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Conditions are found for classes $\mathcal{K}$ of continua under which absolute retracts for $\mathcal{K}$ are locally connected. A number of corollaries is obtained.

One of the fundamental observations of theory of retracts says that any absolute retract for the class of all compacta is a locally connected continuum. However, when more restricted classes of spaces are considered, as for example hereditarily unicoherent continua, tree-like continua, dendroids, or hereditarily indecomposable continua, this is not necessarily the case (see e.g., [**6**, Corollaries 4 and 5, p. 181 and 183]). In this paper we show that, nevertheless, for a large number of classes of continua their absolute retracts remain locally connected.

The concept of an AR space originally had been studied by K. Borsuk, see [**2**]. More recently AR-spaces for some classes of continua had been studied e.g., in [**6**], [**7**] and [**8**]. The aim of this paper is to continue this investigation for various classes $\mathcal{K}$ of spaces. We focus our attention on conditions concerning classes $\mathcal{K}$ implying that each member of $AR(\mathcal{K})$ is locally connected.

By a *space* we mean a topological space, and a *mapping* means a continuous function. Given a space X and its subspace $Y \subset X$, a mapping $r : X \to Y$ is called a *retraction* if the restriction $r|Y$ is the identity. Then Y is called a *retract* of X. Let $\mathcal{K}$ be a class of *compacta*, i.e., of compact metric spaces. Following [**3**, p. 80], we say that a space $Y \in \mathcal{K}$ is an *absolute retract for the class* $\mathcal{K}$ (abbreviated $AR(\mathcal{K})$) if for any space $Z \in \mathcal{K}$ such that Y is a subspace of Z, Y is a retract of Z. The reader is referred to [**2**] and [**3**] for needed information on these concepts.

By a *continuum* we mean a connected compactum. A *curve* means a one-dimensional continuum. For a given continuum X, an *arc component* of X means the union of all arcs A such that $p \in A \subset X$ for some point p of X. A locally connected continuum containing no simple closed curve is called a *dendrite*. A continuum is said to be *decomposable* provided that it can be represented as the union of two its proper subcontinua. Otherwise it is said to be *indecomposable*. A continuum is said to be *hereditarily decomposable* (*hereditarily indecomposable*) provided that each of its subcontinua is decomposable (indecomposable, respectively).

We employ a well-known construction of hereditarily decomposable nondegenerate continua containing no arc; the first such construction is due to Z. Janiszewski, [**4**]; more recent results where such construction can be found are e.g., in [**9**] and in [**10**, (6.1), p. 30]. Fix such a continuum E.

To formulate the main result of the paper (Theorem 1 below) consider two classes $\mathcal{A}$ and $\mathcal{B}$ of nondegenerate continua. Define $\mathcal{A}$ as the class of all continua X such that:

($\mathcal{A}$) Every nonempty open subset of X intersects either an arc or a nondegenerate indecomposable subcontinuum of X.

Observe that all nondegenerate continua that are either arcwise connected or indecomposable belong to $\mathcal{A}$.

Define $\mathcal{B}$ as the class of all continua X such that:

($\mathcal{B}$) Every nonempty open subset of X contains a nondegenerate continuum that is not a pseudo-arc.

Theorem 1. *Let $\mathcal{C}$ be an arbitrary class of continua such that either $\mathcal{C} \subset \mathcal{A}$, or $\mathcal{C} \subset \mathcal{B}$. Denote by $\mathcal{NC}$ the class of all continua containing no member of $\mathcal{C}$. Then each member of $AR(\mathcal{NC})$ is a locally connected continuum.*

Proof. Consider two cases.

Case 1. $\mathcal{C} \subset \mathcal{A}$.

Let $X \in AR(\mathcal{NC})$ and let for some points $x_0, x_1, x_2, \dots \in X$ with $x_0 \neq x_n$ for each $n \in \mathbb{N}$ the sequence $\{x_n\}$ tend to x_0. Take a sequence $\{E_n\}$ of copies of E, and points $a_n, b_n \in E_n$ with $a_n \neq b_n$ such that:

(1.1) $E_n \cap E_m = \emptyset = E_n \cap X$ for each $m, n \in \mathbb{N}$ with $m \neq n$;
(1.2) $\operatorname{Lim} E_n = \{x_0\}$.

