EVEN KAKUTANI EQUIVALENCE VIA $\vec{\alpha}$ AND $\vec{\beta}$ EQUIVALENCE IN $\mathbb{Z}^2$

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We show that an even Kakutani equivalence class of $\mathbb{Z}^2$ actions is “spanned” by $\vec{\alpha}$ and $\vec{\beta}$ equivalence classes where $\vec{\alpha} = \{1 + \alpha_1, 1 + \alpha_2\}$, $\vec{\beta} = \{1 + \beta_1, 1 + \beta_2\}$ and $\{1, \alpha_i^{-1}, \beta_i^{-1}\}$ are rationally independent for $i = 1, 2$. Namely, given such vectors $\vec{\alpha}$ and $\vec{\beta}$ and two evenly Kakutani equivalent $\mathbb{Z}^2$ actions $S$ and $T$, we show that $U$ is $\vec{\alpha}$-equivalent to $S$ and $\vec{\beta}$-equivalent to $T$.

1. Introduction.

In this paper we discuss the relationship between two of the fundamental examples of restricted orbit equivalence: even Kakutani equivalence and $\vec{\alpha}$ equivalence. Both equivalence relations arise in the context of representations of ergodic and measure preserving $\mathbb{R}^d$ actions. The first is related to the Ambrose-Kakutani Theorem [1]: every free, measure preserving and ergodic $\mathbb{R}$ action can be represented as a suspension flow over a free, measure preserving and ergodic $\mathbb{Z}$ action. In the case of $\mathbb{R}^d$ and $\mathbb{Z}^d$ actions, for $d > 1$, this is the Katok Representation Theorem [4]. For all $d \geq 1$ two $\mathbb{Z}^d$ actions are said to be even Kakutani equivalent (denoted $\sim_e$) if they arise as sections of equal frequency in different representations of the same $\mathbb{R}^d$ action.

Rudolph has shown that the representation of a finite entropy $\mathbb{R}^d$ action can be achieved with a restriction on the values which the ceiling function may take [8], [9]. In the one-dimensional case, the return times to the base can be required to be only $\{1, \alpha\}$, with $\alpha > 0$ an irrational. For $d > 1$ a vector $\vec{\alpha} = \{1 + \alpha_1, \ldots, 1 + \alpha_d\}$ is specified, with $\alpha_i > 0$ irrationals, and the suspension is a tiling representation of the $\mathbb{R}^d$ action. The vector $\vec{\alpha}$ determines the sizes and placement rules of the tiles. The equivalence relation in this context analogous to $\sim_e$ is called $\vec{\alpha}$-equivalence ($\sim_{\vec{\alpha}}$).

It is clear, in all dimensions, that $\sim_{\vec{\alpha}}$ implies $\sim_e$. The converse does not hold. In [3], Fieldsteel, del Junco and Rudolph construct a spectral invariant which shows that $\sim_{\vec{\alpha}}$ is a refinement of $\sim_e$. This invariant can easily be extended to higher dimensions.

On the other hand, in the one-dimensional case Park showed that an even Kakutani equivalence class is, in fact, spanned by $\vec{\alpha}$ and $\vec{\beta}$ equivalence
classes, when \( \{1, \alpha^{-1}, \beta^{-1}\} \) are irrationally related [6]. In this paper we extend this result to higher dimensions. The main result of the paper is as follows.

**Theorem 1.1.** Let \((X, \mathcal{M}, \mu)\) and \((Y, \mathcal{G}, \nu)\) be nonatomic Lebesgue probability spaces. Let \(S\) and \(T\) be two free measure preserving ergodic \(\mathbb{Z}^2\) actions of finite entropy acting on \(X\) and \(Y\) respectively.

Let \(\vec{\alpha} = \{1 + \alpha_1, 1 + \alpha_2\}\) and \(\vec{\beta} = \{1 + \beta_1, 1 + \beta_2\}\) where the \(\alpha_i\) and \(\beta_i\) are positive irrationals and \(\{1, \alpha_i^{-1}, \beta_i^{-1}\}\) are rationally independent for \(i = 1, 2\).

If \(S\) and \(T\) are evenly Kakutani equivalent, then there is a nonatomic, Lebesgue probability space \((\mathbb{Z}, \mathcal{F}, \lambda)\) and an ergodic, measure preserving, and free \(\mathbb{Z}^2\) action \(U\) on \(\mathbb{Z}\) so that

\[
S \overset{\vec{\alpha}}{\sim} U \quad \text{and} \quad U \overset{\vec{\beta}}{\sim} T.
\]

In [6] the author proves the result by constructing the action \(U\) and the flows over it explicitly. Extending these methods to higher dimensions quickly becomes an intractable tiling problem. Instead, in this paper, we use the fact that both equivalence relations can be cast as restricted orbit equivalences to prove the result. This characterization will enable us to recognize \(\overset{\varepsilon}{\sim}\) and \(\overset{\vec{\alpha}}{\sim}\) by checking for relationships in the orbit structures of the various \(\mathbb{Z}^d\) actions. Thus we will be able to work with the discrete actions directly, and we won’t be constructing flows or tilings.

Even Kakutani equivalence of \(\mathbb{Z}^d\) actions was cast as a restricted orbit equivalence by Rudolph and del Junco in [2]. The one-dimensional characterization is older (see for example [5]) and relies on the linear ordering of the integers. The higher-dimensional characterization is more complicated due to the more complex geometry in the orbits of a higher-dimensional group action.

The situation for \(\vec{\alpha}\)-equivalence is different. The orbit equivalence characterization for one-dimensional \(\vec{\alpha}\)-equivalence is given in [3] and is extended to dimension two by Şahin in [10]. Surprisingly this higher dimensional formulation does not require more restrictions on the orbit equivalence. Given that tilings of the plane are much more complicated than tilings of the line, it is surprising that new invariants of equivalence do not appear in higher dimensions.

As was discussed above, in this paper we use the results of [2] and [10] to work only with restricted orbit equivalences between discrete actions, and provide a simpler proof of the result in [6]. We note that the arguments in this paper hold in any dimension, but we state the theorem for \(d = 2\), because the techniques in, and hence the results of, [10] hold for \(d = 2\). The paper is self contained in that the next section contains the concepts and definitions from [2] and [10] necessary for our arguments.
Finally, we remark that in [7], the author provides an outline of the proof in the one-dimensional case using the restricted orbit equivalence definition of \( \sim \) equivalence. The results in this paper, of course, subsume the one-dimensional result, but more importantly, the techniques we use here are significantly different to those sketched in [7]. The differences are due to the more complicated geometry and definition of even Kakutani equivalence in higher dimensions.

2. Preliminaries.

2.1. Notation. Throughout the paper \((X, \mathcal{F}, \mu)\) and \((Y, \mathcal{G}, \nu)\) will denote Lebesgue probability spaces. \( S = \{S^n\}_{n \in \mathbb{Z}^d} \) and \( T = \{T^{n}\}_{n \in \mathbb{Z}^d} \) will denote measurable measure preserving ergodic and free \( \mathbb{Z}^d \) actions on \( X \) and \( Y \) respectively.

