic case.

Therefore for $G = O_{2n+1}(\mathbb{C})$, unipotent classes are in 1 to 1 correspondence with partitions $\lambda$ of $2n + 1$ such that $r_i$ is even for even $i$.

Let $u$ be a unipotent element in $G$ and let $S_u$ be its centralizer in $G$. Then we have:
In the orthogonal case (resp. symplectic) case, \(S_u/S_u^0\) is \(k\) product of \(\mathbb{Z}/2\mathbb{Z}\), where \(k\) is the number of odd (resp. even) \(i\) such that \(r_i > 0\).

Here we note that for \(G = GL_N(\mathbb{C})\), the centralizer \(Z_G(S)\) is connected for any subset \(S\) of \(G\).

We say that a unipotent element \(u\) is distinguished if all maximal tori of \(\text{Cent}(u,G)\) are contained in the center of \(G^o\), the connected component of the identity. This is equivalent to the fact that the unipotent orbit \(O\) of \(u\) does not meet any proper Levi subgroup of \(G\) (Spaltenstein [Sp, p. 67]). (I.e., if \(L\) is a Levi subgroup of a parabolic subgroup of \(G\) and \(u \in L\) for a \(u \in O\), then \(L^o = G^o\).) If \(G = O_{2n+1}(\mathbb{C})\), then \(G^o = SO_{2n+1}(\mathbb{C})\) and \(G^o\) has trivial center. By Carter [C], for \(G = O_{2n+1}(\mathbb{C})\) or \(O_{2n}(\mathbb{C})\), if \(u\) is a unipotent element with Jordan blocks \((1_{r_1}, 2_{r_2}, ... )\), then the reductive part of the connected centralizer \(\text{Cent}(u,G)^o\) is of type

\[
\prod_{i \text{ even}} C_{r_i/2} \times \prod_{i \text{ odd}, r_i \text{ even}} D_{r_i/2} \times \prod_{i \text{ odd}, r_i \text{ odd}} B_{(r_i-1)/2}.
\]

Therefore, \(O\) is a distinguished unipotent class if and only if it has Jordan blocks \((1_{r_1}, 3^{r_3}, 5^{r_5}, ... )\), where \(r_i = 0\) or \(1\).

**Jacobson-Morozov Theorem.** Suppose \(u\) is a unipotent element in a semi-simple algebraic group \(G\). Then there exists a homomorphism \(\phi : SL_2 \longrightarrow G\) such that \(\phi \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = u\).

Here, replacing \(\phi\) by a conjugate under \(G\), we can assume that \(\phi \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}\) is in the closure of the positive Weyl chamber in the maximal torus. In fact, by the theory of weighted Dynkin diagrams (cf. Section 5.6 of [C]), \(\phi \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}\) is uniquely determined by the unipotent orbit of \(u\) as follows (Carter [C, p. 395]):

Suppose \(O\) has Jordan blocks \((d_1, d_2, d_3, ... )\). For each \(d_i\), we take the set of integers \(d_i - 1, d_i - 3, ... , 3 - d_i, 1 - d_i\). We then take the union of these sets for all \(d_i\) and write this union as \((\xi_1, \xi_2, \xi_3, ... )\) with \(\xi_1 \geq \xi_2 \geq \xi_3 \geq ... \).

Then

\[
(2.1) \quad \phi \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} = \text{diag}(a^{\xi_1}, a^{\xi_2}, a^{\xi_3}, ... ).
\]

**Lemma ([B-V, Prop. 2.4]).** Let \(u\) be a unipotent element and \(\phi : SL_2 \longrightarrow G\) be a homomorphism such that \(\phi \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = u\). Let \(S_\phi = \text{Cent}(\text{im}\phi, G) \subset S_u = \text{Cent}(u,G)\) and \(U^u\) be the unipotent radical of \(S_u\). Then:

(1) \(S_u = S_\phi \cdot U^u\), a semi-direct product. \(S_\phi\) is reductive.
The inclusion \( S_\varphi \subset S_u \) induces an isomorphism between \( S_\varphi/S_\varphi^0 Z_G \) and \( S_u/S_u^0 Z_G \).

### 3. Quadratic unipotent Arthur parameters.

We follow Moeglin [M3]. Let \( F \) be a number field and let \( W_F \) be the global Weil group of \( F \). Let \( G = Sp_{2n}, SO_{2n+1}, O_{2n} \) and we can take the dual group \( G^* = O_{2n+1}(\mathbb{C}), Sp_{2n}(\mathbb{C}), O_{2n}(\mathbb{C}) \). An Arthur parameter is a homomorphism

\[
\psi : W_F \times SL_2(\mathbb{C}) \mapsto G^*,
\]

with the following properties: (The usual definition of Arthur parameter uses Langlands’ hypothetical group \( L_F \) instead of \( W_F \). But since we are only dealing with Langlands’ quotients which come from principal series, \( W_F \) is enough.)

1. \( \psi(W_F) \) is bounded and included in the set of semi-simple elements of \( G^* \).
2. The restriction of \( \psi \) to \( SL_2(\mathbb{C}) \) is algebraic.
3. Composing \( \psi|_{W_F} \) with the determinant of \( G^* \) gives a quadratic character of \( W_F \), denoted by \( \text{det} \psi \). We want \( \text{det} \psi = 1 \).

We call an Arthur parameter quadratic unipotent if the following condition is satisfied:

4. \( \psi|_{W_F} \) is trivial on the intersection of the kernels of the quadratic characters of \( W_F \).

Because of conditions (1), and (4), the action of \( \psi(W_F) \) gives an orthogonal decomposition:

\[
\mathbb{C}^{2n+1}, \mathbb{C}^{2n} = V_1 \oplus \cdots \oplus V_k \oplus V_0,
\]

where \( \dim V_0 = 2r_0 + 1 \) or \( 2r_0 \), \( \dim V_i = 2r_i, 2r_0 + 1 + 2r_1 + \cdots + 2r_k = 2n + 1, 2n, r_1 \geq \cdots \geq r_k \) and \( V_i \) is the eigenspace with eigenvalue \( \mu_i \). Here \( \mu_1, \ldots, \mu_k \) are non-trivial distinct quadratic grössecharacters of \( F \), viewed as characters of \( W_F \), and \( \dim V_i, i = 1, \ldots, k \), being even comes from condition (3).

The parameter \( \psi \) factors through \( \prod_{i=0}^k G_i^* \), where

\[
G_i^* = \begin{cases} O(V_i), & \text{if } G = Sp_{2n}, O_{2n} \\ Sp(V_i), & \text{if } G = SO_{2n+1} \end{cases}.
\]

\[
\psi : W_F \times SL_2(\mathbb{C}) \mapsto \prod_{i=0}^k G_i^*.
\]

1. \( W_F \) is mapped into the product of centers of \( G_i^* \):

\[
\psi|_{W_F} : w \mapsto 1 \times \mu_1(w) \times \cdots \times \mu_k(w) \in \{\pm 1\} \times \{\pm 1\} \times \cdots \times \{\pm 1\},
\]

where \( \{\pm 1\} \) is the center of \( G_i^* \), for \( i = 0, \ldots, k \).
(2) By Jacobson-Morozov theorem, \( \psi|_{SL_2(\mathbb{C})} \) defines a unipotent orbit of \( G^* \) of the form
\[
\prod_{i=0}^{k} O_i,
\]
where \( O_i \) is a unipotent orbit of \( G_i^* \). Inside \( O_i \) we fix an element \( u_i \) such that
\[
\psi \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \prod_{i=0}^{k} u_i.
\]

Let \( S_\psi = \text{Cent}(im \psi, G^*) \) and
\[
C_\psi = S_\psi / S_\psi^o Z_{G^*}.
\]

We know that \( S_\psi \) is a maximal reductive subgroup of \( \prod_{i=0}^{k} \text{Cent}(u_i, G_i^*) \). Therefore \( S_\psi^o = 1 \), i.e., \( S_\psi \) is finite if and only if each \( u_i \) is a distinguished unipotent element in \( G_i^* \). Especially, since \( O_2(\mathbb{C}) \) has no distinguished unipotent element, we have:

**Lemma.** Let \( \psi \) be a quadratic unipotent Arthur parameter. Suppose \( S_\psi^o = 1 \). Then \( r_k \geq 2 \).

Now it is clear that \( S_\psi / S_\psi^o Z_{G^*} \) is equal to
\[
\text{Cent}(u_0, G_0^*) / \text{Cent}(u_0, G_0^*)^o Z_{G_0^*} \prod_{i=1}^{k} \text{Cent}(u_i, G_i^*) / \text{Cent}(u_i, G_i^*)^o.
\]

Here \( \text{Cent}(u_i, G_i^*) / \text{Cent}(u_i, G_i^*)^o \) is \( t \) product of \( \mathbb{Z}/2\mathbb{Z} \), where \( t \) is the number of \( i \) with \( r_i > 0 \) in Jordan blocks.

For each place \( v \) of \( F \), we have a map \( \psi_v = \psi|_{W_F \times SL_2(\mathbb{C})} \). As in global case, we can then define \( S_{\psi_v} \). But in the local case, \( \mu_{iv} \) may not be distinct. Suppose \( \mu_{1v} = \mu_{2v} = \mu_{iv} \) for \( i = 3, \ldots, k \) and \( \mu_{1v} \neq 1 \). Then in the above formula,
\[
\text{Cent}(u_1, G_1^*) / \text{Cent}(u_1, G_1^*)^o \times \text{Cent}(u_2, G_2^*) / \text{Cent}(u_2, G_2^*)^o
\]
must be replaced by
\[
\text{Cent}(u_1 \times u_2, G_{12}^*) / \text{Cent}(u_1 \times u_2, G_{12}^*)^o,
\]
where \( G_{12}^* = O(V_1 \oplus V_2) \) or \( Sp(V_1 \oplus V_2) \). Now we recall Moeglin’s reformulation of Arthur’s conjecture ([M3]): It is a part of local Arthur’s conjecture that for each irreducible character \( \eta_v \) of \( C_{\psi_v} \), there exists an irreducible representation \( \pi(\psi_v, \eta_v) \). For each \( v \), let \( \Pi_{\psi_v} \) be the set of \( \pi(\psi_v, \eta_v) \).

We define the global Arthur packet \( \Pi_\psi \) to be the set of irreducible representations \( \pi = \otimes_v \pi_v \) of \( G(\mathbb{A}) \) such that for each \( v \), \( \pi_v \) belongs to \( \Pi_{\psi_v} \).
Arthur’s Conjecture (Global).

1. The representations in the packet corresponding to \( \psi \) may occur in the discrete spectrum if and only if \( S_\psi \) is finite, i.e., \( S_\psi = 1 \). We call such an Arthur parameter elliptic.

2. For an elliptic Arthur parameter \( \psi \), any \( \pi \in \Pi_\psi \) occurs in \( L^2_{\text{dis}}(G(F)\backslash G(\mathbb{A})) \) if and only if

\[
\sum_{x \in C_\psi} \prod_v \eta_v(x_v) \neq 0,
\]

where \( \pi = \otimes_v \pi(\psi_v, \eta_v), \ x = (x_v) \). Note that, if \( C_\psi \) is abelian, (3.1) is equivalent to

\[
\prod_v \eta_v|_{C_\psi} = 1,
\]

when \( \psi|_{W_F} \) is trivial.

We define

\[
L^2(G(F)\backslash G(\mathbb{A}))_\psi = \Pi_\psi \cap L^2_{\text{dis}}(G(F)\backslash G(\mathbb{A})).
\]

Remark 3.1. For split classical groups, \( C_\psi \) is abelian and Moeglin [M2] proved the multiplicity formula (3.2). However, in general, \( C_\psi \) is not abelian. For example, in the case of split exceptional group of type \( G_2 \), \( C_\psi \) can be \( S_3 \), the symmetric group on 3 letters. This fact leads to a “bizarre” multiplicity formula in Moeglin-Waldspurger’s work on the residual spectrum of \( G_2 \) [M-W1, Ki2].

Let \( \Pi_{\text{res}_v} \) be the subset of \( \Pi_{\psi_v} \) which consists of the local components of the residual spectrum. It is of great importance to parametrize the elements in \( \Pi_{\text{res}_v} \) and prove the multiplicity formula (3.1), i.e., we will construct a set of characters \( C_{\text{res}_v} \subset \hat{C}_{\psi_v} \) and each character of \( C_{\text{res}_v} \) gives an element in \( \Pi_{\text{res}_v} \).

Remark 3.2. To any Arthur parameter \( \psi \), Arthur associates a Langlands’ parameter \( \phi_\psi : W_F \longrightarrow G^* \) as follows:

\[
\phi_\psi(w) = \psi \left( w, \begin{pmatrix} |w|^{\frac{1}{2}} & 0 \\ 0 & |w|^{-\frac{1}{2}} \end{pmatrix} \right).
\]

Let \( S_{\phi_\psi} = \text{Cent} \left( \text{im} \phi_\psi, G^* \right) \) and \( C_{\phi_\psi} = S_{\phi_\psi}/S_{\phi_\psi}^0 Z_{G^*} \). For each place \( v \), we have \( S_{\phi_{\psi_v}}, C_{\phi_{\psi_v}} \). For each \( v \), there is a natural surjection \( C_{\psi_v} \rightarrow C_{\phi_{\psi_v}} \). The parameter \( \phi_{\psi_v} \) gives a \( L \)-packet \( \Pi_{\phi_{\psi_v}} \) which consists of Langlands’ quotients. It is a part of Arthur’s original local conjecture that for each place \( v \), there is a pairing \( \langle \cdot, \cdot \rangle \) on \( C_{\phi_{\psi_v}} \times \Pi_{\phi_{\psi_v}} \) and an enlargement \( \Pi_{\psi_v} \) of \( \Pi_{\phi_{\psi_v}} \) which allows an extension of \( \langle \cdot, \cdot \rangle \) to \( C_{\psi_v} \times \Pi_{\psi_v} \) such that \( \pi \in \Pi_{\phi_{\psi_v}} \subset \Pi_{\psi_v} \) if and only if the function \( \langle \cdot ; \pi \rangle \) lies in the image of \( \hat{C}_{\phi_{\psi_v}} \) in \( \hat{C}_{\psi_v} \). Since \( C_{\psi_v} \) is
There is a natural pairing \( \Phi(\langle \alpha e \psi \rangle C) \) for the space of functions of the form \( B \) of functions \( \Phi \) on \( G \) with respect to \( (.,.) \) is the standard inner product in \( \Phi(G,T) \). Let \( \omega_i = e_1 + \cdots + e_i \). Then \( \omega_1, \ldots, \omega_n \) are the fundamental weights of \( G \) with respect to \( (G,T) \). (If \( G = Sp_{2n} \), since \( G \) is simply connected, \( X(T) = Z\omega_1 + \cdots + Z\omega_n \) and \( X^+(T) = Z\sigma_1^\vee + \cdots Z\sigma_n^\vee \).) Set \( \mathfrak{a}^* = X(T) \otimes \mathbb{R}, \mathfrak{a}^*_C = X(T) \otimes \mathbb{C}, \) and \( \mathfrak{a} = X^+(T) \otimes \mathbb{R} = \text{Hom}(X(T),\mathbb{R}) \).

The positive Weyl chamber in \( \mathfrak{a}^* \) is

\[
C^+ = \{ \lambda \in \mathfrak{a}^* | \langle \lambda, \alpha^\vee \rangle > 0, \text{ for all } \alpha \text{ positive roots} \}
\]

\[
= \left\{ \sum_{i=1}^{n} a_i \omega_i | a_i > 0 \right\}.
\]

We fix a non-trivial additive character \( \psi_F = \otimes_v \psi_{F_v} \) of \( \mathbb{A}/F \) and let \( L(z, \mu) \) be the Hecke \( L \)-function with the ordinary \( \Gamma \)-factor so that it satisfies the functional equation \( L(z, \mu) = \epsilon(z,\mu) L(1-z, \mu^{-1}) \), where \( \epsilon(z,\mu) = \prod_v \epsilon(z, \mu_v, \psi_{F_v}) \) is the usual \( \epsilon \)-factor. If \( \mu \) is the trivial character \( \mu_0 \), then we write simply \( L(z) \) for \( L(z, \mu_0) \). We have the Laurent expansion of \( L(z) \) at \( z = 1 \):

\[
L(z) = \frac{c(F)}{z - 1} + a + \cdots.
\]

4. Eisenstein series attached to Borel subgroups.

Let \( \alpha^\vee \) be the coroot corresponding to \( \alpha \in \Phi^+(G,T) \). Explicitly, for \( \alpha = e_i - e_j, \alpha^\vee(l) = t(1, \ldots, l_i, \ldots, l_j, \ldots, 1) \in T(F) \) for \( 1 \leq i < j \leq n \). For \( \alpha = e_i + e_j, \alpha^\vee(l) = t(1, \ldots, l_i, \ldots, l_j, \ldots, 1) \), for \( 1 \leq i < j \leq n \). For \( \alpha = 2e_i, \alpha^\vee(l) = t(1, \ldots, l_i, \ldots, 1) \) for \( 1 \leq i \leq n \). Here dots represent 1.

4.1. Definition of Eisenstein series. For \( \mu_1, \ldots, \mu_n \) gröszencharacters of \( F \), we define a character \( \chi = \chi(\mu_1, \ldots, \mu_n) \) of \( T(\mathbb{A}) \) by

\[
\chi(\mu_1, \ldots, \mu_n)(t(l_1, \ldots, l_n)) = \mu_1(l_1) \cdots \mu_n(l_n).
\]

Let \( B = TU \), where \( U \) is the unipotent radical. Let \( I(\chi) \) be the space of functions \( \Phi \) on \( G(\mathbb{A}) \) satisfying \( \Phi(u_1 g) = \chi(t) \Phi(g) \) for any \( u_1 \in U(\mathbb{A}) \), \( t \in T(\mathbb{A}) \) and \( g \in G(\mathbb{A}) \). Then for each \( \lambda \in \mathfrak{a}_C^* \), the representation of \( G(\mathbb{A}) \) on the space of functions of the form

\[
g \mapsto \Phi(g) \exp(\lambda + \rho_B, H_B(g)), \quad \Phi \in I(\chi),
\]
is equivalent to \( I(\lambda, \chi) = \text{Ind}_{B^1 G} \chi \otimes \exp(\lambda, H_B) \). We form the Eisenstein series:

\[
E(g, f, \lambda) = \sum_{\gamma \in B(F) \backslash G(F)} f(\gamma g),
\]

where \( f = \Phi e^{(\lambda + \rho_B, H_B)} \in I(\lambda, \chi) \) and \( \rho_B \) is the half-sum of positive roots, i.e., \( \rho_B = \omega_1 + \cdots + \omega_n \). It converges absolutely for \( \text{Re} \lambda \in \mathbb{C}^+ + \rho_B \) and extends to a meromorphic function of \( \lambda \). It is an automorphic form and the constant term of \( E(g, f, \lambda) \) along \( B \) is given by

\[
E_0(g, f, \lambda) = \int_{U(F) \backslash U(A)} E(ug, f, \lambda) \, du = \sum_{w \in W} M(w, \lambda, \chi) f(g),
\]

where \( W \) is the Weyl group of \( T \) and

\[
M(w, \lambda, \chi)f(g) = \int_{wU(A)w^{-1}U(A) \cap U(A)} f(w^{-1}ug) \, du.
\]

Then \( M(w, \lambda, \chi) \) defines an intertwining map from \( I(\lambda, \chi) \) to \( I(w\lambda, w\chi) \) and satisfies a functional equation of the form

\[
M(w_1 w_2, \lambda, \chi) = M(w_1, w_2 \lambda, w_2 \chi) M(w_2, \lambda, \chi).
\]

Let \( S \) be a finite set of places of \( F \), including all the archimedean places such that for every \( v \notin S, \chi_v, \psi_{F_v} \) are unramified and if \( f = \otimes f_v \) for \( v \notin S \), \( f_v \) is the unique \( K_v \)-fixed function normalized by \( f_v(e_v) = 1 \). We have

\[
M(w, \lambda, \chi) = \otimes_v A(w, \lambda, \chi_v).
\]

Then by applying Gindikin-Karpelevich method, we can see that for \( v \notin S \),

\[
A(w, \lambda, \chi_v)f_v = \prod_{\alpha > 0, w \alpha < 0} \frac{L(\langle \lambda, \alpha^\vee \rangle, \chi_v \circ \alpha^\vee)}{L(\langle \lambda, \alpha^\vee \rangle + 1, \chi_v \circ \alpha^\vee)} \tilde{f}_v,
\]

where \( \tilde{f}_v \) is the \( K_v \)-fixed function in the space of \( I(w\lambda, w\chi) \). For any \( v \), let

\[
r_v(w) = \prod_{\alpha > 0, w \alpha < 0} \frac{L(\langle \lambda, \alpha^\vee \rangle, \chi_v \circ \alpha^\vee)}{L(\langle \lambda, \alpha^\vee \rangle + 1, \chi_v \circ \alpha^\vee)\epsilon(\langle \lambda, \alpha^\vee \rangle, \chi_v \circ \alpha^\vee, \psi_{F_v})}.
\]

We normalize the intertwining operators \( A(w, \lambda, \chi_v) \) for all \( v \) by

\[
A(w, \lambda, \chi_v) = r_v(w)R(w, \lambda, \chi_v).
\]

Let \( R(w, \lambda, \chi) = \otimes_v R(w, \lambda, \chi_v) \) and

\[
r(w) = \prod_v r_v(w) = \prod_{\alpha > 0, w \alpha < 0} \frac{L(\langle \lambda, \alpha^\vee \rangle, \chi \circ \alpha^\vee)}{L(\langle \lambda, \alpha^\vee \rangle + 1, \chi \circ \alpha^\vee)\epsilon(\langle \lambda, \alpha^\vee \rangle, \chi \circ \alpha^\vee)}.
\]

Here \( R(w, \lambda, \chi) \) satisfies the functional equation

\[
R(w_1 w_2, \lambda, \chi) = R(w_1, w_2 \lambda, w_2 \chi) R(w_2, \lambda, \chi),
\]
for any $w_1, w_2$. We know, by Winarsky [W] for $p$-adic cases and Shahidi [Sh3, p. 110] for real and complex cases, that

$$A(w, \lambda, \chi_v) \prod_{\alpha > 0, \omega \alpha < 0} L_v(\langle \lambda, \alpha^\vee \rangle, \chi_v \circ \alpha^\vee)^{-1}$$

is holomorphic for any $v$. So for any $v$, $R(w, \lambda, \chi_v)$ is holomorphic for $\lambda$ with $\text{Re}(\langle \lambda, \alpha^\vee \rangle) > -1$, for all positive $\alpha$ with $\omega \alpha < 0$. For $\chi = \chi(\mu_1, \ldots, \mu_n)$,

$$\chi \circ \alpha^\vee = \begin{cases} \mu_i \mu_j^{-1}, & \text{for } \alpha = e_i - e_j \\ \mu_i \mu_j, & \text{for } \alpha = e_i + e_j \text{ and } i < j \\ \mu_i, & \text{for } \alpha = 2e_i \\ \mu_i^2, & \text{for } \alpha = e_i. \end{cases}$$

For $\alpha \in \Phi^+$, let $S_\alpha = \{\lambda \in a^*_C|\langle \lambda, \alpha^\vee \rangle = 1\}$. We call $S_\alpha$ a singular hyperplane. We say that $E(g, f, \lambda)$ has a pole of order $n$ at $\lambda_0$ if $\lambda_0$ is the intersection of $n$ singular hyperplanes in general position on which the Eisenstein series has a simple pole.

For $\Psi \subset \Phi^+$, we define $r(w, \lambda, \Psi)$ by

$$r(w, \lambda, \Psi) = \prod_{\alpha \in \Psi, \omega \alpha < 0} \frac{L(\langle \lambda, \alpha^\vee \rangle, \chi \circ \alpha^\vee)}{L(\langle \lambda, \alpha^\vee \rangle + 1, \chi \circ \alpha^\vee)\epsilon(\langle \lambda, \alpha^\vee \rangle, \chi \circ \alpha^\vee)}.$$  

Observe that we have suppressed the dependence of $r(w, \lambda, \Psi)$ on $\chi$.

### 4.2. Definition of pseudo-Eisenstein series.

We follow Moeglin [M1] and introduce pseudo-Eisenstein series. For $T$ a maximal torus, a character $\chi$ of $T(A)/T(F)$ defines a cuspidal representation of $T$. For any $w \in W$, $wTw^{-1} = T$ and so $(T, w\chi)$ is conjugate to $(T, \chi)$. Let $I(\chi)$ be the set of entire functions $\phi$ of Paley-Wiener type such that $\phi(\lambda) \in I(\lambda, \chi)$ for each $\lambda$. Let

$$\theta_\phi(g) = \left(\frac{1}{2\pi i}\right)^n \int_{\text{Re}\lambda = \lambda_0} E(g, \phi(\lambda), \lambda) d\lambda,$$

where $\lambda_0 \in \rho_B + C^+$. Let

$$L^2(G(F)\backslash G(A))_{(T, \chi)},$$

be the space spanned by $\theta_\phi$ for all $\phi \in I(w\chi)$ as $w\chi$ runs through all distinct conjugates of $\chi$. Let $L^2_{\text{dis}}(G(F)\backslash G(A))_{(T, \chi)}$ be the discrete part of $L^2(G(F)\backslash G(A))_{(T, \chi)}$. It is the set of iterated residues of $E(g, \phi(\lambda), \lambda)$ of order $n$ and the residual spectrum attached to $(T, \chi)$. In order to decompose $L^2_{\text{dis}}(G(F)\backslash G(A))_{(T, \chi)}$, we use the inner product formula of two pseudo-Eisenstein series: Let $\chi$ and $\chi'$ be conjugate characters and $\phi \in I(\chi)$,
\( \phi' \in I(\chi') \). Then
\[
\langle \theta_\phi, \theta_{\phi'} \rangle = \frac{1}{(2\pi i)^n} \int_{R \lambda = \lambda_0} \sum_{w \in W(\chi, \chi')} (M(w^{-1}, -w\bar{\lambda}, w\chi) \phi'(-w\bar{\lambda}), \phi(\lambda)) \, d\lambda
\]
\[
= \frac{1}{(2\pi i)^n} \int_{R \lambda = \lambda_0} \sum_{w \in W(\chi, \chi')} (M(w, \lambda, \chi) \phi(\lambda), \phi'(-w\bar{\lambda})) \, d\lambda,
\]
where \( W(\chi, \chi') = \{ w \in W | w\chi = \chi' \} \). We can assume, after conjugation, \( \chi = \chi(\mu_1, \mu_2, \ldots, \mu_n) \), \( r_0 + \cdots + r_k = n \), \( r_1 \geq \cdots \geq r_k \), where \( \mu_i \)'s are non-trivial distinct quadratic grôssencharacters. Let \( D \) be the set of distinguished coset representatives in Proposition 6.2.2. Then \( \{ d\chi | d \in D \} \) is the set of distinct conjugates of \( \chi \).

Following Moeglin [M1], we consider the constant term of the pseudo-Eisenstein series:
\[
E_{0}^{PS}(\phi, \lambda, \chi) = \sum_{w \in W} M(w^{-1}, w\lambda, w\chi) \phi(w\lambda).
\]
Here \( \sum_{w \in W} M(w^{-1}, w\lambda, w\chi) \phi(w\lambda) \) signifies
\[
\sum_{d \in D} \left( \sum_{w \in W(\chi, d\chi)} M(w^{-1}, w\lambda, w\chi) \phi_d(w\lambda) \right),
\]
where \( \phi_d \in I(d\chi) \). Here \( M(w^{-1}, w\lambda, w\chi) = r(w^{-1}, w\lambda, w\chi) R(w^{-1}, w\lambda, w\chi) \).
We show the following:

**Lemma 4.1.**

1. \( r(w, \bar{\lambda}, \chi) = r(w, \lambda, \chi) \),
2. \( r(w^{-1}, w\lambda, w\chi) = r(w, -\lambda, \chi^{-1}) \).

**Proof.**

(1) is immediate. We note that if \( \chi = \chi(\mu_1, \ldots, \mu_n) \), then \( \chi^{-1} = \chi(\mu_1^{-1}, \ldots, \mu_n^{-1}) \). Then
\[
r(w^{-1}, w\lambda, w\chi) = \prod_{\alpha > 0, w^{-1} \alpha < 0} L((w\lambda, \alpha^\vee), (w\chi) \circ (w\alpha)) L((w\lambda, \alpha^\vee) + 1, (w\chi)) \circ (w\alpha^\vee).
\]
Let \( \gamma = -w^{-1} \alpha \). Then \( \{ \alpha > 0 | w^{-1} \alpha < 0 \} = -\{ \gamma > 0 | w\gamma < 0 \} \).

If \( \alpha = e_i \pm e_j \), \( (w\lambda, \alpha^\vee) = (w\lambda, \alpha) = (\lambda, w^{-1} \alpha) = -\gamma \), \( \gamma = -\lambda, \gamma \). If \( \alpha = 2e_i \), then \( (w\lambda, \alpha^\vee) = \frac{1}{2}(w\lambda, \alpha) = \frac{1}{2}(\lambda, w^{-1} \alpha) = -\lambda, \gamma^\vee \). Also we can see that \( w\chi \circ (w\alpha)^\vee = \chi \circ \alpha^\vee \). Therefore, \( (w\chi) \circ \alpha^\vee = \chi \circ (w^{-1} \alpha)^\vee = \chi \circ (-\lambda) = \chi^{-1} \circ \gamma^\vee \). Hence (2) follows.

If \( \chi \) is a quadratic character, i.e., \( \chi^{-1} = \chi \), then
\[
E_{0}^{PS}(\phi, \lambda, \chi) = \sum_{w \in W} r(w, -\lambda, \chi) R(w^{-1}, w\lambda, w\chi) \phi(w\lambda).
\]
Even though Moeglin completely analyzed the continuous spectrum as well, we will only be interested in the discrete spectrum. Let \( \langle \cdot, \cdot \rangle_{\text{dis}} \) be the inner product on \( L^2_{\text{dis}}(G(F)\backslash G(\mathbb{A}))_{(T,\chi)} \). The discrete spectrum is spanned by the square integrable iterated residues of order \( n \) of \( E^0_{PS}(\phi, \lambda, \chi) \). More precisely, \( E^0_{PS}(\phi, \lambda, \chi) \) has a simple pole on \( n \) singular hyperplanes \( H_1, \ldots, H_n \) which are the poles of the \( L \)-functions in the numerator of the normalizing factor and whose intersection is \( \lambda_0 \). Then we have to take iterated residue at \( H_1, H_1 \cap H_2, \ldots, H_1 \cap \cdots \cap H_n \).

Let us summarize Moeglin’s arguments. Here \( \chi = 1 \) and we suppress it: Moeglin showed that only the points \( \lambda_p \), where \( p \in P(O) \) and \( O \) is a distinguished unipotent orbit (see Section 5), contribute the square integrable residues. She also found \( n \) singular hyperplanes \( H_1, \ldots, H_n \) whose intersection is \( \lambda_p \). In order to take the iterated residue at \( H_1, H_1 \cap H_2, \ldots, H_1 \cap \cdots \cap H_n \), Moeglin defined the following function, for a certain Weyl group element \( w_p \),

\[
l_p(\phi, \lambda) = \sum_{w \in W} r(w, -\lambda) R(w_p w^{-1}, w \lambda) \phi(w \lambda).
\]

Poles of \( l_p(\phi, \lambda) \) in a neighborhood of \( \lambda_p \) are contained in the local intertwining operators. She showed that \( l_p(\phi, \lambda_p) \) can be defined inductively by restricting \( l_p(\phi, \lambda) \) to \( H_1, H_1 \cap H_2, \ldots, H_1 \cap \cdots \cap H_n \). \( l_p(\phi, \lambda_p) \) is the iterated residue of \( E^0_{PS}(\phi, \lambda) \) and spans the residual spectrum \( L^2_{\text{dis}}(G(F)\backslash G(\mathbb{A}))_{(T,1)} \). \( l_p(\phi, \lambda_p) \) belongs to \( \otimes_v' R_v(w_p, \lambda_p) I_v(\lambda_p) \). She analyzed the image of the local intertwining operator \( R_v(w_p, \lambda_p) I_v(\lambda_p) \) and verified Arthur’s multiplicity formula. We will review her result in detail in Section 5.

In Section 6, we calculate the constant term of the pseudo-Eisenstein series for a general \( \chi \) and show that we can reduce the calculation of the poles to that of the trivial character case and apply Moeglin’s result.

### 5. The trivial character case; summary of Moeglin’s result.

Moeglin [M1] completely analyzed the residual spectrum of \( Sp_{2n} \), \( SO_{2n+1} \) and split \( O_{2n} \), attached to the trivial character of the maximal torus. Her results also completely describe continuous spectrum. Her results are that the residual spectrum of split classical groups attached to the trivial character of the maximal torus is parametrized by distinguished unipotent orbits \( O \) in \( G^* \) and Springer \( (O) \), which is a set of characters of \( A(u) \) which are in the image of the Springer correspondence and \( A(u) = C(u)/C(u)^0 \), \( C(u) = \text{Cent}(u, G^*) \), \( u \in O \). Recall that the Springer correspondence is an injective map from the set of irreducible characters of \( W \), the Weyl group, into the set of pairs \( (O, \eta) \), where \( O \) is a unipotent orbit and \( \eta \) is an irreducible character of \( A(u) \). We refer [Ja-Ki] for the review of her results. We just remark that since we are only interested in discrete spectrum, we
need to consider distinguished unipotent orbits, i.e., \( r = 0 \). In that case we get an explicit order of the set Springer \((\mathcal{O})\).

**Lemma 5.1.** If \( \mathcal{O} = (q_1, \ldots, q_s) \) is a distinguished unipotent orbit in \( O_{2n}(\mathbb{C}) \) or \( O_{2n+1}(\mathbb{C}) \) (thus \( q_i \) odd), then

\[
|\text{Springer}(\mathcal{O})| = sC_{\frac{s}{2}}.
\]

**Proof.** Let \( \text{Symb}(\mathcal{O}) = (a_1, \ldots, a_s) \) (see [M1, p. 663]), where \( a_i = \lceil \frac{q_i}{2} \rceil + i \).

We note that \( a_i \leq a_{i+1} - 2 \) for all \( i \). (If we set \( q_i = 2r_i + 1 \), \( q_{i+1} = 2r_{i+1} + 1 \), \( r_i < r_{i+1} \), then \( a_i = r_i + i \) and \( a_{i+1} = r_{i+1} + i + 1 \).)

Let \( \mathcal{S} = (a'_1, \ldots, a'_s) \) be a class of \( \text{Symb}(\mathcal{O}) \). Then it satisfies: \( (a'_1, \ldots, a'_s) \) coincides with \( \text{Symb}(\mathcal{O}) \) and \( a'_i \leq a'_{i+2} - 2 \) for \( i = 1, \ldots, s-2 \). For such class \( \mathcal{S} \), define \( \eta \), a character of \( A(\mathcal{O}) \) by:

\[
\eta(\sigma(q_i)) = 1 \quad \text{if and only if} \quad \{ j : a'_j = a_i \}
\]

It defines a bijection between classes of \( \text{Symb}(\mathcal{O}) \) and Springer \((\mathcal{O})\). It is clear that the number of classes of \( \text{Symb}(\mathcal{O}) \) is \( sC_{\frac{s}{2}} \).

If \( s \) is odd, let \( s = 2r + 1 \). Then \( sC_{\frac{s}{2}} = 2r + 1 \sim 2 \cdot 2^{2r} \).

