ENTROPY IN TYPE I ALGEBRAS

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It is shown that if \((M, \phi, \alpha)\) is a \(W^*-\)dynamical system with \(M\) a type I von Neumann algebra then the entropy of \(\alpha\) w.r.t. \(\phi\) equals the entropy of the restriction of \(\alpha\) to the center of \(M\). If furthermore \((N, \psi, \beta)\) is a \(W^*-\)dynamical system with \(N\) injective then \(h_{\phi \otimes \psi}(\alpha \otimes \beta) = h_{\phi}(\alpha) + h_{\psi}(\beta)\).

1. Introduction.

In the theory of non-commutative entropy the attention has almost exclusively been concentrated on non-type I algebras. We shall in the present paper remedy this situation by proving the basic facts on entropy of automorphisms of type I C\(^*\)- and von Neumann-algebras. The results are as nice as one can hope. The CNT-entropy of an automorphism of a von Neumann algebra of type I with respect to an invariant normal state is the classical entropy of the restriction of the automorphism to the center of the algebra. If one factor of a tensor product of two von Neumann algebras is of type I and the other injective, then the entropy of a tensor product automorphism with respect to an invariant product state is the sum of the entropies. The results have obvious corollaries to type I C\(^*\)-algebras. The main idea behind our proofs is the use of conditional expectations of finite index, as employed in \([GN]\).

We shall use the notation \(h_{\phi}(\alpha)\) for the CNT-entropy of a C\(^*\)-dynamical system as defined by Connes, Narnhofer and Thirring in \([CNT]\), and \(h'_{\phi}(\alpha)\) for the ST-entropy defined by Sauvageot and Thouvenot in \([ST]\).

2. Main results.

We first prove a general result for the Sauvageot-Thouvenot entropy for the restriction of an automorphism to a globally invariant C\(^*\)-subalgebra of finite index.

**Proposition 1.** Let \((A, \phi, \alpha)\) be a unital C\(^*\)-dynamical system. Let \(B \subset A\) be an \(\alpha\)-invariant C\(^*\)-subalgebra (with \(1 \in B\)). Suppose there exists a conditional expectation \(E: A \rightarrow B\) such that \(E \circ \alpha = \alpha \circ E, \phi \circ E = \phi\) and \(E(x) \geq cx\) for all \(x \in A^+\) for some \(c > 0\). Then \(h'_{\phi}(\alpha) = h'_{\phi}(\alpha|_B)\).
Proof. Let \((C, \mu, \beta)\) be a \(C^*\)-dynamical system with \(C\) abelian. Using \(E\) we can lift any stationary coupling on \(B \otimes C\) to a stationary coupling on \(A \otimes C\). This, together with the property of monotonicity of relative entropy, shows that 
\[
\hphi' (\alpha) \geq \hphi' (\alpha |B).
\]

Conversely, suppose \(\lambda\) is a stationary coupling of \((A, \phi, \alpha)\) with \((C, \mu, \beta)\), and \(P\) is a finite-dimensional subalgebra of \(C\) with atoms \(p_1, \ldots, p_n\). Let 
\[
\phi_i (a) = \frac{\lambda(a \otimes p_i)}{\mu(p_i)} \quad \text{for} \quad a \in A.
\]

Then in the notations of [ST] 
\[
\hphi' (\alpha) = \sup \left\{ H_{\mu}(P | P^*) - H_{\mu}(P) + \sum_{i=1}^n \mu(p_i) S(\phi, \phi_i) \left| (C, \mu, \beta, P) \right\} \right.
\]
\[
\left. \sum_{i=1}^n \mu(p_i) S(\phi, \phi_i) = \sum_{i=1}^n \mu(p_i) (S(\phi | B, \phi_i | B) + S(\phi_i \circ E, \phi_i)) \right.
\]
\[
\leq \sum_{i=1}^n \mu(p_i) S(\phi | B, \phi_i | B) - \log c.
\]

It follows that 
\[
\hphi' (\alpha) \leq \hphi' (\alpha |B) - \log c.
\]

Then for each \(m \in \mathbb{N}\) 
\[
\hphi' (\alpha) = \frac{1}{m} \hphi' (\alpha^m) \leq \frac{1}{m} \hphi' (\alpha^m |B) - \frac{1}{m} \log c = \hphi' (\alpha |B) - \frac{1}{m} \log c.
\]

Thus 
\[
\hphi' (\alpha) \leq \hphi' (\alpha |B). \quad \square
\]

By [ST, Proposition 4.1] the Sauvageot-Thouvenot entropy coincides with the CNT-entropy for nuclear \(C^*\)-algebras. In fact, what is really necessary for the coincidence of the entropies, is the existence of a net of unital completely positive mappings \(\gamma_i\) of finite-dimensional \(C^*\)-algebras into \(A\) such that 
\[
S(\phi, \psi) = \lim_i S(\phi \circ \gamma_i, \psi \circ \gamma_i)
\]
for any positive linear functional \(\psi\) on \(A\), \(\psi \leq \phi\). We therefore have:

Corollary 2. If in the above proposition \(A\) and \(B\) are injective von Neumann algebras and \(\phi\) is normal then \(h_\phi (\alpha) = h_\phi (\alpha |B)\).