For \( \vec{v} \in \mathbb{Z}^d \), if \( \vec{v} = (v_1, \ldots, v_d) \) we set \( \|\vec{v}\| = \max\{|v_1|, \ldots, |v_d|\} \). We denote the \( i \)-th component of the vector \( \vec{v} \) by \( \vec{v}_i \). To avoid confusion, the \( i \)-th component of an indexed vector such as \( \vec{v}_n \) will be denoted by \( [\vec{v}_n]_i \).

Given the action \( S \), we define \[ R_S = \{(x_1, x_2) \in X \times X : \text{there is a } \vec{n} \in \mathbb{Z}^d \text{ with } S^{\vec{n}}x_1 = x_2\}. \]

If \( G \) is an abelian group a \( G \)-valued \( S \) cocycle is defined to be a function \( f : R_S \to G \) satisfying the cocycle condition \( f(x, y) = f(x, z) + f(z, y) \) for every \((x, y), (x, z) \in R_S\). A coboundary is defined in the usual way. We define the \( S \)-ordering cocycle of the \( \mathbb{Z}^d \) action \( S, \vec{S} : R_S \to \mathbb{Z}^d \), by \( \vec{S}(x, y) = \vec{n} \) if and only if \( S^{\vec{n}}x = y \).

Set \( B_n = \{\vec{v} : 0 \leq v_1, \ldots, v_d \leq n\} \) and let \( E \subset X \). If for some integer \( n > 0 \) the sets \( \{S^{\vec{n}}E : \vec{n} \in B_n\} \) are disjoint, we call \( S^{B_n}E = \cup_{\vec{n} \in B_n} S^{\vec{n}}E \) a Rohlin tower of size \( n \) for \( S \). Each set \( S^{\vec{n}}E \), for \( \vec{n} \in B_n \), is called a level of the tower and the level corresponding to \( \vec{0} \) is called the base of the tower. For \( C \subset B_n \) we call \( S^{C}E = \cup_{\vec{n} \in C} S^{\vec{n}}E \) the subtower with shape \( C \). For \( E' \subset C \) we call \( S^{B_n}E' \) a slice of the tower. If \( \mu(S^{B_n}E) > 1 - \delta \) we say the tower has error \( < \delta \).

Let \( P = \{p_1, \ldots, p_k\} \) be a measurable partition of \( X \). By the \((n, P)\)-name of a point \( x \in X \) we mean the map \( P_n(x) : B_n \to P \) defined by \( P_n(x)[\vec{v}] = p_i \) if and only if \( S^{\vec{n}}x \in p_i \). Finally, for \( R \subset \mathbb{Z}^d \), we denote the cardinality of \( R \) by \( |R| \) and the complement of any set \( A \) by \( A^c \).

2.2. Restricted orbit equivalence characterizations. Here we give the restricted orbit equivalence characterizations of the equivalence relations \( \sim \) and \( \overset{\sim}{\vartriangle} \). We refer the reader to [2], [3] and [10] for details.

Definition 2.1. Two \( \mathbb{Z}^d \) actions \( S \) and \( T \) are evenly Kakutani equivalent if there is an orbit equivalence \( \phi : X \to Y \) between \( S \) and \( T \) such that given
\(\epsilon > 0\), there is an \(N(\epsilon) > 0\), and a set \(A \subset X\) with \(\mu A > 1 - \epsilon\) such that for all \(x, y \in A\) and on the same orbit if \(\bar{S}(x, y) > N\) then
\[
\|\bar{S}(x, y) - \bar{T}(\phi x, \phi y)\| < \epsilon \|\bar{S}(x, y)\|.
\]

The set \(A\) will be called an \(\epsilon\) Kakutani pinning set. The constant \(N(\epsilon)\) will be called an \(\epsilon\) Kakutani constant. Property (2) will be referred to as the distortion property of \(\phi\).

**Definition 2.2.** Let \(\vec{\alpha} = \{1 + \alpha_1, 1 + \alpha_2\}\) where \(\alpha_i\) are positive irrationals.

Let \(S\) and \(T\) be measurable, measure preserving, free, and ergodic \(\mathbb{Z}^2\) actions on \((X, \mathcal{M}, \mu)\) and \((Y, \mathcal{G}, \nu)\) respectively. The actions \(S\) and \(T\) are \(\vec{\alpha}\)-equivalent if and only if there exists an orbit equivalence \(\phi : X \to Y\) between \(S\) and \(T\) such that

1) given \(\epsilon > 0\) there is an \(N(\epsilon) > 0\), and a set \(A \subset X\) with \(\mu A > 1 - \epsilon\) such that for all \(x, y \in A\) and on the same orbit if \(\bar{S}(x, y) > N\) then
\[
\|\bar{S}(x, y) - \bar{T}(\phi x, \phi y)\| < \epsilon \|\bar{S}(x, y)\|,
\]

and
2) the function \(d\left(\frac{\bar{S}(x,y) - \bar{T}(\phi x, \phi y)}{\alpha_i}, Z\right)\) is a circle valued coboundary for \(i = 1 \ldots d\).

We will refer to the second condition on the orbit equivalence in Definition 2.2 as the coboundary condition of \(\phi\).

3. **Proof of the main theorem, Theorem 1.1.**

Let \(S, T, \vec{\alpha}, \text{ and } \vec{\beta}\) be as in the statement of Theorem 1.1. Let \(\Theta : X \to Y\) be the orbit equivalence given by applying Definition 2.1 to \(S\) and \(T\). Let \(P = \{p_1, \ldots, p_k\}\) be a generating partition for \(S\).

We will inductively construct a third Lebesgue space \((Z, \mathcal{M}, \lambda)\), a measure preserving ergodic free \(\mathbb{Z}^2\) action \(U\) on \(Z\), and an orbit equivalence \(\psi (\phi)\) between \(S\) and \(U\) \((T \text{ and } U)\) which satisfies Definition 2.2 with \(\vec{\alpha} (\vec{\beta})\). The space \(Z\) will be a subset of \([0, 1]\), and \(\lambda\) will be Lebesgue measure.

3.1. **The First Step of the Construction.** We begin the construction by choosing a Rohlin tower \(\tau_1\) of size \(n_1\) (to be determined later) for \(S\). We begin constructing the space \(Z\) by constructing a subset \(\overline{Z}_1\) of \([0, 1]\) as a copy of a subtower \(\overline{X}_1\) of \(\tau_1\). We will describe the shape \(I_1\) of this subtower in detail below. We construct a partial action \(U_1\) defined on most levels of \(\overline{Z}_1\) by defining set maps \(U_1^{\vec{e}_1} (U_1^{\vec{e}_2})\) so that \(\overline{Z}_1\) is a Rohlin tower of \(U_1\). Note that \(U_1^{\vec{e}_1} (U_1^{\vec{e}_2})\) will be undefined on the rightmost (uppermost) levels of the tower.

