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It is known that the maximal order of a cyclic group of automorphisms admitted by a Klein surface or real algebraic curve of algebraic genus p is $2p$ or $2(p+1)$, depending on whether p is odd or even. In this paper, we classify the automorphism groups of all non-orientable Klein surfaces, without boundary, which admit an automorphism group of order $2p$, or $2(p+1)$. We determine that the automorphism groups are cyclic precisely when the surfaces are hyperelliptic. Defining equations for all but one family of these Klein surfaces are given.

There are certain properties that can be shown to exist for at least one Riemann surface of each genus g or Klein surface of each algebraic genus p. For example, for each $g \geq 2$, there exists a Riemann surface of genus g which possesses $8g+8$ automorphisms [**1**], [**7**]. Similarly, for each $p \geq 2$, there exist orientable and non-orientable Klein surfaces of algebraic genus p which possess $4(p+1)$ and $4p$ automorphisms respectively [**10**]. In each case these bounds are sharp, since there are infinitely many g and p for which there are no Riemann or Klein surfaces which possess more automorphisms. Similarly, for each $p \geq 2$, there exists a Klein surface which admits a cyclic group of automorphisms of order $2(p+1)$ if p is even, or $2p$, if p is odd [**3**], [**9**]. Although Klein surfaces which possess such large cyclic automorphism groups are so numerous, in this paper we prove an interesting converse. Recall that a non-orientable Klein surface without boundary is called a non-orientable Riemann surface. We show that any non-orientable Riemann surface which admits a group G of automorphisms of order $2(p+1)$ or $2p$ has the property that G is either cyclic or an extension of a cyclic group by Z_2. We further determine that G is cyclic if and only if the non-orientable Riemann surface is hyperelliptic.

Recall that the category of Klein surfaces is equivalent to the category of real algebraic curves. Each Klein surface can be realized as an algebraic curve, defined by real equations, upon which complex conjugation acts. Non-orientable Riemann surfaces correspond to algebraic curves whose real locus is empty. We determine the defining equations for the Riemann double covers for all but one family of surfaces found in this paper. We explicitly determine the symmetry which yields each Klein surface and give explicit equations for the automorphism group of order $2p$ or $2(p+1)$. This work, in conjunction with the examination of bordered Klein surfaces in [**6**], provides an extensive analysis of all Klein surfaces which admit automorphism groups of order $2(p+1)$ or $2p$.

1. Preliminaries.

Let U denote the upper half plane and let W be a compact Klein surface of algebraic genus $p \geq 2$. Then W can be realized as U/Γ for some non-euclidean crystallographic (NEC) group Γ. In addition, Γ can be chosen to be a surface group, meaning that it has no nonidentity orientation preserving elements of finite order. If W admits a group of automorphisms G, then there exists an NEC group Λ such that $G \cong \Lambda/\Gamma$, and W/G and U/Λ are equivalent Klein surfaces. Important properties of Λ are contained in its signature

$$(g; \pm; [m_1, m_2, \dots, m_r]; \{(n_{11}, \dots, n_{1s_1}), \dots, (n_{k1}, \dots, n_{ks_k})\}). \tag{1}$$

The above signature indicates that U/Λ has topological genus g and k boundary components. Each m_i is called a proper period, each n_{ij} is called a link period, and each term $(n_{i1}, \dots, n_{is_i})$ is called a period cycle of Λ. It is well-known that Λ, with signature (1), can be generated by one of the following two sets of elements:

$$\{x_u, c_{ij}, e_v, a_w, b_w \mid 1 \leq u \leq r, 1 \leq i \leq k, 0 \leq j \leq s_i, 1 \leq v \leq k, 1 \leq w \leq g\},$$

$$\{x_u, c_{ij}, e_v, d_w \mid 1 \leq u \leq r, 1 \leq i \leq k, 0 \leq j \leq s_i, 1 \leq v \leq k, 1 \leq w \leq g\}. \tag{2}$$

The first set generates Λ if the signature of Λ has a plus sign, and the second set generates Λ if the signature has a minus sign. In either case, the generators satisfy the relations

$$\begin{aligned}\{x_u^{m_u} = 1, c_{i,s_i} = e_i^{-1} c_{i,0} e_i, c_{i,j-1}^2 = c_{ij}^2 = (c_{i,j-1} c_{ij})^{n_{ij}} = 1 \\ \mid 1 \leq u \leq r, 1 \leq i \leq k, 1 \leq j \leq s_i, 1 \leq w \leq g\},\end{aligned} \tag{3}$$

and the additional relation $x_1 \dots x_r e_1 \dots e_k [a_1, b_1] \dots [a_g, b_g] = 1$ if there is a plus sign in the signature, and $x_1 \dots x_r e_1 \dots e_k d_1^2 \dots d_g^2 = 1$ if there is a minus sign in the signature.

Since Γ is a surface group, it has a signature of the form

$$(g'; \pm; [-]; \{(-), \dots, (-)\}). \tag{4}$$

Let k' denote the number of empty period cycles in (4).

The Riemann-Hurwitz formula yields that

$$\frac{\alpha g' + k' - 2}{\mid G \mid} = \alpha g + k - 2 + \sum_{j=1}^{r}(1 - 1/m_j) + \frac{1}{2}\sum_{i=1}^{k}\sum_{j=1}^{s_i}(1 - 1/n_{ij}). \tag{5}$$

The algebraic genus p of a Klein surface is defined to be the genus of its Riemann double cover. If a surface has signature (1), then $p = \alpha g + k - 1$, where α equals 2 or 1, depending on whether there is a $+$ or $-$ in the signature.

A non-orientable Riemann surface is a non-orientable Klein surface without boundary. Note that if the surface group Γ in (4) corresponds to a non-orientable Riemann surface of algebraic genus p, then Γ has the signature

$$(p+1; -; [-]; \{-\}). \tag{6}$$

It is well-known that an NEC group with signature (1) exists if and only if the right hand side of (5) is positive. Therefore, a way to construct Klein surfaces which admit a given automorphism group G is to define a group homomorphism θ from an NEC group Λ, with signature (1), onto a finite group G. If $\Gamma = \ker(\theta)$, then the Klein surface U/Γ admits $G \cong \Lambda/\Gamma$ as a group of automorphisms.

Throughout the paper we will use the following presentation for a group G of order $2n$, where n is odd, which possesses a cyclic group of order n:

$$\langle A, B \mid A^n = B^2 = 1, BAB = A^{\alpha}\rangle. \tag{7}$$

In (7), α and n are relatively prime and $\alpha^2 \equiv 1 \bmod n$. Note that if $\alpha = -1$, then G is the dihedral group D_n, and if $\alpha = 1$, then G is cyclic. However we will use the presentation

$$\langle a \mid a^{2n} = 1\rangle \tag{8}$$

for the cyclic group of order $2n$. In addition, we denote the greatest common divisor of the integers a and b by $\gcd(a, b)$.

We state without proof an elementary result concerning groups of order $2n$ where n is odd.

Proposition 1.1. *Let G be a group of order $2n$, where n is odd. Then G contains a unique normal subgroup H of order n which contains all the elements of G of odd order.*

We now define notation to be used for the rest of the paper. Define $\gamma = p$ if p is odd or $p+1$ if p is even. Let W denote a non-orientable Riemann surface of algebraic genus p which admits a group G of automorphisms of order 2γ and let its corresponding NEC group Γ have signature (6). Let Λ denote an NEC group such that $\Gamma \triangleleft \Lambda$, $G \cong \Lambda/\Gamma$, and $X/G = U/\Lambda$. We assume Λ has signature (1) and generators and relations (2) and (3). Let

$\theta : \Lambda \to G$ denote the canonical map, and let H denote the unique subgroup of G of order γ.

We state two Propositions which allow us to determine the possible signatures for Λ.

Proposition 1.2. *Let W, Λ, G, and H be defined as above. Let Λ have signature* (1)*. Then no proper period m_i in the signature for Λ is divisible by* 4*, and each link period n_{ij} in the signature of Λ must be odd.*

Proof. Let $\theta : \Lambda \to G$ be the canonical epimorphism. Since $\Gamma = \ker(\theta)$, and Γ is a surface group, it cannot contain elliptic elements. Since 4 doesn't divide $\mid G \mid$, no proper period is divisible by 4. We now show that no link period can be even. Assume n_{ij} is an even link period and let $c_{i,j-1}$ and c_{ij} be the elements of order two such that $c_{i,j-1}c_{ij}$ has even order n_{ij}. Since W has no boundary, $c_{i,j-1}$, and c_{ij} are not in $\ker(\theta)$, and $\theta(c_{i,j-1}c_{ij})$ has order n_{ij}. Therefore G contains a dihedral group of order $2n_{ij}$, which implies that 4 divides $\mid G \mid$, a contradiction. Thus each n_{ij} cannot be even.

