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Let k be an imaginary quadratic number field with $C_{k,2}$, the 2-Sylow subgroup of its ideal class group C_k, of rank 4. We show that k has infinite 2-class field tower for particular families of fields k, according to the 4-rank of C_k, the Kronecker symbols of the primes dividing the discriminant Δ_k of k, and the number of negative prime discriminants dividing Δ_k. In particular we show that if the 4-rank of C_k is greater than or equal to 2 and exactly one negative prime discriminant divides Δ_k, then k has infinite 2-class field tower.

Introduction.

Let k denote an algebraic number field and $C_{k,2}$ denote its 2-class group, i.e., the 2-Sylow subgroup of the ideal class group C_k (in the wide sense) of k; denote by k_1 the Hilbert 2-class field of k. Let k_n (for n a nonnegative integer) be defined inductively as $k_0 = k$ and $k_{n+1} = (k_n)_1$. Then $k_0 \subseteq k_1 \subseteq k_2 \subseteq \ldots \subseteq k_n \subseteq \ldots$ is called the 2-class field tower of k. If n is the minimal integer such that $k_n = k_{n+1}$, then n is called the length of the tower. If no such n exists, then the tower is said to be of infinite length.

In 1964, Golod and Shafarevich (cf. [**4**]) established for the first time the existence of infinite p-class field towers, for p prime. In the case $p = 2$, their criterion (as refined by Gaschütz and Vinberg [**10**]) can be stated in the following way, where $E_{k,2}$ denotes the unit group of k mod its squares, E_k/E_k^2: If rank $C_{k,2} \geq 2 + 2(\operatorname{rank} E_{k,2} + 1)^{1/2}$ then k has infinite 2-class field tower. We shall refer to the above inequality as the Golod-Shafarevich inequality. We immediately see that for k imaginary with rank $C_{k,2} \geq 5$, or k real with rank $C_{k,2} \geq 6$, the Golod-Shafarevich inequality is satisfied and k thereby has infinite 2-class field tower. It is well-known that for k imaginary with rank $C_{k,2} = 2$ or 3, the 2-class field tower of k may be finite or infinite, and that if rank $C_{k,2} = 1$ then the 2-class field tower of k is finite and has length 1 (cf. [**3**], [**6**], [**14**], [**17**], [**20**]). It has been conjectured that for k imaginary with rank $C_{k,2} = 4$, k has infinite 2-class field tower (cf. [**17**], [**18**]).

A partial result in this direction, as proved by Hajir, is that if k is an imaginary quadratic number field such that $C_{k,2}$ contains a subgroup isomorphic to $Z_4 \times Z_4 \times Z_4$, then k has infinite 2-class field tower (cf. [**6**], [**7**]). We extend this result to particular fields k with rank $C_{k,2} = 4$ and 4-rank of C_k less than 3. Denoting the discriminant of k by Δ_k, our fields k are classified according to the 4-rank of C_k, the Kronecker symbols (p/q) of the primes dividing Δ_k, and the number of negative prime discriminants dividing Δ_k. We demonstrate that if the 4-rank of C_k is greater than or equal to 2 and exactly one negative prime discriminant divides Δ_k, then k has infinite 2-class field tower.

Preliminaries.

Our initial results are directly related to the following inequality (cf. [**6**], [**7**]):

Proposition 1. *Let F be a totally real field of degree n, E a totally complex quadratic extension of F, and t the number of prime ideals of F which ramify in E. If $t \geq 3 + 2\sqrt{n+1}$ then the 2-class field tower of E is infinite.*

We will also need to utilize the well-known ambiguous class number formula, where for a cyclic extension K/F an ambiguous ideal class is an ideal class of K that remains invariant under the action of $\mathrm{Gal}\,(K/F)$. We denote the subgroup of ambiguous ideal classes by $\mathrm{Am}\,(K/F)$ and its Sylow 2-subgroup by $\mathrm{Am}_2(K/F)$. We state the following two propositions: (cf. [**12**], [**15**]).

Proposition 2. *Let K/F be a cyclic extension of prime degree p. Then $|\mathrm{Am}(K/F)| = h(F) \cdot p^{t-1}/(E : H)$ where t is the number of (finite or infinite) primes of F which are ramified in K/F, $E = E_F$ is the unit group of F, $H = E \cap N_{K/F}K^x$ is the subgroup of units which are norms of elements of K^x, and K^x is the multiplicative group of K.*

Proposition 3. *Let K/F be a quadratic extension of an algebraic number field where $h(F)$ is odd. Then $|\mathrm{Am}_2(K/F)| = 2^e$ where e is the 2-rank of C_K.*

Results.