We interpret Moeglin’s results in terms of Arthur’s parameters:

**Theorem 5.2 (Moeglin).** Let \( G = \text{Sp}_{2n}, \text{SO}_{2n+1}, O_{2n} \). Then the residual spectrum of \( G \) attached to the trivial character of the torus is parametrized by unipotent Arthur parameters

\[
\psi : SL_2(\mathbb{C}) \longrightarrow G^*,
\]

which are given by distinguished unipotent orbits in \( G^* = O_{2n+1}(\mathbb{C}), \text{Sp}_{2n}(\mathbb{C}), \text{O}_{2n}(\mathbb{C}) \). More specifically, for \( \mathcal{O} \) a distinguished unipotent orbit, let \( C_{\text{res}} = \text{Springer}(\mathcal{O}) \subset C_{\psi_v} \) and \( \Pi_{\text{res}} = \text{Unip}_v(\mathcal{O}) \subset \Pi_{\psi_v} \) for all finite places. For each \( X \in \Pi_{\text{res}} \), there is a character \( \eta_X \in \text{Springer}(\mathcal{O}) \) and it satisfies Arthur’s conjecture. i.e.,

\[
L^2_{\text{dis}}(G(F)\backslash G(\mathbb{A}))_{(r,1)} \cap L^2(G(F)\backslash G(\mathbb{A}))\psi,
\]

is the set of \( \pi = \otimes_v X_v \), where \( X_v \) satisfies the following conditions:

1. There exists \( \mathfrak{p} \in P(\mathcal{O}) \) such that \( X_v \in \text{Unip}_v(\mathfrak{p}) \) for all \( v \), i.e., \( \eta_X \) factors through \( \mathbb{A}(\mathfrak{p}) \).
2. \( X_v \) is spherical for almost all \( v \) and archimedean places.
3. \( \prod_v \eta_{X_v} \) is trivial on \( A(\mathcal{O}) \).

**6. Arbitrary character case.**

We first do the simple case to explain our method.
6.1. The case \( G = Sp_{2n} \) and \( \chi = \chi(\mu, \ldots, \mu) \), \( \mu \) non-trivial and quadratic grössencharacter. Let \( \Phi_1 = \{ e_i \pm e_j, \ 1 \leq i < j \leq n \} \) and \( \Phi_2 = \{ 2e_i, \ i = 1, \ldots, n \} \). Then for \( \alpha \in \Phi_1 \), \( \chi \circ \alpha^\vee = 1 \) and for \( \alpha \in \Phi_2 \), \( \chi \circ \alpha^\vee \) is non-trivial. Then the constant term of the pseudo-Eisenstein series is given by
\[
E_0^{PS}(\lambda, \phi, \chi) = \sum_{w \in W} r(w, -\lambda, \chi) R(w^{-1}, w\lambda, w\chi) \phi(w\lambda)
\]
\[
= \sum_{w \in W} r(w, -\lambda, \Phi_1) r(w, -\lambda, \Phi_2) R(w^{-1}, w\lambda, w\chi) \phi(w\lambda).
\]
We note that for \( f \in \text{Ind}_{B}^{O_{2n}} \exp(\lambda, H_B(\_)) \),
\[
E_0^{PS}(\lambda, f) = \sum_{w \in W} r(w, -\lambda, \Phi_1) R(w^{-1}, w\lambda) f(w\lambda),
\]
is a pseudo-Eisenstein series for \( O_{2n} \) attached to the trivial character of the maximal torus. Since \( \chi \circ \alpha^\vee \) is non-trivial for \( \alpha \in \Phi_2 \), \( r(w, -\lambda, \Phi_2) \) is holomorphic.

Let \( O \) be a distinguished unipotent orbit in \( O_{2n}(\mathbb{C}) \) and \( p \in P(O) \). Let, for \( \phi \in PW \),
\[
l_p(\lambda, \phi, \chi) = \sum_{w \in W} r(w, -\lambda, \chi) R(w_p w^{-1}, w\lambda, w\chi) \phi(w\lambda).
\]
Since \( r(w, -\lambda, \Phi_1) \) is identically zero on \( V'(p) \) if \( w \notin W(\uparrow, p) \), the restriction of \( l_p(\lambda, \phi, \chi) \) to \( V'(p) \) is given by
\[
l_p(\phi, \lambda, \chi)|_{V'(p)} = \sum_{w \in W(\uparrow, p)} r(w, -\lambda, \Phi_1) r(w, -\lambda, \Phi_2) R(w_p w^{-1}, w\lambda, w\chi) \phi(w\lambda).
\]
By the definition of \( W(\uparrow, p) \), the poles of \( l_p(\phi, \lambda, \chi)|_{V'(p)} \) is contained in \( r(w, -\lambda, \Phi_1) \). Therefore we apply Moeglin’s global result for \( O_{2n} \) in the trivial character case, i.e.,
\[
\langle \theta_{\phi'}, \theta_{\phi} \rangle_{\text{dis}} = \sum_{O \subset O_{2n}(\mathbb{C})} \sum_{p \in P(O)} c_p \langle (l'_p|_{\Phi})(\phi', \lambda_p, \chi), l_p(\phi, \lambda_p, \chi) \rangle.
\]
We have
\[
l_p(\phi, \lambda_p, \chi) \in \otimes'_v R_v(w_p, \lambda_p, \chi) I(\lambda_p, \chi_v).
\]
From [Ja-Ki], we summarize the result on the image of the local intertwining operator \( R_v(w_p, \lambda_p, \chi) I(\lambda_p, \chi_v) \).

Theorem 6.1.1 (Theorem 3.4.7 of [Ja-Ki]). Let \( O \) be a distinguished unipotent orbit in \( O_{2n}(\mathbb{C}) \) and \( p \in P(O) \). Then \( R_v(w_p, \lambda_p, \chi) I(\lambda_p, \chi_v) \) is semisimple. Let \( \text{Unip}(p, \mu_v) \) be the set of direct summands and \( \text{Unip}(O, \mu_v) \) be the set of union of \( \text{Unip}(p, \mu_v) \). Then for each \( X \in \text{Unip}(O, \mu_v) \), there is a character \( \eta_X \in A(O) \) such that:

1. If \( \mu_v \neq 1 \), then \( C(O, \mu_v) = \{ \eta_X | X \in \text{Unip}(O, \mu_v) \} = \text{Springer}(O) \).
(2) If \( \mu_v = 1 \), then
\[
C(O, \mu_v) = \{ \eta_X | X \in \text{Unip}(O, \mu_v) \}
\]
\[
= \begin{cases} 
\text{Springer}(O), & \text{if } O \text{ does not contain } 1 \\
\text{Springer}(O'), & \text{if } O \text{ contains } 1,
\end{cases}
\]
where \( O' = O - \{ 1 \} \).

So we have:

**Theorem 6.1.2.** Let \( \chi = \chi(\mu, \ldots, \mu) \), \( \mu \) non-trivial quadratic gr"ossencharacter. Then the residual spectrum attached to \( \chi \) is parametrized by the distinguished unipotent orbits in \( O_{2n} \). More specifically, a distinguished unipotent orbit \( O \in O_{2n}(\mathbb{C}) \) and \( \chi \) give a quadratic unipotent Arthur parameter \( \psi \) and let \( C_{\text{res},\psi} = C(O, \mu_\psi) \subset C_{\psi} \) and \( \Pi_{\text{res},\psi} = \text{Unip}(O, \mu_\psi) \subset \Pi_{\psi} \) for all non-archimedean places. For each \( X \in \Pi_{\text{res},\psi} \), there is a character \( \eta_X \in C_{\text{res},\psi} \) and it satisfies Arthur’s conjecture, i.e.,
\[
L^2_{\text{dis}}(G(F) \backslash G(\mathbb{A}))_{(T, \chi)} \cap L^2(G(F) \backslash G(\mathbb{A}))_{\psi},
\]
is the set of \( \pi = \otimes'_v X_v \), where \( X_v \) satisfies the following conditions:

1. There exists \( p \in P(O) \) such that \( X_v \in \text{Unip}_v(p, \mu_v) \) for all \( v \).
2. \( X_v \) is spherical for almost all \( v \) and archimedean places.
3. \( \prod_v \eta_{X_v} \) is trivial on \( C_\psi \).

**6.2. General case.** For \( \chi \) a non-trivial character, we can assume, after conjugation, that \( \chi = \chi(\mu_1, \ldots, \mu_k, \ldots, \mu_k, 1, \ldots, 1), r_0 + \cdots + r_k = n, r_1 \geq \cdots \geq r_k \). We use the following notation throughout this section:

1. If \( G = \text{Sp}(2n) \), \( G' = G'_1 \times \cdots \times G'_k \times G'_0 \), where \( G'_i = O(2r_i) \) for \( i = 1, \ldots, k \), \( G'_0 = \text{Sp}(2r_0) \). Also we denote \( G^*_i = O(2r_i, \mathbb{C}) \) for \( i = 1, \ldots, k \).
2. If \( G = \text{SO}(2n + 1) \), \( G' = G'_1 \times \cdots \times G'_k \times G'_0 \), where \( G'_i = \text{SO}(2r_i + 1) \) for \( i = 1, \ldots, k \), \( G'_0 = \text{SO}(2r_0 + 1) \). Also we denote \( G^*_i = \text{Sp}(2r_i, \mathbb{C}) \) for \( i = 1, \ldots, k \).
3. If \( G = \text{O}(2n) \), \( G' = G'_1 \times \cdots \times G'_k \times G'_0 \), where \( G'_i = \text{O}(2r_i) \) for \( i = 1, \ldots, k \), \( G'_0 = \text{O}(2r_0) \). Also we denote \( G^*_i = \text{O}(2r_i, \mathbb{C}) \) for \( i = 1, \ldots, k \).

We recall some basic facts from [Ki-Sh]. Let \( E(g, \phi, \lambda) \) be the Eisenstein series associated to the character \( \chi \).

**Proposition 6.2.1.** The Eisenstein series has a pole of order \( n \) only if \( r_k \geq 2 \) and \( \mu_i \) is a quadratic gr"ossencharacter for \( i = 1, \ldots, k \).
We divide the set of positive roots $\Phi^+$ as follows:

$$
\Phi_1 = \{e_i \pm e_j, \quad 1 \leq i < j \leq r_1\}, \\
\Phi_2 = \{e_{r_1+i} \pm e_{r_1+j}, \quad 1 \leq i < j \leq r_2\}, \\
\vdots \\
\Phi_k = \{e_{r_1+\cdots+r_{k-1}+i} \pm e_{r_1+\cdots+r_{k-1}+j}, \quad 1 \leq i < j \leq r_k\}, \\
\Phi_0 = \{e_{r_1+\cdots+r_k+i} \pm e_{r_1+\cdots+r_k+j}, \quad 1 \leq i < j \leq r_0, \quad 2e_{r_1+\cdots+r_k+i}, \quad i = 1, \ldots, r_0\},
$$

$$
\Phi_D = \Phi^+ - \bigcup_{i=0}^k \Phi_k.
$$

We note that the above is for $G = Sp_{2n}$. If $G = SO_{2n+1}$, we need to add, to $\Phi_i$, $e_{r_1+\cdots+r_{i-1}+j}$, $j = 1, \ldots, r_i$, for $i = 1, \ldots, k$ and in $\Phi_0$, $2e_{r_1+\cdots+r_k+i}$ should be $e_{r_1+\cdots+r_k+i}$. If $G = O_{2n}$, then $\Phi_0$ does not have the roots $2e_{r_1+\cdots+r_k+i}, \quad i = 1, \ldots, r_0$.

$\Phi_1, \ldots, \Phi_k$ are root systems of type $D_n$ and $\Phi_0$ is a root system of type $C_n$. This corresponds to the decomposition $O(V_1) \times \cdots \times O(V_k) \times O(V_0) \subset O_{2n+1}(\mathbb{C})$.

Let $W_i$ be the Weyl group corresponding to $\Phi_i$ for $i = 1, \ldots, k$ and $W_i$ be the Weyl group of $O_{V_i}$ for $i = 0, \ldots, k$. Then $W_i = W_i e_{r_1+\cdots+r_i}$ for $i = 1, \ldots, k$. Then we have $W(\chi, \chi) = W_1 \times \cdots \times W_k \times W_0$. Let $\lambda = \lambda_1 + \cdots + \lambda_k + \lambda_0$, where $\lambda_i = a_{r_1+\cdots+r_{i-1}+1} e_{r_1+\cdots+r_{i-1}+1} + \cdots + a_{r_1+\cdots+r_i} e_{r_1+\cdots+r_i}$ for $i = 1, \ldots, k$ and $\lambda_0 = a_{r_1+\cdots+r_k+1} e_{r_1+\cdots+r_k+1} + \cdots + a_n e_n$.

We recall the following well-known result. (Carter [C].)

**Proposition 6.2.2.** Let $\Delta$ be a set of simple roots and $W$ be the associated Weyl group. Let $w_\alpha$ be the simple reflection with respect to $\alpha \in \Delta$. Then $W$ is generated by the $w_\alpha$, $\alpha \in \Delta$. Let $\theta$ be a subset of $\Delta$ and $W_\theta$ be the subgroup of $W$ generated by the $w_\alpha$, $\alpha \in \theta$. Then each coset $wW_\theta$ has a unique element $d_\theta$ characterized by any of the following equivalent properties:

1. $d_\theta \theta > 0$.
2. $d_\theta$ is of minimal length in $wW_\theta$.
3. For any $x \in W_\theta$, $l(d_\theta x) = l(d_\theta) + l(x)$.

We apply Proposition 6.2.2 to $\Delta = \{e_1 - e_2, \ldots, e_{n-1} - e_n\}$ and $\theta = \Delta - \{e_{r_1+1} - e_{r_1+2}, e_{r_1+r_2} - e_{r_1+r_2+1}, \ldots, e_{r_1+\cdots+r_k} - e_{r_1+\cdots+r_k+1}\}$. Let $D$ be the set of such distinguished coset representatives. Then for $d \in D$, $w_i \in W_i$, $i = 0, \ldots, k$, we have

$$
\{\alpha > 0 : dw_1 \cdots w_k w_0 \alpha < 0\} \\
= \bigcup_{i=0}^k \{\alpha \in \Phi_i : w_i \alpha < 0\} \cup \{\alpha \in \Phi_D : dw_1 \cdots w_k w_0 \alpha < 0\}.
$$
Then the constant term of pseudo-Eisenstein series is given by
\[
E_0^{PS}(\phi, \lambda, \chi) = \sum_{w \in W} r(w, -\lambda, \chi) R(w^{-1}, w\lambda, w\chi) \phi(w\lambda)
\]
\[
= \prod_{i=0}^k \sum_{w_i \in W_i} r(w_i, -\lambda_i, \Phi_i) \cdot \left( \sum_{d \in D} r(dw_1 \cdots w_kw_0, -\lambda, \Phi_D) R(w_0^{-1}w_k^{-1} \cdots w_1^{-1}d^{-1}, dw_1 \cdots w_kw_0\lambda, d\chi) \phi(dw_1 \cdots w_kw_0\lambda) \right).
\]
We note that \(w_1 \cdots w_k \chi = \chi\). Recall that
\[
E_0^{PS}(\phi, \lambda, \chi) = \sum_{w \in W} r(w, -\lambda, \chi) R(w^{-1}, w\lambda, w\chi) \phi(w\lambda),
\]
actually means
\[
\sum_{d \in D} \left( \sum_{w \in W(\chi, d\chi)} r(w, -\lambda, \chi) R(w^{-1}, w\lambda, w\chi) \phi_d(w\lambda) \right),
\]
where \(\phi_d \in I(d\chi)\). By the cocycle relation, we have
\[
R(w_0^{-1}w_k^{-1} \cdots w_1^{-1}d^{-1}, dw_1 \cdots w_kw_0\lambda, d\chi)
\]
\[
= R(w_0^{-1}w_k^{-1} \cdots w_1^{-1}, w_1 \cdots w_kw_0\lambda, \chi) R(d^{-1}, dw_1 \cdots w_kw_0\lambda, d\chi).
\]
Let
\[
f(w_1 \cdots w_kw_0\lambda) = \sum_{d \in D} r(dw_1 \cdots w_kw_0, -\lambda, \Phi_D)
\]
\[
R(d^{-1}, dw_1 \cdots w_kw_0\lambda, d\chi) \phi(dw_1 \cdots w_kw_0\lambda).
\]
Then we have
\[
E_0^{PS}(\phi, \lambda, \chi) = \prod_{i=0}^k \sum_{w_i \in W_i} r(w_i, -\lambda_i, \Phi_i) \cdot R(w_0^{-1}w_k^{-1} \cdots w_1^{-1}d^{-1}, dw_1 \cdots w_kw_0\lambda, d\chi) f(w_1 \cdots w_kw_0\lambda).
\]
We note that it has the same normalizing factors as the Eisenstein series of \(O_{2r_1}, \ldots, O_{2r_k}\) and \(Sp_{2r_0}\) attached to the trivial character. Let \(O_i\)’s be distinguished unipotent orbits in \(O_{2r_i}(\mathbb{C})\) for \(i = 1, \ldots, k\) and \(O_0\) in \(O_{2r_0+1}(\mathbb{C})\).
Let $p_i \in P(O_i)$ for $i = 0, \ldots, k$ and $p = p_0 \times \cdots \times p_k$. Let
\[
l_p(\phi, \lambda, \chi) = \prod_{i=0}^{k} \sum_{w_i \in W_i} r(w_i, -\lambda_i, \Phi_i) \\
\cdot R(w_p w_k^{i-1} \cdots w_0^{i-1}, w_0 \cdots w_k \lambda, \chi) f(w_0 \cdots w_k \lambda).
\]

Here $w_p = w_{p_0} \cdots w_{p_k}$. Since $r(w_i, -\lambda_i, \Phi_i)$ is identically zero on $V'(p_i)$ if $w_i \notin W(|p_i|)$, the restriction of $l_p(\phi, \lambda, \chi)$ to $V'(p) = V'(p_0) \times \cdots \times V'(p_k)$ is given by
\[
l_p(\phi, \lambda, \chi)|_{V'(p)} = \prod_{i=0}^{k} \sum_{w_i \in W(|p_i|)} r(w_i, -\lambda_i, \Phi_i) \\
\cdot R(w_p w_k^{i-1} \cdots w_0^{i-1}, w_0 \cdots w_k \lambda, \chi) f(w_0 \cdots w_k \lambda).
\]

We note that $f(w_0 \cdots w_k \lambda)$ is holomorphic on $V'(p)$. We apply Moeglin's results and define $l_p(\phi, \lambda_p, \chi)$ inductively. But the order of induction will matter. Among $V'(p_i)$'s, we can shuffle the segments. By shuffling, we mean the following:

Let $p_1 = (; q_1, \ldots, q_s)$ and $p_2 = (; q'_1, \ldots, q'_t)$ be two chains. By shuffling of $p_1 \times p_2$, we mean any permutation on segments so that (1) $(q_1, q_2), \ldots, (q_{2^{[\frac{s}{2}]-1}}, q_{2^{[\frac{s}{2}]}})$ appear in that order and (2) $(q'_1, q'_2), \ldots, (q'_{2^{[\frac{t}{2}]-1}}, q'_{2^{[\frac{t}{2}]}})$ appear in that order.

Take a shuffling of segments in such a way that it satisfies a certain condition (see [Ja-Ki, (3.1)]) to produce a maximum number of components, i.e., a condition on the non-vanishing of the normalized intertwining operators. This amounts to starting with a conjugate of $\chi$. If there is no confusion, we will still write it as $\chi$. Then
\[
\langle \theta_{\phi'}, \theta_\phi \rangle_{\text{dis}} = \sum_{i=0}^{k} \sum_{O_i} \sum_p c_p (\langle l'_p(\phi', \lambda_p, \chi), l_p(\phi, \lambda_p, \chi) \rangle),
\]
where $O_i$ runs through distinguished unipotent orbits in $O_{2r_i}(C)$ for $i = 1, \ldots, k$ and $O_{2r_{i+1}}(C)$ for $i = 0$. $p = p_0 \times \cdots \times p_k \in P(O_0) \times \cdots \times P(O_k)$.

We have
\[
l_p(\phi, \lambda_p, \chi) \in \otimes_v R_v(w_p, \lambda_p, \chi) I_v(\lambda_p, \chi_v).
\]

We recall the local result on the image of intertwining operators $R_v(w_p, \lambda_p, \chi) I_v(\lambda_p, \chi_v)$ from [Ja-Ki]: Let $\text{Unip}(p, \chi_v)$ be the set of direct summands of $R_v(w_p, \lambda_p, \chi) I_v(\lambda_p, \chi_v)$ and $\text{Unip}(O_1, \ldots, O_k, O_0, \chi_v)$ be the set of union of $\text{Unip}(p, \chi_v)$ as $p_i$ runs through $P(O_i)$ for $i = 0, \ldots, k$.

**Theorem 6.2.1** (Theorem 3.4.10 of [Ja-Ki]). $\Pi_{\text{res}_v} = \text{Unip}(O_1, \ldots, O_k, O_0, \chi_v)$ is parametrized by
\[
C(O_1, \ldots, O_k, O_0, \chi_v) = [\text{Springer}(O_1) \times \cdots \times \text{Springer}(O_k) \times \text{Springer}(O_0)],
\]
where \([\ ]\) is defined as follows: If \(\mu_{1v} = \mu_{2v} \neq \mu_{iv}\) for \(i = 0, 3, \ldots, k\), then we replace Springer \((O_1) \times Springer\) by

\[ C(O_1, O_2, \mu_{1v}) = \{ \eta \in Springer(O) : \eta|_{\Lambda(O_1)} \in Springer(O_1), \text{ for } i = 1, 2 \}, \]

where \(O\) is the unipotent orbit of \(G_{12}^s\) obtained by combining \(O_1, O_2\), where

\[
G_{12}^s = \begin{cases} 
O(2(r_1 + r_2), \mathbb{C}), & \text{if } G = Sp(2n), O(2n) \\
Sp(2(r_1 + r_2), \mathbb{C}), & \text{if } G = SO(2n + 1).
\end{cases}
\]

Therefore we have:

**Theorem 6.2.2.** Let \(\chi = \chi(\mu_1, \ldots, \mu_1, \ldots, \mu_k, \ldots, \mu_k, 1, \ldots, 1), r_0 + \cdots + r_k = n, r_1 \geq \cdots \geq r_k\), where \(\mu_i\)'s are non-trivial distinct quadratic grössen-characters. Then the residual spectrum attached to the conjugacy class of \((T, \chi)\) is parametrized by the distinguished unipotent orbits in \(O_{2r_1}(\mathbb{C})\), \(i = 1, \ldots, k\) and \(O_{2r_0+1}(\mathbb{C})\). More specifically, distinguished unipotent orbits \(O_i \in O_{2r_i}(\mathbb{C})\), \(i = 1, \ldots, k\) and \(O_0 \in O_{2r_0+1}(\mathbb{C})\) and \(\chi\) give a quadratic unipotent Arthur parameter \(\psi\) and let \(C_{\text{res}_v} = [\text{Springer}(O_1) \times \cdots \times \text{Springer}(O_k) \times \text{Springer}(O_0)] \subset C_{\psi_v}\) and \(\Pi_{\text{res}_v} = \text{Unip}(O_1, \ldots, O_k, O_0, \chi_v) \subset \Pi_{\psi_v}\) for all non-archimedean places. For each \(X \in \Pi_{\text{res}_v}\), there is a character \(\eta_X \in C_{\text{res}_v}\) and it satisfies Arthur’s conjecture. i.e.,

\[
L^2_{\text{dis}}(G(F)\backslash G(A) \cap L^2(G(F)\backslash G(A))\psi),
\]

is the set of \(\pi = \otimes_v X_v\), where \(X_v\) satisfies the following conditions:

1. There exists \(p_i \in P(O_i), i = 0, \ldots, k\) such that \(X_v \in \text{Unip}_v(p_1, \ldots, p_k, p_0, \chi_v)\) for all \(v\).
2. \(X_v\) is spherical for almost all \(v\) and archimedean places.
3. \(\prod_v \eta_{X_v}\) is trivial on \(C_\psi\).

7. **Relation to Kudla-Rallis example [Ku-Ra].**

7.1. **Trivial character case.** Let \(G = Sp_{2n}\) and \(P = MN\) be the Siegel parabolic subgroup so that \(M = GL_n\). Consider the degenerate principal series \(I(s, \chi) = \text{Ind}_{B}^{G} \chi \otimes \exp(s, H_P(\chi))\), where \(\chi\) is a trivial character of \(GL_n\). Let \(E(s, f, P)\) be the Eisenstein series attached \(f \in I(s, \chi)\).

Then Kudla-Rallis [Ku-Ra] showed the following:

**Theorem ([Ku-Ra]).**

1. \(E(s, f, P)\) has a simple pole at \(s = \{ 0, \ldots, \rho_n - 2, \rho_n - 1, \rho_n \}\), where \(\rho_n = \frac{n+1}{2}\) and \(0\) means that \(0\) is omitted in the case when \(n\) is odd.
2. The residue at \(s = \rho_n\) is constant.
3. The residue at \(s = \rho_n - 1\) is not square integrable.
(4) The residue at \( s_i = \rho_n - i \) is the direct sum \( \oplus R_n(Q) \), where the direct sum ranges over all classes of quadratic forms \( Q \) with \( \dim Q = n + 1 - 2s_i \) and \( \chi_Q = 1 \). Here \( R_n(Q) \) is obtained by theta correspondence, first by local construction and then by being put together. In this case the Hasse invariant \( \chi_Q = 1 \) controls the obstruction to the existence of the global quadratic space.

Let \( O = (q_1, q_2, 1) \), \( q_1 > q_2 > 1, q_1 + q_2 = 2n \), be a distinguished unipotent orbit in \( O_{2n+1}(C) \). There are two inequivalent ordered partitions in \( P(O) \), namely, \( p_1 = (q_1, q_2, 1) \) and \( p_2 = (q_1, 1, q_2) \). We show:

**Proposition 7.1.** The representations \( \oplus R_n(Q) \) are exactly those corresponding to \( p_1 \), i.e., \( l_{p_1}(\phi, \lambda_{p_1}) \) for \( \phi \in PW \). (See Section 5).

**Proof.** In this case, in Moeglin’s notation in Section 5, \( \lambda_{p_1} = (\lambda_1, \ldots, \lambda_n) \), where \( \lambda_i = \begin{cases} \frac{q_1 + 1}{2} - t & \text{if } 1 \leq t < \frac{q_1 + q_2}{2} \Rightarrow w_{p_1}(t) = -t \leq 1 - t < \frac{q_1 - q_2}{2} \text{ and } w_{p_1} \left( \frac{q_1 + q_2 + t}{2} \right) = n + 1 - t \leq 1 \leq q_2. \end{cases} \)

We know that \( S_{p_1} = \{ \alpha > 0 \mid w_{p_1} \alpha < 0, (\lambda_{p_1}, \alpha^\vee) = 1 \} \) has \( n \) elements. But we can see easily that \( S_{p_1} \) contains \( e_i - e_{i+1} \) for \( i = 1, \ldots, n - 1 \). This means that since the simple roots \( e_i - e_{i+1} \) for \( i = 1, \ldots, n - 1 \), generate the Siegel parabolic subgroup, the iterated residue of the Eisenstein series on the singular hyperplanes \( e_i - e_{i+1} = 1, i = 1, \ldots, n - 1 \), is exactly the Eisenstein series attached to the trivial character of the Siegel parabolic subgroup, which Kudla-Rallis considered.

The square integrable residues at \( s = \{ 0, \ldots, \rho_n - 2 \} \) exactly correspond to the partitions \( (q_1, q_2), q_1 + q_2 = 2n, q_1 > q_2 > 1 \) (\( s_i = \rho_n - i \) corresponds to \( 2n - 2i + 1, 2i - 1 \)).

Since \( l_{p_1}(\phi, \lambda_{p_1}) \in \bigotimes_v R_v(w_{p_1}, \lambda_{p_1})I_v(\lambda_{p_1}) \), the local component of \( R_n(Q) \) is the direct summand of the image of the intertwining operator \( R_v(w_{p_1}, \lambda_{p_1})I_v(\lambda_{p_1}) \) which is a sum of two irreducible representations. We note that \( R_v(w_{p_1}, \lambda_{p_1})I_v(\lambda_{p_1}) \) is the subrepresentation of \( \text{Ind}_{G_{L_n}}^G \det |^{-\frac{q_1+q_2}{2}} \). Jantzen [Ja3] gave their Langlands’ data.

**Theorem** ([Ja3, Prop 3.10]). \( R_v(w_{p_1}, \lambda_{p_1})I_v(\lambda_{p_1}) = \pi_1 + \pi_2 \). Here \( \pi_1 \) is a spherical representation which is the Langlands’ quotient of \( \text{Ind}_{M}^G \lambda_O \times \pi \), where \( \lambda_O \) is the conjugate of \( \lambda_p \) which is in the closure of the positive Weyl chamber and \( M = \text{Cent} (\lambda_O, G^*) = F^x \times \cdots \times F^x \times GL_2 \times \cdots \times GL_2 \times SL_2 \) and \( \pi = \text{Ind}_{P_{L+1}}^{SL_2+1} \) is an irreducible representation of \( SL_2 \). And \( \pi_2 \) is the Langlands’ quotient of

\[
\begin{align*}
\text{Ind}_{M}^G & |^{\frac{q_1 - 1}{2}} \times \cdots \times |^{\frac{q_2 + 1}{2}} \times |^{\frac{q_2 - 1}{2}} \times |^{\frac{q_2 - 1}{2}} \times \cdots \times |^2 \times |^2 \times |^1 \times T,
\end{align*}
\]

where \( M = F^x \times \cdots \times F^x \times Sp_4 \) and \( T \) is the unique (irreducible) common component of \( \text{Ind}_{G_{L_2}}^{G_{L_2}} \det |^{\frac{1}{2}} \) and \( \text{Ind}_{F^x \times SL_2}^{F^x \times SL_2} 1 \times St_{SL_2} \). Here \( St_{SL_2} \) is the Steinberg representation of \( SL_2 \). We note that \( T \) is tempered.
Let $\text{Unip}(p_1) = \{\pi_1, \pi_2\}$. Then Proposition 7.1 can be rephrased as follows:

**Proposition 7.2.** The representations $\oplus R_n(Q)$ are the set of $\pi = \bigotimes_v X_v$ which satisfies: (a) $X_v \in \text{Unip}(p_1)$ for all $v$ (b) $X_v$ is spherical almost everywhere (c) $\prod_v \chi_{X_v} = 1$, where $\chi_{X_v}$ is defined in Section 5.

But the non-degenerate Eisenstein series attached to the trivial character of the maximal torus has more residues. We need to consider $p_2 \in P(O)$. In this case $\lambda_{p_2} = (\lambda_1, \ldots, \lambda_n)$, where $\lambda_t = \frac{q_{2t+1}}{2} - t$ for $1 \leq t \leq \frac{q_{2t+1}}{2}$ and $\lambda_{t+\frac{q_{2t+1}}{2}} = \frac{q_{2t+1}}{2} - t$ for $1 \leq t \leq \frac{q_{2t+1}}{2}$. We have $w_{p_2}(t) = -t$ for all $t$ except $t = \frac{q_{2t+1}}{2}$. Then

$$R_v(w_{p_2}, \lambda_{p_2})I_v(\lambda_{p_2}) \subset \text{Ind}_{GL_{q_{2t+1}} \times Sp_{q_{1}-1}} Ind_{GL_{q_{2t+1}} \times Sp_{q_{1}-1}} \det |^{-\frac{q_{2t+1}}{4}} \times \text{tr}.$$ 

We have $R_v(w_{p_2}, \lambda_{p_2})I_v(\lambda_{p_2}) = \pi_1 \oplus \pi_3$.

**Theorem (Ja3).** $\pi_3$ is the Langlands’ quotient of

$$\text{Ind}_{F \times \cdots \times F \times GL_2 \times \cdots \times GL_2} \bigg|^{\frac{q_{1}-1}{2} \times \cdots \times \frac{q_{2t+1}}{2} \times \cdot \cdot \cdot \times \frac{q_{2t+1}}{2}} \times \cdot \cdot \cdot \times \frac{\sigma}{2} \times \cdot \cdot \cdot \times \frac{\sigma}{2},$$

where $\sigma$ is the unique quotient of $\text{Ind}_{F \times \cdots \times F} GL_2 \bigg|^{-\frac{1}{2}} \times \frac{1}{2}$ which is square integrable.

Let $\text{Unip}(p_2) = \{\pi_1, \pi_3\}$. In this special case, we have:

**Theorem 7.1.** The residual automorphic representations attached to the trivial character of the maximal torus and the distinguished unipotent orbit $O = (q_1, q_2, 1)$ are the set of $\pi = \bigotimes_v X_v$ which satisfies:

1. $X_v \in \text{Unip}(p_1)$ for all $v$ or $X_v \in \text{Unip}(p_2)$ for all $v$.
2. $X_v$ is spherical for almost all $v$.
3. $\prod_v \chi_{X_v} = 1$.

**7.2. $\chi = \chi(\mu, \ldots, \mu)$ case, $\mu$ a quadratic grössencharacter.** In this case, Kudla-Rallis showed:

**Theorem (Ku-Ra).** The degenerate Eisenstein series attached to the character $\mu$ of $GL_n$ has a pole at $s = \{0, \ldots, \rho_n - 2, \rho_n - 1\}$ and the residue at $s_i = \rho_n - i$ is square integrable and given by the direct sum $\oplus R_n(Q)$, where the direct sum ranges over all classes of quadratic forms $Q$ with $\dim Q = (n + 1) - 2s_i$ and $\chi_Q = \mu$. Again for this case, Hasse invariant gives obstruction to the existence of the global quadratic space.

These correspond to the distinguished unipotent orbits $O = (q_1, q_2)$ of $O_{2n}(\mathbb{C})$, where $q_1 > q_2 \geq 1$. There is only one element in $P(O)$, namely, $p = (1; q_1, q_2)$. The local components of $R_n(Q)$ are the direct summands of $R_v(w_p, \lambda_p, \chi_v)I(\lambda_p, \chi_v)$. Here $R_v(w_p, \lambda_p, \chi_v)I(\lambda_p, \chi_v)$ is a sum of two irreducible representations unless $\mu_v = 1$ and $q_2 = 1$, in which case it is
irreducible. If \( \mu_v \neq 1 \), then they are the Langlands’ quotients of \( \text{Ind}_M^G \lambda_0 \times \pi_i \), where \( \lambda_0 \) is the conjugate of \( \lambda_p \) which is in the closure of the positive Weyl chamber and \( M = \text{Cent}(\lambda_0, G^s) = F^x \times \cdots \times F^x \times GL_2 \times \cdots \times GL_2 \times SL_2 \) and \( \text{Ind}_{BM}^M \chi_v = \pi_1 \oplus \pi_2 \). If \( \mu_v = 1 \), \( q_2 \neq 1 \), then they are the ones in the case when \( \chi_v = 1 \).

**Remark 7.1.** We can show that the residual spectrum corresponding to the unipotent orbits of the form \( p = (q_1, q_2, q_3) \), \( q_3 > 1 \), can be obtained by considering the degenerate principal series of \( GL_l \times Sp_{2(n-l)} \), where \( l = n - \frac{q_3-1}{2} \). In this case, \( \lambda_p = (\lambda_1, \ldots, \lambda_n) \), where \( \lambda_t = \frac{q_1+q_2}{2} - t \), for \( 1 \leq t \leq \frac{q_1+q_2}{2} \) and \( \lambda_{t+1} = \frac{q_1+q_2}{2} - t \), for \( 1 \leq t \leq \frac{q_1+q_2}{2} - 1 \). \( w_p(t) = t \) for \( 1 \leq t \leq \frac{q_1+q_2}{2} - 1 \) and \( \frac{q_1+q_2}{2} \leq t \leq n \), \( w_p(t) = \frac{q_1+q_2}{2} - t + 1 \) for \( 1 \leq t \leq q_2 \).