Proposition 1.3. *Let W, Λ, Γ, G, and H be defined as above. Assume all of the proper periods, if they exist, in the signature of Λ are odd. Then Λ cannot have one of the following signatures:*

i. $(1; -; [m_1, m_2, \dots, m_r]; \{-\})$,
ii. $(0; +; [m_1, m_2, \dots, m_r]; \{(-)\})$,
iii. $(0; +; [m_1, \dots, m_r]; \{(n_1, n_2, \dots n_s)\})$,
iv. $(0; +; [-]; \{(n_1, \dots, n_s)\})$.

Proof. In each case, let $\theta : \Lambda \to G$ be the canonical map. We say a proper generator of Λ is a generator in (2) which is not in Γ. A proper word of Λ is the product of proper generators of Λ. Recall that $\ker(\theta)$ is non-orientable if and only if a glide reflection (one of the d_w in (2)) or a non-orientable proper word belongs to $\ker(\theta)$ [**5**]. We will show that the above signatures imply that $\ker(\theta)$ is orientable. Considering (2), in Cases (i)-(iv) generators for Λ are:

i. $\{x_1, \dots, x_r, d_1\}$,
ii. $\{x_1, \dots, x_r, e_1, c_0\}$,
iii. $\{x_1, \dots, x_r, e_1, c_0, \dots, c_s\}$,
iv. $\{c_0, \dots, c_s\}$.

Since each x_i has odd order $\theta(x_i) \in H$. In Case (i), this implies that $\theta(d_1) \notin H$, otherwise θ is not onto. In Cases (ii)-(iii), this implies that $\theta(e_1) \in H$, since $x_1 \dots x_r e_1 = 1$. Note that for each j, $c_j \notin \Gamma$, since W has no boundary, therefore $\theta(c_j) \notin H$. For each case, let w be a proper, non-orientable word in Λ and let $w_1 = \theta(w)$. For each case, we consider $w_1 H$ in G/H. For Case (i) we obtain that $w_1 H = \theta(d_1)^q H$, where q is the number of appearances of d_1 in w. Since w is non-orientable, q is odd, therefore $w_1 \notin H$, therefore $w \notin \ker(\theta)$. In Cases (ii)-(iv), $w_1 H = \theta(c_0)^q H$, where q is the number of appearances of $c_0, \ldots, c_s$ in w. Again, since w is non-orientable, q is odd, and therefore $w \notin \ker(\theta)$. Therefore, no proper, non-orientable word in Λ is an element of $\Gamma = \ker(\theta)$. This contradicts that W is non-orientable.

We now determine possible signatures for Λ. Let Λ have the signature (1) and let Γ have the signature (6). From (5), we obtain

$$\frac{1}{2} > \frac{p-1}{2\gamma} = \alpha g + k - 2 + \sum_{i=1}^{r}(1 - 1/m_i) + \frac{1}{2}\sum_{i=1}^{k}\sum_{j=1}^{s_i}(1 - 1/n_{ij}). \tag{9}$$

Therefore $\alpha g + k \leq 2$ and we have the following cases:

(a) $g = 1, \alpha = 1, k = 1$,
(b) $g = 1, \alpha = 1, k = 0$,
(c) $g = 0, k = 2$,
(d) $g = 0, k = 1$,
(e) $g = 2, \alpha = 1, k = 0$.

We consider each case in conjunction with (9). In Case a), if $r > 0$, or if $r = 0$ and $s \geq 2$, then the right hand side of (9) is not less than 1/2. Therefore, for case a), $r = 0$ and $s_1 = 1$. In Case b), r must be two. In Case c), r must be 0, however, it is not possible for both s_1 and s_2 to be greater than 0. In Case d), $r \leq 2$. If $r = 2$, then $s_1 = 0$ or 1. If $r = 1$, then $s_1 = 1$ or 2. Proposition 1.3 yields that $r \neq 0$. Finally, in Case e), the right hand side of (9) is zero or greater than or equal to 1/2. Therefore we arrive at the list of relevant signatures for which the right hand side of (9) is less than 1/2:

(a) $(1; -; [-]; \{(n)\})$,
(b) $(1; -; [m_1, m_2]; \{-\})$,
(c) $(0; +; [-]; \{(-), (n)\})$,
(d1) $(0; +; [m_1, m_2]; \{(n)\})$,
(d2) $(0; +; [m_1, m_2]; \{(-)\})$,
(d3) $(0; +; [m]; \{(n_1, n_2)\})$,
(d4) $(0; +; [m]; \{(n)\})$.

2. Main theorems.

We now determine which signatures for Λ yield non-orientable Riemann surfaces which admit 2γ automorphisms. We also determine all of the possible epimorphisms from Λ onto a group of order 2γ.

Theorem 2.1. *Let W be a non-orientable Riemann surface of even algebraic genus $p \geq 2$ which admits a group of automorphisms G of order $2p+2$.*

Then $G \cong \mathbb{Z}_{2p+2}$ *and* W *is hyperelliptic, or* G *is a non-cyclic group with presentation* (7)*.*

Proof. We continue to use the notation established above. Note that Λ cannot have signatures a) or c) listed above since in these cases (9) implies that $n = (p+1)/2$, which is impossible since p is even.

We now consider Case b). In this case, Λ has signature $(1;-;[m_1, m_2];\{-\})$ and so (9) yields that $1/2 + 1/(p+1) = 1/m_1 + 1/m_2$. If $m_1 = 2$, then $m_2 = p+1$. If m_1 and m_2 are both greater than two, then (9) yields that $2(m_1+m_2)(p+1) = m_1 m_2 (p+3)$, therefore one of m_1 or m_2 is even. However, if m_1 is even, then, since no proper period is divisible by 4, $m_1 \geq 6$. However, $m_1 \geq 6$ and $m_2 \geq 3$ contradicts (9). Therefore, Case b) yields the signature

$$(1; -; [2, p+1]; \{-\}). \tag{10}$$

We now determine the possible signatures for Cases d1)–d4). Case d1) is impossible. To see this, assume $m_1 = m_2 = 2$. Then (9), yields that $n = (p+1)/2$ which is impossible, since p is even. If $m_1 > 2$, then (9) yields that $1/2 > -1/m_1 - 1/m_2 + 3/2 - 1/(2n) \geq -1/3 - 1/2 + 3/2 - 1/6 = 1/2$, a contradiction.

The argument in Case d2) is analogous to Case b) above, and easily yields the signature

$$(0; +; [2, p+1]; \{(-)\}). \tag{11}$$

In Case d3), (9) yields

$$\frac{1}{2} > \frac{p-1}{2(p+1)} = 1 - \frac{1}{m} - \frac{1}{2n_1} - \frac{1}{2n_2}. \tag{12}$$

We will show that $m = 2$. If $m > 2$, Propositions 1.2 and 1.3 yield that $m \geq 6$. However, since n_1 and n_2 are odd, this contradicts (12). Therefore $m = 2$. From (12) we obtain $2n_1 n_2 = (n_1 + n_2)(p+1)$. Let $n_1 \leq n_2$ and assume that $n_2 < p+1$. Then $2n_1 n_2 > n_2(n_1 + n_2)$, so $n_1 > n_2$ which is a contradiction. Thus, since each link period must be odd, $n_2 = p+1$. But then $1/(p+1) = 1/n_1$ which give $n_1 = p+1$ also. Hence Case d3) only yields the signature

$$(0; +; [2]; \{(p+1, p+1)\}). \tag{13}$$

We now examine Case d4). Equation (9) yields that

$$2mn = (p+1)(2n+m). \tag{14}$$

If $m = 2(p+1)$ then $n = p+1$ and so Λ has signature $(0; +; [2(p+1)]; \{(p+1)\})$. However this signature cannot occur, since it implies that the element $x \in \Lambda$ corresponding to the proper period has order $2(p+1)$ in G, which implies that G is cyclic. On the other hand, this signature also implies that G contains two elements of order two, whose product is $p+1$, which is a contradiction. Therefore $m \leq (p+1)$, however, this contradicts (14). Therefore Λ cannot have a signature of the form d4).

Therefore the possible signatures for Λ are

i. $(1; -; [2, p+1]; \{-\})$,
ii. $(0; +; [2, p+1]; \{(-)\})$,
iii. $(0; +; [2]; \{(p+1, p+1)\})$.

Associated to the above signatures are the following presentations for Λ:

i′. $\langle d, x_1, x_2 \mid x_1x_2d^2 = 1, x_1^2 = x_2^{p+1} = 1\rangle$,
ii′. $\langle e, x_1, x_2, c \mid x_1x_2e = 1, x_1^2 = x_2^{p+1} = 1, c^2 = 1, ec = ce\rangle$,
iii′. $\langle e, x, c_0, c_1, c_2 \mid xe = 1, x^2 = 1, c_0^2 = c_1^2 = c_2^2 = 1, ec_2e^{-1} = c_0, (c_0c_1)^{p+1} = (c_1c_2)^{p+1} = 1\rangle$.

