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BOUNDARY CONDITIONS

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Much has been written about various obstacle problems in the context of variational inequalities. In particular, if the obstacle is smooth enough and if the coefficients of associated elliptic operator satisfy appropriate conditions, then the solution of the obstacle problem has continuous first derivatives. For a general class of obstacle problems, we show here that this regularity is attained under minimal smoothness hypotheses on the data and with a one-sided analog of the usual modulus of continuity assumption for the gradient of the obstacle. Our results apply to linear elliptic operators with Hölder continuous coefficients and, more generally, to a large class of fully nonlinear operators and boundary conditions.

Introduction.

For a smooth bounded domain $\Omega \subset \mathbb{R}^n$ with unit inner normal γ , we are concerned with generalizations of the simple obstacle problem of finding a function $u \in W^{1,2}(\Omega)$ which minimizes the functional \mathcal{F} defined on $W^{1,2}$ by

$$\mathcal{F}(v) = \int_{\Omega} |Dv|^2 dx + \int_{\partial\Omega} v^2 d\sigma$$

over the set of all $v \in W^{1,2}$ with $v \geq \psi$ for a given function ψ . From standard results in the theory of variational inequalities and the arguments in [12], it follows that this minimizer has bounded second derivatives if ψ has bounded second derivatives and satisfies the inequality $\partial\psi/\partial\gamma - \psi \geq 0$ on $\partial\Omega$, which is assumed sufficiently smooth. To state our generalization of this problem, we note that the minimizer u will be superharmonic in Ω and harmonic on the set where $u > \psi$; in addition $\partial u/\partial\gamma - u = 0$ on $\partial\Omega$. It is this formulation of the minimization problem that we wish to generalize.

We write \mathbb{S}^n for the set of all $n \times n$ symmetric matrices, and we set $\Gamma = \Omega \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{S}^n$ and $\Gamma' = \partial\Omega \times \mathbb{R} \times \mathbb{R}^n$. For real-valued, differential functions F and G defined on Γ and Γ' , respectively, we consider the problem

$$(0.1) \quad \min\{-F(x, u, Du, D^2u), u - \psi\} = 0 \text{ in } \Omega, \quad G(x, u, Du) = 0 \text{ on } \partial\Omega.$$

(We justify this somewhat nonstandard way of writing the problem by pointing out that, in the special case that F is the Laplace operator, our solution u will be superharmonic with $u \geq \psi$ and u is harmonic on the set $\{u > \psi\}$.) Using subscripts to denote partial derivatives with respect to the variables $z \in \mathbb{R}$, $p \in \mathbb{R}^n$ and $r \in \mathbb{S}^n$, we assume at least that the matrix $F_r(x, z, p, r)$ is positive definite for all $(x, z, p, r) \in \Gamma$ (so that the equation $F(x, u, Du, D^2u) = 0$ is elliptic) and that $G_p(x, z, p) \cdot \gamma(x) > 0$ for all $(x, z, p) \in \Gamma'$ for the unit inner normal γ to $\partial\Omega$ (so the boundary condition is an oblique derivative condition). If $\psi \in C^1$, then the analog of the condition $\partial\psi/\partial\gamma - \psi \geq 0$ would be $G(x, \psi, D\psi) \geq 0$ on $\partial\Omega$ (compare with [12, (0.7)]); however, we shall assume a weaker condition than continuity of $D\psi$ which still implies the continuity of Du , so we shall modify this condition appropriately (see conditions (2.1) and (3.9) below). Under suitable regularity hypotheses on F , G , ψ , and Ω , we shall show that a modulus of continuity for the first derivatives of u can be estimated in terms of known data. In conjunction with known first derivative estimates, our results give a complete description of the regularity of solutions for several problems. As particular examples, we mention here the capillarity obstacle problem from [12] and the Bellman equation problem with linear boundary condition from [23] (strictly speaking, we refer to the problem which the authors of that paper defer to a sequel, listed there as reference [27], which has never appeared in print). In [12], F has the special form

$$(0.2a) \quad F(x, z, p, r) = (1 + |p|^2)^{-1/2} \left(\delta^{ij} - \frac{p_i p_j}{1 + |p|^2} \right) r_{ij} + a(x, z),$$

for a suitable, Lipschitz function a and G has the form

$$(0.2b) \quad G(x, z, p) = \frac{p}{(1 + |p|^2)^{1/2}} \cdot \gamma + \varphi(x, z)$$

for a suitable, smooth function φ such that $\sup |\varphi(x, z)| < 1$. Because of the bound on the gradient of the solution of (0.1) in [12], it follows that our estimates apply to this problem assuming that a is Lipschitz and φ is $C^{1,\alpha}$ for some $\alpha \in (0, 1)$. In [23], F has the form

$$(0.3a) \quad F(x, z, p, r) = \inf_{k \in J} (a_k^{ij}(x) r_{ij} + b_k^i(x) p_i + c_k(x) z + f_k(x)),$$

where J is some index set (assumed to be countable in [23]) and there are uniform (with respect to k) bounds on the C^2 norms of the coefficients a_k^{ij} , b_k^i , c_k , and f_k as well as a positive lower bound (independent of k) on the minimum eigenvalue of the matrix $[a_k^{ij}]$; G has the form

$$(0.3b) \quad G(x, z, p) = \beta(x) \cdot p + b(x) z + g(x)$$

for some vector β such that $\beta \cdot \gamma$ is bounded from below by a positive constant, and the C^2 norms of β , b and g are assumed to be bounded. Again, from the gradient bounds proved in [23], it can be shown that our results apply to such problems if we only assume bounds on the Hölder norms of a_k^{ij} , b_k^i , c_k , and f_k (see Theorem 2.2) and on the Hölder norms of β , b and g (see Theorem 3.2); for second derivative bounds, we need to assume that β , b , and g have Hölder continuous derivatives (see Theorem 3.3). Of course, the uniform lower bounds on the minimum eigenvalue of $[a_k^{ij}]$ and on $\beta \cdot \gamma$ cannot be relaxed for our techniques to work.

In addition to the one-sided condition on ψ , our hypotheses are weaker than those in [11, Section 2], [12], [16, Section 4], [1], and [2] because we relax the smoothness hypotheses on F , G , and Ω .

A basic interpolation inequality appears in Section 1, which allows us to use a weak Harnack inequality rather than the usual Harnack inequality. An interior regularity result is proved in Section 2 using a modification of the technique pioneered by Caffarelli and Kinderlehrer [4]. Specifically, we show (via the weak Harnack inequality) that our one-sided condition on ψ implies a two-sided integral bound for $u - L$ with L a suitable linear function, and then the interpolation inequality from Section 1 gives a two-sided estimate on the first derivatives of u . The corresponding estimates at the boundary are proved in Section 3. Most of our work is to analyze the hypotheses on the obstacle; only some simple elements of the theory of differential equations enters into this analysis. Some similar results, with a Dirichlet boundary condition replacing the oblique derivative boundary condition, appear in a preprint by Jensen [13]. The analysis of the obstacle also provides a straightforward extension to the two obstacle problem, which we present in Section 4, and Section 5 discusses applications of our methods to some degenerate variational inequalities; in particular, problems with the p -Laplacian operators are considered. We close in Section 6 with an outline of the existence theory in a special case.

Our notation follows that in [10]. In addition, we write F^{ij} for the components of the matrix F_r and F^i for the components of the vector F_p . Similarly, G^i denotes the components of the vector G_p . We always assume here that ψ is Lipschitz with

$$(0.4) \quad |\psi| + |D\psi| \leq \Psi_1,$$

and we define

$$\Gamma'_1 = \{(x, z, p) \in \Gamma' : |z| + |p| \leq \Psi_1\}.$$

1. An interpolation lemma.

Our first lemma is an improvement of results on second derivative estimates in terms of estimates on lower order derivatives. For brevity, if $\Sigma \subset \bar{\Omega}$, we use $|u|_{a;\Sigma}^{(b)}$ to denote the norms weighted in terms of distance to $\partial\Sigma \cap \Omega$.

Lemma 1.1. *Let Ω be a bounded Lipschitz domain, let Σ be a subset of $\bar{\Omega}$, and suppose $u \in C^{2+\alpha}(\Sigma)$ for some $\alpha \in (0, 1)$. Suppose that there are positive constants C_1 and C_2 such that*

$$(1.1) \quad |D^2u|_{\alpha;\Sigma \cap B(R)} \leq C_1 R^{-\alpha} |D^2u|_{0;\Sigma \cap B(2R)} + C_2$$

for any two concentric balls $B(R)$ and $B(2R)$, with radii R and $2R$, respectively, such that the boundary of $\Sigma \cap B(2R)$ is disjoint from $\Omega \setminus \Sigma$. Then there is a constant C determined only by C_1 , α , and Ω such that

$$(1.2) \quad |u|_{2+\alpha;\Sigma}^{(0)} \leq C (|u|_{0;\Sigma} + C_2(\text{diam } \Sigma)^{2+\alpha}).$$

In addition, if $\Sigma = \bar{\Omega} \cap B(2R)$ for some ball $B(2R)$, and if $\kappa > 0$, then

$$(1.3) \quad |D^2u|_{0;\Sigma'} \leq C(C_1, \alpha, \kappa, \Omega) \left(R^{-2-(n/\kappa)} \|u\|_{\kappa;\Sigma} + C_2 R^\alpha \right),$$

where $\Sigma' = \bar{\Omega} \cap B(R)$.

Proof. The proof of (1.2) is a simple combination of the interpolation inequality

$$|D^2u|_0^{(2)} \leq C \left(|D^2u|_\alpha^{(2+\alpha)} + |u|_0 \right)^{2/(\alpha+2)} (|u|_0)^{\alpha/(2+\alpha)}$$

and the observation that (1.1) implies that

$$|D^2u|_\alpha^{(2+\alpha)} \leq C \left(C_1 |D^2u|_0^{(2)} + C_2(\text{diam } \Sigma)^{2+\alpha} \right).$$

To prove (1.3), we imitate the proof of [19, Lemma 4.5]. From (1.2), we infer that

$$\rho \sup_{S(\rho)} |Du| \leq C \left(\sup_{S(2\rho)} |u| + C_2 \rho^{2+\alpha} \right),$$

where $S(\rho) = \Sigma \cap B(\rho)$ and the boundary of $S(2\rho)$ is disjoint from $\Omega \setminus \Sigma$. It follows that there is a constant C_0 determined only by C_1 , α , κ , and Ω such that

$$\text{osc}_{S(\theta\rho)} u \leq C_0 \theta \left(\sup_{S(2\rho)} |u| + C_2 \rho^{2+\alpha} \right)$$

for any $\theta \in (0, 1)$. Now we take x_1 so that $d(x_1)^{n/\kappa} |u(x_1)| \geq \frac{1}{2} |u|_0^{(n/\kappa)}$ and we choose our balls to be centered at x_1 with $\rho = \frac{1}{4} d(x_1)$. Then

$$|u(x)| \geq |u(x_1)| [1 - C_0 \theta 2^{n/\kappa}] - C_0 \theta C_2 \rho^{2+\alpha}$$

for $x \in S(\theta\rho)$. If we take θ so small that $C_0\theta 2^{n/\kappa} \leq 1/2$ and $C_0\theta \leq 1$, then rearranging the resulting inequality and integrating over $S(\rho)$ yields

$$\rho^n |u(x_1)|^\kappa \leq C \left(\int_{S(\theta\rho)} |u|^\kappa dx + C_2^\kappa \rho^{(2+\alpha)\kappa+n} \right),$$

and therefore

$$|u|_0^{(n/\kappa)} \leq C \left(\|u\|_\kappa + C_2 R^{2+\alpha+(n/\kappa)} \right).$$

Hence

$$|u|_{0;\Omega \cap B(3R/2)} \leq C \left(R^{-n/\kappa} \|u\|_\kappa + C_2 R^{2+\alpha} \right)$$

for any ball $B(3R/2)$. The desired result follows from this one after applying (1.2) with Σ replaced by $\Omega \cap B(3R/2)$. \square

2. Interior derivative estimates.

Our main ingredient is a pointwise estimate of how fast u moves away from the obstacle near a contact point. In this section, we prove this estimate at an interior point. The argument is a straightforward modification of that in [4], but because of Lemma 1.1, we only need to estimate the L^κ norm of a function related to u ; this estimate is proved quite simply. Our basic assumption on the obstacle ψ is that there are functions Y defined on Ω and ζ defined on $[0, \text{diam } \Omega]$ with ζ continuous and increasing such that

$$(2.1) \quad \psi(x_1) \geq \psi(x_2) + Y(x_2) \cdot (x_1 - x_2) - \zeta(|x_1 - x_2|)|x_1 - x_2|$$

for all x_1 and x_2 in Ω . We have not assumed that $\zeta(0) = 0$, even though this assumption is needed to conclude that Du is actually continuous, because it does not affect the form of our estimates. Note that the usual assumption (from [1, 4, 11, 12, 16, 18]) is that $|D\psi(x_1) - D\psi(x_2)| \leq \zeta(|x_1 - x_2|)$, which is equivalent to the combination of (2.1) and the companion inequality

$$\psi(x_1) \leq \psi(x_2) + Y(x_2) \cdot (x_1 - x_2) + \zeta(|x_1 - x_2|)|x_1 - x_2|.$$

Our condition includes functions which are not continuously differentiable even if $\zeta(0) = 0$. For example, if $(\psi_\alpha)_{\alpha \in I}$ is a family of functions (with arbitrary index set I) satisfying (2.1), then a simple calculation shows that ψ defined by $\psi(x) = \sup_{\alpha \in I} \psi_\alpha(x)$ also satisfies this condition provided we have a uniform L^∞ bound on ψ_α and $D\psi_\alpha$. In particular, condition (2.1) includes the obstacles studied by Troianiello in [30, 31].

