Pacific Journal of Mathematics

PRODUCT FORMULA FOR SELF-INTERSECTION NUMBERS

Yanghyun Byun and Seunghun Yi

Volume 200 No. 2 October 2001

PACIFIC JOURNAL OF MATHEMATICS
Vol. 200, No. 2, 2001

PRODUCT FORMULA FOR SELF-INTERSECTION NUMBERS

Yanghyun Byun and Seunghun Yi

We provide product formulae for self-intersection numbers in various coefficients.

1. Introduction.

Given an immersion $f : M^m \to P^{2m}$, $m \geq 1$, from a closed smooth manifold M to a smooth manifold P, there is a well-known invariant $I(f)$ called the self-intersection number of f. We will consider $I(f)$ in the $\mathbb{Z}_2$-coefficient or in the $\mathbb{Z}$-coefficient if M, P are oriented and m is even or in a $\mathbb{Z}$-module coefficient which is a quotient module of the free $\mathbb{Z}$-module on $\pi_1(P)$ (see §5). These self-intersection numbers can be used to determine whether or not f is regularly homotopic to an embedding if $m \geq 3$. We will consider the problem of what happens to the intersection number if one forms the product of two given immersions.

The problem is much simpler in the case of the type of intersection number which behaves as an obstruction for two submanifolds in an ambient space to get separated from each other by a homotopy (which is not necessarily regular): Let $M_1^{m_1}, M_2^{m_2}$ be submanifolds of $P^{m_1+m_2}$ which intersects transversely and $N_1^{n_1}, N_2^{n_2} \subset Q^{n_1+n_2}$ be another such triple. Then $M_1 \times N_1$ intersects transversely $M_2 \times N_2$ in $P \times Q$ at the points in $(M_1 \cap M_2) \times (N_1 \cap N_2)$. On the other hand, assume we are given two immersions $f : M^m \to P^{2m}, g : N^n \to Q^{2n}$ which are completely regular, that is, which are proper, self-transverse and have no triple points (see §2). In general, $f \times g$ is neither self-transverse nor without triple points. For example, if $p, q \in M$ are such that $f(p) = f(q), p \neq q$, we have $(f \times g)(p, y) = (f \times g)(q, y)$ for any $y \in N$. Therefore we must first transform $f \times g$ into a completely regular immersion through a regular homotopy before we calculate the intersection number.

In fact, a special case of the problem arised in the process of deriving the product formula for surgery obstructions in 1970's (cf. p. 55, [**Mo**]). It essentially concerned the case when g is an embedding, N is orientable and Q is simply connected in the above and the answer was given by, when rewritten in our notation:

$$I(f \times g) = I(f)\chi(\nu_g),$$

where $\chi(\nu_g)$ denotes the integral Euler characteristic of the normal bundle ν_g of g and the intersection numbers should be understood in such coefficients as introduced in §5 below. This of course coincides with our results in this paper. We are motivated by a different reason (cf. [**BY**]) and treat the problem in a complete generality.

The following is one of the two main results of this paper, which concerns the intersection number in the $\mathbb{Z}_2$ or the $\mathbb{Z}$-coefficient.

Theorem A. *Let $f : M^m \to P^{2m}$, $g : N^n \to Q^{2n}$ be immersions where M, N are closed smooth manifolds and P, Q, smooth manifolds. Then,*

(I) *for the mod 2 intersection numbers, we have*

$$I(f \times g) = \chi(\nu_f)I(g) + I(f)\chi(\nu_g) \in \mathbb{Z}_2,$$

where $\chi(\cdot)$ is the Euler characteristic in the $\mathbb{Z}_2$-coefficient.

Furthermore, assume M, N, P, Q are oriented and $m + n$ is even. Then, for the integral intersection numbers, we have

(II) *if both m, n are even,*

$$I(f \times g) = 2I(f)I(g) + \chi(\nu_f)I(g) + I(f)\chi(\nu_g) \in \mathbb{Z},$$

where $\chi(\cdot)$ mean the integral Euler characteristic,

(III) *and, if both m, n are odd, $I(f \times g) = 0 \in \mathbb{Z}$.*

In the above, ν_f, ν_g denote the normal bundles. It must be understood that a normal bundle is given the orientation which is consistent with the orientations of the manifolds. Then the formula in (II) above is invariant under the changes of the orientations of M, N and under those of P, Q.

In general, the mod 2 or the integral intersection number is not sophisticated enough to be an exact obstruction for the immersion in concern to be regularly homotopic to an embedding. Such an intersection number takes its value in a $\mathbb{Z}$-module which is a quotient module of the free $\mathbb{Z}$-module on the fundamental group of the codomain of the immersion in concern. Theorem B in the last section is none other than a generalization of Theorem A to this case. Even if the former unifies the equalities of the latter, it does so only by sacrificing simplicity of the coefficient in which the intersection number takes its values.

The key idea of the proofs of Theorems A and B might be best revealed by the following simple example.

Consider the case when $I(f) = 0$, $\chi(\nu_f) = 0$ and P is simply connected, $m \geq 3$ (cf. [**BY**]): Under the condition, we may assume f is an embedding and that ν_f admits a nowhere vanishing section. This enables us to construct an embedding $F : M \times I \to P$ such that $F(x, 0) = f(x)$ for any $x \in M$, using for instance the exponential map. For simplicity, assume g has only one double point and $p, q \in N, p \neq q$, are such that $g(p) = g(q)$. Choose a smooth function $\varphi : N \to I$ so that $\varphi(p) = 0, \varphi(q) = 1$. Define $\Lambda_t : M \times N \to P \times Q$,

$0 \le t \le 1$, by $\Lambda_t(x,y) = (F(x, t\varphi(y)), g(y))$ for any $(x,y) \in M \times N$. Then it is straightforward to see that Λ_t is a regular homotopy and Λ_1 is an embedding. Note that this observation is consistent with Theorem A.

The two key steps to the proofs of Theorems A and B are to construct carefully a regular homotopy for each of the immersions f, g and subsequently to use them to obtain a completely regular immersion regularly homotopic to $f \times g$ in a way similar to the above.

2. Basic notions and facts.

Throughout this section, let $f : M \to P$ be a smooth map between connected smooth manifolds.

We say f is an *immersion* if f is a proper map and $df : T_xM \to T_{f(x)}P$ is injective for each $x \in M$. Let I denote the closed unit interval in the real line $\mathbb{R}$. A homotopy $f_t : M \to P$, $t \in I$, is *regular* if f_t is an immersion for each $t \in I$.

From now on let M be of dimension m and P, of dimension $2m$.

We say an immersion $f : M \to P$ is *completely regular* if f has no triple points and f is self-transverse, that is, f satisfies the following condition,

$$dfT_pM + dfT_{p'}M = T_{f(p)}P = T_{f(p')}P,$$

for any $p, p' \in M$ such that $f(p) = f(p'), p \neq p'$ (cf. [**A**]). We will call $\{p, p'\}$ a *double pair* of f and $f(p) \in P$ a *double point* of f.

Now assign a metric d on P which induces the topology of P. Then, given any immersion $f : M \to P$ and any continuous function $\delta : M \to \mathbb{R}$, $\delta(x) > 0$, $x \in M$, H. Whitney ([**Wh**]) has shown that there is a regular homotopy f_t, $t \in I$, such that $f_0 = f$ and f_1 is a completely regular immersion and $d(f(x), f_t(x)) < \delta(x)$ for any $t \in I$, $x \in M$.

In the rest of this section, we assume further that M is a closed manifold.

