EXTENSIONS OF TORI IN SL(2)

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Let $\tilde{SL}(2, \mathbb{F})$ be the metaplectic two-fold cover of $SL(2, \mathbb{F})$, the special linear group in two variables over a local field $\mathbb{F}$ of characteristic 0. The inverse image $\tilde{T}$ of a maximal torus $T$ in $SL(2, \mathbb{F})$ is an abelian extension of $T$ by $\pm 1$. We consider the question of whether this extension is trivial. More generally we find the minimal subgroup $A$ of the circle for which the extension is split when considered with coefficients in $A$. We see that $|A| = 2, 4$ or 8 in the p-adic case. We also find an explicit splitting function for the cocycle.

Introduction

Let $\tilde{SL}(2, \mathbb{F})$ be the metaplectic two-fold cover of $SL(2, \mathbb{F})$, the special linear group in two variables over a local field $\mathbb{F}$ of characteristic 0. The inverse image $\tilde{T}$ of a maximal torus $T$ in $\tilde{SL}(2, \mathbb{F})$ is an abelian extension of $T$ by $\pm 1$. We consider the question of whether this extension is trivial. We exclude the case $\mathbb{F} = \mathbb{C}$, which is trivial.

More generally suppose $A$ is a subgroup of the circle $\mathbb{T}$ containing $\pm 1$. The inclusion of $\pm 1$ in $A$ induces a map on cohomology, and defines an extension

$$1 \rightarrow A \rightarrow T_A \rightarrow T \rightarrow 1.$$  

We say $A$ is a splitting group for $\tilde{T}$ if the extension $T_A \rightarrow T$ splits. It is well-known that $\mathbb{T}$ is a splitting group. We say a splitting group $A$ is a minimal splitting group if no proper subgroup of $A$ is a splitting group. It is easy to see the order of a minimal splitting group is a power of 2, and hence unique, if it is finite.

Let $(, )_\mathbb{F}$ be the Hilbert symbol of $\mathbb{F}$, and let $\mu_n$ be the $n^{th}$ roots of unity in $\mathbb{C}$.

Theorem 1. The minimal splitting group $A_{\text{min}}$ for $T$ is given by:

- Suppose $T \simeq \mathbb{F}^*$. Then

$$A_{\text{min}} = \begin{cases} 
\mu_2 & (1, -1)_\mathbb{F} = 1 \\
\mu_4 & (1, -1)_\mathbb{F} = -1. 
\end{cases}$$
Suppose $T \simeq \mathbb{E}^1$ for $\mathbb{E}$ a quadratic extension of $\mathbb{F}$. Then
\[
A_{\text{min}} = \begin{cases} 
\mu_2 & (-1, -1)_F = 1 \\
\mu_4 & (-1, -1)_F = -1, \ F \text{ non-archimedean, } -1 \notin \mathbb{E}^* \\
\mu_8 & (-1, -1)_F = -1, \ F \text{ non-archimedean, } -1 \in \mathbb{E}^* \\
T & F = \mathbb{R}.
\end{cases}
\]

Remark 2. It is well-known that $(-1, -1)_F = 1$ unless $F = \mathbb{R}$, $\mathbb{Q}_2$, or an extension of $\mathbb{Q}_2$ of odd degree.

Theorem 1 is proved in Sections 3, 4 and 5. Here is an alternative realization of $\tilde{T}$. A character of $\tilde{T}$ is said to be genuine if it does not factor to $T$.

Theorem 3. Let $\tau(z) = z^2$ ($z \in \mathbb{C}^*$). Let $\tilde{\alpha}$ be a genuine character of $\tilde{T}$. Then $\tilde{\alpha}^2$ factors to a character $\alpha$ of $T$, and $\tilde{T}$ is isomorphic to the pullback of $\tau$ via $\alpha$. In other words $\tilde{T}$ is isomorphic to the $\sqrt{\alpha}$-extension of $G$.

From this we obtain an interpretation of the minimal splitting group of Theorem 1. Let $n(\tilde{T})$ be the minimal order of a genuine character of $\tilde{T}$. Set $\mu_{\infty} = \mathbb{T}$.

Corollary 4. The minimal splitting group for $\tilde{T}$ is $\mu_{n(\tilde{T})}$.

For the proofs of Theorem 3 and Corollary 4 see Lemma 1.4.

We also give an explicit splitting of this extension, i.e., a function $\zeta : T \to A_{\text{min}}$ whose coboundary is the cocycle defining $\tilde{T}$ (see §3 and Theorem 5.7). These questions arise from the theory of the oscillator representation and dual pairs. The splitting plays a role in this context, for example see [11]. The case of $\mathbb{F}^*$ is well-known ([4], p. 42, attributed to J. Klose), as is the existence of a $\mathbb{T}$-splitting in general [2]. General results about the splitting of the metaplectic cover over subgroups are due to Kudla [7], and a splitting of the extension of an elliptic torus is found in [7], Proposition 4.8 (in the non-archimedean case it is easy to see this can be taken to be a $\mu_8$-splitting).

This paper grew out of an effort to simplify Kudla’s formula. In the case of a $p$-adic field of odd residual characteristic a formula for a $\mu_2$-splitting in some cases may be deduced from [6], cf. ([4], p. 43).

Many of the arguments, especially those of Section 1 apply to other abelian extensions of abelian groups, for example a maximal torus in the two-fold cover of $Sp(2n, \mathbb{F})$. If $\tilde{G}$ is a non-linear $n$-fold cover of the $\mathbb{F}$ points of an algebraic group $G$, then the inverse image $\tilde{T}$ of a maximal torus in $G$ is typically not abelian. However similar arguments apply to the center of $\tilde{T}$.