We will show that there is an epimorphism $\theta : \Lambda \to G$ only if Λ has one of the signatures ii) or iii) above.

Assume that Λ has signature i) with the associated presentation listed above. Let H be the normal subgroup of G of order $p+1$. Clearly $\theta(x_2) \in H$, and $\theta(d^2) \in H$. Since x_1 has order 2, $\theta(x_1) \notin H$. Therefore $\theta(x_1x_2d^2) \notin H$. This contradicts that $x_1x_2d^2 = 1$.

We now determine the epimorphisms if Λ has signature ii) with its associated presentation above. Since Λ has elements of orders two and $p+1$, and G has order $2(p+1)$, we see that G must be a group with presentation (7). Let $\theta : \Lambda \to G$ be an epimorphism. We may assume that $\theta(x_1) = B$ and $\theta(x_2) = A$, therefore $\theta(e) = (BA)^{-1}$. Then $\theta(c) = A^kB$, for some integer k, and $\theta(c^2) = A^kBA^kB = A^kA^{k\alpha} = A^{k(\alpha+1)}$. Therefore

$$k(\alpha+1) \equiv 0 \mod p+1. \tag{15}$$

To satisfy the defining relations of Λ, we must have $\theta(ec) = \theta(ce)$, or equivalently, $BA^{k+1}B = A^kBBA$. This yields that

$$(k+1)(\alpha-1) \equiv 0 \mod p+1 \tag{16}$$

which combined with Equation (15), yields that $k \equiv 2^{-1}(\alpha-1) \mod p+1$. Since $\alpha^2 \equiv 1 \mod p+1$, this value of k satisfies both Equations (15) and (16). Therefore $\theta(c) = A^{(\alpha-1)/2}B$ or $A^{(\alpha+p)/2}B$ depending on whether α is odd or even respectively. We have shown a group G is an epimorphic image of Λ if and only if it has presentation (7). In addition, each epimorphism θ must have the above form.

We now consider signature iii) with its associated presentation for Λ above. It is clear that G must be dihedral since it contains two elements of order two whose product has order $p+1$. Let G have presentation (7) with $\alpha = -1$. We now determine the epimorphisms $\theta : \Lambda \to G$. Clearly we may assume that $\theta(c_0) = B$ and $\theta(c_1) = AB$. Then $\theta(c_2)$ must be of the form A^kB, where $k-1$ and $p+1$ are relatively prime. Since $\theta(x) \notin H$, $\theta(e) \notin H$. Thus $\theta(e) = A^jB$, and the relation $ec_2e^{-1} = c_0$ implies that $2j \equiv k \mod p+1$.

Therefore, given k, it follows that $j = k/2$ if k is even, or $(k+p+1)/2$ if k is odd. We have shown that for each k with $\gcd(k-1, p+1) = 1$, there exists an epimorphism onto G with $\theta(c_2) = A^k B$ and $\theta(x) = \theta(e) = A^{k/2}B$ or $A^{(k+p+1)/2}B$ depending on whether k is even or odd respectively.

Theorem 2.2. *Let W be a non-orientable Riemann surface of odd algebraic genus $p \geq 2$ which admits a group of automorphisms G of order $2p$. Then either $G \cong \mathrm{Z}_{2p}$ and W is hyperelliptic, or $G \cong \mathrm{D}_p$.*

Proof. We again employ Equation (9), with $\gamma = p$, in conjunction with signatures a) through d4).

If Λ has signature a) or c), then (9) easily yields that $n = p$. Therefore Λ has signature $(1; -; [-]; \{(p)\})$ or $(0; +; [-]; \{(-), (p)\})$. If Λ has signature b), then (9) yields that

$$1/2 > \frac{p-1}{2p} = 1 - 1/m_1 - 1/m_2. \tag{17}$$

Assume $m_1 \leq m_2$. Then (17) easily yields that $m_1 \leq 3$. If $m_1 = 2$, then (17) yields that $m_2 = 2p$, thus the signature of Λ is $(1; -; [2, 2p]; \{-\})$. If $m_1 = 3$, then (17) yields that $m_2 = 6 - 18/(p+3)$, therefore $m_2 = 3, 4$, or 5. However $m_2 = 4$ contradicts Proposition 1.2, while $m_2 = 3$ or 5 contradicts Proposition 1.3.

For Case d1), (9) yields $1/2 > (p-1)/(2p) = 3/2 - 1/m_1 - 1/m_2 - 1/(2n)$. This implies that neither m_1 nor m_2 can be greater than 2. This yields the signature $(0; +; [2, 2]; \{(p)\})$ for Λ.

The calculations for Case d2) are exactly the same as for Case b) and yield the signatures $(0; +; [2, 2p]; \{(-)\})$, $(0; +; [3, 3]; \{(-)\})$, and $(0; +; [3, 5]; \{(-)\})$. The last two signatures contradict Proposition 1.3.

For Case d3), (9) yields

$$1/2 > (p-1)/(2p) = 1 - 1/m - 1/(2n_1) - 1/(2n_2). \tag{18}$$

Assume $n_1 \leq n_2$. Clearly $m < 6$, since n_1 and n_2 are odd. From Proposition 1.2, $m \neq 4$, and from Proposition 1.3, $m \neq 3$ or 5. However, if $m = 2$, then (18) yields that $p(n_1 + n_2) = n_1 n_2$ which contradicts that n_1 and n_2 are both odd. Therefore Λ cannot have a signature of the form d3).

For Case d4), (9) yields that $mn = p(2n+m)$, which implies that $m > 2p$, which contradicts that m divides $2p$.

We now summarize the possible signatures for Λ found above. For each signature, we will determine the possible epimorphisms of Λ onto a group of order $2p$.

i. $(1; -; [-]; \{(p)\})$,
ii. $(0; +; [-]; \{(p), (-)\})$,
iii. $(1; -; [2, 2p]; \{-\})$,
iv. $(0; +; [2, 2]; \{(p)\})$,

v. $(0; +; [2, 2p]; \{(-)\})$.

Associated to each of the above signatures are the following presentations for Λ.

i. $\langle e, d, c_0, c_1 \mid ed^2 = 1, c_0^2 = c_1^2 = 1, ec_1e^{-1} = c_0, (c_0c_1)^p = 1\rangle$,
ii. $\langle e_1, e_2, c_{1,0}, c_{1,1}, c_{2,0} \mid e_1e_2 = 1, c_{1,0}^2 = c_{1,1}^2 = c_{2,0}^2 = 1, e_1c_{1,1}e_1^{-1} = c_{1,0}, e_2c_{2,0} = c_{2,0}e_2, (c_{1,0}c_{1,1})^{p+1} = 1\rangle$,
iii. $\langle d, x_1, x_2 \mid x_1x_2d^2 = 1, x_1^2 = x_2^{2p} = 1\rangle$,
iv. $\langle e, x_1, x_2, c_0, c_1 \mid x_1x_2e = 1, x_1^2 = x_2^2 = 1, c_0^2 = c_1^2 = (c_0c_1)^p = 1, ec_1e^{-1} = c_0\rangle$,
v. $\langle e, x_1, x_2, c \mid x_1x_2e = 1, x_1^2 = x_2^{2p} = 1, c^2 = 1, ec = ce\rangle$.

Assume Λ has signature i) with associated presentation i) above. Then $G \cong D_p$, since it contains two elements of order two whose product has order p. Let G have presentation (7) with $\alpha = -1$, and let $H = \langle A\rangle$. We may assume that $\theta(c_0) = B$, and $\theta(c_1) = AB$. Since $\theta(d^2)$ must have odd order, $\theta(d^2) \in H$, therefore, since $ed^2 = 1$, $\theta(e) \in H$, say $\theta(e) = A^k$. From $\theta(ec_1e^{-1}) = \theta(c_0)$, we deduce that $A^{2k+1} = 1$, so $\theta(e) = A^{(p-1)/2}$. If $\theta(d) \notin H$, then $\theta(d^2) = 1$, contradicting $ed^2 = 1$. Therefore $\theta(d) = A^j$, for some integer j. To satisfy the relation $ed^2 = 1$, we must have $2j \equiv (p+1)/2 \bmod p$ therefore, $\theta(d) = A^{(p+1)/4}$ or $A^{(3p+1)/4}$, depending, respectively, on whether $(p+1)/2$ is even or odd.

