ON LOCAL CONNECTEDNESS OF ABSOLUTE RETRACTS

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Conditions are found for classes $\mathcal{K}$ of continua under which absolute retracts for $\mathcal{K}$ are locally connected. A number of corollaries is obtained.

One of the fundamental observations of theory of retracts says that any absolute retract for the class of all compacta is a locally connected continuum. However, when more restricted classes of spaces are considered, as for example hereditarily unicoherent continua, tree-like continua, dendroids, or hereditarily indecomposable continua, this is not necessarily the case (see e.g., [6, Corollaries 4 and 5, p. 181 and 183]). In this paper we show that, nevertheless, for a large number of classes of continua their absolute retracts remain locally connected.

The concept of an AR space originally had been studied by K. Borsuk, see [2]. More recently AR-spaces for some classes of continua had been studied e.g., in [6], [7] and [8]. The aim of this paper is to continue this investigation for various classes $\mathcal{K}$ of spaces. We focus our attention on conditions concerning classes $\mathcal{K}$ implying that each member of $AR(\mathcal{K})$ is locally connected.

By a space we mean a topological space, and a mapping means a continuous function. Given a space $X$ and its subspace $Y \subset X$, a mapping $r : X \to Y$ is called a retraction if the restriction $r|Y$ is the identity. Then $Y$ is called a retract of $X$. Let $\mathcal{K}$ be a class of compacta, i.e., of compact metric spaces. Following [3, p. 80], we say that a space $Y \in \mathcal{K}$ is an absolute retract for the class $\mathcal{K}$ (abbreviated $AR(\mathcal{K})$) if for any space $Z \in \mathcal{K}$ such that $Y$ is a subspace of $Z$, $Y$ is a retract of $Z$. The reader is referred to [2] and [3] for needed information on these concepts.

By a continuum we mean a connected compactum. A curve means a one-dimensional continuum. For a given continuum $X$, an arc component of $X$ means the union of all arcs $A$ such that $p \in A \subset X$ for some point $p$ of $X$. A locally connected continuum containing no simple closed curve is called a dendrite. A continuum is said to be decomposable provided that it can be represented as the union of two its proper subcontinua. Otherwise it is said to be indecomposable. A continuum is said to be hereditarily decomposable (hereditarily indecomposable) provided that each of its subcontinua is decomposable (indecomposable, respectively).
We employ a well-known construction of hereditarily decomposable non-degenerate continua containing no arc; the first such construction is due to Z. Janiszewski, [4]; more recent results where such construction can be found are e.g., in [9] and in [10, (6.1), p. 30]. Fix such a continuum $E$.

To formulate the main result of the paper (Theorem 1 below) consider two classes $A$ and $B$ of nondegenerate continua. Define $A$ as the class of all continua $X$ such that:

(A) Every nonempty open subset of $X$ intersects either an arc or a nondegenerate indecomposable subcontinuum of $X$.

Observe that all nondegenerate continua that are either arcwise connected or indecomposable belong to $A$.

Define $B$ as the class of all continua $X$ such that:

(B) Every nonempty open subset of $X$ contains a nondegenerate continuum that is not a pseudo-arc.

**Theorem 1.** Let $C$ be an arbitrary class of continua such that either $C \subset A$, or $C \subset B$. Denote by $N(C)$ the class of all continua containing no member of $C$. Then each member of $AR(N(C))$ is a locally connected continuum.

**Proof.** Consider two cases.

**Case 1.** $C \subset A$.

Let $X \in AR(N(C))$ and let for some points $x_0, x_1, x_2, \ldots \in X$ with $x_0 \neq x_n$ for each $n \in \mathbb{N}$ the sequence $\{x_n\}$ tend to $x_0$. Take a sequence $\{E_n\}$ of copies of $E$, and points $a_n, b_n \in E_n$ with $a_n \neq b_n$ such that:

1. $E_n \cap E_m = \emptyset = E_n \cap X$ for each $m, n \in \mathbb{N}$ with $m \neq n$;
2. $\lim E_n = \{x_0\}$.

Let $Y = X \cup \bigcup \{E_n : n \in \mathbb{N}\}$. Observe that this union is compact. Define a quotient mapping $q : Y \to Z = q(Y)$ by identifying all pairs $a_n, x_n$ and the set $\{x_0, b_1, b_2, \ldots \}$. Since $q|X$ is a homeomorphism, we can identify $X$ and $q(X)$ assuming that $q|X = id|X$. Let $K_n = q(E_n)$.