Throughout $\mathbb{F}$ denotes a local field of characteristic zero, and $(x, y)_F \in \mu_2$ is the Hilbert symbol. For $x \in \mathbb{F}^*$ and $\psi_\mathbb{F}$ a non-trivial additive character of
\[ F, \gamma_F(x, \psi_F) \in \mu_4 \text{ is the Weil index. We use basic properties of the Hilbert symbol and the Weil index without further comment, see ([10], Appendix) for details. We make repeated use of the identities} \]

\begin{align}
(1) & \quad \gamma_F(x, \psi_F) \gamma_F(y, \psi_F) = (x, y)_F \gamma_F(xy, \psi_F) \\
(2) & \quad \gamma_F(x, \psi_F)^2 = (-1, x)_F.
\end{align}

If \( E \) is a quadratic extension of \( F \) then

\begin{align}
(3) & \quad (x, z)_E = (x, Nz)_F \quad (x \in F^*, z \in E^*) \\
(4) & \quad (x, y)_E = 1 \quad (x, y \in F^*) \\
(5) & \quad \gamma_E(x, \psi_E) \gamma_E(y, \psi_E) = \gamma_E(xy, \psi_E) \quad (x, y \in F^*).
\end{align}

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1. Abstract Groups.

In this section we ignore the topology on \( T \) and consider it as an abstract group. We recall some standard facts from group cohomology and establish some notation. For example see [1].

Suppose \( G \) is a group, \( A \) is an abelian group, and \( G \) acts trivially on \( A \). The equivalence classes of central extensions of \( G \) by \( A \) are parametrized by the group cohomology \( H^2(G, A) \). Given an extension \( p: H \to G \) let \( s: G \to H \) be a section, i.e., \( p \circ s = 1 \). The cohomology class of the extension is represented by the 2-cocycle \( c_s(g, h) = s(gh)s(h)^{-1}s(g)^{-1} \). When there is no danger of confusion we do not distinguish between \( c_s \) and its image \( c_s \) in \( H^2(G, A) \). Any other such splitting \( s' \) is given by \( s'(g) = s(g)\zeta(g) \) for some map \( \zeta: G \to A \), and then \( c_{s'}(g, h) = c(g, h)\zeta(gh)\zeta(h)^{-1}\zeta(g)^{-1} \). Thus \( c_{s'} = c_s d\zeta \), and \( \bar{c}_s = \bar{c}_{s'} \).

Conversely given a cocycle \( c \) we define \( H \) to be equal to \( G \times A \) as a set, with multiplication \( (g, a)(g', a') = (gg, aa'c(g, g')) \). The cocycle \( c \) is trivial in cohomology if and only if

\begin{equation}
(6) \quad c(g, h) = \zeta(g)\zeta(h)\zeta(gh)^{-1}
\end{equation}

for some \( \zeta \), i.e., \( d\zeta = c \). We say \( \zeta \) is a splitting of the cocycle. Equivalently the splitting map \( s(g) = (g, \zeta^{-1}(g)) \) is a homomorphism. Any other splitting is then of the form \( \zeta' = \zeta \alpha \) with \( \alpha: G \to A \) a homomorphism.

Suppose \( A = \mu_2 \), with cocycle \( c \), and \( A \subset \mu_{ab} \) with \( b \) odd. If \( \zeta: G \to \mu_{ab} \) is a splitting of \( c \), then \( \zeta^b \) is a \( \mu_a \) splitting. Therefore we will restrict consideration to \( \mu_n \) with \( n \) a power of 2.

Now suppose \( G \) is abelian. The universal coefficient theorem for group cohomology gives an exact sequence:

\[ 1 \to \text{Ext}(G, A) \to H^2(G, A) \xrightarrow{\delta} \text{Hom}(\Lambda^2 G, A) \to 1. \]
Here $G$ and $A$ are considered as $\mathbb{Z}$-modules, $\text{Hom} = \text{Hom}_{\mathbb{Z}}$, $\text{Ext} = \text{Ext}_{\mathbb{Z}}$, and $\text{Hom}(\Lambda^2 G, A)$ consists of alternating, bilinear maps $G \times G \rightarrow A$.

If $c$ is a 2-cocycle, representing the class $\tilde{c} \in H^2(G, A)$, then $\phi(\tilde{c})(g, h) = c(g, h)c(h, g)^{-1}$. In terms of the group, suppose $p : H \rightarrow G$ is the corresponding extension. For $g, h \in G$ and any section $s$ let $\{g, h\}$ be the commutator $s(g)s(h)s(g)^{-1}s(h)^{-1}$. This is contained in $A$, is independent of the choice of $s$, and $\phi(\tilde{c})(g, h) = \{g, h\}$. In particular $\phi(\tilde{c}) = 1$ if and only if $H$ is abelian, so $\text{Ext}(G, A) \subset H^2(G, A)$ parametrizes the abelian extensions of $G$ by $A$.

Let $G^n = \{g^n | g \in G\}$ and $nG = \{g \in G | g^n = 1\}$. The next result is presumably well-known to the experts.

**Lemma 1.1.** For any positive integer $n$, inclusion $\iota : nG \hookrightarrow G$ induces an isomorphism:

$$\text{Ext}(G, \mu_n) \cong \text{Ext}(nG, \mu_n).$$

**Proof.** Consider the maps

$$G \xrightarrow{\alpha} G^n \xrightarrow{\beta} G$$

where $\alpha(g) = g^n$ and $\beta$ is inclusion. The induced map $\alpha^* \beta^* : \text{Ext}(G, \mu_n) \rightarrow \text{Ext}(G, \mu_n)$ is induced by the $n$th power map $g \rightarrow g^n$ on $G$. This is the same map as that induced by the $n$th power map on $\mu_n$, and therefore $\alpha^* \beta^* = 0$.

Now the long exact cohomology sequence corresponding to $0 \rightarrow G^n \beta \rightarrow G \rightarrow G/G^n \rightarrow 0$ has final two terms $\text{Ext}(G, A) \beta \rightarrow \text{Ext}(G^n, A) \rightarrow 0$. Therefore $\beta^*$ is surjective, which implies $\alpha^* = 0$. On the other hand the short exact sequence

$$0 \rightarrow nG \xrightarrow{\iota} G \xrightarrow{\alpha} G^n \rightarrow 0$$

gives rise to the long exact sequence

$$(7) \quad 0 \rightarrow \text{Hom}(G^n, A) \rightarrow \text{Hom}(G, A) \rightarrow \text{Hom}(nG, A) \rightarrow \text{Ext}(G^n, A) \xrightarrow{\alpha^*} \text{Ext}(G, A) \xrightarrow{\iota^*} \text{Ext}(nG, A) \rightarrow 0.$$

Since $\alpha^* = 0$, $\iota^*$ is an isomorphism. \qed

**Remark 1.2.** In our setting $2T = \pm 1$. For the $\mu_2$ extension $\tilde{T}$ to split it is necessary that it splits over $\pm 1$. Perhaps surprisingly the converse holds as well by the Lemma.