Assume Λ has signature ii) with associated presentation ii) above. Then clearly G is dihedral. Let G have presentation (7) with $\alpha = -1$ and let $H = \langle A\rangle$. We may assume that $\theta(c_{1,0}) = B$, $\theta(c_{1,1}) = AB$, and $\theta(c_{2,0}) = A^kB$ for some integer k. Note that $\theta(e_1) \neq 1$, therefore $\theta(e_2) \neq 1$. The only nonidentity element of G which commutes with A^kB is itself, therefore $\theta(e_2) = A^kB$. This yields that $\theta(e_1) = A^kB$ also. This, in conjunction with $\theta(e_1c_{1,1}e_1^{-1}) = \theta(c_{1,0})$ yields that k must be $(p+1)/2$. Therefore, $\theta(c_{2,0}) = \theta(e_1) = \theta(e_2) = A^{(p+1)/2}B$.

Assume Λ has signature iii) with associated presentation iii) above. In this case $G \cong Z_{2p}$ with presentation (8). We may assume that $\theta(x_1) = a^p$ and $\theta(x_2) = a$. But then $\theta(d) = a^{(p-1)/2}$. Observe that the inverse image of $\langle a^p\rangle$ has signature $(1; -; [2, \overset{p+1}{\ldots}, 2]; \{-\})$. So W is hyperelliptic, by [**5**].

Assume Λ has signature iv) with associated presentation iv) above. In this case, G is again dihedral. Assume G has the presentation (7) with $\alpha = -1$ and let $H = \langle A\rangle$. We may assume that $\theta(c_0) = B$ and $\theta(c_1) = AB$. Since $\theta(x_1)$ and $\theta(x_2)$ have order 2, they do not lie in H, and since $x_1x_2e = 1$, $\theta(e) \in H$. Thus $\theta(e) = A^k$, where k is chosen so that $\theta(ec_1e^{-1}) = c_0$. This yields that $k = (p-1)/2$. Therefore $\theta(x_1) = A^sB$ and $\theta(x_2) = A^tB$ where s and t are chosen so that $A^sBA^tB = A^{(p+1)/2}$. Therefore s and t can be any integers such that $2(s-t) = p+1$.

Assume Λ has signature v) with associated presentation v) above. Then G is cyclic with presentation (8). Therefore $\theta(x_1) = a^p$, $\theta(x_2) = a$, $\theta(e) = a^{p-1}$ and $\theta(c) = a^p$. In this case, the inverse image of $\langle a^p \rangle$ has signature $(0; +; [2, \overset{p+1}{\ldots}, 2]; \{(-)\})$, therefore W is hyperelliptic.

If m is odd, a group with presentation (7) has a nontrivial center if and only if it is cyclic. Therefore we have the following corollary.

Corollary 2.3. *Let W be a non-orientable Riemann surface of algebraic genus $p \geq 2$ which admits a group of automorphisms G of order $2p+2$, if p is even, or $2p$, if p is odd. Then W is hyperelliptic if and only if G is cyclic.*

3. Defining equations.

We now determine the defining equations for the Klein surfaces whose signatures were determined in the previous section. We do this by determining the Riemann double cover of each Klein surface and the conformal automorphism group of order 2γ. We then determine the symmetry of the Riemann double cover which yields the Klein surface as its quotient.

We continue to use the notation established earlier. Let W denote the Klein surface with a group G of 2γ automorphisms whose defining equations we seek. Let $X = W/G$. Since G contains a normal subgroup H of index two, we let Z denote the Klein surface W/H. The double covers of W, Z, and X, will be denoted by $\hat{W}, \hat{Z}$, and $\hat{X}$ respectively. Let σ denote the symmetry acting on $\hat{W}$, such that $\hat{W}/\langle\sigma\rangle = W$. The groups G and G/H induce automorphism groups of $\hat{W}$ and $\hat{Z}$ respectively. We shall use the same notation for the elements of G and their induced actions on the Riemann double covers. Note that Z must be non-orientable, since W is non-orientable and the order of H is odd. In addition, the symmetry σ, acting on $\hat{Z}$ must be fixed point free. To see this, assume it possesses a fixed point z_0. Above z_0, there are an odd number of points of $\hat{W}$, say $w_1, \ldots, w_k$. On $\hat{W}$, σ is fixed point free, therefore σ permutes $w_1, \ldots, w_k$. This contradicts that a permutation of order two cannot act without fixed points on an odd number of objects.

In the following sections we compute defining equations for $\hat{W}$, explicitly determine the symmetry σ such that $W = \hat{W}/\langle\sigma\rangle$, and explicitly determine its automorphism group of order 2γ. The only family of surfaces whose equations we do not compute is the one corresponding to the signature $(0; +; [2, 2]; \{(p)\})$ found in the proof of Theorem 2.2. The computations for this family are quite difficult, due to the fact that this family has real dimension two in Teichmüller space, while the other families listed in the proofs of Theorems 2.1 and 2.2 have real dimension one.

Proposition 3.1. *The following basic results concerning defining equations will be freely used.*

i. *Assume* $w^n - r(x)/s(x)$ *is a defining equation of* $\hat{W}$, *where* $r(x)$ *and* $s(x)$ *are relatively prime polynomials. Then there is a defining equation for* $\hat{W}$ *of the form* $w_1^n - f(x)$, *where* $f(x)$ *is a polynomial. One such equation is obtained by defining* $w_1 := s(x)w$ *and* $f(x) = r(x)s^{n-1}(x)$.

ii. *Assume* $\hat{W}$ *has a defining equation of the form* $w^{dn} - (x-a)^{dm} f(x)$, *where* $(m,n) = 1$, $(n,d) = 1$ *and* $f(x)$ *is a polynomial. Then there is a defining equation for* $\hat{W}$ *of the form* $w_1^{dn} - (x-a)^d f_1(x)$. *One such equation can be obtained in the following way. Since* $(m,n) = 1$, *there exist integers* u *and* v *such that* $mu = 1 - nv$. *We can choose* u *so that* $(d,u) = 1$. *Thus* $w^{dnu} = (x-a)^{dmu} f^u(x) = (x-a)^{d-dnv} f^u(x)$. *Thus define* $w_1 = (x-a)^v w^u$ *and* $f_1(x) = f^u(x)$. *Since* $(u, dn) = 1$, *note that* w *can be expressed in terms of* w_1 *and* x, *thus* $C(w_1, x) = C(w, x) = C(\hat{W})$.

iii. *Let* $w^n - (x-a)^m f(x)$ *be a defining equation for* $\hat{W}$, *where* $(x-a)$ *and* $f(x)$ *are relatively prime polynomials. Assume that* $m = nu + v$, *where* $u \geq 1$. *Then* $\hat{W}$ *has a defining equation of the form* $w_1^n - (x-a)^v f(z)$, *by defining* $w_1 := w/(x-a)^u$.

iv. *Let* $w^n - r(x)/s(x) = 0$ *be a defining equation for* $\hat{W}$, *and assume* $r(x)$ *and* $s(x)$ *are polynomials with no factors in common. Assume* $(x-a)^m$ *divides either* r *or* s *and* $(x-a)^{m+1}$ *does not. Let* d *denote the greatest common factor of* m *and* n. *Then there exists* d *points of* $\hat{W}$ *which lie over the point* $a \in \hat{X}$ *and the ramification index there is* n/d.

v. *Assume* $C(\hat{W})$ *is a cyclic Galois extension of degree* n *of* $C(\hat{X})$. *Assume that* $C(\hat{W}) = C(\hat{X})[w]$, *and that the automorphism* A *is the identity on* $C(\hat{X})$ *but* $A(w) = \epsilon w$, *where* ϵ *is a primitive* n*th unit of unity. Assume* λ *is a symmetry or an automorphism of* $C(\hat{W})$ *of order 2, and assume* $\lambda A \lambda = A^\alpha$. *Let* $\lambda(w) = a_0 + a_1 w + \cdots + a_{n-1} w^{n-1}$, *where each* $a_i \in C(\hat{X})$. *Then*

$$A \circ \lambda(w) = a_0 + a_1(\epsilon w) + \cdots + a_{n-1}(\epsilon w)^{n-1},$$

$$\lambda \circ A^\alpha(w) = \epsilon^{i\alpha}(a_0 + a_1 w + \cdots + a_{n-1} w^{n-1}),$$

where $i = -1$ *if* λ *is a symmetry, but* $i = 1$ *otherwise. We obtain that each* $a_j = 0$ *except for* a_α, *if* $i = 1$, *and* $a_{n-\alpha}$, *if* $i = -1$. *In particular, we have the following cases. If the automorphism* B *has order two and* $BAB = A^{-1}$, *then* $B(w) = h_1/w$, *for some* $h_1 \in C(\hat{X})$. *If* λ *is a symmetry which commutes with* A, *then* $\lambda(w) = h_2/w$ *for some* $h_2 \in C(\hat{X})$.

vi. *Assume that the polynomial* $F(z,w)$ *is a defining equation for a Riemann surface* $\hat{W}$ *and* (r,s) *is a nonsingular solution of* F. *If* $\hat{W}$ *admits a symmetry* σ, *then there exits an induced symmetry* σ *on* $C(\hat{W})$ *such that* $\sigma(i) = -i$. *Note that* (r,s) *is the unique point on* F *which satisfies*

$z - r = 0$ and $w - s = 0$. Then (r, s) is a fixed point of σ if and only if (r, s) is also a solution of $\sigma(z) - \bar{r}$ and $\sigma(w) - \bar{s}$. In particular, we have the following cases. Assume (r, s) is a fixed point of σ. If $\sigma(z) = z$, then r is real. If $\sigma(z) = -z$, then r is pure imaginary. If $\sigma(z) = 1/z$, then r is a complex number with $\mid r \mid = 1$. It is not possible for (r, s) to be a fixed point of σ and $\sigma(z) = -1/z$.