If $f : M \to P$ is a completely regular immersion, one may define the intersection number $I(f)$ of f as follows: (i) For the mod 2 intersection number, one defines $I(f) \in \mathbb{Z}_2$ as the number of the double points mod 2. (ii) Assume that M, P are oriented and m is even. Then one may define the integral intersection number as follows: Let $r = f(p) = f(p')$, $p \neq p'$, be a double point of f. Let $v = (v_1, v_2, \dots, v_m)$, $v' = (v'_1, v'_2, \dots, v'_m)$ be sequences of tangent vectors which represent the orientation of M at p and p', respectively. If the sequence of tangent vectors $(dfv, dfv') = (dfv_1, dfv_2, \dots, dfv_m, dfv'_1, dfv'_2, \dots, dfv'_m)$ represents the orientation of P at r, write $\varepsilon_r = +1$ and, otherwise, write $\varepsilon_r = -1$. Note that ε_r remains unchanged even if we interchange p, p'. Define $I(f) = \sum_r \varepsilon_r \in \mathbb{Z}$, where r runs through all the double points of f.

If f, g are completely regular immersions which are regularly homotopic to each other, then we have $I(f) = I(g)$: According to J. Cerf ([**C**]), for generic regular homotopy, the double points vary continuously except at a finite set of points at each of which a pair of double points appear or disappear. If m is even, the two has opposite values for ε_r. Furthermore, since every immersion is regularly homotopic to a completely regular immersion, it follows that $I(f)$ is well-defined for any immersion f.

Now assume $m \geq 3$ and P is simply connected. Let $I(f)$ denote the mod 2 intersection number if the dimension of M is odd or M is unorientable and, in the remaining case, the integral intersection number. Then $I(f)$ vanishes if and only if the regular homotopy class of f can be represented by an embedding, which is a consequence of the Whitney trick (cf. [**Mi**], [**Wh**] and §5 of this paper).

3. A model case.

Throughout this section, let M^m, N^n, $m, n \geq 1$, be smooth manifolds and f, g, completely regular immersions from M, N, respectively, into P^{2m} and into Q^{2n}, each of which has only one double point. Furthermore, we assume that both ν_f, ν_g admit nowhere vanishing sections. Then we will prove the following.

Proposition 3.1. *The product $f \times g$ is regularly homotopic to a completely regular immersion with exactly two double points. Furthermore, assume M, N, P, Q are oriented and $m+n$ is even. Then the signs of the two double points differ from each other by multiplication by $(-1)^n$. If both m, n are even, then both of the signs for the two double points are the multiplication of the sign of the double point of f with that of g.*

To prove 3.1, we need the following lemma which will be proved later in this section.

Lemma 3.2. *There is a smooth regular homotopy $f_t : M \to P$, $t \in I$, such that $f_0 = f$ and the following conditions hold:*

(i) *f_t is a completely regular immersion with exactly one double pair $\{p_t, p_t'\}$ for each t,*
(ii) *the map $I \times \{0,1\} \to M$ which sends $(t,0)$ to p_t and $(t,1)$ to p_t' is a smooth embedding,*
(iii) *'$f_t(x) = f_s(y)$, $(x,t) \neq (y,s)$' implies that '$(x,y) = (p_s, p_t')$ or $(x,y) = (p_s', p_t)$' and*
(iv) *f_t meets f_s transversely if $t \neq s$.*

Proof of 3.1. Let f_t, $\{p_t, p_t'\}$ be as in Lemma 3.2 and also let $g_t : N \to Q$, $t \in I$, be a smooth regular homotopy for g satisfying the conditions of 3.2 with double pairs $\{q_t, q_t'\}$.

Choose a smooth function $\varphi : M \to I$ which is constantly 1 on a neighborhood of $\{p_t | t \in I\}$ and constantly 0 on a neighborhood of $\{p_t' | t \in I\}$. Likewise choose a smooth function $\psi : N \to I$ satisfying the same condition for the two sets $\{q_t | t \in I\}$, $\{q_t' | t \in I\}$.

Then we define a homotopy $\Lambda_t : M \times N \to P \times Q$, $t \in I$, by

$$\Lambda_t(x, y) = \left(f_{t\psi(y)}(x), g_{t\varphi(x)}(y)\right).$$

Then it is straightforward to see that Λ_t is a smooth homotopy through immersions such that $\Lambda_0 = f \times g$.

We must show that $\Lambda_1 = \Lambda$ has only two double points.

Assume $\Lambda(x, y) = \Lambda(x', y')$ and $(x, y) \neq (x', y')$. Then we have $x \neq x'$ or $y \neq y'$.

First consider the case $x \neq x'$. Then we have from $f_{\psi(y)}(x) = f_{\psi(y')}(x')$ that

$$(x, x') = (p_{\psi(y')}, p'_{\psi(y)}) \text{ or } (x, x') = (p'_{\psi(y')}, p_{\psi(y)}).$$

Assume $(x, x') = (p_{\psi(y')}, p'_{\psi(y)})$, it follows that $\varphi(x) = 1, \varphi(x') = 0$ and that $g_1(y) = g_0(y')$, which means that (y, y') is (q_0, q'_1) or (q'_0, q_1). If $(y, y') = (q_0, q'_1)$, then $(x, x') = (p_0, p'_1)$ and, if $(y, y') = (q'_0, q_1)$, then $(x, x') = (p_1, p'_0)$. Thus we have in this case as the double pairs for Λ

$$\{(p_0, q_0), (p'_1, q'_1)\}, \ \{(p_1, q'_0), (p'_0, q_1)\}.$$

Assume $(x, x') = (p'_{\psi(y')}, p_{\psi(y)})$ and proceed similarly as in the above. Then we obtain the same two double pairs for Λ as in the above.

Now assume $y \neq y'$. Then, from $g_{\varphi(x)}(y) = g_{\varphi(x')}(y')$, we may easily infer that $\psi(y) \neq \psi(y')$. Then, from $f_{\psi(y)}(x) = f_{\psi(y')}(x')$, we conclude that $x \neq x'$. Thus this case reduces to the case when $x \neq x'$.

We conclude that $\{(p_0, q_0), (p'_1, q'_1)\}$, $\{(p_1, q'_0), (p'_0, q_1)\}$ are the only two double pairs for Λ.

That Λ is self-transverse follows from the fact that f_0, f_1 are transverse to each other as well as g_0, g_1 together with the fact that φ, ψ are constant on each of some neighborhoods of $p_i, p'_i, q_i, q'_i, i = 0, 1$.

Finally we prove the last statement of the proposition.

Let $v_t = (v_{1,t}, v_{2,t}, \ldots, v_{m,t})$, $v'_t = (v'_{1,t}, v'_{2,t}, \ldots, v'_{m,t})$ and $w_t = (w_{1,t}, w_{2,t}, \ldots, w_{n,t})$, $w'_t = (w'_{1,t}, w'_{2,t}, \ldots, w'_{n,t})$ be sequences of vectors, continuously parameterized by $t \in I$, representing the given orientations of M and N at p_t, p'_t and at q_t, q'_t, respectively.

Let $\varepsilon(\omega)$ be 1 or -1 for each sequence ω of independent $2(m+n)$ tangent vectors in $T_{(x,y)}P \times Q$, $(x, y) \in P \times Q$, according to whether or not it represents the orientation of $P \times Q$, which is non other than the product orientation.

Then $\varepsilon(df_t v_0, dg_t w_0, df v'_t, dg w'_t)$, $\varepsilon(df v_t, dg_t w'_0, df_t v'_0, dg w_t)$ are constant for $t \in I$ and, by the usual sign convention, we have

$$\varepsilon(df v_0, dg w_0, df v'_0, dg w'_0) = (-1)^{2mn+n^2} \varepsilon(df v_0, dg w'_0, df v'_0, dg w_0).$$

Note that $2mn + n^2 \equiv n \bmod 2$. Thus we conclude that

$$\varepsilon(df_1 v_0, dg_1 w_0, df v'_1, dg w'_1) = (-1)^n \varepsilon(df v_1, dg_1 w'_0, df_1 v'_0, dg w_1).$$

Note that the left hand side of the equality is the intersection number of Λ at $\Lambda(p_0, q_0) = \Lambda(p_1', q_1')$ and the right hand side is the intersection number at $\Lambda(p_1, q_0') = \Lambda(p_0', q_1)$. These observations also proves the last statement of the proposition. $\square$

The rest of this section will be devoted to the proof of Lemma 3.2.

