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In this paper, we compute the second order Taylor expansion of spherical functions on symmetric cones. In order to compute the expansion, we use an expression for the spherical function in terms of a spherical function on a symmetric cone of lower rank.

1. Introduction.

In [**7**], Genkai Zhang gives the asymptotic expansion for the spherical functions on symmetric cones. This is done to prove a central limit theorem for these spaces. The work of Zhang is a natural continuation of the work of Audrey Terras [**6**] (the case of the positive definite matrices of rank 2) and of the work of Donald St.P. Richards [**3**] (the case of the positive definite matrices of all ranks). In each case, the focus is to find the expansion of the spherical function $h_\lambda(e^H)$ of order 2 both in H and λ. Zhang uses a generalized binomial expansion to obtain the first order terms of the expansion, and a recursion formula for the product of spherical polynomials from [**8**] in order to obtain the second order terms of the expansion.

We should point out the work of Piotr Graczyk in [**1**, **2**] who also investigates the central limit theorem and the expansion of the spherical functions on symmetric matrices and, in particular, on the space of positive definite matrices.

In this paper, we prove the same result using a recurrence formula for the spherical functions on symmetric cones we obtained in [**4**]. The interest of this approach is its straightforwardness and the possibilities it opens, as a new method, for other symmetric spaces. In particular, we do not require a product formula to obtain the second order terms in the expansion.

In Section 2, we recall the nature of the problem and some of the notation of [**7**]. In Section 3, we recall our result of [**4**] and explain how it can be used to compute the expansion of the spherical functions. Finally, in Section 4, we find recurrence relations which describe the coefficients of our expansion. Solving these recurrence relations is straightforward.

2. The asymptotic expansion.

We have followed the notation of [**7**] as closely as possible.

In what follows, given a maximal orthogonal system of idempotents in the underlying Jordan algebra, $H = \sum_{i=1}^r h_i\, M_{e_i}$ where M_x corresponds to multiplication by x in the Jordan algebra and $\lambda(H) = \sum_{i=1}^r \lambda_i\, h_i$. We know that when H is close to zero then

$$(1)\quad h_\lambda(e^H) = 1 + a(\lambda) \sum_{i=1}^r h_i + b(\lambda) \sum_{i=1}^r h_i^2 + P(\lambda) \sum_{i<j} h_i\, h_j + O(\|h\|^3)\,.$$

In order to prove the central limit theorem for symmetric cones, we need to compute $a(\lambda)$ and $b(\lambda)$. We note that the terms $a(\lambda)$, $b(\lambda)$ and $P(\lambda)$ are symmetric in λ. For convenience, we will write $f(h) \sim g(h)$ whenever $f(h) = g(h) + O(\|h\|^3)$ for H close to 0.

3. A recursion formula.

Taking into account the different notation, we show in [**5**, Theorem 5.3] that

$$(2)\quad h_\lambda(e^H) = e^{(\lambda_r + a\,(r-1)/4)\,\sum_{i=1}^r h_i}\, \frac{\Gamma(r\,a/2)}{(\Gamma(a/2))^r} \int_\sigma h_{\hat\lambda}(e^\xi)\,(\beta_1 \cdots \beta_r)^{a/2-1}\, d\beta$$

where a is the common multiplicity of the roots, $\sigma = \Big\{t = (\beta_1, \ldots, \beta_r) \colon 0 \le \beta_i \le 1, \sum_{i=1}^r \beta_i = 1\Big\}$, the ξ_i's, $1 \le i \le r-1$, are determined (modulo their order) by the relation

$$(3)\qquad \sum_{i_1<\cdots<i_k} e^{\xi_{i_1}+\cdots+\xi_{i_k}} = \sum_{p=1}^r \beta_p \sum_{\substack{i_1<\cdots<i_k\\ i_k\neq p}} e^{h_{i_1}+\cdots+h_{i_k}}$$

for $1 \le k \le r-1$ and $\hat\lambda(\xi) = \sum_{i=1}^{r-1} (\lambda_i - \lambda_r - a\,r/4)\,\xi_i$. It is understood that when $r=1$, $h_\lambda(e^H) = e^{\lambda_1\, h_1}$ and therefore that $a(\lambda) = a(\lambda_1) = \lambda_1$ and $b(\lambda) = b(\lambda_1) = \lambda_1^2/2$.