For later use we note an explicit formula for a splitting of $\alpha^* \beta^* c$. We drop the assumption that $H$ is abelian, so let $p : H \rightarrow G$ be an extension, with section $s$ and corresponding cocycle $c$.

**Lemma 1.3.** Let

$$\tau(g) = s(g^n)s(g)^{-n} = c(g, g)^{-1}c(g, g^2)^{-1}\ldots c(g, g^{n-1})^{-1} \in A.$$
Then
\[(8) \quad c(g^n, h^n) = \tau(g)\tau(h)\tau(gh)^{-1}\{g, h\}^{n(n-1)/2}.\]

If \(H\) is abelian then \(d\tau = \alpha^*\beta^*c\).

Note that \(\{g, h\}^{n(n-1)/2} = \pm 1\), and is identically 1 if \(n\) is odd. Compare (\([3]\), p. 130) and ([5], \$4).

Proof. This follows from the identity
\[ [s(g)s(h)]^n = s(g)^ns(h)^n\{h, g\}^{n(n-1)/2}. \]

Using \(s(g)s(h) = s(gh)c(g, h)\) and \(s(g)^n = s(g^n)\tau(g)^{-1}\), the left hand side is equal to
\[ s(gh)^n = s(g^n h^n)\tau(gh)^{-1}. \]

The right hand side is
\[ s(g^n)s(h^n)\tau(g)^{-1}\tau(h)^{-1}\{h, g\}^{n(n-1)/2} \]
\[ = s(g^n h^n)c(g^n, h^n)\tau(g)^{-1}\tau(h)^{-1}\{h, g\}^{n(n-1)/2} \]
and the first assertion follows. Since \(\alpha^*\beta^*c(g, h) = c(g^n, h^n)\), the second assertion is equivalent to
\[(9) \quad c(g^n, h^n) = \tau(g)\tau(h)\tau(gh)^{-1}\]
which is (8) for \(H\) abelian. \(\square\)

For \(H\) an abelian extension (9) implies \(\tau\) is a character when restricted to \(nG\). In terms of the exact sequence (7), \(c \in \operatorname{Ext}(G, A), \beta^*c \in \operatorname{Ext}(G^n, A), \alpha^*\beta^*c = 0\), and \(\beta^*c\) is the image of \(\tau \in \operatorname{Hom}(\mathbb{nG}, A)\). Thus \(\beta^*c = 0\) if \(\tau\) extends to an element of \(\operatorname{Hom}(G, A)\).

More generally, we try to find a splitting subgroup \(A\) of \(\beta^*c\), i.e., \(c\) restricted to \(G^n\), together with an explicit formula. Note that (9) does not necessarily define such a splitting since the function \(g^n \rightarrow \tau(g)\) is not necessarily well-defined. Let \(\alpha\) be a character of \(G\) whose restriction to \(nG\) is equal to \(\tau^{-1}\). Then \(\zeta_\alpha(g^n) := \tau(g)\alpha(g)\) is well-defined, and \(d\zeta_\alpha = c\). The minimal splitting subgroup for \(\beta^*c\) is thus the minimal subgroup \(A\) of \(\mathbb{T}\), containing \(\mu_n\), such that \(\tau\) restricted to \(nG\) can be extended to a character of \(G\) with values in \(A\).

**Characters and the \(\sqrt{\alpha}\) extension.**

For the remainder of this section let \(p : \widetilde{G} \rightarrow G\) be a \(\mu_2\) extension of a group \(G\). We do not assume that \(G\) or \(\widetilde{G}\) is abelian. If \(\alpha\) is a character of \(G\), and \(\tau(z) = z^2\) (\(z \in \mathbb{C}^*\)) then the pullback of \(\tau\) via \(\alpha\) is a \(\mu_2\) extension of \(G\), and may be realized as the subgroup of \(G \times \mathbb{C}^*\) given by \(\{(g, z) \mid \alpha(g) = \tau(z)\}\). Projection on the second factor is a genuine character \(\tilde{\alpha}\) of \(G\) satisfying \(\tilde{\alpha}^2 = \alpha \circ p\). This is sometimes denoted the \(\sqrt{\alpha}\)-extension of \(G\). It may or may not be the trivial extension.
We see that $\tilde{G}$ has a $T$-splitting if and only if there is a genuine character of $\tilde{G}$. More precisely:

**Lemma 1.4.** Suppose there is a genuine character $\tilde{\alpha}$ of $\tilde{G}$. Then $\tilde{\alpha}^2$ factors to a character $\alpha$ of $G$, and $\tilde{G}$ is isomorphic to the $\sqrt{\alpha}$ extension of $G$. If $\text{Image}(\tilde{\alpha}) \subset A \subset T$ then $G_A \simeq G \times A$. The minimal splitting group for $G$ is $\mu_{n(\tilde{G})}$ where $n(\tilde{G})$ is the minimal order of a genuine character of $\tilde{G}$.

Conversely if $G_A \simeq G \times A$ then there is a genuine character of $\tilde{G}$ with values in $A$.

Note that there exists a genuine character $\tilde{\alpha}$ of $\tilde{G}$ if and only if $z \notin [\tilde{G}, \tilde{G}]$ where $z$ is the non-trivial element in the inverse image of 1. In particular this holds if $\tilde{G}$ is abelian, which proves the existence of a $T$-splitting (cf. [2]).

**Proof.** The map $\phi : g \to (p(g), \tilde{\alpha}(g)) \subset G \times T$ is an isomorphism of $\tilde{G}$ with the pullback of $\tau$ via $\alpha$. This is a subgroup of $G \times A$, and $\phi$ extends to an isomorphism of $G_A$ with $G \times A$. The final two assertions are immediate. □

**Remark 1.5.** In the setting of the Lemma, suppose $\zeta$ is a $T$-splitting of the cocycle defining $G$ (with respect to a section $s$). Then $\alpha := \zeta^2$ is a character of $G$, and $\tilde{G}$ is isomorphic to the $\sqrt{\alpha}$ cover of $G$.

