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We show that an even Kakutani equivalence class of $\mathbb{Z}^2$ actions is "spanned" by $\vec{\alpha}$ and $\vec{\beta}$ equivalence classes where $\vec{\alpha} = \{1+\alpha_1, 1+\alpha_2\}$, $\vec{\beta} = \{1+\beta_1, 1+\beta_2\}$ and $\{1, \alpha_i^{-1}, \beta_i^{-1}\}$ are rationally independent for $i = 1, 2$. Namely, given such vectors $\vec{\alpha}$ and $\vec{\beta}$ and two evenly Kakutani equivalent $\mathbb{Z}^2$ actions S and T, we show that U is $\vec{\alpha}$ -equivalent to S and $\vec{\beta}$ -equivalent to T.

1. Introduction.

In this paper we discuss the relationship between two of the fundamental examples of restricted orbit equivalence: even Kakutani equivalence and $\vec{\alpha}$ equivalence. Both equivalence relations arise in the context of representations of ergodic and measure preserving $\mathbb{R}^d$ actions. The first is related to the Ambrose-Kakutani Theorem [1]: every free, measure preserving and ergodic $\mathbb{R}$ action can be represented as a suspension flow over a free, measure preserving and ergodic $\mathbb{Z}$ action. In the case of $\mathbb{R}^d$ and $\mathbb{Z}^d$ actions, for $d > 1$, this is the Katok Representation Theorem [4]. For all $d \geq 1$ two $\mathbb{Z}^d$ actions are said to be even Kakutani equivalent (denoted $\overset{e}{\sim}$) if they arise as sections of equal frequency in different representations of the same $\mathbb{R}^d$ action.

Rudolph has shown that the representation of a finite entropy $\mathbb{R}^d$ action can be achieved with a restriction on the values which the ceiling function may take [8], [9]. In the one-dimensional case, the return times to the base can be required to be only $\{1, \alpha\}$, with $\alpha > 0$ an irrational. For $d > 1$ a vector $\vec{\alpha} = \{1+\alpha_1, \ldots, 1+\alpha_d\}$ is specified, with $\alpha_i > 0$ irrationals, and the suspension is a tiling representation of the $\mathbb{R}^d$ action. The vector $\vec{\alpha}$ determines the sizes and placement rules of the tiles. The equivalence relation in this context analogous to $\overset{e}{\sim}$ is called $\vec{\alpha}$-equivalence ($\overset{\vec{\alpha}}{\sim}$).

It is clear, in all dimensions, that $\overset{\vec{\alpha}}{\sim}$ implies $\overset{e}{\sim}$. The converse does not hold. In [3], Fieldsteel, del Junco and Rudolph construct a spectral invariant which shows that $\overset{\vec{\alpha}}{\sim}$ is a refinement of $\overset{e}{\sim}$. This invariant can easily be extended to higher dimensions.

On the other hand, in the one-dimensional case Park showed that an even Kakutani equivalence class is, in fact, spanned by $\vec{\alpha}$ and $\vec{\beta}$ equivalence classes, when $\{1, \alpha^{-1}, \beta^{-1}\}$ are irrationally related [6]. In this paper we extend this result to higher dimensions. The main result of the paper is as follows.

Theorem 1.1. *Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{G}, \nu)$ be nonatomic Lebesgue probability spaces. Let S and T be two free measure preserving ergodic $\mathbb{Z}^2$ actions of finite entropy acting on X and Y respectively.*

Let $\vec{\alpha} = \{1+\alpha_1, 1+\alpha_2\}$ and $\vec{\beta} = \{1+\beta_1, 1+\beta_2\}$ where the α_i and β_i are positive irrationals and $\{1, \alpha_i^{-1}, \beta_i^{-1}\}$ are rationally independent for $i = 1, 2$.

If S and T are evenly Kakutani equivalent, then there is a nonatomic, Lebesgue probability space $(Z, \mathcal{F}, \lambda)$ and an ergodic, measure preserving, and free $\mathbb{Z}^2$ action U on Z so that

$$S \overset{\vec{\alpha}}{\sim} U \qquad \text{and} \qquad U \overset{\vec{\beta}}{\sim} T. \tag{1}$$

In [6] the author proves the result by constructing the action U and the flows over it explicitly. Extending these methods to higher dimensions quickly becomes an intractable tiling problem. Instead, in this paper, we use the fact that both equivalence relations can be cast as restricted orbit equivalences to prove the result. This characterization will enable us to recognize $\overset{e}{\sim}$ and $\overset{\vec{\alpha}}{\sim}$ by checking for relationships in the orbit structures of the various $\mathbb{Z}^d$ actions. Thus we will be able to work with the discrete actions directly, and we won't be constructing flows or tilings.

Even Kakutani equivalence of $\mathbb{Z}^d$ actions was cast as a restricted orbit equivalence by Rudolph and del Junco in [2]. The one-dimensional characterization is older (see for example [5]) and relies on the linear ordering of the integers. The higher-dimensional characterization is more complicated due to the more complex geometry in the orbits of a higher-dimensional group action.

The situation for $\vec{\alpha}$-equivalence is different. The orbit equivalence characterization for one-dimensional $\vec{\alpha}$-equivalence is given in [3] and is extended to dimension two by Şahin in [10]. Surprisingly this higher dimensional formulation does not require more restrictions on the orbit equivalence. Given that tilings of the plane are much more complicated than tilings of the line, it is surprising that new invariants of equivalence do not appear in higher dimensions.

As was discussed above, in this paper we use the results of [2] and [10] to work only with restricted orbit equivalences between discrete actions, and provide a simpler proof of the result in [6]. We note that the arguments in this paper hold in any dimension, but we state the theorem for $d = 2$, because the techniques in, and hence the results of, [10] hold for $d = 2$. The paper is self contained in that the next section contains the concepts and definitions from [2] and [10] necessary for our arguments.

Finally, we remark that in [7], the author provides an outline of the proof in the one-dimensional case using the restricted orbit equivalence definition of $\vec{\alpha}$ equivalence. The results in this paper, of course, subsume the one-dimensional result, but more importantly, the techniques we use here are significantly different to those sketched in [7]. The differences are due to the more complicated geometry and definition of even Kakutani equivalence in higher dimensions.

2. Preliminaries.

2.1. Notation. Throughout the paper $(X, \mathcal{F}, \mu)$ and $(Y, \mathcal{G}, \nu)$ will denote Lebesgue probability spaces. $S = \{S^{\vec{n}}\}_{\vec{n}\in\mathbb{Z}^d}$ and $T = \{T^{\vec{n}}\}_{\vec{n}\in\mathbb{Z}^d}$ will denote measurable measure preserving ergodic and free $\mathbb{Z}^d$ actions on X and Y respectively.

For $\vec{v} \in \mathbb{Z}^d$, if $\vec{v} = (v_1, \dots, v_d)$ we set $\|\vec{v}\| = \max\{|v_1|, \dots, |v_d|\}$. We denote the i^{th} component of the vector $\vec{v}$ by $\vec{v}_i$. To avoid confusion, the i^{th} component of an indexed vector such as $\vec{v}_n$ will be denoted by $[\vec{v}_n]_i$.

