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Dedicated to Professor Sa Ge Lee on his 60th birthday

For a row finite directed graph E, Kumjian, Pask, and Raeburn proved that there exists a universal C^*-algebra $C^*(E)$ generated by a Cuntz-Krieger E-family. In this paper we consider two density problems of invertible elements in graph C^*-algebras $C^*(E)$, and it is proved that $C^*(E)$ has stable rank one, that is, the set of all invertible elements is dense in $C^*(E)$ (or in its unitization when $C^*(E)$ is nonunital) if and only if no loop of E has an exit. We also prove that for a locally finite directed graph E with no sinks if the graph C^*-algebra $C^*(E)$ has real rank zero ($RR(C^*(E)) = 0$), that is, the set of invertible self-adjoint elements is dense in the set of all self-adjoint elements of $C^*(E)$ then E satisfies a condition (K) on loop structure of a graph, and that the converse is also true for $C^*(E)$ with finitely many ideals. In particular, for a Cuntz-Krieger algebra $\mathcal{O}_A$, $RR(\mathcal{O}_A) = 0$ if and only if A satisfies Cuntz's condition (II).

1. Introduction.

Given an $n \times n$ $\{0,1\}$-matrix A with no zero row or column, a family of n partial isometries S_i satisfying the relation

$$(*) \qquad S_i^* S_i = \sum_{j=1}^{n} A(i,j) S_j S_j^*$$

is called a *Cuntz-Krieger A-family*. In [**CK**], under a condition (I) on the matrix A, it is proved that any two such families generate isomorphic C^*-algebras, thus the Cuntz-Krieger algebra $\mathcal{O}_A$ is well-defined. Furthermore when A satisfies condition (II) which is stronger than (I) the ideal structure of $\mathcal{O}_A$ was analysed by Cuntz in [**C**].

As a generalization of Cuntz-Krieger algebras one may consider a C^*-algebra generated by a family of partial isometries satisfying the relation $(*)$ for some infinite $\{0,1\}$-matrix A, provided every row of A contains only finitely many 1's, and this has been done in [**KPRR**] and [**KPR**] with directed graphs. For any row finite directed graph E with countable vertices

$\{v \mid v \in E^0\}$ and edges $\{e \mid e \in E^1\}$, the associated graph C^*-algebra $C^*(E)$ is defined to be a universal C^*-algebra generated by a family of partial isometries $\{s_e \mid e \in E^1\}$ and a family of mutually orthogonal projections $\{p_v \mid v \in E^0\}$ subject to the relations:

$$s_e^* s_e = p_{r(e)},\ p_v = \sum_{s(f)=v} s_f s_f^*,$$

where $r(e)$ (respectively, $s(e)$) denotes the range (respectively, source) vertex of the edge e. If $\{A(e,f)\}$ is the edge matrix of E then these relations give a generalized form of $(*)$, that is, $s_e^* s_e = \sum_{s(f)=r(e)} A(e,f) s_f s_f^*$.

If E has no sinks then there is a locally compact r-discrete groupoid $\mathcal{G}_E$ associated with E whose unit space ${\mathcal{G}_E}^0$ is identified with the infinite path space of E. Furthermore it is shown in [**KPRR**], Theorem 4.2 that the groupoid C^*-algebra $C^*(\mathcal{G}_E)$ is isomorphic to $C^*(E)$, and hence those useful results on groupoid C^*-algebras in [**Rn1**] and [**Rn2**] could be used to analyse the structure of $C^*(E)$. One important therem in [**KPRR**] is about the ideal structure of graph C^*-algebras; there is an inclusion preserving one-to-one map of saturated hereditary vertex subsets of E into the ideals of $C^*(E)$ and moreover if E satisfies a condition (K) then the map is also bijective.

A graph-theoretic condition (L) analogous to Cuntz-Krieger's condition (I) was given in [**KPR**], where it was shown that if E is a locally finite directed graph with no sinks and satisfies (L) then a C^*-algebra generated by a Cuntz-Krieger E-family of non-zero elements is isomorphic to $C^*(E)$. One interesting result among others in [**KPR**] is that $C^*(E)$ is AF if and only if E has no loops. It is also shown in [**D**] that every AF-algebra arises as the C^*-algebra of a locally finite pointed directed graph in the sense of [**KPRR**]. Recall that every AF algebra A has stable rank one ($sr(A) = 1$); the set of invertible elements is dense in A (or $\widetilde{A}$ if A is nonunital). In Section 3, we give a necessary and sufficient graph-theoretic condition on E for the graph algebra $C^*(E)$ to have stable rank one; $sr(C^*(E)) = 1$ if and only if no loop of E has an exit.