3.1. Defining equations for even p.

3.1.1. The signature $(0; +; [2, p+1]; \{(-)\})$. Assume that Λ from the proof of Theorem 2.1 has the above signature. Then its associated Fuchsian group Λ^+ has signature $(0; +; [2, 2, p+1, p+1]; \{-\})$ and the four points of $\hat{X} := U/\Lambda^+$ fixed by the elliptic elements of Λ^+ lie above interior points of $X := U/\Lambda$. Assume that G satisfies presentation (7). Note that $\hat{Z} := \hat{W}/\langle A \rangle$ has genus zero, since two points of $\hat{X}$ are ramified in it. From the remark at the beginning of this section, the induced action of σ on $\hat{Z}$ is fixed point free.

Let us choose coordinates for $C(x) = C(\hat{X})$ so that the induced action of σ on $C(x)$ is conjugation. Assume that $x = a \pm bi$ are the points of $\hat{X}$ with ramification index two in $\hat{Z}$, where a and b are real and $b \neq 0$. By the real change of coordinates $x \mapsto (2/b)(x - a)$, we may assume that $x = \pm 2i$ are the points ramified in the covering of $\hat{X}$ by $\hat{Z}$. Let us choose coordinates for $\hat{Z}$ so that $z = i$ and $z = -i$ lie over $x = 2i$ and $x = -2i$ respectively. In addition, let us choose coordinates so that $z = \infty$ is one of the points lying over $x = \infty$. Note that B is an automorphism of order two such that $\hat{Z}/\langle B \rangle = \hat{X}$. Since $z = \pm i$ are the ramified points, they are fixed by B. This uniquely identifies B as the map $B(z) = -1/z$. Note that $x_0 := z - 1/z$ is fixed by B, therefore $x_0 \in C(x)$. However, $x_0 = x$, since each function agrees at the points $z = i, z = -i$, and $z = \infty$. Note that z satisfies the minimal polynomial $z^2 - zx - 1 = 0$. The roots of this are z and $-1/z$. Since $\sigma(x) = x$, we have that $\sigma(z) = z$ or $\sigma(z) = -1/z$. The first of these yields that $\hat{Z}$ has fixed points under σ, therefore $\sigma(z) = -1/z$.

Let $c + di$, with c and d real, be one of the points of $\hat{Z}$ with ramification index $p + 1$. Note that $d \neq 0$, otherwise $c + di$ lies over a point on the boundary of X. Note that a real transformation of the form

$$z \mapsto \frac{\beta z - 1}{z + \beta} \tag{19}$$

fixes both i and $-i$. If $c \neq 0$, define

$$\beta = -\frac{\sqrt{c^4 + 2c^2(d^2+1) + (d^2-1)^2} + c^2 + d^2 - 1}{2c}. \tag{20}$$

Then β is a real number, and one can check that the map (19) maps $c+di$ to a complex number with real part equal to 0. Thus we may assume that the ramified points are $z = ki, -ki, 1/(ki)$ and $-1/(ki)$. The above change of coordinates still yields that $B(z) = -1/z$ and $\sigma(z) = -1/z$.

Since $C(\hat{W})$ is a cyclic extension of $C(z)$, Proposition 3.1 yields that there is a defining equation of $\hat{W}$ of the form

$$w^{p+1} - (z-ki)(z+ki)^v(z-1/ki)^r(z+1/ki)^s = 0 \tag{21}$$

where each of v, r, and s are between 0 and $p+1$ and relatively prime to $p+1$. In addition, since ∞ is not ramified, $p+1$ divides $1+v+r+s$. Let A be the map which is the identity on $C(\hat{Z})$ and which maps w to ϵw. Since σ and A commute, from Proposition 3.1 we have that $\sigma(w) = h_1(z)/w$ for some $h_1(z) \in C(z)$. Applying this to (21) yields that

$$h_1(z)^{p+1} = w^{p+1}(-1/z+ki)(-1/z-ki)^v(-1/z+1/ki)^r(-1/z-1/ki)^s.$$

Thus $h_1^{p+1} z^{1+v+r+s} =$

$$(-1)^{v+s}(ki)^{1+v-r-s}(z-ki)^{1+r}(z+ki)^{v+s}(z-1/ki)^{1+r}(z+1/ki)^{s+v}, \tag{22}$$

therefore $p+1$ divides both $1+r$ and $v+s$, so $r = p$ and $s = p+1-v$. This implies that $(-1)^{v+s}(ki)^{1+v-r-s} = (-1)^{v+1}k^{2v-2p}$ and $h_1 = \eta_1(z-ki)(z+ki)(z-1/ki)(z+1/ki)/z^2$, where η_1 is a $p+1$st root of $k^{2v-2p}(-1)^{v+1}$. Since σ has order two and fixes $(z-ki)(z+ki)(z-1/ki)(z+1/ki)/z^2$, we deduce that η_1 must be real. A defining equation for $\hat{W}$ is

$$w^{p+1} = f(z) := (z-ki)(z+ki)^v(z-1/ki)^p(z+1/ki)^{p+1-v} = 0. \tag{23}$$

Recall that $AB = BA^\alpha$ and that $B(z) = -1/z$. We may assume that $0 \le \alpha < p+1$. From (v) of Proposition 3.1 we deduce that $B(w) = h_2(z)w^\alpha$ for some $h_2(z) \in C(z)$. Applying this to (23) above yields

$$\begin{aligned}&(h_2(z))^{p+1}\\ &= f(z)^{-\alpha}(-1/z-ki)(-1/z+ki)^v(-1/z-1/ki)^p(-1/z+1/ki)^{p+1-v}.\end{aligned}$$

Thus $h_2(z)^{p+1}$ equals

$$\frac{(z-ik)^{p+1-v-\alpha}(z+ik)^{p-v\alpha}(z-1/ik)^{v-p\alpha}(z+1/ik)^{1-(p+1-v)\alpha}}{(-1)^{v+1}k^{2p-2v}z^{2p+2}}. \tag{24}$$

Considering the power of $z-ik$, this implies that $v = p+1-\alpha$. Since $\alpha^2 \equiv 1 \bmod p+1$, (24) is a $(p+1)$st power. If k_2 is chosen so that $(p+1)k_2 = 1-\alpha^2$, then

$$h_2(z) = \frac{\epsilon^j \eta_1}{z^2}(z+ik)^{1-\alpha-k_2}(z-1/ik)^{1-\alpha}(z+1/ik)^{k_2} \tag{25}$$

where $j \in \mathbb{Z}$ and ϵ is a primitive $(p+1)$st root of unity. A tedious calculation shows that $B^2(w) = \epsilon^{j(\alpha+1)}w$, so $j(\alpha+1) \cong 0 \bmod p+1$. By redefining B as

$A^{-j}B$, we may assume $j = 0$. With this definition of $h_2(z)$, $B(w) = h_2(z)w^\alpha$ and $BAB = A^\alpha$.

We now check that B and σ commute. Recall that $\sigma(w) = h_1/w$, where $h_1 = \eta_1(z - ik)(z + ik)(z - 1/ik)(z + 1/ik)$. However, since $v = p + 1 - \alpha$, we see that $\eta_1 = \eta_2$. Since $B\sigma(z) = \sigma B(z)$, it is sufficient to show that $B\sigma(w) = \sigma B(w)$. However $B\sigma(w) = B(h_1(z)/w) = B(h_1(z))/(h_2(z)w^\alpha)$. On the other hand, $\sigma B(w) = \sigma(h_2(z)w^\alpha) = \sigma(h_2)h_1^\alpha/w^\alpha$. A tedious calculation shows that $B(h_1) = h_2\sigma(h_2)h_1^\alpha$. Therefore, $B\sigma(w) = \sigma B(w)$, thus B and σ commute.

We have shown that a defining equation for $\hat{W}$ is

$$w^{p+1} - (z - ik)(z + ik)^{p+1-\alpha}(z - 1/ik)^p(z + 1/ik)^\alpha = 0.$$

In addition to A, it possesses the following symmetries and automorphisms:

$$\sigma(z) = z, \quad \sigma(w) = \eta_1(z^2 + k^2)(z^2 + 1/k^2)/z^2w,$$
$$B(z) = -1/z, \quad B(w) = h_2(z)w^\alpha,$$

where $h_2(z)$ is defined in (25) with $j = 0$, and η_1 is the real $p + 1$st root of $(-1)^\alpha k^{2-2\alpha}$. Since σ is fixed point free on $C(z)$, σ is fixed point free on $\hat{W}$, therefore the Klein surface $W = \hat{W}/\langle\sigma\rangle$ has no boundary.

