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QUASIHOMEOMORPHISMS AND UNIVALENT HARMONIC MAPPINGS ONTO PUNCTURED BOUNDED CONVEX DOMAINS

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Dedicated to the memory of Professor Ahmad Shamsuddin

This paper deals with univalent harmonic mappings of annuli onto punctured bounded convex domains. Several aspects of these mappings are investigated; in particular, boundary functions, existence and uniquenss questions, and the geometry of their analytic and (co-analytic) parts. The paper also considers univalence criteria for Dirichlet solutions in annuli of boundary functions that are a generalized type of homeomorphisms, called quasihomeomorphisms, on one boundary component and constants on the other.

1. Introduction.

A *harmonic mapping* f of a region $\mathcal{D}$ is a complex-valued function of the form $f = h + \overline{g}$, where h and g are analytic functions in $\mathcal{D}$, unique up to an additive constant, that are single-valued if $\mathcal{D}$ is simply connected and possibly mutiple-valued otherwise. We call h and g the *analytic* and *co-analytic* parts of f, respectively. If f is (locally) injective, then f is called (*locally*) *univalent*. Note that every conformal and anti-conformal function is a univalent harmonic mapping. The *Jacobian* and *second complex dilatation* of f are given by the functions $J(z) = |h'(z)|^2 - |g'(z)|^2$ and $\omega(z) = g'(z)/h'(z)$, $z \in \mathcal{D}$, respectively. Note that ω is either a nonconstant meromorphic function or a (possibly infinite) constant. A result of Lewy [**13**] states that if f is a locally univalent mapping, then its Jacobian J is never zero; namely, for $z \in \mathcal{D}$, either $J(z) > 0$ or $J(z) < 0$. In the first case $|\omega(z)| < 1$ and f is sense-preserving, and in the second $|\omega(z)| > 1$ and f is sense-reversing.

A *ring domain* is a doubly-connected open subset of the plane. Denote by $\mathbb{A}(\rho, 1)$ the annulus $\{z : \rho < |z| < 1\}$, $0 \le \rho < 1$. It seems that Nitsche [**16**] was the first to consider univalent harmonic mappings of $\mathbb{A}(\rho, 1)$ onto $\mathbb{A}(R, 1)$. Indeed, Nitsche observed that, unlike with conformal mappings, R can possibly be zero as with the harmonic mapping

$$f(z) = (z - \rho^2/\overline{z})/(1 - \rho^2) \tag{1.1}$$

which can be easily shown to map $\mathbb{A}(\rho,1)$ univalently onto the punctured disc $\mathbb{A}(0,1)$. Later, Nitsche [**17**, §879] posed the following question.

Question (Nitsche). All univalent harmonic mappings from $\mathbb{A}(\rho,1)$ onto $\mathbb{A}(0,1)$, up to a rotation, are of form (1.1).

A negative answer to this question was given by Hengartner and Schober [**9**]. In their paper, the authors also investigated existence and uniqueness theorems for univalent harmonic mappings with given dilatations between annuli. Subsequently, Hengartner and Szynal [**10**] and Bshouty and Hengartner [**1**] gave a representation for harmonic mappings f defined on an annulus $\mathbb{A}(\rho,1)$ and constant on the inner circle as follows.

Theorem A. *Let f be a harmonic function of $\mathbb{A}(\rho,1)$, $0<\rho<1$, that extends continuously across $|z|=\rho$ with f identically ζ there. Then there exists a constant c and a function h analytic in $\mathbb{A}(\rho^2,1)$ such that*

$$f(z)=h(z)-\overline{h(\rho^2/\overline{z})}+\zeta+2c\log(|z|/\rho). \tag{1.2}$$

Further, if f extends continuously across $|z|=1$, and f^* is the restriction of f on $|z|=1$, then $c=0$ if and only if ζ equals

$$\zeta_0=\frac{1}{2\pi}\int_0^{2\pi}f^*(e^{it})\,dt. \tag{1.3}$$

Using Theorem A, Bshouty and Hengartner [**1**] proved the following result.

Theorem B. *Suppose that the following are true:*

(i) *$\mathbb{G}$ is a bounded convex domain.*
(ii) *f^* is a sense-preserving homeomorphism between the unit circle and $\partial\mathbb{G}$, and the constant $\zeta_0\in\mathbb{G}$ given by Equation* (1.3) *on $|z|=\rho$.*
(iii) *f is the Dirichlet solution of f^* in $\mathbb{A}(\rho,1)$.*

Then $f:\mathbb{A}(\rho,1)\to\mathbb{G}\setminus\{\zeta_0\}$ is a homeomorphism.

The author [**14**, Theorem 2] observed that Theorem B remains true under the weaker condition $f(\mathbb{A}(\rho,1))\subset\mathbb{G}$ rather than the convexity of $\mathbb{G}$.

In this paper, we investigate univalent harmonic mappings of ring domains onto bounded punctured convex domains. Throughout the paper we shall use the following notation: $\mathbb{C}$ for the complex plane, $\widehat{\mathbb{C}}$ for the extended complex plane, $\mathbb{D}$ for the open unit disc $\{z\in\mathbb{C}:|z|<1\}$, $\mathbb{T}$ for the unit circle $\{z\in\mathbb{C}:|z|=1\}$, $0<\rho<1$, $\mathbb{T}_\rho$ for the circle $\{z\in\mathbb{C}:|z|=\rho\}$, $\mathbb{A}(\rho,1)$ for the annulus $\{z\in\mathbb{C}:\rho<|z|<1\}$, $\mathbb{G}$ for a bounded convex domain. Also, for a subset $S\subset\mathbb{C}$, we denote by ∂S and $\overline{S}$ the boundary and closure of S in $\mathbb{C}$, respectively.

The paper is organized as follows. In Section 2, we describe the boundary functions, called quasihomeomorphisms, of univalent harmonic mappings onto punctured convex domains, and extend Theorem B to sense-preserving quasihomeomorphisms. Section 3 is devoted to investigate the geometry of the analytic parts of univalent harmonic mappings in $\mathbb{A}(\rho, 1)$ onto punctured convex domains. One result of this section asserts that these (analytic parts) have nonvanishing derivatives on $\mathbb{T}_\rho$, and that they map $\mathbb{T}_\rho$ univalently onto Jordan convex curves. Another concludes that these can be written as univalent close-to-convex functions of homeomorphisms in $\mathbb{A}(\rho, 1)$. In Section 4, we study univalence criteria for Dirichlet solutions in $\mathbb{A}(\rho, 1)$ of boundary functions that are sense-preserving quasihomeomorphisms between $\mathbb{T}$ and $\partial\mathbb{G}$ and constants on $\mathbb{T}_\rho$— a study which was motivated by Hengartner [**2**, Problem 15]. In Section 5, we prove a uniqueness result implying that the function f defined by (1.1) is the only univalent harmonic mapping, up to rotation, of $\mathbb{A}(\rho, 1)$ onto $\mathbb{A}(0, 1)$ having zero as an average value on $\mathbb{T}$ and with analytic part that extends analytically throughout $\mathbb{D}$. This somehow corrects Nitsche's question above and sheds light on Nitsche's insight in that direction.

2. Quasihomeomorphisms and Univalent Harmonic Mappings.

The purpose of this section is to characterize the boundary functions of univalent harmonic mappings, and to extend Theorem B to "quasihomeomorphisms". For this purpose, we need the following definition.

Definition 2.1. Let f be a function of $\mathbb{T}$ into a Jordan curve C of $\mathbb{C}$. We say f is a sense-preserving quasihomeomorphism of $\mathbb{T}$ into C if it is a pointwise limit of a sequence of sense-preserving homeomorphisms of $\mathbb{T}$ onto C. If in addition, f is a continuous function onto C, then f is called a sense-preserving weak homeomorphism.

The definition is based on Bshouty, Hengartner and Naghibi-Beidokhti [**3**, Definitions 3.1, 3.2]. Sense-preserving quasihomeomorphisms and sense-preserving weak homeomorphisms are characterized as follows.

Proposition 2.1. *Let f be a function of $\mathbb{T}$ into a Jordan curve C, and let F be a sense-preserving homeomorphism of $\mathbb{T}$ onto C.*

(i) *If f is a sense-preserving quasihomeomorphism of $\mathbb{T}$ onto C, then there is a real-valued nondecreasing function φ on $\mathbb{R}$ such that $\varphi(t + 2\pi) = \varphi(t) + 2\pi$ and $f(e^{it}) = F(e^{i\varphi(t)})$.*

(ii) *If $f(e^{it}) = F(e^{i\varphi(t)})$, where φ is a real-valued nondecreasing function on $\mathbb{R}$ such that $\varphi(t + 2\pi) = \varphi(t) + 2\pi$, and if E is the countable set of points $e^{i\varphi(t)}$ where φ is discontinuous, then f coincides on $\mathbb{T} \setminus E$ with a sense-preserving quasihomeomorphism of $\mathbb{T}$. In this case, f is the pointwise limit in $\mathbb{T} \setminus E$ of a sequence of sense-preserving homeomorphisms $f_n(e^{it}) = F(e^{i\varphi_n(t)})$ of $\mathbb{T}$ onto C, where each φ_n is a* *real-valued infinite differentiable function on* $\mathbb{R}$ *such that* $\varphi_n(t+2\pi) = \varphi_n(t) + 2\pi$ *and* ${\varphi_n}'(t)$ *is always positive.*

(iii) f *is a sense-preserving weak homeomorphism of* $\mathbb{T}$ *onto* C *if and only if there is a real-valued continuous nondecreasing function* φ *on* $\mathbb{R}$ *such that* $\varphi(t+2\pi) = \varphi(t) + 2\pi$ *and* $f(e^{it}) = F(e^{i\varphi(t)})$*. In this case,* f *is the uniform limit of a sequence of sense-preserving homeomorphisms* $f_n(e^{it}) = F(e^{i\varphi_n(t)})$ *of* $\mathbb{T}$ *onto* C*, where each* $\{\varphi_n\}$ *is a real-valued infinite differentiable function on* $\mathbb{R}$ *such that* $\varphi_n(t+2\pi) = \varphi_n(t) + 2\pi$ *and* ${\varphi_n}'(t)$ *is always positive.*

Proof. (i) There is a sequence $\{f_n\}$ of sense-preserving homeomorphisms of $\mathbb{T}$ onto C that converges pointwise to f. We can write $f_n(e^{it}) = F(e^{i\varphi_n(t)})$, where each $\{\varphi_n\}$ is a real-valued increasing function on $\mathbb{R}$ such that $0 \leq \varphi_n(0) < 2\pi$ and $\varphi_n(t+2\pi) = \varphi_n(t)+2\pi$. Then, by Helly's selection theorem, there is a real-valued nondecreasing function φ on $\mathbb{R}$ such that $\varphi(t+2\pi) = \varphi(t) + 2\pi$ and $\varphi_n \to \varphi$ pointwise in $\mathbb{R}$. Therefore, $f(e^{it}) = F(e^{i\varphi(t)})$ and (i) follows.

(ii) The function $\varphi(t) - t$ is bounded, a.e. differentiable, and has period 2π. For fixed $n = 1, 2, \ldots$, define the function

$$\varphi_n(t) = t + \frac{1}{2\pi}\int_0^{2\pi} P(1-1/n, t-\theta)[\varphi(\theta) - \theta]\, d\theta, \tag{2.1}$$

where $P(r,\theta)$ is the Poisson kernel. Then φ_n is an infinite differentiable function such that $\varphi_n(t+2\pi) = \varphi_n(t) + 2\pi$. Also,

$${\varphi_n}'(t) = \frac{1}{2\pi}\int_0^{2\pi} P(1-1/n, t-\theta) d\varphi(\theta) > \frac{1}{2n-1}$$

since $P(1-1/n, t-\theta) < 1/(2n-1)$, $-\infty \leq t, \theta \leq \infty$. Denote by E the set of points of $\mathbb{T}$ where $e^{i\varphi(t)}$ is discontinuous; then E is countable since φ is a nondecreasing function. But by a Schwarz's theorem, $\varphi_n \to \varphi$ pointwise in the set of continuity of φ. Therefore, $f_n \to f$ pointwise in $\mathbb{T} \setminus E$.

(iii) If f is a sense-preserving weak homeomorphism of $\mathbb{T}$ onto C, then, by (i), $f(e^{it}) = F(e^{i\varphi(t)})$ where $\varphi(t)$ is a real-valued nondecreasing function on $\mathbb{R}$ such that $\varphi(t+2\pi) = \varphi(t)+2\pi$. Since $F : \mathbb{T} \to C$ is a homeomorphism and f is continuous, $e^{i\varphi(t)} = F^{-1} \circ f(e^{it})$ is also continuous in $\mathbb{R}$. This, together with the nonconstancy of f, implies that φ is also continuous in $\mathbb{R}$.

Suppose now that $f(e^{it}) = F(e^{i\varphi(t)})$ where φ is a real-valued continuous nondecreasing function on $\mathbb{R}$ such that $\varphi(t+2\pi) = \varphi(t) + 2\pi$. Define the functions φ_n as in the proof of (ii), and recall that $\varphi_n(t+2\pi) = \varphi_n(t) + 2\pi$ and that φ_n' is always positive. Observe that, since φ is continuous, $\varphi_n \to \varphi$ uniformly in $\mathbb{R}$. Hence, with $f_n(e^{it}) = F(e^{i\varphi_n(t)})$, each f_n is a sense-preserving homeomorphism of $\mathbb{T}$ onto C and $f_n \to f$ uniformly on $\mathbb{T}$. This concludes (iii). $\square$

Let f be a function of $\mathbb{A}(\rho,1)$ into $\widehat{\mathbb{C}}$, and let $\xi\in\mathbb{T}$. We say that f has the *unrestricted limit* $a\in\widehat{\mathbb{C}}$ at if

$$f(z)\to a \qquad z\to\xi, \qquad z\in\mathbb{A}(\rho,1);$$

by defining $f(\xi)=a$ the function f becomes continuous at ξ as a function in $\mathbb{A}(\rho,1)\cup\{\xi\}$. We shall use $f(\xi)$ to denote the unrestricted limit whenever it exists, and call the resulting function, on its domain of definition in $\mathbb{T}$, the *unrestricted limit* function f. We also define the *cluster set* $C(f,\xi)$ of f at ξ as the set of all $b\in\widehat{\mathbb{C}}$ for which there are sequences $\{z_n\}$ such that

$$z_n\in\mathbb{A}(\rho,1), \qquad z_n\to\xi, \qquad f(z_n)\to b \qquad \text{as } n\to\infty.$$

Moreover, If F is a subset of $\mathbb{T}$, then we define the *cluster set* $C(f,F)$ of f at F as the set-union of the cluster sets $C(f,\xi)$ for $\xi\in E$.