Write $r = f(p) = f(p')$, $p, p' \in M, p \neq p'$. Let D^m_ρ denote the open disk in $\mathbb{R}^m$ of radius $\rho > 0$ centered at the origin.

To prove 3.2, we will make use of Lemma 3.3 and Lemma 3.4 below.

Lemma 3.3. *There is a coordinate neighborhood $\psi : V \to \mathbb{R}^{2m} = \mathbb{R}^m \times \mathbb{R}^m$ of r, such that $\psi(r) = 0$ and there are disjoint open neighborhoods $U, U' \subset M$ of p, p' so that $f^{-1}V = U \cup U'$ and $\psi f U = \mathbb{R}^m \times \{0\}$, $\psi f U' = \{0\} \times \mathbb{R}^m$.*

Proof. Let $\psi_0 : V_0 \to \mathbb{R}^{2m}$ be a coordinate neighborhood of r. Also let $\varphi_0 : U_0 \to \mathbb{R}^m$, $\varphi_0' : U_0' \to \mathbb{R}^m$ be coordinate neighborhoods of p, p', respectively, such that $U_0, U_0' \subset f^{-1}V_0$, $U_0 \cap U_0' = \emptyset$. We choose $\psi_0, \varphi_0, \varphi_0'$ so that $\psi_0(r) = 0$ and $\varphi_0(p) = \varphi_0'(p') = 0$.

Consider $h : \mathbb{R}^{2m} = \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^{2m}$, defined by

$$h(x, y) = \psi_0 f \varphi_0^{-1}(x) + \psi_0 f {\varphi_0'}^{-1}(y).$$

Then it is straightforward to see that $dh_0 : T_0\mathbb{R}^{2m} \to T_0\mathbb{R}^{2m}$ is an isomorphism exploiting the fact that f is self-transverse. Therefore, there is an $\epsilon > 0$ such that h restricts to a diffeomorphism

$$h_1 : D^m_\epsilon \times D^m_\epsilon \to h(D^m_\epsilon \times D^m_\epsilon).$$

Then we consider the coordinate neighborhood of r,

$$\psi_1 = h_1^{-1}\psi_0 : \psi_0^{-1}(h(D^m_\epsilon \times D^m_\epsilon)) \to D^m_\epsilon \times D^m_\epsilon.$$

Now choose $\delta > 0$, exploiting the fact that f is proper, so that

$$\psi_1^{-1}(D^m_\delta \times D^m_\delta) \cap f(M - (\varphi_0^{-1}D^m_\epsilon \cup {\varphi_0'}^{-1}D^m_\epsilon)) = \emptyset.$$

Then we choose ψ as the restriction $\psi_1^{-1}(D^m_\delta \times D^m_\delta) \to D^m_\delta \times D^m_\delta$ of ψ_1 followed by a diffeomorphism $\alpha \times \alpha : D^n_\delta \times D^n_\delta \to \mathbb{R}^m \times \mathbb{R}^m$, where α is a diffeomorphism. $\square$

Note that, for any riemannian manifold, there is the exponential map defined in terms of the geodesics, which we denote by exp. In general, the map exp is a smooth map from an open neighborhood of the zero section in the tangent vector bundle into the manifold. On the other hand, once a riemannian metric $\langle \cdot, \cdot \rangle$ is introduced on P, we will identify the normal bundle ν_f with the subspace

$$\{(x, v) | x \in M, v \in T_{f(x)}P \text{ and } \langle v, w \rangle = 0 \text{ for any } w \in df_x T_x M\}$$

of $X \times TP$. Let $\pi : \nu_f \to M$ denote the projection. Then there is a map from a neighborhood of the zero section in ν_f into P which maps $(x, v) \in \nu_f$ to $\exp(v)$. We denote this map again by exp slightly abusing the notation.

Note that exp is an embedding on a neighborhood of the zero section of $\nu_f|_A$ if $f|_A : A \to P$ is an embedding for a subspace $A \subset M$.

Lemma 3.4. *Consider P with any riemannian metric. Let $V \subset P$ any open neighborhood of r. Then, for any open neighborhoods $U, U' \subset M$ of p, p' such that $U \cap U' = \emptyset$ and $f(\bar{U} \cup \bar{U}') \subset V$, there is an open neighborhood T of the zero section in ν_f satisfying the following conditions:*

(i) $T \cap \pi^{-1}U, T \cap \pi^{-1}U' \subset \exp^{-1} V$.

(ii) exp *is an embedding on each of* $T \cap \pi^{-1}(M - U'), T \cap \pi^{-1}(M - U)$.

Proof. Let U_1, U_1' be open neighborhoods of p, p', respectively, such that $\bar{U}_1 \subset U,\ \bar{U}_1' \subset U'$.

Then, since f embeds each of $M - U_1'$, $M - U_1$ into P (note that f is proper), there are open neighborhoods T_1, T_1' of the zero sections respectively in $\pi^{-1}(M - U_1')$ and in $\pi^{-1}(M - U_1)$ so that exp is an embedding. Furthermore, since $U \subset M - U_1'$, $U' \subset M - U_1$ and $f(\bar{U} \cup \bar{U}') \subset V$, we may choose T_1, T_1' so that $\exp(T_1 \cap \pi^{-1}U), \exp(T_1' \cap \pi^{-1}U') \subset V$.

Then we define T as follows:

$$T = ((T_1 \cap \pi^{-1}(M - \bar{U}_1')) \cap (T_1' \cap \pi^{-1}(M - \bar{U}_1))) \cup (T_1 \cap \pi^{-1}U) \cup (T_1' \cap \pi^{-1}U').$$

It is straightforward to see that T satisfies all the conditions of the lemma. □

Proof of 3.2. Cover P by a locally finite collection of coordinate neighborhoods $\psi_i : U_i \to \mathbb{R}^{2m}$, $i = 1, 2, \ldots$, such that (a) $r \in U_1$ and $\psi_1 : U_1 \to \mathbb{R}^{2m} = \mathbb{R}^m \times \mathbb{R}^m$ satisfies the conditions of Lemma 3.3 and (b) $\psi_1^{-1}(D_2^m \times D_2^m)$ does not intersect U_i for any $i > 1$. Write $C_2 = \psi_1^{-1}(D_2^m \times D_2^m)$.

Construct a riemannian metric on P by piecing together the pull-back metrics on U_i's of the standard metric on $\mathbb{R}^{2m}$ using a partition of unity for $\{U_i | i = 1, 2, \ldots\}$.

Then, by the condition (b), $\psi_1 : C_2 \to D_2^m \times D_2^m$ is an isometry.

Let T be an open neighborhood in ν_f of the zero section which satisfies the conditions (i), (ii) of Lemma 3.4 with respect to C_2 and with the open neighborhoods U, U' of p, p' defined by $f(U) = D_1^m \times \{0\}, f(U') = \{0\} \times D_1^m$. Write $\pi_T : T \to M$ for the restriction of the projection $\pi : \nu_f \to M$.

By assumption, there is a section $\alpha : M \to \nu_f$ such that $\alpha(x) \neq 0$ for any $x \in M$. We may assume $\alpha(x) \in T$ for any $x \in M$.

Allow ourselves a slight abuse of notation so that we may mean by $\alpha(x)$ the vector $v \in T_{f(x)}P$ for which (x, v) is the value of α at x. We may choose α so that $\alpha(x)$'s, $x \in U$, are parallel in C_2 as well as $\alpha(x')$'s, $x' \in U'$, and also so that $\langle \alpha(x), \alpha(x) \rangle^{\frac{1}{2}} < 1$ for any $x \in U \cup U'$.

The following proves Lemma 3.2.

Claim. The homotopy $f_t : M \to V$ defined by the rule

$$f_t(x) = \exp(t\alpha(x)),\ 0 \leq t \leq 1,$$

satisfies all the conditions of Lemma 3.2.

Proof. For any tangent vector $v \in T_xP$, $x \in C_2$, let v_0 denote the the tangent vector at $r \in P$ parallel to v in C_2.