We begin by obtaining some conditions on the Kronecker symbols of the primes dividing Δ_k, directly related to Proposition 1, to insure that an imaginary quadratic number field k with rank $C_{k,2} = 4$ has infinite 2-class field tower.

Lemma 1. *Let k be an imaginary quadratic number field such that rank $C_{k,2} = 4$. If for some prime $p_j \equiv 1 \mod 4$, or $p_j = 2$ in which case we further assume that 8 is a fundamental discriminant dividing Δ_k, we have*

$\left(\frac{p_j}{p_k}\right) = \left(\frac{p_j}{p_l}\right) = \left(\frac{p_j}{p_m}\right) = 1$, *$p_j, p_k, p_l, p_m$ distinct primes, $p_j p_k p_l p_m | \Delta_k$, then k has infinite 2-class field tower.*

Proof. We proceed in a similar way to Hajir in [**6**]. Let $F = Q(\sqrt{p_j})$, $E = k(\sqrt{p_j})$. We see that E is a CM field with maximal field subfield F such that either 7 or 8 primes ramify from F to E. Since $7 \geq 3 + 2\sqrt{2+1}$, it follows from Proposition 1 that E has infinite 2-class field tower. Since E is an umramified quadratic extension of k, k has infinite 2-class field tower as well. $\square$

We utilize the following notational convenience: If d_j is a negative prime discriminant we let p_j denote the prime dividing d_j if $d_j \neq -4$, and $p_j = 1$ if $d_j = -4$.

Lemma 2. *Let k be an imaginary quadratic number field such that rank $C_{k,2} = n$, $n \geq 1$. Let $L = Q(\sqrt{d_j})$ and $F = k(\sqrt{d_j})$, where d_j is a negative prime discriminant, $p_j | \Delta_k$. Then exactly $2n$ prime ideals in L are ramified in F if and only if $\left(\frac{-p_j}{p_i}\right) = 1$, $i \neq j$, for all primes $p_i | \Delta_k$, $1 \leq i \leq n+1$.*

Proof. Assume $\left(\frac{-p_j}{p_i}\right) = 1$, $i \neq j$, for all primes $p_i | \Delta_k$, $1 \leq i \leq n+1$. It follows that there are exactly n primes p_i dividing Δ_k that split in $L = Q(\sqrt{d_j})$. Since $F = k(\sqrt{d_j})$ is an unramified quadratic extension of k, these n primes p_i each have ramification index 2 in F. We therefore can conclude that each of these primes p_i must ramify from L to F. There are no other prime ideals that ramify from L to F, since if there were a prime ideal P_m in L that ramifies in F such that $P_m \cap Q = p_m \neq p_i$, $1 \leq i \leq n+1$, it would imply that p_m ramifies in F. But p_m does not divide Δ_k unless $p_m = p_j$, and F is an unramified quadratic extension of k. Since p_j has ramification index 2 in F, we therefore conclude that exactly $2n$ prime ideals in L are ramified in F. The converse is proved in a similar way and is left to the reader. $\square$

We note that in our proof of Lemma 1 we were able to utilize the full strength of Proposition 1 by requiring only 7 primes to ramify from F to E, whereas Hajir, in his original proof that if the 4-rank of C_k is greater than or equal to 3 then k has infinite 2-class field tower (cf. [**6**]), assumed that $p_i \equiv 1 \mod 4$, $\left(\frac{p_j}{p_i}\right) = 1$, $j \neq i$, and therefore 8 primes ramified from F to E.

We illustrate Hajir's method of proof of the above result in the case where a negative prime discriminant d_j divides Δ_k, $\left(\frac{-p_j}{p_i}\right) = 1$, $j \neq i$, as follows (cf. [**7**]):

Lemma 3. *Let k be an imaginary quadratic number field such that rank $C_{k,2} = 4$. Assume there exists a negative prime discriminant d_j, $d_j | \Delta_k$,* *such that* , $\left(\frac{-p_j}{p_i}\right) = 1$, $j \neq i$, *for all primes* $p_i | \Delta_k$, $1 \leq i \leq 5$. *Then* k *has infinite* 2*-class field tower.*