Lemma 2.1. *Suppose that $u \in W_{\text{loc}}^{2,n}$ satisfies*

$$(2.2) \quad \min\{-F(x, u, Du, D^2u), u - \psi\} = 0 \text{ in } \Omega$$

and that there are positive constants λ, Λ , and μ_0 such that

$$(2.3) \quad \lambda|\xi|^2 \leq F^{ij}(x, u, Du, D^2u)\xi_i\xi_j$$

for all $x \in \Omega$ and $\xi \in \mathbb{R}^n$,

$$(2.4a) \quad F(x, u, Du, 0) \geq -\mu_0\lambda,$$

$$(2.4b) \quad |F_r(x, u, Du, tD^2u)| \leq \Lambda,$$

for all $x \in \Omega$ and all $t \in [0, 1]$. Suppose also that (2.1) holds and that x_0 is a point such that $u(x_0) = \psi(x_0)$. Then there are constants κ and C determined only by n and Λ/λ such that the function \bar{u} defined by

$$(2.5) \quad \bar{u}(x) = u(x) - u(x_0) - Y(x_0) \cdot (x - x_0)$$

satisfies the estimate

$$(2.6) \quad \left(R^{-n} \int_{B(x_0, R/2)} |\bar{u}|^\kappa dx \right)^{1/\kappa} \leq C[\mu_0 R^2 + \zeta(R)R]$$

for all $R \leq d(x_0)/2$.

Proof. Since $u \geq \psi$ and $u(x_0) = \psi(x_0)$, it follows from (2.1) that $\bar{u} \geq -\zeta(R)R$ in $B(x_0, R)$. Next, we note that $a^{ij}D_{ij}u \leq -F(x, u, Du, 0)$ for

$$a^{ij}(x) = \int_0^1 F^{ij}(x, u(x), Du(x), tD^2u(x)) dt.$$

It follows that $v = \bar{u} + \zeta(R)R$ satisfies the conditions $a^{ij}D_{ij}v \leq \lambda\mu_0$ and $v \geq 0$ in $B(x_0, R)$. Therefore [10, Theorem 9.22] and the obvious inequality $\inf_{B(R/2)} v \leq v(0) = \zeta(R)R$ yield

$$\left(R^{-n} \int_{B(R/2)} |v|^\kappa dx \right)^{1/\kappa} \leq C[\mu_0 R^2 + \zeta(R)R],$$

and the triangle inequality gives

$$\left(R^{-n} \int_{B(R/2)} |\bar{u}|^\kappa dx \right)^{1/\kappa} \leq C(\kappa) \left[\left(R^{-n} \int_{B(R/2)} |v|^\kappa dx \right)^{1/\kappa} + \zeta(R)R \right].$$

We complete the proof by combining these last two inequalities. □

Note Lemma 2.1 continues to hold if we only assume that the minimum in (2.2) is nonnegative; however, our full regularity result will use that the minimum is zero.

The regularity of the derivatives of u at an arbitrary point follows from this estimate and Lemma 1.1 by a simple variation of the argument in [4].

Theorem 2.2. *Suppose that u, ψ , and F satisfy conditions (2.1)-(2.4) with ζ a continuous increasing function on $[0, \text{diam } \Omega]$ satisfying*

$$(2.7) \quad \frac{\zeta(t_1)}{t_1} \geq \frac{\zeta(t_2)}{t_2} \text{ if } t_1 \leq t_2.$$

Suppose also that there are constants $\alpha \in (0, 1)$ and μ_1 such that

$$(2.8) \quad |F(x, z, p, r) - F(y, w, q, r)| \leq (\mu_0 + \mu_1|r|)\lambda|x - y|^\alpha,$$

and that F is convex or concave with respect to r . Then there is a constant C determined only by $n, \alpha, \mu_1, \Lambda/\lambda, |u|_1, |\psi|_1$, and $\text{diam } \Omega$ such that

$$(2.9) \quad |Du(x_1) - Du(x_2)| \leq C \left[\zeta(|x_1 - x_2|) + \left(\mu_0 + \frac{\sup |Du|}{d(x_1)} \right) |x_1 - x_2| \right]$$

for all x_1 and x_2 in Ω with $|x_1 - x_2| \leq \frac{1}{4} \min\{d(x_1), d(x_2)\}$.

Proof. Using I to denote the contact set $I = \{x \in \Omega : \psi(x) = u(x)\}$, we consider three cases:

- (i) both points are in I ,
- (ii) one point is in I ,
- (iii) neither point is in I .

In all cases, we set $\rho = |x_1 - x_2|$ and $Z = \zeta(\rho) + \mu_0\rho$.

In the first case, we use (2.6) twice, first with $R = \rho$ and $x_0 = x_1$ and then with $R = 2\rho$ and $x_0 = x_2$ to infer that

$$\left(\rho^{-n} \int_{B(x_1, \rho)} |u(x) - \psi(x_i) - Y(x_i) \cdot (x - x_i)|^\kappa dx \right)^{1/\kappa} \leq CZ\rho$$

for $i = 1, 2$ because $B(x_1, \rho) \subset B(x_2, 2\rho) \subset \Omega$. Next, we use the observation that

$$\begin{aligned} & \psi(x_1) - \psi(x_2) - Y(x_2) \cdot (x_1 - x_2) + [Y(x_1) - Y(x_2)] \cdot (x - x_1) \\ &= [u(x) - \psi(x_1) - Y(x_1) \cdot (x - x_1)] - [u(x) - \psi(x_2) - Y(x_2) \cdot (x - x_2)] \end{aligned}$$

along with the triangle inequality to infer that

$$\begin{aligned} & \left(\rho^{-n} \int_{B(x_1, \rho)} |\psi(x_1) - \psi(x_2) - Y(x_2) \cdot (x_1 - x_2) \right. \\ & \qquad \qquad \qquad \left. + V \cdot (x - x_1)|^\kappa dx \right)^{1/\kappa} \leq CZ\rho, \end{aligned}$$

where $V = Y(x_1) - Y(x_2)$. In addition, (2.1) implies that

$$\psi(x_1) - \psi(x_2) - Y(x_2) \cdot (x_1 - x_2) + V \cdot (x - x_1) \geq V \cdot (x - x_1) - \zeta(\rho)\rho$$

in $B(\rho)$. We therefore infer that

$$\left(\rho^{-n} \int_{B(x_1, \rho)} ([V \cdot (x - x_1)]^+)^\kappa dx \right)^{1/\kappa} \leq CZ\rho,$$

and this inequality easily gives $|V| \leq CZ$. It follows that

$$(2.10) \quad |u(x_1) - u(x_2) - Y(x_2) \cdot (x_1 - x_2)| \leq C[\zeta(|x_1 - x_2|) + \mu_0|x_1 - x_2|]|x_1 - x_2|$$

for any x_1 and x_2 in I , so u is differentiable in the interior of I with $Du = Y$ there.

In the second case, we may assume without loss of generality that $x_1 \in I$, and we write ξ_2 for the closest point to x_2 in I . Note that u is a solution of the equation $F(x, u, Du, D^2u) = 0$ in $\Sigma_0 = B(x_2, |x_2 - \xi_2|)$, so (1.1) holds with Σ any subset of Σ_0 . (This estimate is proved in [28], but the precise form used here does not appear in that reference; see [19, Theorem 14.7] for a proof of the corresponding parabolic estimate.) If x is on the line segment between x_2 and ξ_2 , it follows from (1.3) with $R = 2|x - \xi_2|$ (applied to \bar{u} defined with ξ_2 in place of x_0) that

$$|D\bar{u}(x)| \leq C[R^{-1-(n/\kappa)}\|u\|_\kappa + \mu_0R^{1+\alpha}]$$

and hence

$$|Y(\xi_2) - Du(x_2)| \leq C|x_2 - \xi_2| \leq CZ.$$

Since $|x_1 - \xi_2| \leq 2\rho$, it follows from Case (i) that $|Y(x_1) - Y(\xi_2)| \leq CZ$, and hence (2.10) holds if $x_1 \in I$ and $x_2 \notin I$. Therefore u is also differentiable on ∂I with $Du = Y$ there. Now that we know $Du = Y$ on I , our estimates imply (2.9) for $x_1 \in I$ and $x_2 \in \Omega$.

In the third case, we set $d^*(x) = \text{dist}(x, I)$ and $m_0 = \min\{d^*(x_1), d^*(x_2)\}$, and we consider three possibilities. If $2\rho \geq m_0$, then, with ξ_i denoting the closest point to x_i in I , we have

$$|Du(x_1) - Du(x_2)| \leq |Du(x_1) - Du(\xi_1)| + |Du(\xi_1) - Du(\xi_2)| + |Du(\xi_2) - Du(x_2)|,$$

and the three terms on the right-hand side of this inequality are estimated either by Case (i) or Case (ii) along with the observation that

$$|x_1 - \xi_1| \leq C\rho, \quad |\xi_1 - \xi_2| \leq C\rho, \quad |x_2 - \xi_2| \leq C\rho.$$

If $2\rho < m_0$ and $d(x_1) \leq m_0$, then we can use Lemma 1.1 as in Case (ii) and (2.7) to infer that

$$|D^2u| \leq C \left[\frac{\zeta(m_0)}{m_0} + \mu_0 m_0^\alpha \right] \leq C \left[\frac{\zeta(\rho)}{\rho} + \mu_0 \right]$$

on the line segment joining x_1 and x_2 . An easy integration of this inequality yields (2.9) in this case as well. Finally, if $2\rho < m_0$ and $d(x_1) > m_0$, then (1.3) with $\kappa = \infty$ gives the desired result. \square

Note that the hypothesis (2.7) really involves no loss of generality. Specifically if ζ is a continuous, increasing function, then the function ζ_1 , defined by

$$\zeta_1(t) = t \sup_{s \geq t} \frac{\zeta(s)}{s},$$

satisfies (2.7) and $\zeta_1 \geq \zeta$, so (2.1) holds with ζ_1 in place of ζ . In addition ζ_1 is clearly continuous. To see that ζ_1 is increasing, we let $t_1 < t_2$ and choose s_i so that $\zeta_1(t_i) = (t_i/s_i)\zeta(s_i)$. If $\zeta(s_1) = \zeta(s_2)$, then $\zeta_1(t_1)/\zeta_1(t_2) = t_1/t_2 < 1$. If $\zeta(s_1) < \zeta(s_2)$, then $s_1 < t_2$, so $\zeta_1(t_1) = (t_1/s_1)\zeta(s_1) \leq (t_1/s_1)\zeta(t_2) \leq \zeta(t_2) \leq \zeta_1(t_2)$. Moreover, if $\zeta(0) = 0$, then $\zeta_1(t) \rightarrow 0$ as $t \rightarrow 0$, as we see by considering two cases. If $\zeta(s)/s$ is bounded as $s \rightarrow 0$, say by S , then $\zeta_1(t) \leq St \rightarrow 0$. If $\zeta(s)/s$ is unbounded as $s \rightarrow 0$, let (s_j) be a sequence tending to zero with $\zeta(s_j)/s_j \geq j$ and $\zeta(s_j)/s_j \geq \zeta(s)/s$ if $s \geq s_j$. Then $\zeta_1(s_j) = \zeta(s_j)$ so $\zeta_1(s_j) \rightarrow 0$, and then $\zeta(t) \rightarrow 0$ as $t \rightarrow 0$ because ζ_1 is increasing. In addition, we note (see [24, Section 3.5] for details) that the modulus of continuity for a function defined on an open set satisfies (2.7).

