EXTENSIONS OF TORI IN SL(2)

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Let $\tilde{SL}(2, F)$ be the metaplectic two-fold cover of $SL(2, F)$, the special linear group in two variables over a local field $F$ of characteristic 0. The inverse image $\tilde{T}$ of a maximal torus $T$ in $SL(2, F)$ is an abelian extension of $T$ by $\pm 1$. We consider the question of whether this extension is trivial. More generally we find the minimal subgroup $A$ of the circle for which the extension is split when considered with coefficients in $A$. We see that $|A| = 2, 4$ or $8$ in the p-adic case. We also find an explicit splitting function for the cocycle.

Introduction

Let $\tilde{SL}(2, F)$ be the metaplectic two-fold cover of $SL(2, F)$, the special linear group in two variables over a local field $F$ of characteristic 0. The inverse image $\tilde{T}$ of a maximal torus $T$ in $SL(2, F)$ is an abelian extension of $T$ by $\pm 1$. We consider the question of whether this extension is trivial. We exclude the case $F = \mathbb{C}$, which is trivial.

More generally suppose $A$ is a subgroup of the circle $\mathbb{T}$ containing $\pm 1$. The inclusion of $\pm 1$ in $A$ induces a map on cohomology, and defines an extension

$$1 \rightarrow A \rightarrow \tilde{T}_A \rightarrow \tilde{T} \rightarrow 1.$$  

We say $A$ is a splitting group for $\tilde{T}$ if the extension $\tilde{T}_A \rightarrow \tilde{T}$ splits. It is well-known that $\mathbb{T}$ is a splitting group. We say a splitting group $A$ is a minimal splitting group if no proper subgroup of $A$ is a splitting group. It is easy to see the order of a minimal splitting group is a power of 2, and hence unique, if it is finite.

Let $(\cdot, \cdot)_F$ be the Hilbert symbol of $F$, and let $\mu_n$ be the $n^{th}$ roots of unity in $\mathbb{C}.$

**Theorem 1.** The minimal splitting group $A_{\min}$ for $T$ is given by:

(a) Suppose $T \simeq F^*$. Then

$$A_{\min} = \begin{cases} \mu_2 & (-1, -1)_F = 1 \\ \mu_4 & (-1, -1)_F = -1. \end{cases}$$
(b) Suppose $T \simeq \mathbb{E}^1$ for $\mathbb{E}$ a quadratic extension of $\mathbb{F}$. Then

$$A_{\text{min}} = \begin{cases} 
\mu_2 & (-1,-1)_{\mathbb{F}} = 1 \\
\mu_4 & (-1,-1)_{\mathbb{F}} = -1, \text{ } \mathbb{F} \text{ non-archimedean}, \text{ } -1 \notin \mathbb{E}^* \\
\mu_8 & (-1,-1)_{\mathbb{F}} = -1, \text{ } \mathbb{F} \text{ non-archimedean}, \text{ } -1 \in \mathbb{E}^* \\
T & \mathbb{F} = \mathbb{R}.
\end{cases}$$

**Remark 2.** It is well-known that $(-1,-1)_{\mathbb{F}} = 1$ unless $\mathbb{F} = \mathbb{R}, \mathbb{Q}_2$, or an extension of $\mathbb{Q}_2$ of odd degree.

Theorem 1 is proved in Sections 3, 4 and 5. Here is an alternative realization of $\tilde{T}$. A character of $\tilde{T}$ is said to be **genuine** if it does not factor to $T$.

**Theorem 3.** Let $\tau(z) = z^2$ ($z \in \mathbb{C}^*$). Let $\tilde{\alpha}$ be a genuine character of $\tilde{T}$. Then $\tilde{\alpha}^2$ factors to a character $\alpha$ of $T$, and $\tilde{T}$ is isomorphic to the pullback of $\tau$ via $\alpha$. In other words $\tilde{T}$ is isomorphic to the $\sqrt{\alpha}$-extension of $G$.

From this we obtain an interpretation of the minimal splitting group of Theorem 1. Let $n(\tilde{T})$ be the minimal order of a genuine character of $\tilde{T}$. Set $\mu_\infty = T$.

**Corollary 4.** The minimal splitting group for $\tilde{T}$ is $\mu_{n(\tilde{T})}$.

For the proofs of Theorem 3 and Corollary 4 see Lemma 1.4.

We also give an explicit splitting of this extension, i.e., a function $\zeta : T \to A_{\text{min}}$ whose coboundary is the cocycle defining $\tilde{T}$ (see §3 and Theorem 5.7).

These questions arise from the theory of the oscillator representation and dual pairs. The splitting plays a role in this context, for example see [11]. The case of $\mathbb{F}^*$ is well-known ([4], p. 42, attributed to J. Klose), as is the existence of a $T$-splitting in general [2]. General results about the splitting of the metaplectic cover over subgroups are due to Kudla [7], and a splitting of the extension of an elliptic torus is found in [7], Proposition 4.8 (in the non-archimedean case it is easy to see this can be taken to be a $\mu_8$-splitting).

This paper grew out of an effort to simplify Kudla's formula. In the case of a $p$-adic field of odd residual characteristic a formula for a $\mu_2$-splitting in some cases may be deduced from [6], cf. ([4], p. 43).

Many of the arguments, especially those of Section 1 apply to other abelian extensions of abelian groups, for example a maximal torus in the two-fold cover of $Sp(2n, \mathbb{F})$. If $\tilde{G}$ is a non-linear $n$-fold cover of the $\mathbb{F}$ points of an algebraic group $G$, then the inverse image $\tilde{T}$ of a maximal torus in $G$ is typically not abelian. However similar arguments apply to the center of $\tilde{T}$.