Sense-preserving quasihomeomorphisms are essential for describing the boundary behaviour of univalent harmonic mappings of ring domains onto bounded convex domains. Suppose f is a univalent harmonic mapping of $\mathbb{A}(\rho,1)$ onto a ring domain $G\setminus\{\zeta\}$, $\zeta\in\mathbb{G}$. Then either $\lim_{|z|\uparrow 1}f(z)=\zeta$ and $C(f,\mathbb{T}_\rho)=\partial\mathbb{G}$, or $\lim_{|z|\downarrow\rho}f(z)=\zeta$ and $C(f,\mathbb{T})=\partial\mathbb{G}$. In the first case, $f(1/z)$ becomes a univalent harmonic mapping of $\mathbb{A}(\rho,1)$ onto $G\setminus\{\zeta\}$ with $\lim_{|z|\downarrow\rho}f(1/z)=\zeta$ and $C(f(1/z),\mathbb{T})=\partial\mathbb{G}$. For our study, this leads us to consider, without loss of generality, only univalent harmonic mappings of $\mathbb{A}(\rho,1)$ onto ring domains $G\setminus\{\zeta\}$, $\zeta\in\mathbb{G}$, with $\lim_{|z|\downarrow\rho}f(z)=\zeta$.

Definition 2.2. Denote by $\mathcal{H}_u(\rho,\mathbb{G})$ the class of all univalent harmonic mappings f of $\mathbb{A}(\rho,1)$ onto ring domains $G\setminus\{\zeta\}$, $\zeta\in\mathbb{G}$, with $f(\mathbb{T}_\rho)=\zeta$.

The boundary behavior of functions $f\in\mathcal{H}_u(\rho,\mathbb{G})$ is given as follows.

Theorem 2.1. *Let $f\in\mathcal{H}_u(\rho,\mathbb{G})$. Then there is a countable set $E\subset\mathbb{T}$ such that the following hold:*

(i) *For each $e^{i\theta}\in\mathbb{T}\setminus E$, the unrestricted limit $f(e^{i\theta})$ exists and belongs to $\partial\mathbb{G}$. Furthermore, f is continuous in $\overline{\mathbb{A}(\rho,1)}\setminus E$.*

(ii) *For each $e^{i\theta_0}\in E$, the side-limits* $\lim_{\theta\uparrow\theta_0}f(e^{i\theta})$ *and* $\lim_{\theta\downarrow\theta_0}f(e^{i\theta})$ *exist in $\partial\mathbb{G}$ and are distinct.*

(iii) *For each $e^{i\theta_0}\in E$, the cluster set $C(f,e^{i\theta_0})$ lies in $\partial\mathbb{G}$ and is the straight-line segment joining the side-limits* $\lim_{\theta\uparrow\theta_0}f(e^{i\theta})$ *and* $\lim_{\theta\downarrow\theta_0}f(e^{i\theta})$.

(iv) $\overline{co}(f(\mathbb{T}\setminus E))=\overline{\mathbb{G}}$.

(v) *There is a sense-preserving quasihomeomorphism of $\mathbb{T}$ into $\partial\mathbb{G}$ that coincides with the unrestricted limit function f on $\mathbb{T}\setminus E$.*

(vi) *f is the Dirichlet solution in $\mathbb{A}(\rho,1)$ of the function f^* defined by the unrestricted limit function of f on $\mathbb{T}$ and the value of f on $\mathbb{T}_\rho$.*

The fact that f^* is not defined on E in (vi) is insignificant. Indeed, Dirichlet solutions in multiply connected domains coincide whenever their boundary functions coincide almost everywhere.

Proof. Applying [**8**, Theorem 4.3] to f locally at each $e^{i\theta}$ yields (i), (ii), and (iii) except for the inclusion $C(f, e^{i\theta_0}) \subset \partial\mathbb{G}$ which follows because $f : \mathbb{A}(\rho, 1) \to \partial\mathbb{G}$ is onto. Also, (vi) follows from the maximum principle.

(iv) Since $\overline{\mathbb{G}}$ is convex and each unrestricted limit $f(e^{i\theta})$ belongs to $\partial\mathbb{G}$, $\overline{co}(f(\mathbb{T} \setminus E)) \subset \overline{\mathbb{G}}$. Let $w \in \partial\mathbb{G}$. Because $f : \mathbb{A}(\rho, 1) \to \mathbb{G}$ is onto, w belongs to the cluster set of some point $\xi \in \mathbb{T}$. If $\xi \in E$, then w is the unrestricted limit $f(\xi)$. If $\xi \notin E$, then w belongs to the boundary straight-line segment joining the side-limits at ξ of the unrestricted function f. Note that these limits belong to $\overline{co}(f(\mathbb{T} \setminus E))$; consequently $w \in \overline{co}(f(\mathbb{T} \setminus E))$. In either case $\overline{\mathbb{G}} \subset \overline{co}(f(\mathbb{T} \setminus E))$, and (iv) follows.

(v) Since $\mathbb{G}$ is a Jordan domain, there is a homeomorphism F of $\overline{\mathbb{D}}$ onto $\overline{\mathbb{G}}$ which maps $\mathbb{D}$ conformally onto $\mathbb{G}$. Let $G(z) = F^{-1} \circ f(z)$, $z \in \mathbb{A}(\rho, 1)$. Observe that G is sense-preserving homeomorphism of $\mathbb{A}(\rho, 1)$ into $\mathbb{D}$ which extends continuously to a mapping, also denoted by G, from $\mathbb{A}(\rho, 1) \cup (\mathbb{T} \setminus E)$ to $\mathbb{T}$. Let $I = \{t : -\infty < t < \infty,\ e^{it} \in \mathbb{T} \setminus E\}$. We conclude that there is a continuous nondecreasing function φ on I such that $G(e^{it}) = e^{i\varphi(t)}$, $t \in I$, and

$$\sup\{\varphi(t) : t \in I \cap [0, 2\pi)\} - \inf\{\varphi(t) : t \in I \cap [0, 2\pi)\} \leq 2\pi.$$

Extend φ to $(-\infty, \infty)$ by defining $\varphi(\tau) = \inf\{\varphi(t) : t \in I \cap [0, 2\pi), t > \tau\}$ if $\tau \in [0, 2\pi) \setminus I$, and by letting $\varphi(t + 2\pi) = \varphi(t) + 2\pi$. It is immediate that the new φ is a nondecreasing function on $(-\infty, \infty)$ with period 2π that is continuous only on I. Using Proposition 2.1(ii), the function $F(e^{i\varphi(t)})$ coincides with a sense-preserving quasihomeomorphism of $\mathbb{T}$ into $\partial\mathbb{G}$ on $\mathbb{T} \setminus E$. □

Now Let f^* be a sense-preserving quasihomeomorphism of $\mathbb{T}$ into $\partial\mathbb{G}$. Throughout the paper we denote by $\widetilde{E}(f^*)$ the set of points $e^{i\theta}$ at which f^* is continuous. Our second purpose in this section is to show that if $\overline{\mathbb{G}}$ is the closed convex hull of $f^*(\widetilde{E}(f^*))$, then f^* yields a univalent harmonic mapping of $\mathbb{A}(\rho, 1)$ onto the convex domain $\mathbb{G}$ minus one point. This extends Theorem B to sense-preserving quasihomeomorphisms f^* of $\mathbb{T}$ into $\partial\mathbb{G}$.

Theorem 2.2. *Suppose that the following are true:*

(i) *f^* is a sense-preserving quasihomeomorphism of $\mathbb{T}$ into $\partial\mathbb{G}$, and the constant ζ_0 defined by* (1.3) *on $\mathbb{T}_\rho$.*
(ii) $\overline{co} f^*(\widetilde{E}(f^*)) = \overline{\mathbb{G}}$.
(iii) *f is the Dirichlet solution of f^* in $\mathbb{A}(\rho, 1)$.*

Then $\zeta_0 \in \mathbb{G}$, $f \in \mathcal{H}_u(\rho, \mathbb{G})$, *and there is an analytic function* h *of* $\mathbb{A}(\rho^2, 1)$ *such that*

$$f(z) = h(z) - h(\rho^2/\overline{z}) + \zeta_0, \qquad (z \in \mathbb{A}(\rho, 1)). \tag{2.2}$$

The proof of the theorem requires two lemmas. Let f^* be a sense-preserving quasihomeomorphism of $\mathbb{T}$ into $\partial\mathbb{G}$, and let F be a homeomorphism of $\overline{\mathbb{D}}$ onto $\overline{\mathbb{G}}$ that maps $\mathbb{D}$ conformally onto $\mathbb{G}$. By Proposition 2.1(i), there is a real-valued nondecreasing function φ on $\mathbb{R}$ such that $\varphi(\theta + 2\pi) = \varphi(\theta) + 2\pi$ and $f^*(e^{i\theta}) = F(e^{i\varphi(\theta)})$. If E is the set of points of discontinuity of $e^{i\varphi(\theta)}$ in $\mathbb{T}$, then Proposition 2.1(ii) yields a sequence $\{\varphi_n\}$ of real-valued infinite-differentiable functions on $\mathbb{R}$ such that $\varphi_n(\theta + 2\pi) = \varphi_n(\theta) + 2\pi$, ${\varphi_n}'(\theta) > 0$, and $F(e^{i\varphi_n(\theta)}) \to f(e^{i\theta})$ pointwise on $\mathbb{T}\backslash E$. Let $\{r_n\}$, $\rho < r_n \leq 1$, be a sequence converging to 1, and let $f_n^*(e^{i\theta}) = F(r_n e^{i\varphi_n(\theta)})$. Since F is uniformly continuous on $\overline{\mathbb{D}}$, we conclude that $f_n^*(e^{i\theta}) \to f(e^{i\theta})$ pointwise on $\mathbb{T} \setminus E$. Note that since F is a convex univalent function, if $r_n < 1$ then f_n^* is an infinite-differentiable sense-preserving homeomorphism of $\mathbb{T}$ onto a convex curve in $\mathbb{G}$, and $(f_n^*)'(e^{i\theta})$ is nonvanishing. Define f^* and each f_n^* on $\mathbb{T}_\rho$ by the constants ζ_0 and ζ_n respectively, where ζ_0 is given by (1.3) and

$$\zeta_n = \frac{1}{2\pi} \int_0^{2\pi} f_n^*(e^{it})\, dt. \tag{2.3}$$

By the bounded convergence theorem, $\zeta_n \to \zeta_0$. Now let f and f_n be the solutions of the Dirichlet problems of f^* and f_n^* in $\mathbb{A}(\rho, 1)$ respectively. By Theorem A, we can represent f by (1.2) and write each f_n as

$$f_n(z) = h_n(z) - h_n(\rho^2/\overline{z}) + \zeta_n \tag{2.4}$$

where h_n is analytic in $\mathbb{A}(\rho^2, 1)$. Moreover, Theorem B implies that each $f_n : \mathbb{A}(\rho, 1) \to \mathbb{G} \setminus \{\zeta_n\}$ is a homeomorphism.

Under the above assumptions, we prove the requisite lemmas.

Lemma 2.1. $f_n \to f$ *locally uniformly in* $\mathbb{A}(\rho, 1)$.

Proof. Let Φ be a local homeomorphism of $\mathbb{D} \setminus \{\pm 1\}$ onto $\overline{\mathbb{A}(\rho, 1)}$ that maps $\mathbb{D}$ conformally onto $\mathbb{A}(\rho, 1)$, the upper semi-circle: $|\xi| = 1, \Im\xi > 0$ onto $\mathbb{T}$, and the lower semi-circle: $|\xi| = 1, \Im\xi < 0$ onto $\mathbb{T}_\rho$. Put $T_n^* = f_n^* \circ \Phi$, $T^* = f^* \circ \Phi$, $T_n = f_n \circ \Phi$, and $T = f \circ \Phi$. Note that T_n and T are the Dirichlet solutions of T_n^* and T^* in $\mathbb{D}$ respectively, and that $T_n^* \to T^*$ pointwise *a.e.* in $\mathbb{T}$ since $\zeta_n \to \zeta_0$. Hence, for $\eta = Re^{i\Theta}$, we can write

$$T_n(\eta) = \frac{1}{2\pi} \int_0^{2\pi} P(R, \tau - \Theta) T_n^*(e^{i\tau})\, d\tau$$

and

$$T(\eta) = \frac{1}{2\pi} \int_0^{2\pi} P(R, \tau - \Theta) T^*(e^{i\tau})\, d\tau.$$

Let $K \subset \mathbb{A}(\rho, 1)$ be a compact disc, and let $\widetilde{K}$ be a connected component of $\Phi^{-1}(K)$. Then $\widetilde{K}$ is also compact with a distance $\sigma > 0$ from $\mathbb{T}$. Then, for $\eta \in \widehat{K}$,

$$|T_n(\eta) - T(\eta)| \le \frac{1}{(1-\sigma)\pi} \int_0^{2\pi} |T_n^*(e^{i\tau}) - T(e^{i\tau})|\, d\tau,$$

and $T_n \to T$ uniformly on $\widetilde{K}$ by the bounded convergence theorem. It follows at once that $f_n \to f$ uniformly on K. □

Remark 2.1. The above proof uses only the pointwise convergence a.e. of T_n^* to T^* in $\mathbb{T}$ which follows at once from the the pointwise convergence of f_n^* to f^* in $\widetilde{E}(f^*)$. We conclude that if f^* and f_n^*, $n = 1, 2, \ldots$, are sense-preserving quasihomeomorphisms of $\mathbb{T}$ into $\partial\mathbb{G}$ such that $f_n^* \to f^*$ pointwise *a.e.* in $\mathbb{T}$, then $f_n \to f$ locally uniformly in $\mathbb{A}(\rho, 1)$ where f and each f_n are as defined above.

Lemma 2.2. (a) *$h_n \to h$ locally uniformly in $\mathbb{A}(\rho^2, 1)$.*
(b) *For $z \in \mathbb{A}(\rho, 1)$,*

$$f(z) = h(z) - h(\rho^2/\overline{z}) + \zeta_0 = \zeta_0 + \sum_{k\neq 0} c_k(f^*) \frac{r^{2k} - \rho^{2k}}{1 - \rho^{2k}} r^{-k} e^{ik\theta} \tag{2.5}$$

where $c_k(f^*)$, $k = \pm 1, \pm 2, \ldots$, is the k-th Fourier coefficient of f^*.