We can show that \( S_p = \{ \alpha > 0 | w_p \alpha < 0, (\lambda_0, \alpha^\vee) = 1 \} \) contains all simple roots except \( e_n - \frac{q_1-1}{2} - e_n - \frac{q_1+1}{2} \). This means that the iterated residue of the Eisenstein series on the singular hyperplanes in \( S_p \) is the residue of the degenerate Eisenstein series attached to the trivial character of the parabolic subgroup \( P = MN, M = GL_l \times Sp_{2(n-l)} \).

We note that the degenerate Eisenstein series attached to \( GL_{n-1} \times SL_2 \) parabolic subgroups has been studied by Jiang [Ji].

**8. Relation to our previous work [Ki-Sh].**

Let \( \chi = \chi(\mu_1, \ldots, \mu_k, 1, \ldots, 1) \), \( r_0 + \cdots + r_k = n \), \( r_1 \geq \cdots \geq r_k \geq 2 \). In our previous work [Ki-Sh], we are concerned only with unipotent orbits of the form \( O_i = (2r_i - 1, 1) \) of \( O_{2r_i}(\mathbb{C}) \), \( i = 1, \ldots, k \) and \( O_0 = (2r_0 + 1) \) of \( O_{2r_0+1}(\mathbb{C}) \), which correspond to the half-sum of positive roots of \( O_{2r_i} \) and \( Sp_{2r_0} \), respectively. We showed:

**Theorem ([Ki-Sh]).** \( C_{\phi_v} \) is the Knapp-Stein R-group of the unitary principal series \( \text{Ind}_{BM}^M \chi_v \), where \( M \) is conjugate to \( GL_{n_1} \times \cdots \times GL_{n_r} \times Sp_{2k} \). Here \( n_1, \ldots, n_r \) are determined by \( O_i \)’s. \( \Pi_{\phi_v} \) is the set of the Langlands’ quotients of \( \text{Ind}_M^G \pi_{iv} \times \lambda_p \), where \( \text{Ind}_M^G \chi_v = \oplus \pi_{iv} \). \( \lambda_p \) is in the positive Weyl chamber of the split component of \( M \).

Let \( \mu_{i_1,v}, \ldots, \mu_{i_{n_1},v} \) be the set of non-trivial distinct quadratic characters. Then \( C_{\phi_v} \) is spanned by the order two elements \( c_{r_1 + \cdots + r_1}, \ldots, c_{r_1 + \cdots + r_k} \). On the other hand, by Theorem 6.2.1 in Section 6, \( C_{\text{res}_v} = C(\mu_{i_1}) \times \cdots \times C(\mu_{i_{n_1}}) \times C_0 \), where \( C(\mu_{i_j}) \simeq \mathbb{Z}/2\mathbb{Z} \) and \( C_0 \) is a set determined by \( O_i \)’s. We note that \( C_{\phi_v} \simeq C(\mu_{i_1,v}) \times \cdots \times C(\mu_{i_{n_1},v}) \).

**Example 8.1.** Let \( \chi = \chi(\mu, \mu, 1, 1) \), where \( \mu \) is a non-trivial quadratic grössencharacter. Let \( O_1 = (3, 1) \) be a distinguished unipotent orbit of \( O_4(\mathbb{C}) \) and \( O_0 = (5) \) of \( O_5(\mathbb{C}) \). Then if \( \mu_v \neq 1 \), \( C_{\text{res}_v} = C_{\phi_v} \simeq \mathbb{Z}/2\mathbb{Z} \).
9. Correction to our paper [Ki-Sh].

p. 401: In the Abstract, the Introduction, and also in Theorem 5.1, by the technique used in that paper, we can only claim that $\Pi \in \Pi_{\phi_v}$ appears with multiplicity, not equal to $d_{\psi} |C_{\phi_v}| \sum_{x \in C_{\phi_v}} \langle x, \Pi \rangle$, but greater than or equal to $d_{\psi} |C_{\phi_v}| \sum_{x \in C_{\phi_v}} \langle x, \Pi \rangle$. We need to use the technique of pseudo-Eisenstein series in this paper to claim the assertion.

p. 415: In Theorem 4.5, $w_0$ is different from $w_0$ in Proposition 4.4. In fact, $w_0$ in Theorem 4.5 is given by $w_0 = w_{1,0} \cdots w_{k,0} w_{0,0}$, where $w_{i,0}$ is the longest element in $W_i$ for $i = 0, \ldots, k$.

p. 415: The local intertwining operator $R(w_0, \Lambda, \chi_v)$ is not holomorphic at $\Lambda_0$. We need to define $R(w_0, \Lambda_0, \chi_v)$ as the intertwining operator on $\text{Ind}_P^G \Lambda_0 \otimes \text{Ind}_B^M \chi_v$. We assumed this implicitly there.

p. 418: In Lemma 4.8, $w_{A,0}$ should be $w_{1,0}$, the longest element in $W_1$.

p. 418: In the proof of Lemma 4.8 and in Lemma 4.9, $w_1$ should be $w_{1,0}$.

p. 420: The sentence “The parameter $\Lambda_0$ may not be in the positive Weyl chamber of the split component of $M$,” is false. The parameter $\Lambda_0$ is in the positive Weyl chamber of the split component of $M$.

References


[Ki3] The residual spectrum of U(n,n); contribution from Borel subgroups, preprint.


RESIDUAL SPECTRUM OF SPLIT CLASSICAL GROUPS


Received June 15, 1999. This author was partially supported by NSF grant DMS-9610387.

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QUADRATIC BASE CHANGE FOR $p$-ADIC SL(2) AS A THETA CORRESPONDENCE II: JACQUET MODULES

David Manderscheid

Let $F$ be a $p$-adic field and let $O$ be the orthogonal group attached to a quaternary quadratic form with coefficients in $F$ and of Witt rank one over $F$. We determine, up to one possible exception, which nonsupercuspidal representations of $O(F)$ occur in the theta correspondences attached to $(\text{SL}_2(F), O(F))$.

This paper is the second in a series of papers examining in detail the local theta correspondences attached to reductive dual pairs $(\text{SL}_2(F), O(F))$ where $F$ is a $p$-adic field of characteristic zero and $O$ is the orthogonal group attached to a quaternary quadratic form with coefficients in $F$ and of Witt rank one over $F$. In this paper we determine, up to one possible exception, which nonsupercuspidal representations of $O(F)$ occur in the correspondences. The determination is explicit and in terms of parabolic inducing data.

The results we obtain are consistent with the first occurrence in towers conjecture [KR1] and Prasad’s conjectures [P2]. They are sharper and more explicit than the corresponding results in Cognet’s thesis [C] and complement the results of Roberts [Ro2] in the case of $p$ odd, which are stated in terms of distinguished representations. In future papers in this series, we will examine which supercuspidal representations of $O(F)$ occur in the correspondence and the explicit correspondence.

To explain our method, we first recall the general setting of theta correspondences for symplectic and orthogonal groups; see, e.g., [MVW], [H]. For $i = 1, 2$, let $V_i$ be a finite-dimensional vector space over $F$ equipped with a nondegenerate bilinear form $\langle \ , \rangle_i$; assume that $\langle \ , \rangle_1$ is skew-symmetric while $\langle \ , \rangle_2$ is symmetric. Equip $W = V_1 \otimes V_2$ with the skew-symmetric form $\langle \ , \rangle$ coming from tensoring the $\langle \ , \rangle_i$. Let $G_1, G_2$ and $G$ be the isometry groups of $\langle \ , \rangle_1, \langle \ , \rangle_2$ and $\langle \ , \rangle$, respectively, and identify $G_1$ and $G_2$ with subgroups of $G$ via their usual actions on $W$; then $(G_1, G_2)$ is called a reductive dual pair in $G$. Let $\chi$ be a nontrivial additive character of $F$ and let $\omega_\chi^\infty$ denote the (smooth) oscillator representation of $\tilde{G}$ attached to $\chi$ where $\tilde{G}$ is the (unique) nontrivial two-fold cover of $G$. For $H$ a closed subgroup of $G$, let $\tilde{H}$ denote the inverse image of $H$ in $G$ and let $\mathcal{R}_\chi(\tilde{H})$
denote the set of irreducible admissible representations of $\tilde{H}$ which occur as quotients of $\omega^\infty_{\chi}|_{\tilde{H}}$. Then $\tilde{G}_1$ and $\tilde{G}_2$ commute and $\mathcal{R}_\chi(\tilde{G}_1 \tilde{G}_2)$ gives rise to a correspondence between $\mathcal{R}_\chi(\tilde{G}_1)$ and $\mathcal{R}_\chi(\tilde{G}_2)$. These correspondences are called theta correspondences. We denote these correspondences by $\theta : \mathcal{R}_\chi(\tilde{G}_1) \rightarrow \mathcal{R}_\chi(\tilde{G}_2)$ and $\theta : \mathcal{R}_\chi(\tilde{G}_2) \rightarrow \mathcal{R}_\chi(\tilde{G}_1)$; the direction of $\theta$ will be clear from context. Theta correspondences are known in general to be bijections for $p$ odd [Wa] and for all $p$ in the cases considered in this paper [R2]. Furthermore, in all cases considered here, the space $V_2$ will be even-dimensional and thus the $\tilde{G}_1$ and $\tilde{G}_2$ are trivial covers so that we write, in an abuse of notation, $\mathcal{R}_\chi(G_1)$ and $\mathcal{R}_\chi(G_2)$ instead of $\mathcal{R}_\chi(\tilde{G}_1)$ and $\mathcal{R}_\chi(\tilde{G}_2)$. Elements of these sets will be considered as representations of $G_1$ and $G_2$, respectively.

Then our argument and organization for this paper are as follows. In the first section, we establish notation and recall briefly known results that will be necessary in what follows. These results include the parameterizations of the admissible duals of $GL_2(F)$ and $SL_2(F)$, the results of the first paper in this series [M] on which representations of $SL_2(F)$ occur in $\mathcal{R}_\chi(SL_2(F))$ for the pair $(SL_2(F), O(F))$, quadratic base change for $GL_2(F)$ and $SL_2(F)$ and finally the results of Cognet’s thesis [C] that will be necessary.

We begin the second section by showing that the representations of $O(F)$ that occur in $\mathcal{R}_\chi(O(F))$ must restrict irreducibly to $SO(F)$. This is a standard seesaw duality [K1] argument. We then parameterize the irreducible admissible nonsupercuspidal parameterizations of $O(F)$ with this property. The parameterization is explicit and in terms of inducing data from the (unique up to conjugacy) maximal parabolic subgroup of $SO(F)$. This parabolic has Levi, $M$ say, isomorphic to $F^\times \times E^1$, where $F^\times$ denotes the multiplicative group of $F$ and $E^1$ is the kernel of the norm map from $E^\times$ to $F^\times$ where $E/F$ is the quadratic extension of $F$ attached to the anisotropic part of the quadratic form, $Q$ say, giving rise to $O$.

In the third section, we consider $\omega^\infty_{\chi}$ in the Schroedinger model. In this model, $O(F)$ acts linearly on $S(V)$ where $V$ is the space on which $Q$ is defined and $S(V)$ is the space of locally constant compactly supported function on $V$. We use this to determine necessary conditions on the representations of $M$ that can be used for inducing data of representations occurring in $\mathcal{R}_\chi(O(F))$. The argument is by calculation of Jacquet modules.

In the fourth and final section, we show that the necessary conditions of the third section are also sufficient with one possible exception. The argument involves our previous results, Cognet’s results and results on base change. It also involves determining which representation of $O(F)$ pairs with the trivial representation of $SL_2(F)$. The pairing representation is also one-dimensional and the argument involves the determination of which orbits in $V$ under $O(F)$ can support this representation. We plan on returning to the
possible exception, a generalized Steinberg representation, in a future paper in this series.

Finally, we would like to thank the referee of this paper for pointing out to us that we needed to modify the proof of Lemma 4.2.

1. Notation and known results.

In this section, we establish notation, recall the parameterization of the admissible dual of \( G_1 = SL_2(F) \) and recall some other known results necessary for this paper. We will be brief in our discussion.

Let \( F \) be a nonarchimedean local field of characteristic 0. Let \( p \) denote the residual characteristic of \( F \) and let \( \mathcal{O} = \mathcal{O}_F, P = P_F, \omega = \omega_F, k = k_F, q = q_F \) and \( | \cdot | = | \cdot |_F \) denote, respectively, the ring of integers, the prime ideal, a uniformizing parameter, the residue field and the absolute value on \( F \) normalized so that \( | x | = q^{-\nu(x)} \) where \( \nu = \nu_F \) denotes the order function on \( F \). Let \( U = U_F = \mathcal{O}_F^\times \) and \( U^n = U^n_F = 1 + P^n_F \) for \( n \) a positive integer. Further, for \( K/F \) a Galois extension of fields, let \( \Gamma(K/F) \) denote the associated Galois group and if, in addition, \( [K:F] < \infty \), let \( N_{K/F} = N \) denote the norm map and let \( K^1 = K^1_F \), the norm one elements in \( K \). Finally, fix an algebraic closure \( \bar{F} \) of \( F \) and a Weil group \( W_F \); let the associated Weil group notation be as in [T].

For \( G \) a group and \( \sigma \) a representation (all representations assumed smooth unless stated otherwise) of a subgroup \( H \) of \( G \), let \( \text{Ind}(G, H; \sigma) \) denote the representation of \( G \) induced by \( \sigma \) (form of induction determined by context) and for \( g \) in \( G \), let \( \sigma^g \) denote the representation of \( H^g = gHg^{-1} \) defined by \( \sigma^g(h) = \sigma(g^{-1}hg) \) for \( h \) in \( H^g \). If \( J \) is a subgroup of \( H \), we let \( \sigma|_J \) denote the restriction of \( \sigma \) to \( J \). Further, if \( J \triangleleft H \) and \( \sigma \) is a representation of \( H/J \), then we also view \( \sigma \) as a representation of \( H \) via inflation. If \( \sigma \) and \( \tau \) representations of \( G \), then we let \( \text{Hom}_G(\sigma, \tau) \) denote the set of \( G \)-intertwining operators from \( \sigma \) to \( \tau \) with the category, once again, specified by context. Finally, we let \( G^\wedge \) denote the admissible dual of \( G \).

By a character, we mean a (not necessarily unitary) one-dimensional representation. If \( \chi \) is a character of \( F^\times \), we also view \( \chi \) as a character of \( W_F \) via local class field theory and as a character of \( GL_2(F) \) by composition with \( \det \), the determinant map. Further, if \( K/F \) is a finite-dimensional Galois extension, we view \( \chi \) as a character \( \chi_K \) of \( K^\times \) via composition with \( N_{K/F} \).

If \( \chi \) is a character of \( F \) and \( a \) is an element of \( F \), we let \( \chi_a \) denote the character of \( F \) defined by \( \chi_a(y) = \chi(ay) \). Finally, we say representations \( \pi_1 \) and \( \pi_2 \) of \( GL_2(F) \) are twist equivalent if there exists a character \( \eta \) of \( F^\times \) such that \( \pi_1 \cong \pi_2 \otimes \eta \).

We now briefly recall the parameterization of the admissible dual of \( G_1(F) = SL_2(F) \) in [LL]. To do this we first recall the parameterization of the admissible dual of \( G'_1(F) = GL_2(F) \) in [JL] in a form suitable for our
purposes. If \( \mu \) and \( \nu \) are characters of \( F^\times \) such that \( \mu(x)\nu^{-1}(x) \neq |x| \) or \( |x|^{-1} \), let \( \pi(\mu, \nu) \) denote the irreducibly induced (normalized induction) principal series representation of \( G'_1 \) attached to \( \mu \) and \( \nu \). Note that \( \pi(\mu, \nu) \cong \pi(\nu, \mu) \). If \( \mu(x)\nu^{-1}(x) = |x| \), write \( \mu = \chi | \chi |^{1/2} \) and \( \nu = \chi | \chi |^{-1/2} \) and let \( \sigma(\mu, \nu) \) denote the special representation corresponding to the unique invariant subspace of the space of the associated induced representation from the Borel subgroup of \( G'_1 \) and let \( \pi(\mu, \nu)(\cong \chi) \) denote the corresponding quotient. Similarly, if \( \mu(x)\nu^{-1}(x) = |x|^{-1} \), let \( \sigma(\mu, \nu) \) denote the corresponding special representation (now the quotient) and \( \pi(\mu, \nu) \) the corresponding one-dimensional. Note that \( \sigma(\mu, \nu) \cong \sigma(\nu, \mu) \) and \( \pi(\mu, \nu) \cong \pi(\nu, \mu) \). Further, if \( K/F \) is quadratic and \( \theta \) is a character of \( K^\times \), let \( \pi(\theta) = \pi(\rho) \) denote the corresponding irreducible representation of \( G'_1 \) associated to \( \rho = \text{Ind}(W_F, W_K; \theta) \); note that \( \pi(\theta) \cong \pi(\theta^{-1}) \) and note also that \( \pi \) is supercuspidal if and only if \( \theta \) does not factor through \( N_K/F \) which in turn happens if and only if \( \rho \) is irreducible. We call representations of the form \( \pi(\theta) \) Weil representations. The irreducible representations of \( G'_1 \) not of one of the above forms are called exceptional and occur only if \( p = 2 \). These representations are supercuspidal and can be parameterized naturally in terms of the primitive (i.e., not induced from a proper subgroup) two-dimensional representations of \( W_F \) [Ku]; for \( \sigma \) such a representation of \( W_F \), we write \( \pi(\sigma) \) for the corresponding exceptional representation. Finally, we note that the representations enjoy no other equivalences with the exception that if \( \mu \) and \( \nu \) are characters of \( F^\times \) with \( \mu\nu^{-1} \) of order two, then \( \pi(\mu, \nu) \cong \pi(\mu_K) \) where \( K/F \) is the quadratic extension of \( F \) associated to \( \mu\nu^{-1} \) by local class field theory.

Now let \( G_1 = G_1(F) = \text{SL}_2(F) \) viewed as a subgroup of \( G'_1 \). Then we have:

**Theorem 1.1 ([LL]).** Let \( \pi_1 \) be an irreducible representation of \( G_1 \); then there exists an irreducible representation \( \pi \) of \( G'_1 \), unique up to twist equivalence, which contains \( \pi_1 \) upon restriction to \( G_1 \). The L-packet of \( \pi_1 \) is of the form \( \{ \pi_1, \ldots, \pi_s \} \) where the \( \pi_i \) are distinct irreducible representations of \( G_1 \) and the restriction of \( \pi \) to \( G_1 \) decomposes as \( \bigoplus_{i=1}^s \pi_i \). Further, given \( 1 \leq i, j \leq s \) there exists \( g \) in \( G_1 \) such that \( \pi_i \cong \pi_j \). Moreover, with \( \chi \) a character of \( F^\times \):

(i) If \( \pi \) is not a Weil representation, then \( s = 1 \). Further, if \( \pi \otimes \chi \cong \pi \), then \( \chi \) is trivial.

(ii) If \( \pi = \pi(\theta) \) with \( \theta \) a character of \( K^\times \) such that \( \theta|_{K^1} \) is not of order two, then \( s = 2 \) and \( \pi_i \cong \pi_j \) if and only if \( g \) is a norm from \( K^\times \). Further, \( \pi \otimes \chi \cong \pi \) if and only if \( \chi \) is trivial or \( \chi = \omega_{K/F} \), the character of \( F^\times \) associated to \( K \) by local class field theory. If \( \pi \) is supercuspidal in this setting, then \( \rho \) is singly imprimitive (i.e., can only be induced nontrivially from \( W_K \)).
(iii) If \( \pi = \pi(\theta) \) with \( \theta \) a character of \( K^\times \) such that \( \theta|_{K_1} \) is of order two, then \( s = 4 \). In this case, \( \rho \) is triply imprimitive and if \( K_i, i = 1, 2, 3, \) are the fields such that \( \rho \) may be induced from \( W_{K_i} \) and \( L \) is their composite, then \( \Gamma(L/F) \cong \mathbb{Z}/2\mathbb{Z} \otimes \mathbb{Z}/2\mathbb{Z} \) and \( \pi_i^0 \cong \pi_i \) if and only if \( \det g \) is a norm from \( L^\times \). Further, \( \pi \cong \pi \otimes \chi \) if and only if \( \chi \) is trivial or \( \chi = \omega_{K_i/F} \) for some \( i \).

(iv) The collection of distinct \( L \)-packets partitions \( G_1^\wedge \). Further, another representation \( \pi' \) in \( (G_1^\wedge)^\wedge \) gives rise to the same \( L \)-packet as \( \pi \) if and only if \( \pi \) and \( \pi' \) can be realized as follows:

\begin{enumerate}
  \item \( \pi = \pi(\mu, \nu) \) and \( \pi' = \pi(\mu', \nu') \) with \( \mu \nu^{-1} = (\mu')(\nu')^{-1} \);
  \item \( \pi = \sigma(\mu, \nu) \) and \( \pi' = \sigma(\mu', \nu') \) with \( \mu \nu^{-1} = (\mu')(\nu')^{-1} \);
  \item \( \pi = \pi(\theta) \) and \( \pi = \pi'(\theta') \) with \( \theta \) and \( \theta' \) on \( K^\times \) with \( \theta(\theta')^{-1}|_{K_1} = 1 \);
  \item \( \pi = \pi(\sigma) \) and \( \pi' = \pi(\sigma') \) with \( \sigma \) and \( \sigma' \) primitive projectively equivalent representations.
\end{enumerate}

In what follows we will distinguish among the \( \pi_i \) by their Whittaker models. In particular, recall that if \( \pi \) is an infinite-dimensional irreducible representation of \( G' \) and \( \eta \) is a nontrivial character of \( F \), then \( \pi \) has, up to scaling, a unique Whittaker model with respect to \( \eta \) and, of course, if \( \pi \) is finite-dimensional, then it has no Whittaker models. Thus, for infinite-dimensional \( \pi \), we let \( g(\mu, \nu; \eta) \) denote the component of \( \pi(\mu, \nu) \) with \( \eta \)-Whittaker model and similarly for \( \sigma(\mu, \nu), \pi(\theta) \) and \( \pi(\sigma) \). The only remaining representation is the trivial representation which we denote by \( 1 \). Finally, for \( a \in F^\times \) and \( \pi \) an irreducible representation of \( G_1 \), let \( \pi^a = \pi^g \) where \( g \) is an element of \( G_1' \) with \( \det g = a \). Then one checks that if \( \pi \) has an \( \eta \)-Whittaker model then \( \pi^a \) has an \( \eta_a \)-Whittaker model.

We continue by recalling the result of [M] that will be necessary for this paper. To this end, let \( E/F \) be a quadratic extension and set \( V = \{ A \in M_2(E) \mid \tilde{A}^t = A \} \) where \( \tilde{A} \) denotes the matrix obtained from \( A \) by applying \( \sigma \) to each entry where \( \Gamma(E/F) = \langle \sigma \rangle \). Now the negative of the determinant map \( \det : M_2(E) \to E \) when restricted to \( V \) maps to \( F^\times \) and defines a quadratic form, \( Q \) say, on \( V \) viewed as an \( F \) vector space. Let \( H_1 \) denote the isometry group of this form. Further, for \( \chi \) a nontrivial additive character of \( F \), let \( \mathcal{R}_\chi(G_1) \) denote the representations in the admissible dual of \( G_1 \) that occur in the theta correspondence attached to \( \chi \) and the reductive dual pair \( (G_1, H_1) \); see [M] and [MVW] for more details concerning theta correspondences.

**Theorem 1.2** ([M]). If \( \pi \) is an irreducible representation of \( G_1 \) such that either, for some \( b \) in \( N_{E/F}(E^\times) \), \( \pi^b \) has a Whittaker model with respect to \( \chi \), or \( \pi \) is trivial, then \( \pi \) is in \( \mathcal{R}_\chi(G_1) \).

Theorem 1.1 and Theorem 1.2 have the following consequences: If \( \pi' \) is an irreducible representation of \( G_1' \) that cannot be realized as a \( \pi(\theta) \) with \( \theta \) a character of \( E^\times \), then the entire \( L \)-packet for \( G_1 \) associated to \( \pi \) occurs in
$\mathcal{R}_\chi(G_1)$. On the other hand if $\pi$ can be realized as a $\pi(\theta)$ with $\theta$ a character of $E^\times$, then at least half of the representations in the associated $L$-packet occur.

We now recall the results on base change from $\text{SL}_2(F)$ to $\text{SL}_2(E)$ [LL] that will be necessary in what follows. To begin, we first recall base change from $G_1^1(F) = \text{GL}_2(F)$ to $G_1^1(E) = \text{GL}_2(E)$ [L]. In particular, if $\pi$ is an irreducible representation of $G_1^1(F)$, let $\Pi$ denote its (Langlands-Saito-Shintani) base change to $G_1^1(E)$.

**Theorem 1.3.** Let $\pi$ be an irreducible representation of $G_1^1(F)$.

(i) If $\pi \cong \pi(\mu, \nu)$ with $\mu$ and $\nu$ characters of $F^\times$, then $\Pi \cong \pi(\mu E, \nu E)$.

(ii) If $\pi \cong \sigma(\mu, \nu)$ with $\mu$ and $\nu$ characters of $F^\times$, then $\Pi \cong \sigma(\mu E, \nu E)$.

(iii) If $\pi \cong \pi(\theta)$ with $\theta$ a character of $K^\times$, $K/F$ quadratic, then $\Pi \cong \Pi(\theta|_{W_E})$. In particular, if $K \neq E$, then $\Pi \cong \pi(\theta_{KE})$, and if $K = E$, then $\Pi \cong \pi(\theta, \theta^\sigma)$ where $\langle \sigma \rangle = \Gamma(E/F)$.

(iv) If $\pi$ is exceptional, then so is $\Pi$.

**Proof.** (i) through (iii) follow directly from [L] (see also [GL] for a convenient summary).

(iv) Suppose $\Pi$ is not exceptional. Then there exists a nontrivial character, $\eta$ say, of $E^\times$ such that $\Pi \otimes \eta \cong \Pi$, by Theorem 1.1. First assume $\eta$ factors through $N_{E/F}$, i.e., that $\eta = \chi E$ for some nontrivial character $\chi$ of $F^\times$. Then the base change of $\pi \otimes \chi$ to $G_1^1(E)$ is $\Pi \otimes \chi_E \cong \Pi$ [L] and thus, also by [L], $\pi \cong \pi \otimes \chi$ or $\pi \cong \pi \otimes \chi \otimes \omega_{E/F}$ where $\omega_{E/F}$ is the character of $F^\times$ associated to $E/F$ by local class field theory. But then since $\pi$ is exceptional, either $\chi$ is trivial or $\chi \otimes \omega_{E/F}$ is. By assumption $\chi$ is nontrivial so $\chi = \omega_{E/F}$ but then $\eta$ is trivial, a contradiction. Thus, assume $\eta$ does not factor through $N_{E/F}$, i.e., that $\eta^\sigma \neq \eta$ where $\langle \sigma \rangle = \Gamma(E/F)$ and $\eta^\sigma$ is the character of $E^\times$ defined by $\eta^\sigma(x) = \eta(x^\sigma)$. Now define the representation $\Pi^\sigma$ of $G_1^1(E)$ by $\Pi^\sigma(g) = \Pi(g^\sigma)$ where $\sigma$ acts coordinatewise. Then, by [L] since $\Pi$ is in the image of base change, $\Pi \cong \Pi^\sigma$. Thus, $\Pi \otimes \eta \cong \Pi$ implies that $\Pi \otimes \eta^\sigma \cong \Pi$ whence $\Pi \otimes \eta^\sigma \cong \Pi$. Now, $\eta^\sigma$ does factor through $N_{E/F}$ so it follows from the first part of this argument that $\eta^\sigma \cong 1$. But $\eta$ is of order two (see Theorem 1.1) and thus $\eta = \eta^\sigma$, a contradiction. \(\square\)

**Remark 1.4.** We only include a proof of (iv) above since we know of no place where it occurs in the literature. We, however, make no claim to the result.

Quadratic base change for $\text{SL}_2$ is then at the level of $L$-packets and can be summarized by the following theorem. Note that, with notation as in the theorem, the representations $\{\Pi_i\}_{i=1}^S$ actually factor to $\text{PSL}_2(E)$ since the central character of $\Pi$ is that of $\pi$ composed with $N_{E/F}$ [L].

**Theorem 1.5.** If $\{\pi_i\}_{i=1}^S$ is an $L$-packet for $\text{SL}_2(F)$, then the base change of $\{\pi_i\}_{i=1}^S$ to $\text{SL}_2(E)$ is the packet $\{\Pi_i\}_{i=1}^S$ obtained by restricting $\Pi$ to $\text{SL}_2(E)$.
where $\Pi$ is the base change of $\pi$, a representation of $\text{GL}_2(F)$ restricting to $\text{SL}_2(F)$ to give the $L$-packet $\{\pi_i\}_{i=1}^s$.

Finally, we need to recall a result of Cognet. To this end, we first recall the structure of the orthogonal group $H_1$. Let $H'_1$ denote the generalized orthogonal group attached to $Q$ and $V$. Then (see [D] for further details of this discussion) the map $\Psi : G'_1(E) \times F^x \to \text{End}_F(V)$ defined by

$$\Psi(g,u)(A) = \begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix} gA\bar{g},$$

where $-$ denotes Galois conjugation coordinatewise, is a homomorphism into $H'_1$. It has kernel

$$\left\{ \left( \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} , N(a)^{-1} \right) \bigg| a \in E^x \right\}$$

and image of index two. Further, $H'_1 \cong \text{Im } \Psi \rtimes \langle \sigma \rangle$ where $\sigma$ is the element of $V$ corresponding to the isometry of $V$ given by conjugation and $\text{Im } \Psi$ consists of those elements in $H'_1$ whose determinant is the square of their similitude factor. Now consider the restriction of $\Psi$ to those elements of the form $(g,u)$ with $N(\det g)u^2 = 1$; call this group $H_1 = H'_1 \rtimes \langle \sigma \rangle$. We map $G_1(E)$ to $H_1$ via the map $k(g) = \Psi((g,1))$ for $g$ in $G_1$. The kernel of $k$ is $\pm I$ and thus, in a slight abuse of notation, we can use $k$ to identify $\text{PSL}_2(E)$ with a subgroup of $H_1$. Then $k(\text{PSL}_2(E))$ is the commutator subgroup of $H_1$ and has index $2^{n+3}$ where $n = 0$ unless $p = 2$ in which case $n = [F : \mathbb{Q}_2]$. Indeed identifying $F^x$ with a subgroup of $H'_1$ via the map $i : F^x \to H'_1$ defined by

$$i(a) = \Psi \left( \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} , a^{-1} \right)$$

for $a$ in $F^x$ we get that $i(F^x)k(\text{PSL}_2(E)) \cong H'_1$ and $H'_1/k(\text{PSL}_2(E)) \cong i(F^x)/i((F^x)^2)$.

**Theorem 1.6 ([C]).** If $\pi$ is an irreducible infinite-dimensional representation of $G_1 = \text{SL}_2(F)$, then there exists $\pi'$ in the $L$-packet of $\pi$ such that $\pi'$ occurs in the theta correspondence attached to $\chi$ and $(G_1, H_1)$ and such that the corresponding representation of $H_1$, upon restriction to $\text{PSL}_2(E)$, decomposes as a sum of representations in the $L$-packet for $\text{SL}_2(E)$ obtained from that of $\pi$ by base change.

**Proof.** Cognet’s statements are at the level of the similitude groups $G'_1$ and $H'_1$. However, it is a straightforward argument using results relating similitude theta correspondences to regular theta correspondences (see, e.g., [B] or [Ro1]) to obtain the result above from [C].

$\square$

Although the emphasis in this section will be on representations which are not supercuspidal, our first result will apply to all representations in $\mathcal{R}_\chi(H_1)$. To begin, let $\det : H_1 \to \mathbb{C}^\times$ be the representation of $H_1$ defined by the determinant map. Then our first result is fairly standard but we provide a complete proof since we know of no good reference to the literature.

**Lemma 2.1.** If $\pi$ is an irreducible admissible representation of $H_1$ which is in $\mathcal{R}_\chi(H_1)$, then $\pi \otimes \det$ is not in $\mathcal{R}_\chi(H_1)$. In particular, $\pi$ and $\pi \otimes \det$ are not isomorphic and $\pi|_{H_0^1}$ is irreducible with $(\pi|_{H_0^1})^\sigma \cong \pi|_{H_0^1}$.

**Proof.** It suffices to show that $\pi \otimes \det$ does not occur. Suppose the contrary. Let $W'$ be a four-dimensional symplectic vector space over $F$ with form $\langle \cdot, \cdot \rangle'$ and identify $W'$ with two transverse copies of the space giving rise to $G_1$.

Then, since both $\pi$ and $\pi \otimes \det$ are in $\mathcal{R}_\chi(H_1)$ for the reductive dual pair $(G_1, H_1)$, it follows that $\pi \otimes (\pi \otimes \det)$ occurs in $\mathcal{R}_\chi(H_1 \times H_1)$ (defined relative to the pair $(G_1 \times G_1, H_1 \times H_1)$). But then since irreducible representations of orthogonal groups are self-contragredient [N], it follows that $\pi \otimes (\pi \otimes \det)$ restricted to $H_1$ has $\det$ as a quotient. Then from the reciprocity formula for seesaw reductive dual pairs (see, e.g., [P1] or [M]), it follows that $\det$ is in $\mathcal{R}_\chi(H_1)$ relative to the pair $(G(W'), H_1)$. But this contradicts [R1, Appendix] since $\dim (W')/2 \geq \dim V_1$ does not hold. \[\Box\]

An immediate consequence of the above lemma is that representations in $\mathcal{R}_\chi(H_1)$ are parameterized by their restriction, which is Galois invariant, to $H_0^1$. Our next step is to parameterize such representations in the nonsupercuspidal case.

Recall that we have $i(F^\times)k(\text{PSL}_2(E)) \cong H_0^1$. Let $j : E^1 \to H_0^1$ be the imbedding of $E^1$ in $H_0^1$ defined by

$$j(a) = \Psi \left( \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}, 1 \right)$$

with $\Psi$ also as in the previous section. We note that $i(-1) \neq j(-1)$ and that $i(-1)j(-1) = -I$, the nontrivial element of the center of $H_0^1$. We note that for $a$ in $E^\times$

$$i(N(a))j(a/\bar{a}) = k \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix},$$

(2.1)
as is easily checked.

Now let \( V' \) denote the subspace of \( V \) consisting of those matrices which are zero with the possible exception of the \((1, 1)\) entry. Let \( P \) denote the parabolic subgroup of \( H^0_1 \) which stabilizes \( V' \). Then one checks that \( P = MN \) where

\[
N = \left\{ k \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \mid a \in E \right\}
\]

and \( M = i(F^\times)j(E^1) \). Moreover, all proper parabolic subgroups of \( H^0_1 \) are conjugate to \( P \) and thus all irreducible nonsupercuspidal representations of \( H^0_1 \) may be realized as subrepresentations (or subquotients) of representations induced from \( P \). We parametrize these representations below. Since we use normalized induction, we note that the modulus function of \( P \), \( \delta_P \) say, is given by \( \delta_P(i(a)j(b)) = |a|^2_F \), as is easily checked. Also, note that

\[
\delta_P \left( k \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \right) = \delta_P(i(N(a))) = |N(a)|^2_F = |a|^2_E.
\]

Let \( T \) denote the subgroup of \( \text{PSL}_2(E) \) obtained by considering the diagonal matrices in \( \text{SL}_2(E) \) modulo \( \pm I \). We identify \( E^\times/(-1) \) with \( T \) via the map \( j' : E^\times \to T \) defined by

\[
j'(a) = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}.
\]

We view characters of \( M \) as characters of \( P \), as usual, by inflation. We also note that \( M/k(T) \cong H^0_1/k(\text{PSL}_2(E)) \cong F^\times/(F^\times)^2 \) where the first isomorphism is via the map induced by inclusion as can be checked using (2.1) and the second isomorphism is as was noted above. We will thus view characters of these groups interchangeably. Further, for any character of \( M \), we view \( \lambda|_{k(T)} \) as a character of \( E^\times \) via pullback along \( j' \circ k \).