First of all, we observe the following: Assume $\exp(v) = \exp(w)$, $v, w \in T$, $v \neq w$. Then, from our choice of T, it follows that $(v, w) \in \pi_T^{-1}U \times \pi_T^{-1}U'$ or $(v, w) \in \pi_T^{-1}U' \times \pi_T^{-1}U$. And exploiting the flatness of C_2, we may easily conclude that $f\pi_T(v) = \exp(w_0), f\pi_T(w) = \exp(v_0)$.

It is clear that f_t is an immersion for each $t \in I$ since it is the immersion $t\alpha : M \to T$ followed by the local diffeomorphism $\exp : T \to P$.

Assume $f_t(x) = f_t(y)$ for some $x, y \in M, x \neq y$. Then $\exp(t\alpha(x)) = \exp(t\alpha(y))$. Since $t\alpha(x), t\alpha(y) \in T$, we must have: $(x, y) \in U \times U'$ or $(x, y) \in U' \times U$.

If $(x, y) \in U \times U'$, we have

$$f(x) = \exp(t\alpha(y)_0) = \exp(t\alpha(p')),\ f(y) = \exp(t\alpha(x)_0) = \exp(t\alpha(p')).$$

Similarly, if $(x, y) \in U' \times U$, we have $x = \exp(t\alpha(p))$, $y = \exp(t\alpha(p'))$. Therefore, let $p_t \in U$, $p'_t \in U'$, $t \in I$, be defined by:

$$f(p_t) = \exp(t\alpha(p')), f(p'_t) = \exp(t\alpha(p)).$$

Then f_t has only one double pair, $\{p_t, p'_t\}$, for each $t \in I$. Note that $p_0 = p$, $p'_0 = p'$.

Also note that f_t is self-transverse since $(df_t)_{p_t}T_{p_t}M$ is parallel to df_pT_pM in C_2 and $(df_t)_{p'_t}T_{p'_t}M$ is also parallel to $df_{p'}T_{p'}M$ in C_2. This proves that the homotopy f_t, $t \in I$, satisfies (i).

It is clear that the homotopy f_t, $t \in I$, satisfies (ii) with the double pairs $\{p_t, p'_t\}$, $t \in I$, since we have $\{p_t\}_{t \subset I} \subset U$, $\{p'_t\}_{t \subset I} \subset U'$, $U \cap U' = \emptyset$.

Assume $f_t(x) = f_s(y), (x, t) \neq (y, s)$. If $t = s$, then (iii) follows from (i). If $t \neq s$, then from the equality $\exp(t\alpha(x)) = \exp(s\alpha(y))$ it follows that we must have that $(x, y) \in U \times U'$ or $(x, y) \in U' \times U$ and that $f(x) = \exp(s\alpha(y)_0)$ and $f(y) = \exp(t\alpha(x)_0)$. If $(x, y) \in U \times U'$, then $\alpha(x)$, $\alpha(y)$ are parallel respectively to $\alpha(p)$, $\alpha(p')$ in C_2, which leads to the conclusion $(f(x), f(y)) = (\exp(s\alpha(p')_0), \exp(t\alpha(p)_0))$. Thus we have $(x, y) = (p_s, p'_t)$. Likewise we conclude that if $(x, y) \in U' \times U$ then $(x, y) = (p'_s, p_t)$. This proves that the homotopy f_t, $t \in I$, satisfies (iii).

Assume $t \neq s$. Note that $(df_t)_{p_s}T_{p_s}M$ is parallel to df_pT_pM in C_2 and $(df_t)_{p'_t}T_{p'_t}M$, to $df_{p'}T_{p'}M$. Thus f_t meets f_s transversely at $f_t(p_s) = f_s(p'_t)$. Likewise we may conclude that f_t meets f_s transversely at $f_t(p'_s) = f_s(p_t)$ as well. Thus the homotopy f_t, $t \in I$, satisfies (iv). $\square$

4. Proof of Theorem A.

We begin this section by recalling the following well-known fact.

Lemma 4.1. *Let m be a positive odd integer. Then any orientable vector bundle of rank m over an orientable manifold M^m admits a nowhere vanishing section.*

Proof. The Euler class of an oriented vector bundle of an odd rank is 2-torsion (cf. p. 98, [**MS**]). Since $H^m(M;\mathbb{Z})$ has no torsion, this means the Euler class of the bundle vanishes. However the Euler class is the exact obstruction for an oriented vector bundle in concern to admit a nowhere vanishing section. This completes the proof. $\square$

Assume M^m is oriented and m is even. Let ξ be a smooth oriented vector bundle of rank m over M. We will denote the total space of ξ again by ξ. Then ξ itself is an oriented manifold with the orientation determined by those of the bundle ξ and M. Assume s is a smooth section of ξ which meets the zero section transversely. If $s(p) = 0$, let $\varepsilon(p)$ be the sign of the intersection p between the two embeddings of M into ξ, that is, between the zero section and s. Then the integral Euler characteristic $\chi(\xi)$ satisfies the equality, $\chi(\xi) = \sum_p \varepsilon(p)$, in which p runs through all the zero points of s.

The proof of Theorem A is immediate from the following.

Proposition 4.2. *Let $f : M^m \to P^{2m}$, $g : N^n \to Q^{2n}$ be completely regular immersions with respective double points $r_1, r_2, \dots \in P$, $s_1, s_2, \dots \in Q$. Assume there are sections $\alpha : M \to \nu_f$ and $\beta : N \to \nu_g$ which meet the zero sections transversely respectively at $a_1, a_2, \dots \in M$, $\{a_1, a_2, \dots\} \cap f^{-1}\{r_1, r_2, \dots\} = \emptyset$, and at $b_1, b_2, \dots \in N$, $\{b_1, b_2, \dots\} \cap g^{-1}\{s_1, s_2, \dots\} = \emptyset$. Then,*

(a) *$f \times g : M \times N \to P \times Q$ is regularly homotopic to a completely regular immersion Λ which has, as its double points, two for each of the ordered pairs (r_i, s_j) and one for each of (a_k, s_j), (r_i, b_l), all of which are distinct among themselves.*

Furthermore, assume $m + n$ is even and M, N, P, Q are oriented. Then we have that

(b) *if $x_{i,j}, x_{i,j'}$ are the two double points of Λ corresponding to each of (r_i, s_j), we have $\varepsilon_{x_{i,j}} = (-1)^n \varepsilon_{x'_{i,j}}$ and,*

(c) *if in addition both m, n are even, we have $\varepsilon_{x_{i,j}} = \varepsilon_{x'_{i,j}} = \varepsilon_{r_i}\varepsilon_{s_j}$ and if $y_{k,j}$, $z_{i,l}$ denote the double points of Λ corresponding respectively to (a_k, s_j), (r_i, b_l), we have $\varepsilon_{y_{k,j}} = \varepsilon(a_k)\varepsilon_{s_j}$, $\varepsilon_{z_{i,l}} = \varepsilon_{r_i}\varepsilon(b_l)$.*

The proof of 4.2 will be given in the later of this section. Here we provide:

Proof of Theorem A. We may assume f, g are completely regular, say, with respective double points $r_1, r_2, \dots, r_\kappa \in P$ and $s_1, s_2, \dots, s_\lambda \in Q$. Also let

$\alpha : M \to \nu_f$, $\beta : N \to \nu_g$ be the sections which meet the zero section transversely, say, respectively at $a_1, a_2, \dots, a_\mu \in M$ and at $b_1, b_2, \dots, b_\nu \in N$. We may assume $\{a_1, a_2, \dots, a_\mu\} \cap f^{-1}\{r_1, r_2, \dots, r_\kappa\} = \emptyset$, $\{b_1, b_2, \dots, b_\nu\} \cap g^{-1}\{s_1, s_2, \dots, s_\lambda\} = \emptyset$. Then let $\Lambda : M \times N \to P \times Q$ be a completely regular immersion regularly homotopic to $f \times g$ as in 4.2.

Then the statement (I) is clear since Λ has $(2\kappa\lambda + \mu\lambda + \kappa\nu)$ double points by (a) of 4.2.

Now assume M, N, P, Q are oriented and $m + n$ is even.