Given the action S, we define

$$R_S = \{(x_1, x_2) \in X \times X : \text{there is a } \vec{n} \in \mathbb{Z}^d \text{ with } S^{\vec{n}}x_1 = x_2\}.$$

If G is an abelian group a G-valued S cocycle is defined to be a function $f : R_S \to G$ satisfying the cocycle condition $f(x, y) = f(x, z) + f(z, y)$ for every $(x, y), (x, z) \in R_S$. A coboundary is defined in the usual way. We define the S-*ordering cocycle* of the $\mathbb{Z}^d$ action S, $\vec{S}$:$R_S \longrightarrow \mathbb{Z}^d$, by $\vec{S}(x, y) = \vec{n}$ if and only if $S^{\vec{n}}x = y$.

Set $B_n = \{\vec{v} : 0 \le v_1, \dots, v_d \le n\}$ and let $E \subset X$. If for some integer $n > 0$ the sets $\{S^{\vec{v}}E : \vec{v} \in B_n\}$ are disjoint, we call $S^{B_n}E = \cup_{\vec{v}\in B_n} S^{\vec{v}}E$ a Rohlin tower of size n for S. Each set $S^{\vec{v}}E$, for $\vec{v} \in B_n$, is called a level of the tower and the level corresponding to $\vec{0}$ is called the base of the tower. For $C \subset B_n$ we call $S^C E = \cup_{\vec{v}\in C} S^{\vec{v}}E$ the subtower with shape C. For $E' \subset E$ we call $S^{B_n}E'$ a slice of the tower. If $\mu(S^{B_n}E) > 1 - \delta$ we say the tower has error $< \delta$.

Let $P = \{p_1, \dots, p_k\}$ be a measurable partition of X. By the (n, P)-name of a point $x \in X$ we mean the map $P_n(x) : B_n \to P$ defined by $P_n(x)[\vec{v}] = p_i$ if and only if $S^{\vec{v}}x \in p_i$. Finally, for $R \subset \mathbb{Z}^d$, we denote the cardinality of R by $|R|$ and the complement of any set A by A^c.

2.2. Restricted orbit equivalence characterizations. Here we give the restricted orbit equivalence characterizations of the equivalence relations $\overset{e}{\sim}$ and $\overset{\vec{\alpha}}{\sim}$. We refer the reader to [2], [3] and [10] for details.

Definition 2.1. Two $\mathbb{Z}^d$ actions S and T are evenly Kakutani equivalent if there is an orbit equivalence $\phi : X \to Y$ between S and T such that given

$\epsilon > 0$, there is an $N(\epsilon) > 0$, and a set $A \subset X$ with $\mu A > 1 - \epsilon$ such that for all $x, y \in A$ and on the same orbit if $\vec{S}(x,y) > N$ then

$$\|\vec{S}(x,y) - \vec{T}(\phi x, \phi y)\| < \epsilon \|\vec{S}(x,y)\|. \tag{2}$$

The set A will be called an ϵ **Kakutani pinning set**. The constant $N(\epsilon)$ will be called an ϵ **Kakutani constant**. Property (2) will be referred to as the **distortion property of** ϕ.

Definition 2.2. Let $\vec{\alpha} = \{1 + \alpha_1, 1 + \alpha_2\}$ where α_i are positive irrationals. Let S and T be measurable, measure preserving, free, and ergodic $\mathbb{Z}^2$ actions on $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{G}, \nu)$ respectively. The actions S and T are $\vec{\alpha}$-equivalent if and only if there exists an orbit equivalence $\phi : X \to Y$ between S and T such that

1) given $\epsilon > 0$ there is an $N(\epsilon) > 0$, and a set $A \subset X$ with $\mu A > 1 - \epsilon$ such that for all $x, y \in A$ and on the same orbit if $\vec{S}(x,y) > N$ then
$$\|\vec{S}(x,y) - \vec{T}(\phi x, \phi y)\| < \epsilon \|\vec{S}(x,y)\|,$$
and
2) the function $d\left(\frac{\vec{S}_i(x,y) - \vec{T}_i(\phi x, \phi y)}{\alpha_i}, \mathbb{Z}\right)$ is a circle valued coboundary for $i = 1 \ldots d$.

We will refer to the second condition on the orbit equivalence in Definition 2.2 as the **coboundary condition of** ϕ.

3. Proof of the main theorem, Theorem 1.1.

Let S, T, $\vec{\alpha}$, and $\vec{\beta}$ be as in the statement of Theorem 1.1. Let $\Theta : X \to Y$ be the orbit equivalence given by applying Definition 2.1 to S and T. Let $P = \{p_1, \ldots, p_k\}$ be a generating partition for S.

We will inductively construct a third Lebesgue space $(Z, \mathcal{M}, \lambda)$, a measure preserving ergodic free $\mathbb{Z}^2$ action U on Z, and an orbit equivalence ψ (ϕ) between S and U (T and U) which satisfies Definition 2.2 with $\vec{\alpha}$ ($\vec{\beta}$). The space Z will be a subset of $[0,1]$, and λ will be Lebesgue measure.

3.1. The First Step of the Construction. We begin the construction by choosing a Rohlin tower τ_1 of size n_1 (to be determined later) for S. We begin constructing the space Z by constructing a subset $\overline{Z}_1$ of $[0,1]$ as a copy of a subtower $\overline{X}_1$ of τ_1. We will describe the shape I_1 of this subtower in detail below. We construct a partial action U_1 defined on most levels of $\overline{Z}_1$ by defining set maps $U_1^{\vec{e}_1}$ ($U_1^{\vec{e}_2}$) so that $\overline{Z}_1$ is a Rohlin tower of U_1. Note that $U_1^{\vec{e}_1}$ ($U_1^{\vec{e}_2}$) will be undefined on the rightmost (uppermost) levels of the tower.

The key ingredients of the construction are the properties of τ_1, and we describe them first without technical detail. We select ϵ_i-Kakutani pinning sets $A(i)$ for Θ (where the ϵ_i will be determined later) and set $A = \cap A(i)$. Using the ergodic theorem we will choose τ_1, so that its base E_1 is entirely contained in $A(1)$ and most of its levels are well covered in measure by the set A. By partitioning the base E_1, if necessary, we can assume that a level $S^{\vec{v}}(E_1)$ is entirely contained in A or in A^c. If a level is contained in A, we call it a good level. We define ψ_1 on a good level $S^{\vec{v}}E_1$ by mapping it to level $\vec{v}+\vec{c}_1$ in $\overline{Z}_1$, where $\vec{c}_1$ will be chosen using the following standard lemma with $\epsilon = \epsilon_1$:

Lemma 3.1. *Suppose $\alpha, \beta \in \mathbb{R}$ are such that $\{1, \alpha^{-1}, \beta^{-1}\}$ are rationally independent, and that $\epsilon > 0$ is given. Let $\mathbb{T}^2 = S^1 \times S^1$ denote the 2-torus, and $B_\epsilon(0) \subset \mathbb{T}^2$ denote the ϵ neighbourhood of 0 in $\mathbb{T}^2$.*

Then there exists $K(\epsilon) > 0$ such that for all $(x, y) \in \mathbb{T}^2$, there is $0 \leq k \leq K(\epsilon)$ such that $R^k(x, y) \in B_\epsilon(0)$ where $R^k(x, y) = (x + \frac{k}{\alpha}, y + \frac{k}{\beta})$.