Proof. For $z = re^{i\theta}$, $\rho^2 < r < 1$, and $n = 1, 2, \ldots$, we have

$$h_n(z) = \sum_{k=-\infty}^{\infty} a_k(f_n^*) r^k e^{ik\theta}$$

which, with (2.4), yields

$$f_n(z) = \zeta_n + \sum_{k\neq 0} a_k(f_n^*) \left(r^k - \frac{\rho^{2k}}{r^k} \right) e^{ik\theta}. \tag{2.6}$$

The uniqueness of the Fourier series of $f_n(re^{i\theta})$ gives

$$a_k(f_n^*) \left(r^k - \frac{\rho^{2k}}{r^k} \right) = \frac{1}{2\pi} \int_0^{2\pi} f_n^*(re^{it}) e^{-ikt}\, dt, \qquad (k \neq 0).$$

Letting $r \to 1$, the bounded convergence theorem yields

$$a_k(f_n^*)(1 - \rho^{2k}) = \frac{1}{2\pi} \int_0^{2\pi} f_n^*(e^{it}) e^{-ikt}\, dt = c_k(f_n^*), \qquad (k \neq 0).$$

Hence

$$a_k(f_n^*) = \frac{c_k(f_n^*)}{1 - \rho^{2k}}, \qquad (k \neq 0). \tag{2.7}$$

Substituting this in (2.6) yields

$$f_n(z) = \zeta_n + \sum_{k\neq 0} c_k(f_n^*)\frac{r^{2k}-\rho^{2k}}{1-\rho^{2k}} r^{-k} e^{ik\theta}.$$

Proceeding likewise for h, we conclude that if

$$h(z) = \sum_{k=-\infty}^{\infty} a_k(f^*) r^k e^{ik\theta},$$

then

$$a_k(f^*) = \frac{c_k(f^*)}{1-\rho^{2k}}, \qquad (k \neq 0). \tag{2.8}$$

Now since $f_n^*(e^{it}) \to f^*(e^{it})$ pointwise in $\mathbb{T} \setminus E$, $c_k(f_n^*) \to c_k(f^*)$ uniformly relative to k as $n \to \infty$. It follows, by (2.8) and (2.7), that $a_k(f_n^*) \to a_k(f^*)$ uniformly relative to k as $n \to \infty$. This proves (a). Now since $h_n(z) \to h(z)$ and $f_n(z) \to f(z)$ uniformly in $\mathbb{A}(\rho, 1)$, and $\zeta_n \to \zeta$, we conclude (2.5) by taking the limits of both sides in (2.4). $\square$

Proof of Theorem 2.2. First, we show that $\zeta_0 \in \mathbb{G}$. Obviously, $\zeta_0 \in \overline{\mathbb{G}}$. Suppose that $\zeta_0 \in \partial\mathbb{G}$. Since $\overline{\mathbb{G}}$ is convex, there is a real θ_0 such that

$$\Re\left\{e^{i\theta_0}[f^*(e^{i\theta}) - \zeta_0]\right\} \geq 0, \qquad (0 \leq \theta \leq 2\pi).$$

By virtue of (ii), we conclude that this inequality must be strict in some open interval (α, β), where $0 \leq \alpha < \beta \leq 2\pi$. This implies that

$$\frac{1}{2\pi}\int_0^{2\pi} \Re\left\{e^{i\theta_0}[f^*(e^{i\theta}) - \zeta_0]\right\} d\theta > 0,$$

and consequently

$$\frac{1}{2\pi}\int_0^{2\pi} e^{i\theta_0}[f^*(e^{i\theta}) - \zeta_0]\, d\theta \neq 0.$$

This yields at once

$$\zeta_0 \neq \frac{1}{2\pi}\int_0^{2\pi} f^*(e^{i\theta})\, d\theta$$

which gives a contradiction. Hence, $\zeta_0 \in \mathbb{G}$.

In view of Lemma 2.2(b), it remains to show that $f : \mathbb{A}(\rho, 1) \to \mathbb{G} \setminus \{\zeta_0\}$ is a homeomorphism.

We show that f is univalent. Let f_n^* and f_n, $n = 1, 2, \ldots$, be the functions defined in the first paragraph succeeding the statement of the theorem but with each $r_n = 1$. Using (2.4), the Jacobian of f_n can be written as

$$J_n(z) = [|zh_n'(z)|^2 - |\rho^2 h_n'(\rho^2/\overline{z})|^2]/|z|^2 > 0, \qquad z \in \mathbb{A}(\rho, 1).$$

Since f_n is univalent and sense-preserving, Lewy's theorem [**13**] implies $J_n(z) > 0$; so $h_n'(z) \neq 0$ for $z \in \mathbb{A}(\rho, 1)$. But, by Lemma 2.2(a), $h_n' \to h'$

locally uniformly in $\mathbb{A}(\rho,1)$. This implies, by Hurwitz's theorem, that either h' is nonvanishing or is identically zero in $\mathbb{A}(\rho,1)$. If the latter case holds, then f is constant. This yields, by [**21**, Theorem IV.3] and (ii), that $\partial\mathbb{G}$ is a singleton which contradicts (ii). Hence $h'(z)\neq 0$ for $z\in\mathbb{A}(\rho,1)$. Now the Jacobian of f is given by

$$J(z)=[|zh'(z)|^2-|\rho^2h'(\rho^2/\overline{z})|^2]/|z|^2, \qquad z\in\mathbb{A}(\rho,1).$$

For $z\in\mathbb{A}(\rho,1)$, $J(z)\geq 0$ since $J_n(z)\to J(z)$. If $J(z)=0$ for some z, then J is identically zero; this follows by applying the maximum principle to the dilatation of f given by

$$\omega(z)=\frac{\overline{(\rho^2/\overline{z})h'(\rho^2/\overline{z})}}{zh'(z)}, \qquad (z\in\mathbb{A}(\rho,1)). \tag{2.9}$$

This implies, by [**14**, Lemma 2], that f maps $\mathbb{A}(\rho,1)$ into a straight-line L. It follows that the unrestricted limits of f lie in L. By [**21**, Theorem IV.3] and (ii), $f^*(\widetilde{E}(f^*))\subset L$ which yields $\partial\mathbb{G}\subset L$. This gives a contradiction. Hence $J(z)>0$ for $z\in\mathbb{A}(\rho,1)$, and Lewy's theorem [**13**] yields f locally univalent. Now the univalence of f_n, together with Lemma 2.1, yields f univalent in $\mathbb{A}(\rho,1)$.

Next, we show that $f:\mathbb{A}(\rho,1)\to\mathbb{G}\setminus\{\zeta_0\}$ is onto. Let $\xi\in\mathbb{T}$. The cluster set $C(f,\xi)$ of f at ξ is the singleton $f(\xi)\in\partial\mathbb{G}$ if f has an unrestricted limit at ξ, or the straight-line segment ℓ joining the points $\lim_{\theta\uparrow\theta_0}f(e^{i\theta})$ and $\lim_{\theta\downarrow\theta_0}f(e^{i\theta})$, where $\xi=e^{i\theta_0}$, which belong to $\partial\mathbb{G}$ by (ii). Suppose that the latter case holds. If $\ell\not\subset\partial\mathbb{G}$, then ℓ is a crosscut of $\mathbb{G}$ which separates $\mathbb{G}$ into two Jordan domains of which one contains $f(\mathbb{A}(\rho,1))$. If L is the straight-line containing ℓ, then the cluster set $C(f,\mathbb{T})$ of f on $\mathbb{T}$ lies completely in the closed half-plane bounded by L and containing $f(\mathbb{A}(\rho,1))$. Consequently, $\overline{co}f^*(\widetilde{E}(f^*))$ is a proper subset of $\overline{\mathbb{G}}$ which contradicts (ii). Hence, $\ell\subset\partial\mathbb{G}$, and $C(f,\xi)\subset\mathbb{T}$ for every $\xi\in\mathbb{T}$. Now if $f:\mathbb{A}(\rho,1)\to\mathbb{G}\setminus\{\zeta_0\}$ is not onto, then there is $\xi\in\mathbb{T}$ such that $C(f,\xi)\cap\mathbb{G}\neq\emptyset$ which yields a contradiction. Therefore, $f:\mathbb{A}(\rho,1)\to\mathbb{G}\setminus\{\zeta_0\}$ is a homeomorphism. This completes the proof. $\square$

Remark 2.2. The last paragraph of the above proof is indeed a proof for the following result: Let f be the Dirichlet solution in $\mathbb{A}(\rho,1)$ of a boundary function f^* defined on $\mathbb{T}$ by a sense-preserving quasihomeomorphism into $\partial\mathbb{G}$ with $\overline{co}f^*(\widetilde{E}(f^*))=\overline{\mathbb{G}}$, and on $\mathbb{T}_\rho$ by a constant $\zeta\in\mathbb{G}$. If f is univalent, then $f:\mathbb{A}(\rho,1)\to G\setminus\{\zeta\}$ is a homeomorphism.

Theorems 2.1 and 2.2 provide an interesting relationship between sense-preserving quasihomeomorphisms of $\mathbb{T}$ into $\partial\mathbb{G}$ and univalent harmonic mapping of $\mathbb{A}(\rho,1)$ onto once punctured $\mathbb{G}$. View two sense-preserving quasi-homeomorphisms f^* of $\mathbb{T}$ into $\partial\mathbb{G}$ *equivalent* if they coincide almost everywhere. Let f^* and k^* be sense-preserving quasihomeomorphisms of $\mathbb{T}$ into

$\partial\mathbb{G}$. Using Proposition 2.1(i), it is easily seen that f^* and k^* are equivalent if and only if $\widetilde{E}(f^*) = \widetilde{E}(k^*)$ and f^* and k^* are identical on $\widetilde{E}(f^*)$. Denote by $\mathcal{Q}(\mathbb{G})$ the class of all (equivalence classes of) sense-preserving quasihomeomorphisms f^* of $\mathbb{T}$ into $\partial\mathbb{G}$ satisfying (ii) of Theorem 2.2. It is immediate that if $f^* \in \mathcal{Q}(\mathbb{G})$ and k^* is equivalent to f^*, then $k^* \in \mathcal{Q}(\mathbb{G})$.

Definition 2.3. Denote by $\mathcal{H}_0(\rho, \mathbb{G})$ be the class of all Dirichlet solutions f satisfying (i), (ii) and (iii) of Theorem 2.2.

The classes $\mathcal{Q}(\mathbb{G})$ and $\mathcal{H}_0(\rho, \mathbb{G})$ are related as follows.

Theorem 2.3. *Define $T : \mathcal{Q}(\mathbb{G}) \to \mathcal{H}_0(\rho, \mathbb{G})$ by $T(f^*) = f$, where f is the Dirichlet solution in $\mathbb{A}(\rho, 1)$ of the boundary function which is f^* on $\mathbb{T}$ and the average of f^* on $\mathbb{T}_\rho$. Then T is bijective. Furthermore, for a sequence $\{f_n^*\}$ in $\mathcal{Q}(\mathbb{G})$, the following statements are equivalent:*
(a) $f_n^* \to f^*$ *a.e..*
(b) $f_n^* \to f^*$ *in L^1.*
(c) $f_n \to f$ *locally uniformly in $\mathbb{A}(\rho, 1)$.*

Proof. Suppose that for $f_1^*, f_2^* \in \mathcal{Q}(\mathbb{G})$, $T(f_1^*) = f_1$, $T(f_2^*) = f_2$, and $f_1 = f_2$. Then, by Theorem 2.1 (i), $f_1^* = f_2^*$ everywhere except possibly on a countable set. This makes T injective. Also, by Theorem 2.1, T is surjective. Hence T is bijective.

The implication (a) $\Rightarrow$ (b) follows by the bounded convergence theorem. Conversely, by Proposition 2.1 (i) and Helly's selection theorem, there is a subsequence $\{n_j\}$ of positive integers such that the sequence $\{f_{n_j}^*\}$ converges pointwise to a bounded function k^*. So, $|f_{n_j}^* - f^*| \to |k^* - f^*|$ pointwise. Then, by (b) and the bounded convergence theorem, $\int_0^{2\pi} |k^* - f^*| = 0$. This gives (a), and we conclude (a) $\Leftrightarrow$ (b). On the other hand, by Remark 2.1, (a) $\Rightarrow$ (c). It remains to show (c) $\Rightarrow$ (a). Using Theorem 2.1 (i), there exist sense-preserving quasihomeomorphisms f^* and f_n^* of $\mathbb{T}$ into $\mathbb{G}$ that coincide with the boundary functions of f and f_n everywhere except possibly on countable sets, respectively. By Helly's selection theorem, every subsequence of $\{f_n^*\}$ contains a subsequence $\{f_{n_j}^*\}$ that converges pointwise to some bounded function k^*. Denote by k the Dirichlet solution in $\mathbb{A}(\rho, 1)$ of the boundary function defined on $\mathbb{T}$ by k^* and on $\mathbb{T}_\rho$ by the average of k^* on $\mathbb{T}$. Then, by Remark 2.1, $f_{n_j} \to k$ locally uniformly on $\mathbb{A}(\rho, 1)$. Hence $k = f$. This implies $\widetilde{E}(f^*) = \widetilde{E}(k^*)$ and $f^*(e^{i\theta}) = k^*(e^{i\theta})$ for $e^{i\theta} \in \widetilde{E}(f^*)$. Hence $f_{n_j}^* \to f^*$ pointwise *a.e.*. It follows that $f_n^* \to f^*$ pointwise *a.e.* and (c) $\Rightarrow$ (a). This ends the proof. □

3. Geometry of Analytic Parts of Univalent Harmonic Mappings onto Punctured Convex Domains

Let h be the analytic part of $f \in \mathcal{H}_u(\rho, \mathbb{G})$. The purpose of this section is two-fold: First, to show that h has a nonvanishing derivative on $\mathbb{T}_\rho$, and that it maps $\mathbb{T}_\rho$ homeomorphically onto a sense-preserving convex Jordan curve whose diameter admits a universal upper bound, and second, to prove that h is a composition of a univalent close-to-convex function and a homeomorphism of $\mathbb{A}(\rho, 1) \cup \mathbb{T}$ onto a ring subdomain of $\overline{D}$ that maps $\mathbb{T}$ homeomorphically onto itself.

Our first result in this section relates univalent harmonic maps in $\mathcal{H}_u(\rho, \mathbb{G})$ to their average associates in $\mathcal{H}_0(\rho, \mathbb{G})$.