Throughout $\mathbb{F}$ denotes a local field of characteristic zero, and $(x, y)_{\mathbb{F}} \in \mu_2$ is the Hilbert symbol. For $x \in \mathbb{F}^*$ and $\psi_{\mathbb{F}}$ a non-trivial additive character of
$F$, $\gamma_F(x, \psi_F) \in \mu_4$ is the Weil index. We use basic properties of the Hilbert symbol and the Weil index without further comment, see ([10], Appendix) for details. We make repeated use of the identities
\begin{align*}
(1) \quad \gamma_F(x, \psi_F) \gamma_F(y, \psi_F) &= (x, y)_F \gamma_F(xy, \psi_F) \\
(2) \quad \gamma_F(x, \psi_F)^2 &= (-1, x)_F.
\end{align*}
If $E$ is a quadratic extension of $F$ then
\begin{align*}
(3) \quad (x, z)_E &= (x, Nz)_F \quad (x \in F^*, z \in E^*) \\
(4) \quad (x, y)_E &= 1 \quad (x, y \in F^*) \\
(5) \quad \gamma_E(x, \psi_E) \gamma_E(y, \psi_E) &= \gamma_E(xy, \psi_E) \quad (x, y \in F^*).
\end{align*}

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1. Abstract Groups.

In this section we ignore the topology on $T$ and consider it as an abstract group. We recall some standard facts from group cohomology and establish some notation. For example see [1].

Suppose $G$ is a group, $A$ is an abelian group, and $G$ acts trivially on $A$. The equivalence classes of central extensions of $G$ by $A$ are parametrized by the group cohomology $H^2(G, A)$. Given an extension $p : H \rightarrow G$ let $s : G \rightarrow H$ be a section, i.e., $p \circ s = 1$. The cohomology class of the extension is represented by the 2-cocycle $c_s(g, h) = s(gh)s(h)^{-1}s(g)^{-1}$. When there is no danger of confusion we do not distinguish between $c_s$ and its image $c_s$ in $H^2(G, A)$. Any other such splitting $s'$ is given by $s'(g) = s(g)\zeta(g)$ for some map $\zeta : G \rightarrow A$, and then $c_{s'}(g, h) = c(g, h)\zeta(gh)\zeta(h)^{-1}\zeta(g)^{-1}$. Thus $c_{s'} = c_s d\zeta$, and $\tau_s = \tau_{s'}$.

Conversely given a cocycle $c$ we define $H$ to be equal to $G \times A$ as a set, with multiplication $(g, a)(g', a') = (gg, aa'c(g, g'))$. The cocycle $c$ is trivial in cohomology if and only if
\begin{equation}
(6) \quad c(g, h) = \zeta(g)\zeta(h)\zeta(gh)^{-1}
\end{equation}
for some $\zeta$, i.e., $d\zeta = c$. We say $\zeta$ is a splitting of the cocycle. Equivalently the splitting map $s(g) = (g, \zeta^{-1}(g))$ is a homomorphism. Any other splitting is then of the form $\zeta' = \zeta\alpha$ with $\alpha : G \rightarrow A$ a homomorphism.

Suppose $A = \mu_2$, with cocycle $c$, and $A \subset \mu_{ab}$ with $b$ odd. If $\zeta : G \rightarrow \mu_{ab}$ is a splitting of $c$, then $\zeta^b$ is a $\mu_a$ splitting. Therefore we will restrict consideration to $\mu_n$ with $n$ a power of 2.

Now suppose $G$ is abelian. The universal coefficient theorem for group cohomology gives an exact sequence:
\[1 \rightarrow \text{Ext}(G, A) \rightarrow H^2(G, A) \xrightarrow{\phi} \text{Hom}(\Lambda^2 G, A) \rightarrow 1.\]
Here $G$ and $A$ are considered as $\mathbb{Z}$-modules, $\text{Hom} = \text{Hom}_{\mathbb{Z}}$, $\text{Ext} = \text{Ext}_{\mathbb{Z}}$, and $\text{Hom}(\Lambda^2 G, A)$ consists of alternating, bilinear maps $G \times G \to A$.

If $c$ is a 2-cocycle, representing the class $\overline{c} \in H^2(G, A)$, then $\phi(\overline{c})(g, h) = c(g, h)c(h, g)^{-1}$. In terms of the group, suppose $p : H \to G$ is the corresponding extension. For $g, h \in G$ and any section $s$ let $\{g, h\}$ be the commutator $s(g)s(h)s(g)^{-1}s(h)^{-1}$. This is contained in $A$, is independent of the choice of $s$, and $\phi(\overline{c})(g, h) = \{g, h\}$. In particular $\phi(\overline{c}) = 1$ if and only if $H$ is abelian, so $\text{Ext}(G, A) \subset H^2(G, A)$ parametrizes the abelian extensions of $G$ by $A$.

Let $G^n = \{g^n \mid g \in G\}$ and $nG = \{g \in G \mid g^n = 1\}$. The next result is presumably well-known to the experts.

**Lemma 1.1.** For any positive integer $n$, inclusion $\iota : nG \hookrightarrow G$ induces an isomorphism:

$$\text{Ext}(G, \mu_n) \cong \text{Ext}(nG, \mu_n).$$

**Proof.** Consider the maps

$$G \xrightarrow{\alpha} G^n \xrightarrow{\beta} G$$

where $\alpha(g) = g^n$ and $\beta$ is inclusion. The induced map $\alpha^*\beta^* : \text{Ext}(G, \mu_n) \to \text{Ext}(G, \mu_n)$ is induced by the $n$th power map $g \to g^n$ on $G$. This is the same map as that induced by the $n$th power map on $\mu_n$, and therefore $\alpha^*\beta^* = 0$.

Now the long exact cohomology sequence corresponding to $0 \to G^n \xrightarrow{\beta} G \to G/G^n \to 0$ has final two terms $\text{Ext}(G, A) \xrightarrow{\beta^*} \text{Ext}(G^n, A) \to 0$. Therefore $\beta^*$ is surjective, which implies $\alpha^* = 0$. On the other hand the short exact sequence

$$0 \to nG \xrightarrow{\iota} G \xrightarrow{\alpha} G^n \to 0$$

gives rise to the long exact sequence

$$0 \to \text{Hom}(G^n, A) \to \text{Hom}(G, A) \to \text{Hom}(nG, A) \to \text{Ext}(G^n, A) \xrightarrow{\alpha^*} \text{Ext}(G, A) \xrightarrow{\iota^*} \text{Ext}(nG, A) \to 0.$$

Since $\alpha^* = 0$, $\iota^*$ is an isomorphism. \hfill $\square$

**Remark 1.2.** In our setting $2T = \pm 1$. For the $\mu_2$ extension $\tilde{T}$ to split it is necessary that it splits over $\pm 1$. Perhaps surprisingly the converse holds as well by the Lemma.