Note that computing Harish-Chandra c-function using this formulation is straightforward.

In view of (2), it is natural to point out that (1) is equivalent to

$$(4)\qquad \begin{aligned} h_\lambda(e^H) \sim 1 &+ (2\,b(\lambda) - P(\lambda)) \sum_{i=1}^r (e^{h_i} - 1) \\ &+ (b(\lambda) - a(\lambda)/2) \left(\sum_{i=1}^r (e^{h_i}-1)\right)^2 \\ &+ (a(\lambda) - 2\,b(\lambda) + P(\lambda))\, \left(e^{\sum_{i=1}^r h_i} - 1\right). \end{aligned}$$

Note that the relation $e^x - 1 = x + x^2/2 + O(x^3)$ (when x is close to 0) will be used repeatedly.

Remark 3.1. It is clear that any Taylor expansion of $h_\lambda(e^H)$ which would be in terms of symmetric polynomials of the h_i's can be expressed as an expansion of the elementary symmetric polynomials of the e^{h_i}'s as in (4).

Before we use (4) with $\hat{\lambda}$ and ξ, we must make sure that the integration in (2) preserves the relation $\sim$. It suffices to refer to [**4**, Corollary 2.9] to see that the relation (3) implies that the ξ_i's are squeezed between h_i's and hence that $O(\|\xi\|^3) = O(\|h\|^3)$.

The following result, whose proof is straightforward, will allow us to make the necessary computations.

Lemma 3.2. *For any integers $k_i \geq 0$, $1 \leq i \leq r$, we have*

$$\frac{\Gamma(r\,a/2)}{(\Gamma(a/2))^r} \int_\sigma \Pi_{i=1}^r {\beta_i}^{k_i} (\beta_1 \cdots \beta_r)^{a/2-1}\, d\beta = \frac{\Pi_{i=1}^r (a/2)_{k_i}}{(a\,r/2)_{\sum_{i=1}^r k_i}}$$

where $(\alpha)_k = \prod_{i=1}^k (\alpha + i - 1)$ (the empty product is equal to 1*).*

4. Computing the expansion.

The idea is simple: We are to use Formula (2) with $h_{\hat{\lambda}}(e^\xi)$ replaced by its expansion (such as the one in (4)) and then use (3) and Lemma 3.2 to do the rest.

Once we use (4) with $\hat{\lambda}$ and ξ, the integrand in (2) becomes

$$\begin{aligned}
h_\lambda(e^\xi) &\sim 1 + (2\,b(\hat{\lambda}) - P(\hat{\lambda})) \sum_{p=1}^r \beta_p \sum_{i\neq p} (e^{h_i} - 1) \\
&\quad + (b(\hat{\lambda}) - a(\hat{\lambda})/2) \left(\sum_{p=1}^r \beta_p \sum_{i \neq p} (e^{h_i} - 1) \right)^2 \\
&\quad + (a(\hat{\lambda}) - 2\,b(\hat{\lambda}) + P(\hat{\lambda})) \sum_{p=1}^r \beta_p \left(e^{\sum_{i\neq p} h_i} - 1 \right) \\
&\sim 1 + (2\,b(\hat{\lambda}) - P(\hat{\lambda})) \sum_{p=1}^r \beta_p \left(\sum_{i\neq p} h_i + \sum_{i \neq p} h_i^2/2 \right) \\
&\quad + (b(\hat{\lambda}) - a(\hat{\lambda})/2) \left(\sum_{p=1}^r \left(\sum_{i\neq p} \beta_i \right)^2 h_p^2 \right. \\
&\quad \left. +2 \sum_{p<q} \left(\sum_{i \neq p} \beta_i \right) \left(\sum_{j\neq q} \beta_j \right) h_p\, h_q \right)
\end{aligned}$$