**Claim 1.** Any nondegenerate indecomposable subcontinuum of $Z$ is contained in $X$.

Indeed, let $M$ be a nondegenerate indecomposable subcontinuum of $Z$, and let $M_n = M \cap (K_n \setminus \{x_0, x_n\})$. Suppose on the contrary that $M_n \neq \emptyset$ for some $n \in \mathbb{N}$. Since $K_n$ is homeomorphic to $E$, it does not contain $M$. Thus each component of $M \cap K_n$ intersects $bd K_n = \{x_0, x_n\}$. Let $U_0$ be the union of all components of $M \cap K_n$ that contain $x_0$, and $U_n$ be the union of all components of $M \cap K_n$ that contain $x_n$. Note that $U_0$ and $U_n$ are continua whose union contains a nonempty open subset $M_n$ of $M$. Thus either $U_0$ or $U_n$ has nonempty interior in $M$. Therefore by the indecomposability of $M$ either $M = U_0$ or $M = U_n$. Hence $M \subset K_n$, a contradiction. Therefore the claim is proved.
Since $X$ contains all arcs in $Z$ and all nondegenerate indecomposable continua in $Z$, it follows that $Z \in \mathcal{NC}$.

By the assumption $X \in AR(\mathcal{NC})$, so there is a retraction $r : Z \to X$. The continua $K_n$ join $x_n$ and $x_0$, and $\lim \text{diam } K_n = \lim \text{diam } r(K_n) = 0$, therefore $L_n = r(K_n) \subset X$ are continua such that $x_0, x_n \in L_n$ for each $n \in \mathbb{N}$, and $\lim \text{diam } L_n = 0$. Since the point $x_0$ and the sequence $\{x_n\}$ that tends to $x_0$ have been chosen arbitrarily, the continuum $X$ is locally connected. So the proof is complete for Case 1.

**Case 2.** $\mathcal{C} \subset \mathcal{B}$. 

In this case take the pseudo-arc $P$ in place of the continuum $E$ and follow the procedure (and the notation) from the previous case. Instead of Claim 1 we easily see that the following claim holds.

**Claim 2.** Each subcontinuum of $X$ that intersects some set $K_n \setminus X$ does not belong to $\mathcal{B}$.

This implies that $Z \in \mathcal{NC}$ as previously. The last part of the proof in this case is identical with the one of the previous case. The theorem is proved.

There are many applications of the above theorem for particular classes $\mathcal{C}$ of spaces. First note that for $\mathcal{C} \subset \mathcal{A} \cap \mathcal{B}$ being the empty class we obtain as $\mathcal{NC}$ the class of all continua, and thus we get the known result that absolute retracts for all continua are locally connected ones. Another extremal result is obtained for $\mathcal{C} \subset \mathcal{A}$ consisting of all arcs. Then we see that the only absolute retract for continua containing no arc is the singleton. The same conclusion holds if $\mathcal{C} \subset \mathcal{B}$ is the class of all hereditarily decomposable continua: The singleton is the only absolute retract for continua containing no hereditarily decomposable subcontinua. Further, we have the following application of Theorem 1.

**Corollary 2.** A continuum $X$ is an absolute retract for the class of hereditarily decomposable continua if and only if $X$ is hereditarily decomposable and locally connected.

**Proof.** One implication is Theorem 1 applied for the class $\mathcal{C}$ of all nondegenerate indecomposable continua (note that $\mathcal{C} \subset \mathcal{A}$). On the other hand, each hereditarily decomposable continuum has dimension at most 1, [5, §48, V, Remark 2, p. 206], and each locally connected curve is an absolute retract for all curves (see [5, §53, IV, Theorems 1 and 1′, p. 347], and apply them for $n = 1$).

**Corollary 3.** Let $\mathcal{K}$ be the class of all continua containing no nondegenerate hereditarily indecomposable continuum. Then a continuum $X$ is an absolute retract for the class $\mathcal{K}$ if and only if $X \in \mathcal{K}$ and $X$ is locally connected.
Proof. Observe that the class $C$ of all nondegenerate hereditarily indecomposable continua is contained in $A$. We apply Theorem 1 for the class $C$ to prove that members of $AR(K)$ are locally connected. To see the other implication note that continua in $K$ have dimension at most 1. This is a consequence of [1, Theorem 6, p. 271] (see also [1, Theorem 5, p. 270]). Finally we recall again that locally connected curves are absolute retracts for all curves (take $n = 1$ in [5, §53, IV, Theorems 1 and 1′, p. 347]). □

The next corollary is a consequence of Theorem 1 and of the fact that the dendrites are absolute retracts for all continua.