For later use we note an explicit formula for a splitting of $\alpha^*\beta^*c$. We drop the assumption that $H$ is abelian, so let $p : H \to G$ be an extension, with section $s$ and corresponding cocycle $c$.

**Lemma 1.3.** Let

$$\tau(g) = s(g^n)s(g)^{-n} = c(g, g)^{-1}c(g, g^2)^{-1}\ldots c(g, g^{n-1})^{-1} \in A.$$
Then
\begin{equation}
(8) \quad c(g^n, h^n) = \tau(g)\tau(h)\tau(gh)^{-1}\{g, h\}^{n(n-1)/2}.
\end{equation}

If $H$ is abelian then $d\tau = \alpha^* \beta^* c$.

Note that $\{g, h\}^{n(n-1)/2} = \pm 1$, and is identically 1 if $n$ is odd. Compare ([3], p. 130) and ([5], §4).

**Proof.** This follows from the identity
\[
[s(g)s(h)]^n = s(g)^ns(h)^n\{h, g\}^{n(n-1)/2}.
\]
Using $s(g)s(h) = s(gh)c(g, h)$ and $s(g)^n = s(g^n)\tau(g)^{-1}$, the left hand side is equal to
\[
s(gh)^n = s(g^n h^n)\tau(gh)^{-1}.
\]
The right hand side is
\[
\begin{align*}
(s(g^n)s(h^n))\tau(h)^{-1}\{h, g\}^{n(n-1)/2} \\
= s(g^n h^n)c(g^n, h^n)\tau(h)^{-1}\{h, g\}^{n(n-1)/2}
\end{align*}
\]
and the first assertion follows. Since $\alpha^* \beta^* c(g, h) = c(g^n, h^n)$, the second assertion is equivalent to
\begin{equation}
(9) \quad c(g^n, h^n) = \tau(g)\tau(h)\tau(gh)^{-1}
\end{equation}
which is (8) for $H$ abelian.

For $H$ an abelian extension (9) implies $\tau$ is a character when restricted to $nG$. In terms of the exact sequence (7), $c \in \text{Ext}(G, A)$, $\beta^* c \in \text{Ext}(G^n, A)$, $\alpha^* \beta^* c = 0$, and $\beta^* c$ is the image of $\tau \in \text{Hom}(nG, A)$. Thus $\beta^* c = 0$ if $\tau$ extends to an element of $\text{Hom}(G, A)$.

More generally, we try to find a splitting subgroup $A$ of $\beta^* c$, i.e., $c$ restricted to $G^n$, together with an explicit formula. Note that (9) does not necessarily define such a splitting since the function $g^n \to \tau(g)$ is not necessarily well-defined. Let $\alpha$ be a character of $G$ whose restriction to $nG$ is equal to $\tau^{-1}$. Then $\zeta_\alpha(g^n) := \tau(g)\alpha(g)$ is well-defined, and $d\zeta_\alpha = c$. The minimal splitting subgroup for $\beta^* c$ is thus the minimal subgroup $A$ of $\mathbb{T}$, containing $\mu_n$, such that $\tau$ restricted to $nG$ can be extended to a character of $G$ with values in $A$.

**Characters and the $\sqrt{\alpha}$ extension.**

For the remainder of this section let $p : \tilde{G} \to G$ be a $\mu_2$ extension of a group $G$. We do not assume that $G$ or $\tilde{G}$ is abelian. If $\alpha$ is a character of $G$, and $\tau(z) = z^2$ ($z \in \mathbb{C}^*$) then the pullback of $\tau$ via $\alpha$ is a $\mu_2$ extension of $G$, and may be realized as the subgroup of $G \times \mathbb{C}^*$ given by $\{(g, z) \mid \alpha(g) = \tau(z)\}$. Projection on the second factor is a genuine character $\tilde{\alpha}$ of $\tilde{G}$ satisfying $\tilde{\alpha}^2 = \alpha \circ p$. This is sometimes denoted the $\sqrt{\alpha}$-extension of $G$. It may or may not be the trivial extension.
We see that $\widetilde{G}$ has a $T$-splitting if and only if there is a genuine character of $\widetilde{G}$. More precisely:

**Lemma 1.4.** Suppose there is a genuine character $\tilde{\alpha}$ of $\widetilde{G}$. Then $\tilde{\alpha}^2$ factors to a character $\alpha$ of $G$, and $\widetilde{G}$ is isomorphic to the $\sqrt{\alpha}$ extension of $G$. If $\text{Image}(\tilde{\alpha}) \subset A \subset T$ then $G_A \simeq G \times A$. The minimal splitting group for $G$ is $\mu_{n(\widetilde{T})}$ where $n(\widetilde{G})$ is the minimal order of a genuine character of $\widetilde{G}$.

Conversely if $G_A \simeq G \times A$ then there is a genuine character of $\widetilde{G}$ with values in $A$.

Note that there exists a genuine character $\tilde{\alpha}$ of $\widetilde{G}$ if and only if $z \not\in [\widetilde{G}, \widetilde{G}]$ where $z$ is the non-trivial element in the inverse image of $1$. In particular this holds if $\widetilde{G}$ is abelian, which proves the existence of a $T$-splitting (cf. [2]).

**Proof.** The map $\phi : g \rightarrow (p(g), \tilde{\alpha}(g)) \subset G \times T$ is an isomorphism of $\widetilde{G}$ with the pullback of $\tau$ via $\alpha$. This is a subgroup of $G \times A$, and $\phi$ extends to an isomorphism of $\widetilde{G}_A$ with $G \times A$. The final two assertions are immediate. □

**Remark 1.5.** In the setting of the Lemma, suppose $\zeta$ is a $T$-splitting of the cocycle defining $\widetilde{G}$ (with respect to a section $s$). Then $\alpha := \zeta^2$ is a character of $G$, and $\widetilde{G}$ is isomorphic to the $\sqrt{\alpha}$ cover of $G$.