Let $Y = X \cup \bigcup\{E_n : n \in \mathbb{N}\}$. Observe that this union is compact. Define a quotient mapping $q : Y \to Z = q(Y)$ by identifying all pairs a_n, x_n and the set $\{x_0, b_1, b_2, \dots\}$. Since $q|X$ is a homeomorphism, we can identify X and $q(X)$ assuming that $q|X = \mathrm{id}|X$. Let $K_n = q(E_n)$.

Claim 1. Any nondegenerate indecomposable subcontinuum of Z is contained in X.

Indeed, let M be a nondegenerate indecomposable subcontinuum of Z, and let $M_n = M \cap (K_n \setminus \{x_0, x_n\})$. Suppose on the contrary that $M_n \neq \emptyset$ for some $n \in \mathbb{N}$. Since K_n is homeomorphic to E, it does not contain M. Thus each component of $M \cap K_n$ intersects $\operatorname{bd} K_n = \{x_0, x_n\}$. Let U_0 be the union of all components of $M \cap K_n$ that contain x_0, and U_n be the union of all components of $M \cap K_n$ that contain x_n. Note that U_0 and U_n are continua whose union contains a nonempty open subset M_n of M. Thus either U_0 or U_n has nonempty interior in M. Therefore by the indecomposability of M either $M = U_0$ or $M = U_n$. Hence $M \subset K_n$, a contradiction. Therefore the claim is proved.

Since X contains all arcs in Z and all nondegenerate indecomposable continua in Z, it follows that $Z \in \mathcal{NC}$.

By the assumption $X \in AR(\mathcal{NC})$, so there is a retraction $r : Z \to X$. The continua K_n join x_n and x_0, and $\lim \operatorname{diam} K_n = \lim \operatorname{diam} r(K_n) = 0$, therefore $L_n = r(K_n) \subset X$ are continua such that $x_0, x_n \in L_n$ for each $n \in \mathbb{N}$, and $\lim \operatorname{diam} L_n = 0$. Since the point x_0 and the sequence $\{x_n\}$ that tends to x_0 have been chosen arbitrarily, the continuum X is locally connected. So the proof is complete for Case 1.

Case 2. $\mathcal{C} \subset \mathcal{B}$.

In this case take the pseudo-arc P in place of the continuum E and follow the procedure (and the notation) from the previous case. Instead of Claim 1 we easily see that the following claim holds.

Claim 2. Each subcontinuum of X that intersects some set $K_n \setminus X$ does not belong to $\mathcal{B}$.

This implies that $Z \in \mathcal{NC}$ as previously. The last part of the proof in this case is identical with the one of the previous case. The theorem is proved. □

There are many applications of the above theorem for particular classes $\mathcal{C}$ of spaces. First note that for $\mathcal{C} \subset \mathcal{A} \cap \mathcal{B}$ being the empty class we obtain as $\mathcal{NC}$ the class of all continua, and thus we get the known result that absolute retracts for all continua are locally connected ones. Another extremal result is obtained for $\mathcal{C} \subset \mathcal{A}$ consisting of all arcs. Then we see that the only absolute retract for continua containing no arc is the singleton. The same conclusion holds if $\mathcal{C} \subset \mathcal{B}$ is the class od all hereditarily decomposable continua: The singleton is the only absolute retract for continua containing no hereditarily decomposable subcontinua. Further, we have the following application of Theorem 1.

Corollary 2. *A continuum X is an absolute retract for the class of hereditarily decomposable continua if and only if X is hereditarily decomposable and locally connected.*

Proof. One implication is Theorem 1 applied for the class $\mathcal{C}$ of all nondegenerate indecomposable continua (note that $\mathcal{C} \subset \mathcal{A}$). On the other hand, each hereditarily decomposable continuum has dimension at most 1, [**5**, §48, V, Remark 2, p. 206], and each locally connected curve is an absolute retract for all curves (see [**5**, §53, IV, Theorems 1 and 1′, p. 347], and apply them for $n = 1$). □

Corollary 3. *Let $\mathcal{K}$ be the class of all continua containing no nondegenerate hereditarily indecomposable continuum. Then a continuum X is an absolute retract for the class $\mathcal{K}$ if and only if $X \in \mathcal{K}$ and X is locally connected.*

Proof. Observe that the class $\mathcal{C}$ of all nondegenerate hereditarily indecomposable continua is contained in $\mathcal{A}$. We apply Theorem 1 for the class $\mathcal{C}$ to prove that members of $AR(\mathcal{K})$ are locally connected. To see the other implication note that continua in $\mathcal{K}$ have dimension at most 1. This is a consequence of [**1**, Theorem 6, p. 271] (see also [**1**, Theorem 5, p. 270]). Finally we recall again that locally connected curves are absolute retracts for all curves (take $n = 1$ in [**5**, §53, IV, Theorems 1 and 1′, p. 347]). □

The next corollary is a consequence of Theorem 1 and of the fact that the dendrites are absolute retracts for all continua.