Condition (2.8) can be weakened, say to

$$|F(x, z, p, r) - F(y, w, q, r)| \leq (\mu_0 + \mu_1|r|)\lambda|x - y|^\alpha + \mu'_0\lambda|p - q|^\alpha,$$

since this condition is only used to infer the appropriate form of the Hölder for second derivatives of solutions of the equation $F(x, u, Du, D^2u) = 0$ (see [28]). In particular, our results apply to the operator F defined by (0.3a) if we assume uniform Hölder estimates on the functions a_k^{ij}, b_k^j, c_k , and f_k along with a uniform lower bound on the minimum eigenvalue of $[a_k^{ij}]$; this structure was considered in [23]. Moreover, we can infer condition (2.8) for more general classes of fully nonlinear, uniformly elliptic operators F once we have a Hölder gradient estimate for u . Such an estimate follows by virtue of the following variant of Theorem 2.2, which is also important for our study of oblique derivative problems.

Theorem 2.3. *Suppose u, ψ , and F satisfy conditions (2.1)-(2.4) with ζ a continuous, increasing function on $[0, \text{diam } \Omega]$. Suppose also that F is concave or convex with respect to r . Suppose finally that there are a positive constant ν_1 and a continuous, increasing function ζ_1 with $\zeta_1(0) = 0$ such that*

$$(2.11) \quad |F(x, z, p, r) - F(y, w, q, r)| \leq \mu_0\lambda + \lambda(\nu_1|p - q| + \zeta_1(|x - y|))|r|.$$

Then there are positive constants $\alpha(n, \Lambda/\lambda, \nu_1)$ and $C(n, \zeta_1, \Lambda/\lambda, \nu_1, \text{diam } \Omega)$ such that

$$(2.12) \quad \frac{\zeta(t_1)}{t_1^\alpha} \geq \frac{\zeta(t_2)}{t_2^\alpha} \text{ for } t_1 \leq t_2$$

implies

(2.13)

$$|Du(x_1) - Du(x_2)| \leq C \left[\zeta(|x_1 - x_2|) + \left(\mu_0 + \frac{\sup |Du|}{d(x_1)^\alpha} \right) |x_1 - x_2|^\alpha \right]$$

for all x_1 and x_2 in Ω with $|x_1 - x_2| \leq \frac{1}{4} \min\{d(x_1), d(x_2)\}$.

Proof. We basically follow the proof of Theorem 2.2. The main notational change is that we set $Z = \zeta(\rho) + \mu_0\rho^\alpha$. From the argument in [19, Lemma 12.13] (see also [3, Theorem 2] and [32]), we infer that

$$[Du]_{\alpha; B(R)} \leq C[R^{-\alpha}|Du|_{0; B(2R)} + \mu_0]$$

if $B(2R) \subset \Omega$ and $F(x, u, Du, D^2u) = 0$ in $B(2R)$. The proof is completed by using this inequality in the obvious modification of Lemma 1.1. \square

Note that if ζ satisfies (2.7), then ζ_2 defined by $\zeta_2(t) = (\sup \zeta)^{1-\alpha}(\zeta(t))^\alpha$ satisfies (2.12), so Theorem 2.3 also does not restrict our choice of obstacles.

Condition (2.11) is certainly satisfied for quasilinear operators, that is, $F(x, z, p, r) = a^{ij}(x, z, p)r_{ij} + a(x, z, p)$ provided $[a^{ij}]$ is elliptic, continuous with respect to x and z , and Lipschitz with respect to p . In particular (after using the gradient bound from [12]), this result applies when F is given by (0.2a). Moreover, we can remove the hypothesis that F be either concave or convex with respect to r in Theorem 2.3 by considering viscosity solutions as in [3, 32] and suitably modifying the arguments. Finally, as noted before, we can replace condition (2.11) by any condition which yields the Hölder gradient estimate

$$\text{osc}_{B(x_0, r)} Du \leq C \left[\left(\frac{r}{R} \right)^\alpha \text{osc}_{B(x_0, R)} Du + \mu_0 r^\alpha \right].$$

See [5] for an alternative structure condition which provides such an estimate.

3. Estimates for the oblique derivative problem.

To prove a modulus of continuity estimate for the gradient up to the boundary for the oblique derivative problem, we use a slight variation of the ideas in the proof of Theorem 2.2. We begin with a preliminary estimate which is related to the boundary condition in which we write v' for the first $n - 1$ components of the vector v . The connection of this lemma to our original problem will be made clear in Theorem 3.2.

Lemma 3.1. *Let ω_0, ω_1 , and r be positive constants with $\omega_0 > \omega_1$, and define*

$$(3.1) \quad K = \{x^n \geq \omega_0|x'|, |x| \leq r\}, E = \{x^n \geq \omega_1|x'|, r/4 < |x| \leq r\}.$$

Let $\bar{\psi}$ be a Lipschitz function defined in K and suppose that there are positive constants z and κ along with a vector-valued function \bar{Y} such that

$$(3.2) \quad \bar{\psi}(x) \geq \bar{\psi}(x_1) + \bar{Y}(x_1) \cdot (x - x_1) - z$$

for all x and x_1 in K and

$$(3.3) \quad \left(\int_E |\bar{\psi}|^\kappa dx \right)^{1/\kappa} \leq zr^{n/\kappa}.$$

Suppose also that there is a Lipschitz function g defined on \mathbb{R}^n with

$$(3.4a) \quad \left| \frac{\partial g}{\partial p'} \right| \leq \mu_2 \chi_0,$$

$$(3.4b) \quad \frac{\partial g}{\partial p_n} \geq \chi_0$$

for some positive constants χ_0 and μ_2 with $\mu_2 \omega_1 < 1$. Then

$$(3.5) \quad g(0) \geq g(\bar{Y}(0)) - C(n, \kappa, \omega_0, \omega_1, \mu_2) \chi_0 z / r.$$

Proof. The first step is to prove a pointwise upper bound for $\bar{\psi}$ in

$$E' = \{x^n \geq |x'|/\mu_2, 3r/8 \leq |x| \leq 3r/4\}.$$

To prove this estimate, let x_1 be a point in E' at which the maximum of $\bar{\psi}$ is attained and suppose that $\bar{\psi}(x_1) > 2z$. Then

$$(3.6) \quad \left(\int_{B(x_1, \rho)} |\bar{\psi}|^\kappa dx \right)^{1/\kappa} \leq \left(\int_E |\bar{\psi}|^\kappa dx \right)^{1/\kappa} \leq zr^{n/\kappa}$$

for any ρ such that $B(x_1, \rho) \subset E$. In particular, we can take $\rho = C(\omega_1, \mu_2)r$. With this choice for ρ , we set

$$E^+ = \{x \in B(x_1, \rho) : \bar{Y}(x_1) \cdot (x - x_1) \geq 0\},$$

and note that $|E^+| \geq \frac{1}{2}|B(x_1, \rho)| \geq Cr^n$. In addition, for $x \in E^+$, we have

$$\bar{\psi}(x) \geq \bar{\psi}(x_1) + \bar{Y}(x_1) \cdot (x - x_1) - z \geq \bar{\psi}(x_1) - z,$$

and therefore

$$\begin{aligned} \left(\int_{B(x_1, \rho)} |\bar{\psi}(x)|^\kappa dx \right)^{1/\kappa} &\geq \left(\int_{E^+} \bar{\psi}(x)^\kappa dx \right)^{1/\kappa} \geq (|E^+|(\bar{\psi}(x_1) - z)^\kappa)^{1/\kappa} \\ &\geq C[\bar{\psi}(x_1) - z]r^{n/\kappa}. \end{aligned}$$

In conjunction with (3.6), this inequality implies that $\bar{\psi} \leq Cz$ on E' .

Next, we note that there is a point x_2 with $|x_2| \in (7r/16, 9r/16)$ and $x_2^n > 2\mu_2|x_2'|$ such that $\bar{\psi}(x_2) \geq -Cz$. In addition, if $\bar{Y}(x_2) \neq 0$, then

there is a positive constant c_2 such that $x_3 = x_2 + c_2 r \bar{Y}(x_2)/|\bar{Y}(x_2)| \in E'$. Therefore

$$Cz \geq \bar{\psi}(x_3) \geq \bar{\psi}(x_2) + \bar{Y}(x_2) \cdot (x_3 - x_2) - z = \bar{\psi}(x_2) - c_2 r |\bar{Y}(x_2)| - z.$$

It follows that $|\bar{Y}(x_2)| \leq Cz/r$ and hence

$$(3.7) \quad \bar{\psi}(x) \geq \bar{\psi}(x_2) + \bar{Y}(x_2) \cdot (x - x_2) - z \geq -Cz$$

for any $x \in K$.

To continue, we define ξ to be the unit vector in the direction of

$$\int_0^1 g_p(t\bar{Y}(0)) dt,$$

so

$$g(\bar{Y}(0)) - g(0) = \int_0^1 g_p(t\bar{Y}(0)) \cdot \bar{Y}(0) dt \leq C\chi_0 \xi \cdot \bar{Y}(0).$$

Now set $\rho = r/(2\xi^n)$. It is easy to see that $r/2 \leq \rho \leq Cr$. In addition, we infer from our estimate $\bar{\psi} \leq Cz$ on E' along with (3.2) and (3.7) that

$$Cz \geq \bar{\psi}(\rho\xi) \geq \bar{\psi}(0) + \bar{Y}(0) \cdot (\rho\xi) - z \geq -Cz + \rho\bar{Y}(0) \cdot \xi.$$

It follows that $\bar{Y}(0) \cdot \xi \leq Cz/r$, which yields (3.5). \square

To state our gradient estimate for the oblique derivative problem, we use Γ'_2 to denote the set of all $(x, z, p) \in \Gamma'$ with $|z| + |p| \leq \max\{|u|_1, \Psi_1\}$. Because of the way that a Hölder gradient estimate is used to prove second derivative estimates for the oblique derivative problem without an obstacle, we first prove our estimate in a situation analogous to that in Theorem 2.3.

Theorem 3.2. *Let $u \in W_{\text{loc}}^{2,n} \cap C^1(\bar{\Omega})$ solve (0.1) with $\partial\Omega \in C^{1,\alpha}$ for some $\alpha \in (0, 1)$ and F either convex or concave with respect to r . Suppose that there are positive constants λ , Λ , μ_0 , and ν_1 along with a continuous, increasing function ζ_1 with $\zeta_1(0) = 0$ such that conditions (2.3), (2.4), and (2.11) hold. Suppose also that there are positive constants χ_0 , μ_2 , and μ_3 such that*

$$(3.8a) \quad G_p(x, z, p) \cdot \gamma(x) \geq \chi_0,$$

$$(3.8b) \quad |G_p(x, z, p) \cdot \tau(x)| \leq \mu_2 \chi_0,$$

$$(3.8c) \quad |G(x, z, p) - G(y, w, p)| \leq \mu_0 \chi_0 (|x - y| + |z - w|)^\alpha$$

for all (x, z, p) and (y, w, p) in Γ'_2 and any vector field $\tau(x)$ with $\tau \cdot \gamma = 0$. Suppose further that there is a continuous increasing function ζ on $[0, \text{diam } \Omega]$ satisfying (2.12) such that ψ satisfies (2.1) and

$$(3.9) \quad G(x, \psi, Y) \geq 0$$

for all $x \in \partial\Omega$. Then there are constants $\alpha_0(n, \mu_2, \nu_1, \Lambda/\lambda)$ and C determined only by $n, \alpha, \Lambda/\lambda, \mu_2, \Psi_1, \zeta_1$, and Ω such that $\alpha \leq \alpha_0$ implies

$$(3.10) \quad |Du(x_1) - Du(x_2)| \leq C[\zeta(|x_1 - x_2|) + (\mu_0 + \sup |Du|)|x_1 - x_2|^\alpha]$$

for all x_1 and x_2 in Ω .