The key ingredients of the construction are the properties of \(\tau_1\), and we describe them first without technical detail. We select \(\epsilon\)-Kakutani pinning
sets $A(i)$ for $\Theta$ (where the $\epsilon_i$ will be determined later) and set $A = \cap A(i)$. Using the ergodic theorem we will choose $\tau_1$, so that its base $E_1$ is entirely contained in $A(1)$ and most of its levels are well covered in measure by the set $A$. By partitioning the base $E_1$, if necessary, we can assume that a level $S^\theta(E_1)$ is entirely contained in $A$ or in $A^c$. If a level is contained in $A$, we call it a good level. We define $\psi_1$ on a good level $S^\theta E_1$ by mapping it to level $\vec{v} + \vec{c}_1$ in $\mathbb{Z}_1$, where $\vec{c}_1$ will be chosen using the following standard lemma with $\epsilon = \epsilon_1$:

**Lemma 3.1.** Suppose $\alpha, \beta \in \mathbb{R}$ are such that $\{1, \alpha^{-1}, \beta^{-1}\}$ are rationally independent, and that $\epsilon > 0$ is given. Let $T^2 = S^1 \times S^1$ denote the 2-torus, and $B_\epsilon(0) \subset T^2$ denote the $\epsilon$ neighbourhood of 0 in $T^2$.

Then there exists $K(\epsilon) > 0$ such that for all $(x, y) \in T^2$, there is $0 \leq k \leq K(\epsilon)$ such that $R^k(x, y) \in B_\epsilon(0)$ where $R^k(x, y) = (x + \frac{k}{\alpha}, y + \frac{k}{\beta})$.

This lemma guarantees that we can choose $\vec{c}_1$ so that $\psi_1$ will satisfy a coboundary condition with the $\alpha_i$ (condition (15) below). Since $\epsilon_1$ will be determined at the start of the construction we can choose $n_1$ so that it is much larger than $K(\epsilon_1)$. Then $\psi_1$ will automatically satisfy a distortion property (condition (14) below) on good levels of $\tau_1$.

To construct an orbit equivalence between $U_1$ and $T$, we set $\sigma_1 = \bigcup_{\vec{v} \in B_{\epsilon_1}} T^\vec{v}(\Theta(E_1))$ and make the following observation. Normally an orbit equivalence will not necessarily preserve Rohlin towers, but using the fact that $\Theta$ is an even Kakutani equivalence we can choose $n_1$ so that if a level $S^\theta(E_1)$ of $\tau_1$ is far enough away from the base, and is entirely contained in $A(1)$ then $\Theta(S^\theta(E_1)) \subset \sigma_1$. Thus, if $\tau_1$ is also chosen so that most of its levels are well covered by $A(1)$ we can guarantee that most levels of $\tau_1$ get mapped into $\sigma_1$.

We then define a set map $\phi_1$ from $\mathbb{Z}_1$ to $\sigma_1$ by sending $\psi_1(S^\theta(E_1))$ to $\Theta(S^\theta(E_1))$, when the latter is contained in $\sigma_1$, and to an arbitrary subset of $\sigma_1$ otherwise. Because $n_1$ can be chosen to be very large compared to $K(\epsilon_1)$, $\phi_1$ will inherit a distortion property (with different parameters) from $\Theta$ in spite of the vector $\vec{c}_1$ involved in the definition of $\psi_1$ (condition (14) below).

In addition, since we know a priori where $\Theta$ sends levels of $\tau_1$, using Lemma 3.1 the vector $\vec{c}_1$ can be chosen to additionally guarantee that $\phi_1$ satisfies a coboundary condition with the $\beta_i$ (condition (15) below).

Now for the details. Fix $\epsilon > 0$ and a sequence $\{\epsilon_n\}$ of positive real numbers such that $\sum \epsilon_n < \epsilon$. Let $N(\frac{\epsilon}{32})$ be an increasing sequence of $\frac{\epsilon}{32}$ Kakutani constants, and let $A(n)$ denote the $\frac{\epsilon}{32}$ Kakutani pinning sets. Let $K(n)$ be an increasing sequence chosen to satisfy Lemma 3.1 for $\frac{\epsilon}{4}$ and $\{\alpha, \beta\}$, for both $i = 1, 2$. Let $N(n)$ be an increasing sequence of integers chosen so that

$$N(n) \geq N\left(\frac{\epsilon_n}{32}\right) \quad \text{and}$$

$$N(\epsilon) \geq N\left(\frac{\epsilon}{32}\right) \quad \text{and}$$
(4) \[
\frac{4\sum_{i=1}^{n} K(i)}{N(n)} < \frac{\epsilon_n}{16}.
\]
Let \( A = \cap_{n=1}^{\infty} A(n) \) and note that \( \mu A > 1 - \epsilon. \)
We pick \( k \in \mathbb{N} \) such that
(5) \[
\frac{4N(1)}{k} < \frac{\epsilon_1}{8}.
\]
We use the Rohlin Lemma and the Ergodic Theorem to choose \( n_1 \) large enough so that the tower \( \tau_1 \) has error \( < \frac{\epsilon_1}{16} \) and satisfies:
1) \( E_1 \subset A(1), \)
2) for all \( x \in E_1 \)
(6) \[
\frac{|\bar{v} \in B_{n_1} : S^x_{\bar{v}} x \in A|}{n_1^2} > 1 - 2\epsilon, \quad \frac{|\bar{v} \in B_{n_1} : S^x_{\bar{v}} x \in A(1)|}{n_1^2} > 1 - \frac{\epsilon_1}{16},
\]
3) and finally
(7) \[
\frac{4(k + K(2) + N(2))}{n_1} < \frac{\epsilon_1}{32}.
\]
To define the subtower \( \bar{X}_1 \) of \( \tau_1 \) we let \( b_1 = K(2) + N(2) + \frac{\epsilon_1}{16} n_1 \) and \( \bar{b}_1 = (b_1, b_1). \) We set \( I_1 = B_{n_1 - 2b_1} + \bar{b}_1, C_1 = B_{n_1} \setminus I_1, \) \( X_1 = S^{\bar{b}_1} E_1 \) and \( \bar{X}_1 = X_1 \cup E_1. \) Note that by (7)
(8) \[
\mu X_1 > 1 - \frac{\epsilon_2}{10} - \frac{4(k + K(2) + N(2) + \frac{\epsilon_1}{16} n_1)n_1}{n_1^2} > 1 - \frac{\epsilon_1}{2}.
\]
Suppose \( \mu(\bar{X}_1) = \ell_1 \) and set \( \bar{Z}_1 = [0, \ell_1). \) We slice \( \bar{Z}_1 \) into subintervals to make a copy of \( \bar{X}_1 \) and we denote the base of this tower by \( F_1. \) We define the partial action \( U_1 \) as discussed above and we set \( Z_1 = U_1^{\ell_1}(F_1), \) the subtower of shape \( I_1. \) Thus \( [0, \ell_1) = Z_1 = Z_1 \cup F_1. \)