This lemma guarantees that we can choose $\vec{c}_1$ so that ψ_1 will satisfy a coboundary condition with the α_i (condition (15) below). Since ϵ_1 will be determined at the start of the construction we can choose n_1 so that it is much larger than $K(\epsilon_1)$. Then ψ_1 will automatically satisfy a distortion property (condition (14) below) on good levels of τ_1.

To construct an orbit equivalence between U_1 and T, we set $\sigma_1 = \cup_{\vec{v} \in B_{n_1}} T^{\vec{v}}(\Theta(E_1))$ and make the following observation. Normally an orbit equivalence will not necessarily preserve Rohlin towers, but using the fact that Θ is an even Kakutani equivalence we can choose n_1 so that if a level $S^{\vec{v}}(E_1)$ of τ_1 is far enough away from the base, and is entirely contained in $A(1)$ then $\Theta(S^{\vec{v}}(E_1)) \subset \sigma_1$. Thus, if τ_1 is also chosen so that most of its levels are well covered by $A(1)$ we can guarantee that most levels of τ_1 get mapped into σ_1.

We then define a set map ϕ_1 from $\overline{Z}_1$ to σ_1 by sending $\psi_1(S^{\vec{v}}(E_1))$ to $\Theta(S^{\vec{v}}(E_1))$, when the latter is contained in σ_1, and to an arbitrary subset of σ_1 otherwise. Because n_1 can be chosen to be very large compared to $K(\epsilon_1)$, ϕ_1 will inherit a distortion property (with different parameters) from Θ in spite of the vector $\vec{c}_1$ involved in the definition of ψ_1 (condition (14′) below).

In addition, since we know a priori where Θ sends levels of τ_1, using Lemma 3.1 the vector $\vec{c}_1$ can be chosen to additionally guarantee that ϕ_1 satisfies a coboundary condition with the β_i (condition (15′) below).

Now for the details. Fix $\epsilon > 0$ and a sequence $\{\epsilon_n\}$ of positive real numbers such that $\sum \epsilon_n < \epsilon$. Let $N(\frac{\epsilon_n}{32})$ be an increasing sequence of $\frac{\epsilon_n}{32}$ Kakutani constants, and let $A(n)$ denote the $\frac{\epsilon_n}{32}$ Kakutani pinning sets. Let $K(n)$ be an increasing sequence chosen to satisfy Lemma 3.1 for $\frac{\epsilon_n}{4}$ and $\{\alpha_i, \beta_i\}$, for both $i = 1, 2$. Let $N(n)$ be an increasing sequence of integers chosen so that

$$N(n) \geq N\left(\frac{\epsilon_n}{32}\right) \qquad \text{and} \tag{3}$$

$$\frac{4\sum_{i=1}^{n} K(i)}{N(n)} < \frac{\epsilon_n}{16}. \tag{4}$$

Let $A = \cap_{n=1}^{\infty} A(n)$ and note that $\mu A > 1 - \epsilon$.

We pick $k \in \mathbb{N}$ such that

$$\frac{4N(1)}{k} < \frac{\epsilon_1}{8}. \tag{5}$$

We use the Rohlin Lemma and the Ergodic Theorem to choose n_1 large enough so that the tower τ_1 has error $< \frac{\epsilon_2}{10}$ and satisfies:

1) $E_1 \subset A(1)$,
2) for all $x \in E_1$

$$\frac{|\vec{v} \in B_{n_1} : S^{\vec{v}} x \in A|}{n_1^2} > 1 - 2\epsilon, \qquad \frac{|\vec{v} \in B_{n_1} : S^{\vec{v}} x \in A(1)|}{n_1^2} > 1 - \frac{\epsilon_1}{16}, \tag{6}$$

3) and finally

$$\frac{4(k + K(2) + N(2))}{n_1} < \frac{\epsilon_1}{32}. \tag{7}$$

To define the subtower $\overline{X}_1$ of τ_1 we let $b_1 = K(2) + N(2) + \frac{\epsilon_1}{16} n_1$ and $\vec{b}_1 = (b_1, b_1)$. We set $I_1 = B_{n_1 - 2b_1} + \vec{b}_1$, $C_1 = B_{n_1} \setminus I_1$, $X_1 = S^{I_1} E_1$ and $\overline{X}_1 = X_1 \cup E_1$. Note that by (7)

$$\mu X_1 > 1 - \frac{\epsilon_2}{10} - \frac{4(k + K(2) + N(2) + \frac{\epsilon_1}{16} n_1) n_1}{n_1^2} > 1 - \frac{\epsilon_1}{2}. \tag{8}$$

Suppose $\mu(\overline{X}_1) = \ell_1$ and set $\overline{Z}_1 = [0, \ell_1)$. We slice $\overline{Z}_1$ into subintervals to make a copy of $\overline{X}_1$ and we denote the base of this tower by F_1. We define the partial action U_1 as discussed above and we set $Z_1 = U_1^{I_1}(F_1)$, the subtower of shape I_1. Thus $[0, \ell_1) = \overline{Z}_1 = Z_1 \cup F_1$.

3.1.1. Constructing ψ_1. Let $x \in E_1$ and suppose that $\vec{v} \in I_1$ is such that $S^{\vec{v}} x \in A(1)$. Since $E_1 \subset A(1)$ we claim that our choice of n_1 and b_1 guarantees that

$$\Theta(S^{\vec{v}} x) \in T^{B_{n_1}}(\Theta x). \tag{9}$$

To see this, note that for $\vec{v} \in I_1$ we have $N(2) + \frac{\epsilon_1}{16} n_1 < \|\vec{v}\| < n_1(1 - \frac{\epsilon_1}{16})$. So $\|\vec{v} - \vec{T}(\Theta x, \Theta S^{\vec{v}} x)\| < \frac{\epsilon_1}{32} \|\vec{v}\|$, thus $\|\vec{T}(\Theta x, \Theta S^{\vec{v}} x)\| < (1 + \frac{\epsilon_1}{32}) \|\vec{v}\| < n_1$. Also, since for $s = 1, 2$ we have $\vec{v}_s > \frac{\epsilon_1}{16} n_1$ we have $\vec{T}_s(\Theta x, \Theta S^{\vec{v}} x) > 0$, and (9) follows.

Without loss of generality suppose that the dimensions of I_1 are integer multiples of k. For a fixed $x \in E_1$, take the k grid of $S^{I_1} x$ starting at the lower left hand index of I_1. We say $y \in S^{I_1} x$ is a *good element* of $S^{I_1} x$ if

1) $y \in A$ and
2) y lies at least a distance $N(1)$ away from the boundary of its k-grid box.