Proof. We imitate the proof of Theorem 2.2. First, we show (as in Lemma 2.1) that, if $x_0 \in \Omega$ is a point at which $u(x_0) = \psi(x_0)$ and if \bar{u} is defined by (2.5), then

$$(3.11) \quad \left(R^{-n} \int_{B(x_0, R/2) \cap \Omega} |\bar{u}|^\kappa dx \right)^{1/\kappa} \leq C[(\mu_0 + \mu_2)R^{1+\alpha} + \zeta(R)R]$$

for any sufficiently small R (that is, R is smaller than a constant determined only by μ_2 and Ω). If $d(x_0) \geq R$, then this inequality is just (2.6). If $d(x_0) < R$, then we first prove an estimate for $G(x, u(x), Y(x_0))$ by appropriate application of Lemma 3.1.

Let x^* be a closest point to x_0 in $\partial\Omega$. By rotation and translation, we may assume that x^* is the origin and that x_0 is on the positive x^n -axis. Then $K \subset \Omega$ provided $\omega_0 > 1/\mu_2$ and R is sufficiently small (determined only by Ω and μ_2), and $g(p) = G(x^*, \psi(x^*), p + Y(x_0))$ satisfies (3.4). Next, we define $\bar{\psi}$ by $\bar{\psi}(x) = \psi(x) - \psi(x_0) - Y(x_0) \cdot (x - x_0)$ and we set $\bar{Y}(x) = Y(x) - Y(x_0)$. For $z = C[\mu_0 R^\alpha + \zeta(R)]R$ and $r = 2d(x_0)$, we have (3.2) directly from (2.1) because $r \leq R$. Now, we note that using a chaining argument in the proof of [10, Theorem 9.22] allows us to replace $B(x_0, R/2)$ by E and R by r in the proof of (2.6). Thus, we obtain

$$\left(r^{-n} \int_E |\bar{u}|^\kappa dx \right)^{1/\kappa} \leq Cz,$$

which yields (3.3) because $\bar{u} \geq \bar{\psi} \geq -C\zeta(r)r$ in E . It then follows from Lemma 3.1 that

$$G(x^*, \psi(x^*), Y(x_0)) = g(0) \geq -Cz$$

because $g(\bar{Y}(0)) = G(x^*, u(x^*), Y(x^*)) \geq 0$. For $x \in B(x_0, R) \cap \partial\Omega$, we have

$$|\psi(x^*) - u(x)| \leq |\psi(x^*) - \psi(x_0)| + |u(x) - u(x_0)| \leq (\Psi_1 + |Du|_0)R,$$

and therefore

$$G(x, u(x), Y(x_0)) \geq -\mu_0\chi_0(\Psi_1 + |Du|_0 + 1)^\alpha R^\alpha - C\chi_0 z \geq -Cz/R.$$

It follows that

$$\beta \cdot D\bar{u} \leq C\chi_0[\zeta(R) + \mu_0 R^\alpha]$$

on $B(x_0, R) \cap \partial\Omega$ for

$$\beta(x) = \int_0^1 G_p(x, u, tDu + (1-t)Y(x_0)) dt.$$

We then infer (3.11) by arguing as in Lemma 2.1 but with [20, Theorem 4.2] in place of [10, Lemma 9.22].

To prove the modulus of continuity estimate for Du , we consider the three cases from Theorem 2.2 with $Z = \zeta(\rho) + \mu_0\rho^\alpha$. In addition, we set $\Omega[y, R] = B(y, R) \cap \Omega$. In Case (i), we note that there is a cone Q with height ρ , opening angle θ (determined only by Ω), and vertex 0 such that $x_i + Q \subset \Omega[x_i, \rho]$ for $i = 1, 2$. It follows that

$$\left(\rho^{-n} \int_{\Omega[x_1, \rho]} ([V \cdot (x - x_1)]^+)^{\kappa} dx \right)^{1/\kappa} \leq CZ\rho,$$

and similar reasoning gives

$$\left(\rho^{-n} \int_{\Omega[x_2, \rho]} ([V \cdot (x - x_2)]^-)^{\kappa} dx \right)^{1/\kappa} \leq CZ\rho.$$

Combining these two estimates gives

$$\left(\rho^{-n} \int_Q |V \cdot x|^{\kappa} dx \right)^{1/\kappa} \leq CZ\rho,$$

which again implies $|V| \leq CZ$. For Cases (ii) and (iii), we proceed as in Theorem 2.3 with [19, Lemma 13.22] in place of [19, Lemma 12.13] to prove (3.10). \square

The remarks from Section 2 show that this result applies to the examples from [1, 2, 12, 16, 23]. The function G given by (0.2b) satisfies conditions (3.8) by virtue of the gradient bound in [12] and G from (0.3b) clearly satisfies these conditions if β , b and g are Hölder continuous. Moreover, if $(\psi_\alpha)_{\alpha \in I}$ is a family of C^1 functions which satisfy conditions (2.1) and (3.9) with $Y = D\psi_\alpha(x)$, then it is immediate that there is a vector field Y such that $\psi = \sup_{\alpha \in I} \psi_\alpha$ satisfies these conditions.

We note that this result is a purely local one. Hence if the hypotheses of the theorem are satisfied only in a neighborhood N of some point x^* , then we obtain a modulus of continuity estimate for the first derivatives of u in $N' \cap \Omega$ for any compact subset N' of N . The corresponding local result was proved by B. Huisken [11] although she only considered quasilinear equations and her hypotheses are stronger than ours.

In addition, we have the following result which corresponds to Theorem 2.2.

Theorem 3.3. *Let $u \in W_{\text{loc}}^{2,n} \cap C^1(\bar{\Omega})$ solve (0.1) with $\partial\Omega \in C^{1,\alpha}$ for some $\alpha > 0$. Suppose that there are positive constants λ , λ_1 , Λ , μ_0 , and μ_1 such that conditions (2.1), (2.3), (2.4), (2.7), (2.8), (3.8) are satisfied. Suppose also that $G \in C^{1,\alpha}(\Gamma'_2)$ and that $\zeta(t) \leq z_0 t^\alpha$ for some z_0 . Then there is*

a constant C determined only by $n, z_0, \alpha, \Lambda/\lambda, \mu_0, \mu_1, \mu_2, \mu_3, \Psi_1, \zeta_1, \sup |Du|, |G|_{1,\alpha}$, and Ω such that

$$(3.12) \quad |Du(x_1) - Du(x_2)| \leq C(\zeta(|x_1 - x_2|) + |x_1 - x_2|).$$

Proof. We observe that our hypotheses imply a Hölder estimate for Du . With this estimate, we can follow the proof of Theorem 2.2 with [29] (as modified in [19, Theorem 14.22] to deal with nonlinear boundary conditions; this step uses the Hölder gradient estimate) in place of [28]. \square

In particular, if $\zeta(t) = t$, Theorem 3.3 gives a bound on the second derivatives of u .

4. The double obstacle problem.

The crucial new element in our study of double obstacle problems is a Harnack-type inequality for the difference between the upper obstacle and the lower obstacle. The basic ideas for this inequality were used in Lemma 3.1, but, here, we shall use some precise information on how fast the ratio of the maximum of the difference to its minimum goes to one on a ball of shrinking radius, provided the obstacles are defined in a ball of fixed radius. Specifically, we have the following result.

Lemma 4.1. *Suppose ψ_1 and ψ_2 are two functions defined in $B(x_0, r)$ with $\psi_1 \leq \psi_2$ and that there are two vector fields Y_1 and Y_2 such that*

$$(4.1a) \quad \psi_1(x_1) \geq \psi_1(x_2) + Y_1(x_2) \cdot (x_1 - x_2) - \zeta(|x_1 - x_2|)|x_1 - x_2|,$$

$$(4.1b) \quad \psi_2(x_1) \leq \psi_2(x_2) + Y_2(x_2) \cdot (x_1 - x_2) + \zeta(|x_1 - x_2|)|x_1 - x_2|$$

for all x_1 and x_2 in $B(x_0, r)$. Then for any $\varepsilon \in (0, 1)$, we have

$$(4.2) \quad \sup_{B(x_0, \varepsilon r)} (\psi_2 - \psi_1) \leq \frac{1 + \varepsilon}{1 - \varepsilon} \inf_{B(x_0, \varepsilon r)} (\psi_2 - \psi_1) + 4\varepsilon(\zeta(2\varepsilon r) + \zeta(r))r.$$

Proof. Set $\psi = \psi_2 - \psi_1$ and $I = \inf_{B(x_0, \varepsilon r)} \psi$. Then choose x_2 so that $|x_0 - x_2| \leq \varepsilon r$ and $\psi(x_2) = I$. Our first step is to show that

$$(4.3) \quad |Y(x_2)| \leq \frac{I}{(1 - \varepsilon)r} + 2\zeta(r),$$

so let us assume that $Y(x_2) \neq 0$ and set $\xi = Y(x_2)/|Y(x_2)|$. Then for $R < (1 - \varepsilon)r$, we have that $x_2 - R\xi \in B(x_0, r)$, so (4.1) implies that

$$\begin{aligned} \psi(x_2 - R\xi) &\leq \psi(x_2) - Y(x_2) \cdot (R\xi) + 2\zeta(R)R \\ &= I - R|Y(x_2)| + 2\zeta(R)R. \end{aligned}$$

We infer (4.3) from this inequality by sending $R \rightarrow (1 - \varepsilon)r$ and noting that $\psi(x_2 - R\xi) \geq 0$.

Now let $x \in B(x_0, \varepsilon r)$ and use (4.1) to infer that

$$\begin{aligned}\psi(x) &\leq \psi(x_2) + Y(x_2) \cdot (x - x_2) + 2(|x - x_2|)|x - x_2| \\ &\leq I + 2\varepsilon r|Y(x_2)| + 4\zeta(2\varepsilon r)\varepsilon r.\end{aligned}$$

Simple algebra then completes the proof. \square

We also shall use the following simple variant of (4.2).

Corollary 4.2. *In addition to the hypotheses of Lemma 4.1, suppose that $\psi_1(x_0) = 0$ and $Y_1(x_0) = 0$. Then*

$$(4.4) \quad \sup_{B(x_0, \varepsilon r)} \psi_2 \leq \frac{1 + \varepsilon}{1 - \varepsilon} \inf_{B(x_0, \varepsilon r)} \psi_2^+ + \frac{4\varepsilon}{1 - \varepsilon} (\zeta(2\varepsilon r) + \zeta(r))r.$$

If also $\psi_2 \geq 0$ on $B(x_0, \varepsilon r)$, then

$$(4.5) \quad \sup_{B(x_0, \varepsilon r)} \psi_1 \leq \frac{4\varepsilon}{1 - \varepsilon^2} \inf_{B(x_0, \varepsilon r)} \psi_2 + \frac{6\varepsilon}{1 - \varepsilon^2} (\zeta(2\varepsilon r) + \zeta(r))r.$$

Proof. Because $\psi_2 \geq -\zeta(r)r$, we can follow the proof of Lemma 4.1 with ψ replaced by $\psi_2 + \zeta(r)r$ to infer that

$$\begin{aligned}\sup_{B(x_0, \varepsilon r)} (\psi_2 + \zeta(r)r) &\leq \frac{1 + \varepsilon}{1 - \varepsilon} \inf_{B(x_0, \varepsilon r)} (\psi_2 + \zeta(r)r) + 2\varepsilon(\zeta(2\varepsilon r) + \zeta(r))r \\ &\leq \frac{1 + \varepsilon}{1 - \varepsilon} \inf_{B(x_0, \varepsilon r)} (\psi_2^+ + \zeta(r)r) + 2\varepsilon(\zeta(2\varepsilon r) + \zeta(r))r,\end{aligned}$$

so

$$\sup_{B(x_0, \varepsilon r)} \psi_2 \leq \frac{1 + \varepsilon}{1 - \varepsilon} \inf_{B(x_0, \varepsilon r)} \psi_2^+ + \left(\frac{1 + \varepsilon}{1 - \varepsilon} - 1 \right) \zeta(r)r + 2\varepsilon(\zeta(2\varepsilon r) + \zeta(r))r.$$

Since $(1 + \varepsilon)/(1 - \varepsilon) - 1 = 2\varepsilon/(1 - \varepsilon)$ and $1 < 1/(1 - \varepsilon)$, this inequality gives (4.4).