We see from [**KPR**] that if E is a cofinal graph with no sinks and satisfies (L) then the universal C^*-algebra $C^*(E)$ is simple and it is either AF or purely infinite. It is also well-known that all AF algebras and purely infinite simple C^*-algebras have real rank zero, that is, every self-adjoint element can be arbitrarily closely approximated by invertible self-adjoint elements (or in the unitized algebra for a nonunital C^*-algebra). So it would be interesting to know when a non-simple graph C^*-algebra can have real rank zero, and we prove in Section 4 that for a locally finite directed graph E with no sinks if the graph algebra $C^*(E)$ has real rank zero ($RR(C^*(E)) = 0$) then the graph must satisfy condition (K). Conversely we also show that for any locally finite graph E with no sinks if E satisfies condition (K) and

$C^*(E)$ has finitely many ideals then $RR(C^*(E)) = 0$. In particular, if E is a locally finite graph with no sinks and has finitely many vertices then $RR(C^*(E)) = 0$ if and only if E satisfies condition (K). Therefore, for a Cuntz-Krieger algebra $\mathcal{O}_A$ associated with a $\{0,1\}$-matrix A satisfying (I), $RR(\mathcal{O}_A) = 0$ if and only if A satisfies condition (II) since A can be viewed as a vertex matrix of a finite graph E which has no sinks and satisfies (L) and that the finite graph E satisfies condition (K) is equivalent to that its vertex matrix A satisfies condition (II).

2. Preliminaries.

We recall some definitions and notations from [**KPR**] and [**KPRR**] on directed graphs, graph C^*-algebras, and groupoids associated with graphs. A *directed graph* $E = (E^0, E^1, r, s)$ consists of countable sets E^0 of vertices and E^1 of edges, and the range, source maps $r, s : E^1 \to E^0$. E is *row finite* (*locally finite*) if for each vertex $v \in E^0$, $s^{-1}(v)$ is (both $r^{-1}(v)$ and $s^{-1}(v)$ are) finite. We call a locally finite graph E *finite* if E^0 is finite. If $e_1, \dots, e_n$ ($n \geq 2$) are edges with $r(e_i) = s(e_{i+1})$, $1 \leq i \leq n-1$, then we can form a (finite) path $\alpha = (e_1, \dots, e_n)$ of *length* $|\alpha| = n$, and extend the maps r, s by $r(\alpha) = r(e_n), s(\alpha) = s(e_1)$.

Let E^n be the set of all finite paths of length n and

$$\begin{aligned} E^* &:= \cup_{n\geq 0} E^n, \ r(v) = s(v) = v \text{ for } v \in E^0, \\ E^\infty &:= \{(\alpha_i)_{i=1}^\infty | \alpha_i \in E^1, r(\alpha_i) = s(\alpha_{i+1})\}. \end{aligned}$$

A vertex $v \in E^0$ with $s^{-1}(v) = \emptyset$ is called a *sink*.

Given a row finite directed graph E, a *Cuntz-Krieger E-family* consists of a set $\{P_v \,|\, v \in E^0\}$ of mutually orthogonal projections and a set $\{S_e \,|\, e \in E^1\}$ of partial isometries satisfying the relations

$$S_e{}^* S_e = P_{r(e)}, \ e \in E^1, \text{ and } \ P_v = \sum_{s(e)=v} S_e S_e{}^*, \ v \in s(E^1).$$

From these relations, one can show that every non-zero word in S_e, P_v and S_f^* is a partial isometry of the form $S_\alpha S_\beta^*$ for some $\alpha, \beta \in E^*$ with $r(\alpha) = r(\beta)$ ([**KPR**], Lemma 1.1).

Theorem 2.1 ([**KPR**, Theorem 1.2])**.** *For a row finite directed graph $E = (E^0, E^1)$, there exists a C^*-algebra $C^*(E)$ generated by a Cuntz-Krieger E-family $\{s_e, p_v \,|\, v \in E^0, e \in E^1\}$ of non-zero elements such that for any Cuntz-Krieger E-family $\{S_e, P_v \,|\, v \in E^0, e \in E^1\}$ of partial isometries acting on a Hilbert space $\mathcal{H}$, there is a representation $\pi : C^*(E) \to B(\mathcal{H})$ such that*

$$\pi(s_e) = S_e, \ \textit{and} \ \ \pi(p_v) = P_v$$

for all $e \in E^1, v \in E^0$.

A finite path α with $|\alpha| > 0$ is called a *loop* at v if $s(\alpha) = r(\alpha) = v$. If the vertices $\{r(\alpha_i) \,|\, 1 \le i \le |\alpha|\}$ are distinct, the loop α is *simple*.

E is said to satisfy a condition (L) if every loop in E has an exit, and a condition (K) if for any vertex v on a loop there exist at least two distinct loops α, β based at v, that is, $r(\alpha) = r(\beta) = s(\alpha) = s(\beta) = v$, $r(\alpha_i) \ne v$ for $1 \le i < |\alpha|$, and $r(\beta_j) \ne v$ for $1 \le j < |\beta|$. Note that the condition (K) is stronger than (L) and if E has no loops then the two conditions are trivially satisfied.

If E has no sinks then $E^\infty \ne \emptyset$ and we have the following groupoid associated with E

$$\mathcal{G}_E = \{(x, k, y) \in E^\infty \times \mathbb{Z} \times E^\infty \,|\, x_i = y_{i+k} \text{ for sufficiently large } i\,\}$$

$$(x, k, y)^{-1} := (y, -k, x),$$

$$(x, k, y) \cdot (y, l, z) := (x, k + l, z).$$

Then the range and source maps $r, s : \mathcal{G}_E \to {\mathcal{G}_E}^0$ are given by

$$r(x, k, y) = x,\ \ s(x, k, y) = y.$$

$\mathcal{G}_E$ is a locally compact r-discrete groupoid with respect to a suitable topology and ${\mathcal{G}_E}^0$ is identified with E^∞. Furthermore the groupoid algebra $C^*(\mathcal{G}_E)$ is isomorphic to the graph C^*-algebra $C^*(E)$ by Theorem 4.2 of [**KPRR**].