If both m, n are odd, then by 4.1 we may assume that $\{a_1, a_2, \dots, a_\mu\} = \emptyset$, $\{b_1, b_2, \dots, b_\nu\} = \emptyset$. Then by (a), (b) of 4.2, Λ has $2\kappa\lambda$ double points, two for each (r_i, s_j), $i = 1, 2, \dots, \kappa$, $j = 1, 2, \dots, \lambda$ whose signs are opposite to each other. This proves the clause (III).

Also if both m, n are even, then by (a), (c) of 4.2, we have

$$I(\Lambda) = 2\sum_{i,j} \varepsilon_{r_i}\varepsilon_{s_j} + \sum_{k,j} \varepsilon(a_k)\varepsilon_{s_j} + \sum_{i,l} \varepsilon_{r_i}\varepsilon(b_l) \ .$$

Thus it follows that $I(f \times g) = I(\Lambda) = 2I(f)I(g) + \chi(\nu_f)I(g) + I(f)\chi(\nu_g) \in \mathbb{Z}$ as claimed in the clause (II). $\square$

To prove 4.2, we need the following generalization of Lemma 3.2. For more details of the proof, one must refer to the Proof of 3.2.

Lemma 4.3. *Let $f : M \to P$, $r_1, r_2, \dots \in P$, $\alpha : M \to \nu_f$, $a_1, a_2, \dots \in M$ be as in 4.2. Then there is a smooth regular homotopy $f_t : M \to P$, $t \in I$, such that $f_0 = f$ and satisfying the following conditions:*

(i) *f_t is a completely regular immersion with exactly one double pair $\{p_{i,t}, p'_{i,t}\}$ for each $t \in I$ and for each $i = 1, 2, \dots$,*
(ii) *the map $I \times \{0, 1\} \times \{1, 2, \dots\} \to M$ which sends $(t, 0, i)$ to $p_{i,t}$ and $(t, 1, i)$ to $p'_{i,t}$ is a smooth embedding,*
(iii) *'$f_t(x) = f_s(y)$, $(x, t) \neq (y, s)$' implies that '$(x, y) = (p_{i,s}, p'_{i,t})$ or $(x, y) = (p'_{i,s}, p_{i,t})$, for some $i = 1, 2, \dots$, or $x = y = a_j$, for some $j = 1, 2, \dots$,'*
(iv) *and f_t meets f_s transversely if $t \neq s$.*

Proof. First choose disjoint coordinate neighborhoods $\psi_i : V_i \to \mathbb{R}^{2m}$ of r_i, $i = 1, 2, \dots$ so that each of them satisfies the conditions of Lemma 3.3 with some neighborhoods U_i, U'_i of p_i, p'_i, where $f(p_i) = f(p'_i) = r_i$, $p_i \neq p'_i$. Then it is straightforward to construct a riemannian metric for which the restriction of ψ_i to $\psi_i^{-1}(D_2^m \times D_2^m) \to D_2^m \times D_2^m$ is an isometry for each i.

Write $U_{i,1}, U'_{i,1} \subset M$ for the open neighborhoods of p_i, p'_i such that $\psi_i f U_{i,1} = D_1^m \times \{0\}$, $\psi_i f U'_{i,1} = \{0\} \times D_1^m$. Then, by slightly generalizing Lemma 3.4 above, we may construct an open neighborhood T of the zero section in ν_f such that $\exp : T \to P$ embeds each of $\pi_T^{-1}(M - \cup_i U_{i,1})$, $\pi_T^{-1}(M - \cup_i U'_{i,1})$ and

$\exp \pi_T^{-1}(U_{i,1} \cup U'_{i,1}) \subset \psi_i^{-1}(D_1^m \times D_1^m)$, where $\pi_T : T \to M$ is the restriction of the projection $\pi : \nu_f \to M$.

We may assume the section α is chosen so that $\alpha M \subset T$. Then we define a homotopy $f_t : M \to P, t \in I$ by $f_t(x) = \exp(t\alpha(x))$. It is straightforward to see that $f_t : M \to P, t \in I$, is a regular homotopy satisfying all the conditions of the lemma. □

Proof of 4.2. Let $f_t, t \in I$, be as in 4.3 and choose $g_t, t \in I$, so that it satisfies the conditions of 4.3 with the double pairs $\{q_{j,t}, q'_{j,t}\}$, $j = 1, 2, \dots$.

Let $\varphi : M \to I$ be a smooth function which is constantly 1 on a neighborhood of $\cup_i \{p_{i,t} | t \in I\} \cup \{a_1, a_2, \dots\}$ and constantly 0 on a neighborhood of $\cup_i \{p'_{i,t} | t \in I\}$. Likewise choose a smooth function $\psi : N \to I$ satisfying the same condition for the two sets, $\cup_j \{q_{j,t} | t \in I\} \cup \{b_1, b_2, \dots\}$, $\cup_j \{q'_{j,t} | t \in I\}$.

Now we define a homotopy $\Lambda_t : M \times N \to P \times Q$, $t \in I$, as before, by

$$\Lambda_t(x, y) = (f_{t\psi(y)}(x), g_{t\varphi(x)}(y)).$$

Then it is straightforward to see that Λ_t, $t \in I$, is a regular homotopy such that $\Lambda_0 = f \times g$.

Write $\Lambda = \Lambda_1$ and assume $\Lambda(x, y) = \Lambda(x', y')$, $(x, y) \neq (x', y')$.

Then, we obtain, as the double pairs of Λ,

$$\{(p_{i,0}, q_{j,0}), (p'_{i,1}, q'_{j,1})\},\ \{(p_{i,1}, q'_{j,0}), (p'_{i,0}, q_{j,1})\},$$

for each i, j, and

$$\{(a_k, q_{j,1}), (a_k, q'_{j,1})\},\ \{(p_{i,1}, b_l), (p'_{i,1}, b_l)\},$$

for each k, j and for each i, l. The former are the two double pairs corresponding to (r_i, s_j), which are obtained essentially by 3.1. Note that in this case we have $x \neq x', y \neq y'$. The latter are the double pairs respectively corresponding to (a_k, s_j), (r_i, b_l), for which we have $x = x', y \neq y'$ or $x \neq x', y = y'$.

Since Λ is clearly self-transverse, this proves the statement (a). The clause (b) has been essentially proved by Lemma 3.2. The first part of (c) also has been proved by Lemma 3.2 and its last part is clear. □

5. The non-simply connected case.

We begin this section with a detailed description of the intersection number which behaves as the exact obstruction for a given immersion to be regularly homotopic to an embedding even when the relevant manifolds are not simply connected. In what follows, the usual notational conventions for the paths must be understood.

Let $f : M^m \to P^{2m}$ be a completely regular immersion between connected smooth manifolds. Assume M is closed.

First of all, we recall when the Whitney trick can be applied to cancel two double points (cf. [**Mi**], [**Wh**]). Let $r_0, r_1 \in P$ be two double points of f and $\{p_0, p_0'\} = f^{-1}\{r_0\}$, $\{p_1, p_1'\} = f^{-1}\{r_1\}$. Assume there are paths, $\alpha, \alpha' : I \to M$, such that:

(i) $\alpha(0) = p_0, \alpha(1) = p_1$, $\alpha'(0) = p_0', \alpha'(1) = p_1'$,
(ii) $(f\alpha) \cdot (f\alpha')^{-1}$ is a contractible loop in P and,
(iii) for continuously parameterized orientations ω_t, ω_t' respectively of $T_{\alpha(t)}M$, $T_{\alpha'(t)}M$, the signs $\varepsilon(df\omega_0, df\omega_0')$, $\varepsilon(df\omega_1, df\omega_1')$ are opposite, when the signs are determined with respect to the orientations of $T_{r_0}P, T_{r_1}P$ which are the restrictions of a continuously parameterized orientation of $T_{f\alpha(t)}P$, $t \in I$.