To prove our main result we need also two simple lemmas. The first lemma is more or less well-known.

Lemma 3. Let \((M, \phi, \alpha)\) be a \(W^*\)-dynamical system. Then 

(i) if \(p\) is an \(\alpha\)-invariant projection in \(M\) such that \(\text{supp} \phi \leq p\), then 
\[
h_\phi (\alpha) = h_\phi (\alpha |M_p);
\]

(ii) if \(\phi\) is normal in the GNS-representation of \(\phi\), then 
\[
h_\phi (\alpha) = \sup \left\{ \frac{1}{m} \sum_{i=1}^m S(\phi_i \circ E, \phi) \mid \phi_i \right\}.
\]
(ii) if \( \{p_i\}_{i \in \mathcal{I}} \) is a set of mutually orthogonal \( \alpha \)-invariant central projections in \( M \), \( \sum_i p_i = 1 \), then
\[
h_{\phi}(\alpha) = \sum_i \phi(p_i)h_{\phi_i}(\alpha_i),
\]
where \( \phi_i = \frac{1}{\phi(p_i)}\phi \) is the normalized restriction of \( \phi \) to \( Mp_i \), and \( \alpha_i = \alpha|_{Mp_i} \).

Proof.
(i) easily follows from the definitions; (ii) follows from [CNT, VII.5(iii)], (i) and [SV, Lemma 3.3] applied to the subalgebras \( M(p_i + \cdots + p_n) + \mathbb{C}(1 - p_1 - \cdots - p_n) \).

The proof of the following lemma is left to the reader.

**Lemma 4.** Let \( T \) be an automorphism of a probability space \( (X, \mu) \), \( f \in L^\infty(X, \mu) \) a \( T \)-invariant function such that \( f \geq 0 \) and \( \int_X f \, d\mu = 1 \). Let \( \mu_f \) be the measure on \( X \) such that \( d\mu_f / d\mu = f \). Then \( h_{\mu_f}(T) \leq \|f\|_\infty h_\mu(T) \).

**Theorem 5.** Let \((M, \phi, \alpha)\) be a \( W^* \)-dynamical system with \( M \) a von Neumann algebra of type I. Let \( Z \) denote the center of \( M \). Then \( h_\phi(\alpha) = h_\phi(\alpha|_Z) \).

Proof. By Lemma 3(i) we may suppose that \( \phi \) is faithful. Then \( M \) is a direct sum of homogeneous algebras of type \( I_n \), \( n \in \mathbb{N} \cup \{\infty\} \). By Lemma 3(ii) we may assume that \( M \) is homogeneous of type \( I_n \). We first assume that \( n \in \mathbb{N} \). Then \( Z = L^\infty(X, \mu) \), where \( (X, \mu) \) is a probability space and \( \phi|_Z = \mu \).

Thus
\[
M \cong Z \otimes \text{Mat}_n(\mathbb{C}) = L^\infty(X, \text{Mat}_n(\mathbb{C})), \quad \phi = \int_X \phi_x d\mu(x),
\]
where \( \phi_x = \text{Tr}(.Q_x) \) is a state on \( \text{Mat}_n(\mathbb{C}) \), \( \text{Tr} \) the canonical trace on \( \text{Mat}_n(\mathbb{C}) \). We first assume \( Q_x \geq c > 0 \) for all \( x \).

If \( s \in M^+ \), \( s \) is a function in \( L^\infty(X, \text{Mat}_n(\mathbb{C})) \). Define the \( \phi \)-preserving conditional expectation \( E: M \to Z \) by \( E(s)(x) = \phi_x(s(x)) \). Then
\[
E(s)(x) = \text{Tr}(s(x)Q_x) \geq c\text{Tr}(s(x)) \geq cs(x),
\]
so \( E(s) \geq cs \), and it follows from Corollary 2 that \( h_\phi(\alpha) = h_\phi(\alpha|_Z) \).