3.1.2. The signature $(0; +; [2]; \{(p + 1, p + 1)\})$. Assume Λ from the proof of Theorem 2.1 has the above signature. In this case $G \cong D_{p+1}$ and two boundary points of X and one interior point of X are ramified in W. Note that $\hat{X}$ has genus 0 and assume coordinates are chosen so that the action of σ on $\hat{X}$ is conjugation. By a real change of coordinates, we may assume ∞ is one of the fixed points of $\hat{X}$ which has ramification index $p + 1$ in $\hat{W}$. Note that two points of the form $a \pm bi$ of $\hat{X}$ are ramified with index 2 in $\hat{W}$, where a and b are real and $b \neq 0$. By making the change of coordinates $x \mapsto 2(x - a)/b$, we may assume that the points $\pm 2i$ are ramified with ramification index 2 in $\hat{W}$. Let $\hat{Z}$ be the orbit space of $\hat{W}$ under the action of $\langle A\rangle$. Exactly as in Section 3.1.1 we may assume that $C(\hat{Z}) = C(z)$, where $x = z - 1/z$, $B(z) = -1/z$ and $\sigma(z) = -1/z$. Note that $z = 0$ and $z = \infty$ are the two points lying over $x = \infty$. There is another point of $\hat{X}$ which has ramification index $p + 1$ in $\hat{W}$, and let $z = k$ be one of the points of $\hat{Z}$ which lies over it. Then the point must be $x = k - 1/k$ and the two points lying over it are $z = k$ and $z = -1/k$. It is easy to see, since $k - 1/k$ is real, that k is real.

To obtain the equation of W, we note that the $C(\hat{W})$ is a field extension of $C(z)$ of index $p + 1$ in which $z = 0, k, -1/k$, and ∞ are all ramified. Therefore a defining equation of W is of the form

$$w^{p+1} - z(z - k)^u(z + 1/k)^v$$

where u and v are between 0 and $p+1$, relatively prime to $p+1$, and $1+u+v$ is also relatively prime to $p+1$. Let $A(w) = \epsilon w$, where ϵ is a primitive $p+1$st root of unity. We must show that the maps $B(z) = -1/z$ and $\sigma(z) = -1/z$ lift to $\hat{W}$ with the properties that $BA = A^{-1}B$ and σ commutes with both A and B.

From Proposition 3.1, we have that $\sigma(w) = f/w$ and $B(w) = g/w$ for some f and $g \in C(z)$. Note that both B and σ map $z-k \mapsto -1/z-k = (-k/z)(z+1/k)$ and $z+1/k \mapsto -1/z+1/k = (1/zk)(z-k)$. Therefore letting $h = f$ or g depending on whether we are considering σ or B, we obtain

$$\begin{aligned} 0 &= h^{p+1} - w^{p+1}(-1/z)(-k/z)^u(1/zk)^v(z+1/k)^u(z-k)^v \\ &= h^{p+1} - (-1)^{1+u}k^{u-v}(z-k)^{u+v}(z+1/k)^{u+v}/z^{u+v}. \end{aligned}$$

Therefore $v = p+1-u$. Let λ be the real $p+1$st root of $(-1)^{1+u}k^{2u-p-1}$, then $f = \epsilon_1\lambda(z-k)(z+1/k)/z$ and $g = \epsilon_2\lambda(z-k)(z+1/k)/z$, where ϵ_1 and ϵ_2 are $p+1$st roots of unity. Since σ is a symmetry, $w = \sigma^2(w)$ yields that $\epsilon_1 = 1$. By redefining B as A^jB for an appropriate integer j, we may assume that $\epsilon_2 = 1$. Thus a defining equation for $\hat{W}$ is $w^{p+1} - z(z-k)^u(z+1/k)^{p+1-u}$, where u is relatively prime to $p+1$. In addition, $\hat{W}$ possesses the following automorphisms and symmetries:

$$A(w) = \epsilon w, A(z) = z, B(w) = \lambda(z-k)(z+1/k)/(zw), B(z) = -1/z,$$

$$\sigma(w) = \lambda(z-k)(z+1/k)/(zw), \sigma(z) = -1/z, \sigma(i) = -i,$$

where λ is a real number such that $\lambda^{p+1} = (-1)^{1+u}k^{2u-p-1}$.

3.2. Defining equations for odd p.

3.2.1. The signatures $(1; -; [-]; \{(p)\})$ and $(0; +; [-]; \{(-), (p)\})$.

Let Λ have one of the above signatures. Note that $\hat{X}$ has genus one and recall that $G \cong D_p$. The two signatures are distinguished by the number of fixed ovals of σ acting on $\hat{X}$; the first yields one fixed oval and the latter yields two. If we define $\hat{Z} = \hat{W}/\langle A\rangle$, we see that $\hat{Z}$ is an unramified cover of $\hat{X}$, therefore $\hat{Z}$ has genus one also. There is a distinguished boundary point of X which has ramification index p in W. Lying above this point there is a unique point of $\hat{X}$ and there are two points of $\hat{Z}$. Let us choose coordinates for $\hat{Z}$ so that one of these two points is the point at infinity of a defining equation of the form $y^2 - f(z) = 0$, where $f(z)$ has three distinct zeros. Recall that B is an automorphism of order 2 acting on $\hat{Z}$, such that $\hat{X} = \hat{Z}/\langle B\rangle$. Since B is fixed point free, B is a translation of order two in the group structure of $\hat{Z}$, therefore ∞ is mapped to one of the roots of $f(z)$.

We may make a change of coordinates to assume that $z = 0$ is the root of $f(z)$ to which ∞ is mapped under B. Thus $B(z) = c^2/z$ for some complex number c^2. By making the change of coordinates $z := z/c$, we may assume that $c = 1$.

Therefore $B(z) = 1/z$ is an automorphism of the defining equation

$$y^2 - z(z-a)(z-b), \tag{26}$$

of $\hat{Z}$. Note that $B(y)$ must be of the form $r(z)y/s(z)$, where $r(z)$ and $s(z)$ are relatively prime polynomials. Therefore $z^3r^2y^2 - abs^2(z-1/a)(z-1/b) = 0$ and so $z^4r^2(z-a)(z-b) - abs^2(z-1/a)(z-1/b) = 0$. If $z-a$ does not divide $(z-1/a)(z-1/b)$, it must divide $s(z)$. But then $(z-a)$ divides $r(z)$, a contradiction. Therefore we may assume that $z-a$ divides $(z-1/a)(z-1/b)$. If $a = 1/a$, then $a = \pm 1$, and similarly, $b = 1/b$, so $b = \pm 1$. Therefore $y^2 - z(z^2-1)$, and $B(y) = \pm iy/z^2$. However, in this case, the point corresponding to $z = 1, y = 0$ is fixed by B, thus B is not fixed point free. Therefore, $a = 1/b$, and we may assume a defining equation for $\hat{Z}$ is

$$y^2 - z(z-a)(z-1/a), \tag{27}$$

with $B(y) = -y/z^2$.

Before we compute the symmetry acting on $\hat{Z}$, we first determine the orbit space of $\hat{X} = \hat{Z}/\langle B \rangle$. This will help us determine the action of σ on $\hat{Z}$ that will produce one or two fixed ovals in $\hat{X}$. Note that both $t := z + 1/z$ and $u := y - y/z^2$ are fixed by B. It is easy to verify that they satisfy the defining equation

$$u^2 - (t-2)(t+2)(t-(a^2+1)/a) \tag{28}$$

and that $C(\hat{X}) = C(t,u)$.

Recall that σ commutes with B and is fixed point free on $\hat{Z}$. Since $z = \infty$ lies above a boundary point of X, we deduce that σ interchanges ∞ and $B(\infty)$. Therefore $\sigma(z) = c/z$, where, since σ is a symmetry, $\bar{c}/c = 1$, therefore c is real. However, $\sigma B(z) = \sigma(1/z) = z/c$, while $B\sigma(z) = cz$. Therefore $c = \pm 1$ and $\sigma(z) = 1/z$ or $-1/z$. We now show that $\sigma(z) = -1/z$ if the signature of Λ is $(1; -; [-]; \{(p)\})$ and $\sigma(z) = 1/z$ if the signature of Λ is $(0; +; [-]; \{(-), (p)\})$.