Proof. Let $L = Q(\sqrt{-p_j})$ and $F = k(\sqrt{-p_j})$. By Lemma 2 we see that exactly 8 prime ideals in L are ramified in F. By Proposition 2 and Proposition 3, it follows that rank $C_{F,2} \geq 6$. Since rank $E_{F,2} = 2$, we obtain the Golod-Shafarevich inequality: rank $C_{F,2} \geq 6 \geq 2 + 2\sqrt{2+1}$ and therefore F has infinite 2-class field tower. Since F is an unramified quadratic extension of k, k has infinite 2-class field tower as well. ☐

We state the following corollaries of Lemma 3:

Corollary 1. *Let* k *be an imaginary quadratic number field such that rank* $C_{k,2} = 4$, *exactly one negative prime discriminant divides* Δ_k, *and* $\Delta_k \equiv 4 \mod 8$. *Then* k *has infinite* 2*-class field tower.*

Proof. Since k has exactly one negative prime discriminant and $\Delta_k \equiv 4 \mod 8$, all the odd primes dividing Δ_k are congruent to 1 mod 4. We therefore have $\left(\frac{-1}{p_i}\right) = 1$ for all odd primes $p_i | \Delta_k$ and our result follows immediately from Lemma 3. ☐

Corollary 2. *Let* k *be an imaginary quadratic number field, rank* $C_{k,2} = 4$, *such that five negative prime discriminants divide* Δ_k. *Then the following fields* k *have infinite* 2*-class field tower, where* q_i, $1 \leq i \leq 5$, *is a prime congruent to* 3 *mod* 4*:*

$$Q\left(\sqrt{-q_1q_2q_3q_4q_5}\right), \quad \left(\frac{-q_i}{q_j}\right) = \left(\frac{-q_i}{q_k}\right) = \left(\frac{-q_i}{q_l}\right) = \left(\frac{-q_i}{q_m}\right) = 1,$$
$$\{i,j,k,l,m\} = \{1,2,3,4,5\}$$
$$Q\left(\sqrt{-q_1q_2q_3q_4}\right), \quad \left(\frac{-q_i}{q_j}\right) = \left(\frac{-q_i}{q_k}\right) = \left(\frac{-q_i}{q_l}\right) = \left(\frac{-q_i}{2}\right) = 1,$$
$$\{i,j,k,l\} = \{1,2,3,4\}.$$

Proof. It is immediate by applying Lemma 3 to each field k that we have infinite 2-class field tower. We note that these are all the possible fields satisfying the assumptions of our corollary for which we are able to apply Lemma 3. ☐

For the cases when exactly one negative prime discriminant divides Δ_k where $\Delta_k \not\equiv 4 \mod 8$, and exactly three negative prime discriminants divide Δ_k, we utilize the following lemma:

Lemma 4. *Let* k *be an imaginary quadratic number field such that rank* $C_{k,2} = 4$, *at least two of the prime discriminants dividing* Δ_k *are positive, and* $\left(\frac{p_1}{p_3}\right) = \left(\frac{p_2}{p_3}\right) = 1$ *where* p_1 *and* p_2 *are distinct primes dividing positive* *prime discriminants dividing Δ_k, and p_3 is a prime dividing a positive or negative prime discriminant dividing Δ_k, $p_1 \neq p_3 \neq p_2$. Then k has infinite 2-class field tower.*

Proof. By the assumptions of our lemma, we can write $k = Q(\sqrt{-p_1p_2p_3p_4p_5})$ or $k = Q(\sqrt{-p_1p_2p_3p_4})$ where $p_1 \equiv p_2 \equiv 1 \mod 4$, or $p_1 = 2$ and $p_2 \equiv 1 \mod 4$. By Martinet (cf. [**17**], Proposition 5) we see immediately that k has infinite 2-class field tower. ☐

We now let $k = Q(\sqrt{-p_1p_2p_3p_4p_5})$ where $\Delta_k \not\equiv 4 \mod 8$ and exactly one negative prime discriminant divides Δ_k. We define a Kronecker symbol configuration of k to be a complete list of Kronecker symbols $\left(\frac{p_i}{p_j}\right)$, $i \leq j$, $l \leq i \leq 5$, $l \leq j \leq 5$. We denote a Kronecker symbol configuration by listing all the Kronecker symbols $\left(\frac{p_i}{p_j}\right)$ as above with $\left(\frac{p_i}{p_j}\right) = 1$ (respectively -1), where the remaining Kronecker symbols $\left(\frac{p_i}{p_j}\right)$, $i < j$, are assumed to be -1 (respectively 1).