**Lemma 2.2.** Let \( \lambda \) be a character of \( M \).

(i) If \( \lambda|_{k(T)} \) is not \( \mid | \) or \( \mid |^{-1} \) and is not of order two, then \( \text{Ind}(H^0_1, P; \lambda) \) is irreducible. It is Galois invariant if and only if \( \lambda^2 \) is trivial upon restriction to \( j(E^1) \) or \( i(F^\times) \); in these cases, set \( \pi(\lambda) = \text{Ind}(H^0_1, P; \lambda) \).

(ii) If \( \lambda|_{k(T)} = \mid | \), then \( \text{Ind}(H^0_1, P; \lambda) \) has a unique irreducible subrepresentation, \( \sigma(\lambda) \) say, and unique irreducible quotient, \( \pi(\lambda) \) say, both of which are Galois invariant. Further, \( \pi(\lambda) = \lambda \mid |^{-1} \).

(iii) Similarly, if \( \lambda|_{k(T)} = \mid |^{-1} \), then \( \text{Ind}(H^0_1, P; \lambda) \) has a unique irreducible subrepresentation, \( \pi(\lambda) \) say, and unique irreducible quotient, \( \sigma(\lambda) \) say, both of which are Galois invariant. Further, \( \pi(\lambda) = \lambda \cdot | | \).

(iv) Assume \( \lambda|_{k(T)} \) is of order two. Let \( \omega_\lambda \) be the associated character of \( E^\times \) and let \( E(\lambda)/E \) be the quadratic extension associated to \( \omega_\lambda \) by local class field theory. Then if \( E(\lambda)/F \) is biquadratic, then \( \text{Ind}(H^0_1, P; \lambda) \)
is the direct sum of two distinct irreducible Galois-invariant representations, \( \pi^+(\lambda) \) and \( \pi^-(\lambda) \) say, each of which remains irreducible upon restriction to \( k(\text{PSL}_2(E)) \) with the sign being determined by requiring that \( \pi^+(\lambda) \cong \pi(\lambda, 1; \chi \circ \text{tr}_{E/F}) \) as a representation of \( \text{PSL}_2(E) \). Furthermore, in this case \( \lambda \) itself is of order two. If \( E(\lambda)/F \) is cyclic, then \( \pi(\lambda) = \text{Ind}(H_1^0, P; \lambda) \) is irreducible and Galois invariant. In this case \( \lambda \) is not of order two but \( \lambda^2(j(E^1)) = \lambda^2(i(N_{E/F}(E^\times))) = 1 \). Finally, if \( E(\lambda)/F \) is not Galois, then \( \text{Ind}(H_1^0, P; \lambda) \) is either irreducible or the direct sum of two distinct irreducible representations. In either case, none of the irreducible representations obtained is Galois invariant.

(v) The \( \pi(\lambda), \pi^+\sigma(\lambda) \) and \( \sigma(\lambda) \) constructed above exhaust the nonsupercuspidal Galois-invariant portion of the admissible dual of \( H_1^0 \). Further, \( \pi(\lambda) \cong \pi(\lambda') \) if and only if \( \lambda' = \lambda \) or \( \lambda' = \lambda^{-1} \) and similarly for \( \sigma(\lambda) \). Finally, the representations enjoy no other equivalences.

Proof. The composition series statements in (i), (ii) and (iii) follow readily from the background material of the first section, in particular, Theorem 1.1, since \( i(F^\times)k(\text{PSL}_2(E)) = H_1^0 \) and

\[
i(a)k(g)i(a^{-1}) = k \left( \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} g \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}^{-1} \right)
\]

for \( g \) in \( \text{PSL}_2(E) \) and \( a \) in \( F^\times \). Now consider the composition series when \( \lambda|_{K(T)} \) is of order two. In this case, it follows from the material of the first section that \( \text{Ind}(H_1^0, P; \lambda) \) is irreducible or the direct sum of two irreducible representations with the latter occurring if and only if \( F^\times \) is contained in \( N_{E(\lambda)/E}(E(\lambda)^\times) \). If \( E(\lambda)/F \) is biquadratic, then the decomposition follows since \( N_{E(\lambda)/E}(E(\lambda)^\times) = \{ x \in E^\times \mid N_{E/F}(x) \in N_{E'/F}((E')^\times) \} \) where \( E' \) is any quadratic extension of \( F \) such that \( EE' = E(\lambda) \), see, e.g., [I, Theorem 7.6]. On the other hand if \( E(\lambda)/F \) is cyclic, then \( F^\times/N_{E(\lambda)/F}(E(\lambda)^\times) \) is cyclic of order four by local class field theory and thus \( N_{E(\lambda)/E}(E(\lambda)^\times) \) cannot contain \( (F^\times)^2 \) whence \( N_{E(\lambda)/E}(E(\lambda)^\times) \) cannot contain \( F^\times \). Finally, if \( E(\lambda)/F \) is not Galois, then the composition series statement follows from Theorem 1.1.

We now consider equivalences amongst the \( \pi(\lambda), \pi^\pm(\lambda) \) and \( \sigma(\lambda) \). We consider here only the infinite-dimensional representations, the other cases being trivial. Restricting to \( k(\text{PSL}_2(E)) \), we see that \( \pi(\lambda) \) can only be isomorphic to some \( \pi(\lambda') \) and similarly for \( \pi^\pm(\lambda), \pi^\mp(\lambda) \) and \( \sigma(\lambda) \). Suppose \( \pi(\lambda) \cong \pi(\lambda') \). Then, also by restricting to \( k(\text{PSL}_2(E)) \), we get that \( \lambda'|_{k(T)} = \lambda|_{k(T)} \) or \( \lambda'|_{k(T)} = \lambda^{-1}|_{k(T)} \). Suppose the former. Write \( \lambda' = \lambda\lambda'' \) with \( \lambda'' \) a character of \( M \) trivial on \( k(T) \). Now it follows from Frobenius reciprocity that \( \pi(\lambda) \otimes \lambda'' \cong \pi(\lambda\lambda'') \) with \( \lambda'' \) on the left-hand side viewed as a character of \( H_1^0 \). Thus, \( \pi(\lambda) \otimes \lambda'' \cong \pi(\lambda) \). This either implies \( \lambda'' = 1 \) or, by Theorem 1.1,
\(\lambda_{|k(T)}\) is of order two and \(\lambda'|_{i(F^\times)} \circ i = \omega_{E(\lambda)/E}|_{F^\times} \). If \(\lambda'' = 1\), we are done. Thus, assume the latter. In this case, \(\lambda''\) is completely determined by (2.1) since \(\lambda''|_{k(T)}\) is trivial. Now, also by (2.1), since \(\lambda|_{k(T)} = \lambda^{-1}|_{k(T)}\), it suffices to show that \(\lambda|_{i(F^\times)} = (\lambda')^{-1}|_{i(F^\times)}\). But for \(a \in F^\times\), we have
\[
\lambda^2(i(a)) = \lambda \left( k \left( \begin{array}{cc} a & 0 \\ 0 & a^{-1} \end{array} \right) \right)
\]
as is easily checked and thus
\[
\lambda \lambda'(i(a)) = \lambda^2 \lambda''(i(a)) \\
= \lambda^2(i(a)) \lambda''(i(a)) \\
= \lambda \left( k \left( \begin{array}{cc} a & 0 \\ 0 & a^{-1} \end{array} \right) \right) \omega_{E(\lambda)/E}(a) \\
= \omega_{E(\lambda)/E}(a) \omega_{E(\lambda)/E}(a) \\
= 1.
\]

Now in general considering
\[
\pi(\lambda)^q \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right)
\]
one sees from Frobenius Reciprocity that \(\pi(\lambda) \cong \pi(\lambda^{-1})\). Thus, the case \(\lambda'|_{k(T)} \cong \lambda^{-1}|_{k(T)}\) follows from the previous case. The arguments for the remaining equivalences are similar, using that the relevant composition series are multiplicity free.

Finally, we consider Galois invariance. Let \(\pi\) be a nonsupercuspidal irreducible representation of \(H_1^0\). Then \(\text{Hom}_{H_1^0} (\pi, \text{Ind}(H_1^0, P; \lambda)) \neq 0\) for some \(\lambda\), a one-dimensional representation of \(M\). Now one checks that \(\text{Ind}(H_1^0, P; \lambda)^\sigma \cong \text{Ind}(H_1^0, P; \lambda^\sigma)\) since the modulus character is invariant. Then it follows from a similar argument to that for the equivalences (invariance was not used) that \(\lambda = \lambda^\sigma\) or \(\lambda = (\lambda^{-1})^\sigma\). Now one checks that \(\lambda = \lambda^\sigma\) if and only if \(\lambda^2(j(E^1)) = 1\) and \(\lambda = (\lambda^{-1})^\sigma\) if and only if \(\lambda^2(i(F^\times)) = 1\).

The invariance portions of (i), (ii), (iii) then follow. Thus, assume \(\lambda|_{k(T)}\) is of order two. Now by local class field theory \(E(\lambda)/F\) is Galois if and only if \(E(\lambda) = E(\lambda^\sigma)\). Then since \(\lambda|_{k(T)}\) is of order two, it follows that \(E(\lambda) = E(\lambda^\sigma)\) if and only if \(\lambda|_{k(T)} = \lambda^\sigma|_{k(T)}\) and \(\lambda|_{k(T)} = (\lambda^{-1})^\sigma|_{k(T)}\). But these are equivalent to \(\lambda^2(j(E^1)) = 1\) and \(\lambda^2(i(N_{E/F}(E^\times))) = 1\). Finally, since
\[
\lambda^2(i(a)) = \lambda \left( k \left( \begin{array}{cc} a & 0 \\ 0 & a^{-1} \end{array} \right) \right)
\]
for all \(a \in F^\times\), our determination of the order of \(\lambda\) follows from our discussion of reducibility.

We now turn to Jacquet modules to find restrictions, in addition to Galois invariance, on the nonsupercuspidal representations of $H_1$ that can occur in $R_X(H_1)$. We realize $\omega_X$ on $S(V)$, the space of locally constant compactly supported functions on $V$ with the action of $H_1$ being the natural linear action, i.e., a Schrodinger model. Set $S(V)_N = S(V)/(f - \omega_X(n)f \mid f \in S(V), n \in N)$ and view $S(V)_N$ as an $M$-module (unnormalized) as usual. Finally, for $f$ in $S(V)$, let \( f \) denote the image of $f$ in $S(V)_N$.

For $l$ an integer, let $U_l$ denote the neighborhood of 0 in $V$ consisting of those $v$ such that $v_{i1}$ is in $P^l_F$ for $i = 1,2$ and $v_{12}$ is in $O_E P^l_F$ where $v_{ij}$ denotes the $(i,j)$-entry of $v$. Then the $U_l$ form a neighborhood basis of 0 in $V$. For $A$ in $V$ and $l$ an integer, define $f_{Ai,l}$ in $S(V)$ by setting

$$f_{Ai,l}(v) = \begin{cases} 1 & \text{if } A - v \text{ is in } U_l, \\ 0 & \text{otherwise.} \end{cases}$$

Then the $f_{Ai,l}$ span $S(V)$.

**Theorem 3.1.** Suppose $\lambda$ in $M^\wedge$ occurs as a quotient of $(\omega_X)_N$. Set $\lambda' = \lambda'_{1,1}$ and $\chi(E)$ trivial. Then either $\lambda'|_{1,1}$ is trivial or $\lambda'|_{k(T)}$ is trivial. Further, if $\lambda'|_{1,1}$ is trivial, then $\lambda'$ is determined by $\lambda'|_{k(T)}$. Finally, if $\lambda'|_{1,1}$ is trivial, then $\lambda'$ is determined by $\lambda'|_{k(T)}$ up to (possibly) twisting by $\omega_{E/F}$.

**Proof.** Let $T : S(V)_N \to \mathbb{C}$ be a nonzero element of $\text{Hom}_M((\omega_X)_N, \lambda)$. Then there exist $A$ and $l$ such that $T f_{Ai,l} \neq 0$.

We proceed by cases. First, suppose $\nu_F(a_{22}) < l$ and that $\nu_F(a_{12} a_{22}^{-1}) \geq 0$.

Let

$$n = \begin{pmatrix} 1 & -a_{12} a_{22}^{-1} \\ 0 & 1 \end{pmatrix}.$$ 

Then one checks that $\omega_X(k(n)) f_{Ai,l} = f_{Ai,l}$ with $A_1 = \begin{pmatrix} a_{11} & 0 \\ 0 & a_{22} \end{pmatrix}$. It follows that $T f_{Ai,l} \neq 0$ and thus $\lambda'(j(E^1)) = 1$ since $T \omega_X(j(a)) f_{Ai,l} = \lambda(j(a)) T f_{Ai,l}$ for all $a$ in $E^1$ as can be checked. Then by (2.1),

$$\lambda(i(N(a))) = \lambda \left( \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \right)$$

for $a$ in $F^\times$ and the result follows.

Now suppose that $\nu(a_{22}) < l$ but $\nu_F(a_{12} a_{22}^{-1}) < 0$. Choose $m$ such that $\nu_F((a_{12} a_{22}^{-1}) \omega_{E/F}^m) \geq 0$.
Let $U_{l,m} = \{ A \in U_l \mid a_{12} \in \mathcal{O}_E P_{F}^{l+m}, a_{22} \in P_{F}^{l+2m} \}$. For $B$ in $U_l$, set

$$f_{A,l,B}(v) = \begin{cases} 1 & \text{if } A + B - v \text{ is in } U_{l,m} \\ 0 & \text{otherwise.} \end{cases}$$

Then since $T \bar{f}_{A,l} \neq 0$, it follows that there exists $B$ in $U_l$ such that $T \bar{f}_{A,l,B} \neq 0$. Let

$$n = \begin{pmatrix} 1 & -(a_{12} + b_{12})(a_{22} + b_{22})^{-1} \\ 0 & 1 \end{pmatrix}.$$ 

Then one checks that $\omega_n(k(n)) f_{A,l,B} = f_{A_1,l,B_1}$ where $(A_1 + B_1)_{12} = 0$ and $(A_1 + B_1)_{ii} = (A + B)_{ii}$. Then as in the previous case, one checks that \( \lambda(j(E^1)) = 1 \) and

$$\lambda(i(N(a))) = \lambda \left( k \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \right)$$

for $a$ in $F^\times$, whence the result.

Finally, suppose that $\nu(a_{22}) \geq l$. In this case, we may assume $a_{22} = 0$. Now if $a_{12}$ is in $\mathcal{O}_E P_{F}^l$, then arguing as above we obtain that $\lambda(j(E^1)) = 1$ and

$$\lambda(i(N(a))) = \lambda \left( k \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \right)$$

for $a$ in $F^\times$ and we are done. Thus, suppose $a_{12}$ is not in $\mathcal{O}_E P_{F}^l$. Now suppose $a_{11}$ is in $P_{F}^l$. Then since $E/F$ is separable, there exists an integer $l'$ such that $\text{tr}_{E/F}(P_{E}^{l'} a_{12}) = P_{F}^l$. Similarly, if $a_{11}$ is not in $P_{E}^{l'}$, then there exists $l'$ such that $\text{tr}_{E/F}(P_{E}^{l'} a_{12}) = P_{F}^l$. Let $m' \geq l$ be an integer such that $N(P_{E}^{m'}) \subseteq P_{F}^{l+1}$ and $\text{tr}(P_{E}^{m'} P_{F}^{l'}) \subseteq P_{F}^{l+1}$. Further, let $m \geq m'$ be an integer such that $P_{E}^{m'} P_{F}^m \subseteq \mathcal{O}_E P_{F}^{m'}$. Finally, let $U'$ denote the neighborhood of $0$ in $V$ of $v$ such that $\nu_F(v_{11}) \geq l, \nu_F(v_{22}) > m$ and $v_{12}$ is in $\mathcal{O}_E P_{F}^{m'}$. Then by an argument similar to the above in the case $a_{22} = 0$, we may assume that $T \bar{f}_{A}^l$ is nonzero where $A'$ is in $V$ such that $A - A'$ is in $U_l$, and for any $B$ in $V$, $f_{B}'$ in $S(V)$ is defined by

$$f_{B}'(v) = \begin{cases} 1 & \text{if } v - B \text{ is in } U', \\ 0 & \text{otherwise.} \end{cases}$$

Without loss of generality, we assume $A' = A$ so that $T \bar{f}_{A}^l \neq 0$.

Now, by our choice of $l'$, there exists $x$ in $P_{E}^{l'}$ such that $\text{tr}_{E/F}(xa_{12}) = a_{11}$. Let

$$n = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}.$$ 

Then one checks that, by virtue of our choices of $m$ and $m', \omega_n(k(n)) f_{A}^l = f_{B}'$ with $b_{11} = 0, b_{12} = a_{12}$ and $b_{22} = a_{22} = 0$. Thus, we may assume that $a_{11} = \ldots$
0 in considering $Tf_A' \neq 0$. It then follows that, for $b$ in $O_F^\times, \omega_\chi(i(b))f_A' = f_A'$ and thus $\lambda(i(b)) = \lambda'(i(b)) = 1$. Hence to complete the proof, it suffices to show that $\lambda(i(\omega_F^{-1})) = q$.

For $B$ in $V$, define $f_B''$ in $S(V)$ by

$$f_B''(v) = \begin{cases} 1 & \text{if } v - A \text{ is in } i(\omega_F^{-1})U', \\ 0 & \text{otherwise.} \end{cases}$$

Then it is clear that $\omega_\chi(i(\omega_F^{-1}))f_A' = f_A''$. Now

$$f_A'' = \sum_{B \in R} g_{A+B}$$

where $R$ is a set of coset representatives for $U''/i(\omega_F^{-1})U'$ with $U'' = \{v \in U' : v_F(v_{22}) \geq m + 1\}$ and $g_{A+B}$ is defined as follows:

$$g_{A+B}(v) = \begin{cases} 1 & \text{if } v - A - B \text{ is in } U'', \\ 0 & \text{otherwise.} \end{cases}$$

Then arguing as above in the case $a_{11} \neq 0$, one shows that $\bar{g}_{A+B} = \bar{g}_A$ for all $B$ in $R$. Thus,

$$\omega_\chi(i(\omega_F^{-1}))f_A' = q\bar{g}_A$$

so that

$$T\omega_\chi(i(\omega_F^{-1}))f_A' = qT\bar{g}_A$$

with both sides nonzero. But now

$$g_A = \sum_{C \in S} f_A'' + C$$

with $S$ a set of coset representatives for $U'/U''$. Then by an argument similar to that in case $a_{22} \neq 0$, the result of the theorem follows if $Tf_{A+C}'$ is nonzero for any $C$ not in $U''$. Thus, we may assume $T\bar{g}_A = T\bar{f}_A$ and then the theorem follows.

As an immediate consequence of the above results, the exactness of the Jacquet functor and the adjointness of the Jacquet functor and induction, we have the following.

**Corollary 3.2.** Let $\pi$ be an irreducible nonsupercuspidal representation of $H_1$ which is in $\mathcal{R}_\chi(H_1)$. Then $\pi_0 = \pi|_{H_0}$ is irreducible and $\pi$, as an element of $\mathcal{R}_\chi(H_1)$, is determined by $\pi_0$. Moreover, $\pi_0$ is of the form $\pi(\lambda), \sigma(\lambda)$, or $\pi^\pm(\lambda)$ for some $\lambda$ with $\lambda|_{i(F^\times)}$ trivial or $\lambda|_{j(E')}$ trivial. In particular, with $\lambda$ as above, $\lambda$ is determined by $\lambda|_{k(T)}$ if $\lambda|_{i(F^\times)}$ is trivial and is determined by $\lambda|_{k(T)}$ up to a (possible) twist of $\omega_{E/F}$ if $\lambda|_{j(E')}$ is trivial.
To close this section we prove an elementary lemma that will be useful in what follows. The statements on restriction in the lemma can easily be made more precise but we leave that to the reader since the following suffices, for our purposes.

**Lemma 3.3.** Let $\pi$ be an irreducible representation of $H_1$ such that $\pi_0 = \pi|_{H_1^0}$ is irreducible. Moreover, assume $\pi_0$ is of the form $\pi(\lambda), \sigma(\lambda)$ or $\pi^\pm(\lambda)$ for some $\lambda$ with $\lambda|_{i(F)}$ or $\lambda|_{j(E^1)}$ trivial.

(i) Suppose $\lambda|_{i(F^\times)}$ is trivial. Then $\pi_0$ is of the form $\pi(\lambda)$ or $\pi^\pm(\lambda)$ and when restricted to $k(\text{PSL}_2(E))$ decomposes as a sum of representations in the $L$-packet associated to $\pi(\rho, \rho^\sigma)$ where $\rho$ is any character of $E^\times$ such that $\rho(a/\bar{a}) = \lambda(j(a/\bar{a}))$ for $a$ in $F^\times$.

(ii) Suppose $\lambda|_{j(E^1)}$ is trivial. Then if $\pi_0$ is of the form $\sigma(\lambda)$, then $\pi_0$ restricts to $\sigma(\lambda \circ i \circ N)$ on $k(\text{PSL}_2(E))$. If $\pi_0$ is of the form $\pi(\lambda)$ or $\pi^\pm(\lambda)$, then, when restricted to $\text{PSL}_2(E)$, $\pi_0$ decomposes as a sum of representations in the $L$-packet attached to $\pi(\lambda \circ i \circ N, 1)$.

**Proof.** Let $\pi_1$ be an irreducible representation of $\text{PSL}_2(E)$ that appears in the restriction of $\pi$ to $k(\text{PSL}_2(E))$. Then since $i(F^\times)k(\text{PSL}_2(E)) = H_1^0$, any other representation appearing in the restriction of $\pi$ to $k(\text{PSL}_2(E))$ must be of the form $\pi_1^{i(a)}$ for $a$ in $F^\times$. But then

$$i(a)k(g)i(a^{-1}) = k \left( \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} g \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}^{-1} \right)$$

and

$$k \begin{pmatrix} b & 0 \\ 0 & b^{-1} \end{pmatrix} = i(N(b))j(b/\bar{b})$$

imply the result. \hfill $\square$


In the previous section, we found some necessary conditions, Corollary 3.2, for nonsupercuspidal representations of $H_1$ to occur in the correspondence. In this section, we will show these conditions are also sufficient, with one possible exception. The one possible exception is a generalized Steinberg representation as is explained in (ii) and (iii) of the following theorem.

**Theorem 4.1.** Let $\pi_0$ be an irreducible representation of $H_1^0$.

(i) If $\pi_0$ is of the form $\pi(\lambda)$ or $\pi^\pm(\lambda)$ with $\lambda|_{i(F^\times)}$ trivial or $\lambda|_{j(E^1)}$ trivial, then $\pi_0$ has a unique extension to $H_1$ which occurs in $\mathcal{R}_\chi(H_1)$.

(ii) If $\pi_0 = \sigma(\cdot | \cdot)$ or $\sigma(\omega_{E/F} | \cdot)$ then at most one extension of $\pi_0$ to $H_1$ occurs in $\mathcal{R}_\chi(H_1)$. 


(iii) At least one extension of $\sigma(\lambda |\lambda|$ or $\sigma(\omega_{E/F} |\lambda|$ occurs in $\mathcal{R}_\chi(H_1)$ and pairs with the Steinberg representation $\sigma(\lambda |\lambda$) of $G_1$.

(iv) No other nonsupercuspidal representations of $H_1$ can occur.

Proof. It suffices to show (i) through (iii) since then (iv) would follow from Corollary 3.2.

First consider (i) and suppose that $\lambda_{i(E^\times)}$ is trivial. Let $\theta$ be a character of $E^\times$ such that $\theta|_{E^1} = \lambda|_{j(E^1)} \circ j$. Now suppose further that $\lambda$ is not of order two. Then it follows from Theorem 1.2 that $\pi(\theta;\chi)$ occurs in $\mathcal{R}_\chi(G_1)$. Further, by Kudla’s perseverance result [K2], it must pair with $\pi(\lambda)$. Likewise, if $\lambda_{i(E^\times)}$ is trivial but $\lambda$ is of order two, then $\pi(\theta;\chi)$ and $\pi(\theta;\chi_b)$ occur in $\mathcal{R}_\chi(G_1)$ and pair with $\pi^+(\lambda)$ where $b$ is in $N_{E/F}(E^\times)$ such that $\pi(\theta;\chi)$ and $\pi(\theta;\chi_b)$ are distinct. Therefore, in proving the theorem, we may assume that $\lambda_{i(E^\times)}$ is nontrivial.

Now consider $\lambda$ with $\lambda_{i(E^\times)}$ nontrivial. Let $\mu$ be a character of $F^\times$. Assume for the moment that $\mu_E \neq 1$ and that $\mu_E^2$ is not trivial. Then $\mu^2$ is also nontrivial and thus $\pi(\mu,1)$ and $\pi(\mu_E,1)$ restrict irreducibly to $G_1(F)$ and $G_1(E)$, respectively, with $\pi(\mu_E,1)$ the base change of $\pi(\mu,1)$.

Now by Theorem 1.2, $\pi(\mu,1)$ is in $\mathcal{R}_\chi(G_1)$. Then since $\mu \neq 1$, $\pi(\mu,1)$ is infinite-dimensional and thus, by Theorem 1.6, the image of $\pi(\mu,1)$ under the theta correspondence, $\theta(\pi(\mu,1))$ say, must restrict to $k(PSL_2(E))$ as a sum of copies of $\pi(\mu_E,1)$. Now since $\mu^2$ is nontrivial, $\pi(\mu,1)$ does not occur in the theta correspondence attached to $\chi$ and $(G_1, H_0)$ where $H_0$ is the orthogonal group attached to the anisotropic part of $(V,Q)$. Further, since the correspondence attached to $\chi$ and $(G_1, H_1)$ is a bijection, even in $p = 2$ [R2], it follows from the argument of the previous paragraph that a $\lambda$ giving rise to $\theta(\pi(\mu,1))$ must satisfy $\lambda_{i(E^1)}$ is trivial. Then it follows from Lemma 3.3 that we may assume $\lambda$ must satisfy $\mu_E = \lambda \circ i \circ N$, whence $\lambda = \lambda_1$ or $\lambda_2$ where

$$\lambda_l(i(a)j(b/b)) = \mu(a)\omega_{E/F}(a)$$

for $l = 1, 2$, $b \in E^\times$ and $a \in F^\times$. It follows that both $\pi(\mu,1)$ and $\pi(\mu_E \omega_{E/F},1)$ occur in $\mathcal{R}_\chi(G_1)$ and pair with either $\pi(\lambda_1)$ or $\pi(\lambda_2)$ in $\mathcal{R}_\chi(H_1)$. Then, once again, since the correspondence is a bijection, we get that the theorem holds for all $\pi(\lambda)$ with $(\lambda_{i(N(E^\times))})^2$ nontrivial and $\lambda_{i((N(E)^\times))} \neq 1$.

If $\lambda$ is trivial on $j(E^1)$ and $\lambda(i(a)) = |a|$, then $\pi(\lambda)$ is the trivial representation and, as is well-known, $\pi(\lambda)$ occurs in $\mathcal{R}_\chi(H_1)$ and pairs with $\pi(\omega_{E/F} | \lambda, 1)$ in $\mathcal{R}_\chi(G_1)$, see [KR2]. Further, by Theorem 1.6 and Lemma 3.3, $\sigma(\lambda |\lambda)$ is in $\mathcal{R}_\chi(G_1)$ and pairs with $\sigma(\lambda |\lambda)$ or $\sigma(\omega_{E/F} | \lambda)$ in $\mathcal{R}_\chi(H_1)$.

Now let $\mu$ be a character of $F^\times$ of order two with $\mu_E$ nontrivial. Then the $L$-packets for $G_1(F)$ and $G_1(E)$ associated to $\pi(\mu,1)$ and $\pi(\mu_E,1)$, respectively, each have two components as does the $L$-packet for $G_1(F)$ attached
to $\pi(\omega_{E/F}\mu, 1)$. By Theorem 1.2, the four representations in the $L$-packets associated to $\pi(\mu, 1)$ and $\pi(\mu\omega_{E/F}, 1)$ occur. Then, by arguments similar to those above, they must pair with the four representations of $H_1$, as in Lemma 3.3, attached to $\lambda$ and $\omega_{E/F}\lambda$ where $\lambda$ is the character of $M$ defined by

$$\lambda(i(a)j(b/\bar{b})) = \mu(a).$$

Further, consider the representation $\pi(1, 1)$ of $G'_1(F)$. It restricts irreducibly to $G_1(F)$ and by arguments also similar to those above it occurs and pairs with the representation $\pi(\lambda)$ of $H_1$ with

$$\lambda(i(a)j(b/\bar{b})) = \omega_{E/F}(a).$$

Finally, consider those nontrivial characters of $\mu$ on $F^\times$ such that $\mu^2$ is nontrivial while $\mu^2_{E}$ is trivial. Such a character has order four and by local class field theory is associated to a cyclic extension of degree four of $F$ with the quadratic subfield being $E$. Further, if $\mu$ is such a character, then so is $\mu\omega_{E/F}$. Then arguments such as those above show that $\pi(\mu, 1)$ and $\pi(\mu\omega_{E/F}, 1)$ are in $R_\chi(H_1)$ and pair with $\pi(\lambda_1)$ and $\pi(\lambda_2)$ in $R_\chi(H_1)$ where

$$\lambda_l(i(a)j(b/\bar{b})) = \mu(a)\omega_{E/F}^l(a),$$

$l = 1, 2$.

To summarize, at this point we have shown that the theorem holds for all representations of $H'_1$ with the possible exception of the one-dimensional representation $\pi(\omega_{E/F}|\chi)$. Let $\pi_+(\omega_{E/F}|\chi)$ denote the extension of $\pi(\omega_{E/F}|\chi)$ to $H_1$ with $\sigma$ acting as the identity. Then the following lemma completes the proof of the theorem.

**Lemma 4.2.** The one-dimensional representation $\pi_+(\omega_{E/F}|\chi)$ occurs in $R_\chi(H_1)$ and pairs with the trivial representation of $G_1$.

**Proof.** It suffices to show that $\text{Hom}_{H_1}(\omega_{E/F}^\infty, \pi_+(\omega_{E/F}|\chi))$ is one-dimensional. Write $\pi_+(\omega_{E/F}|\chi) = \omega_{E/F}^+$. Let

$$x = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

in $V$. Then one checks that the stabilizer of $x$ in $H_1$ is $H_x = j(E^1)k(N)\times\langle\sigma\rangle$. One checks further that $H_x$ is unimodular and that $\omega_{E/F}^+|_{H_x}$ is trivial since the image of the spinor norm restricted to $H_x \cap H'_1$ is $N(E^\times)$ because

$$k\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} = i(N(a))j(a/\bar{a}).$$
Thus, it follows from [W] that there exists a unique, up to scalar, linear map $T : \mathcal{S}(H_1/H_x) \to \mathbb{C}$ such that

\[
Tf^g = \omega^+_{E/F}(g)Tf
\]

where $f^g(x) = f(gx)$. Now let $Y = Q^{-1}(0)$ and $Y' = Y - \{0\}$. Then by Witt’s Theorem, $Y'$ can be identified with $H_1/H_x$ and thus we can view $T$ as a distribution on $Y'$.

We claim further that $T$ can be extended to a distribution on $\mathcal{S}(Y)$ satisfying (4.1) for all $f$ in $\mathcal{S}(Y)$. To see this we recall that, up to a nonzero scalar, $T$ can be written

\[
Tf = \int_{H_1/H_x} \omega^+_{E/F}(y)f(y)dy
\]

for all $f$ in $\mathcal{S}(H_1/H_x)$ where $\int_{H_1/H_x} dy$ is a left-invariant positive regular Borel measure on $C_c(H_1/H_x)$, the space of compactly supported functions on $H_1/H_x$ and $\omega^+_{E/F}(y)$ is well-defined since $\omega^+_{E/F}$ is trivial on $H_x$. Moreover, the measure is finite on compact sets and may be normalized so that

\[
\int_{H_1/H_x} \int_{H_x} f(yh)dhdy = \int_{H_1} f(g)dg
\]

(4.2)

where $\int_{H_1} dg$ and $\int_{H_x} dh$ denote Haar measure on $H_1$ and $H_x$ respectively and $f$ is any function in $C_c(H_1)$. We claim that $T$ can be extended to $\mathcal{S}(Y)$ by setting $Tf = Tf|_Y$, for all $f$ in $\mathcal{S}(Y)$. To show this it suffices to show if $U$ is a compact neighborhood of 0 in $Y$ then $U \cap Y'$ is of finite volume with respect to $\int_{H_1/H_x} dy$. To this end, using a Bruhat decomposition, $H_1/H_x$ can be identified with

\[
i(F^X) \cup k(N)k(w)i(F^X)
\]

where $w$ is the standard Weyl element $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ for $SL_2(F)$. Now consider the effect of conjugation by $i(a), a \in F^X$, on (4.2). Since $H_1$ is unimodular we have

\[
\int_{H_1} f(y)dg = \int_{H_1} f(i(a)g(i(a))^{-1})dg
\]

(4.3)

\[= \int_{H_1/H_x} \int_{H_x} f(i(a)y(i(a))^{-1}i(a)h(i(a))^{-1})dhdy\]

\[= \int_{H_1/H_x} |a|^{-1} \int_{H_x} f(y(i(a))^{-1}h)dhdy\]

where the last equality follows from explicit realization of the measure $\int_{H_x} dh$ and left-invariance of $\int_{H_1/H_x} dy$. Finite volume then follows from explicitly realizing $\int_{H_1/H_x} dy$ taking into account the $|a|^{-1}$. It is immediate that the extension satisfies (4.1) since $Y'$ is $H_1$ invariant.
Now let $T : S(Y) \to \mathbb{C}$ denote the extension constructed above. Then since $Y$ is closed in $V$ we can extend $T$ to $S(V)$ by setting $Tf = T(f|_Y)$. Further since $Y$ is $H_1$ invariant $T$ still satisfies (4.1), whence $T$ is a nonzero intertwining map. Furthermore, $T$ is the unique, up to scalar, nonzero intertwining map with image in $D(V - Y)$, the space of distributions on $S(V - Y)$, equal to zero since $\{0\}$ only supports the trivial representation.

Now suppose $S : S(V) \to \mathbb{C}$ is an arbitrary nonzero element in $\text{Hom}_{H_1}(\omega^{\infty}_E, \omega^+_E/F)$. Then it suffices to show that the image of $S$ in $D(V - Y)$ is zero. Suppose that the image is nonzero. Then by [BZ], there exists an orbit $Q^{-1}(a), a \in F^\times$ which supports a nontrivial intertwining operator. Let $X = Q^{-1}(a), a \in F^\times$, be such an orbit and let $y \in X$. Then since $Q(X) \neq 0$, we can write $V = \langle y \rangle \oplus W$ with $W = \langle y \rangle^\perp$. Now let $O(W)$ be the orthogonal group associated to $W$ and $Q|_W$. Then $X = H_1/O(W)$. Now since $W$ is three-dimensional, the spinor norm restricted to $O(W)$ is onto $F^\times$. Further, both $H_1$ and $O(W)$ are unimodular and thus any intertwining operator which is trivial when restricted to $O(W)$ is onto $F^\times$. Further, both $H_1$ and $O(W)$ are unimodular and thus any intertwining operator which is trivial when restricted to $O(W)$ must have been trivial on $H_1$ [W]—contradicting the surjectivity of the spinor norm. □

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Received June 1, 1999 and revised July 11, 2000. This research was supported in part by NSF grant DMS-9003213 and NSA grant MDA904-97-1-0046.