$$+ (a(\hat\lambda) - 2\,b(\hat\lambda)$$
$$+ P(\hat\lambda)) \sum_{p=1}^r \beta_p \left(\sum_{i\neq p} h_i + \sum_{i\neq p} h_i^2/2 + \sum_{i<j, i\neq p, j\neq p} h_i\,h_j \right)$$

for $t = (\beta_1, \ldots, \beta_r) \in \sigma$, and by taking (3) into account. With Lemma 3.2, integrating this last expression is straightforward. It is important to remember that $\sum_{p=1}^r \beta_p = 1$. From (2),

$$\begin{aligned} h_\lambda(e^H) \sim &\left(1 + (\lambda_r + a\,(r-1)/4) \sum_{i=1}^r h_i + (\lambda_r + a\,(r-1)/4)^2 \sum_{i=1}^r h_i^2/2 \right. \\ &\left. + (\lambda_r + a\,(r-1)/4)^2 \sum_{i<j} h_i\,h_j \right) \\ &\cdot \left[1 + (2\,b(\hat\lambda) - P(\hat\lambda)) \sum_{p=1}^r \frac{1}{r} \left(\sum_{i\neq p} h_i + \sum_{i\neq p} h_i^2/2 \right) \right. \\ &+ (b(\hat\lambda) - a(\hat\lambda)/2) \left(\frac{(r-1)\,(a\,(r-1)+2)}{r\,(a\,r+2)} \sum_{p=1}^r h_p^2 \right. \\ &\quad \left. +2\,\frac{a\,(r-1)^2 + 2\,(r-2)}{r\,(a\,r+2)} \sum_{p<q} h_p\,h_q \right) \\ &\left. + (a(\hat\lambda) - 2\,b(\hat\lambda) + P(\hat\lambda)) \sum_{p=1}^r \frac{1}{r} \left(\sum_{i\neq p} (h_i + h_i^2/2) + \sum_{i<j, i\neq p, j\neq p} h_i\,h_j \right) \right] \\ \sim\; & 1 + a(\lambda) \sum_{i=1}^r h_i + b(\lambda) \sum_{i=1}^r h_i^2 + P(\lambda) \sum_{i<j} h_i\,h_j . \end{aligned}$$

This gives us

$$a(\lambda) = \frac{r-1}{r}\,a(\hat\lambda) + \lambda_r + a\,(r-1)/4, \tag{5}$$

$$\begin{aligned} b(\lambda) = &\left[\frac{a/2}{a\,r+2} + \lambda_r + a\,(r-1)/4 \right] \frac{r-1}{r}\,a(\hat\lambda) \\ &+ b(\hat\lambda)\,\frac{(r-1)\,(a\,(r-1)+2)}{r\,(a\,r+2)} + \frac{(\lambda_r + a\,(r-1)/4)^2}{2}, \end{aligned} \tag{6}$$

$$\begin{aligned} P(\lambda) = &\frac{r-2}{r}\,P(\hat\lambda) + (\lambda_r + a\,(r-1)/4)^2 + (2\,b(\hat\lambda) - a(\hat\lambda))\,\frac{a}{r\,(a\,r+2)} \\ &+ 2\,(\lambda_r + a\,(r-1)/4)\,\frac{r-1}{r}\,a(\hat\lambda), \end{aligned}$$

which leads us to the main result of the paper.