Theorem 3 and Corollary 4 are immediate consequences of the Lemma. Theorem 1 also follows from the Lemma, from a computation of $n = n(T^\alpha)$: $n$ is the minimal power of 2 such that $z \notin T^n$. We follow a different approach, by giving explicit formulas for the minimal splitting $\zeta$ in Sections 3-5.

2. Moore cohomology and $\tilde{SL}(2, \mathbb{F})$.

Now suppose $G$ and $A$ are locally compact topological groups, $A$ is abelian, and $G$ acts continuously on $A$. In our applications $A$ will either be $\mu_n$ or $T$, with trivial $G$ action. C. Moore has defined cohomology groups $H^2_{\text{top}}(G, A)$ using measurable cochains [8]. In the case of a totally disconnected group it is equivalent to use continuous cochains. Viewing $G$ and $A$ as abstract groups, there is a natural homomorphism $H^2_{\text{top}}(G, A) \to H^2(G, A)$. In general it is neither surjective nor injective.

Now let $G = GL(2, \mathbb{F})$. We recall the definition of the standard cocycle on $G$ [6], cf. ([4], p. 41). For $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ let $x(g) = c$ (resp. $d$) if $c \neq 0$ (resp. $c = 0$). Then

$$c(g, h) = (x(g)x(gh), x(h)x(gh))_F\text{det}(g), x(g)x(gh))_F.$$

(10)
This defines a $\mu_2$-extension $\widehat{GL(2, F)}$ of $GL(2, F)$, with distinguished section $s$. We write $\widehat{GL(2, F)}$ in cocycle notation as usual. The restriction to $SL(2, F)$ is isomorphic to $SL(2, F)$.

Up to conjugation $GL(2, F)$ contains one hyperbolic torus isomorphic to $F^* \times F^*$, and for each quadratic extension $E$ of $F$ one elliptic torus isomorphic to $E^*$. The commutator of two elements $z, w$ of $E^*$ is given by ([3], p. 128)

\begin{equation}
\{z, w\} = (z, \overline{w})_E
= (z, w)_E(Nz, Nw)_E.
\end{equation}

Here and elsewhere we suppress the map $\iota: E^* \hookrightarrow GL(2, F)$ from the notation, and write $\{z, w\} := \{\iota(z), \iota(w)\}$. The commutator is trivial when restricted to $E^1 = E^* \cap SL(2, F)$.

3. The hyperbolic torus.

Let $\iota: F^* \hookrightarrow SL(2, F)$, so $T = \iota(F^*)$ is a hyperbolic torus. After conjugation we may assume $\iota(x) = \text{diag}(x, x^{-1})$. We drop $\iota$ from the notation and identify $x$ with $\iota(x)$. We prove Theorem 1 in this case, together with a formula for the minimal splitting. While these results are well-known they are not easy to find in the literature, and the calculation illustrates some of the ideas in the next section.

The restriction of $SL(2, F)$ to $T$ defines a $\mu_2$ extension $\tilde{T}$ of $T$ with cocycle

$$c(x, y) = (x, y)_F.$$

In particular $(-I, e)^2 = (I, (-1, -1)_F)$, so the restriction of the extension to $\pm I$ is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ if $(-1, -1)_F = 1$, or $\mathbb{Z}/4\mathbb{Z}$ if $(-1, -1)_F = -1$. By Lemma 1.1 the minimal splitting group for $\tilde{T}$, considered as an abstract group, is $\mu_2$ (resp. $\mu_4$) if $(-1, -1)_F = 1$ (resp. $-1$). It remains to show this splitting is measurable. We do this by computing it explicitly.

Fix $\psi$ and write $\gamma(x) = \gamma_F(x, \psi)$. The key point is that properties (1) and (2) of the Weil index shows that $d\gamma = c$ and $\gamma^4 = 1$, so $\gamma$ is a measurable $\mu_4$ splitting of $c$. This completes the proof in case $(-1, -1)_F = -1$, so assume $(-1, -1)_F = 1$. Let $\alpha$ be a character of $F^*$ satisfying $\alpha(x^2) = (-1, x)_F$. To see that such a character exists, define $\alpha$ restricted to $F^{*2}$ by this formula; it is well-defined since $\alpha((-x)^2) = (-1, -x)_F = (-1, -1)_F\alpha(x^2) = \alpha(x^2)$.

Extend arbitrarily from $F^{*2}$ to $F^*$. By (2) $\alpha(x)^2\gamma(x)^2 = (-1, x)_F^2 = 1$. Let $\zeta_\alpha = \gamma\alpha$, i.e.,

$$\zeta_\alpha(x) = \gamma(x, \psi_F)\alpha(x).$$

Then $d\zeta_\alpha = d\gamma = c$ and $\zeta_\alpha$ is a $\mu_2$-splitting of $c$.

Choose representatives $a_1, a_2, \ldots, a_m \in F^*$ of generators of $F^*/F^{*2} \simeq (\mathbb{Z}/2\mathbb{Z})^m$. (By [4], Lemma 0.3.2, $2^m = 4/|\psi_F|$.) Given any choice of signs
$\epsilon_i$ we may choose $\alpha$ so that $\zeta(a_i) = \epsilon_i$, and $\alpha(x^2) = (1, x)_F$ for all $x \in F$. Then $\zeta$ extends uniquely to a splitting.

For example, if $-1 \in F^{x^2}$ then we may take $\alpha = 1$ and $\zeta(x) = \gamma_F(x, \psi_F) = \pm 1$. On the other hand, suppose $-1 \not\in F^{x^2}$ and the residual characteristic of $F$ is odd. We may take representatives $\pm 1, \pm \varpi$ for $F^*/F^{x^2}$ ($\varpi$ is a uniformizing parameter) and then choose $\zeta$ satisfying:

$$
\zeta(\pm x^2) = (-1, x)_F
$$

$$
\zeta(\pm \varpi x^2) = \mp(-1, x)_F.
$$

4. Elliptic tori.

Let $T$ be an elliptic torus of $SL(2, F)$ as in §2. Thus $E$ is a quadratic extension of $F$, $\iota : E^1 \hookrightarrow SL(2, F)$ is an embedding, and $T = \iota(E^1)$. As in §3 we fix $\iota$ and drop it from the notation.