If there is no \( c > 0 \) such that \( Q_x \geq c \) for all \( x \), let \( X_c = \{x \in X \mid Q_x \geq c\} \), \( (c > 0) \),
\[
N_c = L^\infty(X_c, \text{Mat}_n(\mathbb{C})) \quad \text{and} \quad M_c = N_c + \mathbb{C}\chi_{X \setminus X_c},
\]
where \( \chi_{X \setminus X_c} \) is the characteristic function of \( X \setminus X_c \). Since \( \phi \) is \( \alpha \)-invariant so is \( M_c \), so by the above argument and Lemma 3, letting \( \phi_c = \frac{1}{\mu(X_c)}\phi|_{N_c} \) and \( \mu_c = \frac{1}{\mu(X_c)}\mu|_{X_c} \), we obtain
\[
h_\phi(\alpha|_{M_c}) = \mu(X_c)h_{\phi_c}(\alpha|_{N_c}) = \mu(X_c)h_{\mu_c}(T|_{X_c}) \leq h_\mu(T),
\]
where $T$ is the automorphism of $(X, \mu)$ induced by $\alpha$. Letting $c \to 0$ and using [SV, Lemma 3.3] we obtain the Theorem when $M$ is finite.

If $M$ is homogeneous of type $I_{\infty}$, we have $M \cong L^\infty(X, \mu) \otimes B(H)$, where $H$ is a separable Hilbert space. Let $\text{Tr}$ denote the canonical trace on $B(H)$.

Write again
\[ \phi = \int_X \phi_x d\mu(x), \quad \phi_x = \text{Tr}(\cdot Q_x), \]
and let $E_x(U)$ denote the spectral projection of $Q_x$ corresponding to a Borel set $U$. Let $P_c \in M = L^\infty(X, B(H))$ be the projection defined by $P_c(x) = E_x([c, +\infty))$, where $c > 0$. Then $P_c$ is an $\alpha$-invariant finite projection. Let
\[ M_c = P_c MP_c + \mathbb{C}(1 - P_c). \]
Then $M_c$ is a finite type I von Neumann algebra. Its center is isomorphic to $L^\infty(X_c, \mu_c) \oplus \mathbb{C}$, and the restriction of $\phi$ to it is $\phi(P_c)\mu_c \oplus \phi(1 - P_c)$, where $X_c = \{ x \in X \mid P_c(x) \neq 0 \}$ and
\[ \int_{X_c} f(x) d\mu_c(x) = \frac{1}{\phi(P_c)} \int_{X_c} f(x) \phi_x(P_c(x)) d\mu(x). \]
So we can apply the first part of the proof to $M_c$. Since $d\mu_c/d\mu \leq 1/\phi(P_c)$, applying Lemma 4 we get
\[ h_\phi(\alpha|_{M_c}) = \phi(P_c) h_{\mu_c}(T|_{X_c}) \leq h_\mu(T). \]
Now letting $c \to 0$ we conclude that $h_\phi(\alpha) = h_\mu(T)$. \hfill \Box

It should be remarked that in a special case the above theorem was proved in [GS, Proposition 2.4].

If $A$ is a $C^*$-algebra and $\phi$ a state on $A$, the central measure $\mu_{\phi}$ of $\phi$ is the measure on the spectrum $\hat{A}$ of $A$ defined by $\mu_{\phi}(F) = \phi(\chi_F)$, where $\phi$ is regarded as a normal state on $A''$, see [P, 4.7.5]. Thus by Theorem 5 and [P, 4.7.6] we have the following:

**Corollary 6.** Let $(A, \phi, \alpha)$ be a $C^*$-dynamical system with $A$ a separable unital type I $C^*$-algebra. Then $h_\phi(\alpha) = h_{\mu_{\phi}}(\hat{\alpha})$, where $\hat{\alpha}$ is the automorphism of the measure space $(\hat{A}, \mu_{\phi})$ induced by $\alpha$.

Since inner automorphisms act trivially on the center we have:

**Corollary 7.** If $(M, \phi, \alpha)$ is a $W^*$-dynamical system with $M$ of type I and $\alpha$ an inner automorphism then $h_\phi(\alpha) = 0$.

Note that in the finite case the above corollary also follows from a result of N. Brown [Br, Lemma 2.2].

The next result was shown in [S] when $\phi$ is a trace.
Corollary 8. Let $R$ denote the hyperfinite $\text{II}_1$-factor. Let $A$ be a Cartan subalgebra of $R$ and $u$ a unitary operator in $A$. If $\phi$ is a normal state such that $u$ belongs to the centralizer of $\phi$ then $h_\phi(\text{Ad} u) = 0$.