3. Stable rank of $C^*(E)$.

Recall that a C^*-algebra A has stable rank one ($sr(A) = 1$) if the set A^{-1} of all invertible elements is dense in A (in $\widetilde{A}$ if A is non-unital). One can show that every C^*-algebra A with $sr(A) = 1$ is stably finite, and so there is no infinite projection in A. If two C^*-algebras A and B are strong Morita equivalent, in particular if they are stably isomorphic, then $sr(A) = 1$ if and only if $sr(B) = 1$ ([**BP2**], [**Rf**]).

Lemma 3.1 ([**BP2**, Proposition 6.4])**.** *Let I be an ideal of a C^*-algebra A. Then $sr(A) = 1$ if and only if $sr(I) = sr(A/I) = 1$ and every invertible element lifts (that is, $(\widetilde{A}/I)^{-1} = \widetilde{A}^{-1}/I$).*

We say that a subgraph H of E has *no exit* if $e \in E^1$, $s(e) \in H^0$ implies $e \in H^1$.

Lemma 3.2 ([**KPR**, Proposition 2.1])**.** *If H is a subgraph of a directed graph E with no exit then*

$$I := \overline{\text{span}}\{s_\alpha {s_\beta}^* \,|\, \alpha, \beta \in E^*, r(\alpha) = r(\beta) \in H^0\}$$

is a closed ideal of $C^*(E)$ strong Morita equivalent to the hereditary C^*-subalgebra $B := \overline{\text{span}}\{s_\alpha s_\beta^* \,|\, \alpha, \beta \in H^*\}$.

We call a vertex v *cofinal* if for any infinite path $x = (x_1, x_2, \dots) \in E^\infty$ there is a finite path $\alpha \in E^*$ with $s(\alpha) = v$ and $r(\alpha) = s(x_n)$ for some n ([**KPRR**]). A directed graph E is said to be *cofinal* if every vertex is cofinal.

Theorem 3.3. *Let $E = (E^0, E^1, r, s)$ be a row finite directed graph. Then E has no loop with an exit if and only if $sr(C^*(E)) = 1$.*

Proof. If E has no loops then $C^*(E)$ is AF and so $sr(C^*(E)) = 1$. Assume that E has loops and every loop has no exit. Let H be the subgraph of E consisting of all the loops. Since H has no exit, by Lemma 3.2,

$$I = \overline{\text{span}}\{s_\beta {s_\gamma}^* \mid \beta, \gamma \in E^*,\ r(\beta) = r(\gamma) \in H^0\}$$

is an ideal of $C^*(E)$ which is strong Morita equivalent to the hereditary subalgebra $B = \overline{\text{span}}\{s_\beta {s_\gamma}^* \mid \beta, \gamma \in H^*\}$. Let α be a simple loop in E, then $v = s(\alpha)$ is cofinal in the subgraph H_α consisting only of α, and H_α has no sinks. Thus $C^*(H_\alpha) \cong C^*(\mathcal{G}_{H_\alpha})$ ([**KPRR**], Theorem 4.2). Let $N = \{x \in H_\alpha^\infty \mid s(x) = v\}$, and ${\mathcal{G}_{H_\alpha}}_N^N$ be the reduction of $\mathcal{G}_{H_\alpha}$ to N. Then by [**KPRR**], Theorem 3.1, $C^*({\mathcal{G}_{H_\alpha}}_N^N)$ is isomorphic to the full corner of $C^*(\mathcal{G}_{H_\alpha})$, so they are strong Morita equivalent. Since N consists of only one path, say x, and ${\mathcal{G}_{H_\alpha}}_N^N = \{(x, kn, x) | k \in \mathbb{Z}\} \cong \mathbb{Z}$, $C^*(H_\alpha)$ is strong Morita equivalent to the group C^*-algebra $C^*(\mathbb{Z}) \cong C(\mathbb{T})$. Since $C(\mathbb{T})$ has stable rank 1, it follows that $sr(C^*(H_\alpha)) = 1$, and so $sr(B_\alpha) = 1$, where $B_\alpha := \overline{\text{span}}\{s_\beta {s_\gamma}^* | \beta, \gamma \in H_\alpha^*\}$, because B_α is a quotient algebra of $C^*(H_\alpha)$. Thus $sr(I_\alpha) = sr(B_\alpha) = 1$, where

$$I_\alpha := \overline{\text{span}}\{s_\beta {s_\gamma}^* \mid \beta, \gamma \in E^*,\ r(\beta) = r(\gamma) \in H_\alpha^0\}.$$

Therefore $sr(I) = 1$ since I is the direct sum of the ideals I_α.