Assume $\sigma(z) = 1/z$ and let σ act on the defining equation (27). Using that $\sigma(y) = r(z)y/s(z)$ for some relatively prime polynomials $r(z)$ and $s(z)$, we deduce that $z^3r^2y^2 - s^2(z-1/\bar{a})(z-\bar{a}) = 0$. As before, we deduce that $r/s = \pm 1/z^2$, so $\sigma(y) = \pm y/z^2$, and that $a = 1/\bar{a}$ or $a = \bar{a}$. Since $\sigma(z) = 1/z$, a point (β, γ) satisfying (27) will have its first coordinate fixed by σ if and only if $|\beta| = 1$. If $a = 1/\bar{a}$, the point $(a, 0)$ is fixed by σ. Therefore, $a \neq 1/\bar{a}$ and thus $a = \bar{a}$. We know that $\sigma(y) = \pm y/z^2$; we now determine which sign should be chosen. Let r be real and let $P = (e^{ir}, \gamma)$ be a point satisfying (27). We must define σ so that P is not a fixed point. However

P is fixed if and only if $\pm y/z^2 - \bar{\gamma}$ has a zero at P. This occurs if and only if $\pm\gamma/e^{2ir} - \bar{\gamma} = 0$, which implies $\pm\gamma^2/e^{2ir} - |\gamma|^2 = 0$, which implies that $\pm(e^{ir} - a)(e^{ir} - 1/a)/e^{ir} = |(e^{-ir} - a)(e^{-ir} - 1/a)|$. Simplifying the last equality yields $\pm(2\cos(r) - (a^2+1)/a) = |2\cos(r) - (a^2+1)/a|$. If $a > 0$, the left hand side is negative, so the plus sign should be chosen. If $a < 0$, the left hand side is positive, so the minus sign should be chosen. In this way, σ will have no fixed points on $\hat{Z}$. In summary, we have that if $\sigma(z) = 1/z$, then a is real and $\sigma(y) = y/z^2$ if $a > 0$ while $\sigma(y) = -y/z^2$ if $a < 0$.

We now show that if $\sigma(z) = 1/z$, then $\hat{X}$ has two fixed ovals. From the definitions $t = z + 1/z$ and $u = y - y/z^2$, we see that t is fixed by σ and $\sigma(u) = \pm u$, where the sign depends on whether a is positive or negative. Therefore fixed points of $\hat{X}$ are the points (r, γ), satisfying (28) such that r is real and γ is real or pure imaginary, depending on whether $\sigma(u) = u$, or $-u$ respectively. Considering (28), we see that there will always be two intervals for t for which $(t-2)(t+2)(t-(a^2+1)/a)$ is positive and two intervals for which it is negative. Therefore, independent of the definition of $\sigma(u)$, there are two ovals of $\hat{X}$ fixed by σ.

Now assume $\sigma(z) = -1/z$. We will determine what this implies about a, $\sigma(y)$, and the number of fixed ovals of $\hat{X}$. As before, applying σ to (27) yields that $\sigma(y) = ry/s$ and $r^2y^2 + s^2(1/z)(-1/z - \bar{a})(-1/z - 1/\bar{a}) = 0$. This yields $z^4r^2(z-a)(z-1/a) + s^2(z+\bar{a})(z+1/\bar{a}) = 0$. Since $a = -1/\bar{a}$ implies $|a| = -1$, we conclude that $a = -\bar{a}$, so $a = ik$ for some real number k and $\sigma(y) = \pm iy/z^2$. Since $\sigma(z) = -1/z$, we conclude that no point satisfying (27) will have the same z coordinate under σ, therefore σ is fixed point free on $\hat{Z}$. We now examine its action on $\hat{X}$. Note that $\sigma(t) = -t$ and $\sigma(u) = \pm iu$. Let $\eta^2 = i$. Then σ fixes $t_1 := it$ and $u_1 := \eta u$ or $\eta^3 u$, depending on whether $\sigma(u)$ equals iu or $-iu$ respectively. When expressed in terms of t_1 and u_1, (28) becomes

$$u_1^2 \pm (t_1^2 + 4)(t_1 + (k^2 - 1)/k).$$

Since $t_1^2 + 4 > 0$, this yields only one fixed oval, regardless of the plus or minus sign. In summary, if $\sigma(z) = -1/z$, then $a = ik$, where k is real, $\sigma(y) = \pm iy/z^2$, and one oval of $\hat{X}$ is fixed by σ.

Recall that the point at infinity of (28) is ramified in $\hat{W}$ with ramification index p. Therefore the points $Q := (0,0)$ and the point at infinity P of $\hat{Z}$ are ramified in $\hat{W}$. Since the covering of $\hat{Z}$ by $\hat{W}$ is cyclic, the field extension $C(\hat{W})$ of $C(z,y)$ has a defining equation of the form

$$w^p - h(z,y) = 0, \tag{29}$$

where the multiplicity of each pole and zero of $h(z,y)$ is a multiple of p except at P and Q. At these points, the multiplicity must be relatively prime to p. Note that the function z has the divisor $(z) = 2Q - 2P$ on $\hat{Z}$. We will show that, up to isomorphism, powers of z are the only functions we need to consider for $h(z, y)$ in (29).

Assume the divisor for h is $(h) = p(k_1R_1 + k_2R_2 + \cdots + k_nR_n) + d_1P + d_2Q$, where d_1 and d_2 are relatively prime to p and each $k_i \in Z$. Assume that a pole occurs at P, and that $d_1 = -(rp + s)$ where r and s are positive. Let $g \in C(\hat{Z})$ be a function which has a simple pole at P and which does not have a zero or pole at Q. Then

$$(w/g^r)^p - h/g^{rp} = 0$$

is also a defining equation for X. In addition, the pole divisor at P is s, where $s < p$. A similar result holds for Q, and zeros at P or Q can also be handled in this manner. In this way, we may assume that h has a pole at P and a zero at Q and the pole and zero each have degree less than p. Assume h is not a power of z. Since the divisor for h has degree zero we may now assume that the defining equation for $C(\hat{W})$ is (29), where the divisor for h is of the form

(30)
$$(h) = p(j_1R_1 + j_2R_2 + \dots j_mR_m) - p(k_1S_1 + k_2S_2 + \dots k_nS_n) - dP + dQ,$$

and a zero occurs at each of the R_i's and a pole occurs at each of the S_i's. Consider the divisor $D := -(j_1 - 1)R_1 - j_2R_2 - j_3R_3 \cdots - j_mR_m + k_1S_1 \cdots + k_nS_n$. This divisor has degree 1. By the Riemann Roch theorem, there is a function $g \in C(\hat{Z})$ such that $(g) + D \geq 0$, in particular, g has a zero of order at least $j_1 - 1$ at R_1 and j_i at each R_i for $i > 1$. In addition, it's pole divisor is contained in the pole divisor of D. There are two possibilities. Either the multiplicity of the zeros of g are precisely the same as the zeros in D, (and the pole divisor has degree $k_1 + k_2 + \dots k_n - 1$), or g also has a zero at a point T and the pole divisor of g agrees precisely with that of D. Note that T may be one of the points of (30). In either case, redefining $h = h/g^p$ and $w = w/g$ we have that

$$(h) = pR_1 - pT - dP + dQ. \tag{31}$$

Recall, from (v) of Proposition 3.1, that $B(w) = k(z, y)/w$ for some $k(z, y) \in C(z, y)$. Since B as order 2, we obtain

$$w = B^2(w) = \frac{B(k(z, y))}{k(z, y)} w,$$

so B fixes $k(z, y)$, so $k(z, y) \in C(\hat{X})$. Using the defining equation (29) we obtain that $k^p(z, y)/w^p = B(h)$, thus $k^p(z, y) = hB(h)$. Thus (31) yields that the divisor for $k(z, y)$ is $R_1 + B(R_1) - T - B(T)$. However, since $k(z, y) \in C(\hat{X})$ and $\hat{X}$ is elliptic, we know that $[C(\hat{X}) : C(k)] \geq 2$, thus $[C(\hat{Z}) : C(k)] \geq 4$. On the other hand, since the pole divisor of $k(z, y)$ has degree at most 2, we have $[C(\hat{Z}) : C(k)] \leq 2$ unless $k(z, y)$ is a constant. Thus $k(z, y)$ is a constant, so B switches R_1 and T. If $R_1 \in \{P, Q\}$, then so is T

and therefore h is a constant multiple of a power of z. We now assume that $R \notin \{P, Q\}$.