As in the case of the hyperbolic torus, $(-I, \epsilon)$ has order 2 or 4 depending on whether $(-1, -1)_F = 1$ or $-1$. By Lemma 1.1 this proves $\mu_2$ is a splitting group if and only if $(-1, -1)_F = 1$, so assume $(-1, -1)_F = -1$.

Let $\mathbb{F} = \mathbb{R}, E = \mathbb{C}$. Since $T = T^2$, it is enough to find a splitting of $\beta^c$ as in §2. Choose a character $\alpha$ of $E^1$ such that $\alpha(-1) = -1$, i.e., $\alpha(z) = z^n$ for $n$ odd. It is easy to see $c(-z, -z) = -c(z, z)$, and the discussion in §2 shows that

$$
\zeta_{\alpha}(z^2) := c(z, z)\alpha(z)
$$

is a well-defined (measurable) $T$-splitting. Since $\zeta_{\alpha}$ is surjective onto $T$ for any $\alpha$, this shows that there is no $A$-splitting for any proper subgroup $A$ of $T$.

We now assume $F$ is non-archimedean. Since $(-1, -1)_F = -1$, $F$ is an extension of $Q_2$ of odd degree, and $-1 \not\in F^{x^2}$. If $-1 \not\in E^{x^2}$ then $4T = \{\pm 1\}$. Since $(-I, \zeta)^4 = (I, \pm \zeta^2)^2 = I$ for all $\zeta \in \mu_4$, the $\mu_4$-extension splits over $4T$.

Suppose $-1 = \delta^2$ ($\delta \in E^*$. We claim $4T = S\{\pm 1, \pm \delta\}$. It is enough to show $E^*$ does not contain a primitive eighth root of unity, or equivalently $\delta \not\in E^{x^2}$. Since $F$ is an extension of $Q_2$ of odd degree, $2 \not\in F^{x^2}$. But then $(a + b\delta)^2 = \delta$ implies $a^2 = \pm \frac{1}{2}$, which is a contradiction.

For any $\zeta \in \mu_8$ we compute $(\delta, \zeta)^8 = (-I, \pm \zeta^2)^4 = (I, -\zeta^4)^2 = I$, which implies the extension splits over $S\{T$.

Therefore $\mu_4$ (resp. $\mu_8$) is a minimal splitting group for $T$ if $F$ is non-archimedean, $(-1, -1)_F = -1$, and $-1 \not\in E^{x^2}$ (resp. $-1 \not\in E^{x^2}$). It remains to show these splittings can be chosen to be measurable. In the next section we give explicit such splittings.

Remark 4.1. We have shown the cohomology class $\tau \in H^2_{\text{top}}(G, A)$ has image 0 in $H^2(G, A)$ in the given cases. An argument due to Jonathan
Rosenberg shows that the map $H^2_{\text{top}}(G, A) \to H^2(G, A)$ is injective in this situation. Since we are interested in explicit formulas for the splittings in any case, we do not pursue this approach. For $G$ perfect (not at all the case here!) the injectivity of $\phi$ is known ([9], Theorem 2.3).

5. Explicit splittings for elliptic tori.

We continue with the notation of the previous section. The embedding $\iota : \mathbb{E}^1 \hookrightarrow SL(2, \mathbb{F})$ extends to an embedding $\iota : \mathbb{E}^* \hookrightarrow GL(2, \mathbb{F})$. We will make use of the non-abelian $\mu_2$ extension of $\mathbb{E}^*$ obtained by restricting the extension $GL(2, \mathbb{F})$ of $GL(2, \mathbb{F})$.

We proceed as follows. Since $\{z, w\} = (z, w)_{\mathbb{E}}$, the commutator is trivial on $\mathbb{E}^2$. By the method of §1 we find a splitting of this extension. As in §3 we also find a splitting of the extension of $\mathbb{F}^* \subset \mathbb{E}^*$. The extension of $\mathbb{E}^2 \mathbb{F}^*$ is abelian, and by an explicit version of the Mayer-Vietoris sequence we obtain a splitting of this extension. Finally $\mathbb{E}^1$ is contained in $\mathbb{E}^2 \mathbb{F}^*$, and we restrict to obtain a splitting of the extension of $\mathbb{E}^1$.

Fix non-trivial additive characters $\psi_\mathbb{F}$ of $\mathbb{F}$ and $\psi_\mathbb{E}$ of $\mathbb{E}$. Recall (5) the restriction of $\gamma_{\mathbb{E}}(., \psi_\mathbb{E})$ to $\mathbb{F}^*$ is a quadratic character.

Lemma 5.1. Suppose $\lambda$ (respectively $\mu$) is a splitting of the cocycle restricted to $\mathbb{E}^* \mathbb{F}^*$ (respectively $\mathbb{F}^*$). Assume $\lambda(x) = \mu(x)$ for $x \in \mathbb{E}^* \cap \mathbb{F}^*$.

Conversely if $\zeta$ is any splitting of the cocycle restricted to $\mathbb{E}^2 \mathbb{F}^*$ then $\zeta = \zeta_{\lambda, \mu}$ with $\lambda = |_{\mathbb{E}^2 \mathbb{F}^*}$ and $\mu = |_{\mathbb{F}^*}$.

Proof. Let $\zeta$ be any splitting of the cocycle. Then $\zeta(z^2 x) = \zeta(z^2)\zeta(x)c(x, z^2)$ by (6) and the second assertion is immediate.