Theorem 3 and Corollary 4 are immediate consequences of the Lemma. Theorem 1 also follows from the Lemma, from a computation of $n = n(\tilde{T})$: $n$ is the minimal power of 2 such that $z \not\in \tilde{T}^n$. We follow a different approach, by giving explicit formulas for the minimal splitting $\zeta$ in Sections 3-5.

2. Moore cohomology and $\widetilde{SL}(2, \mathbb{F})$.

Now suppose $G$ and $A$ are locally compact topological groups, $A$ is abelian, and $G$ acts continuously on $A$. In our applications $A$ will either be $\mu_n$ or $T$, with trivial $G$ action. C. Moore has defined cohomology groups $H^2_{\text{top}}(G, A)$ using measurable cochains [8]. In the case of a totally disconnected group it is equivalent to use continuous cochains. Viewing $G$ and $A$ as abstract groups, there is a natural homomorphism $H^2_{\text{top}}(G, A) \rightarrow H^2(G, A)$. In general it is neither surjective nor injective.

Now let $G = GL(2, \mathbb{F})$. We recall the definition of the standard cocycle on $G$ [6], cf. ([4], p. 41). For $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ let $x(g) = c$ (resp. $d$) if $c \neq 0$ (resp. $c = 0$). Then

$$c(g, h) = (x(g)x(gh), x(h)x(gh))_F(\det(g), x(g)x(gh))_F.$$
This defines a $\mu_2$-extension $\tilde{GL}(2, \mathbb{F})$ of $GL(2, \mathbb{F})$, with distinguished section $s$. We write $GL(2, \mathbb{F})$ in cocycle notation as usual. The restriction to $SL(2, \mathbb{F})$ is isomorphic to $SL(2, \mathbb{F})$.

Up to conjugation $GL(2, \mathbb{F})$ contains one hyperbolic torus isomorphic to $\mathbb{F}^* \times \mathbb{F}^*$, and for each quadratic extension $E$ of $\mathbb{F}$ one elliptic torus isomorphic to $E^*$. The commutator of two elements $z, w$ of $E^*$ is given by ([3], p. 128)

\begin{equation}
\{z, w\} = (z, \overline{w})_F
= (z, w)_{\mathbb{E}}(Nz, Nw)_F.
\end{equation}

Here and elsewhere we suppress the map $\iota : \mathbb{E}^* \hookrightarrow GL(2, \mathbb{F})$ from the notation, and write $\{z, w\} := \{\iota(z), \iota(w)\}$. The commutator is trivial when restricted to $E^1 = E^* \cap SL(2, \mathbb{F})$.

3. The hyperbolic torus.

Let $\iota : \mathbb{F}^* \hookrightarrow SL(2, \mathbb{F})$, so $T = \iota(\mathbb{F}^*)$ is a hyperbolic torus. After conjugation we may assume $\iota(x) = \text{diag}(x, x^{-1})$. We drop $\iota$ from the notation and identify $x$ with $\iota(x)$. We prove Theorem 1 in this case, together with a formula for the minimal splitting. While these results are well-known they are not easy to find in the literature, and the calculation illustrates some of the ideas in the next section.

The restriction of $SL(2, \mathbb{F})$ to $T$ defines a $\mu_2$ extension $\tilde{T}$ of $T$ with cocycle $c(x, y) = (x, y)_F$.

In particular $(-I, e)^2 = (I, (-1, -1)_F)$, so the restriction of the extension to $\pm I$ is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ if $(-1, -1)_F = 1$, or $\mathbb{Z}/4\mathbb{Z}$ if $(-1, -1)_F = -1$. By Lemma 1.1 the minimal splitting group for $\tilde{T}$, considered as an abstract group, is $\mu_2$ (resp. $\mu_4$) if $(-1, -1)_F = 1$ (resp. $-1$). It remains to show this splitting is measurable. We do this by computing it explicitly.

Fix $\psi$ and write $\gamma(x) = \gamma_F(x, \psi)$. The key point is that properties (1) and (2) of the Weil index shows that $d\gamma = c$ and $\gamma^4 = 1$, so $\gamma$ is a measurable $\mu_4$ splitting of $c$. This completes the proof in case $(-1, -1)_F = -1$, so assume $(-1, -1)_F = 1$. Let $\alpha$ be a character of $\mathbb{F}^*$ satisfying $\alpha(x^2) = (-1, x)_F$. To see that such a character exists, define $\alpha$ restricted to $\mathbb{F}^{\ast 2}$ by this formula; it is well-defined since $\alpha((-x)^2) = (-1, -x)_F = (-1, -1)_F\alpha(x^2) = \alpha(x^2)$. Extend arbitrarily from $\mathbb{F}^{\ast 2}$ to $\mathbb{F}^*$. By (2) $\alpha(x)^2\gamma(x)^2 = (-1, x)_F^2 = 1$. Let $\zeta_\alpha = \gamma\alpha$, i.e.,

$$
\zeta_\alpha(x) = \gamma(x, \psi_F)\alpha(x).
$$

Then $d\zeta_\alpha = d\gamma = c$ and $\zeta_\alpha$ is a $\mu_2$-splitting of $c$.

Choose representatives $a_1, a_2, \ldots, a_m \in \mathbb{F}^*$ of generators of $\mathbb{F}^*/\mathbb{F}^{\ast 2} \simeq (\mathbb{Z}/2\mathbb{Z})^m$. (By [4], Lemma 0.3.2, $2^m = 4/|2|_F$.) Given any choice of signs
ε_i we may choose α so that ζ(α_i) = ε_i, and α(x^2) = (−1, x)_F for all x ∈ F. Then ζ extends uniquely to a splitting.

For example, if −1 ∈ F^*^2 then we may take α = 1 and ζ(x) = γ_F(x, ω_F) = ±1. On the other hand, suppose −1 ∉ F^*^2 and the residual characteristic of F is odd. We may take representatives ±1, ±ω for F^*/F^*^2 (ω is a uniformizing parameter) and then choose ζ satisfying:

\[ ζ(±x^2) = (−1, x)_F \]
\[ ζ(±ωx^2) = ±(−1, x)_F. \]

4. Elliptic tori.

Let T be an elliptic torus of SL(2, F) as in §2. Thus E is a quadratic extension of F, \( \iota : E^{1} \hookrightarrow SL(2, F) \) is an embedding, and \( T = \iota(E^{1}) \). As in §3 we fix \( \iota \) and drop it from the notation.