Now, let D be the C^*-subalgebra of $C^*(E)$ generated by

$$\{s_e \mid e \in E^1 \setminus H^1\} \cup \{p_v \mid v \in E^0\},$$

which is a Cuntz-Krieger G-family for the subgraph $G = (E^0, E^1 \setminus H^1)$ of E. Thus by Theorem 2.1 there is a $*$-homomorphism from $C^*(G)$ onto D. Since G has no loops at all, $C^*(G)$ is an AF algebra having stable rank one, so we have $sr(D) = 1$ by Lemma 3.1.

It is clear that under the canonical projection $\pi : C^*(E) \to C^*(E)/I$ the subalgebra D of $C^*(E)$ maps onto $C^*(E)/I$ and hence the stable rank of $C^*(E)/I$ is one as a homomorphic image of an algebra of stable rank one. Also, every invertible element in the AF algebra $\pi(\widetilde{D})(= \widetilde{C^*(E)/I})$ is connected to the unit, whence it lifts to an invertible element in $\widetilde{C^*(E)}$. Then by Lemma 3.1, $sr(C^*(E)) = 1$.

Conversely, suppose that E has a simple loop $\alpha = (\alpha_1, \dots, \alpha_n)$ with an exit at $v = s(\alpha)$. It is easy to see that the projection p_v is infinite, so the algebra $C^*(E)$ is not stably finite, whence $sr(C^*(E)) \neq 1$.

Lemma 3.4. *If V is the set of all sinks in E then*

$$I := \overline{\operatorname{span}}\{s_\alpha s_\beta^* \mid \alpha, \beta \in E^*, r(\alpha) = r(\beta) = v \text{ for some } v \in V\}$$

is a closed two-sided ideal of $C^*(E)$. With $E^*(v) = \{\alpha \in E^* \mid r(\alpha) = v\}$, we have

$$I \cong \oplus_{v\in V}\mathcal{K}(\ell^2(E^*(v))).$$

Proof. For each $v \in V$, let

$$I_v := \overline{\operatorname{span}}\{s_\alpha s_\beta{}^* \mid \alpha, \beta \in E^*, r(\alpha) = r(\beta) = v\}.$$

Then by Corollary 2.2 of [**KPR**], I_v is a closed ideal of $C^*(E)$ and isomorphic to $\mathcal{K}(\ell^2(E^*(v)))$. If $\beta, \gamma \in E^*$, with $r(\beta) = v_i, r(\gamma) = v_j$, then $s_\beta^* s_\gamma = 0$ when $i \neq j$, whence the ideals are mutually orthogonal.

If a (locally finite) directed graph E has sinks then it might not contain any infinite paths so that we can not directly apply results on groupoid C^*-algebras since the groupoid $\mathcal{G}_E$ associated with E was invented to have its unit space consisting of infinite paths in E. In case E has no sinks, in [**KPRR**], an isomorphism of lattice of saturated hereditary subsets V of E^0 into the lattice of ideals $I(V)$ in $C^*(E)(\cong C^*(\mathcal{G}_E))$ was established and it is shown that the quotient algebra $C^*(E)/I(V)$ is isomorphic to the graph algebra $C^*(G)$ for a certain subgraph G of E. The proof applies the results on ideal structure of groupoid algebras obtained in [**Rn1**, **Rn2**]. See Section 4 for this isomorphism. In the following we show a similar assertion when V is the set of all sinks in E. For this, we need to recall that a vertex subset H of E^0 is *saturated* if whenever $v \in E^0$ emits only edges e with $r(e) \in H$, we have $v \in H$. The smallest saturated vertex subset containing V is called the *saturation* of V.

Theorem 3.5. *Let $E = (E^0, E^1, r, s)$ be a locally finite directed graph with the set V of sinks. Then there is a subgraph $G = (E^0 \setminus H, \{e \in E^1 \mid r(e) \notin H\})$ of E with no sinks such that $C^*(E)/I(V)$ is isomorphic to $C^*(G)$, where H is the saturation of V and $I(V) = \overline{\operatorname{span}}\{s_\alpha s_\beta{}^* \mid \alpha, \beta \in E^*,\ r(\alpha) = r(\beta) \in V\}$.*

Proof. Note that the ideal $I(= I(V))$ contains the projections p_v, for $v \in V$. If $e \in E^1, r(e) = v$ for some $v \in V$ then $s_e \in I$ because $s_e = s_e s_e{}^* s_e = s_e p_v \in I$. For an edge $e \in E^1$ with $r(e) \notin V$ we have

$$s_e = s_e p_{r(e)} = \sum_{s(f)=r(e)} s_e s_f s_f{}^* p_{r(e)} \in I$$

whenever the vertex $r(e)$ emits only edges f with $s_f \in I$. If $r(e)$ emits an edge f with $s_f \notin I$ then $s_f s_f{}^* \notin I$ $(s_f = s_f s_f{}^* s_f)$. From $s_e{}^* s_e = p_{r(e)} \geq s_f s_f{}^* \notin I$, we see that $s_e{}^* s_e \notin I$, so $s_e \notin I$. Thus

$$s_e \in I \Longleftrightarrow \text{ either } r(e) \in V \text{ or } r(e) \text{ emits only edges } f \text{ with } s_f \in I.$$