Proposition 3.1. *Suppose that the following are true:*

(i) *f^* is a sense-preserving quasihomeomorphism of $\mathbb{T}$ into $\partial\mathbb{G}$ such that $\overline{co}(f^*(\widehat{E}(f^*))) = \overline{\mathbb{G}}$.*

(ii) *f is the Dirichlet solution in $\mathbb{A}(\rho, 1)$ of the function defined by f^* on $\mathbb{T}$ and a constant $\zeta \in \mathbb{G}$ on $\mathbb{T}_\rho$.*

(iii) *f_0 is the Dirichlet solution of the function defined by f^* on $\mathbb{T}$ and the average ζ_0 of f^* on $\mathbb{T}$.*

Then there is an analytic function h in $\mathbb{A}(\rho^2, 1)$ such that

$$f(z) = h(z) - h(\rho^2/\overline{z}) + \zeta + 2c_\zeta \log(|z|/\rho) \tag{3.1}$$

$$= f_0(z) + 2c_\zeta \log|z|, \tag{3.2}$$

where c_ζ is given by (5.4)*.*

Proof. By Theorem A, there is a constant c and an analytic function h of $\mathbb{A}(\rho^2, 1)$ such that

$$f(z) = h(z) - h(\rho^2/\overline{z}) + \zeta + 2c \log(|z|/\rho), \qquad (z \in \mathbb{A}(\rho, 1)).$$

By Theorem 2.2, there is an analytic function h_0 of $\mathbb{A}(\rho^2, 1)$ such that

$$f_0(z) = h_0(z) - h_0(\rho^2/\overline{z}) + \zeta_0, \quad (z \in \mathbb{A}(\rho, 1)).$$

Then

$$\begin{aligned}(f - f_0)(z) &= (h - h_0)(z) - (h - h_0)(\rho^2/\overline{z}) \\ &\quad + \zeta - \zeta_0 + 2c \log(|z|/\rho)\end{aligned} \tag{3.3}$$

is a bounded harmonic mapping in $\mathbb{A}(\rho, 1)$. We conclude, by Schwarz's theorem, that the unrestricted limit function of $(f - f_0)$ exists everywhere on $\mathbb{T}$ except possibly on a countable subset E. Furthermore, it is identically zero on $\mathbb{T} \setminus E$ by the definition of f and f_0. Since $(f - f_0)(\mathbb{T}_\rho) = \zeta - \zeta_0$, the maximum principle yields

$$(f - f_0)(z) = \zeta - \zeta_0 + 2c_\zeta \log(|z|/\rho) \tag{3.4}$$

where c_ζ is as given in (5.4). By comparing (3.3) and (3.4), it follows that $c = c_\zeta$ and $h - h_0$ is constant. This yields (3.1) and (3.2), and the proof is complete. □

Note that Proposition 3.1 does not require f to be in $\mathcal{H}_u(\rho, \mathbb{G})$. If this however is the case, then we obtain Corollary 3.1.

Corollary 3.1. *Let f_0 be the average associate of $f \in \mathcal{H}_u(\rho, \mathbb{G})$ with $f(\mathbb{T}_\rho) = \zeta$ and $f_0(\mathbb{T}_\rho) = \zeta_0$. Then there is an analytic function h in $\mathbb{A}(\rho^2, 1)$ such that* (3.1) *and* (3.2) *hold simultaneously.*

Suppose now that $f \in \mathcal{H}_u(\rho, \mathbb{G})$ has form (1.2). According to Proposition 3.1, f and its average associate f_0 have the same analytic and co-analytic part h. Since our interest in this section is exclusively in h, we restrict ourselves to functions $f \in \mathcal{H}_0(\rho, \mathbb{G})$ of form (2.2).

We shall need the notion of the *module* $M(R)$ of a ring domain R [**18**]. It is known that R is conformally equivalent to a unique annulus $A(r, 1)$, $0 < r < 1$. In this case $M(R)$ is defined by $\log(1/r)$ if $r \neq 0$ and by ∞ if $r = 0$. It is immediate that M is a conformal invariant, and that if $R \subset R'$ where R' is also a ring domain, then $M(R) \leq M(R')$ with equality if and only if $R = R'$. The *Grötzsch's ring domain*, $B(t)$, $0 < t < 1$, of R is the doubly-connected open subset of $\mathbb{D}$ whose boundary components are $\mathbb{T}$ and the segment $\{x : 0 \leq x \leq t\}$. Observe that $B(t)$ is unique. The module of $B(t)$ is usually denoted by $\mu(t)$. It follows that if $B(s)$ is the Grötzsch's ring domain of $\mathbb{A}(\rho, 1)$, then $\mu(s) = \log(1/\rho)$. It is known that μ is a strictly decreasing function of $[0, 1)$.

Let S be a subset of $\mathbb{C}$. The *diameter* of S is the least upper bound of the distances between any two points of S. If ℓ_α, $\alpha \in \mathbb{R}$, is a straight-line in the direction of $e^{i\alpha}$ perpendicular to two support lines π and π' of S, then we call the distance between $\ell_\alpha \cap \pi$ and $\ell_\alpha \cap \pi'$ the *width of S in the direction of $e^{i\alpha}$*. It is known that if S is compact, then the diameter of S is equal to its maximum width [**6**, p. 77]. In what follows, we denote by d the diameter of $\mathbb{G}$ and by d_α its diameter in the direction of $e^{i\alpha}$. We call a Jordan curve *convex* if it is the boundary of a bounded convex domain.

Using these notions, our result states as follows.

Theorem 3.1. *Suppose $f \in \mathcal{H}_0(\rho, \mathbb{G})$ has form* (2.2). *Then*

(a) *h' is nonvanishing on $\mathbb{T}_\rho$ and h maps $\mathbb{T}_\rho$ homeomorphically onto a convex curve whose diameter is bounded above by*

$$D = (4d/\pi)\tanh^{-1}\left(\mu^{-1}(\log(1/\rho))\right).$$

(b) *If $h(z) = \sum_{-\infty}^{\infty} a_n z^n$, $z \in \mathbb{A}(\rho^2, 1)$, then*

$$\sum_{n=1}^{\infty} n|a_{-n}|^2\rho^{-2n} < \sum_{n=1}^{\infty} n|a_n|^2\rho^{2n} \leq D^2/4 + \sum_{n=1}^{\infty} n|a_{-n}|^2\rho^{-2n}.$$

The proof of the theorem needs two lemmas. The first is due to Bshouty and Hengartner [**1**, Theorem 2.5]. To state this result, we call a ring domain Ω a *slit domain convex in the direction of the real axis* if it is obtained by removing a horizontal slit from a domain convex in the direction of the real axis.

Lemma 3.1. *Suppose* $f \in \mathcal{H}_u(\rho, \mathbb{G})$ *has form* (2.2), *and let*

$$\Phi_\alpha(z) = e^{i\alpha} h(z) + e^{-i\alpha} \overline{h(\rho^2/\overline{z})}, \qquad (z \in \mathbb{A}(\rho^2, 1)). \tag{3.5}$$

Then Φ_α *is univalent in* $\mathbb{A}(\rho, 1)$ *and it maps* $\mathbb{A}(\rho, 1)$ *onto a slit domain convex in the direction of the real axis.*

Our second lemma is intuitive and geometric in nature, and it needs some basic notions. A *closed curve* is a continuous image of $\mathbb{T}$; we use the same notation for the curve and its defining function. Let γ be a closed curve, and let ℓ be a straight line. A point $w \in \gamma \cap \ell$ is called a *meeting point* of γ and ℓ of *multiplicity* n if $|\gamma^{-1}(w)| = n$. For a meeting point w of γ and ℓ, we call w a *crossing point* of γ *and* ℓ if there is an open subarc I of $\mathbb{T}$ such that $\gamma^{-1}(w) \cap \ell$ is a singleton and ℓ separates $\gamma(I) \setminus \{w\}$.

Lemma 3.2. *If every straight-line through the origin meets a closed curve* γ *exactly twice, counting multiplicity, and at crossing points only, then* γ *is a Jordan curve whose inner domain is starlike with respect to the origin.*

Proof. We show first that γ is a Jordan curve. Suppose that there are points $z_1, z_2 \in \mathbb{T}$ such that $\gamma(z_1) = \gamma(z_2) = w$. If $w = 0$, then any straight-line passing through the origin and some other point of γ meets γ in at least three points, counting multiplicity. If $w \neq 0$, for convenience $w > 0$, then γ does not meet the negative real axis. This implies, by the compactness of γ, that γ lies within a minimal sector vertexed at the origin whose sides meet γ without crossing. In either case, we have a contradiction and the claim holds.

Next, we show that the winding number $n(\gamma, 0)$ is ± 1. We consider two cases.

(i) $0 \in \gamma$: In this case γ meets only one of the positive and negative real axes.

(ii) $0 \notin \gamma$: In this case γ meets $\mathbb{R}$ in two distinct points a and b, say $a < b$. Here also we consider two cases.

(a) $0 < a < b$ *or* $a < b < 0$: In the first case γ does not meet the negative real axis, and in the second it does not meet the positive real axis.

(b) $a < 0 < b$.

In (i) and (ii.a), the above compactness argument yield a contradiction. Thus only (ii.b) holds. It is immediate then that $n(\gamma, x) = 0$ for all $x \in (-\infty, a) \cup (b, \infty)$. Because $\mathbb{R} \cap \gamma = \{a, b\}$ for all $a < x < b$, either $n(\gamma, x) \neq 0$ or $n(\gamma, x) = 0$. In the latter case $\gamma \setminus \{a, b\}$ lies completely in one of the upper- or lower-half planes and $\mathbb{R}$ fails to cross γ at a or b. Hence $|n(\gamma, 0)| = 1$.

We further conclude that any straight-line passing through the origin meets the inner domain of γ in an open segment containing the origin. Therefore, the inner domain of γ is starlike with respect to the origin. $\square$

Proof of Theorem 3.1. (a) Fix $\alpha \in \mathbb{R}$ and let Φ_α be given as in (3.5). Then we can write

$$\Phi_\alpha(\rho e^{i\theta}) = 2\Re\{e^{i\alpha} h(\rho e^{i\theta})\}, \qquad (0 \le \theta \le 2\pi).$$

Let $m_\alpha = \min_\theta \Phi_\alpha(\rho e^{i\theta})$, $M_\alpha = \max_\theta \Phi_\alpha(\rho e^{i\theta})$, and Γ be the curve defined by $\Gamma(\theta) = h(\rho e^{i\theta})$, $0 \le \theta \le 2\pi$. Observe that $M_\alpha - m_\alpha$ is the width of Γ in the direction of $e^{-i\alpha}$, and that Φ_α maps $\mathbb{T}_\rho$ onto the real interval $I_\alpha = [m_\alpha, M_\alpha]$ which is the inner boundary of the ring domain $\Phi_\alpha(\mathbb{A}(\rho, 1))$. Since Φ_α is univalent by Lemma 3.1, Φ'_α admits two simple zeros $\rho e^{i\alpha_1}$ and $\rho e^{i\alpha_2}$, where $\alpha_1 < \alpha_2 < \alpha_1 + 2\pi$, such that $\Phi_\alpha(\rho e^{i\alpha_1}) = m_\alpha$ and $\Phi_\alpha(\rho e^{i\alpha_2}) = M_\alpha$. Letting $\Psi(\theta) = \Phi_\alpha(\rho e^{i\theta})$, $0 \le \theta \le 2\pi$, we obtain

$$\Psi'(\theta) = i\rho e^{i\theta}\Phi'_\alpha(\rho e^{i\theta}) = -2\Im\left\{e^{i\alpha}[\rho e^{i\theta} h'(\rho e^{i\theta})]\right\}.$$

The first equality yields $\Psi'(\alpha_1) = \Psi'(\alpha_2) = 0$, $\Psi'(\theta) > 0$ for $\alpha_1 < \theta < \alpha_2$, and $\Psi'(\theta) < 0$ for $\alpha_2 < \theta < \alpha_1 + 2\pi$. Denote by γ the curve defined by $\gamma(\theta) = \rho e^{i\theta} h'(\rho e^{i\theta})$, $0 \le \theta \le 2\pi$. The second equality implies that the real axis meets the curve $e^{i\alpha}\gamma$ exactly twice, counting multiplicity, and only at crossing points; namely $\rho e^{i\alpha_1} h'(\rho e^{i\alpha_1})$ and $\rho e^{i\alpha_2} h'(\rho e^{i\alpha_2})$. This means that the line in the direction of $e^{-i\alpha}$ meets γ exactly twice and only at crossing points. Since α is arbitrary, this property also holds for all straight-lines passing through origin. Using Lemma 3.2, we conclude that γ is a Jordan curve that bounds a starlike domain with respect to the origin. Thus h' is nonvanishing on $\mathbb{T}_\rho$ and

$$\frac{d}{d\theta} \arg \rho e^{i\theta} h'(\rho e^{i\theta}) = \Re\left\{1 + \rho e^{i\theta}\frac{h''(\rho e^{i\theta})}{h'(\rho e^{i\theta})}\right\}$$

is always either nonpositive or nonnegative. Hence Γ is a convex curve as claimed.

Now we show that the diameter of Γ is bounded by D. With a fixed α again, we can write

$$\Phi_\alpha(z) = e^{i\alpha}(f(z) - \zeta_0) + 2\Re\{e^{i\alpha}\overline{h(\rho^2/\overline{z})}\}.$$

Geometrically, this means that for every $z \in \mathbb{A}(\rho, 1)$ the value $\Phi_\alpha(z)$ can be obtained from the point $e^{i\alpha}(f(z) - \zeta_0)$ by a horizontal shift by $2\Re\{e^{i\alpha}\overline{h(\rho^2/\overline{z})}\}$. Recall d_α, d, and I_α. We conclude that the ring domain $\Phi_\alpha(\mathbb{A}(\rho, 1))$ is properly contained in a horizontal strip of width d_β, $\beta = \pi/2 - \alpha$, and with a slit along I_α. Let S_α and S be the horizontal strips symmetric about $\mathbb{R}$ and of widths $2d_\beta$ and $2d$ respectively. Obviously, $\Phi_\alpha(\mathbb{A}(\rho, 1))$ is a proper subset of $S_\alpha \setminus I_\alpha$, $S_\alpha \setminus I_\alpha \subset S \setminus I_\alpha$, and

$S \setminus I_\alpha$ is conformally equivalent to $S \setminus [(m_\alpha - M_\alpha)/2, (M_\alpha - m_\alpha)/2]$. Observe that the length of the boundary slit of the Grötzsch's domain of $S \setminus [(m_\alpha - M_\alpha)/2, (M_\alpha - m_\alpha)/2]$ is $\tanh[\pi(M_\alpha - m_\alpha)/(4d)]$. Then

$$\begin{aligned} \log(1/\rho) &= M(\mathbb{A}(\rho,1)) < M(S_\alpha \setminus I_\alpha) \\ &\leq M(S \setminus [(m_\alpha - M_\alpha)/2, (M_\alpha - m_\alpha)/2]) \\ &= \mu(\tanh[\pi(M_\alpha - m_\alpha)/(4d)]). \end{aligned}$$

Since μ is a decreasing function, we obtain

$$\tanh[\pi(M_\alpha - m_\alpha)/(4d)] < \mu^{-1}(\log(1/\rho)),$$

or

$$M_\alpha - m_\alpha < \frac{4d}{\pi} \tanh^{-1}\left(\mu^{-1}(\log(1/\rho))\right) = D.$$

Note that α may be chosen so that $d = M_\alpha - m_\alpha$. This concludes (a).