As in the case of the hyperbolic torus, \((-I, 1, 1)\) has order 2 or 4 depending on whether \((-1, -1)_F = +1 \) or \(-1 \). By Lemma 1.1 this proves \( μ_2 \) is a splitting group if and only if \((-1, -1)_F = 1 \), so assume \((-1, -1)_F = -1 \).

Let \( F = \mathbb{R}, E = \mathbb{C} \). Since \( T = T^2 \), it is enough to find a splitting of \( β^*c \) as in §2. Choose a character \( α \) of \( E^{1} \) such that \( α(-1) = -1 \), i.e., \( α(z) = z^n \) for \( n \) odd. It is easy to see \( c(-z, -z) = -c(z, z) \), and the discussion in §2 shows that

\[ ζ_α(z^2) := c(z, z)α(z) \]

is a well-defined (measurable) \( T \)-splitting. Since \( ζ_α \) is surjective onto \( T \) for any \( α \), this shows that there is no \( A \)-splitting for any proper subgroup \( A \) of \( T \).

We now assume \( F \) is non-archimedean. Since \((-1, -1)_F = -1 \), \( F \) is an extension of \( Q_2 \) of odd degree, and \(-1 ∉ F^*^2 \). If \(-1 ∉ E^*^2 \) then \( 4T = \{ ±1 \} \).

Since \((-I, 1)^4 = (I, ±ζ^2)^4 = I \) for all \( ζ ∈ μ_4 \), the \( μ_4 \)-extension splits over \( 4T \).

Suppose \(-1 = δ^2 \) (\( δ ∈ E^* \)). We claim \( 4T = 8T = \{ ±1, ±δ \} \). It is enough to show \( E^* \) does not contain a primitive eighth root of unity, or equivalently \( δ ∉ E^*^2 \). Since \( F \) is an extension of \( Q_2 \) of odd degree, \( 2 ∉ E^*^2 \). But then \( (a + bδ)^2 = δ \) implies \( a^2 = ±\frac{1}{2} \), which is a contradiction.

For any \( ζ ∈ μ_8 \) we compute \( δ^4 = (−I, ±ζ^2)^4 = (I, −ζ^4)^2 = I \), which implies the extension splits over \( 8T \).

Therefore \( μ_4 \) (resp. \( μ_8 \)) is a minimal splitting group for \( T \) if \( F \) is non-archimedean, \((-1, -1)_F = -1 \), and \(-1 ∉ E^*^2 \) (resp. \(-1 ∈ E^*^2 \)). It remains to show these splittings can be chosen to be measurable. In the next section we give explicit such splittings.

Remark 4.1. We have shown the cohomology class \( τ ∈ H^2_{top}(G, A) \) has image 0 in \( H^2(G, A) \) in the given cases. An argument due to Jonathan
Rosenberg shows that the map $H^2_{\text{top}}(G, A) \rightarrow H^2(G, A)$ is injective in this situation. Since we are interested in explicit formulas for the splittings in any case, we do not pursue this approach. For $G$ perfect (not at all the case here!) the injectivity of $\phi$ is known ([9], Theorem 2.3).

5. Explicit splittings for elliptic tori.

We continue with the notation of the previous section. The embedding $\iota : E^1 \hookrightarrow SL(2, \mathbb{F})$ extends to an embedding $\iota : E^* \hookrightarrow GL(2, \mathbb{F})$. We will make use of the non-abelian $\mu_2$ extension of $E^*$ obtained by restricting the extension $GL(2, \mathbb{F})$ of $GL(2, \mathbb{F})$.

We proceed as follows. Since $\{z, w\} = (z, w)_G$, the commutator is trivial on $E^*$. By the method of §1 we find a splitting of this extension. As in §3 we also find a splitting of the extension of $F^* \subset E^*$. The extension of $E^* \otimes F^*$ is abelian, and by an explicit version of the Mayer-Vietoris sequence we obtain a splitting of this extension. Finally $E^1$ is contained in $E^* \otimes F^*$, and we restrict to obtain a splitting of the extension of $E^1$.

Fix non-trivial additive characters $\psi_F$ of $F$ and $\psi_E$ of $E$. Recall (5) the restriction of $\gamma_E(\cdot, \psi_E)$ to $F^*$ is a quadratic character.

Lemma 5.1. Suppose $\lambda$ (respectively $\mu$) is a splitting of the cocycle restricted to $E^* \otimes F^*$ (respectively $F^*$). Assume $\lambda(x) = \mu(x)$ for $x \in E^* \otimes F^*$. For $z \in E^*, x \in F^*$ let $\zeta_{\lambda, \mu}(z^2 x) := \lambda(z^2)\mu(x)c(x, z^2)$. Then $\zeta_{\lambda, \mu}$ is a well-defined splitting of the cocycle restricted to $E^* \otimes F^*$.

Conversely if $\zeta$ is any splitting of the cocycle restricted to $E^* \otimes F^*$ then $\zeta = \zeta_{\lambda, \mu}$ with $\lambda = \zeta|_{E^*}$ and $\mu = \zeta|_{F^*}$.

Proof. Let $\zeta$ be any splitting of the cocycle. Then $\zeta(z^2 x) = \zeta(z^2)\zeta(x)c(x, z^2)$ by (6) and the second assertion is immediate.