Now let $\pi : C^*(E) \to C^*(E)/I$ be the canonical surjective homomorphism. Then $\pi(C^*(E))$ is generated by $\pi(s_f),\ s_f \notin I$. Let G be the subgraph of E obtained from E by deleting the vertices w with $p_w \in I$ and edges f with $s_f \in I$, that is,

$$(**) \qquad w \in G^0 \Longleftrightarrow p_w \notin I, \quad e \in G^1 \Longleftrightarrow s_e \notin I.$$

Then $\pi(C^*(E))$ is generated by $\pi(s_f),\ f \in G^1$. Let $w \in G^0$. Then $w \notin V$ and hence w emits edges $e_1, \ldots, e_m$ in E. If w is a sink in G then $s_{e_i} \in I,\ i = 1, \ldots, m$, and so $p_w = \sum_i s_{e_i} {s_{e_i}}^* \in I$, a contradiciton. Therefore the subgraph G has no sinks.

Let $\pi(s_f) \neq 0$, then f appears in G by $(**)$. If the vertex $w = r(f)$ emits edges $e_1, \ldots, e_k, \ldots, e_m$ in E such that $s_{e_1}, \ldots, s_{e_k} \notin I$, and $s_{e_{k+1}}, \ldots, s_{e_m} \in I$ then

$$\begin{aligned} \pi(s_f^*)\pi(s_f) &= \pi\left(\sum_{s(e)=r(f)=w} s_e s_e^*\right) \\ &= \sum_{i=1}^{k} \pi(s_{e_i})\pi(s_{e_i})^* = \sum_{\substack{s(g)=w=r(f) \\ g \in G^1}} \pi(s_g)\pi(s_g)^*, \end{aligned}$$

which means that the partial isometries $\{\pi(s_f) | f \in G^1\}$ is a Cuntz-Krieger G-family in $\pi(C^*(E)) = C^*(E)/I$. Therefore there exists a homomorphism $\phi : C^*(G) \to C^*(E)/I$ such that

$$\phi(t_f) = \pi(s_f), f \in G^1 \text{ and } \phi(q_w) = \pi(p_w),\ w \in G^0,$$

where $\{t_f, q_w\}$ is a Cuntz-Krieger G-family generating $C^*(G)$. On the other hand, one can form a Cuntz-Krieger E-family in $C^*(G)$ by adding $t_e = 0$ for $e \in E^1 \setminus G^1$, and $q_v = 0$ for $v \in E^0 \setminus G^0$ to the family $\{t_f, q_w\}$. Then we have a homomorphism $\rho : C^*(E) \to C^*(G)$ such that

$$\rho(s_e) = t_e,\ \rho(p_v) = q_v, \quad e \in E^1,\ v \in E^0.$$

Clearly, $I \subset Ker(\rho)$. Now let $x = \sum \lambda_{\alpha,\beta} s_\alpha s_\beta^* \in Ker(\rho)$. Then

$$\pi\left(\sum \lambda_{\alpha,\beta} s_\alpha s_\beta^*\right) = \phi\left(\sum \lambda_{\alpha,\beta} t_\alpha t_\beta^*\right) = \phi \circ \rho(x) = 0.$$

Thus $x \in Ker(\pi) = I$. Therefore $Ker(\rho) = I$ and the map ρ induces an isomorphism from $C^*(E)/I$ onto $C^*(G)$.

Recall that a C^*-algebra A is said to be *purely infinite* if every non-zero hereditary C^*-subalgebra of A has an infinite projection.

If an r-discrete groupoid $\mathcal{G}$ is essentially free and locally contracting then $C^*(\mathcal{G})$ is purely infinite ([**A**], Proposition 2.4). From Lemma 3.4 of [**KPR**], we see that the groupoid $\mathcal{G}_E$ associated with a locally finite graph E with no sinks is essentially free if and only if E satisfies condition (L). It is also known from the same paper that if every vertex connects to a loop with an exit then $\mathcal{G}_E$ is locally contracting, so that $C^*(E)(\cong C^*(\mathcal{G}_E))$ is purely infinite. Moreover there is a dichotomy for simple graph C^*-algebras.

Proposition 3.6 ([**KPR**, Corollary 3.11])**.** *Let E be a locally finite graph which has no sinks, is cofinal, and satisfies condition* (L)*. Then $C^*(E)$ is simple, and*

(i) *if E has no loops, then $C^*(E)$ is AF;*
(ii) *if E has a loop, then $C^*(E)$ is purely infinite.*

Proposition 3.7. *Let E be a locally finite directed graph. If E is cofinal then either $sr(C^*(G)) = 1$ or it is purely infinite simple.*

Proof. If E has no loop with an exit then $\mathrm{sr}(C^*(E)) = 1$ by Theorem 3.3. Suppose E has a loop with an exit. Since E is cofinal, E can not have a sink. If E has precisely one loop then E satisfies (L) and so $C^*(E)$ is purely infinite simple by the previous proposition. Let E have two distinct loops, α, β. If γ is a loop of E then consider the infinite path $x = \alpha\alpha\cdots\alpha = (x_1, x_2, \dots)$ assuming $\gamma \neq \alpha$. Since E is cofinal the vertex $v = s(\gamma)$ connects to x by a finite path, and this shows that the loop γ has an exit. Therefore E satisfies (L) and $C^*(E)$ is purely infinite simple by Proposition 3.6.