Given $\lambda$ and $\mu$, choose any splitting $\zeta$. Then $\lambda$ and $|_{\mathbb{E}^2 \mathbb{F}^*}$ both define splittings, so $\lambda(z^2) = \zeta(z^2)\alpha(z^2)$ for some character $\alpha$ of $\mathbb{E}^* \mathbb{F}^*$. Similarly $\mu(x) = \zeta(x)\beta(x)$ for some character $\beta$ of $\mathbb{F}^*$. Let $\tau$ be a character of $\mathbb{E}^*$ extending $\alpha$ and $\beta$; this exists since, for $x \in \mathbb{E}^* \cap \mathbb{F}^*$, $\lambda(x) = \mu(x)$ implies $\alpha(x) = \beta(x)$. Then

$$\lambda(z^2)\mu(x)c(x, z^2) = \zeta(z^2)\alpha(z^2)\zeta(x)\beta(x)c(x, z^2)$$

$$= \zeta(z^2 x)\alpha(z^2)\beta(x)$$

$$= \zeta(z^2 x)\tau(z^2 x).$$

This shows that $\zeta_{\lambda, \mu}$ is well-defined and is a splitting of the cocycle. \hfill \Box

Lemma 5.2. (1) Choose a character $\alpha$ of $\mathbb{E}^*$ satisfying

$$\alpha(-1) = (-1, -1)_{\mathbb{F}}\gamma_{\mathbb{E}}(-1, \psi_\mathbb{E})$$

and let

$$\lambda_\alpha(z^2) = c(z, z)\gamma_{\mathbb{E}}(z, \psi_\mathbb{E})\gamma_{\mathbb{F}}(Nz, \psi_\mathbb{F})\alpha(z).$$
Then \( \lambda_\alpha \) is a well-defined splitting of the cocycle restricted to \( \mathbb{E}^2 \).
Furthermore every splitting of the cocycle restricted to \( \mathbb{E}^2 \) is equal to \( \lambda_\alpha \) for some \( \alpha \) satisfying (13).

(2) Let \( \beta \) be a character of \( \mathbb{F}^* \) and let
\[
\mu_\beta(x) = \gamma_\beta(x, \psi_\beta)\beta(x).
\]
Then \( \mu_\beta \) is a splitting of the cocycle restricted to \( \mathbb{F}^* \), and every splitting of the cocycle restricted to \( \mathbb{F}^* \) is equal to \( \mu_\beta \) for some \( \beta \).

(3) Suppose \( \alpha \in \mathbb{F}^*, \beta \in \mathbb{F}^* \) satisfy
\[
\alpha(z) = \gamma_\beta(z^2, \psi_\beta)\gamma_\beta(Nz, \psi_\beta)\gamma_\beta(z, \psi_\beta)c(z, z)\beta(z^2) \quad (z^2 \in \mathbb{F}^*).
\]
In particular \( \alpha \) satisfies (13). For \( z \in \mathbb{E}^*, x \in \mathbb{F}^* \) define
\[
\zeta_{\alpha, \beta}(z^2x) := \lambda_\alpha(z^2)\mu_\beta(x)c(x, z^2)
\]
\[
= \gamma_\beta(z, \psi_\beta)\gamma_\beta(Nz, \psi_\beta)\gamma_\beta(x, \psi_\beta)\alpha(z)\beta(x)c(z, z)c(x, z^2).
\]
Then \( \zeta \) is a well-defined splitting of the cocycle restricted to \( \mathbb{E}^2 \mathbb{F}^* \).
Furthermore every splitting of the cocycle restricted to \( \mathbb{E}^2 \mathbb{F}^* \) is equal to \( \zeta_{\alpha, \beta} \) for some \( \alpha, \beta \) satisfying (14).

**Proof.** Part (1) is an extension of (8) to the case of a non-abelian group.
Thus by (8) and (11),
\[
c(z^2, w^2) = c(z, z)c(w, w)c(zw, zw)\{z, w\}
\]
\[
= c(z, z)c(w, w)c(zw, zw)(Nz, Nz)\beta(Nz, Nz).
\]
Replacing \((z, w)\in\mathbb{E}\) by \(\gamma_\beta(z, \psi_\beta)\gamma_\beta(w, \psi_\beta)\gamma_\beta(zw, \psi_\beta)^{-1}\), and similarly \((Nz, Nz)\in\mathbb{E}\) gives
\[
c(z^2, w^2) = \tau(z)\tau(w)\tau(zw)^{-1}
\]
with \(\tau(z) = c(z, z)\gamma_\beta(z, \psi_\beta)\gamma_\beta(Nz, \psi_\beta)\). The same relation holds with \(\tau(z)\) replaced by \(\tau(z)\alpha(z)\).

We check the condition that \(\lambda_\alpha(z^2) := \tau(z)\alpha(z)\) be well-defined:
\[
\tau(-z)\alpha(-z)
\]
\[
= c(-z, z)\gamma_\beta(-z, \psi_\beta)\gamma_\beta(Nz, \psi_\beta)\alpha(-z)
\]
\[
= c(-z, z)\gamma_\beta(-1, \psi_\beta)\gamma_\beta(z, \psi_\beta)\alpha(-1, z)\gamma_\beta(Nz, \psi_\beta)\alpha(z)\alpha(-1).
\]
A simple calculation using (10) gives
\[
(16) \quad c(-z, -z) = (-1, -1)\mathbb{F}(-1, Nz)\mathbb{E}(z, z)
\]
and inserting this gives
\[
\tau(-z)\alpha(-z) = (-1, -1)\mathbb{F}(-1, \psi_\beta)\alpha(-1)\tau(z)\alpha(z)
\]
\[
= \tau(z)\alpha(z) \quad \text{by (13)}.
\]
Fix $\alpha$ satisfying (13). If $\lambda$ is any splitting of the cocycle restricted to $E^* \times E$ then $\lambda = \lambda_\alpha \delta$ for some character $\delta$ of $E^*$. Extend $\delta$ to a character $\delta^*$ of $E^*$. Then $\lambda_\alpha \delta = \lambda_\alpha \delta^*$, and $\alpha \delta^*$ satisfies (13). This proves (1).

By (10), $c(\text{diag}(x, x), \text{diag}(y, y)) = (x, y)_F$, and (2) follows as in Section 3.

For (3) apply Lemma 5.1. Inserting $\lambda_\alpha, \mu_\beta$ in the condition of the Lemma gives (14). The final assertion follows as in the proof of (1). This completes the proof. \qed

Let $\zeta$ be any splitting of the cocycle restricted to $E^* \times F^*$. Then $\zeta = \zeta_{\alpha, \beta}$ for some $\alpha, \beta$ satisfying (14). This implies:

**Lemma 5.3.** The map $\alpha(z) := \gamma_E(z^2, \psi_E)\gamma_E(Nz, \psi_E)\gamma_E(z, \psi_E)c(z, z)$ is a character of $\{ z \in E^* \mid z^2 \in F^* \}$.