Next, we set $\psi = \psi_2 - \psi_1$, $I = \inf_{B(x_0, \varepsilon r)} \psi$ and $I_2 = \inf_{B(x_0, \varepsilon r)} \psi_2$ to see that

$$I_2 \leq \psi_2(x_0) = \psi(x_0) \leq \sup_{B(x_0, \varepsilon r)} \psi \leq \frac{1 + \varepsilon}{1 - \varepsilon} I + 4\varepsilon(\zeta(2\varepsilon r) + \zeta(r))r$$

and hence

$$\begin{aligned}\sup_{B(x_0, \varepsilon r)} \psi_1 &\leq \sup_{B(x_0, \varepsilon r)} \psi_2 - I \\ &\leq \left(\frac{1 + \varepsilon}{1 - \varepsilon} - \frac{1 - \varepsilon}{1 + \varepsilon} \right) I_2 + \left(\frac{4\varepsilon}{1 + \varepsilon} + \frac{2\varepsilon}{1 - \varepsilon} \right) (\zeta(2\varepsilon r) + \zeta(r))r.\end{aligned}$$

The desired inequality follows from this one by simple algebra. \square

These lemmata allow us to imitate the argument in [18, Lemma 1.1] to prove an analog of Lemma 2.1 when u is a solution of the double obstacle problem:

$$(4.6a) \quad \psi_1 \leq u \leq \psi_2 \text{ in } \Omega,$$

$$(4.6b) \quad \min\{-F(x, u, Du, D^2u), u - \psi_1\} = 0 \text{ if } u < \psi_2,$$

$$(4.6c) \quad \min\{F(x, u, Du, D^2u), \psi_2 - u\} = 0 \text{ if } u > \psi_1.$$

Lemma 4.3. *Suppose u , ψ_1 , and ψ_2 are as above, and suppose that there are positive constants λ , Λ , and μ_0 such that conditions (2.3), (2.4b), and*

$$(4.7) \quad |F(x, u, Du, 0)| \leq \mu_0 \lambda$$

are satisfied. If x_0 is a point such that $u(x_0) = \psi_1(x_0)$, then there are positive constants C , δ , and κ determined only by n and Λ/λ such that \bar{u} , defined by

$$(4.8) \quad \bar{u}(x) = u(x) - u(x_0) - Y_1(x_0) \cdot (x - x_0),$$

satisfies the estimate

$$(4.9) \quad \left(R^{-n} \int_{B(x_0, \delta R)} |\bar{u}|^\kappa dx \right)^{1/\kappa} \leq C[\mu_0 R^2 + \zeta(R)R]$$

for all $R \leq d(x_0)$.

Proof. We first note that the hypotheses of this lemma are unchanged if we subtract the same linear function from u , ψ_1 and ψ_2 , so we may assume that $\psi_1(x_0) = 0$ and $Y_1(x_0) = 0$. Then, for $\varepsilon \in (0, 1)$ to be chosen, we set $I_2 = \inf_{B(x_0, \varepsilon R)} \psi_2$.

If $I_2 \leq 12\zeta(R)R + \mu_0 R^2$, then (4.4) implies that $\psi_2 \leq C(\varepsilon)[\mu_0 R^2 + \zeta(R)R]$ in $B(x_0, \varepsilon R)$, and (4.9) follows for any $\delta \leq \varepsilon$.

On the other hand, if $I_2 > 12\zeta(R)R + \mu_0 R^2$, we set

$$\begin{aligned} M &= (1 - \varepsilon)I_2, \\ M_1 &= \frac{5\varepsilon}{1 - \varepsilon^2}I_2 + 2\zeta(R)R, \\ U &= \min\{u, M\} + M_1, \end{aligned}$$

and we note that $U \geq 0$. Now, for $\eta > 0$, define f_η by

$$f_\eta(t) = (\max\{t, 0\}^3 + \eta^3)^{1/3} - \eta,$$

and set

$$U_\eta = M + M_1 - f_\eta(M - u).$$

Then $U_\eta \rightarrow U$ uniformly as $\eta \rightarrow 0$ and $U_\eta \geq 0$ in $B(x_0, \varepsilon R)$. Moreover, because $a^{ij}D_{ij}u \leq \lambda\mu_0$ wherever $u \leq M$ and f_η is C^2 with $f''_\eta \geq 0$, it follows

that $a^{ij}D_{ij}U_\eta \leq \lambda\mu_0$. It follows from the weak Harnack inequality [10, Theorem 9.22] that

$$\left((\varepsilon R)^{-n} \int_{B(x_0, \varepsilon R/2)} |U_\eta|^\kappa dx \right)^{1/\kappa} \leq C_1 \left[\inf_{B(\varepsilon R/2)} U_\eta + \mu_0(\varepsilon R)^2 + \zeta(\varepsilon R)(\varepsilon R) \right]$$

for some $C_1(n, \Lambda/\lambda)$ and $\kappa(n, \Lambda/\lambda)$. Sending $\eta \rightarrow 0$, we infer that

$$(4.10) \quad \left((\varepsilon R/2)^{-n} \int_{B(x_0, \varepsilon R/2)} |U|^\kappa dx \right)^{1/\kappa} \leq C_1 [M_1 + \varepsilon\mu_0 R^2 + \varepsilon\zeta(R)R]$$

because $U(x_0) = M_1$.

Next, we set $M_2 = \sup_{B(x_0, \varepsilon R)} \psi_1 + \zeta(R)R$ and $V = \max\{u, M_2\}$. By a similar approximation argument, we infer from the local maximum principle [10, Theorem 9.20] that there is a constant $C_2(n, \Lambda/\lambda)$ so that

$$\begin{aligned} & \sup_{B(x_0, \varepsilon R/4)} V \\ & \leq C_2 \left[\left(\left(\frac{\varepsilon R}{2} \right)^{-n} \int_{B(x_0, \varepsilon R/2)} V^\kappa dx \right)^{1/\kappa} + \mu_0(\varepsilon R)^2 + \zeta(\varepsilon R)\varepsilon R \right]. \end{aligned}$$

Now we note that $u \leq U$ (because $M_1 \geq 0$ and $M + M_1 \geq \sup_{B(x_0, \varepsilon R)} \psi_2$) and $M_2 \leq M_1 - \zeta(R)R$ (because $M_2 \leq 5I_2\varepsilon/(1 - \varepsilon^2)$), so $U_2 \leq U_1$ provided $\varepsilon \leq 1/2$. It follows that

$$\begin{aligned} \sup_{B(x_0, \varepsilon R/4)} u & \leq C_2(C_1[M_1 + \mu_0\varepsilon R^2 + \varepsilon\zeta(R)R] + \mu_0\varepsilon R^2 + \varepsilon\zeta(R)R) \\ & \leq C_1C_2 \frac{2\varepsilon}{1 - \varepsilon} \psi_2(x_2) + \left(\frac{5C_1}{1 - \varepsilon^2} + C_1 + 1 \right) C_2\varepsilon I_2. \end{aligned}$$

By taking ε sufficiently small, we conclude that $u < \psi_2$ on $B(x_0, \varepsilon R/4)$. Therefore, $u + \zeta(R)R$ is a positive supersolution on $B(x_0, \varepsilon R/4)$ and we can use the weak Harnack inequality directly to infer (4.9) with $\delta = \varepsilon/8$. \square

Note that the arguments of Lemmata 3.1 and 4.3 can be combined to prove pointwise decay of \bar{u} near a contact point. Specifically, suppose u satisfies (2.2) with F satisfying (2.3), (2.4b), and (4.7). If $u(x_0) = \psi(x_0)$, then (2.6) and the proof of $\bar{\psi} \leq Cz$ in E' (from Lemma 3.1) give a constant c_1 such that $\bar{\psi} \leq c_1[\zeta(R) + \mu_0R]R$ in $B(R)$. The local maximum principle applied to $\max\{\bar{u}, (1 + c_1)[\zeta(R) + \mu_0R]R\}$ then yields $\bar{u} \leq C[\zeta(R) + \mu_0R]R$ in $B(R/4)$. With this pointwise estimate in hand, we can imitate the proof of [4, Theorem 2.3] to obtain a modulus of continuity estimate for solutions of obstacle problems with linear equations when ζ does not necessarily satisfy condition (2.7).

For our purposes, the next important step is to obtain a modulus of continuity for Du .

Theorem 4.4. *Suppose that u , ψ_1 , ψ_2 , and F satisfy conditions (4.6), (4.7), (2.3), and (2.4b) with ζ a continuous increasing function on $[0, \text{diam } \Omega]$. Suppose also that there are constants α , ν_0 and ν_1 along with a continuous increasing function ζ_1 with $\zeta_1(0) = 0$ such that conditions (2.11) and (2.12) hold. If F is convex or concave with respect to r , then there are constants $\alpha_0(n, \Lambda/\lambda, \nu_1)$ and C determined only by n , α , ν_1 , Λ/λ , $|u|_1$, and $\text{diam } \Omega$ such that $\alpha \leq \alpha_0$ implies (3.12) for all x_1 and x_2 in Ω with $|x_1 - x_2| \leq \frac{1}{4} \min\{d(x_1), d(x_2)\}$.*

Of course, the double obstacle analog of Theorem 2.2 holds with $\alpha = 1$ if condition (2.11) is replaced by (2.8).

We can use the same ideas for oblique derivative problems, but the proofs are more complicated. In place of Lemma 4.1, we have a similar, but more subtle, inequality. To state our results more simply, we let ω be a C^1 function in some $(n - 1)$ -dimensional ball $B(0, R)$ with $R > 0$ and $\omega(0) = 0$, and we set $\omega_0 = \sup |D\omega|$. We also define

$$K[r] = \{x \in \mathbb{R}^n : x^n < r - (\omega_0 + 1)|x'|, x^n > \omega(x')\},$$

$$\Sigma[r] = \{x \in \mathbb{R}^n : x^n < r - (\omega_0 + 1)|x'|, x^n = \omega(x')\}$$

for $r \in (0, R)$, where here and below we abbreviate $x' = (x^1, \dots, x^{n-1})$.

Lemma 4.5. *Let ψ_1 and ψ_2 be two functions defined in $K[r] \cup \Sigma[r]$ for some $r \in (0, R)$ with $\psi_1 \leq \psi_2$ there. Suppose that there are vector fields Y_1 and Y_2 such that conditions (4.1) hold for all x_1 and x_2 in $K[r] \cup \Sigma[r]$. Suppose also that there are positive constants $\alpha \leq 1$, $\mu_3 < 1/\omega_0$, μ_3 , and χ_0 along with a function G such that*

$$(4.11a) \quad \left| \frac{\partial G}{\partial p'}(x, z, p) \right| \leq \mu_3 \chi_0,$$

$$(4.11b) \quad G^n(x, z, p) \geq \chi_0,$$

$$(4.11c) \quad |G(x, z, p) - G(x, w, p)| \leq \mu_0 \chi_0 |w - z|^\alpha$$

for any $(x, z, p) \in \Sigma[r] \times \mathbb{R} \times \mathbb{R}^n$ and any $w \in \mathbb{R}$, and set

$$(4.12) \quad g_0 = \frac{1}{\chi_0} \inf_{K[\eta r]} (G(x, \psi_2(x), Y_2(x)) - G(x, \psi_1(x), Y_1(x)))^+.$$

Then, for any $\varepsilon \in (0, 1)$, there is a constant $\eta(\varepsilon, \mu_3, \omega_0)$ such that

$$(4.13) \quad \sup_{K[\eta r]} (\psi_2 - \psi_1) \leq (1 + \varepsilon) \inf_{K[\eta r]} (\psi_2 - \psi_1) + 3\varepsilon \zeta(r)r + C_1 \varepsilon r^{2/(2-\alpha)} + \varepsilon g_0 r$$

for $C_1 = 3(\mu_0 \sup[\psi_2 - \psi_1]^{\alpha/2})^{2/(2-\alpha)}$.

Proof. To simplify the notation, we shall set $\psi = \psi_2 - \psi_1$, $Y = Y_2 - Y_1$, and $K = K[\eta r]$. In addition, we write I for the infimum of ψ over K and we let x_2 be a point in the closure of K at which $\psi(x_2) = I$. We now consider several cases.