3.1.1. Constructing \( \psi_1. \) Let \( x \in E_1 \) and suppose that \( \bar{v} \in I_1 \) is such that \( S^{\bar{v}} x \in A(1). \) Since \( E_1 \subset A(1) \) we claim that our choice of \( n_1 \) and \( b_1 \) guarantees that
(9) \[
\Theta(S^\bar{v} x) \in T^{B_{n_1}}(\Theta x).
\]
To see this, note that for \( \bar{v} \in I_1 \) we have \( N(2) + \frac{\epsilon_1}{16} n_1 < \|\bar{v}\| < n_1(1 - \frac{\epsilon_1}{16}). \) So \( \|\bar{v} - \bar{T}(\Theta x, \Theta S^{\bar{v}} x)\| < \frac{\epsilon_1}{32} \|\bar{v}\|, \) thus \( \|\bar{T}(\Theta x, \Theta S^{\bar{v}} x)\| < (1 + \frac{\epsilon_1}{32}) \|\bar{v}\| < n_1. \) Also, since for \( s = 1, 2 \) we have \( \tilde{v}_s > \frac{\epsilon_1}{16} n_1 \) \( \Theta(S^{\tilde{v}} x) \) we have \( \tilde{T}_s(\Theta x, \Theta S^\tilde{v} x) > 0, \) and (9) follows.
Without loss of generality suppose that the dimensions of \( I_1 \) are integer multiples of \( k. \) For a fixed \( x \in E_1, \) take the \( k \) grid of \( S^{\bar{b}_1} x \) starting at the lower left hand index of \( I_1. \) We say \( y \in S^{\bar{b}_1} x \) is a \textit{good element} of \( S^{\bar{b}_1} x \) if
1) \( y \in A \) and
2) \( y \) lies at least a distance \( N(1) \) away from the boundary of its \( k \)-grid box.
Choose a good element from each grid box which contains one and call this set $A_1(x)$. Set $A_1 = \cup_{x \in E_1} A_1(x)$. It follows from (5), (6) and (7) that $\mu(A_1) > \frac{1}{2^s}$. 

For each $x \in E_1$, let the set $\{x_1, \ldots, x_{m(x)}\}$ denote the elements of $A_1(x)$ in lexicographic order. Define $\tilde{V}_1(x) = \{\tilde{v}_1, \ldots, \tilde{v}_{m(x)}\} \subset I_1$ by $\tilde{v}_j = \tilde{S}(x, x_j)$ for $j = 1, \ldots, m(x)$. Thus, $\tilde{V}_1(x)$ is a list of the levels of $\tau_1$ containing the elements of $A_1(x)$.

We first define $\psi_1$ on the levels in $\tilde{V}_1(x)$. To this end we partition $E_1$ into subsets $E_1^i$ such that:

1) The set $V_1$ is constant over each set $E_i$. Namely, if $x, y \in E^i_1$ then $\tilde{V}_1(x) = \tilde{V}_1(y) = \tilde{V}_1(i)$.

2) For each $\tilde{v} \in B_{n_1}$, the level $S^{\tilde{v}}(E_1^i)$ is contained entirely in $A \ (A(1))$ or $A^c \ (A(1)^c)$. Namely, $S^{\tilde{v}}(E_1^i) \cap A(1)$ is either empty or all of $S^{\tilde{v}}(E_1^i)$ and $S^{\tilde{v}}(E_1^i) \cap A$ is either empty or all of $S^{\tilde{v}}(E_1^i)$.

3) The map $\Theta$ is constant on the levels of $I_1$ which lie in $A(1)$. Namely if $\tilde{v} \in I_1$ is such that $S^{\tilde{v}}(E_1^i) \subset A(1)$ then

\[ (10) \quad \tilde{T}(\Theta x, \Theta S^{\tilde{v}} x) = \tilde{T}(\Theta y, \Theta S^{\tilde{v}} y) \]

for all $x, y \in E_1^i$.

4) Finally, every $x \in E_1^i$ has the same $(n_1, P)$-name.

Let $i(1)$ denote the number of sets $E_1^i$. Note that by (9) we are guaranteed that $i(1)$ is finite, even with condition (10). Partition $F_1$ into $i(1)$ measurable subsets with $\lambda(F_1^i) = \mu(E_1^i)$ and for each $i = 1, \ldots, i(1)$ set

$\psi_1(E_1^i) = F_1^i$.

For each $i$, we can now define $\psi_1$ on $S^{\tilde{v}_1(i)} E_1^i$. Fix $x \in E_1^i$. By our choice of $K(1)$, for each $x_j \in A_1(x)$ we can find a vector $\tilde{c}_1(i, x_j)$ with

\[ (11) \quad \|\tilde{c}_1(i, x_j)\| < K(1) \]

such that for $s = 1, 2$

\[ (12) \quad d \left( \frac{(\tilde{c}_1(i, x_j))_s}{\alpha_s}, Z \right) < \frac{\epsilon_1}{4} \quad \text{and} \]

\[ d \left( \frac{\tilde{T}_s(\Theta(x), \Theta(x_j)) - S_s(x, x_j) - (\tilde{c}_1(i, x_j))_s}{\beta_s}, Z \right) < \frac{\epsilon_1}{4} \]

For each $j$ set

\[ (13) \quad \psi_1(S^{\tilde{v}_j}(E_1^i)) = U_1^{\tilde{v}_j + \tilde{c}_1(i, x_j)}(F_1^i). \]

For a vector $\tilde{v} \in I_1$ which is not in $\tilde{V}_1(i)$ if there is no conflict arising from (13) we set $\psi_1(S^{\tilde{v}} E_1^i) = U_1^\tilde{v}(F_1^i)$. If there is a conflict, then we map $S^{\tilde{v}} E_1^i$ to an empty level in $U_1^{i_1}(F_1^i)$. We do not define $\psi_1$ on $S^{C_1}(E_1)$ at this stage.
We claim that $\psi_1$ is well-defined on the levels of $X_1$ and has range $Z_1$. To see this note that $\vec{v}_j + \vec{c}_1(i, x_j) \in I_1$, so $\psi_1(S^{\vec{v}_j} E_1^i) \subset Z_1$. Also note that if $j \neq k$ then since
\[
\|\vec{S}(x, x_j) - \vec{S}(x, x_k)\| = \|\vec{S}(x_j, x_k)\| > 2N(1)
\]
it follows from (4) and (11) that $\vec{v}_j + \vec{c}_1(i, x_j) \neq \vec{v}_k + \vec{c}_1(i, x_k)$.

If $y_1, y_2 \in A_1(x)$ then by construction $\|\vec{S}(y_1, y_2)\| > 2N(1)$. Using the properties of the set $A_1(x)$, (4), and (11) we have
\[
\|\vec{S}(y_1, y_2) - \vec{U}(\psi_1(y_1), \psi_1(y_2))\| \leq \frac{\|c_1(i, y_1) + c_1(i, y_2)\|}{\|\vec{S}(y_1, y_2)\|} < \frac{\epsilon_1}{8}\|\vec{S}(y_1, y_2)\|.
\]
In addition, by (12) for $s = 1, 2$ and $y_1$ and $y_2$ as above
\[
d\left(\frac{\vec{S}(y_1, y_2) - (\vec{U}(\psi_1(y_1), \psi_1(y_2)))_s}{\alpha_s}, \mathbb{Z}\right) = d\left(\frac{c_1(i, y_1)}{\alpha_s}, \mathbb{Z}\right) + d\left(\frac{c_1(i, y_2)}{\alpha_s}, \mathbb{Z}\right) < \frac{\epsilon_1}{2}.
\]

3.1.2. Defining $\phi_1$. For every $i = 1, \ldots, i(1)$ we set $\phi_1(F_1^i) = \Theta(E_1^i)$ and if $\vec{v} \in I_1$ is such that the level $\psi_1^{-1}(U_1^\vec{v} F_1^i)$ in $X_1$ is contained in $A(1)$, we set
\[
\phi_1(U_1^\vec{v} F_1^i) = \Theta(\psi_1^{-1}(U_1^\vec{v} (F_1^i))).
\]
We do not define $\phi_1$ on the rest of the levels of $Z_1$ at this stage.