Corollary 4. *Let $\mathcal{K}$ be the class of all continua containing no simple closed curve. Then $X \in AR(\mathcal{K})$ if and only if X is a dendrite.*

A *simple triod* (a simple *n-od*) means the cone over a 3-point (an n-point) discrete space. We say that a sequence of compacta A_n converges to a compactum A *homeomorphically* provided there exists a sequence of homeomorphisms $h_n : A \to A_n$ converging to the identity on A. This last condition means that $\lim(\sup\{d(h_n(x), x) : x \in A\}) = 0$.

Corollary 5. *An arc is the only nondegenerate absolute retract for the class of all continua containing no simple triod.*

Proof. Only one implication needs a proof. By Theorem 1 members of the considered class are locally connected, so an arc or a simple closed curve are the only nondegenerate ones. To eliminate a simple closed curve take an arc-like continuum X without arcs (a pseudo-arc, for example) that is irreducible between points a and b. Let S^1 be the unit circle. In the product $X \times S^1$ shrink the circle $\{b\} \times S^1$ to a point, and denote by Y the resulting continuum. Thus $Y = (X \times S^1)/(\{b\} \times S^1)$ contains no simple triod. Let $q : X \times S^1 \to Y$ be the quotient mapping. Then the circle $C = q(\{a\} \times S^1) \subset Y$ is not a retract of Y. To see this, consider a continuous monotone decomposition (in the sense of [**5**, §43, V, p. 67]) $\mathbf{D}$ of Y into continua $q(\{x\} \times S^1)$ for $x \in X$ which all are the circles except the one, which is the singleton $q(\{b\} \times S^1)$. Thus $\mathbf{D}$, if equipped with the quotient topology, [**5**, §43, III, p. 64], is a continuum (homeomorphic to X). Observe further that the decomposition $\mathbf{D}$ is continuous with respect to the homeomorphical convergence (as defined above).

Suppose there is a retraction $r : Y \to C$. Let $\mathbf{D}(e)$ (let $\mathbf{D}(i)$) denote the subfamily of $\mathbf{D}$ composed of those members D of $\mathbf{D}$ for which the restriction $r|D : D \to C$ is essential (is inessential, respectively). Thus $C \in \mathbf{D}(e)$ and $q(\{b\}\times S^1) \in \mathbf{D}(i)$, so $\mathbf{D}(e)$ and $\mathbf{D}(i)$ are nonempty, mutually disjoint subsets of $\mathbf{D}$ whose union is $\mathbf{D}$. By continuity of the decomposition with respect to the homeomorphical convergence they are also open, which contradicts the connectedness of $\mathbf{D}$. □

By an *n-dimensional umbrella* we understand the union of an n-dimensional cube $Q = [0,1]^n$ and of an arc A such that the set $A \cap Q$ consists of only one point which is an end point of the arc A and an inner point of Q.

One can also observe, applying Theorem 1, the following corollary.

Corollary 6. *Let n be a positive integer and $\mathcal{K}$ be a class of all continua defined by any of the following conditions:*

(6.1) *Each member of $\mathcal{K}$ contains no simple n-od.*
(6.2) *Each member of $\mathcal{K}$ contains no n-dimensional cube.*
(6.3) *Each member of $\mathcal{K}$ contains no n-dimensional sphere.*
(6.4) *Each member of $\mathcal{K}$ contains no n-dimensional umbrella.*
(6.5) *Each member of $\mathcal{K}$ contains no Hilbert cube.*

Then any member of $AR(\mathcal{K})$ is a locally connected continuum.

Remark 7. Indecomposability and arcwise connectedness plays the crucial role in Theorem 1 and in its applications. We admit other properties of continua to produce similar results. Namely, given a property $\mathcal{P}$ we define the class $\mathcal{NP}$ of continua containing no subspaces having property $\mathcal{P}$. Then, if it is possible to find an appropriate continuum instead of the continuum E (see the paragraph preceding Theorem 1), or instead of the pseudo-arc P, one can prove that members of $AR(\mathcal{NP})$ are locally connected.

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