Choose a good element from each grid box which contains one and call this set $A_1(x)$. Set $A_1 = \cup_{x\in E_1} A_1(x)$. It follows from (5), (6) and (7) that $\mu(A_1) > \frac{1}{2k^2}$.

For each $x \in E_1$, let the set $\{x_1, \ldots, x_{m(x)}\}$ denote the elements of $A_1(x)$ in lexicographic order. Define $\vec{V}_1(x) = \{\vec{v}_1, \ldots, \vec{v}_{m(x)}\} \subset I_1$ by $\vec{v}_j = \vec{S}(x, x_j)$ for $j = 1, \ldots, m(x)$. Thus, $\vec{V}_1(x)$ is a list of the levels of τ_1 containing the elements of $A_1(x)$.

We first define ψ_1 on the levels in $\vec{V}_1(x)$. To this end we partition E_1 into subsets E_1^i such that:

1) The set V_1 is constant over each set E_i. Namely, if $x, y \in E_1^i$ then $\vec{V}_1(x) = \vec{V}_1(y) = \vec{V}_1(i)$.
2) For each $\vec{v} \in B_{n_1}$, the level $S^{\vec{v}}(E_1^i)$ is contained entirely in A $(A(1))$ or A^c $(A(1)^c)$. Namely, $S^{\vec{v}}(E_1^i) \cap A(1)$ is either empty or all of $S^{\vec{v}}(E_1^i)$ and $S^{\vec{v}}(E_1^i) \cap A$ is either empty or all of $S^{\vec{v}}(E_1^i)$.
3) The map Θ is constant on the levels of I_1 which lie in $A(1)$. Namely if $\vec{v} \in I_1$ is such that $S^{\vec{v}}(E_1^i) \subset A(1)$ then
$$\vec{T}(\Theta x, \Theta S^{\vec{v}} x) = \vec{T}(\Theta y, \Theta S^{\vec{v}} y) \tag{10}$$
for all $x, y \in E_1^i$.
4) Finally, every $x \in E_1^i$ has the same (n_1, P)-name.

Let $i(1)$ denote the number of sets E_1^i. Note that by (9) we are guaranteed that $i(1)$ is finite, even with condition (10). Partition F_1 into $i(1)$ measurable subsets with $\lambda(F_1^i) = \mu(E_1^i)$ and for each $i = 1, \ldots, i(1)$ set
$$\psi_1(E_1^i) = F_1^i.$$
For each i, we can now define ψ_1 on $S^{\vec{V}_1(i)} E_1^i$. Fix $x \in E_1^i$. By our choice of $K(1)$, for each $x_j \in A_1(x)$ we can find a vector $\vec{c}_1(i, x_j)$ with
$$\|\vec{c}_1(i, x_j)\| < K(1) \tag{11}$$
such that for $s = 1, 2$
$$d\left(\frac{(\vec{c}_1(i, x_j))_s}{\alpha_s}, \mathbb{Z}\right) < \frac{\epsilon_1}{4} \qquad \text{and} \tag{12}$$
$$d\left(\frac{\vec{T}_s(\Theta(x), \Theta(x_j)) - S_s(x, x_j) - (\vec{c}_1(i, x_j))_s}{\beta_s}, \mathbb{Z}\right) < \frac{\epsilon_1}{4}.$$
For each j set
$$\psi_1(S^{\vec{v}_j}(E_1^i)) = U_1^{\vec{v}_j + \vec{c}_1(i, x_j)}(F_1^i). \tag{13}$$
For a vector $\vec{v} \in I_1$ which is not in $\vec{V}_1(i)$ if there is no conflict arising from (13) we set $\psi_1(S^{\vec{v}} E_1^i) = U_1^{\vec{v}}(F_1^i)$. If there is a conflict, then we map $S^{\vec{v}} E_1^i$ to an empty level in $U_1^{I_1}(F_1^i)$. We do not define ψ_1 on $S^{C_1}(E_1)$ at this stage.

We claim that ψ_1 is well-defined on the levels of X_1 and has range Z_1. To see this note that $\vec{v}_j + \vec{c}_1(i, x_j) \in I_1$, so $\psi_1(S^{\vec{v}_j} E_1^i) \in Z_1$. Also note that if $j \neq k$ then since

$$\|\vec{S}(x, x_j) - \vec{S}(x, x_k)\| = \|\vec{S}(x_j, x_k)\| > 2N(1)$$

it follows from (4) and (11) that $\vec{v}_j + \vec{c}_1(i, x_j) \neq \vec{v}_k + \vec{c}_1(i, x_k)$.

If $y_1, y_2 \in A_1(x)$ then by construction $\|\vec{S}(y_1, y_2)\| > 2N(1)$. Using the properties of the set $A_1(x)$, (4), and (11) we have

$$\begin{aligned} \|\vec{S}(y_1, y_2) - \vec{U}(\psi_1(y_1), \psi_1(y_2))\| &\leq \frac{\|-\vec{c}_1(i, y_1) + \vec{c}_1(i, y_2)\|}{\|\vec{S}(y_1, y_2)\|} \|\vec{S}(y_1, y_2)\| \\ &< \frac{\epsilon_1}{8} \|\vec{S}(y_1, y_2)\|. \end{aligned} \tag{14}$$

In addition, by (12) for $s = 1, 2$ and y_1 and y_2 as above

$$\begin{aligned} &d\left(\frac{\vec{S}_s(y_1, y_2) - (\vec{U}_1(\psi_1 y_1, \psi_1 y_2))_s}{\alpha_s}, \mathbb{Z}\right) \\ &\qquad = d\left(\frac{\vec{c}_1(i, y_1)}{\alpha_s}, \mathbb{Z}\right) + d\left(\frac{\vec{c}_1(i, y_2)}{\alpha_s}, \mathbb{Z}\right) < \frac{\epsilon_1}{2}. \end{aligned} \tag{15}$$

3.1.2. Defining ϕ_1. For every $i = 1, \ldots, i(1)$ we set $\phi_1(F_1^i) = \Theta(E_1^i)$ and if $\vec{v} \in I_1$ is such that the level $\psi_1^{-1}(U_1^{\vec{v}} F_1^i)$ in X_1 is contained in $A(1)$, we set

$$\phi_1(U_1^{\vec{v}}(F_1^i)) = \Theta(\psi_1^{-1}(U_1^{\vec{v}}(F_1^i))). \tag{16}$$

We do not define ϕ_1 on the rest of the levels of Z_1 at this stage.

Recall that by construction Θ is constant on the levels of $S^{I_1} E_1$, for all i every level $S^{\vec{v}} E_1^i$ is either entirely in $A(1)$ or in $A(1)^c$ so ϕ_1 is well-defined. For levels $\vec{v}$ where (16) holds (9) is satisfied and the range of ϕ_1 is contained in σ_1.

The map ϕ_1 is then defined on $Z_1'' = \psi_1(A(1) \cap X_1)$. By (6) and (7)

$$\lambda Z_1' = \mu(A(1) \cap X_1) > 1 - \frac{\epsilon_1}{32} - \frac{\epsilon_1}{16} > 1 - \epsilon_1. \tag{17}$$

We let $Y_1 = \phi_1(Z_1'')$, and note that since ψ_1 and Θ are measure preserving, so is ϕ_1.