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UNIFORM INTEGRABILITY OF APPROXIMATE GREEN FUNCTIONS OF SOME DEGENERATE ELLIPTIC OPERATORS

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We prove the uniform integrability of the approximate Green functions of some degenerate elliptic operators in divergence form with lower order term coefficients satisfying a Kato type condition. Some further properties of the approximate Green functions of such operators are also established.

1. Introduction.

In this paper, we study the approximate Green functions of certain degenerate elliptic operators $L$ on balls in $\mathbb{R}^n$, $n > 2$, when $L$ has the divergence form

$$L := -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left( a_{ij}(x) \frac{\partial}{\partial x_j} \right) + \sum_{i=1}^{n} b_i(x) \frac{\partial}{\partial x_i} + V(x).$$

The coefficients $a_{ij}$ are real-valued measurable functions whose coefficient matrix $A(x) := (a_{ij}(x))$ is symmetric and satisfies

$$\omega(x)|\xi|^2 \leq \langle A(x)\xi, \xi \rangle \leq \upsilon(x)|\xi|^2. \tag{1.1}$$

Here $\langle ., . \rangle$ denotes the usual Euclidean inner product, and $\upsilon, \omega$ are weight functions that will be stipulated below.

Throughout, we will use the following notations. For functions $f$ and $g$, we shall write $f \lesssim g$ to indicate that $f \leq Cg$ for some positive constant $C$. We write $f \approx g$ if $f \lesssim g$ and $g \lesssim f$. We shall use $B_t(x)$ to designate a ball of radius $t$ centered at $x$. Also, $tB$ will be used to represent the ball concentric with the ball $B$, but with radius $t$ times as big. Given a locally integrable function $f$, we shall let $f(B)$ denote the Lebesgue integral of $f$ over the set $B$. If $f \in L_{loc}(d\mu)$, where $d\mu := \gamma(x) dx$ is a weighted measure, then we denote by

$$\int_B f(x)\gamma(x) dx := \frac{1}{\gamma(B)} \int_B f(x)\gamma(x) dx,$$

the $\mu$-average of $f$ over $B$. This average shall also be denoted by $f_B, \gamma$. 

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A non-negative locally integrable function $\omega$ on $\mathbb{R}^n$ is said to be in the class $A_2$ if $1/\omega$ is also locally integrable and there is a constant $C$ such that for all balls $B$,

$$\left( \int_B \omega(x) \, dx \right) \left( \int_B \frac{1}{\omega(x)} \, dx \right) \leq C.$$

A non-negative locally integrable function $\upsilon$ on $\mathbb{R}^n$ is said to satisfy a doubling condition if there is a constant $C$ such that $\upsilon(2B) \leq C\upsilon(B)$ for all balls $B$. Here $C$ is independent of the center and radius of $B$. We denote this by writing $\upsilon \in D\infty$. It is known that $A_2 \subset D\infty$.

It is also known (see [9]) that if $\upsilon$ satisfies a doubling condition, then it satisfies

$$\upsilon(tB) \leq C_1 t^k \upsilon(B), \quad \text{and} \quad \upsilon(B) \leq C_2 t^{-m}(tB), \quad t > 1,$$

for some positive constants $C_1$, $C_2$, $k$, and $m$. The second condition is called a reverse doubling condition.

The following assumptions will be made on $\omega$ and $\upsilon$.

$\omega$ and $\upsilon$ are non-negative locally integrable functions on $\mathbb{R}^n$ that satisfy the following conditions:

\begin{align}
(1.2) \quad &\quad \omega \in A_2, \quad \upsilon \in D\infty; \\
(1.3) \quad &\quad \omega \text{ and } \upsilon \text{ are related by the existence of some } q > 2 \text{ such that} \\
&\quad \frac{s}{t} \left[ \frac{\upsilon(B_s(x))}{\upsilon(B_t(x))} \right] \frac{1}{q} \leq C \left[ \frac{\omega(B_s(x))}{\omega(B_t(x))} \right] \frac{1}{2}, \quad 0 < s < t, \quad x \in \mathbb{R}^n,
\end{align}

for some constant $C$ independent of $x$, $s$, and $t$.

We shall use the notation $\sigma = q/2$ so that $\sigma > 1$. Note that when $\upsilon$ and $\omega$ are positive constants, as in the strongly elliptic case, the value of $q$ in (1.3) is $q = 2n/(n-2)$, so that $\sigma = n/(n-2)$.

Let now $L_0$ be the principal part of $L$; that is

$$L_0 := -\sum_{i,j=1}^n \partial \frac{\partial}{\partial x_i} \left( a_{ij}(x) \frac{\partial}{\partial x_j} \right).$$

Let $B_0$ be a ball of radius $R$ that will be fixed in the sequel. Under the conditions (1.2) and (1.3), Chanillo and Wheeden have established, in [3] the existence and integrability properties of the Green function of $L_0$. Among other important properties, they have shown that if $G(x, y)$ is the Green function of $L_0$ on $2B_0$, then for $0 < p < \sigma$,

\begin{align}
(1.4) \quad &\quad \sup_{y \in B_0} \int_{2B_0} G(x, y)^p \upsilon(x) \, dx < \infty.
\end{align}
Let $B \subset B_0$. In analogy with the way the usual Kato class is defined, we introduce a class of functions $K^n(B)$ as

$$K^n(B) := \left\{ h \in L^1_{\text{loc}}(B) : \lim_{r \to 0^+} \eta(h)(r) = 0 \right\},$$

where

$$\eta(h)(r) := \sup_{x \in B} \int_{B_r(x) \cap B} G(y,x) |h(y)| dy.$$

If $L^p_\mu(B)$ denotes the usual $L^p$ space with respect to the measure $\mu$, then for $B \subset B_0$, and $p > \sigma/(\sigma - 1)$, the following inclusion holds:

$$L^p_\upsilon(B) \subset K^n(B).$$

To see this let $h \in L^p_\upsilon(B)$, and $x \in B$. We pick $\sigma/(\sigma - 1) < s < p$. Define $s'$ by $1/s + 1/s' = 1$ (we will use this notation throughout). Then, by Hölder inequality

$$\int_{B_r(x) \cap B} G(y,x) |h(y)| \, dy \leq \left( \int_{2B_0} G(y,x)^s \upsilon(y) \, dy \right)^{\frac{1}{s'}} \cdot \left( \int_{B_r(x) \cap B} |h(y)|^{s} \upsilon^{1-s} \, dy \right)^{\frac{1}{s}}.$$

Since $\upsilon$ satisfies a reverse doubling condition, there exist positive constants $C$ and $d$ such that $\upsilon(B_{R}(x)) \geq C(R/r)^d \upsilon(B_r(x))$ for any $0 < r < R$. Therefore

$$\int_{B_r(x) \cap B} \left( \frac{|h(y)|}{\upsilon} \right)^s \upsilon \, dx \leq \left( \int_{B_r(x) \cap B} \left( \frac{|h(y)|}{\upsilon} \right)^p \, dx \right)^{\frac{s}{p}} \left( \int_{B_r(x)} \upsilon \, dx \right)^{\frac{p-s}{p}} \leq \left( \int_{B} \left( \frac{|h(y)|}{\upsilon} \right)^p \, dx \right)^{\frac{s}{p}} \left[ C \left( \frac{r}{R} \right)^d \right]^{\frac{p-s}{p}} \left[ \upsilon(2B_0) \right]^{\frac{p-s}{p}}.$$

Thus, from this last inequality and (1.4), we get the desired conclusion.

For notational simplicity, we shall use $K$ for the function space $K^n(B_0)$.

**Remark 1.1.** We should remark that when $\upsilon$ and $\omega$ are identically equal to positive constants, as in the strongly elliptic case, the class of functions $K$ coincides with the usual Kato class (see [1], or [4] for definition). Also, if $\upsilon$ and $\omega$ are constant multiples of each other, then again $K$ is the same as the one introduced in [6].

The following assumptions will be made of the lower order coefficients $b := (b_1, b_2, \cdots, b_n)$, and $V$ of the degenerate elliptic operator $L$.

$$|b|^2 \omega^{-1}, \quad V \in K.$$
The paper is organized as follows. As the work here relies heavily on the results of the important works of S. Chanillo and R. Wheeden in their papers [2], and [3], we will recall several of their results that are relevant to our discussion in Section 2. We start Section 3 by proving the boundedness of certain linear functionals on some Hilbert spaces. These functionals are associated with elements of the Kato type class defined above. Some properties related to the approximate Green function of $L_0$ will also be obtained. The main result in this paper is Theorem 3.2 which establishes the uniform integrability of approximate Green functions of $L_0$ on balls. Uniform integrability of approximate Green functions is a useful tool in proving existence and size estimates of the Green function. See [3], [5] and [8] for such applications. In a forthcoming paper, we will use this uniform integrability result to derive Harnack’s inequality for functions naturally associated with non-negative solutions of the operator $L$.

2. Preliminaries and background.

Let $\Omega \subset \mathbb{R}^n$ be a bounded open set. Using a standard notation, let Lip$(\overline{\Omega})$ denote the class of Lipschitz continuous functions on the closure $\overline{\Omega}$. We say that $\phi \in$ Lip$_0(\Omega)$ if $\phi \in$ Lip$(\Omega)$ and $\phi$ has compact support contained in $\Omega$.

The following two-weight Sobolev inequality has been proved in [2].

Let $\omega, \upsilon$ be non-negative locally integrable functions that satisfy (1.2), (1.3), and $q$ be the constant that appears in (1.3). Then, for any ball $B$,

$$\left( \int_B |f|^{q} \upsilon \, dx \right)^{\frac{1}{q}} \leq C |B|^{\frac{1}{2}} \left( \int_B |\nabla f|^{2} \omega \, dx \right)^{\frac{1}{2}}, \quad f \in \text{Lip}_0(B).$$

The constant $C$ is independent of both the ball $B$ and $f$.

Now let us consider the inner product

$$a_0(u, \varphi) := \int_{\Omega} \langle A \nabla u, \nabla \varphi \rangle, \quad u, \varphi \in \text{Lip}_0(\Omega).$$

The completion of Lip$_0(\Omega)$ with respect to the norm $a_0(u, u)^{1/2}$ is denoted by $H_0(\Omega)$. An element of $H_0(\Omega)$ is thus an equivalence class of Cauchy sequences $\{u_k\}$, $u_k \in$ Lip$_0(\Omega)$. If $u, \varphi \in H_0(\Omega)$, with $u = \{u_k\}$, $\varphi = \{\varphi_k\}$, $u_k, \varphi_k \in$ Lip$_0(\Omega)$, then $a_0(u_k, \varphi_k)$ is convergent, and we define

$$a_0(u, \varphi) = \lim_{k} a_0(u_k, \varphi_k).$$

In this way, $\|u\|_0 := a_0(u, u)^{1/2}$ defines a norm on the Hilbert space $H_0(\Omega)$. Lip$_0(B)$ is included in $H_0(B)$ by considering $\{\varphi_k\}$ with all $\varphi_k = \varphi \in$ Lip$_0(B)$. As a consequence of the Sobolev inequality (2.1), it is possible to associate with each $\varphi \in H_0(\Omega)$ a unique pair $(\tilde{\varphi}, \nabla \tilde{\varphi})$ so that if $\varphi = \{\varphi_k\}$, then $\varphi_k \rightarrow \tilde{\varphi}$ in $L^2_\upsilon(\Omega)$, and $\nabla \varphi_k \rightarrow \nabla \tilde{\varphi}$ in $L^2_\omega(\Omega)$. We shall refer to $(\tilde{\varphi}, \nabla \tilde{\varphi})$
as the pair of functions associated with \( \varphi \). This pair is independent of the particular representation \( \{ \varphi_k \} \) of \( \varphi \). If \( \varphi \in \text{Lip}_0(\Omega) \), then \( \tilde{\varphi} = \varphi \), and \( \nabla \tilde{\varphi} = \nabla \varphi \). Furthermore, it can be shown that given \( \varphi \in H_0(\Omega) \), \( \nabla \tilde{\varphi} \) is the distributional gradient of \( \tilde{\varphi} \). See [3] for proofs of these assertions.

For future reference, we record the following inequality that can be easily verified using the Cauchy-Schwarz inequality.

\[
\|\varphi \varphi\|_0 \leq \|\varphi\|_\infty \|\varphi\|_0 + \|\varphi\|_\infty \|\varphi\|_0, \quad \varphi, \phi \in \text{Lip}_0(\Omega).
\]

We will also consider the Hilbert space \( H(\Omega) \) which is the completion of \( \text{Lip}(\Omega) \) under the inner product

\[
a(u, \varphi) := a_0(u, \varphi) + \int_\Omega u \varphi v, \quad u, \varphi \in \text{Lip}(\Omega).
\]

If \( u \in H(\Omega), u = \{u_k\}, u_k \in \text{Lip}(\Omega) \), then \( u_k \) converges in \( L^2(\Omega) \) to a function \( \tilde{u} \), and \( \nabla u_k \) converges in \( L^2(\Omega) \) to a vector \( \nabla \tilde{u} \). If \( \varphi = \{\varphi_k\}, \varphi_k \in \text{Lip}(\Omega) \), then the limits \( a(u, \varphi) = \lim_k a(u_k, \varphi_k) \) and \( a_0(u, \varphi) = \lim_k a_0(u_k, \varphi_k) \) exist, and satisfy

\[
a(u, \varphi) := a_0(u, \varphi) + \int_\Omega \tilde{u} \tilde{\varphi} v.
\]

In this way, \( a(u, \varphi) \) defines an inner product on \( H(\Omega) \), and \( \|u\| := a(u, u)^{1/2} \) defines a norm. By the Sobolev inequality (2.1), \( H_0(\Omega) \) is continuously embedded in \( H(\Omega) \).

For \( u \in H(\Omega) \) we say that \( u \geq 0 \) on \( \Omega \), if \( u_k \geq 0 \) for all \( k \) and some \( \{u_k\} \) representing \( u \). If \( u \geq 0 \) on \( \Omega \), then \( \tilde{u} \geq 0 \) a.e. on \( \Omega \). The following, proved in [3], will be useful to us.

Let \( u, \varphi \in H(\Omega) \), and \( \nabla \tilde{u}, \nabla \tilde{\varphi} \) be the associated gradients respectively. If \( u = \{u_k\}, \varphi = \{\varphi_k\}, \) then as \( k \to \infty \)

\[
\int_\Omega |\langle A \nabla u_k, \nabla \varphi_k \rangle - \langle A \nabla \tilde{u}, \nabla \tilde{\varphi} \rangle| \to 0.
\]

In particular

\[
a_0(u, \varphi) = \int_\Omega \langle A \nabla \tilde{u}, \nabla \tilde{\varphi} \rangle, \quad \text{and} \quad a(u, \varphi) = \int_\Omega \langle A \nabla \tilde{u}, \nabla \tilde{\varphi} \rangle + \int_\Omega \tilde{u} \tilde{\varphi} v.
\]

Before we proceed further, we should perhaps make two remarks. Let \( B \subset B_0 \) be a ball.

**Remark 2.1.** If \( f \nu^{-1} \in L^{(2\sigma)^{\prime}}(B) \), then

\[
\varphi \mapsto \int_B f \tilde{\varphi}
\]

defines a continuous linear functional on \( H_0(B) \). This follows from Hölder’s inequality and the Sobolev inequality (2.1). Therefore, by the Lax-Milgram
theorem there is a unique \( u \in H_0(B) \) such that

\[
a_0(u, \varphi) = \int_B f \tilde{\varphi}.
\]

We shall refer to \( u \) as the Lax-Milgram solution of \( L_0u = f \) in \( B \) and \( u = 0 \) on \( \partial B \).

By the above Remark, given \( x \in B \), and \( \rho > 0 \) with \( B_\rho(x) \subset B \) there is a unique \( \tilde{G}_\rho \in H_0(B) \) such that

\[
a_0(\tilde{G}_\rho, \varphi) = -\int_{B_\rho(x)} \tilde{\varphi} \nu, \quad \varphi \in H_0(B).
\]

\( \tilde{G}_\rho \) is called an approximate Green’s function of \( L_0 \) on \( B \) with pole \( x \).

**Remark 2.2.** Let \( f \in L^1(B) \) such that the map in (2.4) is a continuous linear functional on \( H_0(B) \). Suppose \( \{\varphi_k\} \) is a bounded sequence in \( H_0(B) \). Then since \( |a_0(u, \varphi)| \leq \|u\| \|\varphi\|_0 \), we have

\[
a_0(u, \varphi) = \lim a_0(u, \varphi_k) = \lim \left[ a_0(u_k, \varphi_k) + a_0(u - u_k, \varphi_k) \right]
\]

\[
= \lim a_0(u_k, \varphi_k).
\]

Therefore

\[
a_0(u, \varphi) - \int_B f \tilde{\varphi} = \lim_{j \to \infty} \left[ a_0(u_k, \varphi_k) - \int_B f \tilde{\varphi}_k \right].
\]

We shall need several lemmas from [3], and we will state them below for the readers’ convenience.

**Lemma 2.1.** Suppose \( u \) is a supersolution in \( H_0(\Omega) \); that is \( u \in H_0(\Omega) \), and \( a_0(u, \varphi) \geq 0 \) whenever \( 0 \leq \varphi \in \text{Lip}_0(B) \). Then \( u \geq 0 \).

**Proof.** The proof that the approximate Green function \( G^\rho \) of \( L_0 \) is non-negative is given on page 323 of [3]. It depends on properties of the inner product \( a_0(\ldots) \) and the fact that \( a_0(G^\rho, \varphi) \geq 0 \) for \( 0 \leq \varphi \in H_0(\Omega) \). Exactly the same proof applies in our case. In fact, if \( u := \{u_k\} \), then also \( u = \{|u_k|\} \).

See Lemma 3.6 below for a detailed proof.

**Lemma 2.2.** Let \( G(x, y) \) be the Green function of \( L_0 \) on \( 2B \) and \( f \nu^{-1} \in L^t_v(2B) \) for some \( t < \sigma \). If \( u \) is the Lax-Milgram solution of \( L_0u = f \) in \( 2B \), then

\[
\tilde{u}(y) = \int_{2B} G(x, y)f(x) \, dx, \quad \text{for a.e. } y \in B.
\]

Another useful Lemma is the following weak maximum principle (cf. Lemma 2.1 above).
Let \( u \in H(\Omega) \) satisfy \( a_0(u, \varphi) \geq 0 \) if \( \varphi \in \text{Lip}_0(\Omega) \), \( \varphi \geq 0 \). Let \( u = \{u_k\} \), \( u_k \in \text{Lip}(\Omega) \) and assume that \( u_k \geq 0 \) in some neighborhood (depending on \( k \)) of \( \partial \Omega \). Then \( \bar{u} \geq 0 \) a.e in \( \Omega \).

Let \( \tilde{G}^\rho \) be the approximate Green function of \( L_0 \) on a ball \( B \) with pole \( x \in B \), and \( G \) be the corresponding Green function. In [3], it was shown that for an appropriate subsequence, \( \tilde{G}^{\rho_k}(y) \to G(x, y) \) pointwise a.e. on \( B \) for a.e. \( x \in \frac{1}{2}B \). If \( B \subset B^* \), then the weak maximum principle, Lemma 2.3 shows that \( \tilde{G}^\rho \leq \tilde{G}^\rho_k \) a.e on \( B \) if \( \tilde{G}^\rho_k \) is the approximate Green function of \( L_0 \) on \( B^* \) with pole \( x \). Consequently, the inequality \( G \leq G^* \) holds a.e. on \( \frac{1}{2}B \times \frac{1}{2}B \), where \( G^* \) is the Green function of \( L_0 \) on \( B^* \).

We also need Lemma (2.7) of [3] in the following slightly modified form. To accommodate this change, we shall indicate the minor alterations needed in the proof of Lemma (2.7) of [3].

**Lemma 2.4.** Let \( B_j := B_j(x_0) \) be balls of radius \( r_j \) for \( j = 1, 2, 3 \), with \( r_j < r_{j+1} \). If \( \varphi \in H(B_3) \) and \( \tilde{\varphi} \leq m \) a.e. in \( B_2 \), then given any \( M > m \), and \( r_1 < r < r_2 \), there exist \( \varphi_k \in \text{Lip}(B_3) \) such that \( \varphi_k \to \varphi \) in \( H(B_3) \) and \( \varphi_k \leq M \) a.e. on \( B^*(x_0) \), a ball of radius \( r \).

**Proof.** As in [3], we pick \( h_k \in \text{Lip}(\overline{B}_3) \) with \( h_k \to \varphi \) in \( H(B_3) \). Thus \( h_k \to \tilde{\varphi} \) in \( L^2_v(B_3) \), and by using a subsequence, we may assume that \( h_k \to \tilde{\varphi} \) a.e. on \( B_3 \). By hypothesis, \( \tilde{\varphi} \leq m \) a.e. on \( B_2 \). By Egorov’s theorem, given \( M > m \), and \( \delta > 0 \), there exist \( E \subset B_2 \), and \( k_0 \) such that \( |B_2 \setminus E| < \delta \) and \( h_k \leq M \) on \( E \), if \( k \geq k_0 \). Let \( \chi \in C_0^\infty(B_2), 0 \leq \chi \leq 1 \), and \( \chi \equiv 1 \) on \( B^* \), where \( B^* := B^*(x_0) \) is a ball of radius \( r \). We now define \( \varphi_k := h_k \chi \land M + h_k(1 - \chi) \). Clearly \( \varphi_k \in \text{Lip}(\overline{B}_3) \), and \( \varphi_k \leq M \) a.e. on \( B^* \). It now remains to show that \( \varphi_k \to \varphi \) in \( H(B_3) \). Noting that \( \varphi_k - h_k \) is supported on \( B_2 \), the rest of the proof proceeds in the same way as that of Lemma (2.7) of [3]. \[ \square \]

**Remark 2.3.** If \( \varphi \geq 0 \) in the sense of \( H(B_3) \), then the \( \varphi_k \) in Lemma 2.4 can be taken to be non-negative, as can be seen from the definition of \( \varphi_k \) in the proof.

**Remark 2.4.** Let \( 0 \leq m \), and \( \varphi \in H_0(B) \) such that \( \tilde{\varphi} \leq m \) a.e. on \( B \). Then given \( M > m \), we can choose \( \varphi_k \in \text{Lip}_0(B) \) such that \( \varphi_k \to \varphi \) in \( H_0(B) \), and \( \varphi_k \leq M \) a.e. on \( B \). This follows from the proof of Lemma 2.4 by extending \( \tilde{\varphi} \) to be zero outside \( B \).

### 3. Approximate Green functions.

The following embedding lemma is useful in the subsequent development. In proving the Lemma, we adapt a method used in [6], in the case of equal weights.
Lemma 3.1. If \( f \in K \), and \( B \subset \subset B_0 \) is a ball of radius \( r \), then for any \( u \in H_0(B) \) the following holds.

\[
\int_B |f| \bar{u}^2 \, dx \lesssim \eta(f)(3r) \int_B \langle A \nabla \bar{u}, \nabla \bar{u} \rangle.
\]

Proof. Let \( G^* \) and \( G \) be the Green functions of \( L_0 \) on \( 2B^* \) and \( 2B_0 \), respectively, where \( B^* \) is a concentric slight enlargement of \( B \). As pointed out in the remark following Lemma 2.3, we first observe that \( G^* \leq G \) a.e. on \( B^* \times B^* \). Let \( f_k = |f| \wedge (kv), \ k = 1, 2, \ldots \), and note that \( f_k v^{-1} \in L^1_0(2B) \) for any \( t \). Since \( \omega \in A_2 \), and \( \omega \leq v \), we see that \( v \) can not vanish on a set of positive Lebesgue measure. Therefore \( f_k \rightarrow |f| \) a.e. on \( B \). Thus once the inequality in the Lemma is shown to hold for \( f_k \), then by Fatou’s Lemma, it will also hold for \( f \). So there is no loss of generality in assuming that \( f v^{-1} \in L^1_0(2B) \) for some \( t < \sigma \).

First, let us suppose that \( u \in \text{Lip}_0(B) \). Let \( \zeta := \{ \zeta_k \} \in H_0(2B^*) \) be the Lax-Milgram solution of \( L_0 \zeta = |f| \chi_B \) in \( 2B^* \) and \( \zeta = 0 \) on \( \partial(2B^*) \). Let \( \zeta \) be the associated function. Then by the representation theorem in Lemma 2.2, we know that for a.e. \( x \in B^* \)

\[
\tilde{\zeta}(x) = \int_B G^*(x, y)|f(y)| \, dy \leq \int_B G(x, y)|f(y)| \, dy
\]

\[
\leq \int_{3B^*(x) \cap 2B_0} G(x, y)|f(y)| \, dy.
\]

Therefore, \( \tilde{\zeta}(x) \leq \eta(f)(3r) \) for a.e \( x \in B^* \). By Lemma 2.1, and Lemma 2.4 we pick a sequence \( \zeta_k \in \text{Lip}_0(2B^*) \) such that \( 0 \leq \zeta_k \to \zeta \) in \( H_0(2B^*) \) and \( \zeta_k \lesssim \eta(f)(3r) \) a.e. on \( B \). By extending \( u \) to be zero outside \( B \), we consider the element \( \varphi = \{ u^2 \} \in H_0(2B^*) \). Then, we write

\[
\delta_k + \int_{2B^*} |f| u^2 = \int_{2B^*} \langle A \nabla \zeta_k, \nabla u^2 \rangle,
\]

where

\[
\delta_k = \int_{2B^*} \langle A \nabla \zeta_k, \nabla u^2 \rangle - \int_{2B^*} |f| u^2.
\]

By Cauchy-Schwarz inequality, we have

\[
\langle A \nabla \zeta_k, \nabla u^2 \rangle = 2 \langle A(u \nabla \zeta_k), \nabla u \rangle \leq 4 \eta \langle A \nabla u, \nabla u \rangle + \frac{1}{4 \eta} \langle A(u \nabla \zeta_k), u \nabla \zeta_k \rangle,
\]

where \( \eta := \eta(f)(3r) \). But

\[
\langle A(u \nabla \zeta_k), u \nabla \zeta_k \rangle = \langle A \nabla \zeta_k, u^2 \rangle - \langle A \nabla \zeta_k, \nabla (u^2 \zeta_k) \rangle - 2 \langle A(u \nabla \zeta_k), \zeta_k \nabla u \rangle
\]

\[
\leq \langle A \nabla \zeta_k, \nabla (u^2 \zeta_k) \rangle + \frac{1}{2} \langle A(u \nabla \zeta_k), u \nabla \zeta_k \rangle + 2 \langle A \zeta_k \nabla u, \zeta_k \nabla u \rangle.
\]
That is,
\[
(3.3) \quad \langle A(u \nabla \zeta_k), u \nabla \zeta_k \rangle \leq 2 \langle A \nabla \zeta_k, \nabla (u^2 \zeta_k) \rangle + 4 \langle A \nabla u, \nabla u \rangle \zeta_k^2.
\]
Using (3.3) in (3.2), we obtain
\[
\int_{2B^*} \langle A \nabla \zeta_k, \nabla u^2 \rangle \\
\leq 4\eta \int_{2B^*} \langle A \nabla u, \nabla u \rangle + \frac{1}{2\eta} \int_{2B^*} \langle A \nabla \zeta_k, \nabla (u^2 \zeta_k) \rangle \\
+ \frac{1}{\eta} \int_{2B^*} \langle A \nabla u, \nabla u \rangle \zeta_k^2 \\
= 4\eta \int_{2B^*} \langle A \nabla u, \nabla u \rangle + \frac{1}{2\eta} \int_{2B^*} |f| u^2 \zeta_k + \frac{1}{\eta} \int_{2B^*} \langle A \nabla u, \nabla u \rangle \zeta_k^2 + \frac{1}{2\eta} \gamma_k,
\]
where
\[
\gamma_k := \int_{2B^*} \langle A \nabla \zeta_k, \nabla (u^2 \zeta_k) \rangle - \int_{2B^*} |f| u^2 \zeta_k.
\]
Using this in (3.1), and recalling that supp(u) ⊂ B, and 0 ≤ \zeta_k ≤ \eta a.e. on B, we obtain
\[
(3.4) \quad \delta_k + \int_B |f| u^2 \lesssim 5\eta \int_B \langle A \nabla u, \nabla u \rangle + \frac{1}{2} \int_B |f| u^2 + \frac{1}{2\eta} \gamma_k.
\]
By (2.2), \{\phi_k\} := \{u^2 \zeta_k\} is easily seen to be bounded in \(H_0(2B^*)\). Therefore there is a weakly convergent subsequence which we continue to denote by \{\phi_k\}. Using this subsequence, and recalling that \(\zeta \in H_0(2B^*)\) is the Lax-Milgram solution of \(L_0 \zeta = |f| \chi_B\) in \(2B^*\) and \(\zeta = 0\) on \(\partial(2B^*)\), we see by Remark 2.2 that \(\delta_k \to 0\), and \(\gamma_k \to 0\) as \(k \to \infty\). Therefore, taking the limit as \(k \to \infty\) in the inequality (3.4), we conclude
\[
\int_B |f| u^2 \lesssim 5\eta \int_B \langle A \nabla u, \nabla u \rangle + \frac{1}{2} \int_B |f| u^2,
\]
from which follows the desired result when \(u \in \text{Lip}_0(B)\).

To prove the Lemma for \(u \in H_0(B)\), suppose \(u = \{u_k\}, u_k \in \text{Lip}_0(B)\). For each \(k\), we have
\[
\int_B |f| u_k^2 \lesssim 10\eta \int_B \langle A \nabla u_k, \nabla u_k \rangle.
\]
Take a subsequence of \(\{u_k\}\) that converges pointwise a.e. to \(\tilde{u}\) on \(B\). By appealing to (2.3), and Fatou’s Lemma we get the desired result after taking the limit as \(k \to \infty\). \(\square\)

**Remark 3.1.** Let \(f \in K\), and \(B \subset \subset B_0\) be a ball. Using Hölder inequality, followed by an application of Lemma 3.1, the map
\[
\varphi \mapsto \int_B f \varphi
\]
is seen to be continuous on \( H_0(B) \). Therefore, by the Lax-Milgram theorem there is a unique \( \zeta \in H_0(B) \) such that
\[
a_0(\zeta, \varphi) = \int_B f \tilde{\varphi}, \quad \text{for } \varphi \in H_0(B).
\]
We will also refer to \( \zeta \) as the Lax-Milgram solution of \( L_0 \zeta = f \) on \( B \), \( \zeta = 0 \) on \( \partial B \).

**Lemma 3.2.** Let \( f \in K \), and \( B \subset B_0 \) be a ball of radius \( r \).

1. If \( \tilde{G}^0 \) is the approximate Green function of \( L_0 \) on \( B \), then
\[
\int_B |f| \tilde{G}^0 \lesssim \eta(f)(2r).
\]
2. If \( \xi \in H_0(B) \) is the Lax-Milgram solution of \( L_0 \xi = |f| \) in \( B \), then
\[
\tilde{\xi}(x) \lesssim \eta(f)(2r), \quad \text{for a.e. } x \in B.
\]

**Proof.** First we show that if \( \zeta \in H_0(2B) \) is the Lax-Milgram solution of \( L_0 \zeta = |f| \chi_B \) in \( 2B \), then \( \zeta(x) \lesssim \eta(f)(2r) \) for a.e. \( x \in 2B \). To this end, let us write \( |f|^{(k)} := |f| \wedge (kv), \ k = 1, 2, 3, \cdots \), and \( |f|^{(0)} := |f| \). As in the proof of Lemma 3.1, we can argue that \( |f|^{(k)} \rightarrow |f| \) pointwise a.e. on \( B \). By (2.1) and Lemma 3.1 the map
\[
\varphi \in H_0(2B) \mapsto \int_{B^*} |f|^{(k)} \tilde{\varphi},
\]
is seen to be continuous on \( H_0(2B) \) for all \( k = 0, 1, 2, \cdots \). Let \( \zeta^{(k)} \in H_0(2B) \) be the Lax-Milgram solution of \( L_0 \zeta = |f|^{(k)} \chi_{B^*} \) on \( 2B \). Here \( B^* \) is a ball concentric to \( B \) but with radius \((1-\epsilon)r\) for small \( \epsilon > 0 \). For \( k = 1, 2, 3, \cdots \), note that \( |f|^{(k)} v^{-1} \in L^1_t(B), \ t < \sigma \). Then, by the representation formula of Lemma 2.2, we have for a.e. \( x \in B \), and \( k = 1, 2, \cdots \),
\[
\tilde{\zeta}^{(k)}(x) = \int_{B^*} G_r(x,y) |f|^{(k)}(y) \, dy \leq \int_{B^*} G(x,y) |f|^{(k)}(y) \, dy
\]
\[
\leq \int_{2B_r(x) \cap 2B_0} G(x,y) |f|(y) \, dy \leq \eta(f)(2r),
\]
where \( G_r \) and \( G \) denote the Green functions of \( L_0 \) on \( 2B \) and \( 2B_0 \) respectively. We have used the fact that \( G_r \leq G \) on \( B \), which is valid by the weak maximum principle, Lemma 2.3. By Lemma 2.4, there is a sequence \( \{\zeta_{m}^{(k)}\} \) in \( \text{Lip}_0(2B) \) such that \( \zeta_{m}^{(k)} \rightarrow \zeta^{(k)} \) in \( H_0(2B) \), and \( \zeta_{m}^{(k)} \lesssim \eta(f)(2r) \) a.e. on a ball concentric with \( B \), and of radius strictly between that of \( B^* \), and \( B \). Now let us observe that \( \zeta^{(k)} \), for \( k = 1, 2, 3, \cdots \), is the weak solution of \( L_0 \zeta = 0 \) on \( 2B \setminus B^* \) such that \( C \eta(f)(2r) - \zeta_{m}^{(k)} \geq 0 \) on a neighborhood of \( \partial(2B \setminus B^*) \). Therefore by the weak maximum principle, Lemma 2.3, we conclude that \( \zeta^{(k)} \lesssim \eta(f)(2r) \) on \( 2B \setminus B^* \), and hence on \( 2B \). To show
that the same bound holds for $\zeta := \zeta^{(0)}$, let $\tilde{G}_*^\rho$ be the approximate Green function of $L_0$ on $2B$, and let us observe that
\[
\int_{B_\rho} (\bar{\zeta} - \bar{\zeta}^{(k)})v = a_0(G_*^\rho, \zeta - \zeta^{(k)}) = a_0(\zeta - \zeta^{(k)}, G_*^\rho) = \int_B (|f| - |f|^{(k)}\chi_{B^c})\tilde{G}_*^\rho,
\]
for all $k$. Since $|f|\tilde{G}_*^\rho \in L^1(B)$, we invoke the Lebesgue dominated convergence theorem to conclude that
\[
\int_{B_\rho} \bar{\zeta}v = \lim_{k \to \infty} \int_{B_\rho} \bar{\zeta}^{(k)}v + \int_B |f|(1 - \chi_{B^c})\tilde{G}_*^\rho \leq C\eta(f)(2r) + \int_{B \sim B^c} |f|\tilde{G}_*^\rho.
\]
After letting $\epsilon \to 0$, we obtain
\[
\int_{B_\rho} \bar{\zeta}v \lesssim \eta(f)(2r).
\]
This leads to the claimed estimate after taking the limit as $\rho \to 0$, namely we get,
\[
\hat{\zeta}(x) \lesssim \eta(f)(2r), \quad \text{for a.e. } x \in 2B.
\]
We now use this result to prove the statement in (1). To see this, let us take the Lax-Milgram solution $\zeta$ of $L_0\zeta = |f|\chi_B$ in $H_0(2B)$. Let $\tilde{G}^\rho$, and $\tilde{G}_*^\rho$ be the approximate Green functions of $L_0$ on $B$, and $2B$ respectively. Since $G_*^\rho - G^\rho$ is a solution of $L_0$ in $B$, and $\tilde{G}_*^\rho - \tilde{G}^\rho \geq 0$ near $\partial B$, by Lemma 2.3 we note that $\tilde{G}^\rho \leq \tilde{G}_*^\rho$ on $B$. Therefore, by (3.5) above,
\[
\int_B |f|\tilde{G}^\rho \leq \int_B |f|\tilde{G}_*^\rho = a_0(\zeta, G_*^\rho) = a_0(G_*^\rho, \zeta) = \int_{B_\rho} \bar{\zeta}v \lesssim \eta(f)(2r).
\]
The statement in (2) is now an easy consequence of (1). To see this, let $\xi \in H_0(B)$ be the Lax-Milgram solution of $L_0\xi = |f|$ in $B$. Let $\tilde{G}^\rho$ be the approximate Green function of $L_0$ on $B$. Then
\[
\int_{B_\rho} \bar{\xi}v = a_0(G^\rho, \xi) = a_0(\xi, G^\rho) = \int_B |f|\tilde{G}^\rho \lesssim \eta(2r).
\]
Taking the limit as $\rho \to 0$, we obtain the desired result. \quad \square

The next Lemma is a slight extension of (2.3), and we will use it repeatedly.