Suppose first that $x_2 \in \Sigma[r]$. Then we infer from (4.11) and (4.12) that

$$(4.14) \quad g_0\chi_0 \geq -\mu_0\chi_0|\psi(x_2)|^\alpha + v \cdot Y(x_2)$$

for some vector v with $|v'| \leq \mu_3v^n$ and $v^n \geq \chi_0$. Now we set $\xi = v/|v|$ and we set

$$x_1 = x_2 + \frac{\varepsilon}{8}r\xi, \quad x_3 = x_2 + \frac{1}{2}r\xi.$$

If $\eta < 1/2$, it follows that x_1 and x_3 are in $\Omega[r]$. Setting $I_1 = \psi(x_1)$, we see that

$$\begin{aligned} I_1 &\leq I + Y(x_2) \cdot (x_1 - x_2) + \frac{\varepsilon}{8}\zeta(r)r = I + \frac{\varepsilon r}{8}Y(x_2) \cdot \xi + \frac{\varepsilon}{8}\zeta(r)r \\ &\leq I + \frac{\mu_0\varepsilon}{8}I^\alpha r + \frac{\varepsilon}{8}\zeta(r)r + \frac{\varepsilon}{8}g_0r \\ &\leq \left(1 + \frac{\varepsilon}{2}\right)I + \frac{C_1}{3}\varepsilon r^{2/(2-\alpha)} + \frac{\varepsilon}{8}\zeta(r)r + \frac{\varepsilon}{8}g_0r \end{aligned}$$

by virtue of (4.14) and Young's inequality. Now we obtain two estimates for $Y(x_1)$. First, there is a constant $k(\omega_0, \mu_3)$ such that $B(x_1, k\varepsilon r) \subset K[r]$ and then the proof of (4.3) with $\varepsilon = 1/2$ shows that

$$|Y(x_1)| \leq \frac{2I_1}{k\varepsilon r} + 4\zeta(r) \leq \frac{3}{2\varepsilon k r}I + \frac{2C_1}{3k}r^{\alpha/(2-\alpha)} + \left(\frac{2}{k} + 4\right)\zeta(r) + \frac{g_0}{4k}.$$

Moreover,

$$0 \leq \psi(x_3) \leq I_1 + \left(\frac{1}{2} - \frac{\varepsilon}{8}\right)rY(x_1) \cdot \xi + \zeta(r)r$$

because $x_3 - x_1 = (1/2 - \varepsilon/8)r\xi$ and hence

$$-rY(x_1) \cdot \xi \leq \frac{1 + \varepsilon/2}{1/2 - \varepsilon/8}I + 8\zeta(r)r + \frac{4}{3}C_1\varepsilon r^{2/(2-\alpha)} + \frac{\varepsilon}{2}g_0r.$$

Now we note that, for any $x \in K$, we have $|x| < 2\eta r$, and hence

$$\psi(x) \leq \psi(x_1) + Y(x_1) \cdot (x - x_1) + 4\eta\zeta(4\eta r)r.$$

To analyze the right hand side of this inequality, we first observe that $x - x_1 = (x - x_2) + (x_2 - x_1)$ and that $|x - x_2| \leq 4\eta r$. It follows that

$$\begin{aligned} & Y(x_1) \cdot (x - x_1) \\ & \leq -\frac{\varepsilon}{8} r Y(x_1) \cdot \xi + 4\eta r |Y(x_1)| \\ & \leq \left((1 + \varepsilon/2) \frac{\varepsilon/8}{1/2 - \varepsilon/8} + \frac{6\eta}{2\varepsilon k} \right) I + \left(\left(\frac{2}{k} + 4 \right) 2\eta + \varepsilon \right) \zeta(r)r \\ & + C_1 \left(\frac{\varepsilon^2}{3} + \frac{8\eta}{3k} \right) r^{2/(2-\alpha)} + \left(\frac{\varepsilon}{2} + \frac{2\eta}{2k} \right) g_0 r, \end{aligned}$$

and

$$\begin{aligned} \psi(x) & \leq \left((1 + \varepsilon/2) \frac{1/2}{1/2 - \varepsilon/8} + \frac{6\eta}{k\varepsilon} \right) I + \frac{C_1}{3} \left(2\varepsilon + \frac{8\eta}{k} \right) r^{2/(2-\alpha)} \\ & + \left(2\varepsilon + \left(\frac{2}{k} + 6 \right) 2\eta \right) \zeta(r)r + \left(\frac{\varepsilon}{2} + \frac{2\eta}{2k} \right) g_0 r \end{aligned}$$

provided $4\eta \leq 1$. By simple calculation, $(1 + \varepsilon/2)(1/2)/((1/2) - \varepsilon/8) < 1 + \varepsilon$, so we can take η sufficiently small to infer (4.13) in this case.

If $x_2 \notin \Sigma[r]$, then $Y^n(x_2) \geq 0$ and we can imitate the calculations of the preceding case with $\xi = (0, \dots, 0, 1)$, to see that

$$\psi(x) \leq \left(\frac{1/2}{1/2 - \varepsilon/8} + \frac{4\eta}{k\varepsilon} \right) I + \left[\left(\frac{1}{2k} + 8 \right) 2\eta + \left(\frac{5/8}{1/2 - \varepsilon/8} + 1 \right) \varepsilon \right] \zeta(r)r,$$

which implies (4.13) if η is sufficiently small. □

Our next step is to prove a corresponding estimate for our general geometric situation.

Lemma 4.6. *Let ψ_1 and ψ_2 be two functions defined in $\bar{\Omega}$ with $\psi_1 \leq \psi_2$. Suppose conditions (4.1) and (3.8) are satisfied. Let $x_0 \in \Omega$ and set*

$$(4.15) \quad g_1(r) = \frac{1}{\chi_0} \sup_{\partial\Omega[r]} (G(x, \psi_2(x), Y_2(x)) - G(x, \psi_1(x), Y_1(x)))^+.$$

If $\partial\Omega \in C^1$, then for any $\varepsilon > 0$, there are constants $R(\mu_2, \Omega)$, $\delta(\varepsilon, \mu_2, \Omega)$ and $C(\mu_2, \mu_0, \alpha, \sup(\psi_2 - \psi_1))$ such that

$$(4.16) \quad \sup_{\Omega[\delta r]} (\psi_2 - \psi_1) \leq (1 + \varepsilon) \inf_{\Omega[\delta r]} (\psi_2 - \psi_1) + C\varepsilon[\zeta(r)r + r^{2/(2-\alpha)}] + \varepsilon g_1(r)r$$

for any $x_0 \in \Omega$ and $r \in (0, R)$.

Proof. Let x_1 be a closest point to x_0 in $\partial\Omega$, which we can take to be the origin, and rotate axes so that $x'_0 = 0$ and $x^n_0 > 0$. Then there is a

constant R_1 determined only by μ_2 and Ω so that there is a function ω with $\mu_2 \sup |D\omega| < 1/2$ such that

$$\Omega[R] = \{x \in \mathbb{R}^n : |x - x_0| < R, x^n > \omega(x')\}$$

and $\omega(0) = 0$ and $D\omega(0) = 0$. By choosing $R < R_1$ sufficiently small, we can also arrange that $|G^n - G_p \cdot \gamma| \leq \frac{1}{2}\chi_0$ and $|D\omega| < 1/2$. It follows that conditions (4.11a,b) hold with $\mu_3 = 2\mu_2$.

Now take η to be the constant from Lemma 4.5 and note that there is a constant η_1 such that $d(x_0) \leq \eta_1 r$ implies that $\Omega[\eta_1 r] \subset K[\eta r]$. Therefore (4.16) holds in this case with any $\delta \leq \eta_1$. On the other hand, if $d(x_0) > \eta_1 r$, then Lemma 4.1 (with $\eta_1 r$ in place of r) implies (4.16) in this case with $\delta = \eta_1 \varepsilon/3$ because $(1 + \varepsilon/3)/(1 - \varepsilon/3) \leq 1 + \varepsilon$. Combining this two cases yields the desired result with $\delta = \eta_1 \varepsilon/2$. □

As before, we then have the following estimates.

Corollary 4.7. *In addition to the hypotheses of Lemma 4.6, suppose that $\psi_1(x_0) = 0$ and $Y_1(x_0) = 0$, and set*

(4.17)

$$g_2(r) = \frac{1}{\chi_0} \sup_{\partial\Omega[r]} (G(x, \psi_2(x), Y_2(x)) - \min\{G(x, 0, 0), G(x, \psi_1(x), Y_1(x))\})^+.$$

Then

(4.18)

$$\sup_{B(x_0, \delta r)} \psi_2 \leq (1 + \varepsilon) \inf_{\Omega[\delta r]} \psi_2^+ + C\varepsilon[\zeta(r) + r^{\alpha/(2-\alpha)} + g_2(r) + (\zeta(r)r)^\alpha]r.$$

If also $\psi_2 \geq 0$ in $\Omega[\delta r]$, then

(4.19)

$$\sup_{B(x_0, \delta r)} \psi_1 \leq 2\varepsilon \inf_{\Omega[\delta r]} \psi_2 + C\varepsilon[\zeta(r) + r^{\alpha/(2-\alpha)} + g_2(r) + (\zeta(r)r)^\alpha]r.$$

Proof. We follow the proof of Corollary 4.2, noting that

$$G(x, -\zeta(r)r, 0) \geq G(x, 0, 0) - \mu_0(\zeta(r)r)^\alpha$$

and that $(1 + \varepsilon) - 1/(1 + \varepsilon) \leq 2\varepsilon$. □

The estimate on the modulus of continuity for the gradient of the solution of the double obstacle problem follows easily.

Theorem 4.8. *Let $\partial\Omega \in C^{1,\alpha}$ for some $\alpha \in (0, 1)$, and let ψ_1 and ψ_2 be two functions satisfying condition (4.1) in Ω for some continuous increasing function ζ . Suppose also that $\psi_1 \leq \psi_2$ in Ω . Let $u \in W_{loc}^{2,n} \cap C^1(\bar{\Omega})$ satisfy (4.6) and $G(x, u, Du) = 0$ on $\partial\Omega$. Suppose there are constants λ, μ_0, μ_2 , and ν_1 , along with a continuous increasing function ζ_1 with $\zeta_1(0) = 0$ such*

that conditions (2.11), (2.12), (2.3), (2.4b), (4.7), and (3.8) hold. Suppose finally that

$$(4.20) \quad G(x, \psi_1(x), Y_1(x)) \geq 0, \quad G(x, \psi_2(x), Y_2(x)) \leq 0,$$

for all $x \in \partial\Omega$. If F is concave or convex with respect to r , then there are constants $\alpha_0(n, \mu_2, \Lambda/\lambda, \nu_1)$ and $C(n, \alpha, \Lambda/\lambda, \mu_1, \mu_2, |D\psi_1|_1, |D\psi_2|_1, \zeta_1, \Omega)$ such that if $\alpha \leq \alpha_0$, then (3.10) holds for all x_1 and x_2 in Ω .

Proof. We proceed by combining the proof of Theorem 3.2 with that of Theorem 4.4, taking Corollary 4.7 into account. \square

We omit the obvious two-obstacle analog of Theorem 3.3.

5. Variational inequalities.

Our methods also apply to certain types of variational inequalities. In particular, let H be a convex, C^2 function defined on $[0, \infty)$ with $H(0) = 0$ and suppose that $h = H'$ satisfies the conditions

$$(5.1) \quad \delta \leq \frac{th'(t)}{h(t)} \leq g_0$$

for some positive constants δ and g_0 , and all $t > 0$; we also assume that $H(1) = 1$ for simplicity. The model such function is $H(t) = t^m$ with $m > 1$. Let $W^{1,H}$ denote the set of all functions $v \in W^{1,1}$ with $H(|Dv|) \in L^1(\Omega)$, and write K for a convex subset of $W^{1,H}$ such that $v \geq \psi$ for all $v \in K$. (For example if $H(t) = t^m$, then $W^{1,H} = W^{1,m}$ and we can take K to be the set of all $v \in W^{1,m}$ with $v \geq \psi$.) We then consider the problem of finding a function $u \in K$ such that

$$(5.2) \quad \int_{\Omega} [A(x, u, Du) \cdot D(u - v) - B(x, u, Du)(u - v)] dx \leq 0$$

for all $v \in K$, where A is a vector-valued function (for example $A(x, z, p) = h(|p|)p/|p|$) and B is a scalar-valued function, which we shall assume to be bounded. Such problems have a long history for various choices of h provided A and B satisfy suitable structure conditions; see, for example, [14, Section III.4], [18], [6], [8], [9], [22], [25], [27]. We note, however, that all of these works assume that ψ has Hölder continuous gradient when trying to prove a modulus of continuity estimate for the gradient of u .