We now show that ϕ_1 satisfies a distortion and coboundary condition on the set $D_1 = \psi_1\left(\bigcup_{i=1}^{i(1)} S^{\vec{V}(i)} E_1^i\right)$, the image of the special levels in $\vec{V}_1(i)$.

Pick $z_j, z_k \in D_1$ and identify which level of in τ_1 they came from. Namely, choose $i \in \{1, \ldots, i(1)\}$ and $\vec{v}_j, \vec{v}_k \in \vec{V}_1(i)$ such that $z_j \in \psi_1(S^{\vec{v}_j} E_1^i)$ and $z_k \in \psi_1(S^{\vec{v}_k} E_1^i)$. Then for $x \in E_1^i$ we can choose representative points from these levels of τ_1. Namely, we can find $x_j, x_k \in S^{\vec{V}_1(i)} E_1^i$ such that $S^{\vec{v}_j} x = x_j$, $S^{\vec{v}_k} x = x_k$. Note that by construction we have $\|\vec{S}(x_j, x_k)\| > 2N(1)$ and

$$\vec{U}_1(z_j, z_k) = (\vec{v}_j - \vec{v}_k) + (\vec{c}_1(i, x_j) - \vec{c}_1(i, x_k)).$$

Thus, $\|\vec{U}_1(z_j,z_k)\| > N(1)$ and by (4) we have:

$$\begin{aligned}
&\|\vec{U}_1(z_j,z_k) - \vec{T}(\phi_1 z_j, \phi_1 z_k)\| \\
&= \|\vec{v}_j - \vec{v}_k + \vec{c}_1(i,x_j) - \vec{c}_1(i,x_j) - \vec{T}(\Theta x_j, \Theta x_k)\| \\
&\le \|\vec{S}(x_j,x_k) - \vec{T}(\Theta x_j, \Theta x_k)\| + 2K(1) \\
&\le \frac{\epsilon_1}{32}\|\vec{S}(x_j,x_k)\| + 2K(1) \\
&\le \|\vec{U}_1(z_j,z_k)\|\left(\frac{\epsilon_1}{32} + \frac{4K(1)}{N(1)}\right) \\
&< \frac{\epsilon_1}{8}\|\vec{U}_1(z_j,z_k)\|.
\end{aligned}$$

So we have

$$\|\vec{U}_1(z_j,z_k) - \vec{T}(\phi_1 z_j, \phi_1 z_k)\| < \frac{\epsilon_1}{8}\|\vec{U}_1(z_j,z_k)\|. \tag{14'}$$

Now notice that

$$\begin{aligned}
&d\left(\frac{\vec{T}_s(\Theta x, \Theta x_k) - (\vec{U}_1(z_j,z_k))_s}{\beta_s}, \mathbb{Z}\right) \\
&\le d\left(\frac{\vec{T}_s(\phi_1 z, \phi_1 z_j) - \vec{S}_s(x,x_j) - (\vec{c}_1(i,x_j))_s}{\beta_s}, \mathbb{Z}\right) \\
&\quad + d\left(\frac{\vec{T}_s(\Theta x, \Theta x_k) - \vec{S}_s(x,x_k) - (\vec{c}_1(i,x_k))_s}{\beta_s}, \mathbb{Z}\right)
\end{aligned}$$

for any $z \in F_1^i$. Thus by (12)

$$d\left(\frac{\vec{T}_s(\phi_1 z_j, \phi_1 z_k) - (\vec{U}_1(z_j,z_k))_s}{\beta_s}, \mathbb{Z}\right) < \frac{\epsilon_1}{2}. \tag{15'}$$

3.2. The Induction Step of the Construction. We will again begin by choosing a Rohlin tower τ_2 for S, and we will construct Z_2, a copy of a subtower X_2 of τ_2 in $[0,1]$. The key issue is to ensure that Z_2 refines Z_1, and that the map ψ_2 (ϕ_2) respects ψ_1 (ϕ_1) on most of X_1 (Z_1). We first briefly describe the part of the construction which is parallel to the first step.

We choose $n_2 \in \mathbb{N}$ such that there is a Rohlin tower τ_2 for S with shape B_{n_2}, base E_2, and error $\frac{\epsilon_3}{10}$ such that E_2 is entirely contained in $A(2)$, and for all $x \in E_2$ we have

$$\frac{|\vec{v} \in B_{n_2} : S^{\vec{v}}x \in \tau_1|}{n_2^2} > 1 - \frac{\epsilon_2}{5}, \qquad \frac{|\vec{v} \in B_{n_2} : S^{\vec{v}}x \in A(2)|}{n_2^2} > 1 - \frac{\epsilon_2}{16}, \tag{18}$$

and

$$\frac{4(K(3)+N(3)+n_1)}{n_2} < \frac{\epsilon_2}{32}. \tag{19}$$

We define $b_2, \vec{b}_2, I_2$, and C_2 as in the first step (with the index in each parameter increased by one). Set $X_2 = S^{I_2} E_2$ and $\overline{X}_2 = X_2 \cup E_2$ and note that by (19) we have

$$\mu X_2 > 1 - \frac{\epsilon_3}{10} - \frac{\epsilon_2}{32} > 1 - \frac{\epsilon_2}{2}. \tag{20}$$

As in the first step we will partition E_2 into subsets so that the levels of X_2 are entirely covered by special sets or their complements. The special sets are a little different this time. For $x \in E_2$ we set $A_2(x) = S^{I_2}x \cap A_1$ and $A_2 = \cup_{x \in E_2} A_2(x)$. Note that

$$\mu A_2 > (1 - \epsilon_2)\mu A_1. \tag{21}$$

We let $\{x_1, \ldots, x_{m(2)}\}$ denote the elements of $A_2(x)$ in lexicographic order and we define $\vec{V}_2(x) = \{\vec{v}_1, \ldots, \vec{v}_{m(x)}\}$ as before: the list of the levels based at x containing the x_i.

We then partition E_2 into subsets E_2^j such that each level of τ_2 is either entirely contained in $A(2)$ (A_2), or in $A(2)^c$ (A_2^c), every $x \in E_2^j$ has the same (n_2, P)-name, the list $V_2(x)$ is constant on each E_2^j, and Θ is constant on the levels of $S^{I_2} E_2$ which lie in $A(2)$. By a computation parallel to the one given in the first step of the construction we can show that

$$\Theta(S^{\vec{v}} x) \in T^{B_{n_2}}(\Theta x), \tag{22}$$

so E_2 is partitioned into finitely many subsets, in spite of the last condition. Let $j(2)$ denote the number of sets E_2^j.

We impose one new condition on the partitioning of E_2: we require that each level of X_2 lies entirely in E_1^c or in exactly one subset E_1^i of E_1.

To begin copying X_2 in $[0,1]$ we first cut an interval of length μE_2 from $[0,1] \setminus \overline{Z}_1$. This will be the base of the tower $\overline{Z}_2$ and will be labelled F_2.