(b) Let Ω be the closed region bounded by the curve Γ defined in the proof of (a). We show first that the area $A(\Omega)$ of Ω is at most $\pi D^2/4$. By [**6**, Theorem 54], Ω is contained in a convex region Ω' of constant width D in every direction. Then Cauchy's theorem [**6**, p. 127] implies that the perimeter of Ω' is πD. But the area of Ω' is at most $\pi D^2/4$ by the isoperimetric inequality [**6**, p. 108]. This proves our claim.

On the other hand,

$$\begin{aligned} A(\Omega) &= \frac{1}{2i}\int_{|z|=\rho} \overline{h(z)} h'(z)\, dz \\ &= \frac{1}{2}\int_0^{2\pi} \left\{ \sum_{n=-\infty}^{\infty} \overline{a}_n \rho^n e^{-in\theta} \right\} \left\{ \sum_{n=-\infty}^{\infty} n a_n \rho^n e^{in\theta} \right\} d\theta \\ &= \pi \left\{ \sum_{n=1}^{\infty} n|a_n|^2 \rho^{2n} - \sum_{n=1}^{\infty} n|a_{-n}|^2 \rho^{-2n} \right\}. \end{aligned}$$

Therefore,

$$\sum_{n=1}^{\infty} n|a_n|^2 \rho^{2n} - \sum_{n=1}^{\infty} n|a_{-n}|^2 \rho^{-2n} < D^2/4$$

and (b) follows. □

Next, we embark on proving that the analytic part of every harmonic mapping in $\mathcal{H}_u(\rho, \mathbb{G})$ is a univalent close-to-convex function of $\mathbb{D}$ precomposed with a homeomorphism of $\mathbb{A}(\rho,1) \cup \mathbb{T}$ onto a ring subdomain of $\overline{D}$ that maps $\mathbb{T}$ homeomorphically onto itself. As above, it suffices to consider harmonic mappings $f \in \mathcal{H}_0(\rho, \mathbb{G})$.

Theorem 3.2. *Suppose $f \in \mathcal{H}_0(\rho, \mathbb{G})$ has form (2.2). Then there is a univalent close-to-convex function H of $\mathbb{D}$ and a homeomorphism ϕ of $\mathbb{A}(\rho,1) \cup \mathbb{T}$ into $\overline{\mathbb{D}}$ with $\phi(\mathbb{T}) = \mathbb{T}$ such that $h = H \circ \phi$.*

Observe that if $f \in \mathcal{H}_0(\rho, \mathbb{G})$ is given by (2.2), then the dilatation of f is given by (2.9).

The proof of the theorem needs two lemmas. The first states as follows.

Lemma 3.3. *Fix p, $p = 2, 3, \ldots$. Suppose $f \in \mathcal{H}_0(\rho, \mathbb{G})$ has form* (2.2) *and an unrestricted limit function that satisfies the following properties:*

(i) *f is a sense-preserving local homeomorphism of $\mathbb{T}$ onto $\partial\mathbb{G}$.*
(ii) *$f^{(p)}$ exists and is absolutely continuous on $\mathbb{T}$.*
(iii) *f' is nonvanishing on $\mathbb{T}$.*

Then

(a) *h extends to $\overline{\mathbb{A}(\rho^2, 1)}$ such that $h(e^{i\theta})$ and $h(\rho^2 e^{i\theta})$ are continuously $(p-1)$-differentiable with*

$$\lim_{z \to re^{i\theta}} h^{(k)}(z) = h^{(k)}(re^{i\theta}), \qquad (z \in \mathbb{A}(\rho^2, 1)), \tag{3.6}$$

where r is either 1 or ρ^2.

(b) *$h'(e^{i\theta}) \neq 0$ and $h'(\rho^2 e^{i\theta}) \neq 0$ for all θ.*

(c) *ω extends continuously to $\overline{\mathbb{A}(\rho, 1)}$ such that $\omega(e^{i\theta}) \neq -1$ and $|\omega(z)| \leq 1$ for $z \in \overline{\mathbb{A}(\rho, 1)}$.*

Proof. (a) If

$$h(z) = \sum_{n=-\infty}^{\infty} a_n z^n, \qquad (z \in \mathbb{A}(\rho^2, 1)),$$

then for $z = re^{i\theta}$, $\rho < r < 1$,

$$f(z) = \zeta_0 + \sum_{n \neq 0} a_n [r^n - (\rho^2/r)^n] e^{in\theta}. \tag{3.7}$$

Since $f'(e^{i\theta})$ exists for all θ, [**7**, Theorem 55] gives

$$f(e^{i\theta}) = \zeta_0 + \sum_{n \neq 0} c_n e^{in\theta} \tag{3.8}$$

where, by the bounded convergence theorem and (3.7),

$$\begin{aligned} c_n &= \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) e^{-in\theta}\, d\theta = \lim_{r \to 1} \frac{1}{2\pi} \int_0^{2\pi} f(re^{i\theta}) e^{-in\theta}\, d\theta \\ &= \lim_{r \to 1} a_n [r^n - (\rho^2/r)^n] = a_n (1 - \rho^{2n}). \end{aligned}$$

Using this in (3.8), we obtain

$$f(e^{i\theta}) = \zeta_0 + \sum_{n \neq 0} a_n (1 - \rho^{2n}) e^{in\theta}.$$

Since $f^{(p)}$ is absolutely continuous, [**7**, Theorem 40] yields

$$a_n (1 - \rho^{2n}) = o(|n|^{-(p+1)}).$$

This gives for $k = 1, 2, \dots, p-1$,

$$n(n-1)\cdots(n-k+1)a_n = o(|n|^{-p+k-1}) = o(|n|^{-2}). \tag{3.9}$$

Define, for $r = 1$ or ρ^2,

$$h(re^{i\theta}) = \sum_{n=-\infty}^{\infty} a_n r^n e^{in\theta}.$$

Observe that term by term differentiation of the latter series yields, by (3.9), a uniformly convergent series. Now term by term integration of the resulting series yields $h(re^{i\theta})$ continuously differentiable. Repeating the same procedure with $h'(re^{i\theta})$ yields $h(re^{i\theta})$ continuously 2-differentiable. Observe, again because of (3.9), that the same procedure can be repeated $p-1$ times proving $h(re^{i\theta})$ continuously $(p-1)$-differentiable. Using (3.9) once again, together with the uniform convergence of k-th, $k = 1, 2, \dots, p-1$, derivatives of $h(re^{i\theta})$ and the above Laurent's series of $h(z)$, yields (3.6).

(b) The Jacobian of f is given by

$$J(z) = [|zh'(z)|^2 - |\rho^2 h'(\rho^2/\overline{z})|^2]/|z|^2, \qquad (z \in \mathbb{A}(\rho, 1)).$$

Since f is univalent in $\mathbb{A}(\rho, 1)$, Lewy's theorem [**13**] yields $J(z) > 0$ for $z \in \mathbb{A}(\rho, 1)$; that is,

$$\rho^2 |h'(\rho^2/\overline{z})| < |zh'(z)|, \qquad (z \in \mathbb{A}(\rho, 1)),$$

which implies $h'(z) \neq 0$ in $\mathbb{A}(\rho, 1)$. Using (a), we conclude

$$\rho^2 |h'(\rho^2 e^{i\theta})| \leq |h'(e^{i\theta})|, \qquad (0 \leq \theta \leq 2\pi).$$

We infer that if $h'(e^{i\theta}) = 0$ for some θ, then $\rho^2 e^{2i\theta} h'(\rho^2 e^{i\theta}) = 0$. Note that

$$f'(e^{i\theta}) = h'(e^{i\theta}) + \overline{\rho^2 e^{2i\theta} h'(\rho^2 e^{i\theta})}. \tag{3.10}$$

Thus $f'(e^{i\theta}) = 0$ which gives a contradiction. Hence $h'(e^{i\theta}) \neq 0$ for all θ. On the other hand, Theorem 3.1 yields $h'(\rho e^{i\theta}) \neq 0$ for all θ. This concludes (b).

(c) It is immediate from (a), (b), and (2.9) that ω extends continuously to $\overline{\mathbb{A}(\rho, 1)}$. If $\omega(e^{i\theta}) = -1$ for some θ, then (2.9) and (3.10) give $f'(e^{i\theta}) = 0$ which leads to a contradiction. Now since f is univalent, Lewy's theorem [**13**] implies $|\omega(z)| < 1$ for $z \in \mathbb{A}(\rho, 1)$. Using (2.9) once more, with (b), gives $|\omega(z)| \leq 1$ for $z \in \overline{\mathbb{A}(\rho, 1)}$. This completes the proof. □

Our second lemma is a weaker form of Theorem 3.2.

Lemma 3.4. *Suppose $f \in \mathcal{H}_0(\rho, \mathbb{G})$ has form (2.2), $\partial\mathbb{G}$ an analytic curve, and $f(e^{i\theta})$ an infinite-differentiable function with a nonvanishing derivative. Let Γ be the convex curve defined by $\Gamma(\theta) = h(\rho e^{i\theta})$, $0 \leq \theta \leq 2\pi$ (see Theorem* 3.1(a)*). Then h is a sense-preserving homeomorphism of $\mathbb{T}_\rho$ onto*

Γ, and $h = H \circ \phi$ where H is a univalent close-to-convex function of $\mathbb{D}$ and ϕ is a homeomorphism of $\mathbb{A}(\rho,1) \cup \mathbb{T}$ into $\overline{\mathbb{D}}$ with $\phi(\mathbb{T}) = \mathbb{T}$.

Proof. From Lemma 3.3, we infer that $h(e^{i\theta})$ is infinite-differentiable with a nonvanishing derivative. Using (3.10) and Lemma 3.3(c), we can write

$$e^{i\theta} h'(e^{i\theta}) = \frac{e^{i\theta} f'(e^{i\theta})}{1+\omega(e^{i\theta})}. \tag{3.11}$$

Differentiation of both sides yields

$$\Re\left\{1 + e^{i\theta}\frac{h''(e^{i\theta})}{h'(e^{i\theta})}\right\} = \Re\left\{1 + e^{i\theta}\frac{f''(e^{i\theta})}{f'(e^{i\theta})}\right\} - \Re\left\{\frac{e^{i\theta}\omega'(e^{i\theta})}{1+\omega(e^{i\theta})}\right\}. \tag{3.12}$$

Denote by F a conformal map of $\mathbb{D}$ onto $\mathbb{G}$. Since $\partial\mathbb{G}$ is an analytic curve, F extends to a conformal map of $\overline{\mathbb{D}}$ onto $\overline{\mathbb{G}}$. Using the bounded convergence theorem and [**18**, p. 65], we obtain

$$\Re\left\{1 + e^{i\Theta}\frac{F''(e^{i\Theta})}{F'(e^{i\Theta})}\right\} \geq 0, \qquad (\Theta \in (-\infty,\infty)). \tag{3.13}$$

Observe that we can write $f(e^{i\theta}) = F(e^{i\Theta(\theta)})$ where $\Theta(\theta)$ is an increasing differentiable function of $(-\infty,\infty)$ such that $\Theta(\theta+2\pi) = \Theta(\theta)+2\pi$. It is easy to verify that

$$\Re\left\{1 + e^{i\theta}\frac{f''(e^{i\theta})}{f'(e^{i\theta})}\right\} = \Theta'(\theta)\Re\left\{1 + e^{i\Theta}\frac{F''(e^{i\Theta})}{F'(e^{i\Theta})}\right\}. \tag{3.14}$$

Thus

$$\int_0^{2\pi} \Re\left\{1 + e^{i\theta}\frac{f''(e^{i\theta})}{f'(e^{i\theta})}\right\} d\theta = \int_0^{2\pi} \Re\left\{1 + e^{i\Theta}\frac{F''(e^{i\Theta})}{F'(e^{i\Theta})}\right\} d\Theta = 2\pi, \tag{3.15}$$

and

$$\Re\left\{1 + e^{i\theta}\frac{f''(e^{i\theta})}{f'(e^{i\theta})}\right\} \geq 0, \qquad (\theta \in (-\infty,\infty)) \tag{3.16}$$

since $\Theta'(\theta) > 0$. On the other hand, by Lemma 3.3(c), $\Re[1+\omega(e^{i\theta})] > 0$. Since

$$\Re\left\{\frac{e^{i\theta}\omega'(e^{i\theta})}{1+\omega(e^{i\theta})}\right\} = \frac{d}{d\theta}\arg[1+\omega(e^{i\theta})],$$

we conclude

$$\int_0^{2\pi} \Re\left\{\frac{e^{i\theta}\omega'(e^{i\theta})}{1+\omega(e^{i\theta})}\right\} d\theta = 0 \tag{3.17}$$

and, for $\theta_1 \leq \theta_2 < \theta_1 + 2\pi$,

$$\int_{\theta_1}^{\theta_2} \Re\left\{\frac{\omega'(e^{i\theta})}{1+\omega(e^{i\theta})}\right\} d\theta \quad = \quad \arg\left\{\frac{1+\omega(e^{i\theta_2})}{1+\omega(e^{i\theta_1})}\right\} > -\pi. \tag{3.18}$$

Using (3.12), with (3.15) and (3.17), we get

$$\int_0^{2\pi} \Re\left\{1 + e^{i\theta}\frac{h''(e^{i\theta})}{h'(e^{i\theta})}\right\} d\theta = 2\pi, \tag{3.19}$$

and, with (3.16) and (3.18), we get

$$\int_{\theta_1}^{\theta_2} \Re\left\{1 + e^{i\theta}\frac{h''(e^{i\theta})}{h'(e^{i\theta})}\right\} d\theta > -\pi, \qquad (\theta_1 \leq \theta_2 < \theta_1 + 2\pi). \tag{3.20}$$

Using Lemma 3.3(b) and (3.19), the argument principle gives

$$\int_0^{2\pi} \Re\left\{1 + \rho e^{i\theta}\frac{h''(\rho e^{i\theta})}{h'(\rho e^{i\theta})}\right\} d\theta = 2\pi. \tag{3.21}$$

This, together with Theorem 3.1(a), implies

$$\Re\left\{1 + \rho e^{i\theta}\frac{h''(\rho e^{i\theta})}{h'(\rho e^{i\theta})}\right\} \geq 0 \tag{3.22}$$

for all θ, and consequently the convex curve Γ is positively-oriented.