From the proof of the above proposition, we see that for a cofinal graph E with no sinks $C^*(E)$ is simple unless E has precisely one loop and the loop has no exit.

4. Real rank of $C^*(E)$.

Recall that a unital C^*-algebra A is said to have *real rank zero* ($RR(A) = 0$) if every self-adjoint element can be arbitrarily closely approximated by invertible self-adjoint elements, that is, A_{sa}^{-1} is dense in A_{sa}. For a nonunital C^*-algebra A, we say that A has real rank zero if $\widetilde{A}$ has real rank zero ([**BP1**]). Then $RR(A) = 0$ if and only if $RR(A \otimes \mathcal{K}) = 0$. Also it is well-known that $RR(A) = 0$ is equivalent to that A satisfies a condition (FS), that is, the set of self-adjoint elements with finite spectra is dense in A_{sa}, so $RR(A) = 0$ implies that A contains fairly many projections so that the linear span of its projections is dense in A. Graph C^*-algebras $C^*(E)$ are basically generated by their partial isometries, and thus they would have plenty of projections and one might expect that most of them have real rank zero. In fact, if $C^*(E)$ is simple then it is either AF or purely infinite simple and in both cases it is well-known that these algebras have real rank zero; for real rank of a purely infinite simple C^*-algebra, see [**Z**].

In this section, we first find a necessary condition for a graph C^*-algebra $C^*(E)$ to have real rank zero. We need to review the ideal theory of a graph C^*-algebra $C^*(E)$ for a directed graph E with no sinks. Recall that $C^*(E)$

can be identified with its infinite path space groupoid model $C^*(\mathcal{G})$ and $C^*(\mathcal{G}) \cong C^*_r(\mathcal{G})$ since the groupoid associated with a locally finite directed graph E is amenable ([**KPRR**], Corollary 5.3). A subset H of the vertex set E^0 is *hereditary* if $v \in H$ and $w \in E^0$ with $s(\alpha) = v$, $r(\alpha) = w$ for some $\alpha \in E^*$ then $w \in H$.

For a hereditary and saturated vertex set $H \subset E^0$, let

$$U(H) = \{x \in E^\infty \mid r(x_n) \in H \text{ for some } n\}.$$

Then $U(H)$ is an open invariant subset of E^∞ (which is identified with the unit space $\mathcal{G}^0$ of the groupoid $\mathcal{G}$ associated with the graph E). The map $H \mapsto U(H)$ is an isomorphism between the lattices of saturated hereditary subsets of E^0 and open invariant subsets $\mathcal{O}(\mathcal{G})$ of E^∞ ([**KPRR**], Lemma 6.5). On the other hand, for each open invariant subspace $U \subset E^\infty (= \mathcal{G}^0)$, the space

$$C_c(\mathcal{G}_U^U) := \left\{ f \in C_c(\mathcal{G}) : \operatorname{supp} f \subset \mathcal{G}_U^U \right\}$$

is an ideal of $C_c(\mathcal{G})$, hence its closure is an ideal $I(U)$ of $C^*(\mathcal{G})$. We see from [**Rn1**], Proposition 4.5 that the correspondence $U \mapsto I(U)$ is a one-to-one order preserving map between $\mathcal{O}(\mathcal{G})$ and the lattice of ideals $\mathcal{J}(C^*(\mathcal{G}))$ of $C^*(\mathcal{G})$. Thus $H \to I(U(H))$ is an order preserving isomorphism from the lattice of hereditary saturated vertex subsets into $\mathcal{J}(C^*(\mathcal{G}))$. It is proved in the proof of [**KPRR**], Theorem 6.6 that the ideals $I(H)$ and $I(U(H))$ coincide, where

$$I(H) := \overline{\operatorname{span}}\{1_{Z(\alpha,\beta)} \mid \alpha, \beta \in E^*, r(\alpha) = r(\beta) \in H\},$$

and $1_{Z(\alpha,\beta)}$ is the characteristic function on the compact open subset $Z(\alpha, \beta)$ of the groupoid $\mathcal{G}$.

The isomorphism from $C^*(\mathcal{G})$ onto $C^*(E)$ obtained in [**KPRR**] maps the functions $1_{Z(\alpha,\beta)}$ $(\alpha, \beta \in E^*,\ r(\alpha) = r(\beta) \in H)$ onto $s_\alpha s_\beta^*$. Therefore we have

$$I(H) = \overline{\operatorname{span}}\{s_\alpha {s_\beta}^* \mid \alpha, \beta \in E^*,\ r(\alpha) = r(\beta) \in H\}.$$

Furthermore the following is known.