The rest of $\overline{Z}_2$ will consist of slices of Z_1, and some new intervals cut from the remaining part of $[0,1]$. The new intervals will form the levels of Z_2 not covered by Z_1. The set maps $U_2^{\vec{e}_1}$ and $U_2^{\vec{e}_2}$ are defined as before and we have $\overline{Z}_2 = F_2 \cup Z_2$ where $Z_2 = U_2^{I_2} F_2$, the subtower corresponding to X_2.

Since the definition of the map ψ_2 will depend heavily on how we locate the various slices of Z_1 inside Z_2 we finish constructing $\overline{Z}_2$ as we define ψ_2.

3.2.1. Constructing ψ_2. As before we will construct ψ_2 only on $E_2 \cup X_2$ so we first eliminate from consideration those slices of X_1 in τ_2 which don't lie entirely in X_2. Denote these slices by X_1'. For ease of notation we continue to call the partitioned base of this new subtower E_1^i. By (8) and (19) we have

$$\mu(X_1') > 1 - \frac{\epsilon_1}{2} - \frac{\epsilon_2}{2}. \tag{23}$$

To construct Z_2 we will first slice F_1 and Z_1 into subsets corresponding to the various slices of E_1 and X_1' appearing in X_2. Label these slices in some way to keep track of where the X_1' slices appear in X_2. Specifically, the subset $E_1^{i,j,k}$ of E_1^i is the base of the k^{th} slice of X_1 appearing in X_2^j. So we have

$$X_1' = \bigcup_{i=1}^{i(1)} \bigcup_{j=1}^{j(2)} \bigcup_{k} X_1^{i,j,k} \qquad \text{and} \qquad Z_1' = \bigcup_{i=1}^{i(1)} \bigcup_{j=1}^{j(2)} \bigcup_{k} Z_1^{i,j,k}.$$

We will place $F_1^{i,j,k}$ in the same location in Z_2 as $E_1^{i,j,k}$ appears in X_2. The set $Z_1^{i,j,k}$, however, will be shifted to a different location relative to its base than $X_1^{i,j,k}$ sits relative to $E_1^{i,j,k}$. Recall that ψ_1 was left undefined on $S^{C_1}(E_1^i)$, for every i, so the C_1-collar around $Z_1^{i,j,k}$ does not have a preimage in X. The set $Z_1^{i,j,k}$ will be placed in Z_2 starting at a location in this collar.

The translation of $Z_1^{i,j,k}$ relative to its base will be by a vector $\vec{c}_2$ obtained from Lemma 3.1, this time applied with $\epsilon = \epsilon_2$. The map ψ_2 will then respect ψ_1 on X_1', and will map each slice $X_1^{i,j,k}$ to a location which is a shift by the vector $\vec{c}_2$ from its original position in τ_2. Again, n_2 is very large compared to $K(2)$, so the distortion property is guaranteed, and Lemma 3.1 will guarantee the coboundary property.

To choose the vectors $\vec{c}_2$ let $\vec{v}_k \in \vec{V}_2(j)$ be such that $x_k = S^{\vec{v}_k}x \in A_2(x)$ is the first lexicographic occurrence of A_2 in $X_1^{i,j,1}$. Using Lemma 3.1 choose $\vec{c}_2(j, x_k)$ with

$$\|\vec{c}_2(j, x_k)\| < K(2) \tag{24}$$

such that for $s = 1, 2$ we have

$$d\left(\frac{(\vec{c}_2(j,x_k))_s}{\alpha_s}, \mathbb{Z}\right) < \frac{\epsilon_2}{4} \qquad \text{and} \tag{25}$$
$$d\left(\frac{\vec{T}_s(\Theta(x), \Theta(x_k)) - \vec{S}_s(x, x_k) - (\vec{c}_2(j,x_k))_s}{\beta_s}, \mathbb{Z}\right) < \frac{\epsilon_2}{4}.$$

Place $Z_1^{i,j,1}$ in Z_2 so that its location in I_2 relative to F_2, is a shift of the position of $X_1^{i,j,1}$ in I_2 by the vector $\vec{c}_2(j, x_k) - \vec{c}_1(i, x_k)$. In particular, for $z \in F_2$ and $z_k \in \psi_2(S^{\vec{v}_k}x)$ we have

$$\vec{U}_2(z, z_k) = \vec{S}(x, x_k) + \vec{c}_2(j, x_k).$$

Since C_1 is a collar of width greater than $K(2)$ around $S^{I_1}E_1$, (24) guarantees that the images of distinct slices of $S^{I_1}E_1$ under ψ_2 will not intersect.

We repeat this procedure until all the slices of X_1' are taken care of, hence all of Z_1' is placed in Z_2.

We will also define ψ_2 on $S^{C_1}(E_1^{i,j,k})$ at this stage by mapping this subtower level by level into locations in $U_1^{C_1}(F_1^{i,j,k})$ vacated by the translation of $Z_1^{i,j,k}$.

We complete the tower Z_2 by slicing intervals of the appropriate length from $[0,1]\setminus(Z_1'\cup F_2)$ and placing these in the empty positions of I_2. Finally, for the remaining $\vec{v}\in I_2$ we set

$$\psi_2(S^{\vec{v}}E_2^i)=U_2^{\vec{v}}(F_2^i).$$

The map ψ_2 is now defined on all of X_2 with image Z_2. Further, ψ_2 refines ψ_1 on $X_1'\subset X_2$.

We will now show that for $y_1,y_2\in A_2$, if $\|\vec{S}(y_1,y_2)\|>N(2)$ then

$$\|\vec{S}(y_1,y_2)-\vec{U}_2(\psi_2y_1,\psi_2y_2)\|<\frac{\epsilon_2}{8}\|\vec{S}(y_1,y_2)\|, \tag{26}$$

and that regardless of the value of $\|\vec{S}(y_1,y_2)\|$ we always have

$$d\biggl(\frac{\vec{S}_s(y_1,y_2)-(\vec{U}_2(\psi_2y_1,\psi_2y_2))_s}{\alpha_s},\mathbb{Z}\biggr)<\epsilon_1+\epsilon_2. \tag{27}$$

Pick such a pair y_1 and y_2 and suppose they lie in $A_2(x)$, $x\in E_2^j$. Then either y_1 and y_2 lie in the same slice, $X_1^{i,j,k}$ of X_1, or there exist $k_1\neq k_2$ and i_1,i_2 such that $y_1\in X_1^{i_1,j,k_1}$ and $y_2\in X_1^{i_2,j,k_2}$. In the first case, by construction we have

$$\vec{U}_2(\psi_2y_1,\psi_2y_2)=\vec{U}_1(\psi_1y_1,\psi_1y_2)$$

and

$$\vec{U}_1(\psi_1y_1,\psi_1y_2)=\vec{S}(y_1,y_2)+\vec{c}_1(i,y_1)+\vec{c}_1(i,y_2). \tag{28}$$