Recall that by construction $\Theta$ is constant on the levels of $S^{1} E_1$, for all $i$ every level $S^{\vec{v}} E_1^i$ is either entirely in $A(1)$ or in $A(1)^c$ so $\phi_1$ is well-defined. For levels $\vec{v}$ where (16) holds (9) is satisfied and the range of $\phi_1$ is contained in $\sigma_1$.

The map $\phi_1$ is then defined on $Z_1'' = \psi_1(A(1) \cap X_1)$. By (6) and (7)
\[
\lambda Z_1'' = \mu(A(1) \cap X_1) > 1 - \frac{\epsilon_1}{32} - \frac{\epsilon_1}{16} > 1 - \epsilon_1.
\]
We let $Y_1 = \phi_1(Z_1'')$, and note that since $\psi_1$ and $\Theta$ are measure preserving, so is $\phi_1$.

We now show that $\phi_1$ satisfies a distortion and coboundary condition on the set $D_1 = \psi_1\left(\bigcup_{i=1}^{i(1)} S^{\vec{v}}(i) E_1^i\right)$, the image of the special levels in $\vec{V}_1(i)$.

Pick $z_j, z_k \in D_1$ and identify which level of in $\tau_1$ they came from. Namely, choose $i \in \{1, \ldots, i(1)\}$ and $\vec{v}_j, \vec{v}_k \in \vec{V}_1(i)$ such that $z_j \in \psi_1(S^{\vec{v}_j} E_1^i)$ and $z_k \in \psi_1(S^{\vec{v}_k} E_1^i)$. Then for $x \in E_1^i$ we can choose representative points from these levels of $\tau_1$. Namely, we can find $x_j, x_k \in S^{\vec{V}(i)} E_1^i$ such that $S^{\vec{v}_j} x = x_j, S^{\vec{v}_k} x = x_k$. Note that by construction we have $\|\vec{S}(x_j, x_k)\| > 2N(1)$ and
\[
\vec{U}_1(z_j, z_k) = (\vec{v}_j - \vec{v}_k) + (c_1(i, x_j) - c_1(i, x_k)).
\]
Thus, $\|\vec{U}_1(z_j, z_k)\| > N(1)$ and by (4) we have:

$$
\|\vec{U}_1(z_j, z_k) - \vec{T}(\phi_1 z_j, \phi_1 z_k)\|
$$

$$
= \|\vec{v}_j - \vec{v}_k + \vec{c}_1(i, x_j) - \vec{c}_1(i, x_j) - \vec{T}(\Theta x_j, \Theta x_k)\|
$$

$$
\leq \|\vec{S}(x_j, x_k) - \vec{T}(\Theta x_j, \Theta x_k)\| + 2K(1)
$$

$$
\leq \|\vec{U}_1(z_j, z_k)\| \left(\frac{\epsilon_1}{32} + \frac{4K(1)}{N(1)}\right)
$$

$$
< \frac{\epsilon_1}{8}\|\vec{U}_1(z_j, z_k)\|.
$$

So we have

(14')

$$
\|\vec{U}_1(z_j, z_k) - \vec{T}(\phi_1 z_j, \phi_1 z_k)\| < \frac{\epsilon_1}{8}\|\vec{U}_1(z_j, z_k)\|.
$$

Now notice that

$$
d\left(\frac{\vec{T}_s(\Theta x, \Theta x_k) - (\vec{U}_1(z_j, z_k))_s}{\beta_s}, \vec{Z}\right)
$$

$$
\leq d\left(\frac{\vec{T}_s(\phi_1 z, \phi_1 z_j) - \vec{S}_s(x, x_j) - (\vec{c}_1(i, x_j))_s}{\beta_s}, \vec{Z}\right)
$$

$$
+ d\left(\frac{\vec{T}_s(\Theta x, \Theta x_k) - \vec{S}_s(x, x_k) - (\vec{c}_1(i, x_k))_s}{\beta_s}, \vec{Z}\right)
$$

for any $z \in F^i$. Thus by (12)

(15')

$$
d\left(\frac{\vec{T}_s(\phi_1 z_j, \phi_1 z_k) - (\vec{U}_1(z_j, z_k))_s}{\beta_s}, \vec{Z}\right) < \frac{\epsilon_1}{2}.
$$

3.2. The Induction Step of the Construction. We will again begin by choosing a Rohlin tower $\tau_2$ for $S$, and we will construct $Z_2$, a copy of a subtower $X_2$ of $\tau_2$ in $[0, 1]$. The key issue is to ensure that $Z_2$ refines $Z_1$, and that the map $\psi_2$ respects $\psi_1$ on most of $X_1$ ($Z_1$). We first briefly describe the part of the construction which is parallel to the first step.

We choose $n_2 \in \mathbb{N}$ such that there is a Rohlin tower $\tau_2$ for $S$ with shape $B_{n_2}$, base $E_2$, and error $\frac{\epsilon_3}{10}$ such that $E_2$ is entirely contained in $A(2)$, and for all $x \in E_2$ we have

(18)

$$
\frac{|\vec{v} \in B_{n_2} : S^{\vec{v}} x \in \tau_1|}{n_2^2} > 1 - \frac{\epsilon_2}{5}, \quad \frac{|\vec{v} \in B_{n_2} : S^{\vec{v}} x \in A(2)|}{n_2^2} > 1 - \frac{\epsilon_2}{16},
$$

and

(19)

$$
\frac{4(K(3) + N(3) + n_1)}{n_2} < \frac{\epsilon_2}{32}.
$$
We define $b_2, \bar{b}_2, I_2$, and $C_2$ as in the first step (with the index in each parameter increased by one). Set $X_2 = S^{I_2}E_2$ and $\overline{X}_2 = X_2 \cup E_2$ and note that by (19) we have
\begin{equation}
\mu X_2 > 1 - \frac{\epsilon_3}{10} - \frac{\epsilon_2}{32} > 1 - \frac{\epsilon_2}{2}.
\end{equation}

As in the first step we will partition $E_2$ into subsets so that the levels of $X_2$ are entirely covered by special sets or their complements. The special sets are a little different this time. For $x \in E_2$ we set $A_2(x) = S^{I_2}x \cap A_1$ and $A_2 = \cup_{x \in E_2} A_2(x)$. Note that
\begin{equation}
\mu A_2 > (1 - \epsilon_2)\mu A_1.
\end{equation}

We let \{\(x_1, \ldots, x_{m(2)}\}\} denote the elements of $A_2(x)$ in lexicographic order and we define $\bar{V}_2(x) = \{\bar{v}_1, \ldots, \bar{v}_{m(x)}\}$ as before: the list of the levels based at $x$ containing the $x_i$.

We then partition $E_2$ into subsets $E_2^j$ such that each level of $\tau_2$ is either entirely contained in $A(2) (A_2)$, or in $A(2)^c (A_2^c)$, every $x \in E_2^j$ has the same \((n_2, P)\)-name, the list $V_2(x)$ is constant on each $E_2^j$, and $\Theta$ is constant on the levels of $S^{I_2}E_2$ which lie in $A(2)$. By a computation parallel to the one given in the first step of the construction we can show that
\begin{equation}
\Theta(S^{\bar{v}}x) \in T^{B_{n_2}}(\Theta x),
\end{equation}
so $E_2$ is partitioned into finitely many subsets, in spite of the last condition. Let $j(2)$ denote the number of sets $E_2^j$.