Given $\lambda$ and $\mu$, choose any splitting $\zeta$. Then $\lambda$ and $\zeta|_{E^*}$ both define splittings, so $\lambda(z^2) = \zeta(z^2)\alpha(z^2)$ for some character $\alpha$ of $E^*$. Similarly $\mu(x) = \zeta(x)\beta(x)$ for some character $\beta$ of $F^*$. Let $\tau$ be a character of $E^*$ extending $\alpha$ and $\beta$; this exists since, for $x \in E^* \otimes F^*$, $\lambda(x) = \mu(x)$ implies $\alpha(x) = \beta(x)$. Then

$$\lambda(z^2)\mu(x)c(x, z^2) = \zeta(z^2)\alpha(z^2)\zeta(x)\beta(x)c(x, z^2) = \zeta(z^2 x)\alpha(z^2)\beta(x) = \zeta(z^2 x)\tau(z^2 x).$$

This shows that $\zeta_{\lambda, \mu}$ is well-defined and is a splitting of the cocycle. \qed

Lemma 5.2. (1) Choose a character $\alpha$ of $E^*$ satisfying

$$\alpha(-1) = (-1, -1)_{F\gamma_E}(-1, \psi_E)$$

and let

$$\lambda_{\alpha}(z^2) = c(z, z)\gamma_E(z, \psi_E)\gamma_F(Nz, \psi_F)\alpha(z).$$
Thus by (8) and (11),

\[
\text{Proof.}
\]

Part (1) is an extension of (8) to the case of a non-abelian group. Furthermore every splitting of the cocycle restricted to \( \mathbb{E}^2 \) is equal to \( \lambda_\alpha \) for some \( \alpha \) satisfying (13).

(2) Let \( \beta \) be a character of \( \mathbb{F}^* \) and let

\[
\mu_\beta(x) = \gamma_\beta(x, \psi_\beta)\beta(x).
\]

Then \( \mu_\beta \) is a splitting of the cocycle restricted to \( \mathbb{F}^* \), and every splitting of the cocycle restricted to \( \mathbb{F}^* \) is equal to \( \mu_\beta \) for some \( \beta \).

(3) Suppose \( \alpha \in \mathbb{E}^*, \beta \in \mathbb{F}^* \) satisfy

\[
(14) \quad \alpha(z) = \gamma_\alpha(z^2, \psi_\alpha)\gamma_\alpha(Nz, \psi_\alpha)\gamma_\alpha(z, \psi_\alpha)c(z, z)\beta(z^2) \quad (z^2 \in \mathbb{F}^*).
\]

In particular \( \alpha \) satisfies (13). For \( z \in \mathbb{E}^*, x \in \mathbb{F}^* \) define

\[
(15) \quad \zeta_{\alpha, \beta}(z^2x) := \lambda_\alpha(z^2)\mu_\beta(x)c(x, z^2) = \gamma_\alpha(z, \psi_\alpha)\gamma_\alpha(Nz, \psi_\alpha)\gamma_\alpha(x, \psi_\alpha)\alpha(z)\beta(x)c(z, z)c(x, z^2).
\]

Then \( \zeta \) is a well-defined splitting of the cocycle restricted to \( \mathbb{E}^2 \mathbb{F}^* \). Furthermore every splitting of the cocycle restricted to \( \mathbb{E}^2 \mathbb{F}^* \) is equal to \( \zeta_{\alpha, \beta} \) for some \( \alpha, \beta \) satisfying (14).

**Proof.** Part (1) is an extension of (8) to the case of a non-abelian group. Thus by (8) and (11),

\[
c(z^2, w^2) = c(z, z)c(w, w)c(zw, zw)\{z, w\}
= c(z, z)c(w, w)c(zw, zw)(Nz, Nw)_E.
\]

Replacing \((z, w)_E\) by \(\gamma_E(z, \psi_E)\gamma_E(w, \psi_E)\gamma_E(wz, \psi_E)^{-1}\), and similarly \((Nz, Nw)_E\) gives

\[
c(z^2, w^2) = \tau(z)\tau(w)\tau(zw)^{-1}
\]

with \(\tau(z) = c(z, z)\gamma_E(z, \psi_E)\gamma_E(Nz, \psi_E)\). The same relation holds with \(\tau(z)\) replaced by \(\tau(z)\alpha(z)\).

We check the condition that \(\lambda_\alpha(z^2) := \tau(z)\alpha(z)\) be well-defined:

\[
\tau(-z)\alpha(-z)
= c(-z, -z)\gamma_E(-z, \psi_E)\gamma_E(N(-z), \psi_E)\alpha(-z)
= c(-z, -z)\gamma_E(-1, \psi_E)\gamma_E(z, \psi_E)(-1, z)\gamma_E(Nz, \psi_E)\alpha(z)\alpha(-1).
\]

A simple calculation using (10) gives

\[
(16) \quad c(-z, -z) = (-1, -1)_F(-1, Nz)_Fc(z, z)
\]

and inserting this gives

\[
\tau(-z)\alpha(-z) = (-1, -1)_F\gamma_E(-1, \psi_E)\alpha(-1)\tau(z)\alpha(z)
= \tau(z)\alpha(z) \quad \text{by (13)}.
\]
Fix $\alpha$ satisfying (13). If $\lambda$ is any splitting of the cocycle restricted to $E^s\mathbb{F}^s$ then $\lambda = \lambda_\alpha \delta$ for some character $\delta$ of $E^s\mathbb{F}^s$. Extend $\delta$ to a character $\delta^*$ of $E^\ast$. Then $\lambda_\alpha \delta = \lambda_\alpha \delta^*$, and $\alpha \delta^2$ satisfies (13). This proves (1).

By (10), $c(\text{diag}(x,x), \text{diag}(y,y)) = (x,y)_\mathbb{F}$, and (2) follows as in Section 3.

For (3) apply Lemma 5.1. Inserting $\lambda_\alpha, \mu_\beta$ in the condition of the Lemma gives (14). The final assertion follows as in the proof of (1). This completes the proof. \hfill $\square$

Let $\zeta$ be any splitting of the cocycle restricted to $E^s\mathbb{F}^s$. Then $\zeta = \zeta_{\alpha, \beta}$ for some $\alpha, \beta$ satisfying (14). This implies:

**Lemma 5.3.** The map $\alpha(z) := \gamma_E(z^2, \psi_E)(Nz, \psi_E)\gamma_E(z, \psi_E)c(z, z)$ is a character of $\{z \in E^s \mid z^2 \in \mathbb{F}^s\}$.