Furthermore, assume $m \geq 3$. Then α, α' can be chosen as smooth embeddings and there is a smoothly embedded disk in P which meets fM on two arcs which extend the arcs $f\alpha I$, $f\alpha' I$ slightly and subsequently one may use these to apply the Whitney trick to cancel the two double points.

Now choose a base point $x_0 \in M$ and write $f(x_0) = z_0$ and fix orientations for $T_{x_0}M$, $T_{z_0}P$. Let $r \in P$ be a double point of f and $\{p, p'\} = f^{-1}\{r\}$. Choose paths $\alpha, \alpha' : I \to M$ such that $\alpha(0) = \alpha'(0) = x_0$ and $\alpha(1) = p, \alpha'(1) = p'$. Then $f\alpha(1) = f\alpha'(1) = r$ and therefore $(f\alpha) \cdot (f\alpha')^{-1} : I \to P$ is a loop based at z_0. Write $\gamma_r = [(f\alpha) \cdot (f\alpha')^{-1}] \in \pi_1(P)$. Also we decide the sign $\varepsilon_r = \pm 1$ as follows: Use the paths α, α' together with the orientation of $T_{x_0}M$ to orient $T_pM, T_{p'}M$ and use the path $f\alpha$ together with the orientation of $T_{z_0}P$ to orient T_rP. We write $\varepsilon_r = 1$ if the orientation of T_rP coincides with the one determined by the ordered pair of oriented subspaces dfT_pM, $dfT_{p'}M$ and $\varepsilon_r = -1$ otherwise. We will consider $\varepsilon_r\gamma_r$ in $\mathbb{Z}\pi_1(P)$, the free $\mathbb{Z}$-module on $\pi_1(P)$.

However, $\varepsilon_r\gamma_r$ depends on the choice of α, α'. Let $w_M : \pi_1(M) \to \{\pm 1\}$, $w_P : \pi_1(P) \to \{\pm 1\}$ be the orientation characters, that is, the homomorphisms which respectively represent the first Stiefel-Whitney classes of M and P. Then for any $a, a' \in \pi_1(M)$, the element

$$w_P(f_*a)w_M(a)w_M(a')\varepsilon_r(f_*a)\gamma_r(f_*a')$$

could have been chosen instead of $\varepsilon_r\gamma_r$ if we chose α, α' differently. On the other hand, if we interchanged p, p', $(-1)^m w_P(\gamma_r)\varepsilon_r\gamma_r^{-1}$ could have been chosen by the same process. Here the multiplication by $w_P(\gamma_r)$ is due to our using the path $f\alpha'$ to orient T_rP instead of $f\alpha$.

Therefore we denote by K_f the submodule of $\mathbb{Z}\pi_1(P)$ generated by

$$\{b - w_P(f_*a)w_M(a)w_M(a')(f_*a)b(f_*a'),$$
$$b - (-1)^m w_P(b)b^{-1} | a \in \pi_1(M), b \in \pi_1(P)\}$$

and consider the quotient module

$$\mathbb{Z}\pi_1(P)/K_f \equiv \Gamma_f.$$

Note that, if there is a homotopy from f to f', then there corresponds a natural isomorphism from Γ_f to $\Gamma_{f'}$. Furthermore, if the homotopy fixes the base point, then $\Gamma_f = \Gamma_{f'}$ and the isomorphism is just the identity.

Write $[\gamma_r]$ to denote the class in Γ_f represented by $\gamma_r \in \pi_1(P)$. Note that $\varepsilon_r[\gamma_r]$ is a well-defined element in Γ_f for any double point r, even if each of ε_r, $[\gamma_r]$ in general depends on the choice of α, α'. Then, the intersection number $I_\Gamma(f)$ of f is defined by

$$I_\Gamma(f) = \sum_r \varepsilon_r[\gamma_r] \in \Gamma_f$$

where r runs through all the double points of f.

It is straightforward to see that $I_\Gamma(f)$ is invariant of the regular homotopy class of f up to the natural isomorphisms and that $I_\Gamma(f)$ is well-defined even if f is only an immersion (see §2 and also p. 46, [**Wa**]). Also it is clear by construction that $I_\Gamma(f) = 0$ if and only if f is regularly homotopic to an embedding, assuming $m \geq 3$.

Let $g : N^n \to Q^{2n}$ be another immersion from a connected closed manifold N to a connected manifold Q, where N, Q have base points y_0, $w_0 = g(y_0)$ and $T_{y_0}N$, $T_{w_0}Q$ are oriented. Consider $M \times N$, $P \times Q$ with the base points (x_0, y_0), (z_0, w_0) and with the product orientations for $T_{(x_0,y_0)}M \times N$, $T_{(z_0,w_0)}P \times Q$. Then, $I_\Gamma(f \times g) \in \Gamma_{f\times g}$ must be well-defined.

Note that there is a natural isomorphism $\pi_1(P) \times \pi_1(Q) \to \pi_1(P \times Q)$. We will write $a \otimes b$ for the image of $(a, b) \in \pi_1(P) \times \pi_1(Q)$ by this map. For simplicity we let w denote the orientation character for any manifold in concern.

Lemma 5.1. *There is a well-defined map $* : \Gamma_f \times \Gamma_g \to \Gamma_{f\times g}$, defined by extending the rule bilinearly*

$$*([a], [b]) \equiv [a] * [b] = [a \otimes b] + (-1)^n w(b)[a \otimes b^{-1}],$$

for any $a \in \pi_1(P), b \in \pi_1(Q)$.

Proof. We have that

$$\begin{aligned}
&((-1)^m w(a)[a^{-1}]) * [b] \\
&= (-1)^m w(a)[a^{-1} \otimes b] + (-1)^{m+n} w(a)w(b)[a^{-1} \otimes b^{-1}] \\
&= (-1)^n w(b)[a \otimes b^{-1}] + [a \otimes b] = [a] * [b].
\end{aligned}$$

Similarly we have $[a] * ((-1)^n w(b)[b^{-1}]) = [a] * [b]$. Also, for any $c, c' \in \pi_1(M), d, d' \in \pi_1(N)$, it is straightforward to see the identities:

$$(w(f_*c)w(c)w(c')[(f_*c)a(f_*c')]) * [b] = [a] * [b],$$

$$[a] * (w(g_*d)w(d)w(d')[(g_*d)b(g_*d')]) = [a] * [b].$$

□

Let e denote the identity element for any fundamental group. Even if in general the rule $([a],[b]) \to [a \otimes b]$ does not provide a well-defined map from $\Gamma_f \times \Gamma_g$ into $\Gamma_{f\times g}$, we have the following, for which we omit the proof:

Lemma 5.2. *The maps, $\iota_1 : \Gamma_f \to \Gamma_{f\times g}$, $\iota_2 : \Gamma_g \to \Gamma_{f\times g}$ respectively defined by extending the rules $\iota_1([a]) = [a \otimes e]$, $\iota_2([b]) = [e \otimes b]$ for any $a \in \pi_1(P)$, $b \in \pi_1(Q)$ linearly is well-defined.*

On the other hand, we need to consider the Euler characteristic of a vector bundle over a manifold in a more general context than before.

Let ξ be a smooth vector bundle of rank l over the connected manifold L^l, which has the base point x_0 and with a fixed orientation for $T_{x_0}L$. Assume ξ_{x_0} is oriented. Let s be a smooth section of ξ which meets the zero section transversely. Let $p \in L$ be such that $s(p) = 0$. Let α be a path from x_0 to p. Then ξ_p, T_pL are oriented subspaces of $T_p\xi$ regarding ξ itself as a manifold, in which the orientations are respectively determined by the orientations of ξ_{x_0}, $T_{x_0}L$ together with the path α. Define the sign $\varepsilon(p) = 1$ if the orientation of $T_p\xi$ determined by the ordered pair of oriented subspaces dsT_pL, T_pM coincides with the one determined by another such pair ξ_p, T_pL and $\varepsilon(p) = -1$ otherwise. Then $\varepsilon(p)$ does not depend on the choice of the path α if and only if $w_1(\xi) = w_1(L) \equiv w_1(TL)$, where w_1 denotes the first Stiefel-Whitney class. If $w_1(\xi) = w_1(L)$, $\chi(\xi)$ is defined by $\chi(\xi) = \sum_p \varepsilon(p) \in \mathbb{Z}$, where p runs through all the zeros of s. In fact, this is the Euler characteristic in the twisted integral coefficient (cf. [**B**]). If $w_1(\xi) \neq w_1(L)$, then $\chi(\xi)$ is defined as the number of the zeros of s modulo 2. We will refer to $\chi(\xi)$ defined in this way as *the Euler characteristic in the twisted coefficient*.