**Lemma 3.3.** Let $u = \{u_k\}, \varphi = \{\varphi_k\}$ be in $H(B)$. If $\{\zeta_k\}$ is a bounded sequence in $L^\infty(B)$ that converges pointwise a.e. to $\zeta \in L^\infty(B)$, then
\[
\int_B \langle A\nabla u_k, \nabla \varphi_k \rangle \zeta_k \to \int_B \langle A\nabla \bar{u}, \nabla \varphi \rangle \zeta, \quad \text{as } k \to \infty.
\]
Proof. Since
\[ \int_B |\langle A \nabla u_k, \nabla \varphi \rangle \zeta_k - \langle A \nabla \tilde{u}, \nabla \tilde{\varphi} \rangle \zeta_k| \]
is not bigger than
\[ \| \zeta_k \|_\infty \int |\langle A \nabla u_k, \nabla \varphi \rangle - \langle A \nabla \tilde{u}, \nabla \tilde{\varphi} \rangle| + \int |\langle A \nabla \tilde{u}, \nabla \tilde{\varphi} \rangle \zeta_k - \zeta_k|, \]
and \( \langle A \nabla \tilde{u}, \nabla \tilde{\varphi} \rangle \zeta \in L^1(B) \) the Lemma follows from (2.3), and the Lebesgue dominated convergence theorem. \( \Box \)

Let us now consider the general elliptic operator:
\[ Mu := -\text{div}(A(x)\nabla u + c(x)u) + b(x) \cdot \nabla u + V(x)u, \]
where, in addition to (1.5) we also assume that \(|c|^2 \omega^{-1} \in K\). With \( M \), and its adjoint operator
\[ M^* u := -\text{div}(A(x)\nabla u + b(x)u) + c(x) \cdot \nabla u + V(x)u, \]
we associate the bilinear forms \( D(.,.) \) and \( D_*(.,.) \) as follows. Fix a ball \( B \subset B_0 \) of radius \( r \), and let
\[ D(u, \varphi) := \int_B \langle A \nabla u, \nabla \varphi \rangle + c(x) \cdot (\nabla \varphi)u + b(x) \cdot \nabla u \varphi + Vu \varphi, \]
and \( D_*(u, \varphi) := D(\varphi, u) \), for all \( u, \varphi \in \text{Lip}_0(B) \). Observe that by Hölder inequality and Lemma 3.1, it follows
\[ |D(u, \varphi) - a_0(u, \varphi)| \leq \vartheta(r) \|u\|_0 \|\varphi\|_0, \quad u, \varphi \in \text{Lip}_0(B), \]
where \( \vartheta(r) := (\eta(|c|^2 \omega^{-1})(3r))^{1/2} + (\eta(|b|^2 \omega^{-1})(3r))^{1/2} + \eta(V)(3r) \). Therefore, we get
\[ |D(u, \varphi)| \leq (1 + \vartheta(r)) \|u\|_0 \|\varphi\|_0, \quad u, \varphi \in \text{Lip}_0(B). \]
Thus if \( u = \{u_k\}, \varphi = \{\varphi_k\}, u_k, \varphi_k \in \text{Lip}_0(B) \) are elements of \( H_0(B) \) then the above inequality shows that \( \{D(u_k, \varphi_k)\} \) is a Cauchy sequence and hence \( \lim_k D(u_k, \varphi_k) \) exists. Therefore we define
\[ D(u, \varphi) := \lim_k D(u_k, \varphi_k). \]
Having defined \( D(u, \varphi) \) for \( u, \varphi \in H_0(B) \), the inequality (3.7) still holds for any \( u, \varphi \in H_0(B) \). As a result of this inequality we see that for a fixed \( u \in H_0(B) \), the map \( \varphi \mapsto D(u, \varphi) \) is a continuous linear functional on \( H_0(B) \).

Using (3.6) one also obtains \( a_0(u, u)(1 - C \vartheta(r)) \lesssim D(u, u) \), for \( u \in \text{Lip}_0(B) \). Therefore for sufficiently small \( r_0 \), and all \( 0 < r \leq r_0 \), we have
\[ \|u\|_0^2 \lesssim D(u, u), \quad u \in H_0(B), \]
so that \( D(.,.) \) is a coercive bilinear form on \( H_0(B) \).
Given \( f \in K \), we shall say that \( u \in H_0(B) \) is a weak solution of \( Mu = f \) in \( B \) if
\[
D(u, \varphi) = \int_B f \tilde{\varphi},
\]
for all \( \varphi \in H_0(B) \). Similar statements and definitions hold for the adjoint operator \( M^* \) and the associated bilinear form \( D_*(.,.) \).

The following remark will be useful at several stages in our subsequent proofs.

**Remark 3.2.** Let \( f \in K \), and \( u = \{u_k\} \in H_0(B) \) be a weak solution of \( Mu = f \) in \( B \). If \( \{v_k\} \) is a bounded, weakly convergent sequence in \( H_0(B) \), then
\[
\lim_{k \to \infty} \left[ D(u_k, v_k) - \int_B f \tilde{v_k} \right] = 0.
\]
This can be verified along the lines of argument given in Remark 2.2, since the linear functionals \( \varphi \mapsto D(u, \varphi) \), and \( \varphi \mapsto \int_B f \tilde{\varphi} \) are continuous on \( H_0(B) \).

**Lemma 3.4.** Suppose \( f \nu^{-1} \in L^p_\nu(B) \) for some \( p > \frac{\sigma}{\sigma - 1} \), and some \( B \subset B_0 \), where \( B \) is a ball of radius \( r \). Then there is a unique solution \( u \) of \( L_0u = f \) in \( H_0(B) \), and the following estimate holds.
\[
\| \tilde{u} \|_{L^\infty(B)} \leq Cr^{2 \nu(B) \frac{1}{\nu}} \| f \nu^{-1} \|_{L^p_\nu(B)}.
\]

*Proof.* The existence and uniqueness follows by the Lax-Milgram Theorem as pointed out in Remark 2.1.

Let \( \tilde{G}^\rho \) be the approximate Green function of \( L_0 \) on \( B \) with pole \( y \in B \). Then
\[
\int_{B_y} \tilde{u}v = a_0(G^\rho, u) = a_0(u, G^\rho) = \int_B f \tilde{G}^\rho,
\]
so that by Hölder’s inequality,
\[
\left| \int_{B_y} \tilde{u}v \right| = \left| \int_B f \tilde{G}^\rho \right| \leq \| f \nu^{-1} \|_{L^p_\nu(B)} \left( \int_B (\tilde{G}^\rho)^{p'}v \right)^{\frac{1}{p'}}
\]
\[
\leq Cr^{2 \nu(B) \frac{1}{p'}} \| f \nu^{-1} \|_{L^p_\nu(B)}.
\]
In the last inequality, we used, (see [3]) the fact that, when \( 1 < p' < \sigma \),
\[
\left( \int_B (\tilde{G}^\rho)^{p'}v \right)^{\frac{1}{p'}} \leq Cr^{2 \nu(B) \frac{1}{p'}} \omega(B),
\]
(3.8)
where $C$ is independent of $\rho$ and the pole of $\tilde{G}^\rho$. If we now let $\rho \to 0$, we conclude

$$|\tilde{u}(y)| \leq C r^2 \frac{v(B)^2}{\omega(B)} \|fv^{-1}\|_{L^q(B)}.$$ 

This and the arbitrariness of $y \in B$ establishes the Lemma. \hfill \Box

**Remark 3.3.** Let $G^\rho$ be the approximate Green function of $L_0$ on $B$. Thus $G^\rho$ is the Lax-Milgram solution of $L_0G^\rho = v\chi_{B_\rho}[v(B_\rho)]^{-1}$. Therefore, by Lemma 3.4, $\|\tilde{G}^\rho\|_{L^\infty(B)} \leq C$ for some constant $C$ depending on $\rho$, the pole of $G^\rho$, and $v$.

For the next Lemma, given $B \subset B_0$ we take $f \in L^1(B)$ such that the map

$$\varphi \mapsto \int_B f\tilde{\varphi}, \quad \varphi \in H_0(B)$$

is continuous on $H_0(B)$. Furthermore, we require that

$$\int_B |f\tilde{G}^\rho = O(1), \quad \text{as} \quad \rho \to 0^+,$$

where $\tilde{G}^\rho$ is the approximate Green function of $L_0$ on $B$.

**Lemma 3.5.** If $u \in H_0(B)$ is the unique solution of $L_0u = f$ on $B$, then $u$ has a representative $u = \{u_k\}, u_k \in \text{Lip}_0(B)$ such that $\|u_k\|_\infty \leq M$ uniformly in $k$ for some constant $M$.

**Proof.** Let $u^{(+)} \in H_0(B)$, and $u^{(-)} \in H_0(B)$ be the solutions of $L_0u^{(+)} = f^+$, and $L_0u^{(-)} = f^-$ respectively. Here $f^+ := \max\{0,f\}$, and $f^- := \max\{0,-f\}$. If $\tilde{u}^{(+)}$, and $\tilde{u}^{(-)}$ are the associated functions, then

$$\int_{B_\rho} \tilde{u}^{(\pm)}v = a_0(G^\rho, u^{(\pm)}) = a_0(u^{(\pm)}, G^\rho) = \int_B f^\pm \tilde{G}^\rho \leq C.$$

Taking the limit as $\rho \to 0$, we conclude that $\tilde{u}^{(\pm)} \leq C$ a.e. on $B$. By Lemma 2.4 (see Remark 2.4), the solutions $u^{(+)}$, and $u^{(-)}$ have representatives $u^{(+)} = \{u_k^{(+)}\}, u_k^{(+)} \in \text{Lip}_0(B)$, and $u^{(-)} = \{u_k^{(-)}\}, u_k^{(-)} \in \text{Lip}_0(B)$ such that

$$u_k^{(\pm)} \leq C, \quad \text{and} \quad u_k^{(\pm)} \leq C$$

a.e. on $B$. By Lemma 2.1, we can in fact choose such representatives to satisfy

$$0 \leq u_k^{(\pm)} \leq C, \quad \text{and} \quad 0 \leq u_k^{(-)} \leq C$$

a.e. on $B$. Now let $u^* = \{u_k^{(+)} - u_k^{(-)}\}$. It is easy to verify that $u^* \in H_0(B)$ is a solution of $L_0u = f$ on $B$. By uniqueness, we must then have $u = \{u_k^{(+)} - u_k^{(-)}\}$, and this representation satisfies the condition stated in the Lemma. \hfill \Box
We want to show that for some constant $C$ such that $0 \leq G$, that is $H$ solution $u$ of $Lu = f$ in $B$ and it satisfies the estimate
\[ \|u\|_{L^\infty(B)} \leq Cr^2 \frac{v(B)}{\omega(B)} \|f^{-1}\|_{L^p(B)}, \]
for some constant $C$.

**Proof.** Choose $r_0$ such that the bounded bilinear form $D(\cdot, \cdot)$ is coercive on $\mathcal{H}_0(B)$, whenever $B$ is a ball of radius $r$, with $0 < r \leq r_0$. Since $\varphi \mapsto \int_B f \tilde{\varphi}$ is a continuous linear functional on $\mathcal{H}_0(B)$, by the Lax-Milgram theorem there is a unique $u \in \mathcal{H}_0(B)$ such that
\[ D(u, \varphi) = \int_B f \tilde{\varphi}, \quad \varphi \in \mathcal{H}_0(B); \]
that is $Lu = f$ in $B$. Moreover, by Hölder’s and Sobolev inequality,
\[ \|u\|_0 \leq C \|f^{-1}\|_{L^p(B)}. \]
We want to show that for some constant $C$,
\[ \|\tilde{u}\|_{L^\infty(B)} \leq Cr^2 \frac{v(B)}{\omega(B)} \|f^{-1}\|_{L^p(B)}. \]

Let $u_{-1} \equiv 0$, and we inductively define $u_j \in \mathcal{H}_0(B)$, $j = 0, 1, 2, \cdots$, as the unique element for which
\[ a_0(u_j, \varphi) = \int_B (f - b \cdot \nabla \bar{u}_{j-1} - V \bar{u}_{j-1}) \tilde{\varphi}, \quad \text{for all } \varphi \in \mathcal{H}_0(B), \]
so that $u_j$ is the solution of $L_0 u + b \cdot \nabla \bar{u}_{j-1} + V \bar{u}_{j-1} = f$ in $\mathcal{H}_0(B)$. This is possible, since for a given $u_{j-1} \in \mathcal{H}_0(B)$, the map
\[ \varphi \mapsto \int_B (f - b \cdot \nabla \bar{u}_{j-1} - V \bar{u}_{j-1}) \tilde{\varphi}, \]
is a continuous linear functional on $\mathcal{H}_0(B)$. Suppose that $\tilde{G}^\rho$ is the approximate Green function of $L_0$ on $B$. We now claim that for each $j = 0, 1, 2, \cdots$, we can choose a representative $u_j = \{u_j^{(k)}\}$, $u_j^{(k)} \in \text{Lip}_0(B)$, such that
\[ \|u_j^{(k)}\|_\infty \leq M_j, \quad \text{and} \quad \int_B |\nabla \bar{u}_j|^2 \tilde{G}^\rho \omega = O(1), \quad \text{as } \rho \to 0^+, \]
for some positive constant $M_j$ independent of $k$. We show this by induction on $j$. Since $\tilde{G}^\rho$ is essentially bounded, and since $G^\rho \geq 0$, by Lemma 2.4 (or see Remark 2.4) we can take a representative $G^\rho = \{G_k^\rho\}$, $G_k^\rho \in \text{Lip}_0(B)$ such that $0 \leq G_k^\rho \leq C$ a.e. on $B$ for some constant $C$ independent of $k$. 
Since \( u_0 \in H_0(B) \) is the solution of \( L_0 u = f \), by Lemma 3.5 we can choose a representative \( u_0 = \{ u_0^{(k)} \} \), \( u_0^{(k)} \in \text{Lip}_0(B) \) such that \( u_0^{(k)} \) is uniformly bounded on \( B \). Consequently, one can use (2.2) to show that \( \{ u_0^{(k)} G_k^\rho \} \), and \( \{ (u_0^{(k)})^2 \} \) are bounded in \( H_0(B) \). Then for some subsequences, \( \varphi_0^{(k)} := u_0^{(k)} G_k^\rho \) and \( \psi_0^{(k)} := (u_0^{(k)})^2 \) are weakly convergent in \( H_0(B) \). Using a further subsequence if necessary, we can assume that \( \varphi_0^{(k)} \to \tilde{u}_0 G^\rho \) a.e. on \( B \). Let us now observe that

\[
\int_B \left< A \nabla u_0^{(k)}, \nabla u_0^{(k)} \right> G_k^\rho \\
= \int_B \left< A \nabla u_0^{(k)}, \nabla \left( u_0^{(k)} G_k^\rho \right) \right> - \frac{1}{2} \int_B \left< A \nabla G_k^\rho, \nabla \left( u_0^{(k)} \right)^2 \right> \\
= \delta_k + \int_B f u_0^{(k)} G_k^\rho - \frac{1}{2} \int_{B_\rho} \left( u_0^{(k)} \right)^2 v \\
\leq \delta_k + \int_B |f| u_0^{(k)} G_k^\rho,
\]

where

\[
\delta_k := \int_B \left< A \nabla u_0^{(k)}, \nabla \left( u_0^{(k)} G_k^\rho \right) \right> - \frac{1}{2} \int_B \left< A \nabla G_k^\rho, \nabla \left( u_0^{(k)} \right)^2 \right> \\
- \int f u_0^{(k)} G_k^\rho + \frac{1}{2} \int_{B_\rho} \left( u_0^{(k)} \right)^2 v,
\]

and the first three integrals are over \( B \). We now take the limit as \( k \to \infty \). By Remark 2.2, we observe that \( \delta_k \to 0 \). Then by Lemma 3.3, Lebesgue dominated convergence theorem, and the fact that \( \delta_k \to 0 \), we obtain

\[
\int_B \left< A \nabla \tilde{u}_0, \nabla \tilde{u}_0 \right> \tilde{G}^\rho \leq \int_B |f| \tilde{u}_0 \tilde{G}^\rho.
\]

Using (1-1), this leads to the estimate

\[
\int_B |\nabla \tilde{u}_0|^2 \tilde{G}^\rho \omega \leq \| \tilde{u}_0 \| \| f \tilde{G}^\rho \| \leq C \| \tilde{u}_0 \| \| f v^{-1} \|_{L^q(B)}
\]

\[
= A_0 \| \tilde{u}_0 \| \| f v^{-1} \|_{L^q(B)},
\]

where we have also used (3.8) in the penultimate inequality and \( A_0 \) stands for the expression \( C r^2 v(B) \frac{1}{\omega(B)} \). This completes the first induction step.

Let us now suppose that \( u_j \) has a representative \( u_j = \{ u_j^{(k)} \} \), \( u_j^{(k)} \in \text{Lip}_0(B) \) and that (3.9) holds for the index \( j \) and some constant \( M_j \). Then by Lemma 3.2, Hölder’s inequality, and assumption (3.9), we see that

\[
\int_B |f - b \cdot \nabla \tilde{u}_j - V \tilde{u}_j| \tilde{G}^\rho \leq C,
\]
for some constant $C$ independent of $\rho$. Thus by Lemma 3.5, we can find a representative $u_{j+1}^{(k)} = \{u_{j+1}^{(k)}\}$, $u_{j+1}^{(k)} \in \text{Lip}(B)$ such that $\|u_{j+1}^{(k)}\|_\infty \leq M_{j+1}$ on $B$ for some positive constant $M_{j+1}$ independent of $k$. The rest of the argument proceeds in exactly the same way as for the $j = 0$ case. This completes the induction, thereby proving the claim that (3.9) holds for all $j$.

Now let $\xi_j := u_j - u_{j-1}$, for $j = 0, 1, 2, \ldots$, where we take the representation $\xi_j := \{\xi_j^{(k)}\}$ with $\xi_j^{(k)} := u_j^{(k)} - u_{j-1}^{(k)}$. Then the $\xi_j$ satisfy

$$a_0(\xi_j, \varphi) = -\int_B \left( b \cdot \nabla \xi_{j-1} + V \xi_{j-1} \right) \varphi,$$

for all $\varphi \in H_0(B)$, and $j = 1, 2, \ldots$.

As a result of (3.9), we have $\|\tilde{\xi}_j\|_\infty < \infty$, and $\int_B |\nabla \tilde{\xi}_j|^2 \tilde{G}^\rho \omega = O(1)$, as $\rho \to 0^+$. For notational convenience, let us introduce the following. For some sufficiently small $\rho_0$, and for $j = 0, 1, 2, \ldots$, let

$$\vartheta := \sqrt{\eta(|b|^2 \omega^{-1})(3r) + \eta(V)(3r)}, \quad \text{and} \quad \tau_j := \sup_{0 < \rho \leq \rho_0} \left( \int_B |\nabla \tilde{\xi}_j|^2 \tilde{G}^\rho \omega \right)^{1/2}.$$

Using these notations, and using Lemma 3.2, we find that

$$\int_{B_\rho} \tilde{\xi}_j \varphi = a_0(G^\rho, \xi_j) = a_0(\xi_j, G^\rho) = -\int_B \left( b \cdot \nabla \tilde{\xi}_{j-1} + V \tilde{\xi}_{j-1} \right) \tilde{G}^\rho$$

$$\leq \left( \int_B |b|^2 \omega^{-1} \tilde{G}^\rho \right)^{1/2} \cdot \left( \int_B |\nabla \tilde{\xi}_{j-1}|^2 \tilde{G}^\rho \omega \right)^{1/2} + \|\tilde{\xi}_{j-1}\|_\infty \int_B |V| \tilde{G}^\rho$$

$$\leq \vartheta \left( \tau_{j-1} + \|\tilde{\xi}_{j-1}\|_\infty \right).$$

After letting $\rho \to 0$, we obtain

(3.10) \[ \|\tilde{\xi}_j\|_\infty \leq \vartheta \left( \tau_{j-1} + \|\tilde{\xi}_{j-1}\|_\infty \right). \]

As a consequence of (2.2), and (3.9) one can see that the sequence $\{\xi_j^{(k)} G_k^\rho\}$ is bounded in $H_0(B)$. Then for an appropriate subsequence, $\varphi_j^{(k)} := \xi_j^{(k)} G_k^\rho$ and $\psi_j^{(k)} := (\xi_j^{(k)})^2$ are weakly convergent in $H_0(B)$. Without loss of generality,
we can assume that \( \varphi_j^{(k)} \to \tilde{\xi}_j \tilde{G}^\rho \) pointwise a.e. on \( B \). We now observe that

\[
\int_B \langle A \nabla \xi_j^{(k)}, \nabla \xi_j^{(k)} \rangle G_k^\rho = \int_B \langle A \nabla \xi_j^{(k)}, \nabla (\xi_j^{(k)} G_k^\rho) \rangle - \frac{1}{2} \int_B \langle A \nabla G_k^\rho, \nabla (\xi_j^{(k)})^2 \rangle \\
= \delta_{j}^{(k)} + \int_B \left( b \cdot \nabla \tilde{\xi}_{j-1} + V \tilde{\xi}_{j-1} \right) \xi_j^{(k)} G_k^\rho - \int_{B_\rho} (\xi_j^{(k)})^2 v \\
\leq \delta_{j}^{(k)} + \int_B \left( b \cdot \nabla \tilde{\xi}_{j-1} + V \tilde{\xi}_{j-1} \right) \xi_j^{(k)} G_k^\rho,
\]

where \( \delta_{j}^{(k)} \) is given by

\[
\int_B \langle A \nabla \xi_j^{(k)}, \nabla (\xi_j^{(k)} G_k^\rho) \rangle - \frac{1}{2} \int_B \langle A \nabla G_k^\rho, \nabla (\xi_j^{(k)})^2 \rangle \\
- \int_B \left( b \cdot \nabla \tilde{\xi}_{j-1} + V \tilde{\xi}_{j-1} \right) \xi_j^{(k)} G_k^\rho + \int_{B_\rho} (\xi_j^{(k)})^2 v.
\]

Notice that by Remark 2.2, \( \delta_{j}^{(k)} \to 0 \) as \( k \to \infty \). Therefore taking the limit in the last inequality, as \( k \to \infty \), applying Lemma 3.3, and the Lebesgue dominated convergence theorem, followed by an application of Hölder inequality and Lemma 3.1, we obtain

\[
\int_B |\nabla \tilde{\xi}_j|^2 \tilde{G}^\rho \omega \leq \theta \| \tilde{\xi}_j \|_\infty \left( \tau_{j-1} + \| \tilde{\xi}_{j-1} \|_\infty \right).
\]

Using (3.10) to estimate \( \| \tilde{\xi}_j \|_\infty \) in the above inequality, we get

\[
\tau_j \leq \theta \left( \tau_{j-1} + \| \tilde{\xi}_{j-1} \|_\infty \right), \quad j = 1, 2, \ldots
\]

The sum \( \| \tilde{\xi}_j \|_\infty + \tau_j \) can thus be estimated as

\[
\| \tilde{\xi}_j \|_\infty + \tau_j \leq 2 \theta \left( \tau_{j-1} + \| \tilde{\xi}_{j-1} \|_\infty \right), \quad j = 1, 2, \ldots
\]

Observe that \( \tau_0 + \| \tilde{\xi}_0 \|_\infty \leq (A_0 \| f v^{-1} \|_{L_\infty(B)} \| \tilde{u}_0 \|_\infty)^{1/2} + \| \tilde{u}_0 \|_\infty \). But by Lemma 3.4, we recall \( \| \tilde{u}_0 \|_\infty \leq A_0 \| f v^{-1} \|_{L_\infty(B)} \). Therefore from (3.10), and (3.11) one obtains by induction

\[
\| \tilde{\xi}_j \|_\infty \leq 2^{j-1} \theta^j \left( \tau_0 + \| \tilde{\xi}_0 \|_\infty \right) \leq (2 \theta)^j A_0 \| f v^{-1} \|_{L_\infty(B)}, \quad j = 1, 2, \ldots
\]
An application of Cauchy-Schwarz inequality, and Lemma 3.1 leads us also, on using (1.1), to observe that
\[
\int_B \langle A \nabla \tilde{\xi}_j, \nabla \tilde{\xi}_j \rangle = a_0(\xi_j, \xi_j) = - \int_B \left( b \cdot \nabla \tilde{\xi}_{j-1} + V \tilde{\xi}_{j-1} \right) \tilde{\xi}_j
\]
\[
\leq \left( \int_B |b|^2 \omega^{-1} \xi_j^2 \right)^{\frac{1}{2}} \cdot \left( \int_B |\nabla \tilde{\xi}_{j-1}|^2 \omega \right)^{\frac{1}{2}}
\]
\[
+ \left( \int_B |V| \xi_j^2 \right)^{\frac{1}{2}} \cdot \left( \int_B |V \tilde{\xi}_{j-1}|^2 \right)^{\frac{1}{2}}
\]
\[
\leq 2\vartheta \int_B \langle A \nabla \tilde{\xi}_j, \nabla \tilde{\xi}_j \rangle + \frac{1}{2} \int_B |\nabla \tilde{\xi}_{j-1}|^2 \omega.
\]
Therefore
\[
\beta_j \leq \frac{1}{\sqrt{2(1 - 2\vartheta)}} \beta_{j-1}, \quad \text{where} \quad \beta_j := \left( \int_B \langle A \nabla \tilde{\xi}_j, \nabla \tilde{\xi}_j \rangle \right)^{\frac{1}{2}}, \quad j = 1, 2, \ldots .
\]
Thus
\[
(3.13) \quad \|\xi_j\|_0 = \beta_j \leq \left( \frac{1}{\sqrt{2(1 - 2\vartheta)}} \right)^j \beta_0 = \left( \frac{1}{\sqrt{2(1 - 2\vartheta)}} \right)^j \|\xi_0\|_0.
\]
Now, from (3.12) we observe that
\[
\|\tilde{u}_m - \tilde{u}_k\|_\infty \leq \sum_{j=k+1}^m \|\tilde{\xi}_j\|_\infty \leq A_0 \|f\|_{L^p(B)} \sum_{j=k+1}^m (2\vartheta)^{j-1}, \quad \text{for} \ m > k.
\]
Also, from (3.13) we obtain
\[
\|u_m - u_k\|_0 \leq \sum_{j=k+1}^m \|\tilde{\xi}_j\|_0 \leq \|\xi_j\|_0 \sum_{j=k+1}^m \left( \frac{1}{\sqrt{2(1 - 2\vartheta)}} \right)^j, \quad \text{for} \ m > k.
\]
Thus, if we further choose \( r_0 \) such that \( 4\vartheta(r) < 1 \) for \( 0 < r \leq r_0 \), then we conclude that \( \{\tilde{u}_m\} \), and \( \{u_k\} \) are Cauchy sequences in \( L^\infty(B) \), and \( H_0(B) \) respectively. So let us take \( u_* \in H_0(B) \) such that \( u_m \rightarrow u_* \) in \( H_0(B) \). Now let \( \varphi \in \text{Lip}_0(B) \) be arbitrary. Then, we have
\[
D(u_* - u, \varphi) = D(u_* - u_m, \varphi) + D(u_m, \varphi) - D(u, \varphi).
\]
But
\[
D(u_m, \varphi) - D(u, \varphi) = a_0(u_m, \varphi) + \int_B (b \cdot (\nabla u_m) \varphi + V u_m \varphi) - \int_B f \varphi
\]
\[
= a_0(u_m, \varphi) - a_0(u_{m+1}, \varphi) + a_0(u_m - u_{m+1}, \varphi).
\]
Therefore, for \( m \geq 1 \)
\[
|D(u_* - u, \varphi)| \lesssim \|u_* - u_m\|_0 \|\varphi\|_0 + |D(u_m, \varphi) - D(u, \varphi)| \\
\lesssim \|u_* - u_m\|_0 \|\varphi\|_0 + |a_0(u_m - u_{m+1}, \varphi)| \\
\lesssim (\|u_* - u_m\|_0 + \|u_{m+1} - u_m\|_0) \|\varphi\|_0.
\]

Taking the limit as \( m \to \infty \), we obtain \( D(u_* - u, \varphi) = 0 \) for \( \varphi \in \text{Lip}_0(B) \).
Since \( \text{Lip}_0(B) \) is dense in \( H_0(B) \), and the bilinear form \( D(\cdot, \cdot) \) is coercive on \( H_0(B) \) we conclude that \( u = u_* \).
Since \( \{\tilde{u}_m\} \) is a Cauchy sequence in \( L^\infty(B) \), by uniqueness of limits we know that \( \tilde{u}_m \rightharpoonup \tilde{u} \) in \( L^\infty(B) \).
But,
\[
\|\tilde{u}_m\|_\infty \leq \sum_{k=1}^m \|\tilde{\xi}_k\|_\infty \leq CA_0\|f_{\psi}^{-1}\|_{L^p_\psi(B)} \sum_{k=1}^m (2\vartheta)^{k-1}.
\]

Therefore, since
\[
\|\tilde{u}\|_\infty \leq \|\tilde{u}_m - \tilde{u}\|_\infty + C(r)A_0\|f_{\psi}^{-1}\|_{L^p_\psi(B)},
\]
letting \( m \to \infty \), and recalling the value of \( A_0 \), gives the desired estimation.

\begin{remark}
Let \( f_{\psi}^{-1} \in L^p_\psi(B) \) for some \( p > \frac{\sigma}{\sigma - 1} \). If \( L^\ast u = f \) for \( u \in H_0(B) \), then \( \tilde{L}u = f \), where \( \tilde{L} := -\text{div}(A(x)\nabla) - b(x) \cdot \nabla + (V - \text{div} b) \).
Therefore, if \( |b|^2\omega^{-1}, \text{div} b, V \in K \), then by the above theorem, we also have the estimate
\[
\|\tilde{u}\|_{L^\infty(B)} \leq C \rho^2 \frac{v(B)}{\omega(B)} \|f_{\psi}^{-1}\|_{L^p_\psi(B)},
\]
for some constant \( C \).
\end{remark}

For the rest of the paper we will require an additional condition on the coefficient \( b \) of the operator \( L \). Thus, in addition to the condition \((1.5)\) on the coefficients \( b \), and \( V \) of \( L \), we impose the following:

\begin{equation}
(3.14) \quad \text{div} b \in K.
\end{equation}

Let \( B \subset B_0 \) be a ball of radius sufficiently small that the bounded bilinear form (with \( c \equiv 0 \)) \( D_s(u, \varphi) \) is coercive on \( H_0(B) \). Let \( y \in B \), and \( \rho > 0 \) such that \( B_\rho := B_\rho(y) \subset B \). Since the map
\[
\varphi \mapsto \int_{B_\rho} \tilde{\varphi} v
\]
is a continuous linear functional on \( H_0(B) \), by Lax-Milgram theorem there is a unique \( G^\rho \in H_0(B) \) such that
\[
D_s(G^\rho, \varphi) = \int_{B_\rho} \tilde{\varphi} v, \quad \varphi \in H_0(B).
\]
Following [3], we call \( \tilde{G}^\rho \) the approximate Green function of \( L \) on \( B \) with pole \( y \). Note that by Theorem 3.1 (see Remark 3.4), \( \| \tilde{G}^\rho \|_{L^\infty(B)} \leq C \) for some constant \( C \) depending on \( \rho \), the pole of \( \tilde{G}^\rho \), and \( v \).

In the following Lemma, \( B \subset B_0 \) is a ball of radius so small that the bilinear form \( D(\cdot,\cdot) \) is coercive.

**Lemma 3.6.** Suppose \( u \in H_0(B) \), and \( D(u, \varphi) \geq 0 \) whenever \( 0 \leq \varphi \in \text{Lip}_0(B) \). Then \( u \geq 0 \).

**Proof.** Let \( u = \{u_k\} \), \( u_k \in \text{Lip}_0(B) \). Since \( \nabla |u_k| = (\text{sgn} \, u_k) \nabla u_k \) where \( u_k \neq 0 \), and \( (\text{sgn} \, u_k) |u_k| = u_k \), it follows that for each \( k \)

\[
D(u_k, |u_k|) = D(|u_k|, u_k), \quad \text{and} \quad D(|u_k|, |u_k|) = D(u_k, u_k).
\]

Since \( a_0(u_k, |u_k|) = a_0(|u_k|, u_k) \) also, the sequence \( \{u_k\} \) is bounded in \( H_0(B) \), and thus a subsequence which we continue to write as \( \{u_k\} \) converges weakly to some \( v \in H_0(B) \). Since \( \varphi \rightarrow D(u, \varphi) \) is continuous on \( H_0(B) \) and \( D(u, |u_k| - u_k) \geq 0 \) by hypothesis, it follows that

\[
0 \leq \lim_{k \to \infty} D(u, |u_k| - u_k) = D(u, v) - D(u, u),
\]

so that \( D(u, u) \leq D(u, v) \). Then \( D(u, u) = \alpha D(u, v) \) for some \( 0 < \alpha \leq 1 \).

Let us now observe that

\[
0 \leq \|u - \alpha |u_k|\|_0^2 \leq D(u - \alpha |u_k|, u - \alpha |u_k|)
= D(u, u) - 2\alpha D(u, |u_k|) + \alpha^2 D(|u_k|, |u_k|)
= D(u, u) - 2\alpha D(u, |u_k|) + \alpha^2 D(u_k, u_k).
\]

Taking the limit as \( k \to \infty \), the last inequality reduces to

\[
0 \leq \lim_{k \to \infty} \|u - \alpha |u_k|\|_0^2 \leq D(u, u) - 2\alpha D(u, v) + \alpha^2 D(u, u)
\leq D(u, u) - 2D(u, u) + \alpha^2 D(u, u) = (\alpha^2 - 1)D(u, u).
\]

Hence

\[
0 \leq \lim_{k \to \infty} \|u - \alpha |u_k|\|_0^2 \leq (\alpha^2 - 1)D(u, u) \leq 0.
\]

That is, \( \alpha |u_k| \to u \) in \( H_0(B) \) as \( k \to \infty \), showing that \( u \) is the limit in \( H_0(B) \) of \( \alpha |u_k| \geq 0 \). From this, it also follows that \( \alpha |u_k| \to \tilde{u} \) in \( L^2_\nu \). But also \( u_k \to \tilde{u} \) in \( L^2_\nu(B) \). Therefore we must have \( \alpha |\tilde{u}| = \tilde{u} \) a.e. on \( B \). Thus \( \alpha = 1 \), and hence \( u = \{|u_k|\} \).