**Corollary 4.** Let $K$ be the class of all continua containing no simple closed curve. Then $X \in AR(K)$ if and only if $X$ is a dendrite.

A simple triod (a simple $n$-od) means the cone over a 3-point (an $n$-point) discrete space. We say that a sequence of compacta $A_n$ converges to a compactum $A$ homeomorphically provided there exists a sequence of homeomorphisms $h_n : A \rightarrow A_n$ converging to the identity on $A$. This last condition means that $\lim (\sup \{d(h_n(x), x) : x \in A\}) = 0$.

**Corollary 5.** An arc is the only nondegenerate absolute retract for the class of all continua containing no simple triod.

Proof. Only one implication needs a proof. By Theorem 1 members of the considered class are locally connected, so an arc or a simple closed curve are the only nondegenerate ones. To eliminate a simple closed curve take an arc-like continuum $X$ without arcs (a pseudo-arc, for example) that is irreducible between points $a$ and $b$. Let $S^1$ be the unit circle. In the product $X \times S^1$ shrink the circle $\{b\} \times S^1$ to a point, and denote by $Y$ the resulting continuum. Thus $Y = (X \times S^1)/\{b\} \times S^1$ contains no simple triod. Let $q : X \times S^1 \rightarrow Y$ be the quotient mapping. Then the circle $C = q(\{a\} \times S^1) \subset Y$ is not a retract of $Y$. To see this, consider a continuous monotone decomposition (in the sense of [5, §43, V, p. 67]) $D$ of $Y$ into continua $q(\{x\} \times S^1)$ for $x \in X$ which all are the circles except the one, which is the singleton $q(\{b\} \times S^1)$. Thus $D$, if equipped with the quotient topology, [5, §43, III, p. 64], is a continuum (homeomorphic to $X$). Observe further that the decomposition $D$ is continuous with respect to the homeomorphical convergence (as defined above).

Suppose there is a retraction $r : Y \rightarrow C$. Let $D(e)$ (let $D(i)$) denote the subfamily of $D$ composed of those members $D$ of $D$ for which the restriction $r|D : D \rightarrow C$ is essential (is inessential, respectively). Thus $C \in D(e)$ and $q(\{b\} \times S^1) \in D(i)$, so $D(e)$ and $D(i)$ are nonempty, mutually disjoint subsets of $D$ whose union is $D$. By continuity of the decomposition with respect to the homeomorphical convergence they are also open, which contradicts the connectedness of $D$. □
By an \( n \)-dimensional umbrella we understand the union of an \( n \)-dimensional cube \( Q = [0,1]^n \) and of an arc \( A \) such that the set \( A \cap Q \) consists of only one point which is an end point of the arc \( A \) and an inner point of \( Q \).

One can also observe, applying Theorem 1, the following corollary.

**Corollary 6.** Let \( n \) be a positive integer and \( \mathcal{K} \) be a class of all continua defined by any of the following conditions:

- (6.1) Each member of \( \mathcal{K} \) contains no simple \( n \)-od.
- (6.2) Each member of \( \mathcal{K} \) contains no \( n \)-dimensional cube.
- (6.3) Each member of \( \mathcal{K} \) contains no \( n \)-dimensional sphere.
- (6.4) Each member of \( \mathcal{K} \) contains no \( n \)-dimensional umbrella.
- (6.5) Each member of \( \mathcal{K} \) contains no Hilbert cube.

Then any member of \( \text{AR}(\mathcal{K}) \) is a locally connected continuum.

**Remark 7.** Indecomposability and arcwise connectedness plays the crucial role in Theorem 1 and in its applications. We admit other properties of continua to produce similar results. Namely, given a property \( \mathcal{P} \) we define the class \( \mathcal{N}\mathcal{P} \) of continua containing no subspaces having property \( \mathcal{P} \). Then, if it is possible to find an appropriate continuum instead of the continuum \( E \) (see the paragraph preceding Theorem 1), or instead of the pseudo-arc \( P \), one can prove that members of \( \text{AR}(\mathcal{N}\mathcal{P}) \) are locally connected.

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**References**


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