We first observe that, when $h(t)/t$ is bounded from above and below by positive constants and A and B are sufficiently smooth, smooth solutions of this variational inequality are also solutions of (2.2) with

$$F(x, z, p, r) = \frac{\partial A^i}{\partial p_j}(x, z, p)r_{ij} + \frac{\partial A^i}{\partial z}(x, z, p)p_i + \frac{\partial A^i}{\partial x^i}(x, z, p) + B(x, z, p).$$

More generally, we assume that A is differentiable with respect to p and that there are nonnegative constants α , Λ and Λ_1 with $\alpha \in (0, 1)$ such that

$$(5.3a) \quad \frac{\partial A^i}{\partial p_j} \xi_i \xi_j \geq \frac{h(|p|)}{|p|} |\xi|^2,$$

$$(5.3b) \quad |A_p| \leq \Lambda h(|p|)/|p|,$$

$$(5.3c) \quad |B| \leq \Lambda_1,$$

$$(5.3d) \quad |A(x, z, p) - A(y, w, p)| \leq \Lambda_1(|x - y| + |w - z|)^\alpha.$$

These conditions were studied extensively in [17]. In fact, we have simplified the conditions there somewhat by assuming a known bound for the gradient of u .

We begin by proving an estimate like (2.6). As in [18], we first prove the estimate for a simpler problem.

Lemma 5.1. *Let A be a vector valued function defined on \mathbb{R}^n and suppose that there are positive constants δ , g_0 , and Λ along with a function h such that conditions (5.1) and (5.3a,b) are satisfied. Let u and ψ be in $C^{0,1}(B(x_0, r))$ for some ball $B(x_0, r)$ with $u \geq \psi$, and let K be the set of all v with $v - u \in W_0^{1,H}(B(x_0, r))$ and $v \geq \psi$ in $B(x_0, r)$. Then there is a unique solution U of the variational inequality*

$$(5.4) \quad \int_{B(x_0, r)} A(DU) \cdot D(U - v) \, dx \leq 0 \quad \text{for all } v \in K,$$

and there are constants $C_1(n, \delta, g_0, |Du|_0, |D\psi|_0, \Lambda)$, $C_2(\Lambda, n)$, $\theta(\Lambda, n, \delta, g_0)$, and $\kappa(\Lambda, n)$ such that

$$(5.5) \quad [U]_{\theta; B(x_0, r)} \leq C_1 r^{1-\theta}$$

and

$$(5.6) \quad \left(r^{-n} \int_{B(x_0, r/2)} |U - L|^\kappa \, dx \right)^{1/\kappa} \leq C_2 \inf_{B(x_0, r/2)} (U - L)$$

for any linear function L such that $U - L \geq 0$ in $B(x_0, r)$.

Proof. The standard theory of variational inequalities gives the existence and uniqueness of U . In addition, (5.5) follows from the arguments in [18, Lemma 1.3].

To prove (5.6), we proceed by approximation. First, we fix $\alpha \in (0, 1)$ and note (from the proof of [17, Lemma 5.2]) that there is a sequence of $C^{1,\alpha}$ functions (A_k) which converge uniformly to A on compact subsets of $B(x_0, r)$ and which satisfy

$$\frac{\partial A_k^i}{\partial p_j} \xi_i \xi_j \geq \frac{h_k(|p|)}{|p|} |\xi|^2$$

and

$$\left| \frac{\partial A_k^i}{\partial p_j} \right| \leq 2\Lambda h_k(|p|)/|p|$$

for functions h_k satisfying (5.1). For $\varepsilon \in (0, 1)$, define β_ε by $\beta_\varepsilon(t) = (\min\{t, 0\})^2/\varepsilon$, and let U_k solve $\operatorname{div} A_k(DU_k) + \beta_{1/k}(U_k - \psi) = 0$ in $B(x_0, r)$ and $U_k = u$ on $\partial B(x_0, r)$. The existence of a unique solution to this problem is straightforward, and classical regularity theory implies that $U_k \in C^2(B(x_0, r))$. Thus, the weak Harnack inequality [10, Theorem 9.22] implies that

$$\left(r^{-n} \int_{B(x_0, r/2)} |U_k - L|^\kappa dx \right)^{1/\kappa} \leq C(\Lambda, n) \inf_{B(x_0, r/2)} (U_k - L)$$

for any linear function L with $U_k - L \geq 0$ in $B(x_0, r)$. It is not hard to show that U_k converges uniformly to U as $k \rightarrow \infty$ (see, for example, [14, Theorem IV.5.2]), so the desired result follows immediately. \square

From this lemma and a suitable choice for the linear function L , we infer a version of (2.6).

Lemma 5.2. *Under the hypotheses given before Lemma 5.1, suppose that ψ satisfies condition (2.1) for some continuous increasing function ζ . Suppose $u(x_0) = \psi(x_0)$ and define \bar{u} by (2.5). If κ and θ are the constants from Lemma 5.1 and if $r \leq 1$, then there is a constant C determined only by $\Lambda, n, \Lambda_1, |Du|_0$ such that*

$$(5.7) \quad \left(r^{-n} \int_{B(x_0, r/2)} |\bar{u}|^\kappa dx \right)^{1/\kappa} \leq C[r^{\alpha/(2n+2\theta)} + \zeta(r)]r.$$

Proof. Let U be the solution of (5.4) given by Lemma 5.1 with $A(p) = A(x_0, u(x_0), p)$ and set $w = u - U$. Then we can use $v = U$ in (5.2) and $v = u$ in (5.4) to see from [17, (5.8) and Lemma 2.2] that

$$\int_{B(x_0, r)} H(|w|/r) dx \leq C \int_{B(x_0, r)} H(|Dw|) dx \leq Cr^{n+\alpha/2}$$

because H is convex. Then Jensen’s inequality gives

$$(5.8) \quad \int_{B(x_0, r)} |w| dx \leq Cr^{n+1+\alpha/2}$$

because [17, Lemma 1.1(c)] says that $H(r^{\alpha/2})/H(1) \leq r^{\alpha/2}/1$.

To continue, we use a variation of the argument in Lemma 1.1. Choose x_1 so that $d(x_1)|w(x_1)| \geq (1/2)|w|_0^{(n)}$ and set $\rho = \varepsilon d(x_1)$ with $\varepsilon \in (0, 1/2)$. We have from (5.5) that

$$|w(x)| \geq |w(x_1)| - |w(x) - w(x_1)| \geq \frac{1}{2}d(x_1)^{-n}|w|_0^{(n)} - cr^{1-\theta}\rho^\theta$$

for $|x - x_1| \leq \rho$, so

$$\int_{B(x_0,r)} |w| dx \geq \int_{B(x_1,\rho)} |w| dx \geq \frac{\omega_n}{2} \varepsilon^n |w|_0^{(n)} - c\omega_n r^{1-\theta} \rho^{n+\theta},$$

where ω_n is the measure of the n -dimensional unit ball. Therefore

$$|w|_0^{(n)} \leq C\varepsilon^{-n} \int_{B(x_0,r)} w dx + C\varepsilon^\theta d^{n+\theta} r^{1-\theta} \leq Cr^{n+1}(\varepsilon^{-n} r^{\alpha/2} + \varepsilon^\theta)$$

from (5.8). Now we take $\varepsilon = r^{\alpha/(2n+2\theta)}/2$ to conclude that

$$\sup_{B(x_0,r/2)} |w| \leq cr^{-n} |w|_0^{(n)} \leq Cr^{1+\alpha/(2n+2\theta)}.$$

Thus we can take $L(x) = \psi(x_0) - Y(x_0) \cdot (x - x_0) - \zeta(r)r - Cr^{1+\alpha/(2n+2\theta)}$ in Lemma 5.2, and hence (5.7) holds. □

The interior gradient modulus of continuity estimate for such problems follows by using the argument of Theorem 2.3 and the Hölder gradient estimates for weak solutions of divergence structure equations from [17, Section 5]. The correct form of this estimate is an easy consequence of the last inequality on page 346 of [17].

Theorem 5.3. *Let A and B be, respectively, a vector-valued function and a scalar-valued function on $\Omega \times \mathbb{R} \times \mathbb{R}^n$, and let H be a convex, C^2 function on $[0, \infty)$ with $H(0) = 0$, and suppose $h = H'$ satisfies (5.1). Suppose also that conditions (5.3a–d) are satisfied. Let ψ satisfy (2.1), let $u \in C^{0,1}(\bar{\Omega})$ and suppose $u \geq \psi$ in Ω . If u is a solution of (5.2) with K the set of all $v \in W^{1,H}$ with $v - u \in W_0^{1,H}$ and $v \geq \psi$, then there are constants $\sigma_0(\Lambda, \delta, g_0)$ and $C(n, \Lambda, \delta, g_0, \sup |Du|, \Psi_1, \Lambda_1, \alpha, \text{diam } \Omega)$ such that*

$$(5.9) \quad |Du(x_1) - Du(x_2)| \leq C \left[\zeta(|x_1 - x_2|) + \left(1 + \frac{\sup |Du|}{d(x_1)} \right) |x_1 - x_2|^\sigma \right]$$

for all x_1 and x_2 in Ω with $|x_1 - x_2| \leq \frac{1}{4}d(x_1)$, where $\sigma = \min\{\sigma_0, \alpha/(2n + 2\theta)\}$ and θ is the constant from Lemma 5.1.

The corresponding boundary regularity result is similar, and the proof is similar.

Theorem 5.4. *Let $\partial\Omega \in C^{1,\alpha}$ for some $\alpha \in (0, 1)$, let A and B be, respectively, a vector-valued function and a scalar-valued function on $\Omega \times \mathbb{R} \times \mathbb{R}^n$, let a_0 be a scalar valued function on $\partial\Omega \times \mathbb{R}$, and let H be a convex, C^2 function on $[0, \infty)$ with $H(0) = 0$, and suppose $h = H'$ satisfies (5.1). Suppose also that conditions (5.3a–d) are satisfied and that*

$$(5.10) \quad |a_0(x, z) - a_0(y, w)| \leq \Lambda_2(|x - y| + |z - w|)^\alpha$$

for all (x, z) and (y, w) in $\partial\Omega \times \mathbb{R}$. Let ψ satisfy (2.1) and

$$(5.11) \quad A(x, \psi, Y) \cdot \gamma + a_0(x, \psi) \geq 0$$

on $\partial\Omega$. If u is a solution of

$$(5.12) \quad \int_{\Omega} [A(x, u, Du) \cdot D(u - v) - B(x, u, Du)(u - v)] dx \leq \int_{\partial\Omega} a_0(x, u)(u - v) d\sigma$$

with K the set of all $v \in W^{1,H}$ with $v \geq \psi$, then there are constants $\sigma_0(\Lambda, \delta, g_0)$ and $C(n, \Lambda, \delta, g_0, \sup |Du|, \Psi_1, \Lambda_1, \alpha, \Omega)$ such that

$$(5.13) \quad |Du(x_1) - Du(x_2)| \leq C[\zeta(|x_1 - x_2|) + |x_1 - x_2|^\sigma]$$

for all x_1 and x_2 in Ω , where $\sigma = \min\{\sigma_0, \alpha/(2n + 2\theta)\}$.

Proof. To prove the analog of (5.7), we let U solve the variational inequality

$$\int_{\Omega} A(x_0, u(x_0), DU) \cdot D(U - v) dx \leq \int_{\partial\Omega} [a_0(x_0, u(x_0)) + C_0R] d\sigma,$$

where K is the set of all $v \in W^{1,H}$ with $v \geq \psi$ in Ω and $v = u$ on $\Omega \setminus \Omega[R]$; the constant C_0 , which is independent of x_0 and R , is chosen so that

$$A(x_0, \psi(x_0), Y(x)) \cdot \gamma(x) + a_0(x_0, \psi(x_0)) + C_0R \geq 0$$

for all $x \in \partial\Omega[R]$. The appropriate Hölder gradient estimate was proved in [15, Section 4] for the special case that A depends only on p , a_0 is constant, and B is identically zero, and the general estimate follows from the perturbation argument in [17, Section 5]. \square

We leave the straightforward modifications of these results for double obstacle problems to the reader. We do observe that the previous results for double obstacle problems (specifically [7, 18, 26]) all assume that the obstacle has Hölder continuous first derivatives. Thus, we have improved these results by considering general moduli of continuity and also suitable one-sided conditions.