In Table 1 we utilize the Rédei & Reichardt conditions [**19**] to list all possible Kronecker symbol configurations (without loss of generality) according to the 4-rank of C_k.

Table 1.

4-rank of C_k	possible Kronecker symbol configurations
4	all Kronecker symbols equal 1
3	$\left(\frac{p_1}{p_2}\right) = \left(\frac{p_1}{p_3}\right) = \left(\frac{p_1}{p_4}\right) = \left(\frac{p_1}{p_5}\right) = 1$ $\left(\frac{p_1}{p_2}\right) = -1$
2	$\left(\frac{p_1}{p_2}\right) = \left(\frac{p_1}{p_3}\right) = \left(\frac{p_2}{p_3}\right) = 1$ $\left(\frac{p_1}{p_2}\right) = \left(\frac{p_3}{p_4}\right) = \left(\frac{p_1}{p_5}\right) = \left(\frac{p_2}{p_5}\right) = 1$ $\left(\frac{p_1}{p_2}\right) = \left(\frac{p_1}{p_3}\right) = \left(\frac{p_1}{p_4}\right) = \left(\frac{p_1}{p_5}\right) = \left(\frac{p_2}{p_5}\right) = 1$ $\left(\frac{p_1}{p_2}\right) = \left(\frac{p_3}{p_4}\right) = -1$ $\left(\frac{p_1}{p_2}\right) = \left(\frac{p_1}{p_3}\right) = -1$ $\left(\frac{p_1}{p_2}\right) = \left(\frac{p_1}{p_3}\right) = \left(\frac{p_2}{p_3}\right) = -1$ $\left(\frac{p_1}{p_2}\right) = \left(\frac{p_3}{p_5}\right) = \left(\frac{p_1}{p_3}\right) = \left(\frac{p_1}{p_5}\right) = \left(\frac{p_2}{p_3}\right) = 1$ $\left(\frac{p_1}{p_2}\right) = \left(\frac{p_4}{p_5}\right) = \left(\frac{p_1}{p_3}\right) = \left(\frac{p_1}{p_5}\right) = \left(\frac{p_2}{p_3}\right) = 1$

4-rank of C_k	possible Kronecker symbol configurations
1	$\left(\frac{p_1}{p_2}\right)=\left(\frac{p_1}{p_3}\right)=\left(\frac{p_1}{p_4}\right)=\left(\frac{p_2}{p_3}\right)=1$ $\left(\frac{p_1}{p_2}\right)=\left(\frac{p_1}{p_3}\right)=\left(\frac{p_1}{p_4}\right)=\left(\frac{p_2}{p_3}\right)=-1$ $\left(\frac{p_1}{p_2}\right)=\left(\frac{p_1}{p_3}\right)=\left(\frac{p_1}{p_4}\right)=-1$ $\left(\frac{p_1}{p_2}\right)=\left(\frac{p_1}{p_3}\right)=\left(\frac{p_2}{p_4}\right)=-1$ $\left(\frac{p_1}{p_2}\right)=\left(\frac{p_1}{p_3}\right)=1$
0	all Kronecker symbols equal -1 $\left(\frac{p_1}{p_2}\right)=1$ $\left(\frac{p_1}{p_2}\right)=\left(\frac{p_3}{p_4}\right)=1$ $\left(\frac{p_1}{p_2}\right)=\left(\frac{p_1}{p_3}\right)=\left(\frac{p_2}{p_4}\right)=1$ $\left(\frac{p_1}{p_2}\right)=\left(\frac{p_3}{p_4}\right)=\left(\frac{p_1}{p_5}\right)=\left(\frac{p_2}{p_3}\right)=1$ $\left(\frac{p_1}{p_2}\right)=\left(\frac{p_3}{p_4}\right)=\left(\frac{p_1}{p_5}\right)=\left(\frac{p_2}{p_3}\right)=\left(\frac{p_4}{p_5}\right)=1$ $\left(\frac{p_1}{p_2}\right)=\left(\frac{p_1}{p_3}\right)=\left(\frac{p_1}{p_4}\right)=\left(\frac{p_2}{p_5}\right)=\left(\frac{p_3}{p_5}\right)=1$ $\left(\frac{p_1}{p_2}\right)=\left(\frac{p_3}{p_4}\right)=\left(\frac{p_1}{p_5}\right)=1$ $\left(\frac{p_1}{p_2}\right)=\left(\frac{p_1}{p_3}\right)=\left(\frac{p_1}{p_4}\right)=1$ $\left(\frac{p_1}{p_2}\right)=\left(\frac{p_3}{p_4}\right)=\left(\frac{p_1}{p_3}\right)=\left(\frac{p_1}{p_5}\right)=1$ $\left(\frac{p_1}{p_2}\right)=\left(\frac{p_3}{p_4}\right)=\left(\frac{p_1}{p_3}\right)=\left(\frac{p_1}{p_5}\right)=\left(\frac{p_2}{p_3}\right)=1$ $\left(\frac{p_1}{p_2}\right)=\left(\frac{p_3}{p_4}\right)=\left(\frac{p_1}{p_5}\right)=-1$ $\left(\frac{p_1}{p_2}\right)=\left(\frac{p_3}{p_4}\right)=\left(\frac{p_1}{p_5}\right)=\left(\frac{p_2}{p_3}\right)=-1$ $\left(\frac{p_1}{p_2}\right)=\left(\frac{p_3}{p_4}\right)=\left(\frac{p_1}{p_3}\right)=\left(\frac{p_1}{p_5}\right)=-1$ $\left(\frac{p_1}{p_2}\right)=\left(\frac{p_3}{p_4}\right)=\left(\frac{p_1}{p_5}\right)=\left(\frac{p_2}{p_5}\right)=1$ $\left(\frac{p_1}{p_2}\right)=\left(\frac{p_1}{p_3}\right)=\left(\frac{p_1}{p_4}\right)=\left(\frac{p_1}{p_5}\right)=-1$