Theorem 4.1 ([**KPRR**, Theorem 6.6], or [**P**, Theorem 2.2])**.** *Let E be a locally finite directed graph with no sinks. Then the map $H \mapsto I(H)$ described above is injective, and the quotient algebra $C^*(E)/I(H)$ is isomorphic to $C^*(F)$ of the directed graph $F := (E^0 \setminus H, \{e \mid r(e) \notin H\})$. The ideal $I(H)$ is strong Morita equivalent to $C^*(K)$ of the directed graph $K := (H, \{e \mid s(e) \in H\})$. Moreover, if E satisfies the condition* (K) *then the map $H \mapsto I(H)$ is surjective.*

Theorem 4.2 ([**BP1**])**.** *Let A be a C^*-algebra and I be an ideal of A.*

(a) *If $RR(A) = 0$ then $RR(I) = RR(A/I) = 0$.*

Suppose $RR(I) = RR(A/I) = 0$. Then we have the following.

(b) $RR(A) = 0$ *if and only if every projection in* A/I *lifts to a projection in* A*. In particular if* $K_1(I) = 0$ *then every projection lifts.*
(c) *If* B *is a* C^**-subalgebra of* A *with* $RR(B) = 0$ *and* $A = B + I$ *then* $RR(A) = 0$*.*

Now, we can prove our first theorem on real rank of graph C^*-algebras.

Theorem 4.3. *Let* E *be a locally finite directed graph with no sinks. If* $RR(C^*(E)) = 0$ *then* E *satisfies condition* (K).

Proof. Suppose there is a simple loop α with no exit in E. Then the subgraph H_α consisiting of α has no exit and generates an ideal I stably isomorphic to $C(\mathbb{T})$, that is, $I \otimes \mathcal{K} \cong C(\mathbb{T}) \otimes \mathcal{K}$, as in the proof of Theorem 3.3. Since $RR(C(\mathbb{T})) \neq 0$ it follows that $RR(C^*(E)) \neq 0$ by Theorem 4.2(a), a contradiction, which shows that E satisfies condition (L).

To prove condition (K), let v be a vertex such that there is only one loop at v. Let $\beta = (\beta_1, \beta_2, \ldots, \beta_n)$ be the loop and let V be the set of vertices $w \in V$ such that $w = r(e)$ for an exit e of β and H be the smallest hereditary and saturated vertex set containing V. Then $V \neq \emptyset$ because E satisfies (L). Moreover, H is a proper subset of E^0 since vertices on the loop β are not elements in H. Thus there exists a proper ideal $I(H)$ in $C^*(E)$, and the quotient algebra $C^*(E)/I(H)$ is isomorphic to $C^*(F)$ of the directed graph $F = (E^0 \backslash H, \{e \,|\, r(e) \notin H\})$. Hence F has a loop β with no exit in F and by the argument in the first paragraph of the proof $RR(C^*(F)) \neq 0$. Therefore $RR(C^*(E)) \neq 0$ by Theorem 4.2(a).

Corollary 4.4. *Let* E *be a locally finite directed graph with no sinks. If* $sr(C^*(E)) = 1$ *and* $RR(C^*(E)) = 0$ *then* $C^*(E)$ *is AF.*

Proof. By Theorem 3.3 and Theorem 4.3, E has no loops, and the assertion follows from Theorem 2.4 in [**KPR**].

Proposition 4.5. *Let* E *be a locally finite directed graph with no sinks. Then* $C^*(E)$ *is simple if and only if* E *is cofinal and satisfies* (K).

Proof. Suppose E is cofinal and satisfies condition (K) then $C^*(E)$ is simple by the proof of [**KPRR**], Corollary 6.8.

Since the converse has not been proved there in the same proof, we provide one for reader's convenience. To prove the converse, suppose E is not cofinal. Then there exist an infinite path x and a vertex v which cannot connect to x by a finite path. Let H_1 be the set of all vertices w which can be connected from v, that is, there is a finite path $\alpha \in E^*$ with $s(\alpha) = v$, $r(\alpha) = w$. Then H_1 is the smallest hereditary vertex set containing v. Let H be the set of all vertices w satisfying that for any path $\alpha \in E^* \cup E^\infty$ with $s(\alpha) = w$, if $\alpha \in E^*$ then there is another path $\beta \in E^*$ such that $s(\beta) = r(\alpha)$ and $r(\beta) \in H_1$, if $\alpha \in E^\infty$ then $r(\alpha_j) \in H_1$ for some j. Then clearly $v \in H_1 \subset H$. We show that H is a saturated hereditary vertex set which does not contain vertices on the infinite path x. Suppose a vertex w emits edges $e_1, \ldots, e_n$ and $r(e_i) \in H$ for all i. If α is a path with $s(\alpha) = w$ then $\alpha_1 = e_j$ for some j and $\alpha = e_j\gamma$ for some path with $s(\gamma) = r(e_j) \in H$. Since γ is a path with $s(\gamma) = r(e_j) \in H$, if $\gamma \in E^*$ then we can find a path $\beta \in E^*$ such that $s(\beta) = r(\gamma)$ and $r(\beta) \in H_1$. If $\gamma \in E^\infty$ then $r(\gamma_i) \in H_1$ for some i, and hence $r(\alpha_{i+1}) \in H_1$. Thus $w \in H$, and H is saturated. Now let u be a vertex connected by a finite path β from some vertex $w \in H$, that is, $s(\beta) = w, r(\beta) = u$. Then for any path α with $s(\alpha) = u$, the path $\beta\alpha$ starts from w, and it is easy to see that $u \in H$, and H is hereditary. Obviously the infinite path x does not meet any vertex in H_1, hence H is a proper saturated hereditary subset of E^0. Therefore $C^*(E)$ is not simple by Theorem 4.1.