Recall that B is a fixed point free automorphism of the elliptic curve $\hat{Z}$. In the group structure of $\hat{Z}$, B corresponds to the map $S \mapsto S + Q$. Therefore, in the group structure $T = R_1 + Q$ and (31) becomes $(h) = pR_1 - p(R_1 + Q) - dP + dQ$. However, a divisor is the divisor of a function in $C(\hat{Z})$ if and only if it has degree zero and its sum is zero in the group structure of $\hat{Z}$. Since p is odd and Q has order two, we obtain that d is odd. In addition, there is a function h_1 with divisor $(h_1) = R_1 - T + P - Q = R_1 - (R_1 + Q) + P - Q$. But then h/h_1^p has divisor $(p + d)Q - (p + d)P$. Since d is odd, this yields that h/h_1^p is a constant multiple of a power of the function z. We may redefine h_1 to absorb this constant, so (29) yields that $(w/h_1)^p - z^j = 0$, for some positive integer j. From (ii) of Proposition 3.1, we may assume $j = 1$ and $\hat{W}$ has defining equations of the form

$$w^p - z = 0, \quad y^2 - z(z - a)(z - 1/a), \tag{32}$$

where a is real if $\sigma(z) = 1/z$, and $a = ik$, with k real, if $\sigma(z) = -1/z$.

We now determine the automorphisms of $\hat{W}$. The map $A(w) = \epsilon w$, $A(z) = z$, $A(y) = y$, where ϵ is a primitive pth root of unity is clearly an automorphism of $\hat{W}$. From (v) of Proposition 3.1, $\sigma(w) = k_1/w$ and $B(w) = k_2/w$ for some k_1 and k_2 in $C(\hat{Z})$.

If $\sigma(z) = 1/z$, applying σ to (32) yields $k_1^p - 1 = 0$, therefore k_1 is a pth root of unity. We may redefine $w = \epsilon^j w$, where j is an appropriate integer, to obtain $\sigma(w) = 1/w$. This does not change the defining equations (32) or any results concerning $\hat{Z}$. Applying B to (32) yields that $k_2 = \epsilon^j$ for some integer j. Given this, it is trivial to check that $\sigma B(w) = B\sigma(w)$. We now redefine B as $A^j B$ to obtain the simplification $B(w) = 1/w$. This change merely concerns the representation of the dihedral group $\langle A, B\rangle$ and does not change the defining equations or the action of B on $\hat{Z}$.

In a similar manner, if $\sigma(z) = -1/z$, then $\hat{W}$ has defining equations (32), and possesses the automorphism $B(w) = 1/w$ and the symmetry $\sigma(w) = -1/w$.

In summary, if $\hat{W}$ is defined as in (32), then $\hat{W}$ has the automorphisms

$$A(w) = \epsilon w \;\; A(z) = z \;\; A(y) = y \tag{33}$$

$$B(w) = 1/w \;\; B(z) = 1/z \;\; B(y) = -y/z^2.$$

If Λ has signature $(0; +; [-]; \{(-), (p)\})$ then a in (32) is real and $\hat{W}$ has the symmetry

$$\sigma(w) = 1/w, \;\; \sigma(z) = 1/z, \;\; \sigma(y) = \pm y/z^2 \tag{34}$$

where the plus sign is chosen if $a > 0$ and the minus sign is chosen if $a < 0$. If Λ has signature $(1; -; [-]; \{(p)\})$, then in (32), $a = ik$, where k is real. In addition, $\hat{W}$ has the symmetry

$$\sigma(w) = -1/w, \ \ \sigma(z) = -1/z, \ \ \sigma(y) = \pm iy/z^2. \tag{35}$$

In (35), either sign can be chosen in the definition of $\sigma(y)$.

3.2.2. The signature $(1; -; [2, 2p]; \{-\})$.

Assume Λ has the above signature. Then $\hat{X}$ has genus 0, and σ is fixed point free on $\hat{X}$. Thus the points of X of ramification index 2 and $2p$ each have two points of $\hat{X}$ lying over them. Let us choose coordinates so that $x = 0$ and $x = \infty$ are the points of $\hat{X}$ which have ramification index $2p$ and $x = -1$ is one of the points with ramification index 2. Then σ interchanges $x = 0$ and $x = \infty$, so $\sigma(x) = c/x$, where c is a real number. It is easy to see that c must be negative, otherwise σ has fixed points on $\hat{X}$. Defining the real number k by $-k^2 = c$, we redefine x as $x = x/k$. With this change of coordinates, $\sigma(x) = -1/x$, $x = 0$ and $x = \infty$ each have ramification index $2p$ in $\hat{W}$, and $x = -1/k$ has ramification index 2. In addition, the other point with ramification index 2 is $x = k$.

From Proposition 3.1, we may assume a defining equation for $\hat{W}$ is of the form

$$w^{2p} - x(x-k)^p(x+1/k)^p. \tag{36}$$

We note that $\hat{W}$ possess the automorphism A which maps $w \mapsto \epsilon w$ and $x \mapsto x$, where ϵ is a primitive $2p$th root of unity. From (v) of Proposition 3.1 we observe that $\sigma(w) = h/w$, for some $h \in C(x)$. Note that $\sigma(x - k) = -1/x - k = (-k/x)(x + 1/k)$ and $\sigma(x + 1/k) = -1/x + 1/k = (1/(kx))(x - k)$. This yields that

$$\begin{aligned} h^{2p} &= (-1/x)(-1/x - k)^p(-1/x + 1/k)^p x(x-k)^p(x+1/k)^p \\ &= (x-k)^{2p}(x+1/k)^{2p}/x^{2p}. \end{aligned}$$

Therefore, $\sigma(w) = \epsilon^j(x - k)(x + 1/k)/xw$, for some integer j. Since σ has order two and fixes $(x - k)(x + 1/k)/x$, we deduce that $\epsilon = \pm 1$. Regardless of the sign, σ will not have fixed points. Therefore $\hat{W}$ has (36) as a defining equation, possesses the automorphism A defined above, and possesses the symmetry

$$\sigma(i) = -i, \ \ \sigma(x) = -1/x, \ \ \sigma(w) = \pm(x-k)(x+1/k)/wx, \tag{37}$$

where either sign can be chosen in (37).

3.2.3. The signature $(0; +; [2, 2p]; \{(-)\})$.

Let Λ have the above signature. In this case $G \cong Z_{2p} = \langle A \mid A^{2p} = 1\rangle$. Note that $\hat{X}$ has genus 0 and let coordinates be chosen so that the action of σ on $\hat{X}$ is complex conjugation. Through a real transformation similar to that in Section 3.1.1, we may assume that $C(x)$ is ramified at the points $x = \pm i$ and $x = \pm ki$, where $k \neq \pm 1$ is a nonzero real number, and the ramification indices are $2, 2, 2p$ and $2p$ respectively in $C(\hat{W})$. A defining equation for $\hat{W}$ is

$$w^{2p} - (x-i)^p (x+i)^p (x-ki)(x+ki)^{2p-1} = 0, \tag{38}$$

and $\hat{W}$ possesses the automorphism A which maps $w \mapsto \epsilon w$ and $x \mapsto x$, where ϵ is a primitive $2p$th root of unity. It is easy to deduce that $W = \hat{W}/\langle\sigma\rangle$ where

$$\sigma(x) = x, \;\; \sigma(w) = \lambda(x^2+1)(x^2+k^2)/w,$$

for some $2p$th root of unity λ. Since σ is a symmetry, $\sigma^2(w) = w$ implies that $\lambda = \pm 1$. We will show that $\lambda = -1$, since σ is fixed point free on $\hat{W}$. Note that if a point (r, s) satisfying (38) is fixed by σ, then r must be real. Therefore, $s^{2p} = (r^2+1)^p(r-ki)(r+ki)^{2p-1} = (r^2+1)^p(r+ki)^{2p}(r-ki)/(r+ki) = e^{it}(r^2+1)^p(r+ki)^{2p}$, where t is real and $e^{it} = (r-ki)/(r+ki)$. Therefore, points (r, s) with r real, which satisfy (38) have $s = \epsilon\sqrt{r^2+1}e^{it/(2p)}(r+ik)$ for some $2p$th root of unity ϵ. Now assume $\sigma(w) = \lambda(x^2+1)(x^2+k^2)/w$, where $\lambda = \pm 1$. If the point $(r, s) = (r, \epsilon\sqrt{r^2+1}e^{it/(2p)}(r+ik))$ is fixed under σ, then we have that $\sigma(w-s) = \sigma(w) - \bar{s}$ must have a zero at (r, s). This yields that

$$0 = \lambda(r^2+1)(r^2+k^2)/\left(\epsilon\sqrt{r^2+1}e^{it/(2p)}(r+ik)\right) - \bar{\epsilon}\sqrt{r^2+1}e^{-it/(2p)}(r-ik).$$

It is easy to see that $\lambda = 1$ implies that such a point is always a fixed point, and $\lambda = -1$ implies that such a point is never a fixed point. Therefore $\sigma(w) = -(x^2+1)(x^2+k^2)/w$.

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Facultad de Ciencias
UNED
28040 Madrid
Spain
E-mail address: eb@mat.uned.es

University of Gdansk
Wita Stwosza 57
Gdansk
Poland
E-mail address: greggrom@ksinet.univ.gda.pl

Purdue University Calumet
Hammond, IN 46323
E-mail address: turbek@nwi.calumet.purdue.edu