In the second case pick $z\in F_2^j$. Then for $p=1,2$ there exist $x_{m_p}\in A_2(x)\cap X_1^{i_p,j,k_p}$ such that

$$\vec{U}_2(z,\psi_2y_p)=\vec{S}(x,x_{m_p})+\vec{c}_2(j,x_{m_p})+\vec{U}_1(\psi_1x_{m_p},\psi_1y_p). \tag{29}$$

In both cases by (4), (11), (24), and the construction of ψ_1, if $\|\vec{S}(y_1,y_2)\|>N(2)$, then we have

$$\|\vec{S}(y_1,y_2)-\vec{U}_2(\psi_2y_1,\psi_2y_2)\|\le 4K(1)+2K(2)<\frac{\epsilon_2}{8}\|\vec{S}(y_1,y_2)\|.$$

To see that (27) holds we note that in the first case (15) holds, hence, so does (27). In the second case using (29) we see that by (12) and (25)

$$\begin{aligned}
&d\biggl(\frac{\vec{S}_s(y_1,y_2)-(\vec{U}_2(\psi_2y_1,\psi_2y_2))_s}{\alpha_s},\mathbb{Z}\biggr)\\
&\le d\biggl(\frac{(\vec{c}_1(i_1,y_1))_s}{\alpha_s},\mathbb{Z}\biggr)+d\biggl(\frac{(\vec{c}_1(i_1,y_2))_s}{\alpha_s},\mathbb{Z}\biggr)+d\biggl(\frac{(\vec{c}_2(j,x_{m_1}))_s}{\alpha_s},\mathbb{Z}\biggr)\\
&\quad+d\biggl(\frac{(\vec{c}_2(j,x_{m_2}))_s}{\alpha_s},\mathbb{Z}\biggr)+d\biggl(\frac{(\vec{c}_1(i_1,x_{m_1}))_s}{\alpha_s},\mathbb{Z}\biggr)+d\biggl(\frac{(\vec{c}_1(i_2,x_{m_2}))_s}{\alpha_s},\mathbb{Z}\biggr)\\
&<\epsilon_1+\epsilon_2.
\end{aligned}$$

3.2.2. Constructing ϕ_2. The construction here is essentially the same as in the first step of the construction. The map ϕ_2 will be a set map defined on most of the levels of Z_2 with range contained in $T^{B_{n_2}}(\Theta E_2)$.

We start by setting $\phi_2(F_2^j) = \Theta(E_2^j)$ for all $j = 1, \ldots, j(2)$. By construction we are guaranteed that for all j, and $\vec{v} \in I_2$ the level $S^{\vec{v}} E_2^j$ in X_2 is entirely in $A(2)$ or $A(2)^c$. For $\vec{v} \in I_2$ if $\psi_2^{-1}\big(U_2^{\vec{v}}(F_2^j)\big) \subset A(2)$ then (22) holds and we set

$$\phi_2\big(U_2^{\vec{v}}(F_2^j)\big) = \Theta\big(\psi_2^{-1}(U_2^{\vec{v}}(F_2^j))\big).$$

The map ϕ_2 is defined on $Z_2 \cap \psi_2(A(2))$ which has measure $\mu(X_2 \cap A(2)) > 1 - \frac{\epsilon_2}{2} - \frac{\epsilon_2}{16} > 1 - \epsilon_2$. We let $Y_2 = \phi_2(Z_2 \cap \psi_2(A(2)))$. Since ψ_2 is a refinement of ψ_1 on X_1' it is clear that ϕ_2 is a refinement of ϕ_1 on $Z_2'' = \psi_2(X_1' \cap A(1) \cap A(2))$. By (17) and (18) we have

$$\lambda Z_2'' > 1 - \epsilon_1 - \epsilon_2. \tag{30}$$

To see that ϕ_2 satisfies the appropriate properties we argue exactly as before. Let $D_2 = \psi_2\big(\cup_{j=1}^{j(2)} S^{\vec{V}_2(j)} E_2^j\big)$, notice that $D_2 \subset Z_2''$ and pick $z_1, z_2 \in D_2$ such that either z_1 and z_2 lie in the same slice $Z_1^{i,j,k}$ of Z_1' or there exist $k_1 \neq k_2$ and i_1, i_2 such that $z_1 \in Z_1^{i_1,j,k_1}$ and $z_2 \in Z_1^{i_2,j,k_2}$, and further

$$\|\vec{U}_2(z_1, z_2)\| > 2N(2). \tag{31}$$

In the first case, by construction we have

$$\vec{T}(\phi_2 z_1, \phi_2 z_2) = \vec{T}(\phi_1 z_1, \phi_1 z_2)$$

and we know there exist $y_1, y_2 \in X_1^{i,j,k} \cap A_2$ such that (28) holds. Then (31) and (24) guarantee that $\|\vec{S}(y_1, y_2)\| > N(2)$. Since $\vec{T}(\phi_1 z_1, \phi_1 z_2) = \vec{T}(\Theta y_1, \Theta y_2)$ we have

$$\begin{aligned}
\|\vec{T}(\phi_1 z_1, \phi_1 z_2) - \vec{U}_2(z_1, z_2)\| &\leq \frac{\epsilon_2}{32}\|\vec{S}(x_1, x_2)\| + 2K(1) \\
&\leq \frac{\epsilon_2}{32}(\|\vec{U}_1(z_1, z_2)\| + 2K(1)) + 2K(1) \\
&< \frac{\epsilon_2}{8}\|\vec{U}_1(z_1, z_2)\|.
\end{aligned}$$

On the other hand, if z_1 and z_2 lie in different slices of Z_1' then (29) holds, so

$$\begin{aligned}
&\|\vec{T}(\phi_2 z_1, \phi_2 z_2) - \vec{U}_2(z_1, z_2)\| \\
&\leq \|\vec{T}(\Theta y_1, \Theta y_2) - \vec{S}(y_1, y_2)\| + 4K(1) + 2K(2),
\end{aligned}$$

and an argument similar to the previous case then yields

$$\|\vec{T}(\phi_1 z_1, \phi_1 z_2) - \vec{U}_2(z_1, z_2)\| < \frac{\epsilon_2}{8}\|\vec{U}(z_1, z_2)\|.$$

To see that the coboundary property holds, note that if z_1 and z_2 lie in the same slice of Z_1' then by (15′)

$$d\biggl(\frac{(\vec{U}_2(z_1, z_2))_s - \vec{T}_s(\phi_2 z_1, \phi_2 z_2)}{\beta_2}, \mathbb{Z}\biggr) < \epsilon_1.$$