Now let Ω be the convex domain bounded by Γ. Since h is a sense-preserving homeomorphism of $\mathbb{T}$ onto Γ, Schoenflies theorem [**18**, p. 25] extends h to a local homeomorphism of $\mathbb{D}$ which maps the closed disc bounded by $\mathbb{T}_\rho$ homeomorphically onto Ω. Let W be the image surface of h in $\mathbb{D}$. Note that W is a simply connected hyperbolic covering of $\mathbb{C}$. Hence, by the Uniformization theorem, there is a locally univalent function H of $\mathbb{D}$ with image surface W. Define $\phi = H^{-1} \circ h$; ϕ is obviously a conformal map of $\mathbb{A}(\rho, 1)$ onto a ring subdomain of $\mathbb{D}$ that extends conformally between the unit circles. Write $\phi(e^{i\theta}) = e^{i\tau}$, $0 \leq \theta, \tau \leq 2\pi$. Observe that $H(e^{i\tau})$ is infinite-differentiable with $H'(e^{i\tau}) \neq 0$, since both h and ϕ are, and that $\Im[e^{i\theta}\phi'(e^{i\theta})/\phi(e^{i\theta})] = 0$ for all θ. Then direct computation yields

$$\Re\left\{1 + e^{i\tau}\frac{H''(e^{i\tau})}{H'(e^{i\tau})}\right\} d\tau \ = Re\left\{1 + e^{i\theta}\frac{h''(e^{i\theta})}{h'(e^{i\theta})}\right\} d\theta,$$

which, with (3.19), gives

$$\int_0^{2\pi} \Re\left\{1 + e^{i\tau}\frac{H''(e^{i\tau})}{H'(e^{i\tau})}\right\} d\tau \ = 2\pi,$$

and, with (3.20), gives

$$\int_{\theta_1}^{\theta_2} \Re\left\{1 + e^{i\tau}\frac{H''(e^{i\tau})}{H'(e^{i\tau})}\right\} d\tau \ > -\pi, \qquad (\theta_1 \leq \theta_2 < \theta_1 + 2\pi).$$

It follows from Kaplan's proof [**11**, Theorem 2] that H is a univalent close-to-convex function. This completes the proof. □

Proof of Theorem 3.2. Using the ideas in the paragraph succeeding the statement of Theorem 2.2, there exists a sequence $\{f_n\}$ of functions in $\mathcal{H}_0(\rho, \mathbb{G})$ with form (2.4) such that each $f_n(e^{i\theta})$ is an infinite-differentiable function in $\mathbb{T}$, and $f_n \to f$ and $h_n \to h$ locally uniformly in $\mathbb{A}(\rho, 1)$ and $\mathbb{A}(\rho^2, 1)$ respectively. Let Γ be the convex curve defined by $\Gamma(\theta) = h(\rho e^{i\theta})$, $0 \le \theta \le 2\pi$, and let Ω be the convex domain bounded by Γ. Also, let Γ_n be the convex curve defined by the function $h_n(\rho e^{i\theta})$, $0 \le \theta \le 2\pi$. By Lemma 3.4, each h_n is a sense-preserving homeomorphism of $\mathbb{T}_\rho$ onto Γ_n with

$$\Re\left\{1 + \rho e^{i\theta} \frac{h_n''(\rho e^{i\theta})}{h_n'(\rho e^{i\theta})}\right\} \ge 0$$

for all θ; see (3.22). Using this, with Lemma 2.2(a) and Theorem 3.1(a), we conclude that h satisfies (3.22) and, consequently, h is also a sense-preserving homeomorphism of $\mathbb{T}_\rho$ onto Γ. Also, by Lemma 3.4, we have each h_n univalent in $\mathbb{A}(\rho, 1)$. Hence, by Hurwitz's theorem, h is also a univalent function on $\mathbb{A}(\rho, 1)$ or else f is a constant. Define W as above, Ω_n as the convex domain bounded by Γ_n, and $W_n = h_n(\mathbb{A}(\rho, 1)) \bigcup \overline{\Omega_n}$. It is immediate that W and each W_n are simply connected domains in $\mathbb{C}$. Fix a point $\varrho \in \Omega$. We show that

$$W_n \to W \qquad \text{as } n \to \infty \qquad \text{with respect to } \varrho \tag{3.23}$$

in the sense of Carathéodory's kernel convergence [**18**, pp. 13-15]. Let $w_0 \in W$. We show first that $w_0 \in W_n$ for sufficiently large n. Let γ be a separating Jordan curve in $\mathbb{A}(\rho, 1)$ with w_0 in the interior domain of $h(\gamma)$. Since $h_n \to h$ uniformly on γ, w_0 belongs to the interior domain of the Jordan curve $h_n(\gamma)$ for sufficiently large n. Since each W_n is simply connected, $w_0 \in W_n$ for sufficiently large n. Now let $w_0 \in \partial W$. We show that w_0 is the limit point of a sequence $\{w_n\}$ where $w_n \in \partial W_n$. Suppose that this is false. Then there is an increasing sequence of positive integers $\{n_\nu\}$ and an open neighborhood V of w_0 such that $\partial W_{n_\nu} \bigcap V = \emptyset$. Also, choose V so that $\overline{\Omega} \bigcap \overline{V} = \emptyset$; this is possible since $\Gamma_n \to \Gamma$. It follows that, for each n_ν, either $V \bigcap W_{n_\nu} = \emptyset$ or $V \subset W_{n_\nu}$. Suppose that the first case happens infinitely often, say, without loss of generality, for all ν. Then $h_{n_\nu}(z) \notin V$ for $z \in \mathbb{A}(\rho, 1)$. Since $h_{n_\nu}(z) \to h(z)$, $h(\mathbb{A}(\rho, 1)) \bigcap V = \emptyset$ and we have a contradiction. Now suppose, without loss of generality, that $V \subset W_{n_\nu}$ for all ν. Then the inverse function $\psi_{n_\nu}(w) = h_{n_\nu}^{-1}(w)$ is analytic in V with $|\psi_{n_\nu}(w)| < 1$. By Montel's theorem, we can find a subsequence of $\{\psi_{n_\nu}\}$ that converges locally uniformly in V. Suppose, without loss of generality, that $\{\psi_{n_\nu}\}$ converges locally uniformly in V. Then the limit function ψ satisfies $\rho \le |\psi(w)| \le 1$ for $w \in V$. By Hurwitz's theorem, either ψ is a constant or is a univalent function in V. We show that the latter holds. To do so, we show first that $\{\psi_{n_\nu}\}$ converges locally uniformly in $h(\mathbb{A}(\rho, 1))$ even though these functions may not be defined in $h(\mathbb{A}(\rho, 1))$ in the proper sense. Let

Δ be a closed Jordan region in $h(\mathbb{A}(\rho,1))$, and let K be a compact subset of the interior Δ. Since h is univalent, $h^{-1}(\Delta)$ is a closed Jordan region in $\mathbb{A}(\rho,1)$ whose interior contains $h^{-1}(K)$. Since $h_{n_\nu} \to h$ uniformly on $h^{-1}(\Delta)$, an argument using Rouche's theorem implies that $K \subset h_{n_\nu} \circ h^{-1}(\Delta)$ or $K \subset h_{n_\nu}(\mathbb{A}(\rho,1))$ for sufficiently large n_ν. A compactness argument also yields the same conclusion for any compact subset K of $h(\mathbb{A}(\rho,1))$. So, for a given compact subset K of $h(\mathbb{A}(\rho,1))$, the functions ψ_{n_ν} are defined on K for sufficiently large n_ν. Since the range of each ψ_{n_ν} is $\mathbb{A}(\rho,1)$, the sequence $\{\psi_{n_\nu}\}$ is a normal family in $h(\mathbb{A}(\rho,1))$. Since $V \bigcap h(\mathbb{A}(\rho,1)) \neq \emptyset$, $\psi_{n_\nu} \to \psi$ in $h(\mathbb{A}(\rho,1))$. Recall the above curve γ. If ψ is constant, then $\psi_{n_\nu}(h(\gamma))$ admits an arbitrarily small diameter for large n_ν which is impossible since each curve $\psi_{n_\nu}(\gamma)$ separates $\mathbb{A}(\rho,1)$. Hence ψ is univalent in V and $\rho < |\psi(w)| < 1$ for $w \in V$, in particular $\psi(w_0) \in \mathbb{A}(\rho,1)$. It follows that $\{h_{n_\nu}\}$ converges locally uniformly near $\psi(w_0)$. Since $\psi_{n_\nu}(w_0) \to \psi(w_0)$ and $w_0 = h_{n_\nu} \circ \psi_{n_\nu}(w_0)$, we conclude $w_0 = h(\psi(w_0))$. This contradicts $w_0 \in \partial W$ and (3.23) holds.

Now define H as above but with the additional conditions $H(0) = \varrho$ and $H'(0) > 0$. Also, let H_n be the conformal map of $\mathbb{D}$ onto W_n satisfying $H_n(0) = \varrho$ and $H_n{}'(0) > 0$. By Carathéodary's kernel theorem [**18**, pp. 13-15], $H_n \to H$ locally uniformly in $\mathbb{D}$. Since, by Lemma 3.4, each H_n is a univalent close-to-convex function, H is also a univalent close-to-convex function. Letting $\phi = H^{-1} \circ h$. It is easily seen that ϕ satisfies the desired properties. This ends the proof. $\square$

4. Univalent Harmonic Mappings onto Punctured Convex Domains.

Let f be the Dirichlet solution in $\mathbb{A}(\rho,1)$ of a function f^* of $\partial\mathbb{A}(\rho,1)$ defined on $\mathbb{T}$ be a sense-preserving quasihomeomorphism into $\partial\mathbb{G}$ satisfying $\overline{co}f^*(\widetilde{E}(f^*)) = \overline{\mathbb{G}}$, and on $\mathbb{T}_\rho$ by a constant $\zeta \in \mathbb{G}$. Theorem 2.2 asserts that f belongs to $\mathcal{H}_u(\rho,\mathbb{G})$ if $\zeta = \zeta_0$, where ζ_0 is the average of f^* on $\mathbb{T}_\rho$ given by (1.3). Recently however, Duren and Hengartner [**5**, Example 1] observed that this condition is not necessary, and showed that the harmonic mapping

$$F(z) = (z - \rho^2/\overline{z})/(1-\rho^2) + 2c\log|z|, \qquad (z \in \mathbb{A}(\rho,1)), \tag{4.1}$$

belongs to $\mathcal{H}_u(\rho,\mathbb{D})$ with $f(0) = 2c\log\rho$ if $|c| < \rho/(1-\rho^2)$. Note that the boundary function of F is the identity map on $\mathbb{T}$ and the constant $2c\log\rho$ on $\mathbb{T}_\rho$. In view of this, Hengartner [**2**, Problem 15] suggested the problem of finding the set of values $\zeta \in \mathbb{G}$ that yields $f : \mathbb{A}(\rho,1) \to G \setminus \{\zeta\}$ a homeomorphism.

Now, let f^* be a sense-preserving quasihomeomorphism of $\mathbb{T}$ into $\partial\mathbb{G}$ with $\overline{co}f^*(\widetilde{E}(f^*)) = \overline{\mathbb{G}}$. Denote by $\mathcal{H}(\rho,f^*)$ the class of Dirichlet solutions in $\mathbb{A}(\rho,1)$ of functions of $\partial\mathbb{A}(\rho,1)$ defined on $\mathbb{T}$ by f^* and on $\mathbb{T}_\rho$ by some constant $\zeta \in \mathbb{G}$. Also, denote by $\mathcal{H}_u(\rho, f^*)$ the subclass of $\mathcal{H}(\rho, f^*)$ of univalent mappings. Of interest shall be the set $K(\rho, f^*)$ of values $\zeta \in \mathbb{G}$ for which a function $f \in \mathcal{H}(\rho, f^*)$ belongs to $\mathcal{H}_u(\rho, f^*)$.

Our first result in this section states that $K(\rho, f^*)$ is compact. In view of Proposition 3.1, the class $\mathcal{H}(\rho, f^*)$ yields an analytic function h in $\mathbb{A}(\rho^2, 1)$, unique up to an additive constant, such that every $f \in \mathcal{H}_u(\rho, f^*)$ is of the forms (3.1) and (3.2). In our second result, we characterize in terms of h and f^* the boundary points of $K(\rho, f^*)$ in a manner leading to a univalence criterion for functions $f \in \mathcal{H}(\rho, f^*)$. Finally, we provide sufficient conditions on ρ, $\mathbb{G}$, and f^* that warrant a nonempty interior for $K(\rho, f^*)$.

Theorem 4.1. *$K(\rho, f^*)$ is a nonempty compact subset of $\mathbb{G}$.*

Proof. Let ζ_0 be the average of f^* on $\mathbb{T}$. It is immediate from Theorem 2.2 that $\zeta_0 \in K(\rho, f^*)$. Hence $K(\rho, f^*) \neq \emptyset$.