We impose one new condition on the partitioning of $E_2$: we require that each level of $X_2$ lies entirely in $E_1^c$ or in exactly one subset $E_1^j$ of $E_1$.

To begin copying $X_2$ in $[0, 1]$ we first cut an interval of length $\mu E_2$ from $[0, 1] \setminus \overline{Z}_1$. This will be the base of the tower $\overline{Z}_2$ and will be labelled $F_2$.

The rest of $\overline{Z}_2$ will consist of slices of $Z_1$, and some new intervals cut from the remaining part of $[0, 1]$. The new intervals will form the levels of $Z_2$ not covered by $Z_1$. The set maps $U_2^1$ and $U_2^2$ are defined as before and we have $\overline{Z}_2 = F_2 \cup Z_2$ where $Z_2 = U_2^1 F_2$, the subtower corresponding to $X_2$.

Since the definition of the map $\psi_2$ will depend heavily on how we locate the various slices of $Z_1$ inside $\overline{Z}_2$ we finish constructing $\overline{Z}_2$ as we define $\psi_2$.

\textbf{3.2.1. Constructing $\psi_2$.} As before we will construct $\psi_2$ only on $E_2 \cup X_2$ so we first eliminate from consideration those slices of $X_1$ in $\tau_2$ which don’t lie entirely in $X_2$. Denote these slices by $X_1'$. For ease of notation we continue to call the partitioned base of this new subtower $E_1^j$. By (8) and (19) we have
\begin{equation}
\mu(X_1') > 1 - \frac{\epsilon_1}{2} - \frac{\epsilon_2}{2}.
\end{equation}

To construct $Z_2$ we will first slice $F_1$ and $Z_1$ into subsets corresponding to the various slices of $E_1$ and $X_1'$ appearing in $X_2$. Label these slices in
some way to keep track of where the \( X'_1 \) slices appear in \( X_2 \). Specifically, the subset \( E^{i,j,k}_1 \) of \( E^i_1 \) is the base of the \( k^{th} \) slice of \( X_1 \) appearing in \( X^j_2 \). So we have

\[
X'_1 = \bigcup_{i=1}^{i(1)} \bigcup_{j=1}^{j(2)} \bigcup_{k=1}^{k} X^{i,j,k}_1 \quad \text{and} \quad Z'_1 = \bigcup_{i=1}^{i(1)} \bigcup_{j=1}^{j(2)} \bigcup_{k=1}^{k} Z^{i,j,k}_1.
\]

We will place \( F^{i,j,k}_1 \) in the same location in \( Z_2 \) as \( E^{i,j,k}_1 \) appears in \( X_2 \). The set \( Z^{i,j,k}_1 \), however, will be shifted to a different location relative to its base than \( X^{i,j,k}_1 \) sits relative to \( E^{i,j,k}_1 \). Recall that \( \psi_1 \) was left undefined on \( S^{C_1}(E^i_1) \), for every \( i \), so the \( C_1 \)-collar around \( Z^{i,j,k}_1 \) does not have a preimage in \( X \). The set \( Z^{i,j,k}_1 \) will be placed in \( Z_2 \) starting at a location in this collar.

The translation of \( Z^{i,j,k}_1 \) relative to its base will be by a vector \( \vec{c}_2 \) obtained from Lemma 3.1, this time applied with \( \epsilon = \epsilon_2 \). The map \( \psi_2 \) will then respect \( \psi_1 \) on \( X_1 \), and will map each slice \( X^{i,j,k}_1 \) to a location which is a shift by the vector \( \vec{c}_2 \) from its original position in \( \tau_2 \). Again, \( \epsilon_2 \) is very large compared to \( K(2) \), so the distortion property is guaranteed, and Lemma 3.1 will guarantee the coboundary property.

To choose the vectors \( \vec{c}_2 \) let \( \vec{v}_k \in V_2(j) \) be such that \( x_k = S^{\vec{v}_k} x \in A_2(x) \) is the first lexicographic occurrence of \( A_2 \) in \( X^{i,j,1}_1 \). Using Lemma 3.1 choose \( \vec{c}_2(j, x_k) \) with

\[
\|\vec{c}_2(j, x_k)\| < K(2)
\]

such that for \( s = 1, 2 \) we have

\[
\begin{align*}
d\left( \frac{(\vec{c}_2(j, x_k))_s}{\alpha_s}, Z \right) & < \frac{\epsilon_2}{4} \quad \text{and} \\
d\left( \frac{T_s(\Theta(x), \Theta(x_k)) - S_s(x, x_k) - (\vec{c}_2(j, x_k))_s}{\beta_s}, Z \right) & < \frac{\epsilon_2}{4}.
\end{align*}
\]

Place \( Z^{i,j,1}_1 \) in \( Z_2 \) so that its location in \( I_2 \) relative to \( F_2 \), is a shift of the position of \( X^{i,j,1}_1 \) in \( I_2 \) by the vector \( \vec{c}_2(j, x_k) - \vec{c}_1(i, x_k) \). In particular, for \( z \in F_2 \) and \( z_k \in \psi_2(S^{\vec{v}_k} x) \) we have

\[
\tilde{U}_2(z, z_k) = \tilde{S}(x, x_k) + \vec{c}_2(j, x_k).
\]

Since \( C_1 \) is a collar of width greater than \( K(2) \) around \( S^{I_1} E_1 \), (24) guarantees that the images of distinct slices of \( S^{I_1} E_1 \) under \( \psi_2 \) will not intersect.

We repeat this procedure until all the slices of \( X'_1 \) are taken care of, hence all of \( Z'_1 \) is placed in \( Z_2 \).

We will also define \( \psi_2 \) on \( S^{C_1}(E^{i,j,k}_1) \) at this stage by mapping this sub-tower level by level into locations in \( U^{C_1}(F^{i,j,k}_1) \) vacated by the translation of \( Z^{i,j,k}_1 \).
We complete the tower $Z_2$ by slicing intervals of the appropriate length from $[0, 1] \setminus (Z'_1 \cup F_2)$ and placing these in the empty positions of $I_2$. Finally, for the remaining $\bar{v} \in I_2$ we set
\[ \psi_2(S_2^{\bar{v}}E_2^1) = U_2^{\bar{v}}(F_2^1). \]
The map $\psi_2$ is now defined on all of $X_2$ with image $Z_2$. Further, $\psi_2$ refines $\psi_1$ on $X'_1 \subset X_2$.