The following lemma will be useful.

**Lemma 3.7.** Let \( B \subset \subset B_0 \) be a ball of radius \( r \), and \( \tilde{G}^\rho \) be the approximate Green function of \( L \) on \( B \). There is \( r_0 > 0 \) such that if \( 0 < r \leq r_0 \), then

\[
\int_B (|b|^2 \omega^{-1} + |V|) \tilde{G}^\rho \leq C \eta (|b|^2 \omega^{-1} + |V|)(2r),
\]

where \( C \) is a constant independent of \( \rho \) and the pole of \( \tilde{G}^\rho \).
Proof. Since
\[ \varphi \mapsto \int_B \left( |b|^2 \omega^{-1} + |V| \right) \tilde{\varphi}, \]
is a continuous linear functional on \( H_0(B) \), by Lax-Milgram theorem let \( \zeta \in H_0(B) \) be the unique solution of
\[ a_0(\zeta, \varphi) = \int_B \left( |b|^2 \omega^{-1} + |V| \right) \tilde{\varphi}, \quad \varphi \in H_0(B). \]

By Lemma 3.2, \( \tilde{\zeta}(x) \lesssim \eta := \eta(|b|^2 \omega^{-1} + V)(2r) \) for a.e. \( x \in B \). By Lemma 2.1, and Lemma 2.4 (or Remark 2.4), let us pick a sequence \( \zeta_k \in \text{Lip}_0(B) \) such that \( 0 \leq \zeta_k \lesssim \eta \) a.e. on \( B \). Let \( \tilde{G}^\rho \) be the approximate Green function of \( L \) on \( B \). By Remark 3.4, \( \tilde{G}^\rho \) is essentially bounded, and Lemma 3.6 shows \( G^\rho \geq 0 \). Thus by Lemma 2.4 (see Remark 2.4), we can pick a representative \( G^\rho = \{ G^\rho_k \} \), \( G^\rho_k \in \text{Lip}_0(B) \) such that for some constant \( C \) independent of \( k \), we have \( 0 \leq G^\rho_k \leq C \) a.e. on \( B \). Let us now observe that
\[ \int_B \left( |b|^2 \omega^{-1} + |V| \right) G^\rho_k \]
\[ = \delta_k + \int_B \langle A \nabla \zeta_k, \nabla G^\rho_k \rangle \]
\[ = \delta_k + \gamma_k - \int_B (b \cdot (\nabla \zeta_k)G^\rho_k + V \zeta_k G^\rho_k) + \int_{B_{\rho}} \zeta_k v \]
\[ \leq \delta_k + \gamma_k + \sqrt{\eta} \int_B |b|^2 \omega^{-1} G^\rho_k + \frac{1}{\sqrt{\eta}} \int_B \langle A \nabla \zeta_k, \nabla \zeta_k \rangle G^\rho_k \]
\[ + \eta \int_B |V| G^\rho_k + \eta. \]
Here \( \delta_k \) and \( \gamma_k \) are given by
\[ \delta_k := \int_B \left( |b|^2 \omega^{-1} + |V| \right) G^\rho_k - \int_B \langle A \nabla \zeta_k, \nabla G^\rho_k \rangle, \]
and \( \gamma_k := D_*(G^\rho_k, \zeta_k) - \int_{B_{\rho}} \zeta_k v. \)

By Remark 2.2, and Remark 3.2 respectively, we notice that \( \delta_k \to 0 \), and \( \gamma_k \to 0 \) as \( k \to \infty \). We thus take the limit as \( k \to \infty \). By Lemma 3.3, and the Lebesgue dominated convergence theorem, we obtain
\[ \int_B \left( |b|^2 \omega^{-1} + |V| \right) \tilde{G}^\rho \]
\[ \leq \sqrt{\eta} \int_B |b|^2 \omega^{-1} G^\rho + \frac{1}{\sqrt{\eta}} \int_B \langle A \nabla \tilde{\zeta}, \nabla \tilde{\zeta} \rangle \tilde{G}^\rho + \eta \int_B |V| \tilde{G}^\rho + \eta. \]
As a result of (2.2), we see that \( \{ \zeta_k G_{k}^p \} \) is a bounded sequence in \( H_0(B) \). Therefore, we pick a subsequence still denoted by \( \{ \zeta_k G_{k}^p \} \) that converges weakly in \( H_0(B) \), and such that \( \zeta_k G_{k}^p \to \tilde{\zeta} G^p \) pointwise a.e. on \( B \). Using this subsequence we have

\[
\int_B \langle A \nabla \zeta_k, \nabla \zeta_k \rangle G_{k}^p = \int_B \langle A \nabla \zeta_k, \nabla (\zeta_k G_{k}^p) \rangle - \frac{1}{2} \int_B \langle A \nabla G_{k}^p, \nabla \zeta^2_k \rangle \\
= \delta_k + \int_B (|b|^2 \omega^{-1} + |V|) G_{k}^p \zeta_k \\
+ \frac{1}{2} \left( \int_B (b \cdot (\nabla \zeta_k^2) G_{k}^p + V G_{k}^p \zeta_k^2) + \gamma_k - \int_{B_{\rho}} \zeta_k^{2\nu} \right) \\
\leq \delta_k + \frac{1}{2} \gamma_k + \eta \int_B (|b|^2 \omega^{-1} + |V|) G_{k}^p \\
+ \frac{1}{2} \left( \eta^2 \int_B (|b|^2 \omega^{-1} + |V|) G_{k}^p + \int_B \langle A \nabla \zeta_k, \nabla \zeta_k \rangle G_{k}^p \right),
\]

where

\[
\delta_k := a_0(\zeta_k, \zeta_k G_{k}^p) - \int_B (|b|^2 \omega^{-1} + |V|) G_{k}^p \zeta_k,
\]

and \( \gamma_k := -D_s(G_{k}^p, \zeta_k^2) + \int_{B_{\rho}} \zeta_k^{2\nu}. \)

Again by Remark 2.2, and Remark 3.2 respectively, we see that \( \delta_k \to 0 \), and \( \gamma_k \to 0 \) as \( k \to \infty \). We thus take the limit as \( k \to \infty \). By Lemma 3.3, and the Lebesgue dominated convergence theorem, we obtain

\[
\int_B \langle A \nabla \tilde{\zeta}, \nabla \tilde{\zeta} \rangle \tilde{G}^p \leq \eta \int_B (|b|^2 \omega^{-1} + |V|) \tilde{G}^p \\
+ \frac{1}{2} \left( \eta^2 \int_B (|b|^2 \omega^{-1} + |V|) \tilde{G}^p + \int_B \langle A \nabla \tilde{\zeta}, \nabla \tilde{\zeta} \rangle \tilde{G}^p \right).
\]

Putting this last inequality back into (3.15), we see that there is \( r_0 \) such that for all \( 0 < r \leq r_0 \),

\[
\int_B (|b|^2 \omega^{-1} + |V|) \tilde{G}^p \leq C \eta(|b|^2 \omega^{-1} + |V|)(2r),
\]

as required. \( \square \)

We now have all the needed ingredients to demonstrate the uniform integrability of the approximate Green functions of \( L \). We use the methods in [3] (see also [5], [8]) to prove the integrability theorem.

**Theorem 3.2.** Let \( B \subset B_0 \) be a ball of radius \( r \). Suppose \( \tilde{G}^p \) is the approximate Green function of \( L \) on \( B \), where we assume that the coefficients
of $L$ satisfy the conditions (1.5) and (3.14). Then for $0 < p < \sigma$ there is a positive constant, independent of $\rho$ and the pole, such that
\[
\left( \int_B (\tilde{G}^\rho)^{p} \, \upsilon \right)^{\frac{1}{p}} \leq C \frac{r^2}{\omega(B)},
\]
when $r$ is sufficiently small.

**Proof.** Let $\tilde{G}^\rho$ be the approximate Green function of $L$ on $B$. By Lemma 3.6, $G^\rho \geq 0$ on $B$. Therefore we pick a representative $G^\rho = \{G^\rho_k\}$, $G^\rho_k \in \text{Lip}_0(B)$ such that $G^\rho_k \geq 0$.

Now, for $t > 0$, let us define
\[
\varphi_k := \left[ \frac{1}{t} - \frac{1}{G^\rho_k} \right]^+ = \left[ \frac{1}{t} - \frac{1}{G^\rho_k} \right] \chi(G^\rho_k > t).
\]
Then $\varphi_k \in \text{Lip}_0(B)$, and
\[
\nabla \varphi_k = \frac{\nabla G^\rho_k}{(G^\rho_k)^2} \chi(G^\rho_k > t).
\]
Since $\|\varphi_k\|_{0}^2 \leq \|G^\rho_k\|_{0}^2/t^4$, we can pick a subsequence, still denoted by $\{\varphi_k\}$ that converges weakly in $H_0(B)$. With such subsequence, we observe that
\[
\int_B \langle A \nabla G^\rho_k, \nabla \varphi_k \rangle = \delta_k - \int_B (b \cdot \nabla \varphi_k + V \varphi_k) G^\rho_k + \int_{B_\rho} \varphi_k \upsilon, \tag{3.16}
\]
where
\[
\delta_k := D_*(G^\rho_k, \varphi_k) - \int_{B_\rho} \varphi_k \upsilon.
\]
Using the Cauchy-Schwarz inequality, and noting that $\varphi_k \leq 1/t$ on $B$, we estimate
\[
\int_B (b \cdot \nabla \varphi_k + V \varphi_k) G^\rho_k \leq \frac{1}{2t} \int_B |b|^2 \omega^{-1} G^\rho_k + \frac{t}{2} \int_B |\nabla \varphi_k|^2 G^\rho_k \omega + \frac{1}{t} \int_B |V| G^\rho_k
\leq \frac{1}{t} \int_B (|b|^2 \omega^{-1} + |V|) G^\rho_k + \frac{t}{2} \int_{\{G^\rho_k > t\}} \frac{|\nabla G^\rho_k|^2}{(G^\rho_k)^2} G^\rho_k \omega
\leq \frac{1}{t} \int_B (|b|^2 \omega^{-1} + |V|) G^\rho_k + \frac{1}{2} \int_{\{G^\rho_k > t\}} \omega
\leq \frac{1}{t} \int_B (|b|^2 \omega^{-1} + |V|) G^\rho_k + \frac{1}{2} \int_B \langle A \nabla G^\rho_k, \nabla \varphi_k \rangle.
\]
Taking this last estimation, and using again the fact that $\varphi_k \leq 1/t$ on $B$ we get from (3.16) that
\[
\int_B \langle A \nabla G^\rho_k, \nabla \varphi_k \rangle \leq 2|\delta_k| + \frac{2}{t} \int_B (|b|^2 \omega^{-1} + |V|) G^\rho_k + \frac{2}{t}.
\]
We now take the limit as $k \to \infty$. By Remark 3.2 we know that $\delta_k \to 0$ as $k \to \infty$. Using this fact and applying Lemma 3.7, we get

$$\lim_{k \to \infty} \int_B (A \nabla G^\rho_k, \nabla \varphi_k) \leq \frac{2}{t} \int_B (|b|^2 \omega^{-1} + |V|) \tilde{G}^\rho + \frac{2}{t} \leq \frac{2}{t}(1 + \eta(|b|^2 \omega^{-1} + V)(3r)).$$

Hence by the degeneracy condition, we obtain the inequality

$$\limsup_{k \to 0} \int_{\{G^\rho_k > t\}} \frac{|\nabla G^\rho_k|^2}{(G^\rho_k)^2} \omega \leq \frac{2}{t}(1 + \eta(|b|^2 \omega^{-1} + V)(3r)).$$

Now let

$$\psi_k := [\log G^\rho_k - \log t]^+ = [\log G^\rho_k - \log t] \chi_{\{G^\rho_k > t\}}.$$

Then $\psi_k \in H^0_0(B)$, and $\nabla \psi_k = \left( \frac{\nabla G^\rho_k}{G^\rho_k} \right) \chi_{\{G^\rho_k > t\}}$. Therefore, for sufficiently small $r$ the last inequality now reads

$$\limsup_{k \to \infty} \int_B |\nabla \psi_k|^2 \omega \leq C,$$

and hence by Sobolev inequality (2.1), we get

$$\limsup_{k \to \infty} \left( \frac{1}{v(B)} \int_{B \cap \{G^\rho_k > t\}} \log \left( \frac{G^\rho_k}{t} \right)^q v \right)^{\frac{2}{q}} \leq C \frac{r^2}{\omega(B)} \frac{C}{t}.$$

Restricting the integration to $\{G^\rho_k > 2t\}$, we get

$$(\log 2)^2 \limsup_{k \to \infty} \left( \frac{v(\{G^\rho_k > 2t\})}{v(B)} \right)^{\frac{2}{q}} \leq C \frac{r^2}{\omega(B)} \frac{1}{t}.$$

The above inequality remains valid if we replace $2t$ by $t$. Also by using further subsequence if necessary, we may assume that $G^\rho_k \to \tilde{G}^\rho$ pointwise a.e. on $B$. Thus $\chi_{\{G^\rho_k > t\}} \leq \liminf \chi_{\{G^\rho > t\}}$ a.e., and by Fatou’s lemma,

$$v(\{\tilde{G}^\rho > t\}) \leq \liminf_{k \to \infty} v(\{G^\rho_k > t\}).$$

Therefore we obtain,

$$v(\{\tilde{G}^\rho > t\}) \leq C \left( \frac{r^2}{\omega(B)} \right) \frac{1}{t^\sigma} v(B).$$

The theorem then follows from this estimate, and the formula

$$\int_B (\tilde{G}^\rho)^p = p \int_0^\infty t^{p-1} v(\{\tilde{G}^\rho > t\}) \leq pv(B) \int_0^A t^{p-1} + p \int_A^\infty t^{p-1} v(\{\tilde{G}^\rho > t\}),$$

where $A = r^2/\omega(B)$. □
Acknowledgments. I would like to record my indebtedness to Professor C.E. Gutierrez for his constant encouragement and his invaluable advice throughout the preparation of this paper. I would also like to thank the referee for the timely and careful reading of the manuscript.

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Received September 10, 1999 and revised February 15, 2000. This research was partially supported by Kuwait University grant SM-137.

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PERIODIC SUBWORDS IN 2-PIECE WORDS

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We find families of words $W$ where $W$ is a product of $k$ pieces for $k=2$. For $k=3, 4, 6$, $W$ arises in a small cancellation group with single defining relation $W=1$. We assume $W$ involves generators but not their inverses and does not have a periodic cyclic permutation (like $XY...XYX$ for nonempty base word $XY$). We prove $W$ or $W$ written backwards equals $ABCD$ where $ABC$, $CDA$ are periodic words with base words of different lengths. One family includes words of the form $EFGG$ for periodic words $G$, $E$, $F$ with the same base word and increasing lengths. Other $W$ are found using $Mathematica$.

1. Introduction.

A small cancellation condition on a group’s defining relations yields, for example, a solution to the conjugacy problem. See [1, 4]. There are 3 types of such conditions. Each includes a condition $C(k)$ for $k=3, 4$ or 6, depending on the type. For a group $G$ with one defining relation $R=1$, $C(k)$ involves the set $[R]$ of cyclic permutations of $R$ and of $R^{-1}$. A piece is a nonempty, initial subword of 2 distinct members of $[R]$. $C(k)$ requires that no word in $[R]$ is a product of fewer than $k$ pieces.

To study a “large cancellation” group $G$ and avoid all small cancellation types, we can use the condition that $R$ is a product of 2 pieces. What does such a word $R$ look like? For simplicity, in this paper we consider words involving generators but not their inverses. In particular, we study 2-piece words $R$, meaning

\begin{equation}
R \text{ involves generators but not their inverses and } R \text{ is a product of 2 pieces.}
\end{equation}

An attempt to classify these words led to the results in this paper. These results also lie in the field of combinatorics on words which is surveyed in [3].

2. Summary of results.

For convenient exposition, from now on, a word $W$ is a finite sequence of letters taken from some alphabet; $|W| = \text{length of } W$; the empty word $= 1$; $|1| = 0$. Write $W \sim V$ if $W$, $V$ are cyclic permutations. $W$ is 2-piece if
If the minimum length of the words admitting $W$ is periodic for the piece pair $UV$, then $W$ is biperiodic if $(\exists P, Q, R, S, m, n) W = UV, U = P(QP)^m, V = R(SR)^n, SR < U, QP < V, 1 \neq QP \neq SR \neq 1, URS \neq SRU$ and $VPQ \neq QPV$; $m, n \geq 1$.

The main results are: If a 2-piece word $W$ has no periodic cyclic permutation then $W$ or $W$ written backwards is biperiodic. Each biperiodic word is 2-piece. $W$ is biperiodic and $|S| < |RS| < |PR| < |QP| < |RSR|$ is equivalent to $(\exists A, B, a, b, c, m, n) W = A(BA)^b(A(BA)^c)^m(A(BA)^a)^n$ together with $AB \neq BA, 1 < a < b < c \leq 2a, m, n \geq 1$. Such a word $W$ is not periodic for $n \geq 2$. Two other similar equivalences are proved. Other as yet unclassified biperiodic words are found using Mathematica.

The title of the paper refers to the periodic subwords $P(QP)^{m+1}, R(SR)^{n+1}$ which begin word $W = P(QP)^m R(SR)^n$ and its cyclic permutation $R(SR)^n P(QP)^m$, respectively, whenever $W$ is biperiodic and hence 2-piece.

3. Terminology.

Terminology in the previous section is augmented as follows. Let $A, B$ be words over some alphabet. The concatenation of words $A, B$ is written as a product $AB$. The product of $k$ copies of $A$, written $A^k$, is a power of $A$ if $k \geq 0$, with $A^0 = 1$, and a proper power if $k \geq 2$. Call $W$ simple if $W$ is not a proper power. Note the empty word $E = 1$ is not simple since $E = E^2$.

If $W = XYZ$ then $X, Y, Z$ are factors of $W$; write $X, Y, Z \subseteq W$. $X, Z$ are left and right factors; write $X \leq W, W \geq Z$. If $U$ is a factor of $W$ then $U$ is major if $2|U| \geq |W|$ and proper if $U \neq W$. Proper left and right factors of $W$ are indicated by $X < W, W > Z$. Denote $W$ written backwards by $W^*$, the reverse of $W$. As in [4, p. 153], the period $\pi(W)$ of a word $W$ is the minimum length of the words admitting $W$ as a factor of some of their powers. Equivalent definitions of periodic are in Theorem 4.13. Call $W$ plain if $W$ has no periodic cyclic permutation.

We restate a definition to enable later reference to its parts:

Definition 3.1. Word $W$ is biperiodic using $U, V, P, Q, R, S, m, n$ if $W = UV$ for words $U, V$ such that:

(3.1a) $U = P(QP)^m$ for some words $Q \neq 1, P$ and some integer $m \geq 1$.

(3.1b) $V = R(SR)^n$ for some words $S \neq 1, R$ and some integer $n \geq 1$.

(3.1c) $SR < U, QP < V$.

(3.1d) $1 \neq QP \neq SR \neq 1$.

(3.1e) $URS \neq SRU, VPQ \neq QPV$. 
4. Preliminaries.

Let $A, B, \ldots$ denote words and $a, b, \ldots$ denote integers in the following Lemmas. Lemmas 4.1–4.3 are found in Propositions 1.3.4, 8.1.1 and Theorem 8.1.2 in [3]. Lemmas 4.4, 4.5, 4.6, 4.8, 4.9, 4.10, 4.12 are in the following Propositions in [2]: 1.2, 1.3, 1.4, 1.4′, 1.8, 1.16, 1.23. Prove Lemma 4.7 from Lemma 4.6 and Lemma 4.11 from Lemma 4.10 using reverse words. Use $S$ simple if and only if $S^*$ simple.

Lemma 4.1. $Y, Z \neq 1$, $XZ = YX$ imply $(\exists n \geq 0, U, V) Y = UV, Z = VU, X = U(VU)^n$.

Lemma 4.2. $\pi(W) = \min\{|W| - |V|\}$ where $V < W > V$.

Lemma 4.3. $p = \pi(XY), q = \pi(YZ), d = \gcd(|X|, |Y|), |Y| \geq p + q - d$ imply $p = q = \pi(XYZ)$.

Lemma 4.4. If $XY = YX$ then $(\exists S$ simple, $a, b \geq 0) X = a, Y = b$.

Lemma 4.5. If $S, T$ simple, $S^a = T^b, a, b \geq 1$ then $S = T$.

Lemma 4.6. If $S$ is a simple word and $PS \leq S^n, n \geq 1$ then $P$ is a power of $S$.

Lemma 4.7. If $S$ is a simple word and $S^n \geq SP, n \geq 1$ then $P$ is a power of $S$.

Lemma 4.8. If $PBA \leq A(BA)^r, r \geq 1, P \neq 1$ with $BA$ simple then $(\exists e \geq 0) P = A(BA)^e$.

Lemma 4.9. A cyclic permutation of a simple word is a simple word.

Lemma 4.10. $X \leq Y^e X, e > 0, Y \neq 1$ imply $(\exists t \geq 0, E) t, E$ unique, $X = Y^t E, E < Y$.

Lemma 4.11. $XY^e \geq X, e > 0, Y \neq 1$ imply $(\exists t \geq 0, E) t, E$ unique, $X = EY^t, Y > E$.

Lemma 4.12. If $XYZ = ZYX, X \neq 1, Z \neq 1$ then $(\exists a, b, c \geq 0, U, V) X = U(VU)^a, Y = V(UV)^b, Z = U(VU)^c$ and the word $UV$ is simple.

Lemma 4.13. Equivalent conditions on a word $W$ are:

(4.13a) $W$ has a proper major left factor which is also a right factor.

(4.13b) $W = YX = XZ, |X| \geq |Y| > 0$.

(4.13c) $(\exists k \geq 2, U \neq 1) W \leq U^k, |W| \geq 2|U|$.

(4.13d) $W$ is periodic, that is, $(\exists B \neq 1, A, m \geq 2) W = A(BA)^m$.

(4.13e) $|W| \geq 2\pi(W)$ and $W \neq 1$.

Proof. (a) if and only if (b): Use definitions.
(b) implies (c): Deduce $|W| = |YX| \geq |YY| = 2|Y|$ and $W < YYW$. By Lemma 4.10, $W = Y^tE < Y^{t+1}$ for some word $E$ with $t \geq 1$ because $|E| < |Y| < |W|$

(c) implies (d): From (c), $(\exists t \geq 2) t|U| \leq |W| < (1+t)|U|$. So $(\exists B \neq 1, A) U = AB$, $W = U^tA$, proving (d).

(d) implies (b): Use $X = A(AB)^{m-1}$, $Y = AB$, $Z = BA$.

(d) implies (e): From (d), $W < (AB)^{m+1}$ and so $\pi(W) \leq |AB|$. It follows that $|W| = |A(AB)^m| \geq m|BA| \geq 2|AB| \geq 2\pi(W)$, yielding (e).

(e) implies (c): In general, $\pi(W) \leq |W|$. From (e), $(\exists V \neq 1, k \geq 1)$ $|V| = \pi(W)$, $W \leq V^k$. $k \geq 2$ since $2|V| = 2\pi(W) \leq |W| \leq k|V|$. Then $(\exists U \sim V) W \leq U^k$.

Lemma 4.14. If $W = XY^eZ$, $Z \leq Y \geq X$, $Y \neq 1$, $e \geq 1$ then $(\exists B \neq 1, A, p) 0 \leq p \leq 2$, $W = A(AB)^{e+p}$, $|AB| = |Y|$, $XZ = A(AB)^p$.

Proof. $(\exists C, D) Y = ZC = DX$. Let $Y_1 = XD$, $X_1 = XZ$. Then $X_1 \leq W = (Y_1)^eX_1$. Apply Lemma 4.10 to $X_1 \leq (Y_1)^eX_1$. $(\exists p \geq 0, A) X_1 = (Y_1)^pA$, $A < Y_1$. So $(\exists B \neq 1) Y_1 = AB$. Hence $W = (AB)^e(AB)^pA = A(AB)^{e+p}$; $XZ = X_1 = (AB)^pA = A(AB)^p$. Also $p \leq 2$ since $|Y^pA| = |(Y_1)^pA| = |X_1| = |XZ| \leq |Y^2|$. \hfill $\Box$

Lemma 4.15. If $AB \neq BA$ then $(\exists C, D \neq 1, a, b \geq 0) CD$ simple, $A = C(DC)^a$, $B = D(CD)^b$. For $t \geq 0$, $A(AB)^t = C(DC)^{p(t)}$, $(AB)^t = (CD)^{q(t)}$, $p(t) = (a+b+1)t + a$, $q(t) = (a+b+1)t$.

Proof. $(\exists$ simple $S$, $e \geq 1) AB = S^e$. $(\exists a, b \geq 0, C, D) S = CD$, $A = C(DC)^a$, $B = D(CD)^b$. $C, D \neq 1$ else $A, B$ are powers of the same word and $AB = BA$, a contradiction. \hfill $\Box$

Lemma 4.16. $Y, Z \neq 1$, $XZ = YX$ imply $(\exists r \geq 0, s \geq 1, C, D) CD$ simple, $Y = (CD)^s$, $Z = (DC)^s$, $X = C(DC)^r$.

Proof. By Lemma 4.1, $(\exists n \geq 0, U, V) Y = UV$, $Z = VU$, $X = U(VU)^n$. $(\exists$ simple $S) UV$ is a power of $S$. $(\exists i, j \geq 0, C, D) S = CD$, $U = C(DC)^i$, $V = D(CD)^j$. Use $r = i + n(i + j + 1)$, $s = i + j + 1$. \hfill $\Box$

5. General results.

Each 2-piece word is simple (Theorem 5.5). If $W = UV$ is 2-piece then $U, V$ appear again as factors of cyclic permutations of $W$. If $U, V$ appear at least twice in $W$ then $W$ is periodic (Theorem 5.6). If a 2-piece word $W$ is plain then $W$ or $W^*$ is biperiodic (Theorem 5.8). Each biperiodic word is 2-piece (Theorem 5.9). We start with some easily verifiable remarks.
Remark 5.1. If $PQ \neq QP$ and $m \geq 2$ then the periodic word $W = P(QP)^m$ is 2-piece, by definition, using $U = P(QP)^{m-1}$, $V = QP, Y = PQ, Z = QPP(QP)^{m-2}$.

Remark 5.2. A word is simple, periodic, 2-piece or biperiodic if and only if its reverse has the same property.

Remark 5.3. If $(U, V)$ is a 2-piece pair then so are $(V, U)$ and $(V^*, U^*)$.

A word is simple, periodic, 2-piece or biperiodic if and only if its reverse has the same property.

Given a 2-piece word $W, W^*$ inherits properties as follows:

Remark 5.4. If a 5-tuple $(W, U, V, Y, Z)$ of words satisfies $W = UV \sim YU \sim ZV, 1 \neq U \neq Z, 1 \neq V \neq Y$ then so does $(W^*, V^*, U^*, Z^*, Y^*)$.

Theorem 5.5. Each 2-piece word is a simple word.

Proof. Let $W$ be 2-piece word, $W = UV \sim UY \sim VZ, 1 \neq U \neq Z, 1 \neq V \neq Y$. $(\exists A, B) W = AB, UY = BA$. Suppose $W$ is not simple. Then $(\exists \text{ simple } X) W = X^m, m \geq 2$.

If $|X| \leq |U|$ then $(\exists C) U = XC$. So $AX \leq AUY \leq ABAB = X^{2m}$. By Lemma 4.6, $A$ is a power of $X$, so is $W$ and hence so is $B$. Thus $AB = BA$.

Then $UV = AB = BA = UY$ implies $V = Y$, a contradiction.

If $|X| > |U|$ then $|X| \leq |V|$. By Remark 5.3, $W^* = V^* U^*$ is 2-piece and $W^* = (X^*)^m$. Get a contradiction for $W^*, V^*, U^*$ as previously with $W, U, V$.

Theorem 5.6. If $U, V \neq 1$ each appear at least twice in $W = UV$ then $W$ is periodic. In other words, $W = UV = IUJ = KVL$ and $U, V, I, L \neq 1$ imply $W$ is periodic.

Proof. Assume $W = UV = XUT = RVY$ and $U, V, X, Y \neq 1$. The change in letters allows a more pleasing factorization $W = RST$ in Case 1.

Case 1: $|X| < |U|; |Y| < |V|$. Then $(\exists F, G) UGT = XUT = W = RFV = RVY, |U| > |X| = |G| > 0, |V| > |F| = |Y| > 0$. So $U = RF, V = GT, W = RFGT$. Let $S = FG$. Then $W = RST$. Hence $UGT = W = RST = W = RFV$ imply $UG = RS, ST = FG$. Apply Lemma 4.2 to $W_1 = RS, W_2 = ST$.

So $\pi(RS) \leq |RS| - |U|, \pi(ST) \leq |ST| - |V|$ since $RS = UG = XV, ST = RV = VY$. Then $|S| = |F| + |G| = |RS| - |U| + |ST| - |V| \geq \pi(RS) + \pi(ST)$. By Lemma 4.3, $\pi(RST) = \pi(RS) = \pi(ST)$. Since $|W| \geq |S| \geq \pi(RS) + \pi(ST) = 2\pi(W)$, $W$ is periodic by (4.13e) in Lemma 4.13.

Case 2: $|U| \leq |X|; |V| \leq |Y|$. Then $|U| \leq |UT| \leq |V|, |V| \leq |RV| \leq |U|$. So $|U| = |V|, U = V$ and hence $W = UU$ is periodic.

Case 3: $|X| < |U|; |V| \leq |Y|$. Then $(\exists F) UFT = XUT = W$ with $|U| > |X| = |F| > 0$. So $UF = XV$. By Lemma 4.16, $(3C, D, r \geq 0, s \geq 1) CD$ simple, $U = C(DC)^r, F = (DC)^s, X = (CD)^s, r \geq 1$. r
since \(|U| > |X| \geq |CD|\). \(DC \leq V\) since \(V = FT\). Thus \(|W| \geq |UF| \geq 2|CD|\). Also \((\exists P, Q)\ U = PVQ\).

Then \(PDC \leq PV \leq U = C(DC)^\circ\). Also \(DC \sim CD\) implies \(DC\) simple by Lemma 4.9. By Lemma 4.8, \(P = C(DC)^\circ\), \(e \geq 0\) implying \(V \leq (DC)^{\circ -e}\). So \(W \leq C(DC)^{2r - e}\); hence \(\pi(W) \leq |CD|\). Then \(|W| \geq 2|CD| \geq 2\pi(W)\) implies \(W\) is periodic by (4.13e) in Lemma 4.13.

Case 4: \(|U| \leq |X|; |Y| < |V|\). Then \(V^*, U^*\) each appear at least twice in \(W^* = V^*U^*\). Apply Case 3 to \(W^*\); get \(W^*\) periodic. By Remark 5.2, \(W\) is periodic.

\(\square\)

Lemma 5.7. Let \(W = UV\) be a plain word. Assume cyclic \(W\) has 2nd occurrences \(U''\), \(V''\) of the words \(U, V\), respectively. Then (i) \(U, U''\) overlap and \(V, V''\) overlap and (ii) \(U''\) is a factor of one of the words \(UV, VU\) and \(V''\) is a factor of the other.

Proof. First prove results (1)-(7).

(1) Conclusions for \(U, V, U'', V''\) apply to \(V^*, U^*, (V'')^*, (U'')^*\). By Remarks 5.2, 5.3, 5.4, assumptions on \(W, U, V, U'', V''\) apply to \(W^*, V^*, U^*, (V'')^*, (U'')^*\), respectively.

(2) Neither \(UV\) nor \(VU\) has both factors \(U'', V''\). Use Theorem 5.6.

(3) \(U, U''\) overlap or \(V, V''\) overlap. If not then \(U\) has factor \(V'', U\) has factor \(U''\). Therefore \(U = V, W = UU\), \(W\) is periodic, contradicting \(W\) is plain.

(4) \(U, U''\) overlap implies \((U'' \subseteq UV\ or \ U'' \subseteq VU\)\). Suppose \(U, U''\) overlap and \(U''\) is not a factor of \(UV\ or \ VU\). Then \((\exists A, B, C \neq 1)\ W = ABCV, U = ABC, U'' = CV\). So \(CV < ABCV > CV, AB < CVAB > AB\). Then \(ABCV\ or \ CVAB\) has a major left and right factor, namely, \(CV\) or \(AB\), respectively. Thus \(ABCV\ or \ CVAB\) is periodic by Lemma 4.13, contradicting \(W\) is plain. Thus (4) is true.

(5) \(V, V''\) overlap implies \((V'' \subseteq VU\ or \ V'' \subseteq UV)\). Use (1), (4). Get \(V^*, (V'')^*\ overlap implies ((V'')^* \subseteq V^*U^* or (V'')^* \subseteq U^*V^*). This implies (5).

(6) \(U, U''\) overlap implies \(V, V''\) overlap. If not then \(U, U''\) overlap but \(V, V''\) do not. So \(V'' \subseteq U\). Also \(U'' \subseteq UV\ or \ U'' \subseteq VU\). Then \(U, U''\, V, V'' \subseteq UV\ (or \ VU)\), contradicting (2).

(7) \(V, V''\) overlap implies \(U, U''\) overlap. Use (1), (6). Therefore \(V^*, (V'')^* \ overlap implies \ U^*, (U'')^* \ overlap. This implies (7).

Now (i) follows from (3), (6), (7) and (ii) follows from (i), (2), (4), (5). \(\square\)

Theorem 5.8. Each plain, 2-piece word \(W\) is biperiodic or its reverse is biperiodic.
Proof. By Remarks 5.2 and 5.3, $W^*$ is plain, 2-piece. Since $W = UV$ satisfies the 2-piece condition, cyclic $W$ has 2nd occurrences $U'', V''$ of the words $U, V$, respectively. By Lemma 5.7, there are 2 cases:

**Case 1:** $U'' \subseteq UV, V'' \subseteq VU$. Using the 2-piece property and Lemma 5.7 Part (i), it follows that $UV = W = UDB = CU''B, VU = VFG = EV''G, U = FG, V = DB, Y = BC, Z = GE, |U| > |C|, |V| > |E|$ for some words $B, C, D, E, F, G \neq 1$. Since $|U| > |C|, |V| > |E|$, we can apply Lemma 4.1 to $UD = CU$ and $VF = EV$.

$(\exists Q \neq 1, P, m \geq 1) U = P(QP)^m, C = PQ, D = QP$ and $(\exists S \neq 1, R, n \geq 1) V = R(SR)^n, E = RS, F = SR$. Thus $UV = UDB = UQPB, QP < V = R(SR)^n$ and $VU = VFG = VSRG, SR < U = P(QP)^m$, implying Conditions (3.1a), (3.1b) and (3.1c).

If $|PQ| = |RS|$ then $QP < V = R(SR)^n$, implying $QP = RS$. So $W = PX^{m+n}R$ for $X = QP$ and $W$ is periodic by Lemma 4.14, a contradiction. So (3.1d) is true.