Before the statement of the second main result of the paper, we need to observe the following.

Lemma 5.3. *Assume $w_1(\nu_f) \neq w_1(M)$. Then $[e] \in \Gamma_f$ is 2-torsion and $\iota_2(y) \in \Gamma_{f\times g}$ is also 2-torsion for any $y \in \Gamma_g$.*

Proof. Note that under the assumption f^*TP is not orientable. Moreover f^*TP is orientable if and only if $w_P(f_*a) = 1$ for any $a \in \pi_1(M)$. Therefore, there is an $a \in \pi_1(M)$ such that $w_P(f_*a) = -1$. Then the following observations prove the lemma: Firstly we have

$$e - w(f_*a)w(a)w(a^{-1})(f_*a)e(f_*a^{-1}) = 2e$$

is in K_f and secondly for any $b \in \pi_1(Q)$ we have

$$\begin{aligned} e \otimes b - w((f \otimes g)_*(a \otimes e))w(a \otimes e)w(a^{-1} \otimes e) \\ \cdot ((f \times g)_*(a \otimes e))(e \otimes b)((f \times g)_*(a^{-1} \otimes e)) \end{aligned}$$

is $2e \otimes b$ and it is in $K_{f\times g}$. □

Therefore, if $w_1(\nu_f) \neq w_1(M)$, there is an action by mod 2 integers on $\iota_2(y)$ coming from the $\mathbb{Z}$-action and the product $\chi(\nu_f)\iota_2(y)$ in 5.1 below should be understood in this sense for any $y \in \Gamma_g$. Similarly with $\chi(\nu_g)\iota_1(x)$ for any $x \in \Gamma_f$.

In the following, we understand the fiber $(\nu_f)_{x_0}$ is given the consistent orientation in the sense that the orientation of $T_{z_0}P$ determined by the ordered pair of oriented subspaces $\iota(\nu_f)_{x_0}$, $dfT_{x_0}M$ coincides with the fixed orientation, where ι is the natural bundle monomorphism given by a choice of riemannian metric on P. Similarly with $(\nu_g)_{y_0}$.

Then we have the following product formula which unifies the equalities in Theorem A.

Theorem B. *Assume M^m, N^n, P^{2m}, Q^{2n} are connected closed smooth manifolds and assume further that M, N are closed. Let $f : M \to P$, $g : N \to Q$ be immersions. Let x_0, y_0 be the respective base points of M, N and $z_0 = f(x_0)$, $w_0 = g(y_0)$, those of P, Q. Assume $T_{x_0}M$, $T_{z_0}P$ and $T_{y_0}N$, $T_{w_0}Q$ are oriented and $T_{(x_0,y_0)}M \times N$, $T_{(z_0,w_0)}P \times Q$ are given the product orientations. Then, we have*

$$I_\Gamma(f \times g) = (-1)^{mn}\big(I_\Gamma(f) * I_\Gamma(g) + \chi(\nu_g)\iota_1(I_\Gamma(f)) + \chi(\nu_f)\iota_2(I_\Gamma(g))\big) \in \Gamma_{f\times g},$$

where $\chi(\cdot)$ denotes the Euler characteristic in the twisted coefficients.

Proof. We retain the notations and contexts of the statement of 4.2 and its proof. Note that

$$I_\Gamma(f) = \sum_i \varepsilon_{r_i}[\gamma_{r_i}] \in \Gamma_f,\ I_\Gamma(g) = \sum_j \varepsilon_{s_j}[\gamma_{s_j}] \in \Gamma_g$$

where each of $\gamma_{r_i} \in \pi_1(P)$, $\gamma_{s_j} \in \pi_1(Q)$ and ε_{r_i}, ε_{s_j}, are determined by a choice of an ordered pair of paths in M or in N for each of the double points.

Note that we have as the double points of Λ, for each i, j,

$$\begin{aligned} x_{i,j} &= (f_1(p_{i,0}), g_1(q_{j,0})) = (f(p'_{i,1}), g(q'_{j,1})), \\ x'_{i,j} &= (f(p_{i,1}), g_1(q'_{j,0})) = (f_1(p'_{i,0}), g(q_{j,1})), \end{aligned}$$

and also for each j, k and for each i, l,

$$\begin{aligned} y_{k,j} &= (f(a_k), g_1(q_{j,1})) = (f(a_k), g_1(q'_{j,1})), \\ z_{i,l} &= (f_1(p_{i,1}), g(b_l)) = (f_1(p'_{i,1}), g(b_l)). \end{aligned}$$

By definition we have

$$I_\Gamma(\Lambda) = \sum_{i,j}(\varepsilon_{x_{i,j}}[\gamma_{x_{i,j}}] + \varepsilon_{x'_{i,j}}[\gamma_{x'_{i,j}}]) + \sum_{k,j}\varepsilon_{y_{k,j}}[\gamma_{y_{k,j}}] + \sum_{i,l}\varepsilon_{z_{i,l}}[\gamma_{z_{i,l}}]$$

where the expression in the right hand side depends essentially on a choice of an ordered pair of paths in $M \times N$ for each of the double points of Λ.

We may assume that $\varphi(x_0) = 0, \psi(y_0) = 0$. Then note that the homotopy $\Lambda_u : M \times N \to P \times Q$, $u \in I$, which is defined by $\Lambda_u(x, y) = (f_{u\psi(y)}(x), g_{u\varphi(x)}(y))$, preserves the base point. The following completes the proof of the theorem.

Claim. We may arrange so that

$$\gamma_{x_{i,j}} = \gamma_{r_i} \otimes \gamma_{s_j}, \quad \gamma_{x'_{i,j}} = \gamma_{r_i} \otimes \gamma_{s_j}^{-1},$$
$$\gamma_{y_{k,j}} = e \otimes \gamma_{s_j}, \quad \gamma_{z_{i,l}} = \gamma_{r_i} \otimes e$$

and

$$\varepsilon_{x_{i,j}} = (-1)^{mn}\varepsilon_{r_i}\varepsilon_{s_j}, \quad \varepsilon_{x'_{i,j}} = (-1)^{mn+n}w(\gamma_{s_j})\varepsilon_{r_i}\varepsilon_{s_j},$$
$$\varepsilon_{y_{k,j}} = (-1)^{mn}\varepsilon(a_k)\varepsilon_{s_j}, \quad \varepsilon_{z_{i,l}} = (-1)^{mn}\varepsilon(b_l)\varepsilon_{r_i}.$$

Proof. For any paths, $\alpha : I \to M$, $\beta : I \to N$, write $\alpha \otimes \beta : I \to M \times N$ to denote the path defined by $\alpha \otimes \beta(t) = (\alpha(t), \beta(t))$, $t \in I$.

Let $\alpha, \alpha' : I \to M$ be the paths such that $\alpha(0) = \alpha'(0) = x_0$ and $\alpha(1) = p_{i,0}, \alpha'(1) = p'_{i,0}$ and the loop $a = (f\alpha) \cdot (f\alpha')^{-1}$ represents the class $\gamma_{r_i} \in \pi_1(P)$. Similarly let $\beta, \beta' : I \to N$ be the paths such that the loop $b = (g\beta) \cdot (g\beta')^{-1}$ represents the class $\gamma_{s_j} \in \pi_1(Q)$.

First we consider the double point

$$x_{i,j} = (f_1(p_{i,0}), g_1(q_{j,0})) = (f_0(p'_{i,1}), g_0(q'_{j,1})).$$

Let $\delta'_u : I \to M, u \in I$, be defined by $\delta'_u(t) = p'_{i,ut}$ and $\epsilon'_u : I \to N, u \in I$, by $\epsilon'_u(t) = q'_{j,ut}$.