We will now show that for $y_1, y_2 \in A_2$, if $\| \tilde{S}(y_1, y_2) \| > N(2)$ then
\begin{equation}
\| \tilde{S}(y_1, y_2) - \tilde{U}_2(\psi_2 y_1, \psi_2 y_2) \| < \frac{\epsilon_2}{8} \| \tilde{S}(y_1, y_2) \|, \tag{26}
\end{equation}
and that regardless of the value of $\| \tilde{S}(y_1, y_2) \|$ we always have
\begin{equation}
\frac{d}{\alpha_s} \left( \tilde{S}_s(y_1, y_2) - \left( \tilde{U}_2(\psi_2 y_1, \psi_2 y_2) \right)_s, z \right) < \epsilon_1 + \epsilon_2. \tag{27}
\end{equation}

Pick such a pair $y_1$ and $y_2$ and suppose they lie in $A_2(x), x \in E^j_2$. Then either $y_1$ and $y_2$ lie in the same slice, $X_1^{i_1,j,k_1}$ of $X_1$, or there exist $k_1 \neq k_2$ and $i_1, i_2$ such that $y_1 \in X_1^{i_1,j,k_1}$ and $y_2 \in X_1^{i_2,j,k_2}$. In the first case, by construction we have
\[ \tilde{U}_2(\psi_2 y_1, \psi_2 y_2) = \tilde{U}_1(\psi_1 y_1, \psi_1 y_2) \]
and
\begin{equation}
\tilde{U}_1(\psi_1 y_1, \psi_1 y_2) = \tilde{S}(y_1, y_2) + \tilde{c}_1(i, y_1) + \tilde{c}_1(i, y_2). \tag{28}
\end{equation}

In the second case pick $z \in F^j_2$. Then for $p = 1, 2$ there exist $x_{m_p} \in A_2(x) \cap X_1^{i_p,j,k_p}$ such that
\begin{equation}
\tilde{U}_2(z, \psi_2 y_p) = \tilde{S}(x, x_{m_p}) + \tilde{c}_2(j, x_{m_p}) + \tilde{U}_1(\psi_1 x_{m_p}, \psi_1 y_p). \tag{29}
\end{equation}

In both cases by (4), (11), (24), and the construction of $\psi_1$, if $\| \tilde{S}(y_1, y_2) \| > N(2)$, then we have
\[ \| \tilde{S}(y_1, y_2) - \tilde{U}_2(\psi_2 y_1, \psi_2 y_2) \| \leq 4K(1) + 2K(2) < \frac{\epsilon_2}{8} \| \tilde{S}(y_1, y_2) \|. \]
To see that (27) holds we note that in the first case (15) holds, hence, so does (27). In the second case using (29) we see that by (12) and (25)
\begin{align*}
\frac{d}{\alpha_s} \left( \tilde{S}_s(y_1, y_2) - \left( \tilde{U}_2(\psi_2 y_1, \psi_2 y_2) \right)_s, z \right) \\
\leq \frac{d}{\alpha_s} \left( \tilde{c}_1(i_1, y_1) \right)_s + \frac{d}{\alpha_s} \left( \tilde{c}_1(i_2, y_2) \right)_s + \frac{d}{\alpha_s} \left( \tilde{c}_2(j, x_{m_1}) \right)_s + \frac{d}{\alpha_s} \left( \tilde{c}_2(j, x_{m_2}) \right)_s + \frac{d}{\alpha_s} \left( \tilde{c}_1(i_1, x_{m_1}) \right)_s + \frac{d}{\alpha_s} \left( \tilde{c}_1(i_2, x_{m_2}) \right)_s + \frac{d}{\alpha_s} \left( \tilde{c}_1(i_1, x_{m_1}) \right)_s + \frac{d}{\alpha_s} \left( \tilde{c}_1(i_2, x_{m_2}) \right)_s \\
< \epsilon_1 + \epsilon_2.
\end{align*}
3.2.2. Constructing $\phi_2$. The construction here is essentially the same as in the first step of the construction. The map $\phi_2$ will be a set map defined on most of the levels of $Z_2$ with range contained in $T^{(2)}\Theta E_2$.

We start by setting $\phi_2(F_2') = \Theta(E_2')$ for all $j = 1, \ldots, j(2)$. By construction we are guaranteed that for all $j$, and $\bar{v} \in I_2$ the level $S^j E_2'$ in $X_2$ is entirely in $A(2)$ or $A(2)^c$. For $\bar{v} \in I_2$ if $\psi^{-1}_2(U_2^j F_2') \subset A(2)$ then (22) holds and we set

$$\phi_2(U_2^j F_2') = \Theta(\psi^{-1}_2(U_2^j F_2')).$$

The map $\phi_2$ is defined on $Z_2 \cap \psi_2(A(2))$ which has measure $\mu(X_2 \cap A(2)) > 1 - \epsilon_2$. We let $Y_2 = \phi_2(Z_2 \cap \psi_2(A(2)))$. Since $\psi_2$ is a refinement of $\psi_1$ on $X_1'$ it is clear that $\phi_2$ is a refinement of $\phi_1$ on $Z''_2 = \psi_2(X_1' \cap A(1) \cap A(2))$. By (17) and (18) we have

$$\lambda Z''_2 > 1 - \epsilon_1 - \epsilon_2. \tag{30}$$

To see that $\phi_2$ satisfies the appropriate properties we argue exactly as before. Let $D_2 = \psi_2\left( \bigcup_{j=1}^{j(2)} S^j \bar{E}_2' \right)$, notice that $D_2 \subset Z''_2$ and pick $z_1, z_2 \in D_2$ such that either $z_1$ and $z_2$ lie in the same slice $Z_{1}^{i,j,k}$ of $Z_1'$ or there exist $k_1 \neq k_2$ and $i_1, i_2$ such that $z_1 \in Z_{1}^{i_1,j,k_1}$ and $z_2 \in Z_{1}^{i_2,j,k_2}$, and further

$$\|\bar{U}_2(z_1, z_2)\| > 2N(2). \tag{31}$$

In the first case, by construction we have

$$\bar{T}(\phi_2 z_1, \phi_2 z_2) = \bar{T}(\phi_1 z_1, \phi_1 z_2)$$

and we know there exist $y_1, y_2 \in X_{1}^{i,j,k} \cap A_2$ such that (28) holds. Then (31) and (24) guarantee that $\|\bar{S}(y_1, y_2)\| > N(2)$. Since $\bar{T}(\phi_1 z_1, \phi_1 z_2) = \bar{T}(\Theta y_1, \Theta y_2)$ we have

$$\|\bar{T}(\phi_1 z_1, \phi_1 z_2) - \bar{U}_2(z_1, z_2)\| \leq \frac{\epsilon_2}{2} ||\bar{S}(x_1, x_2)|| + 2K(1)$$

$$\leq \frac{\epsilon_2}{32} (||\bar{U}_1(z_1, z_2)|| + 2K(1)) + 2K(1)$$

$$< \frac{\epsilon_2}{8} ||\bar{U}_1(z_1, z_2)||.$$

On the other hand, if $z_1$ and $z_2$ lie in different slices of $Z_1'$ then (29) holds, so

$$\|\bar{T}(\phi_2 z_1, \phi_2 z_2) - \bar{U}_2(z_1, z_2)\| \leq \|\bar{T}(\Theta y_1, \Theta y_2) - \bar{S}(y_1, y_2)\| + 4K(1) + 2K(2),$$
and an argument similar to the previous case then yields
\[ \| \vec{T}(\phi_1 z_1, \phi_1 z_2) - \vec{U}_2(z_1, z_2) \| < \frac{\epsilon_2}{8} \| \vec{U}(z_1, z_2) \|. \]

To see that the coboundary property holds, note that if \( z_1 \) and \( z_2 \) lie in the same slice of \( Z'_1 \) then by (15')
\[ d\left( \frac{(\vec{U}_2(z_1, z_2))_s - \vec{T}_s(\phi_2 z_1, \phi_2 z_2)}{\beta_2}, Z \right) < \epsilon_1. \]