If $URS = SRU$ then $FU = SRU = URS = UE = FGE = FZ$ and $U = Z$, not true. If $QPV = VPQ$ then $DV = QPV = VQ = DBC = DY$ and $V = Y$, not true. So $URS \neq SRU, QPV \neq VPQ$, (3.1e) is true, making $W$ biperiodic.

**Case 2:** $V'' \subseteq UV, U'' \subseteq VU$. Then $(V'')^* \subseteq V^*U^*, (U'')^* \subseteq U^*V^*$. By Remark 5.4, Case 1 applies to $W^*$ so $W^*$ is biperiodic.

\[ \square \]

**Theorem 5.9.** Each biperiodic word is a 2-piece word.

Proof. Let $W = UV$ be biperiodic using $P, Q, R, S, U, V, m, n$. $UV$ is biperiodic using $R, S, P, Q, V, U, n, m$. By symmetry and Remark 5.3, we may assume $|SR| < |PQ|$. By Conditions (3.1a), (3.1b) and (3.1c), $SR < PQ$ and $QP < R(SR)^n$. Define $F, J$ by $SRF = PQ, QPJ = R(SR)^n$. We now check that $W$ satisfies the definition of being 2-piece by using $Y = JPQ$, $Z = FP(QP)^{m-1}R$.

\[ UY = P(QP)^mJQ \sim P(QP)^{m+1}J = UQPJ = UV \]

\[ VZ = R(SR)^nFP(QP)^{m-1}RS \sim SRFP(QP)^{m-1}R(SR)^n \]

\[ = PQP(QP)^{m-1}V = UV. \]

If $Y = V$ then $VPQ = QPJPQ = QPY = QPV$, a contradiction. If $Z = U$ then we get a contradiction from $SRU = SRZ = PQP(QP)^{m-1}RS = URS$. Thus $W$ is 2-piece.

\[ \square \]

6. Factoring some 2-piece words.

The 2-piece words to be factored are two types of biperiodic words. They are called biperiodic-1 and biperiodic-2 words. They are 2-piece by Theorem 5.9.
Their factorizations are called *binary*-1 and *binary*-2 words and have factors $A, B, AB \neq BA$. See Theorems 6.10 and 6.11. A 3rd type of biperiodic word, called a *biperiodic*-3 word, has a cyclic permutation possessing a 3rd type of factorization, a *binary*-3 word (Theorem 6.12). Each binary-3 word has a binary-2 cyclic permutation (Remark 6.4). Likewise, each biperiodic-3 word has a biperiodic-2 cyclic permutation, using Theorem 6.12, Remark 6.4 and Theorem 6.11.

Results in this section, together with Remark 5.3, will show that 2-piece words can be found using the above factorizations and their reverses. More precisely, we have:

**Theorem 6.1.** *Binary*-1 and *binary*-2 words and their reverses are 2-piece words. Each *binary*-3 word has a 2-piece cyclic permutation and so does its reverse.

The types of biperiodic words and factorizations are defined as follows:

**Definition 6.2.** A word $W$ is *biperiodic*-1, *biperiodic*-2 or *biperiodic*-3 if $W$ satisfies Definition 3.1 together with (6.2a), (6.2b) or (6.2c), respectively.

(6.2a) $|S| < |RS| < |PR| < |PQ| < |RSR|$.  
(6.2b) $P = 1, |SR| < |Q|$.  
(6.2c) $|RS| \leq |P| < |PQ|$.

**Definition 6.3.** A word $W$ is *binary*-1, *binary*-2 or *binary*-3 if (6.3a), (6.3b) or (6.3c), respectively, with $AB \neq BA$. Call such $W$ *binary*. Terminology for later use: $W$ is binary-1 using $A, B : AB \neq BA$ and $W$ is binary-1 for $A, B, h, i, j$. Similar terminology applies to binary-2 and binary-3.

(6.3a) $W = A(BA)^i(A(BA)^j)^m(A(BA)^h)^n, 1 < h < i < j \leq 2h, m, n \geq 1$.  
(6.3b) $W = (A(BA)^i)^m(AB)^j, 1 \leq i, i+1 \leq j, m, n \geq 1$.  
(6.3c) $W = (A(BA)^i)^mA(BA)^j, 1 \leq i, i+2 \leq j, m, n \geq 1$.

**Remark 6.4.** Each binary-3 word $W$ has a cyclic permutation $V$ which is binary-2. In particular, if $W$ satisfies (6.3c), use $V = (A(BA)^i)^{m+1}(AB)^{j-i}$.

The proof that the biperiodic-1 and binary-1 conditions are equivalent for a word $W$ requires 3 lemmas involving closely related conditions defined as follows:

**Definition 6.5.** A word $W$ is *biperiodic*-1* if $W$ satisfies (3.1a)-(3.1d) with $m = n = 1$. Notice the omission of (3.1e). In other words, $W$ satisfies:

(6.5a) $(\exists P, Q, R, S) W = PQPRS, \quad SR < PQ, \quad QP < RSR, \quad |S| < |RS| < |PR| < |PQ|$.

**Definition 6.6.** $W$ is *binary*-1* if (6.3a) with $m = n = 1$. ($AB \neq BA$ not required.)
Lemma 6.7. Word \( W \) is biperiodic-1* if and only if

\[(6.7a)\]
\[\exists F, I, J, P, Q, R, S, T, U \text{ all words except possibly } S \text{ and } W = PQPRSR, \quad SRF = PQ, \quad QP = RSI, \quad R = IJ, \quad P = ST, \quad Q = RU.\]

Proof. Assume \((6.5a)\). Then \( SR < PQ, \quad QP < RSR \) imply \( PQ = SRF, QP = RSI, \quad R = IJ \) for some words \( F, I, J \neq 1 \). Using these inequalities and \( SR < PQ, \quad QP < RSR \) we get \( P = ST, \quad Q = RU \) for some words \( T, U \neq 1 \). Thus \((6.7a)\) is true. \((6.5a)\) follows easily from \((6.7a)\). \qed

Lemma 6.8. If \( W \) is biperiodic-1* then \( W \) is binary-1* using \( A, B : B \neq 1 \).

Proof. By Lemma 6.7, we can assume \( W \) satisfies \((6.7a)\) from which we deduce:

\[(1) \quad (\exists V \neq 1) \quad F = VU \quad \text{since } STRU = PRU = PQ = SRF \quad \text{and so } F > U.\]
\[(2) \quad UP = SI \quad \text{since } RUPJ = QPJ = RSIJ.\]
\[(3) \quad |T| < |I| \quad \text{since } |UST| = |UP| = |SI|, \quad |UT| = |I|.\]
\[(4) \quad RV = TR \quad \text{since (1) and } SRVU = SRF = PQ = STRU.\]
\[(5) \quad (\exists K \neq 1) \quad I = TK \quad \text{since (3), (4) and } R = IJ.\]
\[(6) \quad Q = KJF \quad \text{since } PQ = SRF = SIJF = STKJF = PKJF.\]
\[(7) \quad KJFST = TKJSTK \quad \text{by (5), (6) and } KJFST = PQ = IJSI = TKJSTK.\]
\[(8) \quad TK = KT \quad \text{since, from (7), } K \leq TK \geq T.\]
\[(9) \quad (\exists N \neq 1, r, s \geq 1) \quad K = N^r, \quad T = N^s \quad \text{using (8) and Lemma 4.4.}\]
\[(10) \quad SN^r = SK = US, \quad \text{since } SN^r > S \quad \text{hence } SKT = STK = SI = UP = UST.\]
\[(11) \quad JFS = TJSK \quad \text{since (7) and (8).}\]
\[(12) \quad JVUS = TJSK \quad \text{since (11) and (1).}\]
\[(13) \quad JV = TJ = N^s J, \quad \text{hence } J < N^s J \quad \text{using (12), (10) and (9).}\]
\[(14) \quad (\exists t \geq 0, D) \quad S = DN^t, \quad N > D \quad \text{by applying Lemma 4.11 to (10) } SN^r > S.\]
\[(15) \quad (\exists u \geq 0, L) \quad J = N^u L, \quad L < N \quad \text{by applying Lemma 4.10 to (13) } J < N^s J.\]
\[(16) \quad PR = DN^b L, \quad b = r + 2s + t + u \quad \text{since } PR = STIJ = DN^t N^s N^r + s N^u L.\]
4.14. ∗ 18
4.14. 19
6.3a
19
6.3a
19

Using (18) and (19) with Lemma 4.14 and (16) and (19) with Lemma 4.14, we have:

(∃B ≠ 1, A, p ≥ 0) SR = A(BA)^a+p, |AB| = |N|,
DL = A(BA)^p, |A| < |N| and hence AB = DC.

(∃G ≠ 1, H, q ≥ 0) PR = G(HG)^b+q, |GH| = |N|,
DL = G(HG)^q, |G| < |N| and hence GH = DC.

So p|N| + |A| = |DL| = q|N| + |G|, implying p = q, |A| = |G|. Then A = G, B = H since AB = DC = GH. So PR = A(BA)^b+p. Similarly from (17) and (19) with Lemma 4.14, UPR = A(BA)^r+p. From (16), (17) and (18), 1 < a < b < c ≤ 2a. So 1 < a+p < b+p < c+p ≤ 2a+p ≤ 2a+2p = 2(a+p).

Let h = a + p, i = b + p, j = c + p.

Then W = PQPRSR = (PR)(UPR)(SR) = A(BA)^iA(BA)^jA(BA)^h satisfies (6.3a) for m = n = 1. So W is binary-1∗. □

Theorem 6.10. A word is biperiodic-1 if and only if it is binary-1.
Proof. Assume that $W$ is biperiodic-1 with $P, Q, R, S$ as in (3.1a)-(3.1e) and (6.2a). Word $W_1 = PQPRSR$ is biperiodic-1*. By Lemma 6.7, $(3F, I, J, T, U)$ satisfying Condition (6.7a). By Lemma 6.8, $W_1$ is binary-1* and satisfies (6.3a) for $m = n = 1$ and some $B \neq 1$. As in the proof of Lemma 6.8:

$$W_1 = PQPRSR = PRUPRSR,$$

$$PR = A(BA)^i, UPR = A(BA)^j, SR = A(BA)^h.$$ 

So $W = P(QP)^m R(SR)^n = P(RUP)^m R(SR)^n = PR(UPR)^m (SR)^n$ implying (6.3a).

If $AB = BA$ then by Lemma 4.4, $A, B$ (and hence $W$) are powers of the same word. So $W$ is a proper power, not simple. By Theorems 5.9 and 5.5, $W$ is 2-piece and simple, a contradiction. Thus $AB \neq BA$ and $W$ is a binary-1.

Now assume $W$ is binary-1. Define $P, Q, R, S$ as in proof of Lemma 6.9 so that:

(6.10a) \[ 1 < SR < PQ, \quad QP < RSR. \]

(6.10b) \[ (QP)^m = A(BA)^i(A(BA)^j)^{m-1}(AB)^{j-i}. \]

(6.10c) \[ (QP)^m PR = A(BA)^i(A(BA)^j)^{m-1} A(BA)^j = A(BA)^i(A(BA)^j)^m. \]

(6.10d) \[ (QP)^m PR(SR)^n = A(BA)^i(A(BA)^j)^m (A(BA)^h)^n = W. \]

$W = UV$ for $U = (QP)^m P, V = R(SR)^n$. Then (6.10a) implies (3.1a)-(3.1d) are true.

If $URS = SRU$ then $A(BA)^{h+1} \leq A(BA)^i < PQ < URS$ and $A(BA)^h AB \leq SRP < SRU$ imply $A(BA)^{h+1} = A(BA)^h AB$. So $BA = AB$, a contradiction. Thus $URS \neq SRU$. If $VPQ = QPV$ then $VPQ > Q > (AB)^{j-i} > AB, QPV > V > SR > BA$ imply $AB = BA$, a contradiction. So $VPQ \neq QPV, (3.1e)$ is true and $W$ is biperiodic-1.

**Theorem 11.** A word is biperiodic-2 if and only if it is binary-2.

Proof. Assume $W$ is biperiodic-2. Then $U = Q^m, 1 < SR < Q < R(SR)^n = V$, implying $RS = SR$. By Lemma 4.4, $R = X^r, S = X^s, X \neq 1, r, s \geq 1$. Since $SR < Q < R(SR)^n, (\exists A \neq 1, B, t \geq 0) Q = A(BA)^{r+s+t}, X = AB$. Then $W = (A(BA)^i)^m (AB)^j$ using $i = r + s + t, j = r + n(r + s)$. Also $X^i < X^j A = Q < R(SR)^n = X^j$ implies $i < j, i+1 < j$. Thus (6.3b) is true.

If $AB = BA$ then by Lemma 4.4 $W$ is a proper power, not simple. But $W$ is 2-piece, simple by Theorems 5.9 and 5.5, a contradiction. So $AB \neq BA$ and $W$ is binary-2.

Now assume $W$ is binary-2. Use $Q = A(BA)^i, P = R = 1, S = AB, n = j, U = Q^m, V = R(SR)^n$. Then (3.1a)-(3.1d) and (6.2b) $P = 1, |SR| < |Q|$ are true. Suppose $URS = SRU$. Then

$$(A(BA)^i)^m AB = URS = SRU = AB(A(BA)^i)^m.$$
Hence, $AB = BA$, a contradiction. Suppose $VPQ = QPV$. Then

$$(AB)^n A(AB)^i = VPQ = QPV = A(AB)^i (AB)^n$$

hence, $AB = BA$, a contradiction. Thus (3.1e) is true and $W$ is biperiodic-2.

\begin{proof}
Assume contradiction. So

Each biperiodic-

Hence,

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QPRS < QPV

and

Theorem 6.12. Each biperiodic-3 word has a binary-3 cyclic permutation. Each binary-3 word has a biperiodic-3 cyclic permutation.

Proof. Assume $W$ is binary-3. Then $W \sim W_1 = BA(A(AB)^i)^m A(AB)^j - 1$. Use $P = BA, Q = A(AB)^i - 1, R = A, S = B, n = j$. Thus (3.1a)-(3.1d) imply $AB = BA$, a contradiction. If $VPQ = QPV$ then $A(AB)^i AB = QPRS < QPV$ and $A(AB)^i + 1 < A(AB)^j = V < VPQ$ imply $AB = BA$, a contradiction. So (3.1e) is true and $W$ has a biperiodic-3 cyclic permutation $W_1$.

Now assume $W$ is biperiodic-3. Then (3.1a)-(3.1c), $|RS| \leq |P| < |PQ|$ imply $SR \leq P, RS < QP, QP < R(SR)^n$. Thus $(\exists I, J \neq 1, T) QP = RSI, QPJ = R(SR)^n, P = SRT$.

1. $QRSR = QP = RSI,$

$IJ = R(SR)^{n-1}$

using $R(SR)^n = QPJ = RSIJ$.

2. $I = VT$ for some $V$,

$|V| = |Q|$

from (1).

3. $QRSRTJ = VTJSR$

from (3).

4. $Q = V$

from (3).

5. $SRTJ = TJSR$

from (6).

6. $QRSRTJ = RSIJ$

$RSVTJ = RSRTJ$

using (1), (2) and (4).

7. $RSQ = QSR$

from (7) and Lemma 4.11.

8. $(\exists a, b, c \geq 0, C, D)$

$R = C(DC)^a, S = D(CD)^b,$

$Q = C(DC)^c, CD$ simple

by Lemma 4.9 and (8).

9. $DC$ is simple; $SR = (DC)^{a+b+1}$

from (9), (5) and Lemmas 4.4 and 4.5.

10. $(\exists t \geq 1) T = (DC)^t$

from (10).

11. $(\exists d \geq 0, F \neq 1, G) T = (FG)^d F,$

$DC = FG$

from (11).

12. $R(SR)^n = C(DC)^a((DC)^{a+b+1})^n$

$= C(DC)^r$ with

$r = a + n(a + b + 1) \geq n \geq 2$

from (8).

13. $|C(DC)^{c+1}| \leq |C(DC)^{a+b+c+1}|

= |QRS| \leq |QP|

from (8) and $|RS| \leq |P|$.
(14) \[ |QP| < |R(SR)^n| \]

\[ = |C(DC)^n(a+b+1+a)| \]

from \( QP < R(SR)^n \) and (12).

(15) \[ c + 1 < n(a + b + 1) + a = r \]

from (13) and (14).

(16) \[ QP = C(DC)^eF, \]

\[ e = a + b + c + d + 2 \]

by (1) \( QP = QSRT, (8) \) and (11).

(17) \( (\exists B \neq 1, A, p \geq 0) \]

\[ QP = A(BA)^{e+p}, AB = CD. \]

Let \( i = e + p \) so that \( i \geq 2. \)

by (16), \( QP = X_1(Y_1)^eZ_1, \)

\( X_1 = C, Y_1 = DC = FG, Z_1 = F; \)

apply Lemma 4.14 to

\[ W_1 = X_1(Y_1)^eZ_1. \]

(18) \[ R(SR)^nP = C(DC)^jF, \]

\[ f = r + a + b + d + 1 \]

by \( P = SRT, (8) \) and (11).

(19) \( (\exists M \neq 1, L, q \geq 0) \]

\[ R(SR)^nP = L(ML)^j+q, \]

\[ LM = CD. \]

Let \( j = f + q. \)

apply Lemma 4.14 to

\[ W_2 = X_2(Y_2)^jZ_2. \]

(20) \[ |A| \equiv |CF| \text{ Modulo } |DC|; \]

\[ |A| \leq |DC| \]

from (16), (17) and \( |BA| = |DC|. \)

(21) \[ |L| \equiv |CF| \text{ Modulo } |DC|; \]

\[ |L| \leq |DC| \]

from (18), (19) and \( |ML| = |DC|. \)

(22) \[ |A| = |L|, A = L, B = M, \]

\[ L(ML)^j = A(BA)^j \]

by (20) and (21).

(23) \[ (j - i)|DC| = (j - i)|AB| \]

since (17) \( AB = CD. \)

\[ = |A(BA)^j| - |A(BA)^j| \]

from (19), (22) and (17).

\[ = |R(SR)^nP| - |QP| \]

from (12) and (8).

\[ = |C(DC)^e| - |C(DC)^e| \]

\[ = (r - c)|DC| \]

(24) \[ j - i = r - c > 1 \]

and hence \( i + 2 \leq j \)

from (23) and (15).

(25) \[ W \text{ has cyclic permutation} \]

\[ W_3 = (QP)^m(R(SR)^nP) \]

\[ = (A(BA)^j)^mA(BA)^j \]

from (17), (24) and (25).

If \( AB = BA, \) it follows that \( W_3, \) \( W \) are proper powers, not simple by (25)
and Lemma 4.4. However, \( W \) is 2-piece, simple by Theorems 5.9 and 5.5, a contradiction. So \( AB \neq BA \) and \( W_3 \) is binary-3.

\[ \square \]

### 7. Some nonperiodic binary words.

We prove that binary words, with some restrictions on their exponents, are not periodic. Details are in Theorems 7.5, 7.6 and 7.7. In the proofs, \( AB \)
simple, \( A, B \neq 1 \) can be assumed in (6.3a)-(6.3c) instead of \( AB \neq BA \) because of the following lemma.
Lemma 7.1. For \( k = 1, 2 \) or 3, equivalent properties for a word \( W \) are:

(i) binary-\( k \) using \( A, B : AB \neq BA \);
(ii) binary-\( k \) using \( C, D : C, D \neq 1, CD \) simple.

Proof. Assume (ii) for \( W \). Then \( CD \neq DC \) by Lemma 4.4. Use \( A = C, B = D \) to get (i). Now assume (i) for \( W \). By Lemma 4.14, \( (\exists C, D \neq 1, a, b \geq 0) \) \( CD \) simple, \( A = C(DC)^a, B = D(CD)^b \). Define \( p(t) = (1 + a + b)t + a, q(t) = (1 + a + b)t \) for \( t \geq 0 \).

Assume \( k = 1 \). Then \( p(h) < p(i) < p(j) \) since \( p(t) \) is strictly increasing. Since \( j \leq 2h \), \( p(j) \leq p(2h) = (1 + a + b)2h + a \leq (1 + a + b)2h + 2a = 2p(h) \). Also \( 1 < p(h) \) since \( 1 < h \leq p(h) \). So \( W \) is binary-1 for \( C, D, p(h), p(i), p(j) \).

Assume \( k = 2 \). Then \( p(i) + 1 = (1 + a + b)i + a + 1 \leq (1 + a + b)(j - 1) + a + 1 = q(j) - b \leq q(j) \) since \( i \leq j - 1 \). \( 1 \leq p(i) \) since \( 1 \leq i \leq p(i) \). So \( W \) is binary-2 for \( C, D, p(i), p(j) \).

Assume \( k = 3 \). Then \( 0 < p(i) \) since \( 0 < i \leq p(i) \). Since \( i \leq j - 2 \),

\[
p(i) + 2 = (1 + a + b)i + a + 2 \leq (1 + a + b)(j - 2) + a + 2 = q(j) - a - 2b \leq q(j).
\]

So \( W \) is binary-3 for \( C, D, p(i), p(j) \). Thus (ii) is true for \( W \) for \( k = 1, 2, 3 \).

\( \square \)

By Lemma 7.1, binary-1 and binary-3 words are products of words \( X_k \) defined below. Results about such products appear in the next two lemmas.

Definition 7.2. For fixed words \( A, B \neq 1, AB \) simple, define \( X_k = A(BA)^k, k \geq 0 \).

Lemma 7.3. Let \( G = X_{a_1} \ldots X_{a_m}, H = X_{b_1} \ldots X_{b_n}, 1 \leq a_i, 1 \leq b_j, 1 \leq i \leq m, 1 \leq j \leq n \) with \( 2 \leq m, n \). Assume \( G = H \). Then \( m = n, a_i = b_i \) for \( 1 \leq i \leq m \).

Proof. If not, pick least integer \( k \geq 1 \) with \( a_k \neq b_k \). Assume \( a_k < b_k \) so that \( X_{a_k}BA < X_{b_k} \). Therefore \( k < m, TX_{a_k}AB < TX_{a_k}X_{a_{k+1}} \leq G, TX_{a_k}BA < TX_{b_k} \leq H \) for \( T = X_{a_1} \ldots X_{a_k} \ldots X_{a_{k-1}} \). So \( AB = BA \); hence \( AB \) not simple by Lemma 4.4, a contradiction.

\( \square \)

Lemma 7.4. Let \( W = X_{a_1} \ldots X_{a_m}, 1 \leq a_i, 1 \leq i \leq m, m \geq 2 \). Assume \( (\exists F) F \leq W \geq F \).

(7.4a) If \( |X_{a_1}| < |F| \) then \( (\exists s, b) F = X_{a_1} \ldots X_{a_s}X_b, 1 \leq s < m, 1 \leq b \leq a_{s+1} \).

(7.4b) If \( |X_{a_m}| < |F| \) then \( (\exists t, c) F = X_cX_{a_t} \ldots X_{a_m}, 1 < t \leq m, 1 \leq c \leq a_{t-1} \).

Proof. Assume \( |X_{a_1}| < |F| \). Using \( F < W, W = X_{a_1} \ldots X_{a_m} \) induces a factorization \( F = Y_1 \ldots mY_rZ \) where \( Y_k = A \) or \( Y_k = B \) for \( 1 \leq k \leq r \) and
$Z$ equals $P$ or $Q$ for some words $P, Q$ satisfying $1 \leq P < A$, $1 \leq Q < B$. Also $r \geq 3$ since $|F| > |X_{a_1}|$, $a_1 \geq 1$. Since $AAA$, $BB$ do not appear in $W = X_{a_1} \ldots mX_{a_m}$, $Y = Y_{r-2}Y_{r-1}Y_r$ equals $BAA$, $AAB$, $BAB$ or $ABA$.

To prove (7.4a) it suffices to prove that $Y = ABA$ and $Z = 1$.

The five cases for $YZ$ are $BAAQ$, $AABP$, $BABP$, $ABAP$, $ABAQ$. As shown below, only Case 4 with $P = 1$ and Case 5 with $Q = 1$ can occur. So indeed $Y = ABA$, $Z = 1$.

**Case 1** $BAAQ$: $W > F$, $W > BABA$ imply $BABA > BAAQ$. By Lemma 4.7, $AQ$ is a power of $BA$ but $|AQ| < |BA|$. So $AQ = 1$, contradicting $A \neq 1$.

**Case 2** $AABP$: $W > F$, $W > ABA$ imply $ABA > ABP$. $ABA = RABP$ for some $R \neq 1$ with $|RP| = |A|$. So $ABAB = RABPB$, $RAB < ABAB$, $AB$ simple. By Lemma 4.6, $R$ is a power of $AB$ but $|R| \leq |A| < |AB|$. So $R = 1$, a contradiction.

**Case 3** $BABP$: $W > F$, $W > ABA$ imply $ABA > ABP$ as in Case 2.

**Case 4** $ABAP$: $W > F$, $W > BABA$ imply $BABA > BAP$. By Lemma 4.7, $P$ is a power of $BA$ but $|P| < |BA|$. So $P = 1$.

**Case 5** $BAQ$: $W > F$, $W > BABA$ imply $BABA > BAQ$. By Lemma 4.7, $Q$ is a power of $BA$ but $|Q| < |BA|$. So $Q = 1$.

Now assume $|X_{a_m}| < |F|$. So $W^* = (X_{a_m})^* \ldots m(X_{a_1})^*$ and $(X_{a_1})^* = A^*(B^*A^*)^t$, $t \geq 0$. $AB$ simple implies $BA$ simple by Lemma 4.9. $(BA)^*$ is simple by Remark 5.2. $A^*B^*$ is simple since $A^*B^* = (BA)^*$. Note that $F^* < W^* \geq F^*$. Apply (7.4a) to $W^*$, $F^*$. So $(3t, c) F^* = (X_{a_m})^* \ldots m(X_{a_1})(X_2)^*$, $1 < t \leq m$, $1 \leq c \leq a_{t-1}$. Take reverses to get (7.4b). □

**Theorem 7.5.** Each binary-1 word $W$ with $n \geq 2$ is not periodic.

**Proof.** By Lemma 7.1, Definition 7.2, $W = X_i(X_j)^m(X_h)^n$, $1 < h < i < j \leq 2h$ for some $A, B \neq 1$, $AB$ simple. By (4.13a), it suffices to prove that each major left factor of $W$ which is also a right factor is equal to $W$. Suppose $F \leq W \geq F$, $2|F| \geq |W|$. Then $|F| > |X_i|$, $|F| > |X_h|$ since $|X_i| < |X_j| > |X_h|$. By Lemma 7.4, $F = X_iGHX_h$ where $G$ is a product of one or more $X_j$ and $H$ is a product of one or more $X_h$. It also follows from Lemma 7.4 that:

(7.5a) $X_iGH$ and the start of $W$ have the same $X_k$ factors.

(7.5b) $GHX_h$ and the end of $W$ have the same $X_k$ factors.

$|F| \leq |W|$ implies $|GH| \leq |(X_j)^m(X_h)^{n-1}|$. It follows that:

(7.5c) $GH$ and the start of $(X_j)^m(X_h)^{n-1}$ have the same $X_k$ factors.

(7.5d) $GH$ and the end of $(X_j)^m(X_h)^{n-1}$ have the same $X_k$ factors.

(7.5c) implies $G = (X_j)^m$. (7.5d) implies $H = (X_h)^{n-1}$. Thus $F = W$ as required. □
Theorem 7.6. Each binary-2 word $W$ with $m(i + 1) \leq j$, $3 \leq j$ is not periodic.

Proof. By Lemma 7.1 and Definition 7.2, $W = (X_i)^m(AB)^j$, $1 \leq i$, $i+1 \leq j$, $m \geq 1$ for some $A, B \neq 1$, $AB$ simple. By (4.13a) it suffices to show that $W$ has no proper major left factor which is also a right factor. Suppose $F$ is such a factor, $2|F| \geq |W|$, $F < W > F$. We show this implies $AB = BA$, a contradiction.

Since $m(i + 1) \leq j$, $|(X_i)^m| = |A^m(BA)^m| < |(AB)^m(i+1)| \leq |(AB)^j|$. Then $2|F| \geq |W|$ and $|(X_i)^m| < |(AB)^j|$ imply $|(X_i)^m| < |F|$. Using $F < W$, $F$ has one of the forms:

$$(X_i)^m(AB)^rP, (X_i)^m(AB)^sAQ, (X_i)^mR$$

where $1 \leq r < j$, $0 \leq s < j$, $1 \leq P < A$, $1 \leq Q < B$, $1 < R < A$.

Case 1. $F = (X_i)^m(AB)^rP$: $W = (X_i)^m(AB)^j > F, r < j$ imply $(AB)^j > (AB)^rP$. By Lemma 4.7, $P = 1$. Then $F > BA, W > AB$ imply $AB = BA$.

Case 2. $F = (X_i)^m(AB)^sAQ$: $W > F$ implies $(AB)^j > (AB)^sAQ$ and $(AB)^j > AQ$. By Lemma 4.7, $AQ$ is a power of $AB$. Since $|AQ| < |AB|$, we have $AQ = 1, A = 1$, contradicting $AB \neq BA$. So this case cannot occur.

Case 3. $F = (X_i)^mR$: $W > F, 3 \leq j, 1 \leq i$ imply $(AB)^3 > ABAR$.

By Lemma 4.7, $AR = (AB)^t$ for some $t \geq 0$. Here $t \leq 1$ since $0 < |R| < |A|$. If $t = 0$ then $A = 1$, a contradiction. If $t = 1$ then $R = B$, $F > BA, W > AB$ so that $AB = BA$. 

Theorem 7.7. Each binary-3 word $W$ is not periodic.

Proof. By Lemma 7.1 and Definition 7.2, $W = (X_i)^mX_j$, $1 \leq i$, $i+2 \leq j$, $m \geq 1$ for some $A, B \neq 1$, $AB$ simple. By (4.13a) it suffices to show that $W$ has no proper major left factor which is also a right factor. Suppose $F$ is such a factor, $2|F| \geq |W|$, $F < W > F$. We show this implies $AB = BA$, a contradiction.

Since $|W| \geq |X_iX_j| > |X_iX_j| = 2|X_i|$ we have $2|F| \geq |W| > 2|X_i|$. By (7.4a), $F$ has one of the forms: $(X_i)^r, (X_i)^sX_b, (X_i)^mX_c$ where $1 \leq r \leq m$, $1 \leq s < m$, $1 \leq b < j$. Thus $F$ has a right factor $X_iX_a$ for some $a, 1 \leq a < j$ and hence $F > BAX_a$. Also $W > ABX_a$. Therefore $AB = BA$. 

8. Computing possibly biperiodic words.

Let $W = UV, SR < P(QP)^m = U, QP < R(SR)^n = V, 0 < |RS| < |PQ| < |U|, m, n \geq 1$. $W$ may be biperiodic. $p = |PQ|, q = |SR|, d = |U| - p, e = |V| - q$ satisfy $0 < d, 0 < e, q < p < q + e$. Function $g$ (see below) with
inputs $p, q, d, e$, generates such a $W$ as a list of integers $\geq 1$. Format for $g$
comes from Mathematica software, version 3.0.

\[
\begin{aligned}
g[p, q, d, e_\ldots] &:= \text{Module} \left\{ \{ i, k, n = p + q + d + e, w \} \right. \\
& \text{If} \left[ \!(\!(0 < d) \& \& (0 < e) \& \& (q < p) \& \& (p < q + e)) \right] , \\
& \quad \text{Return} \left[ \text{"Invalid Input"} \right] ; \\
& \quad w = \text{Join}\left[ \text{Range}[q], \text{Range}[n - 2q], \text{Range}[q] \right] ; \\
& \quad \text{For}[k = n - q, k \geq p + d + 1, k --, w[[k]] = w[[k + q]]] ; \\
& \quad \text{For}[k = p + d, k \geq q + 1, k --, w[[k]] = w[[k + p]]] ; \\
& \quad \text{For}[k = q, k \geq 1, k --, i = w[[k + p]]; w = w ./ (k -- i)]; w].
\end{aligned}
\]

The observed output from $g$ is (unpredictably) either biperiodic or a proper
power.

In Mathematica, \text{Range}[q] is the list of positive integers from 1 to $q$. 
\text{Join}[a, b, c] concatenates lists $a, b, c$. The code $k$ indicates integer $k$ is
decreased by 1 after each stage of a loop. $w[[i]]$ is $i$-th element of the list $w$.
The code $w = w ./ (k -- i)$ rewrites list $w$ by replacing each instance of the
current value of $k$ in list $w$ by the current value of $i$.


We give 2 sets of examples of biperiodic words $W = PQPRSR$ over the
alphabet \{a, b\}. In Example (9.1), $|PR| < |RS|$, $P \neq 1$. In Example (9.2),
$|PQ| < |PR|$, $P \neq 1$. Therefore (6.2a)-(6.2c) are not true. These examples
include $g[24, 18, 3, 10]$ and $g[24, 18, 15, 14]$ for the function $g$ defined in the
previous section.

**Example 9.1.** $W = (CCDDCD)^2D(ab)^{j - i}, C = a(ba)^i, D = a(ba)^j, 0 < i < j \leq 2i$.

Let $P = C, Q = CCDDCD, U = PQP = CCDDCD, R = CD(ab)^{j - i},$
$S = CC(ab)^{j - i}, V = RSR = CCDDCD(ab)^{j - i}$. $W$ is biperiodic because:

- $SR < PQ$ since $SR = CCDD(ab)^{j - i}, PQ = CCDDCD = CCDD(ab)^i aD, j \leq 2i$.
- $QP < RSR$ since $C < D, QP = CCDDCD, RSR = CCDDCD(ab)^{j - i}$.
- $|PR| < |RS|$ since $|PR| = |CCD(ab)^{j - i}| < |CCDDCD(ab)^{j - i}| = |RS|$.
- $URS \neq SRU$ else $URCCD = URS = SRUC = SRCCDDCDCC, CCD = DCC$, not true.
- $QPV \neq VQP$ else $QPCDDCDD = QPV = VPCDDCD, DD > DC$, not true.

Using $i = 1, j = 2$ and shorthand $2 = ab, 3 = aba, 5 = ababa,$
$W = (335535)(335535)52$. Rewrite $W$ by replacing the letters $a, b$ with the
symbols 1, 2, respectively. The resulting word, written as a list, is equal to
$g[24, 18, 3, 10]$.
Example 9.2. \( W = X(XZY)^2XXY, X = (abb)^h a b, Y = (abb)^i a b, Z = (abb)^{2h+1} a b, 0 < h < i. \)

Let \( P = XXX, Q = b X (abb)^{i-h}, PQ = XXX (abb)^{i-h}, QP = bXYXX, U = PQP = XXXZYXX, R = bXY, S = (abb)^h a, V = RSR = bXYXXY. \)

\( W \) is biperiodic because:

\[ SR < PQ, \quad PQ < RSR \quad \text{since} \quad SR = XXY. \]

\[ |PQ| < |PR| \quad \text{since} \quad i - h < i \implies Q < R. \]

\[ URS \neq SRU \quad \text{since} \quad URS > S > ba, \quad SRU > X > ab \text{ and } ab \neq ba. \]

\[ VPQ \neq QPV \quad \text{since} \quad VPQ > Q > bb, \quad QPV > Y > ab \text{ and } bb \neq ab. \]

Using \( h = 1, \ i = 2 \) and shorthand \( 5. = X, \ 8. = Y, \ 11. = Z, \ W = 5.5.11.8.5.11.8.5.5.8. \) Rewrite \( W \) by replacing the letters \( a, b \) with the symbols \( 1, 2 \), respectively. The resulting word, written as a list, is equal to \( g[24, 18, 15, 14]. \)

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Received May 17, 1999 and revised April 10, 2000.

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