For completeness we also prove this directly:

\[
\alpha(zw) = \gamma_E(z^2w^2, \psi_E)\gamma_E(Nzw, \psi_E)\gamma_E(zw, \psi_E)c(zw, zw)
\]

\[
= \gamma_E(z^2, \psi_E)\gamma_E(w^2, \psi_E)(z^2, w^2)\gamma_E(Nz, \psi_E)\gamma_E(Nw, \psi_E)(Nz, Nw)_F
\]

\[
\gamma_E(z, \psi_E)\gamma_E(w, \psi_E)(z, w)c(zw, zw)
\]

\[
= \alpha(z)\alpha(w)c(z, z)c(w, w)c(z^2, w^2)\{ z, w \}c(zw, zw) \quad \text{by (11)}
\]

\[
= \alpha(z)\alpha(w) \quad \text{by (8)}.
\]

**Remark 5.4.** Given $\alpha \in \widehat{E}^*$ there exists $\beta \in \widehat{F}^*$ satisfying (14) if and only if (13) holds.

This follows by an argument as in the proof of Lemma 5.2 (1): Define $\beta(z^2) = \alpha(z)\gamma_E(z^2, \psi_E)^{-1}\gamma_E(Nz, \psi_E)^{-1}\gamma_E(z, \psi_E)^{-1}c(z, z)$; this is well-defined if (13) holds, and extends to a character of $F^*$.

Suppose $z \in E^1$. By Hilbert’s Theorem 90, $z = w/N(w)$ for some $w \in E^*$. Then $z = w^2/N(w) \in E^* \times F^*$, so $\zeta$ restricts to a splitting of the extension of $E^1$. We now make explicit choices such that $\zeta$ is a $T$ or $\mu_n$ splitting as in Theorem 1.

**Lemma 5.5.** We may choose $\psi_E$ so that

\[
\gamma_E(x, \psi_E) = 1 \quad (x \in F^*).
\]

*Proof.* Since $\gamma_E(\cdot, \psi_E)$ is a quadratic character of $F^*$, $\gamma_E(x, \psi_E) = (x, y)_F$ for some $y \in F^*$. Suppose $E = F(\sqrt{\Delta})$. Then $\gamma_E(\Delta, \psi_E) = 1$ since $\Delta$ is a square in $E^*$. Therefore $(\Delta, y)_F = 1$, so $y = Nw$ for some $w \in E^*$. Replacing $\psi_E$ by $w\psi_E$ gives (cf. [10], Appendix)

\[
\gamma_E(x, w\psi_E) = (x, w)_E\gamma_E(x, \psi_E)
\]

\[
= (x, Nw)_E(x, y)_F = 1.
\]

This completes the proof. \qed
Lemma 5.6. Fix $\psi_E$ satisfying Lemma 5.5. Choose $\alpha, \beta$ satisfying (14) and let $\zeta = \zeta_{\alpha, \beta}$. Then

$$\zeta(z)^2 = \alpha(z) \quad (z \in E).$$

Proof. Writing $z = w/\bar{w} = w^2/N(w)$ and applying the definition (15) gives

(18) \quad $\zeta(w/\bar{w}) = (-1, Nw)_E \gamma_E(w, \psi_E) \alpha(w) \beta(Nw^{-1}) c(w, w) c(Nw^{-1}, w^2)$

and

(19) \quad $\zeta(w/\bar{w})^2 = (-1, Nw)_E \alpha(w^2) \beta(Nw^{-2})$

\[= (-1, Nw)_E \alpha(w^2) \alpha(Nw^{-2})(-1, Nw)_E \gamma_E(Nw^{-1}, \psi_E) \quad \text{by (14)}
\]

\[= \alpha(w/\bar{w}) \gamma_E(Nw, \psi_E)
\]

\[= \alpha(w/\bar{w}) \quad \text{by (17)}.
\]

$\square$

We see that $\zeta_{\alpha, \beta}$ is a $\mu_{2n}$ splitting if and only if $\alpha(z)^n = 1$ for all $z \in E^1$. We now complete the proof of Theorem 1.

Proof of Theorem 1. By (16) and (17) we have

$$\alpha(-1) = (-1, -1)_F.$$

Therefore we may choose $\alpha = 1$ if $(-1, -1)_F = 1$. Assume $(-1, -1)_F = -1$. If $F = \mathbb{R}$ then $\alpha(z) = z^n$ for $n$ odd as in Section 4, so assume $F$ is non-archimedean. If $-1 \not\in E^{\times 2}$ we may choose $\alpha^2 = 1$. If $-1 \in E^{\times 2}$ then $-1 \not\in E^{\times 4}$ (cf. §4) and we may choose $\alpha^4 = 1$.

We make some explicit choices and summarize the preceding discussion.

If $(-1, -1)_F = -1$ and $-1 \not\in E^{\times 2}$ choose $z_1 \in E^*$ with $(z_1, -1)_E = -1$.

If $(-1, -1)_F = -1$ and $-1 \in E^{\times 2}$, i.e., $E = \mathbb{F}(\sqrt{-1})$, then the norm residue symbol $(w, z)_E,4$ is defined. In particular the map $z \mapsto (w, z)_E,4$ is a character of $E^*$ of order 4. Choose $z_2 \in E^*$ satisfying $(z_2, -1)_E,4 = -1$.

Theorem 5.7. Choose a non-trivial character $\psi_E$ of $E$ such that $\gamma_E(x, \psi_E) = 1$ for all $x \in E^*$ (Lemma 5.5). For $z \in E^*$ let

$$\alpha(z) := \begin{cases} 
  z & F = \mathbb{R} \\
  1 & (-1, -1)_F = 1 \\
  (z_1, z)_E & (-1, -1)_F = -1, -1 \not\in E^{\times 2} \\
  (z_2, z)_E,4 & (-1, -1)_F = -1, -1 \in E^{\times 2}.
\end{cases}$$

Then $\alpha$ is a character of $E^*$ of order $\infty, 1, 2$ or 4 respectively, satisfying (13). Choose a character $\beta$ of $F^*$ satisfying

(20) \quad $\beta(z^2) = \gamma_F(z^2, \psi_F)^{-1} \gamma_F(Nz, \psi_F)^{-1} \gamma_E(z, \psi_E)^{-1} c(z, z) \alpha(z) \quad (z^2 \in F^*)$
In particular
\[ \beta(x^2) = (-1, x)_{\mathbb{F}} \alpha(x) \quad (x \in \mathbb{F}^*). \]