Now suppose E is cofinal but does not satisfy condition (K). Since for a cofinal graph two conditions (K) and (L) are equivalent, E has a loop with no exit. We have already seen from the proof of Theorem 3.3 that such a loop generates an ideal strong Morita equivalent to $C(\mathbb{T})$. Thus $C^*(E)$ can not be simple.

We prove the converse of Theorem 4.3 when $C^*(E)$ has finitely many ideals.

Theorem 4.6. *Let E be a locally finite directed graph with no sinks which satisfies condition* (K). *If $C^*(E)$ has only finitely many ideals then $RR(C^*(E)) = 0$. In particular, if E is a finite graph then $RR(C^*(E)) = 0$.*

Proof. Let n be the number of non-zero ideals in $C^*(E)$. We prove our assertion by induction on n.

For $n = 1$, $C^*(E)$ is simple and $RR(C^*(E)) = 0$ since $C^*(E)$ is either AF or purely infinite simple.

Let $n > 1$. Let $I(H)$ be a maximal ideal of $C^*(E)$ for some hereditary saturated vertex subset H of E^0. By Theorem 4.1 and induction hypothesis, $I(H)$ and the simple C^*-algebra $C^*(E)/I(H)$ have real rank zero. We show that $C^*(E) = I(H) + B$ for some C^*-subalgebra B isomorphic to $C^*(\widetilde{F})$ for a directed subgraph $\widetilde{F}$ (possibly with sinks) of E such that $RR(C^*(\widetilde{F})) = 0$ and then apply Theorem 4.2(c). According to Theorem 4.1, $C^*(E)/I(H) \cong C^*(F)$, where $F = (E^0 \setminus H, \{e \mid r(e) \notin H\})$. Let

$$V := \{v \in H \mid v = r(e) \text{ for some edge } e \in E^1 \text{ with } s(e) \in F^0 = E^0 \setminus H\}.$$

If $V = \emptyset$, then $C^*(E) \cong I(H) \oplus C^*(F)$, and therefore $RR(C^*(E)) = 0$ since two direct summands have real rank zero by induction hypothesis. If $V \neq \emptyset$ we set

$$\widetilde{F} = (F^0 \cup V,\ F^1 \cup \{f \in E^1 \mid r(f) \in V,\ s(f) \in F^0\}).$$

Then V is the set of all sinks of $\widetilde{F}$. By Theorem 3.5, $C^*(\widetilde{F})/I(V)$ is isomorphic to the simple C^*-algebra $C^*(F)$, where

$$I(V) = \overline{\operatorname{span}}\{s_\alpha s_\beta^* \mid \alpha, \beta \in \widetilde{F}^*, r(\alpha) = r(\beta) \in V\}.$$

Thus $RR(C^*(\widetilde{F})/I(V)) = RR(C^*(F)) = 0$. The ideal

$$I(V) \cong \oplus_{v \in V} \mathcal{K}(\ell^2(E^*(v)))$$

also has real rank zero. Furthermore since $K_1(I(V)) = 0$, by Theorem 4.2(b), $RR(C^*(\widetilde{F})) = 0$. Let B be the C^*-subalgebra of $C^*(E)$ generated by the family of nonzero elements $\{p_v, s_f \mid v \in (\widetilde{F})^0,\ f \in (\widetilde{F})^1\}$. Then this is a Cuntz-Krieger $\widetilde{F}$-family and hence B is a quotient of $C^*(\widetilde{F})$. Thus $RR(B) = 0$. Now, it is not hard to see that $C^*(E) = B + I(H)$, and this completes the proof.

Let A be a $\{0, 1\}$-matrix with no zero row or column. Then A can be viewed as a vertex matrix of a finite graph E with no sinks. If A satisfies Cuntz-Krieger's condition (I) in [**CK**] then it clearly follows that E satisfies (L) (or, equivalently condition (I) introduced for graphs in [**KPR**]) from their definitions. By Proposition 4.1 of [**KPRR**], the graph algebra $C^*(E)$ is also generated by a Cuntz-Krieger A-family of partial isometries, hence the Cuntz-Krieger algebra $\mathcal{O}_A$ is isomorphic to the graph algebra $C^*(E)$. On the other hand, the graph algebra $C^*(E)$ is known to be isomorphic to the Cuntz-Krieger algebra $\mathcal{O}_B$ associated with the edge matrix B of E. Therefore those three algebras are all isomorphic. Furthermore by Theorem 4.3, 4.6, and Lemma 6.1 of [**KPRR**], we have the following corollary.

Corollary 4.7. *Let A be a $\{0, 1\}$-matrix with no zero row or column. Suppose A satisfies Cuntz-Krieger's condition* (I) *and let E be the finite graph having A as its vertex matrix. Then the following are equivalent:*

(i) *$RR(\mathcal{O}_A) = 0$,*
(ii) *A satisfies Cuntz's condition* (II)*,*
(iii) *E satisfies condition* (K)*.*

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