For completeness we also prove this directly:

$$
\alpha(zw) = \gamma_E(z^2w^2, \psi_E)(N(zw), \psi_E)\gamma_E(zw, \psi_E)c(zw, zw)
= \gamma_E(z^2, \psi_E)\gamma_E(w^2, \psi_E)(z^2, w^2)\gamma_E(Nz, \psi_E)\gamma_E(Nw, \psi_E)\gamma_E(Nz, Nw)_\mathbb{F}
\gamma_E(z, \psi_E)\gamma_E(w, \psi_E)(z, w)c(zw, zw)
= \alpha(z)\alpha(w)c(z, z)c(w, w)c(z^2, w^2)\{z, w\}c(zw, zw) \quad \text{by (11)}
= \alpha(z)\alpha(w) \quad \text{by (8)}.
$$

**Remark 5.4.** Given $\alpha \in \widehat{E}^s$ there exists $\beta \in \widehat{\mathbb{F}}^s$ satisfying (14) if and only if (13) holds.

This follows by an argument as in the proof of Lemma 5.2 (1): Define $\beta(z^2) = \alpha(z)\gamma_E(z^2, \psi_E)^{-1}\gamma_E(Nz, \psi_E)^{-1}\gamma_E(z, \psi_E)c(z, z)$; this is well-defined if (13) holds, and extends to a character of $\mathbb{F}^s$.

Suppose $z \in E^1$. By Hilbert’s Theorem 90, $z = w/\mathbb{F}$ for some $w \in E^s$. Then $z = w^2 / N(w) \in E^s \mathbb{F}^s$, so $\zeta$ restricts to a splitting of the extension of $E^1$. We now make explicit choices such that $\zeta$ is a $T$ or $\mu_n$ splitting as in Theorem 1.

**Lemma 5.5.** We may choose $\psi_E$ so that

\begin{equation}
\gamma_E(x, \psi_E) = 1 \quad (x \in \mathbb{F}^s).
\end{equation}

**Proof.** Since $\gamma_E(\cdot, \psi_E)$ is a quadratic character of $\mathbb{F}^s$, $\gamma_E(x, \psi_E) = (x, y)_\mathbb{F}$ for some $y \in \mathbb{F}^s$. Suppose $E = \mathbb{F}(\sqrt{\Delta})$. Then $\gamma_E(\Delta, \psi_E) = 1$ since $\Delta$ is a square in $E^s$. Therefore $(\Delta, y)_\mathbb{F} = 1$, so $y = Nw$ for some $w \in E^s$. Replacing $\psi_E$ by $w\psi_E$ gives (cf. [10], Appendix)

$$
\gamma_E(x, w\psi_E) = (x, y)_\mathbb{F}\gamma_E(x, \psi_E)
= (x, Nw)_\mathbb{F}(x, y)_\mathbb{F} = 1.
$$

This completes the proof. \hfill $\square$
Theorem 5.7. Choose a non-trivial character \( \psi_E \) satisfying Lemma 5.5. Choose \( \alpha, \beta \) satisfying (14) and let \( \zeta = \zeta_{\alpha, \beta} \). Then
\[
\zeta(z)^2 = \alpha(z) \quad (z \in \mathbb{E}^1).
\]

Proof. Writing \( z = w/w = w^2/N(w) \) and applying the definition (15) gives
\[
(18) \quad \zeta(w/w) = (-1, Nw)_E \gamma_E(w, \psi_E) \alpha(w) \beta(Nw^{-1}) c(w, w) c(Nw^{-1}, w^2)
\]
and
\[
(19) \quad \zeta(w/w)^2 = (-1, Nw)_E \alpha(w^2) \beta(Nw^{-2})
\]
\[
= (-1, Nw)_E \alpha(w^2) \alpha(Nw^{-2})(-1, Nw)_E \gamma_E(Nw^{-1}, \psi_E) \quad \text{by (14)}
\]
\[
= \alpha(w/w) \gamma_E(Nw, \psi_E)
\]
\[
= \alpha(w/w) \quad \text{by (17)}.
\]

We see that \( \zeta_{\alpha, \beta} \) is a \( \mu_{2n} \) splitting if and only if \( \alpha(z)^n = 1 \) for all \( z \in \mathbb{E}^1 \). We now complete the proof of Theorem 1.

Proof of Theorem 1. By (16) and (17) we have
\[
\alpha(-1) = (-1, -1)_F.
\]
Therefore we may choose \( \alpha = 1 \) if \( (-1, -1)_F = 1 \). Assume \( (-1, -1)_F = -1 \).

If \( F = \mathbb{R} \) then \( \alpha(z) = z^n \) for \( n \) odd as in Section 4, so assume \( F \) is non-
archimedean. If \(-1 \not\in \mathbb{E}^{*2} \) we may choose \( \alpha^2 = 1 \). If \(-1 \in \mathbb{E}^{*2} \) then
\(-1 \not\in \mathbb{E}^{*4} \) (cf. §4) and we may choose \( \alpha^4 = 1 \).

We make some explicit choices and summarize the preceding discussion.

If \( (-1, -1)_F = -1 \) and \(-1 \not\in \mathbb{E}^{*2} \) choose \( z_1 \in \mathbb{E}^{*} \) with \((z_1, -1)_E = -1 \).

If \( (-1, -1)_F = -1 \) and \(-1 \in \mathbb{E}^{*2} \), i.e., \( E = F(\sqrt{-1}) \), then the norm
residue symbol \( (w, z)_{E,4} \) is defined. In particular the map \( z \to (w, z)_{E,4} \) is a
character of \( \mathbb{E}^{*} \) of order 4. Choose \( z_2 \in \mathbb{E}^{*} \) satisfying \((z_2, -1)_{E,4} = -1 \).