6. Existence of solutions.

A suitable existence theory for our obstacle problem is based on known *a priori* estimates and the penalization method of Lions (see [14, Section IV.5]). We assume first that $\partial\Omega \in C^3$ (although this assumption can be relaxed by the remarks at the end of [21, Section 3]), and we assume that ψ satisfies (2.1) with $\zeta(t) = z_0t$. In addition, we assume that (3.9) holds. For ρ a $C^2(\bar{\Omega})$ function such that $D\rho = \gamma$ on $\partial\Omega$ (which always exists), we suppose that there are nonnegative constants M_0 and M_1 such that

$$(6.1a) \quad zF(x, z, -M_1D\rho, -M_1D^2\rho) < 0 \text{ in } \Omega,$$

$$(6.1b) \quad zG(x, z, -M_1\gamma) < 0 \text{ on } \partial\Omega$$

for $z \geq M_0$. Next, we assume that there are increasing functions μ and μ_0 such that

$$(6.2a) \quad \lambda(x, z, p, r)|\xi|^2 \leq F^{ij}(x, z, p, r)\xi_i\xi_j \leq \Lambda(x, z, p, r)|\xi|^2,$$

$$(6.2b) \quad \Lambda(x, z, p, r) \leq \mu(|z|)\lambda(x, z, p, r)$$

$$(6.2c) \quad |F(x, z, p, 0)| \leq \mu_0(|z|)\lambda(x, z, p, r)[1 + |p|^2]$$

for all $(x, z, p, r) \in \Gamma$ and

$$(6.2d) \quad |G(x, z, p')| \leq \mu_0(|z|)G_p(x, z, p) \cdot \gamma[1 + |p'|]$$

for all $(x, z, p) \in \Gamma'$, where $p' = p - (\gamma \cdot p)\gamma$. We also assume that there is an increasing function μ_1 such that

$$(6.3) \quad (1 + |p|)|F_p| + |F_z| + |F_x| \leq \mu_1(|z|)\lambda[1 + |p|^2 + |r|]$$

on Γ and

$$(6.4) \quad (1 + |p|)|G_p| + |G_z| + |F_x| \leq \mu_1(|z|)G_p \cdot \gamma[1 + |p|]$$

on Γ' . Finally we assume that F is concave (or concave) with respect to r and that λ is uniformly bounded above and uniformly positive on bounded sets of Γ , and we assume that $G_p(x, z, p) \cdot \gamma$ is uniformly bounded and uniformly positive on bounded subsets of Γ' . Note that [21, Lemma 7.1] implies the upper bound $u \leq M_1 \sup \rho$ while the obstacle condition imply that $u \geq \min \psi$. Hence, we may assume that conditions (6.2)–(6.4) hold with μ, μ_0 , and μ_1 independent of z by redefining F and G for large $|z|$ as needed. In particular, we may assume that F and G are independent of z for $z \leq \psi(x)$.

Now for $\varepsilon \in (0, 1)$, we define β_ε by

$$\beta_\varepsilon(t) = (\min\{t, 0\})^2/\varepsilon.$$

It then follows from [21, Lemma 7.1, Theorems 3.3, 4.1, and 7.8] along with [29, Theorem 3.3] (see also [19, Theorems 14.22 and 14.23]) that the problem

$$\begin{aligned} F(x, u_\varepsilon, Du_\varepsilon, D^2u_\varepsilon) + \beta_\varepsilon(u_\varepsilon - \psi) &= 0 && \text{in } \Omega, \\ G(x, u_\varepsilon, Du_\varepsilon) + \varepsilon &= 0 && \text{on } \partial\Omega \end{aligned}$$

has a $C^{2,\theta}(\overline{\Omega})$ solution for any $\varepsilon \in (0, 1)$ and some $\theta \in (0, 1)$ upon recalling our previous observations that we may take μ, μ_0 , and μ_1 independent of z . As previously remarked, [21, Lemma 7.1] implies that (u_ε) is uniformly bounded, independent of ε .

Now we estimate $\beta_\varepsilon(u_\varepsilon - \psi)$. If the minimum of $u_\varepsilon - \psi$ is nonnegative, then $\beta_\varepsilon = 0$. In addition, at a boundary minimum,

$$-\varepsilon = G(x, u_\varepsilon, Du_\varepsilon) \geq G(x, u_\varepsilon, Y),$$

so if the minimum of $u_\varepsilon - \psi$ is negative, it must occur at some $x_0 \in \Omega$. In this case, $Du_\varepsilon(x_0) = Y(x_0)$ and $D^2u_\varepsilon \geq -2z_0I$, where I denotes the $n \times n$ identity matrix, so

$$\beta_\varepsilon(u_\varepsilon - \psi)(x_0) = -F(x_0, u_\varepsilon, Du_\varepsilon, D^2u_\varepsilon) \leq F(x_0, \psi(x_0), Y(x_0), -2z_0I).$$

It follows that $\beta_\varepsilon(u_\varepsilon - \psi_\varepsilon) \leq c_1$ for some nonnegative constant c_1 independent of ε . We can then use [21, Theorem 3.3] to infer a global gradient bound for u_ε , which is uniform with respect to ε . We can then apply [3, Theorem 2] (see [19, Lemma 13.21, and Theorems 14.14 and 14.20] for a discussion of the extension to the oblique derivative boundary condition) to infer that $[Du_\varepsilon]_\alpha \leq c_2$ for constants $\alpha \in (0, 1)$ and c_2 independent of ε . Finally, [3, Theorem 1] shows that (D^2u_ε) is bounded in $L^p_{\text{loc}}(\Omega)$ for any $p < \infty$. From these estimates and the argument on pages 44 and 45 of [1], we infer that there is a sequence $(\varepsilon(j))$ such that $(u_{\varepsilon(j)})$ converges to a function $u \in W^{2,n}_{\text{loc}}(\Omega) \cap C^{1,\alpha}$ and that u solves (0.1). Theorem 2.3 then implies that $u \in C^{1,1}(\bar{\Omega})$.

Note that a more thorough existence theory can be derived via approximation of the obstacle; however, the convergence of the approximating solutions to a function in $W^{2,n}_{\text{loc}}(\Omega)$ requires at least that the obstacle be a supremum of $W^{2,n}_{\text{loc}}(\Omega)$ functions. On the other hand, the extension to two-obstacle problems, which we leave to the reader, is very simple.

References

- [1] J. Bao, *The obstacle problems for second order fully nonlinear elliptic equations with Neumann boundary conditions*, J. Partial Differential Equations, **5(3)** (1992), 33-45, MR 93f:35084, Zbl 776.35015.
- [2] ———, *Obstacle problems for second order fully nonlinear elliptic equations with nonlinear oblique boundary conditions*, Beijing Shifan Daxue Xuebao, **30** (1994), 143-149, MR 96d:35050, Zbl 821.35050.
- [3] L.A. Caffarelli, *Interior a priori estimates for solutions of fully non-linear equations*, Ann. Math., **130** (1989), 189-213, MR 90i:35046, Zbl 692.35017.
- [4] L.A. Caffarelli and D. Kinderlehrer, *Potential methods in variational inequalities*, J. Anal. Math., **73** (1980), 285-295, MR 82b:49009, Zbl 455.49010.
- [5] Y.Z. Chen, *$C^{1+\alpha}$ regularity of viscosity solutions of fully nonlinear elliptic PDE's under natural structure conditions*, J. Partial Differential Equations, **6(3)** (1993), 193-216, MR 94k:35055, Zbl 781.35011.
- [6] H.J. Choe, *A regularity theory for a more general class of quasilinear elliptic partial differential equations and obstacle problems*, Arch. Rational Mech. Anal., **114** (1991), 383-394, MR 92b:35068, Zbl 733.35024.
- [7] ———, *Regularity for certain degenerate double obstacle problems*, J. Math. Anal. Appl., **169** (1992), 111-126, MR 93h:35077, Zbl 798.35057.
- [8] H.J. Choe and J.L. Lewis, *On the obstacle problem for p harmonic functions*, SIAM J. Math. Anal., **22** (1991), 623-638, MR 92b:35055, Zbl 762.35035.

- [9] M. Giaquinta, *Remarks on the regularity of weak solutions to some variational inequalities*, Math. Z., **177** (1981), 15-31, MR 83k:49013, Zbl 453.35014.
- [10] D. Gilbarg and N.S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, 2nd. ed., Springer-Verlag, Berlin-Heidelberg-New York, 1983, MR 86c:35035, Zbl 562.35001.
- [11] B. Huisken, *Local $H^{2,\infty}$ -regularity near the boundary for quasilinear variational inequalities*, Manuscripta Math., **63** (1989), 333-342, MR 90d:35116, Zbl 703.49013.
- [12] G. Huisken, *Capillary surfaces over obstacles*, Pac. J. Math., **117** (1985) 121-141, MR 87c:49047, Zbl 561.49009.
- [13] R. Jensen, *Local quadratic growth estimates for elliptic obstacle problems*, preprint, 1990.
- [14] D. Kinderlehrer and G. Stampacchia, *An Introduction to Variational Inequalities*, Academic Press, New York, 1980, MR 81g:49013, Zbl 457.35001.
- [15] G.M. Lieberman, *Boundary regularity for solutions of degenerate elliptic equations*, Nonlinear Anal., **12** (1988), 1203-1219, MR 90a:35098, Zbl 675.35042.
- [16] ———, *Boundary regularity for linear and quasilinear variational inequalities*, Proc. Royal Soc. Edinburgh Sect. A, **112** (1989), 319-326, MR 91a:35035.
- [17] ———, *The natural generalization of the natural conditions of Ladyzhenskaya and Ural'tseva for elliptic equations*, Comm. Partial Differential Equations, **16** (1991), 311-361, MR 92c:35041, Zbl 742.35028.
- [18] ———, *Regularity of solutions to some double obstacle problems*, Indiana Univ. Math. J., **40** (1991), 1009-1028, MR 92m:35102, Zbl 767.35029.
- [19] ———, *Second Order Parabolic Differential Equations*, World Scientific, Singapore, 1996, MR 98k:35003, Zbl 884.35001.
- [20] ———, *Pointwise estimates for oblique derivative problems in nonsmooth domains*, J. Differential Equations, **173** (2001), 178-211.
- [21] G.M. Lieberman and N.S. Trudinger, *Nonlinear oblique derivative problems for nonlinear elliptic equations*, Trans. Amer. Math. Soc., **295** (1986), 509-546, MR 87h:35114, Zbl 619.35047.
- [22] P. Lindqvist, *Regularity for the gradient of the solution to a nonlinear obstacle problem with degenerate ellipticity*, Nonlinear Anal., **12** (1988), 1245-1255, MR 90b:49011.
- [23] P.-L. Lions and N.S. Trudinger, *Linear oblique derivative problems for the uniformly elliptic Bellman operator*, Math. Z., **191** (1986), 1-15, MR 87e:35041, Zbl 593.35046.
- [24] G.G. Lorentz, *Approximation of Functions*, Holt, Reinhart, and Winston, New York, 1966, MR 35 #4642, Zbl 153.38901.
- [25] J. Mu, *Higher regularity of the solution to the p -Laplacian obstacle problem*, J. Differential Equations, **92** (1992), 370-384, MR 93g:35036, Zbl 765.49008.
- [26] J. Mu and W.P. Ziemer, *Smooth regularity of solutions of double obstacle problems involving degenerate elliptic equations*, Comm. Partial Differential Equations, **16** (1991), 821-843, MR 92h:35086, Zbl 742.35010.
- [27] T. Norando, *$C^{1,\alpha}$ local regularity for a class of nonlinear possibly degenerate elliptic variational inequalities*, Boll. Un. Ital. Mat., **5-C** (1986), 281-292, MR 88m:35068, Zbl 639.49009.

- [28] M.V. Safonov, *On the classical solution of nonlinear elliptic equations of second order*, Izv. Akad. Nauk SSSR Ser. Mat., **52** (1988), 1272-1287 [Russian]; English transl. in Math. USSR-Izv., **33** (1989) 597-612, MR 90d:35104, Zbl 682.35048.
- [29] ———, *On the oblique derivative problem for second order elliptic equations*, Comm. Partial Differential Equations, **20** (1995), 1349-1367, MR 96e:35033, Zbl 841.35035.
- [30] G.M. Troianiello, *Maximal and minimal solutions to a class of elliptic quasilinear problems*, Proc. Amer. Math. Soc., **91** (1984), 95-101, MR 86a:35057, Zbl 524.35051.
- [31] ———, *Elliptic Differential Equations and Obstacle Problems*, Plenum Press, New York, 1987, MR 92b:35004, Zbl 655.35002
- [32] N.S. Trudinger, *Hölder gradient estimates for fully nonlinear elliptic equations*, Proc. Roy. Soc. Edinburgh, **108** (1988), 57-65, MR 90b:35041, Zbl 653.35026.

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