We are now able to state the following theorem:

Theorem 1. *Let k be an imaginary quadratic number field with* $\operatorname{rank} C_{k,2}=4$, $\Delta_k\not\equiv 4 \mod 8$, *and exactly one negative prime discriminant dividing* Δ_k. *Then the following fields k have infinite 2-class field tower, where* $\{i,j,k,l,m\}=\{1,2,3,4,5\}$*:*

(A) *4-rank of C_k equal to* $2,3$ *or* 4

(B) 4-*rank of* C_k *equal to* 1 *and Kronecker symbol configuration of* k *not* $(\frac{p_i}{p_j}) = (\frac{p_i}{p_k}) = 1$ *where either* p_j *or* p_k *is the prime dividing the negative prime discriminant dividing* d_k

(C) 4-*rank of* C_k *equal to* 0 (*i.e.,* $C_{k,2}$ *elementary*) *and Kronecker symbol configuration of* k *not one of the following types:*
- *all Kronecker symbols equal* -1
- $(\frac{p_i}{p_j}) = 1$
- $(\frac{p_i}{p_j}) = (\frac{p_k}{p_l}) = 1$
- $(\frac{p_i}{p_j}) = (\frac{p_k}{p_l}) = (\frac{p_i}{p_m}) = 1$ *where either* p_j *or* p_m *is the prime dividing the negative prime discriminant dividing* d_k.

Proof. For case (A) with 4-rank of C_k equal to 3 or 4 the result has been established by Hajir (cf. [**6**], [**7**]). For case (A) with 4-rank of C_k equal to 2, and cases (B) and (C), we apply Lemma 4 to our fields listed in Table 1 to establish our result. ☐

From Table 1 we see that there are 32 possible Kronecker symbol configurations when exactly one negative prime discriminant divides Δ_k, $\Delta_k \not\equiv 4 \mod 8$. From Theorem 1 we find that for 27 of these Kronecker symbol configurations, k has infinite 2-class field tower. The unknown cases can be summarized by means of the 4-rank of C_k as follows:

Table 2.

4-rank of C_k	number of Kronecker symbol configurations where 2-class field tower of k may be finite
4	0
3	0
2	0
1	1
0	4

Remark 1. For the case when exactly three negative prime discriminants divide Δ_k, one can again utilize the Rédei & Reichardt conditions and Lemma 4 to obtain fields with infinite 2-class field tower. We note that in this case the Kronecker symbol configuration $(\frac{p_1}{p_2}) = (\frac{p_1}{p_3}) = (\frac{p_1}{p_4}) = 1$ may not satisfy the requirements of Lemma 4; however, Lemma 1 may be used when $p_1 \not\equiv 1 \mod 4$, or when $p_1 = 2$ if 8 is a fundamental discriminant dividing Δ_k. For the case when five negative prime discriminants divide Δ_k, one can utilize the Rédei & Reichardt conditions and Corollary 2 to obtain

fields with infinite 2-class field tower. In a follow-up paper we will demonstrate that our techniques allow us to conclude that k has infinite 2-class field tower for imaginary quadratic number fields k when rank $C_{k,2} = 4$, C_k has 4-rank equal to 2, and either five negative prime discriminants divide Δ_k or $\Delta_k \not\equiv 4 \bmod 8$ (cf. [**1**]).