Theorem 5.7. Choose a non-trivial character \( \psi_E \) of \( E \) such that \( \gamma_E(x, \psi_E) = 1 \) for all \( x \in \mathbb{E}^{*} \) (Lemma 5.5). For \( z \in \mathbb{E}^{*} \) let
\[
\alpha(z) := \begin{cases} 
  z & F = \mathbb{R} \\
  1 & (-1, -1)_F = 1 \\
  (z_1, z)_{E} & (-1, -1)_F = -1, -1 \not\in \mathbb{E}^{*2} \\
  (z_2, z)_{E,4} & (-1, -1)_F = -1, -1 \in \mathbb{E}^{*2}.
\end{cases}
\]

Then \( \alpha \) is a character of \( \mathbb{E}^{*} \) of order \( \infty, 1, 2 \) or 4 respectively, satisfying (13).
Choose a character \( \beta \) of \( F^{*} \) satisfying
\[
(20) \quad \beta(z^2) = \gamma_F(z^2, \psi_F)^{-1} \gamma_F(Nz, \psi_F)^{-1} \gamma_E(z, \psi_E)^{-1} c(z, z) \alpha(z) \quad (z^2 \in F^{*})
\]
In particular
$$\beta(x^2) = (-1, x)_F \alpha(x) \quad (x \in F^*).$$

Let
$$\zeta(w/\overline{w}) = \gamma_E(w, \psi_E) \alpha(w) \beta(Nw^{-1}) c(w, w)c(Nw^{-1}, w^2).$$

Then \(\zeta\) is a splitting of the cocycle restricted to \(E_1\). Furthermore for \(z \in E^1\)
\(\zeta(x)^2 = \alpha(z)\), and for \(F\) non-archimedean this gives \(\zeta(z)^n = \alpha(z)^{n/2} = 1\) with
$$n = \begin{cases} 2 & (-1, -1)_F = 1 \\ 4 & (-1, -1)_F = -1, -1 \notin E^2 \\ 8 & (-1, -1)_F = -1, -1 \in E^2. \end{cases}$$

**Remark 5.8.** With \(\alpha, \beta\) as in the Theorem,
$$\zeta(w/\overline{w}) = \zeta_{\alpha, \beta}(w/\overline{w})(-1, Nw)_F.$$ 

We have dropped the term \((-1, Nw)\), which is allowed since \(w/\overline{w} \rightarrow (-1, Nw)\) is a quadratic character of \(E^1\).

Henceforth write \(E = F(\delta)\) with \(\Delta := \delta^2 \in F\).

**Remark 5.9.** Condition (20) is equivalent to
$$\beta(x^2) = (-1, x)_F \alpha(x) \quad (x \in F^*)$$
$$\beta(\Delta) = \gamma_F(-1, \psi_F)^{-1} \gamma_E(\delta, \psi_E)^{-1} c(\delta, \delta) \alpha(\delta).$$

The splitting has a simple formula on \(T^2\):
$$\zeta(z^2) = c(z, z) \alpha(z) \quad (z \in E^1),$$
which is independent of \(\beta, \psi_F\) and \(\psi_E\). Note that any two \(\mu_2\) splittings of \(T\) have the same restriction to \(T^2\) since they differ by a quadratic character.

The map \(w/\overline{w} \rightarrow Nw\) induces an isomorphism \(T/T^2 \cong NE^*/F^*\). Choose representatives \(a_1, \ldots, a_n\) of generators of \(NE^*/F^*\), with corresponding elements \(z_1, \ldots, z_n \in T\). Given \(\alpha\) there are two choices of each \(\beta(a_i)\) differing by sign, and these signs may be chosen arbitrarily. The following result follows easily.

**Corollary 5.10.** Choose representatives \(z_1, \ldots, z_m\) of generators of \(T/T^2 \cong NE^*/F^*\). Choose \(\alpha\) as in Theorem 5.7. Define
$$\zeta(z^2) = c(z, z) \alpha(z) \quad (z \in E^1)$$
and for \(1 \leq i \leq m\) let \(\zeta(z_i)\) be either square root of \(\alpha(z_i)\). Then \(\zeta\) extends uniquely to a splitting of the cocycle as in Theorem 5.7.
In the non-archimedean case by ([4], Lemma 0.3.2) $|T/T^2| = 2/|2|_F$, which equals 2 if the residual characteristic of $F$ is odd.

We conclude with a few remarks about the definition of $\zeta$.

From the definition we have for $w \in E^*$:

$$c(w, w) = (-Nw, x(w)x(w^2))_F$$

and

$$c(Nw^{-1}, w^2) = (Nw, x(w^2))_F.$$  

Note that for $\lambda \in F^*, w \in E^*$ we have

$$c(\lambda, w) = (\lambda, x(w))_F.$$  

Fix $u \in E^*$ with trace$(u) = 0$, and define $\text{Tr}_u : \mathbb{E}^* \to \mathbb{F}^*$ by

$$\text{Tr}_u(z) = \begin{cases} \text{trace}(z) & \text{trace}(z) \neq 0 \\ uz & \text{trace}(z) = 0. \end{cases}$$

Up to conjugation by $SL(2, \mathbb{F})$ we may assume $\iota(x + y\delta) = \begin{pmatrix} x & y\Delta/a \\ ya & x \end{pmatrix}$ for some $a \in \mathbb{F}^*$. If $w = x + y\delta$ with $xy \neq 0$ we have $x(w)x(w^2) = 2xy^2a^2$.

Considering the cases with $xy = 0$ separately gives

$$c(w, w) = (-Nw, \text{Tr}_{a\delta}(w))_F \quad (z \in \mathbb{E}^*)$$

and

$$\zeta(z^2) = (-1, \text{Tr}_{a\delta}(z))_F \alpha(z) \quad (z \in \mathbb{E}^1).$$

For example if $p$ is odd and $E \neq \mathbb{F}(\sqrt{-1})$ then $m = 1$ (cf. Corollary 5.10), we may take $z_1 = -1$, $\alpha = 1$ and $\zeta(-1) = 1$ which gives

$$\zeta(\epsilon z^2) = c(z, z)c(\epsilon, z^2) \quad (\epsilon = \pm 1, z \in \mathbb{E}_1).$$

For example let $\mathbb{F} = \mathbb{R}$ and define $\iota(x + iy) = \begin{pmatrix} x & y \\ -y & x \end{pmatrix}$. We take $\alpha(z) = z$ and $\beta(x) = \pm \sqrt{|x|}$. Then for $z \in \mathbb{E}_1$,

$$\zeta(z^2) = (-1, \text{Tr}_{-i}(z))_F z = \text{sgn}(\text{Tr}_{-i}(z))z$$

(independent of $\beta$). Note that $a = -1$ and $\text{Tr}_{-i}(iy) = y \quad (y \in \mathbb{R}^*)$.

References


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