Then let $\gamma_{x_{i,j}} \in \pi_1(P \times Q)$ be the class represented by the loop $(\Lambda(\alpha \otimes \beta)) \cdot (\Lambda((\alpha \cdot \delta'_1) \otimes (\beta \cdot \epsilon'_1)))^{-1}$.

Note that

$$\Lambda_u(p_{i,0}, q_{j,0}) = (f_u(p_{i,0}), g_u(q_{j,0})) = (f_0(p'_{i,u}), g_0(q'_{j,u})) = \Lambda_u(p'_{i,u}, q'_{j,u}),$$

for any $u \in I$. Therefore, $(\Lambda_u(\alpha \otimes \beta)) \cdot (\Lambda_u((\alpha' \cdot \delta'_u) \otimes (\beta' \cdot \epsilon'_u)))^{-1}$ is well-defined for any $u \in I$. This gives a loop homotopy from $(f\alpha \otimes g\beta) \cdot (f\alpha' \otimes g\beta')^{-1} = ((f\alpha) \cdot (f\alpha')^{-1}) \otimes ((g\beta) \cdot (g\beta')^{-1})$ to the loop which represents $\gamma_{x_{i,j}}$, which proves that $\gamma_{x_{i,j}}$ can be chosen as $\gamma_{r_i} \otimes \gamma_{s_j}$.

To prove that $\gamma_{x'_{i,j}}$ can be chosen as $\gamma_{r_i} \otimes \gamma_{s_j}^{-1}$, we introduce the paths $\delta_u : I \to M$, $\delta_u(t) = p_{i,ut}$ and $\epsilon_u : I \to N$, $\epsilon_u(t) = q_{j,ut}$, for each $u \in I$. Then consider the homotopy $(\Lambda_u((\alpha \cdot \delta_u) \otimes \beta')) \cdot (\Lambda_u((\alpha' \otimes (\beta \cdot \epsilon_u)))^{-1}$, which provides a loop homotopy from $((f\alpha) \cdot (f\alpha')^{-1}) \otimes ((f\beta') \cdot (f\beta)^{-1})$ to a loop which we let represent $\gamma_{x'_{i,j}} \in \pi_1(P \times Q)$.

To prove that $\gamma_{y_{k,j}}$ can be chosen as $e \otimes \gamma_{s_j}$, we choose a path $\bar{\alpha} : I \to M$ such that $\bar{\alpha}(0) = x_0, \bar{\alpha}(1) = a_k$ and consider the homotopy $(\Lambda_u(\bar{\alpha} \otimes (\beta \cdot \delta_u))) \cdot (\Lambda_u(\bar{\alpha} \otimes (\beta' \cdot \delta'_u)))^{-1}$, $u \in I$. This gives a homotopy from $((f\bar{\alpha}) \cdot (f\bar{\alpha})^{-1}) \otimes ((g\beta) \cdot (g\beta')^{-1})$ to a loop which we let represent the class $\gamma_{y_{k,j}}$. Similarly to prove that $\gamma_{z_{i,l}}$ can be chosen as $\gamma_{r_i} \otimes e$.

It remains to prove the equalities concerning the signs of the double points with respect to the above choices of ordered pairs of paths in $M \times N$ for the double pairs of Λ.

First consider $\varepsilon_{x_{i,j}}$. By definition it is determined as follows: Note that $T_{x_{i,j}} P \times Q$ is oriented by the orientation on $T_{(z_0,y_0)} P \times Q$ and the path $\Lambda(\alpha \otimes \beta)$. Also $T_{(p_{i,0},q_{j,0})} M \times N$, $T_{(p'_{i,1},q'_{j,1})} M \times N$ are oriented by the orientation of $T_{(x_0,y_0)} M \times N$ and the paths $\alpha \otimes \beta$, $(\alpha \cdot \delta'_1) \otimes (\beta \cdot \epsilon'_1)$. Then we compare the orientation of $T_{x_{i,j}} P \times Q$ with the orientation given by the ordered pair of subspaces $d\Lambda T_{(p_{i,0},q_{j,0})} M \times N$, $d\Lambda T_{(p'_{i,1},q'_{j,1})} M \times N$.

Note that, for each $u \in I$, in particular including $u = 0$, the paths $\Lambda_u(\alpha \otimes \beta)$, $\Lambda_u((\alpha' \cdot \delta'_u) \otimes (\beta' \cdot \epsilon'_u))$, also determines a sign for $\Lambda_u(p_{i,0}, q_{j,0})$. Since this assignment of signs is continuous, the sign $\varepsilon_{x_{i,j}}$ is the same as the one given by the following data: i) The orientation of $T_{(f(p_{i,0}),g(q_{j,0}))} P \times Q$ determined by the path $f\alpha \otimes g\beta$ and ii) the orientations of $T_{(p_{i,0},q_{j,0})} M \times N$, $T_{(p'_{i,0},q'_{j,0})} M \times N$ whose orientations are respectively determined by the paths $\alpha \otimes \beta$, $\alpha' \otimes \beta'$. From this it is immediate that $\varepsilon_{x_{i,j}} = (-1)^{mn} \varepsilon_{r_i} \varepsilon_{s_j}$.

Similar considerations establish the other claimed equalities. In particular, the multiplication by $w(\gamma_{s_j})$ in the equality $\varepsilon_{x'_{i,j}} = (-1)^{mn+n} w(\gamma_{s_j}) \varepsilon_{r_i} \varepsilon_{s_j}$ is due to the fact that we use the path $f\alpha \otimes f\beta'$ to orient $T_{(r_i,s_j)} P \times Q$ instead of the path $f\alpha \otimes f\beta$. $\square$

Remark. Theorem A is a corollary of Theorem B only modulo Lemma 4.1.

References

[A] M. Adachi, *Embeddings and Immersions*, Amer. Math. Soc., Providence, R. I., 1993, MR 95a:57039, Zbl 810.57001.

[B] Y. Byun, *Tangent fibration of a Poincaré complex*, J. of London Math. Soc., **59**(**3**) (1999), 1101-1116, MR 2000f:57023, Zbl 935.57032.

[BY] Y. Byun and S. Yi, *Weakly Lagrangian embedding and product manifolds*, Bull. of Kor. Math. Soc., **35** (1998), 809-817, MR 2000f:53107, Zbl 933.53037.

[C] J. Cerf, *Sur les Diffeomorphisms de la Sphère de Dimension Trois* ($\Gamma_4 = 0$), Lecture Notes in Math., no. 53, Springer-Verlag, 1968, MR 37 #4824.

[Mi] J. Milnor, *Lectures on the h-Cobordism Theorem*, Princeton University Press, 1965, MR 32 #8352, Zbl 161.20302.

[MS] J. Milnor and J. Stasheff, *Characteristic Classes*, Princeton University Press, 1974, MR 55 #13428, Zbl 298.57008.

[Mo] J.W. Morgan, *A product formula for surgery obstructions*, Memoirs of the American Mathematical Society, **201**, Amer. Math. Soc., 1978, MR 8 #18496, Zbl 388.57017.

[Wa] C.T.C. Wall, *Surgery on compact manifolds*, Academic Press Inc., London, 1970, MR 55 #4217, Zbl 219.57024.

[Wh] H. Whitney, *The self-intersections of a smooth n-manifold in 2n-space*, Ann. of Math., **45** (1944), 220-246, MR 5,273g, Zbl 063.08237.

Received March 1, 1999. This work was supported by grant no. 2000-2-10100-002-3 from the Basic Research Program of the Korea Science & Engineering Foundation.

Department of Mathematics
Hanyang University
Sungdong-gu Seoul 133-791
Korea
E-mail address: yhbyun@fermat.hanyang.ac.kr

Liberal Arts and Science (Mathematics)
Youngdong University
Youngdong, Chungbuk, 370-701
Korea
E-mail address: seunghun@youngdong.ac.kr