If, instead, (29) holds we have

$$\begin{aligned} d\biggl(&\frac{\vec{T}_s(\phi_2 z_1, \phi_2 z_2) - (\vec{U}_2(z_1, z_2))_s}{\beta_s}, \mathbb{Z}\biggr) \\ &\leq d\biggl(\frac{\vec{T}_s(\Theta x, \Theta x_{m_1}) - \vec{S}_s(x, x_{m_1}) - (\vec{c}_2(j, x_{m_1}))_s}{\beta_2}, \mathbb{Z}\biggr) \\ &\quad + d\biggl(\frac{\vec{T}_s(\Theta x_{m_1}, \Theta y_1) - (\vec{U}_1(\psi_1 x_{m_1}, \psi_1 y_1))_s}{\beta_2}, \mathbb{Z}\biggr) \\ &\quad + d\biggl(\frac{\vec{T}_s(\Theta x, \Theta x_{m_2}) - \vec{S}_s(x, x_{m_2}) - (\vec{c}_2(j, x_{m_2}))_s}{\beta_2}, \mathbb{Z}\biggr) \\ &\quad + d\biggl(\frac{\vec{T}_s(\Theta x_{m_2}, \Theta y_2) - (\vec{U}_1(\psi_1 x_{m_2}, \psi_1 y_2))_s}{\beta_2}, \mathbb{Z}\biggr). \end{aligned}$$

By (25) the first and third summands are bounded by $\frac{\epsilon_2}{4}$. Since the points $\psi_1 x_{m_i}$ and $\psi_1 y_i$ are all in D_1 and in the same slice of Z_1', by (15′) the second and last summands are bounded by $\frac{\epsilon_1}{2}$. We have then

$$d\biggl(\frac{\vec{T}_s(\phi_2 z_1, \phi_2 z_2) - (\vec{U}_2(z_1, z_2))_s}{\beta_s}, \mathbb{Z}\biggr) < \epsilon_1 + \frac{\epsilon_2}{2}.$$

3.3. Conclusion of the proof. Continuing in this fashion, at stage n we define sequences $\{X_n'\}$ and $\{Z_n'\}$ of subsets of X and $[0,1]$ respectively with

$$\mu X_n' = \lambda Z_n' > 1 - \epsilon_n$$

(see for example (23)) and a set map ψ_n from the level sets of X_n' to those of Z_n' such that ψ_n is a refinement of ψ_{n-1}. Thus if we set

$$X' = \cup_{n=1}^{\infty} \cap_{k=n}^{\infty} X_k' \qquad Z' = \cup_{n=1}^{\infty} \cap_{k=n}^{\infty} Z_k'$$

then $\mu X' = \lambda Z' = 1$. In addition, since P was chosen to be a generating partition $\psi = \lim \psi_n : X' \to Z'$ is a well-defined point map. If we set $U = \lim U_n$, this is a $\mathbb{Z}^2$ action on Z' and ψ is an orbit equivalence between S restricted to X' and U on Z'.

We also define at each stage n sets $Z''_n \subset Z_n$ and $Y_n \subset Y$ with

$$\lambda(Z''_n) = \nu(Y_n) > 1 - (\epsilon_{n-1} + \epsilon_n)$$

(see for example (30)). The maps ϕ_n are constructed so that $\phi_n(Z''_n) = Y_n$ and ϕ_n refines ϕ_{n-1} on Z''_n. We set

$$Z'' = \cup_{n=1}^{\infty} \cap_{k=n}^{\infty} Z''_k \qquad Y' = \cup_{n=1}^{\infty} \cap_{k=n}^{\infty} Y_n.$$

Then $\lambda Z'' = \nu Y' = 1$ and the map $\phi = \lim \phi_n$ is a well defined point map on Z''.

Lemma 3.2. *Let ψ be the map described above. Then*

1) *the function*

$$d\left(\frac{\vec{S}_s(x,y) - \vec{U}_s(\psi x, \psi y)}{\alpha_s}, \mathbb{Z}\right)$$

is a circle valued coboundary for $s = 1, 2$, and

2) *for all $\eta > 0$ there exists a set $A \subset X'$ and an integer $M(\eta) > 0$ such that if $x, y \in A$ are on the same orbit and $\|\vec{S}(x,y)\| > M(\eta)$ then*

$$\|\vec{S}(x,y) - \vec{U}(\psi x, \psi y)\| < \eta \|\vec{S}(x,y)\|.$$

Proof. For each n, there exist sets $A_n \subset X'_n$, such that $\mu A_n > (1 - \epsilon_n)\mu A$ and for all $x, y \in A_n$ and on the same orbit

1) $d\big(\frac{\vec{S}_s(x,y) - ((\vec{U}_n)(\psi_n x, \psi_n y))_s}{\alpha_s}, \mathbb{Z}\big) < \sum_{i=1}^{n} \epsilon_n$ for $s = 1, 2$, and
2) if $\|\vec{S}(x,y)\| > N(n)$ then

$$\|\vec{S}(x,y) - \vec{U}_n(\psi_n x, \psi_n y)\| < \frac{\epsilon_n}{2} \|\vec{S}(x,y)\|.$$

We set $A = \cap_n A_n \subset X'$ and notice that $\mu A > 0$. To see the first part of the claim we note that for all $x, y \in A$ and on the same orbit

$$d\left(\frac{\vec{S}_s(x,y) - \vec{U}_s(\psi x, \psi y)}{\alpha_s}, \mathbb{Z}\right) < \epsilon$$

where $\epsilon > 0$ is chosen at the start of the construction. If $\epsilon < \frac{1}{3}$ a standard argument yields that the function is a circle valued coboundary on $R_{S \cap (A \times A)}$, and thus that it is a circle valued S coboundary on all of R_S (see for example [3]).

To see the second part of the claim note that if $x, y \in A$ then $x, y \in A_n$ for all n. Then given $\eta > 0$ we select n such that $\epsilon_n < \eta$ and set $M(\eta) = N(n)$. The result follows. $\square$

A similar argument yields the parallel result for the orbit equivalence ϕ:

Lemma 3.3. *Let ϕ be the map described above. Then*

1) *the function*

$$d\left(\frac{\vec{U}_s(x,y)-\vec{T}_s(\phi x,\phi y)}{\beta_s},\mathbb{Z}\right)$$

is a circle valued coboundary for $s=1,2,$ *and*

2) *for all* $\eta>0$ *there exists a set* $D\subset Z$ *and an integer* $M(\eta)>0$ *such that if* $x,y\in D$ *are on the same orbit and* $\|\vec{U}(x,y)\|>M(\eta)$ *then*

$$\|\vec{U}(x,y)-\vec{T}(\phi x,\phi y)\|<\eta\|\vec{U}(x,y)\|.$$

Proof. For each n there exists a set $D_n\subset Z_n''$ such that $\lambda D_n>(1-\epsilon_n)A$ and for all $z_1,z_2\in D_n$ on the same orbit

1) $d\left(\frac{(\vec{U}_n(z_1,z_2))_s-\vec{T}_s(\phi_n z_1,\phi_n z_2)}{\beta_s},\mathbb{Z}\right)<\sum_{i=1}^n\epsilon_n$ for $s=1,2$, and
2) if $\|\vec{U}_n(z_1,z_2)\|>2N(n)$ then

$$\|\vec{U}_n(z_1,z_2)-\vec{T}(\phi_n z_1,\phi_n z_2)\|<\frac{\epsilon_n}{2}\|\vec{U}_n(z_1,z_2)\|.$$

Again, set $D=\cap D_n\subset Z$ and argue as before.

□

By Proposition 4 in [2] we have that the sets A and D from the previous two results can be made arbitrarily large. This completes the proof of Theorem 1.1.

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