Suppose that a sequence $\{\zeta_n\}_{n=1}^{\infty}$ in $K(\rho, f^*)$ converges to $\zeta \in \overline{\mathbb{G}}$. We show that $\zeta \in K(\rho, f^*)$. Clearly, there is a unique function $f_n \in \mathcal{H}_u(\rho, f^*)$ such that $f_n(\mathbb{T}_\rho) = \zeta_n$. By Proposition 3.1, we can find an analytic function h in $\mathbb{A}(\rho^2, 1)$, unique up to an additive constant, such that

$$f_n(z) = h(z) - h(\rho^2/\overline{z}) + \zeta_n + 2c_n \log(|z|/\rho), \qquad (z \in \mathbb{A}(\rho, 1)), \tag{4.2}$$

where $c_n = (\zeta_n - \zeta_0)/(2 \log \rho)$. Obviously, $c_n \to c = (\zeta - \zeta_0)/(2 \log \rho)$ as $n \to \infty$. Using h and c, we define the harmonic mapping f as in (1.2). If $c = 0$, then $\zeta = \zeta_0 \in K(\rho, f^*)$ by Theorem 2.2. So, suppose that $c \neq 0$. Then $f_n \to f$ (locally) uniformly in $\mathbb{A}(\rho, 1)$. It is easy to see that the Jacobian of f_n is given by

$$J_n(z) = [|zh'(z) + c_n|^2 - |(\rho^2/\overline{z})h'(\rho^2/\overline{z}) + c_n|^2]/|z|^2, \qquad (z \in \mathbb{A}(\rho, 1)). \tag{4.3}$$

Since f_n is univalent and sense-preserving, $J_n(z) > 0$, and consequently $|zh'(z) + c_n| \neq 0$ for $z \in \mathbb{A}(\rho, 1)$. But $zh'(z) + c_n \to zh'(z) + c$ uniformly in $\mathbb{A}(\rho, 1)$. Hence, by Hurwitz's theorem, either $zh'(z) + c \neq 0$ or $zh'(z) = 0$ for $z \in \mathbb{A}(\rho, 1)$. If the latter holds, then $h'(z) = -c/z$ which contradicts the analyticity of h in $\mathbb{A}(\rho, 1)$. Hence $zh'(z) + c \neq 0$. The Jacobian of f is now given by

$$J(z) = [|zh'(z) + c|^2 - |(\rho^2/\overline{z})h'(\rho^2/\overline{z}) + c|^2]/|z|^2, \qquad (z \in \mathbb{A}(\rho, 1)).$$

Clearly, $J_n z \to J(z)$. Since $J_n(z) > 0$, $J(z) \geq 0$. Thus

$$|(\rho^2/\overline{z})h'(\rho^2/\overline{z}) + c| \leq |zh'(z) + c|, \qquad (z \in \mathbb{A}(\rho, 1)).$$

We prove that this inequality must be strict. Suppose that equality holds for some z. Then the maximum principle yields that the dilatation of f

given by

$$\omega(z)=\frac{\overline{(\rho^2/\overline{z})h'(\rho^2/\overline{z})}+\overline{c}}{zh'(z)+c}, \qquad (z\in\mathbb{A}(\rho,1)), \tag{4.4}$$

is a unimodular constant $e^{2i\alpha}$ for some real α. That is,

$$\overline{(\rho^2/\overline{z})h'(\rho^2/\overline{z})}+\overline{c}=e^{2i\alpha}(zh'(z)+c), \qquad (z\in\mathbb{A}(\rho,1)). \tag{4.5}$$

Since h is analytic in $\mathbb{A}(\rho^2,1)$, (4.5) holds for $z\in\mathbb{A}(\rho^2,1)$. In particular, for all θ,

$$\overline{\rho e^{i\theta}h'(\rho e^{i\theta})+c}=e^{2i\alpha}[\rho e^{i\theta}h'(\rho e^{i\theta})+c].$$

This means that the function $zh'(z)$ maps $\mathbb{T}_\rho$ into the straight-line passing through $-c$ in the direction of $e^{-i\alpha}$. We conclude that $h(z)$ maps $\mathbb{T}_\rho$ to a straight-line in the direction of $e^{i}(\pi/2-\alpha)$. This contradicts Theorem 3.1. Therefore,

$$|(\rho^2/\overline{z})h'(\rho^2/\overline{z})+c|<|zh'(z)+c| \qquad (z\in\mathbb{A}(\rho,1)).$$

This yields $J(z)>0$ for $z\in\mathbb{A}(\rho,1)$, and consequently f is locally univalent function by Lewy's theorem [**13**]. Since each f_n is univalent and $f_n\to f$ uniformly in $\mathbb{A}(\rho,1)$, f is univalent in $\mathbb{A}(\rho,1)$. Using this, with the fact $\overline{co}f^*(\widetilde{E}(f^*))=\overline{\mathbb{G}}$, we infer, by Remark 2.2, that $f:\mathbb{A}(\rho,1)\to G\setminus\{\zeta\}$ is a homeomorphism. Therefore $\zeta\in K(\rho,f^*)$ and the proof is complete. $\square$

Our second result is Theorem 4.2.

Theorem 4.2. *Let $f\in\mathcal{H}_u(\rho,f^*)$ be of form* (3.1), *where $f^*:\mathbb{T}\to\partial\mathbb{G}$ is a twice-differentiable function with nonvanishing derivative and absolutely continuous second derivative. Then the dilatation of f and $zh'(z)+c_\zeta$ extend continuously to $\mathbb{A}(\rho,1)\cup\mathbb{T}$ such that $e^{i\theta}h'(e^{i\theta})+c_\zeta\neq 0$ for all θ. Moreover, we have:*

(a) *If $\zeta\in\partial K(\rho,f^*)$, then either $\rho e^{i\theta_1}h'(\rho e^{i\theta_1})+c_\zeta=0$ for some θ_1, or $|\omega(e^{i\theta_2})|=1$ for some θ_2.*

(b) *If $|\omega(e^{i\theta})|=1$ for some θ, then $\zeta\in\partial K(\rho,f^*)$.*

(c) *If in* (a) *and* (b) *the function $|\omega(e^{i\theta})|$ is replaced by the function*

$$2\Re\left\{\frac{e^{i\theta}h'(e^{i\theta})+c_\zeta}{e^{i\theta}f'(e^{i\theta})}\right\},$$

then (a) *and* (b) *continue to hold.*

Regarding (a), a result of Hengartner and Szynal [**10**, Theorem 3.1] asserts that if $\zeta\in\partial K(\rho,f^*)$ then $\rho e^{i\theta_1}h'(\rho e^{i\theta_1})+c_\zeta$ has at most one zero which is of order one.

Proof. The Jacobian of f is given by

$$J(z)=[|zh'(z)+c_\zeta|^2-|(\rho^2/\overline{z})h'(\rho^2/\overline{z})+c_\zeta|^2]/|z|^2, \qquad (z\in\mathbb{A}(\rho,1)).$$

Since f is univalent and sense-preserving, Lewy's theorem [**13**] yields $J(z) > 0$. This implies $zh'(z) + c_\zeta \neq 0$ for $z \in \mathbb{A}(\rho, 1)$. By Lemma 3.3, h has a continuously differentiable extension to $\mathbb{A}(\rho^2, 1)$ such that $h'(e^{i\theta}) \neq 0$ and $h'(\rho^2 e^{i\theta}) \neq 0$ for all θ. It follows that J has a continuous extension to $\overline{\mathbb{A}(\rho, 1)}$ such that

$$J(e^{i\theta}) = |e^{i\theta}h'(e^{i\theta}) + c_\zeta|^2 - |\rho^2 e^{i\theta}h'(\rho^2 e^{i\theta}) + c_\zeta|^2,$$

and $J(e^{i\theta}) \geq 0$ for all θ. If for some θ, $e^{i\theta}h'(e^{i\theta}) + c_\zeta = 0$, then $\rho^2 e^{i\theta}h'(\rho^2 e^{i\theta}) + c_\zeta = 0$, and consequently

$$e^{i\theta}f'(e^{i\theta}) = e^{i\theta}h'(e^{i\theta}) - \rho^2 e^{i\theta}h'(\rho^2 e^{i\theta}) = 0$$

which gives a contradiction. Hence, $e^{i\theta}h'(e^{i\theta}) + c_\zeta \neq 0$ for all θ.

It also follows that the dilatation of f given by

$$\omega(z) = \frac{\overline{(\rho^2/\overline{z})h'(\rho^2/\overline{z})} + \overline{c_\zeta}}{zh'(z) + c_\zeta}, \qquad (z \in \mathbb{A}(\rho, 1)),$$

extends continuously to $\mathbb{A}(\rho, 1) \cup \mathbb{T}$ such that

$$|\omega(e^{i\theta})| = \left| \frac{\rho^2 e^{i\theta}h'(\rho^2 e^{i\theta}) + c_\zeta}{e^{i\theta}h'(e^{i\theta}) + c_\zeta} \right|. \tag{4.6}$$

(a) We proceed to prove (a) by contrapositivity. Suppose that $|\omega(e^{i\theta})| < 1$ for all θ. Then

$$|\rho^2 e^{i\theta}h'(\rho^2 e^{i\theta}) + c_\zeta| < |e^{i\theta}h'(e^{i\theta}) + c_\zeta|.$$

By the compactness of $\mathbb{T}$, we can find $\delta > 0$ such that

$$|\rho^2 e^{i\theta}h'(\rho^2 e^{i\theta}) + c_\zeta| < |e^{i\theta}h'(e^{i\theta}) + c_\zeta| - \delta$$

for all θ. It follows that, for $|\eta - \zeta| < \delta \log(1/\rho)$ and any θ,

$$|\rho^2 e^{i\theta}h'(\rho^2 e^{i\theta}) + c_\eta| < |e^{i\theta}h'(e^{i\theta}) + c_\eta| \tag{4.7}$$

where $c_\eta = (\eta - \zeta_0)/(2\log\rho)$ (see (5.4)).

Suppose now that $\rho e^{i\theta}h'(\rho e^{i\theta}) + c_\zeta \neq 0$ for all θ. Then, in view of the above, $zh'(z) + c_\zeta \neq 0$ for $z \in \overline{\mathbb{A}(\rho, 1)}$. Since $\overline{\mathbb{A}(\rho, 1)}$ is compact, there is $\sigma > 0$ such that $|zh'(z) + c_\zeta| > \sigma$ for $z \in \overline{\mathbb{A}(\rho, 1)}$. It follows that, for $|\eta - \zeta| < 2\sigma \log(1/\rho)$,

$$|zh'(z) + c_\eta| > 0, \qquad (z \in \overline{\mathbb{A}(\rho, 1)}). \tag{4.8}$$

Then (4.7) and (4.8) hold for every η satisfying

$$|\eta - \zeta| < \tau = \min\{2\delta \log(1/\rho), 2\sigma \log(1/\rho)\}.$$

For each such η, let

$$f_\eta(z) = h(z) - h(\rho^2/\overline{z}) + \eta + 2c_\eta \log(|z|/\rho), \qquad (z \in \overline{\mathbb{A}(\rho, 1)}).$$

Then f_η is a harmonic mapping whose dilatation is given by

$$\omega_\eta(z) = \frac{\overline{(\rho^2/\overline{z})h'(\rho^2/\overline{z})} + \overline{c}_\eta}{zh'(z) + c_\eta}, \qquad (z \in \overline{\mathbb{A}(\rho,1)}).$$

Clearly, by (4.7) and (4.8), ω_η is an analytic function that extends continuously to $\overline{\mathbb{A}(\rho,1)}$ such that $|\omega_\eta((e^{i\theta}))| < 1$ and $|\omega_\eta(\rho(e^{i\theta}))| = 1$ for all θ. Hence, by the maximum principle, $|\omega_\eta(z)| < 1$. This yields, because of (4.8), that the Jacobian of f_η is positive in $\mathbb{A}(\rho,1)$, and consequently f_η is a univalent sense-preserving harmonic mapping. Now, by invoking Theorem 2.1 and Remark 2.2, we conclude that each $f_\eta : \mathbb{A}(\rho,1) \to \mathbb{G}\setminus\{\eta\}$ is a homeomorphism. Since this holds whenever $|\eta - \zeta| < \tau$, ζ is an interior point of $K(\rho, f^*)$ and we have a contradiction. This proves (a).

(b) Suppose that $|\omega(e^{i\theta_1})| = 1$ for some θ_1. Then the Möbius transformation

$$T(z) = \frac{\rho^2 e^{i\theta_1} h'(\rho^2 e^{i\theta_1}) + z}{e^{i\theta_1} h'(e^{i\theta_1}) + z}$$

satisfies $|T(c_\zeta)| = 1$. Since, by (5.4), $\eta - \zeta = 2(c_\eta - c_\zeta)\log\rho$, any open neighborhood of ζ contains an η such that $|T(c_\eta)| > 1$, or equivalently, $\omega_\eta(e^{i\theta_1}) > 1$ where ω_η is as defined above. Therefore, $\eta \notin K(\rho, f^*)$ and $\zeta \in \partial K(\rho, f^*)$.

(c) Since $e^{i\theta}h'(e^{i\theta}) + c_\zeta \neq 0$ for all θ, using (3.1), we obtain

$$\begin{aligned} e^{i\theta} f'(e^{i\theta}) &= [e^{i\theta}h'(e^{i\theta}) + c_\zeta] - [\rho^2 e^{i\theta}h'(\rho^2 e^{i\theta}) + c_\zeta] \\ &= [e^{i\theta}h'(e^{i\theta}) + c_\zeta] \\ &\quad \times \Big[1 - [\rho^2 e^{i\theta}h'(\rho^2 e^{i\theta}) + c_\zeta]/[e^{i\theta}h'(e^{i\theta}) + c_\zeta]\Big]. \end{aligned}$$

Since $f'(e^{i\theta}) \neq 0$ for all θ, we obtain

$$\frac{e^{i\theta}h'(e^{i\theta}) + c_\zeta}{e^{i\theta} f'(e^{i\theta})} = \frac{1}{1 - [\rho^2 e^{i\theta}h'(\rho^2 e^{i\theta}) + c_\zeta]/[e^{i\theta}h'(e^{i\theta}) + c_\zeta]}.$$

This implies that

$$2\Re\left\{\frac{e^{i\theta}h'(e^{i\theta}) + c_\zeta}{e^{i\theta} f'(e^{i\theta})}\right\} = 1$$

for some θ if and only if $|\omega(e^{i\theta})| = 1$; see (4.6). This proves (c). $\square$

We apply Theorem 4.2 to a function $f \in \mathcal{H}_u(\rho, f^*)$ of form (2.2). In this case, ζ is the average ζ_0 of f^* on $\mathbb{T}$, $c_\zeta = 0$, $\rho e^{i\theta}h'(\rho e^{i\theta}) \neq 0$ for all θ by Theorem 3.1(a), and $|\omega(e^{i\theta})| = 1$ for some θ if and only if $\rho^2|h'(\rho^2 e^{i\theta})| = |h'(e^{i\theta})|$. We conclude the following Corollary 4.1.

Corollary 4.1. *Let $f \in \mathcal{H}_u(\rho, f^*)$ be of form (2.2), where f^* is as in Theorem 4.2. Then the following statements are equivalent:*

(a) $\zeta_0 \in \partial K(\rho, f^*)$.

(b) $\rho^2|h'(\rho^2 e^{i\theta})| = |h'(e^{i\theta})|$ *for some* θ.
(c) $2\Re\{h'(e^{i\theta})/f'(e^{i\theta})\} = 1$ *for some* θ.

The arguments used in the proof of Theorem 4.2 yield at once sufficient conditions for the univalence of functions in $\mathcal{H}(\rho, f^*)$ where f^* is as in Theorem 4.2.