Theorem 4.1. *When* $r = 1$, $a(\lambda) = a(\lambda_1) = \lambda_1$ *and* $b(\lambda) = b(\lambda_1) = \lambda_1^2/2$ *while* $P(\lambda) = 0$. *For* $r \geq 2$,

$$a(\lambda) = \frac{1}{r} \sum_{i=1}^{r} \lambda_i,$$

$$b(\lambda) = \frac{2+a}{2\,r\,(2+a\,r)} \sum_{i=1}^{r} \lambda_i^2 + \frac{a}{r\,(2+a\,r)} \sum_{i<j} \lambda_i\,\lambda_j - \frac{a^2\,(r^2-1)}{48\,(2+a\,r)},$$

$$P(\lambda) = \frac{a}{r\,(2+a\,r)} \sum_{i=1}^{r} \lambda_i^2 + \frac{2\,(a\,(r-1)+2)}{r\,(r-1)\,(2+a\,r)} \sum_{i<j} \lambda_i\,\lambda_j + \frac{a^2\,(r+1)}{24\,(a\,r+2)}.$$

Proof. Using induction and (5), we find easily that $a(\lambda) = \left(\sum_{i=1}^{r} \lambda_i\right)/r$. From (6), we clearly have $b(\lambda) = \alpha_r + \beta_r \sum_{i=1}^{r} \lambda_i + \delta_r \sum_{i=1}^{r} \lambda_i^2 + \gamma_r \sum_{i<j} \lambda_i\,\lambda_j$ with $\alpha_1 = \beta_1 = \gamma_1 = 0$ and $\delta_1 = 1/2$. By considering the coefficients of λ_1^2, λ_1, $\lambda_1\,\lambda_2$ and the constant term in (6), we derive the following recurrence relations:

$$\delta_r = \frac{(r-1)\,(a\,(r-1)+2)}{r\,(a\,r+2)}\,\delta_{r-1},$$

$$\gamma_r = \frac{(r-1)\,(a\,(r-1)+2)}{r\,(a\,r+2)}\,\gamma_{r-1},$$

$$\beta_r = \left[\frac{a/2}{a\,r+2} + a\,(r-1)/4\right]\frac{1}{r} + \frac{(r-1)\,(a\,(r-1)+2)}{r\,(a\,r+2)}\,[\beta_{r-1} - a\,r\,\delta_{r-1}/2 - (r-2)\,a\,r\,\gamma_{r-1}/4],$$

$$\alpha_r = -\frac{a}{4}\,(r-1)\left[\frac{a/2}{a\,r+2} + a\,(r-1)/4\right] + \frac{(r-1)\,(a\,(r-1)+2)}{r\,(a\,r+2)}\left[\alpha_{r-1} - a\,\frac{r\,(r-1)}{4}\,\beta_{r-1} + a^2\,\frac{(r-1)\,r^2}{16}\,\delta_{r-1} + a^2\,\frac{r^2\,(r-1)\,(r-2)}{32}\,\gamma_{r-1}\right] + a^2\,(r-1)^2/32.$$

All these equations are valid for $r \geq 2$ except the second one which is valid only for $r \geq 3$. One can easily find δ_r and γ_r. This in turn brings us to

$$\beta_r = \frac{(r-1)\,(a\,(r-1)+2)}{r\,(a\,r+2)}\,\beta_{r-1}$$

for $r \geq 2$ which means that $\beta_r = 0$ for all r. Finally, we have

$$\alpha_r = \frac{1}{16}\,\frac{(r-1)\,(a\,(r-1)+2)}{r\,(a\,r+2)}\,\alpha_{r-1} - \frac{1}{16}\,\frac{a^2\,r\,(r-1)}{r\,(a\,r+2)}$$

whose solution is straightforward. We compute $P(\lambda)$ in much the same way. $\square$

Remark 4.2. There is a minor computation error in the last term of [**7**, Page 573] which is why we obtain a different value for α_r and therefore for $b(\lambda)$. The term $P(\lambda)$ is not computed in [**3**, **6**, **7**]. It is given in [**1**] in the case of the real positive definite matrices ($a = 1$) and in [**2**] in the case of the hermitian positive definite matrices ($a = 2$).

Conclusion.

This approach suggests, that once we express a spherical function on a symmetric space in terms of spherical functions on a symmetric space of the same type but lower rank, we become able to compute its expansion. This approach could be applied to the other symmetric spaces which correspond to the classical Lie algebras.

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