Let
\[ (21) \quad \zeta(w/\overline{w}) = \gamma_{\mathbb{E}}(w, \psi_{\mathbb{E}}) \alpha(w) \beta(Nw^{-1}) c(w, w)c(Nw^{-1}, w^2). \]

Then \( \zeta \) is a splitting of the cocycle restricted to \( \mathbb{E}^1 \). Furthermore for \( z \in \mathbb{E}^1 \)
\[ \zeta(x)^2 = \alpha(z), \]
and for \( \mathbb{F} \) non-archimedean this gives \( \zeta(z)^n = \alpha(z)^{n/2} = 1 \) with
\[ n = \begin{cases} 
2 & (-1, -1)_{\mathbb{F}} = 1 \\
4 & (-1, -1)_{\mathbb{F}} = -1, -1 \notin \mathbb{E}^2 \\
8 & (-1, -1)_{\mathbb{F}} = -1, -1 \in \mathbb{E}^2. 
\end{cases} \]

**Remark 5.8.** With \( \alpha, \beta \) as in the Theorem,
\[ \zeta(w/\overline{w}) = \zeta_{\alpha, \beta}(w/\overline{w})(-1, Nw)_{\mathbb{F}}. \]

We have dropped the term \( (-1, Nw) \), which is allowed since \( w/\overline{w} \rightarrow (-1, Nw) \) is a quadratic character of \( \mathbb{E}^1 \).

Henceforth write \( \mathbb{E} = \mathbb{F}(\delta) \) with \( \Delta := \delta^2 \in \mathbb{F} \).

**Remark 5.9.** Condition (20) is equivalent to
\[ \beta(x^2) = (-1, x)_{\mathbb{F}} \alpha(x) \quad (x \in \mathbb{F}^*) \]
\[ \beta(\Delta) = \gamma_{\mathbb{F}}(-1, \psi_{\mathbb{F}})^{-1} \gamma_{\mathbb{E}}(\delta, \psi_{\mathbb{E}})^{-1} c(\delta, \delta) \alpha(\delta). \]

The splitting has a simple formula on \( T^2 \):
\[ \zeta(z^2) = c(z, z) \alpha(z) \quad (z \in \mathbb{E}^1), \]
which is independent of \( \beta, \psi_{\mathbb{F}} \) and \( \psi_{\mathbb{E}} \). Note that any two \( \mu_2 \) splittings of \( T \) have the same restriction to \( T^2 \) since they differ by a quadratic character.

The map \( w/\overline{w} \rightarrow Nw \) induces an isomorphism \( T/T^2 \cong N\mathbb{E}^*/\mathbb{F}^{*2} \). Choose representatives \( a_1, \ldots, a_n \) of generators of \( N\mathbb{E}^*/\mathbb{F}^{*2} \), with corresponding elements \( z_1, \ldots, z_n \in T \). Given \( \alpha \) there are two choices of each \( \beta(a_i) \) differing by sign, and these signs may be chosen arbitrarily. The following result follows easily.

**Corollary 5.10.** Choose representatives \( z_1, \ldots, z_m \) of generators of \( T/T^2 \cong N\mathbb{E}^*/\mathbb{F}^{*2} \). Choose \( \alpha \) as in Theorem 5.7. Define
\[ \zeta(z^2) = c(z, z) \alpha(z) \quad (z \in \mathbb{E}^1) \]
and for \( 1 \leq i \leq m \) let \( \zeta(z_i) \) be either square root of \( \alpha(z_i) \). Then \( \zeta \) extends uniquely to a splitting of the cocycle as in Theorem 5.7.
In the non-archimedean case by ([4], Lemma 0.3.2) \(|T/T^2| = 2/|2|_F|_F\), which equals 2 if the residual characteristic of \(F\) is odd.

We conclude with a few remarks about the definition of \(\zeta\).

From the definition we have for \(w \in E^*:\)

\[
c(w, w) = (-Nw, x(w)x(w^2))_F
\]

and

\[
c(Nw^{-1}, w^2) = (Nw, x(w^2))_F.
\]

Note that for \(\lambda \in F^*, w \in E^*\) we have

\[
c(\lambda, w) = (\lambda, x(w))_F.
\]

Fix \(u \in E^*\) with \(\text{trace}(u) = 0\), and define \(\text{Tr}_u : E^* \rightarrow F^*\) by

\[
\text{Tr}_u(z) = \begin{cases} 
\text{trace}(z) & \text{trace}(z) \neq 0 \\
u z & \text{trace}(z) = 0.
\end{cases}
\]

Up to conjugation by \(SL(2, F)\) we may assume \(\iota(x + y\delta) = \left(x \ y\Delta/a \ a \ x\right)\) for some \(a \in F^*\). If \(w = x + y\delta\) with \(xy \neq 0\) we have \(x(w)x(w^2) = 2xy^2a^2\). Considering the cases with \(xy = 0\) separately gives

\[
c(w, w) = (-Nw, \text{Tr}_{a\delta}(w))_F \quad (z \in E^*)
\]

and

\[
\zeta(z^2) = (-1, \text{Tr}_{a\delta}(z))_F \alpha(z) \quad (z \in E^1).
\]

For example if \(p\) is odd and \(E \neq \mathbb{F}(\sqrt{-1})\) then \(m = 1\) (cf. Corollary 5.10), we may take \(z_1 = -1, \alpha = 1\) and \(\zeta(-1) = 1\) which gives

\[
\zeta(\epsilon z^2) = c(z, z)c(\epsilon, z^2) \quad (\epsilon = \pm 1, z \in E^1).
\]

For example let \(F = \mathbb{R}\) and define \(\iota(x + iy) = \left(x \ y \ -y \ x\right)\). We take \(\alpha(z) = z\) and \(\beta(x) = \pm \sqrt{|x|}\). Then for \(z \in E^1,\)

\[
\zeta(z^2) = (-1, \text{Tr}_{-i}(z))_\mathbb{R} z = \text{sgn}(\text{Tr}_{-i}(z))z
\]

(independent of \(\beta\)). Note that \(a = -1\) and \(\text{Tr}_{-i}(iy) = y \quad (y \in \mathbb{R}^*)\).

References


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