Theorem 4.3. *Let* $f \in \mathcal{H}(\rho, f^*)$ *be of form* (3.1), *where* f^* *be smooth as in Theorem* 4.2. *Then* $f \in \mathcal{H}_u(\rho, f^*)$ *if* $zh'(z) + c_\zeta \neq 0$ *for* $z \in \overline{\mathbb{A}(\rho, 1)}$, *and if one of the following two inequalities holds for all* θ:

(a) $|\omega(e^{i\theta})| \leq 1$.
(b)

$$2\Re\left\{\frac{e^{i\theta}h'(e^{i\theta}) + c_\zeta}{e^{i\theta}f'(e^{i\theta})}\right\} \geq 1.$$

We remark that f^* as defined in Theorem 4.2 yields, by Lemma 3.3, $zh'(z) \neq 0$ for $z \in \overline{\mathbb{A}(\rho, 1)}$. This makes the above sufficiency condition, $zh'(z) + c_\zeta \neq 0$ for $z \in \overline{\mathbb{A}(\rho, 1)}$, easily achievable for functions $f \in \mathcal{H}(\rho, f^*)$ with appropriately small c_ζ.

Finally, we prove the existence of a large family of triplets, $0 < \rho < 1$, $\mathbb{G}_\rho$, f^*, where $\mathbb{G}_\rho$ is a bounded convex domain and $f^*_\rho : \mathbb{T} \to \partial\mathbb{G}_\rho$ is a sense-preserving homeomorphism, such that $K(\rho, f^*)$ has a nonempty interior containing the average of f^*.

Theorem 4.4. *Let* Ω *be a bounded convex domain, and let* h *be a homeomorphism of* $\overline{\mathbb{D}}$ *onto* $\overline{\Omega}$ *that maps* $\mathbb{D}$ *conformally onto* Ω. *Suppose that* h'' *is continuous on* $\overline{\mathbb{D}}$, $h''(e^{i\theta})$ *is absolutely continuous, and*

$$\Re\left\{1 + e^{i\theta}\frac{h''(e^{i\theta})}{h'(e^{i\theta})}\right\} > 0 \tag{4.9}$$

for all θ. *Then there exists* $\delta > 0$ *such that for each* $0 < \rho < \delta$ *we can find a bounded convex domain* $\mathbb{G}_\rho$ *such that the harmonic mapping*

$$f_\rho(z) = h(z) - h(\rho^2/\overline{z}), \qquad (z \in \overline{\mathbb{A}(\rho, 1)}), \tag{4.10}$$

satisfies the following properties:

(i) $f_\rho : \mathbb{T} \to \partial\mathbb{G}_\rho$ *is a sense-preserving homeomorphism.*
(ii) f_ρ *is continuously twice-differentiable on* $\overline{\mathbb{A}(\rho, 1)}$.
(iii) $f_\rho \in \mathcal{H}_0(\rho, \mathbb{G}_\rho)$.
(iv) *There is* $\sigma > 0$, *depending on* ρ, *such that for any* $|\zeta| < \sigma$ *the function*

$$f_\zeta(z) = h(z) - h(\rho^2/\overline{z}) + \zeta + 2c_\zeta \log(|z|/\rho) \tag{4.11}$$

belongs to $\mathcal{H}_u(\rho, \mathbb{G}_\rho)$.

Remark 4.1. (i) Without (4.9), the hypothesis of the theorem yields the following weaker form of (4.9):

$$\Re\left\{1+e^{i\theta}\frac{h''(e^{i\theta})}{h'(e^{i\theta})}\right\}\geq 0. \tag{4.12}$$

To see this, observe that $zh'(z)$ is a univalent starlike function in $\mathbb{D}$ which gives

$$\Re\left\{1+z\frac{h''(z)}{h'(z)}\right\}>0, \qquad (z=re^{i\theta}\in\mathbb{D}). \tag{4.13}$$

Now, because h'' extends continuously to $\overline{\mathbb{D}}$, the integral

$$\int_0^z h''(\zeta)\,d\zeta, \qquad (z\in\overline{\mathbb{D}}),$$

where the differentiable path of integration from 0 to z lies in $\overline{\mathbb{D}}$, yields, by Cauchy's theorem, the continuous extension of $h'(z)$ to $\overline{\mathbb{D}}$. On the other hand, since $zh'(z)$ is univalent in $\mathbb{D}$ and maps the origin to itself, $zh'(z)\neq 0$ for $z\in\overline{\mathbb{D}}$. Then (4.12) follows at once by letting $r\to 1$ in (4.13).

(ii) Using Kellogg and Warschawski [**18**, Theorem 3.6, p. 49], the hypothesis that $h''(z)$ admits a continuous extension to $\overline{\mathbb{D}}$ with absolutely continuous $h''(e^{i\theta})$ follows if $\partial\mathbb{G}$ has a parametrization $w(t)$, $0\leq t\leq 2\pi$, whose first derivative is nonvanishing and second derivative is Lipschitz of order α, $0<\alpha<1$.

Proof of Theorem 4.4. By the compactness of $\mathbb{T}$, there is $q>0$ such that

$$\Re\left\{1+e^{i\theta}\frac{h''(e^{i\theta})}{h'(e^{i\theta})}\right\}>q \tag{4.14}$$

for all θ. For a fixed $0<\rho<1$, let

$$k_\rho(z)=h(z)-h(\rho^2 z), \qquad (z\in\overline{\mathbb{D}}). \tag{4.15}$$

Then k_ρ is an analytic function in $\mathbb{D}$ with $k_\rho(0)=0$. We can write

$$1+e^{i\theta}\frac{k_\rho''(e^{i\theta})}{k_\rho'(e^{i\theta})}=1+e^{i\theta}\frac{h''(e^{i\theta})}{h'(e^{i\theta})}+e^{i\theta}q_\rho(e^{i\theta}), \tag{4.16}$$

where

$$q_\rho(e^{i\theta})=\rho^2e^{i\theta}\frac{h'(\rho^2e^{i\theta})h''(e^{i\theta})-\rho^2h'(e^{i\theta})h''(\rho^2e^{i\theta})}{h'(e^{i\theta})[h'(e^{i\theta})-\rho^2h'(\rho^2e^{i\theta})]}.$$

Let $m_1=\min_\theta|h'(e^{i\theta})|$, $M_1=\max_\theta|h'(e^{i\theta})|$, $M_2=\max_\theta|h''(e^{i\theta})|$, and

$$\delta=\min\left\{\sqrt{\frac{m_1}{2M_1}},\frac{m_1}{2}\sqrt{\frac{q}{M_1M_2}}\right\}.$$

Then for $0 < \rho < \delta$, it is easy to verify that $|q_\rho(e^{i\theta})| < q$ which gives $\Re q_\rho(e^{i\theta}) > -q$. Using (4.14) and (4.16), we obtain

$$\Re\left\{1 + e^{i\theta}\frac{k_\rho''(e^{i\theta})}{k_\rho'(e^{i\theta})}\right\} > 0. \tag{4.17}$$

Using (4.15), we conclude that k' and k'' extend continuously to $\overline{\mathbb{D}}$. Moreover, since $zh'(z)$ is univalent and $0 < \rho < \delta$, $k'(z) \neq 0$ for $z \in \overline{\mathbb{D}}$. It follows by the maximum principle and (4.17) that

$$\Re\left\{1 + z\frac{k_\rho''(z)}{k_\rho'(z)}\right\} > 0, \qquad (z \in \overline{\mathbb{D}}).$$

Let $\mathbb{G}_\rho = k_\rho(\mathbb{D})$. We conclude that $\mathbb{G}_\rho$ is a bounded convex domain, and that k_ρ is a sense-preserving homeomorphism of $\overline{\mathbb{D}}$ onto $\overline{\mathbb{G}_\rho}$ that maps $\mathbb{D}$ conformally onto $\mathbb{G}_\rho$. Now define f_ρ as in (4.10). Then, by (4.15), $f_\rho(e^{i\theta}) = k_\rho(e^{i\theta})$ which yields (i) and (ii). Furthermore,

$$0 = f_\rho(\rho e^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} f_\rho(e^{i\theta})\, d\theta.$$

Then (iii) follows at once from Theorem B. On the other hand, by the definition of δ, we obtain

$$\rho^2|h'(\rho^2 e^{i\theta})| < \rho^2 M_1 < \delta^2 M_1 \leq \frac{m_1}{2} \leq |h'(e^{i\theta})|.$$

This implies $f'(e^{i\theta}) \neq 0$. Since $h''(e^{i\theta})$ is absolutely continuous, $f''(e^{i\theta})$ is also absolutely continuous. Now an application of Corollary 4.1 implies (iv). This completes the proof. □

5. Nitsche's Question Revisited.

In this section, we determine explicitly all harmonic mappings $f \in \mathcal{H}_u(\rho, \mathbb{G})$ whose analytic parts extend analytically throughout $\mathbb{D}$. As a consequence, we conclude that the function f defined by (1.1) is the only harmonic mapping, up to rotation, in $\mathcal{H}_0(\rho, \mathbb{D})$, (here $\mathbb{G}$ is taken as $\mathbb{D}$), of $\mathbb{A}(\rho, 1)$ onto $\mathbb{A}(0, 1)$ whose analytic part is analytic in $\mathbb{D}$. This somehow justifies Nitsche's question above.

Definition 5.1. Let $f \in \mathcal{H}_u(\rho, \mathbb{G})$. Then, by Theorem 2.1, the unrestricted limit function of f coincides with a sense-preserving quasihomeomorphism f^* except possibly on a countable subset of $\mathbb{T}$. We call the value ζ_0 given by (1.3) the average of f on $\mathbb{T}$. Denote by f_0 the Dirichlet solution in $\mathbb{A}(\rho, 1)$ of the boundary function which coincides with f^* on $\mathbb{T}$ and is the constant ζ_0 on $\mathbb{T}_\rho$. (By virtue of Theorem 2.2, $f_0 \in \mathcal{H}_0(\rho, \mathbb{G})$.) We call f_0 the average associate of f.

The result of this section is Theorem 5.1.

Theorem 5.1. *Suppose $f \in \mathcal{H}_u(\rho, \mathbb{G})$ has form* (1.2) *with ζ_0 the average of f on $\mathbb{T}$. If h is analytic in $\mathbb{D}$, then*

$$f(z) = \sum_{n=1}^{\infty} \frac{\lambda^n b_n}{1-\rho^{2n}}[z^n - (\rho^2/\overline{z})^n] + \zeta + 2c_\zeta \log(|z|/\rho) \tag{5.1}$$

$$= \sum_{n=1}^{\infty} \frac{\lambda^n b_n}{1-\rho^{2n}}[z^n - (\rho^2/\overline{z})^n] + \zeta_0 + 2c_\zeta \log|z|, \tag{5.2}$$

where b_n, $n = 1, 2, \ldots$, is the n-th coefficient of the conformal map

$$F(z) = \zeta_0 + \sum_{n=1}^{\infty} b_n z^n \tag{5.3}$$

of $\mathbb{D}$ onto $\mathbb{G}$ satisfying $F(0) = \zeta_0$, and

$$c_\zeta = \frac{\zeta - \zeta_0}{2\log\rho}. \tag{5.4}$$

Proof. By virtue of Proposition 3.1, it suffices to prove the theorem for the average associate f_0 of f. Using the proposition, we write

$$f_0(z) = h(z) - h(\rho^2/\overline{z}) + \zeta_0, \qquad (z \in \mathbb{A}(\rho, 1)). \tag{5.5}$$

Since h is analytic in $\mathbb{D}$, the function

$$q(z) = h(z) - h(\rho^2 z) + \zeta_0 \tag{5.6}$$

is analytic in $\mathbb{D}$, maps the origin to ζ_0, and satisfies

$$\lim_{|z|\to 1} [f_0(z) - q(z)] = \lim_{|z|\to 1} [h(\rho^2 z) - h(\rho^2/\overline{z})] = 0. \tag{5.7}$$

This implies that f_0 and q have the same cluster set at each $\xi \in \mathbb{T}$. But $C(f_0, \xi) \subset \partial\mathbb{G}$ for $\xi \in \mathbb{T}$. Hence, by [**18**, Corollary 2.10], q, and consequently f_0 by (5.7), has a continuous extension to $\overline{\mathbb{D}}$ that assumes every value of $\overline{\mathbb{G}}$ exactly m times in $\overline{\mathbb{D}}$. It follows that $f_0(e^{i(t)}) = F(e^{i\varphi(t)})$ where φ is a continuous increasing function of $(-\infty, \infty)$ with $\varphi(t + 2\pi) = \varphi(t) + 2m\pi$. Using Theorem 2.1(v), we conclude $m = 1$. This implies that q, like F, is a conformal map of $\mathbb{D}$ onto $\mathbb{G}$ with $q(0) = \zeta_0$. By Schwarz's lemma,

$$q(z) = F(\lambda z) \tag{5.8}$$

for some unimodular constant λ.

Suppose

$$h(z) = a_0 + \sum_{n=1}^{\infty} a_n z^n, \qquad (z \in \mathbb{D}).$$

Then (5.3), (5.6) and (5.8) yield

$$q(z) = \zeta_0 + \sum_{n=1}^{\infty} a_n(1-\rho^{2n})z^n = \zeta_0 + \sum_{n=1}^{\infty} \lambda^n b_n z^n.$$

This gives

$$a_n = \frac{\lambda^n b_n}{1-\rho^{2n}} \qquad (n = 1, 2, \dots).$$

Using (5.5), we obtain

$$f_0(z) = \zeta_0 + \sum_{n=1}^{\infty} \frac{\lambda^n b_n}{1-\rho^{2n}} [z^n - (\rho^2/\overline{z})^n].$$

This completes the proof. □

If $\mathbb{G} = \mathbb{D}$, then Theorem 5.1 yields Corollary 5.1 by taking

$$F(z) = \frac{z+\zeta_0}{1+\overline{\zeta}_0 z} = \zeta_0 + (1-|\zeta_0|^2) \sum_{n=2}^{\infty} (-\zeta_0)^{n-1} z^n.$$

Corollary 5.1. *Suppose* $f \in \mathcal{H}_u(\rho, \mathbb{D})$ *has form* (1.2) *with* ζ_0 *the average of* f *on* $\mathbb{T}$ *and* h *analytic in* $\mathbb{D}$. *Then there is a unimodular constant* λ *such that*

$$\begin{aligned} f(z) &= \lambda(1-|\zeta_0|^2) \left\{ \frac{z-\rho^2/\overline{z}}{1-\rho^2} + \sum_{n=2}^{\infty} \frac{(-\lambda\zeta_0)^{n-1}}{1-\rho^{2n}} [z^n - (\rho^2/\overline{z})^n] \right\} \\ &\quad + \zeta + 2c_\zeta \log(|z|/\rho), \qquad (z \in \mathbb{A}(\rho, 1)). \end{aligned}$$

In particular, if $\zeta_0 = 0$, *then*

$$f(z) = \lambda \frac{z-\rho^2/\overline{z}}{1-\rho^2} + 2c_\zeta \log|z|, \qquad (z \in \mathbb{A}(\rho, 1)).$$

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