Examples.

From Lemma 1, Corollary 2, and Theorem 1 we immediately obtain that the following fields k have infinite 2-class field tower. We list our fields according to the 4-rank of C_k.

$$\begin{aligned}
C_{k,2}\ \text{ elementary:}\quad & k = Q\left(\sqrt{-61,620}\right) = Q\left(\sqrt{-3\cdot 5\cdot 13\cdot 79}\right)\\
& k = Q\left(\sqrt{-120,180}\right) = Q\left(\sqrt{-5\cdot 13\cdot 17\cdot 29}\right)\\
& k = Q\left(\sqrt{-122,655}\right) = Q\left(\sqrt{-3\cdot 5\cdot 13\cdot 17\cdot 37}\right)\\
& k = Q\left(\sqrt{-212,135}\right) = Q\left(\sqrt{-5\cdot 7\cdot 11\cdot 19\cdot 29}\right)\\
& k = Q\left(\sqrt{-256,360}\right) = Q\left(\sqrt{-2\cdot 5\cdot 13\cdot 17\cdot 29}\right)\\
& k = Q\left(\sqrt{-430,360}\right) = Q\left(\sqrt{-2\cdot 5\cdot 7\cdot 29\cdot 53}\right)\\
& k = Q\left(\sqrt{-440,115}\right) = Q\left(\sqrt{-3\cdot 5\cdot 13\cdot 37\cdot 61}\right)\\
& k = Q\left(\sqrt{-850,135}\right) = Q\left(\sqrt{-5\cdot 11\cdot 13\cdot 29\cdot 41}\right)\\
& k = Q\left(\sqrt{-2,035,240}\right) = Q\left(\sqrt{-2\cdot 5\cdot 17\cdot 41\cdot 73}\right)\\
& k = Q\left(\sqrt{-5,863,655}\right) = Q\left(\sqrt{-5\cdot 7\cdot 29\cdot 53\cdot 109}\right)
\end{aligned}$$

$$\begin{aligned}
4-\text{rank of } C_k = 1:\quad & k = Q\left(\sqrt{-184,008}\right) = Q\left(\sqrt{-2\cdot 3\cdot 11\cdot 17\cdot 41}\right)\\
& k = Q\left(\sqrt{-531,867}\right) = Q\left(\sqrt{-3\cdot 7\cdot 19\cdot 31\cdot 43}\right)\\
& k = Q\left(\sqrt{-2,657,415}\right) = Q\left(\sqrt{-3\cdot 5\cdot 29\cdot 41\cdot 149}\right)\\
& k = Q\left(\sqrt{-6,425,679}\right) = Q\left(\sqrt{-3\cdot 13\cdot 37\cdot 61\cdot 73}\right)
\end{aligned}$$

$$4-\text{rank of } C_k = 2:\quad k = Q\left(\sqrt{-3,989,095}\right) = Q\left(\sqrt{-5\cdot 11\cdot 29\cdot 41\cdot 61}\right)$$

Remark 2. Since the fields satisfying the conditions of Lemmas 1, 3, and 4 possess an unramified quadratic extension which satisfy the Golod-Shafarevich inequality, (cf. [**19**] in regard to Lemmas 1 and 4) it follows from Theorem 6 of Hajir (cf. [**5**]) that the rank of the 2-class groups of these fields tend to infinity.

In conclusion, we see that the conjecture concerning the 2-class field tower of k being infinite holds in a number of particular fields k when the 4-rank of C_k is equal to 0, 1, or 2, and always holds when the 4-rank of C_k is greater than or equal to 3. Our techniques allow us to obtain families of fields k with 4-rank of C_k equal to 0,1, or 2 and k having infinite 2-class field tower, as well as the rank of the 2-class groups of the fields in the tower of k tending to infinity. However, the complete resolution of the conjecture concerning all imaginary quadratic number fields k with rank $C_{k,2} = 4$ is still a very open question.

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