If, instead, (29) holds we have
\[
\begin{align*}
&d\left( \frac{\vec{T}_s(\phi_2 z_1, \phi_2 z_2) - (\vec{U}_2(z_1, z_2))_s}{\beta_2}, Z \right) \\
&\leq d\left( \frac{\vec{T}_s(\Theta x, \Theta x_{m_1}) - \vec{S}_s(x, x_{m_1}) - (\vec{v}_2(j, x_{m_1}))_s}{\beta_2}, Z \right) \\
&+ d\left( \frac{\vec{U}_1(\psi_1 x_{m_1}, \psi_1 y_1)_s}{\beta_2}, Z \right) \\
&+ d\left( \frac{\vec{U}_1(\psi_1 x_{m_2}, \psi_1 y_2)_s}{\beta_2}, Z \right).
\end{align*}
\]

By (25) the first and third summands are bounded by \( \frac{\epsilon_2}{4} \). Since the points \( \psi_1 x_{m_1} \) and \( \psi_1 y_1 \) are all in \( D_1 \) and in the same slice of \( Z'_1 \), by (15') the second and last summands are bounded by \( \frac{\epsilon_1}{2} \). We have then
\[
\begin{align*}
d\left( \frac{\vec{T}_s(\phi_2 z_1, \phi_2 z_2) - (\vec{U}_2(z_1, z_2))_s}{\beta_2}, Z \right) < \epsilon_1 + \frac{\epsilon_2}{2}.
\end{align*}
\]

3.3. Conclusion of the proof. Continuing in this fashion, at stage \( n \) we define sequences \( \{X'_n\} \) and \( \{Z'_n\} \) of subsets of \( X \) and \( [0, 1] \) respectively with
\[ \mu X'_n = \lambda Z'_n > 1 - \epsilon_n \]
(see for example (23)) and a set map \( \psi_n \) from the level sets of \( X'_n \) to those of \( Z'_n \) such that \( \psi_n \) is a refinement of \( \psi_{n-1} \). Thus if we set
\[
X' = \bigcup_{n=1}^{\infty} \cap_{k=n}^{\infty} X'_k \quad Z' = \bigcup_{n=1}^{\infty} \cap_{k=n}^{\infty} Z'_k
\]
then \( \mu X' = \lambda Z' = 1 \). In addition, since \( P \) was chosen to be a generating partition \( \psi = \lim \psi_n : X' \to Z' \) is a well-defined point map. If we set \( U = \lim U_n \), this is a \( \mathbb{Z}^2 \) action on \( Z' \) and \( \psi \) is an orbit equivalence between \( S \) restricted to \( X' \) and \( U \) on \( Z' \).
We also define at each stage \( n \) sets \( Z_n'' \subset Z_n \) and \( Y_n \subset Y \) with
\[
\lambda(Z_n'') = \nu(Y_n) > 1 - (\epsilon_{n-1} + \epsilon_n)
\]
(see for example (30)). The maps \( \phi_n \) are constructed so that \( \phi_n(Z_n'') = Y_n \) and \( \phi_n \) refines \( \phi_{n-1} \) on \( Z_n'' \). We set
\[
Z'' = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} Z''_k \quad Y' = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} Y_n.
\]
Then \( \lambda Z'' = \nu Y' = 1 \) and the map \( \phi = \lim \phi_n \) is a well defined point map on \( Z'' \).

Lemma 3.2. Let \( \psi \) be the map described above. Then
1) the function
\[
d \left( \frac{S_s(x, y) - U_s(\psi x, \psi y)}{\alpha_s}, Z \right)
\]
is a circle valued coboundary for \( s = 1, 2 \), and
2) for all \( \eta > 0 \) there exists a set \( A \subset X' \) and an integer \( M(\eta) > 0 \) such that if \( x, y \in A \) are on the same orbit and \( \| S(x, y) \| > M(\eta) \) then
\[
\| S(x, y) - U(\psi x, \psi y) \| < \eta \| S(x, y) \|.
\]
Proof. For each \( n \), there exist sets \( A_n \subset X'_n \), such that \( \mu A_n > (1 - \epsilon_n) \mu A \) and for all \( x, y \in A_n \) and on the same orbit
\[
1) d \left( \frac{S_s(x, y) - (U_n(\psi x, \psi y))}{\alpha_s}, Z \right) < \sum_{i=1}^{n} \epsilon_n \quad \text{for } s = 1, 2,
\]
and
\[
2) \| S(x, y) - U_n(\psi_n x, \psi_n y) \| < \frac{\epsilon_n}{2} \| S(x, y) \|.
\]
We set \( A = \bigcap_n A_n \subset X' \) and notice that \( \mu A > 0 \). To see the first part of the claim we note that for all \( x, y \in A \) and on the same orbit
\[
d \left( \frac{S_s(x, y) - U_s(\psi x, \psi y)}{\alpha_s}, Z \right) < \epsilon
\]
where \( \epsilon > 0 \) is chosen at the start of the construction. If \( \epsilon < \frac{1}{2} \) a standard argument yields that the function is a circle valued coboundary on \( R_{S(\cap (A \times A))} \), and thus that it is a circle valued \( S \) coboundary on all of \( R_S \) (see for example [3]).

To see the second part of the claim note that if \( x, y \in A \) then \( x, y \in A_n \) for all \( n \). Then given \( \eta > 0 \) we select \( n \) such that \( \epsilon_n < \eta \) and set \( M(\eta) = N(n) \). The result follows.

A similar argument yields the parallel result for the orbit equivalence \( \phi \):

Lemma 3.3. Let \( \phi \) be the map described above. Then
1) the function
\[ d \left( \frac{\vec{U}_s(x,y) - \vec{T}_s(\phi x, \phi y)}{\beta_s}, Z \right) \]
is a circle valued coboundary for \( s = 1, 2 \), and

2) for all \( \eta > 0 \) there exists a set \( D \subset Z \) and an integer \( M(\eta) > 0 \) such that if \( x, y \in D \) are on the same orbit and \( \|\vec{U}(x, y)\| > M(\eta) \) then
\[ \|\vec{U}(x, y) - \vec{T}(\phi x, \phi y)\| < \eta \|\vec{U}(x, y)\|. \]

Proof. For each \( n \) there exists a set \( D_n \subset Z''_n \) such that \( \lambda D_n > (1 - \epsilon_n)A \) and for all \( z_1, z_2 \in D_n \) on the same orbit

1) \( d \left( \frac{\vec{U}_n(z_1, z_2) - \vec{T}_n(\phi_n z_1, \phi_n z_2)}{\beta_n}, Z \right) < \sum_{i=1}^{n} \epsilon_n \) for \( s = 1, 2 \), and

2) if \( \|\vec{U}_n(z_1, z_2)\| > 2N(n) \) then
\[ \|\vec{U}_n(z_1, z_2) - \vec{T}(\phi_n z_1, \phi_n z_2)\| < \frac{\epsilon_n}{2} \|\vec{U}_n(z_1, z_2)\|. \]

Again, set \( D = \cap D_n \subset Z \) and argue as before.

By Proposition 4 in [2] we have that the sets \( A \) and \( D \) from the previous two results can be made arbitrarily large. This completes the proof of Theorem 1.1.

References

Received May 10, 1999 and revised April 12, 2000. The research of the first author was partially supported by KOSEF 986-0100-001-2, and BK21. The research of the second author was partially supported by the NSF under grant number DMS-9501103.

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