Proof. As in [S], it follows from [CFW] that there exists an increasing sequence of full matrix algebras $N_1 \subset N_2 \subset \ldots$ with union weakly dense in $R$ such that $A \cong A_n \otimes B_n$, where $A_n = N_n \cap A$ and $B_n = (N_n' \cap R) \cap A$ for all $n \in \mathbb{N}$. Let $M_n = N_n \otimes B_n$. Then $M_n$ is of type $I$ and contains $u$. Hence $h_\phi(\text{Ad} u|_{M_n}) = 0$. Since $(\bigcup_n M_n)^- = R$, $h_\phi(\text{Ad} u) = 0$ by [SV, Lemma 3.3].

If $(A, \phi, \alpha)$ and $(B, \psi, \beta)$ are $C^*$-dynamical systems we always have

$$h_{\phi \otimes \psi}(\alpha \otimes \beta) \geq h_\phi(\alpha) + h_\psi(\beta),$$

see [SV, Lemma 3.4]. Equality does not always hold, see [NST] or [Sa]. However, we have:

Theorem 9. Let $(A, \phi, \alpha)$ and $(B, \psi, \beta)$ be $W^*$-dynamical systems. Suppose that $A$ is of type $I$, and $B$ is injective. Then

$$h_{\phi \otimes \psi}(\alpha \otimes \beta) = h_\phi(\alpha) + h_\psi(\beta).$$

Proof. We shall rather prove that $h_{\phi \otimes \psi}(\alpha \otimes \beta) = h_\phi(\alpha|_{Z(A)}) + h_\psi(\beta)$. For this it suffices to consider the case when $A$ is abelian; the general case will follow by the same arguments as in the proof of Theorem 5. (Note that the mapping $x \mapsto \text{Tr}(x) - x$ on $\text{Mat}_n(\mathbb{C})$ is not completely positive, but the mapping $x \mapsto \text{Tr}(x) - \frac{1}{n} x$ is by the Pimsner-Popa inequality. Thus replacing $M$ with $M \otimes B$ and $Z$ with $Z \otimes B$ in the proof of Theorem 5 we have to replace the inequality $E(s) \geq cs$ in the proof with $E(s) \geq \frac{c}{n} s$.)

So suppose that $A$ is abelian. It is clear that it suffices to prove that if $A_1, \ldots, A_n$ are finite-dimensional subalgebras of $A$, and $B_1, \ldots, B_n$ are finite-dimensional subalgebras of $B$, then

$$H_{\phi \otimes \psi}(A_1 \otimes B_1, \ldots, A_n \otimes B_n) = H_\phi(A_1, \ldots, A_n) + H_\psi(B_1, \ldots, B_n).$$

We always have the inequality "$\geq$, [SV, Lemma 3.4]. To prove the opposite inequality consider a decomposition

$$\phi \otimes \psi = \sum_{i_1, \ldots, i_n} \omega_{i_1 \ldots i_n}.$$
Set \( C = \vee_{k=1}^{n} A_k \). Let \( p_1, \ldots, p_r \) be those atoms \( p \) of \( C \) for which \( \phi(p) > 0 \). Define positive linear functionals \( \psi_{m,i_1,\ldots,i_n} \) on \( B \),

\[
\psi_{m,i_1,\ldots,i_n}(b) = \frac{\omega_{i_1,\ldots,i_n}(p_m \otimes b)}{\phi(p_m)}.
\]

Let also \( \phi_m \) be the linear functional on \( C \) defined by the equality \( \phi_m(a) = \phi(ap_m) \). Then

\[
\omega_{i_1,\ldots,i_n} = \sum_{m=1}^{r} \phi_m \otimes \psi_{m,i_1,\ldots,i_n} \text{ on } C \otimes B,
\]

and

\[
\psi = \sum_{i_1,\ldots,i_n} \psi_{m,i_1,\ldots,i_n} \text{ for } m = 1, \ldots, r.
\]

Since the supports of the positive functionals \( \phi_m \) are mutually orthogonal minimal projections in \( C \), we have

\[
\sum_{k=1}^{n} \sum_{i} \left( \phi \otimes \psi|_{A_k \otimes B_k}, \sum_{i_k=i} \omega_{i_1,\ldots,i_n}|_{A_k \otimes B_k} \right)
\]

\[
\leq \sum_{k=1}^{n} \sum_{i} \left( \phi \otimes \psi|_{C \otimes B_k}, \sum_{i_k=i} \omega_{i_1,\ldots,i_n}|_{C \otimes B_k} \right)
\]

\[
= \sum_{k=1}^{n} \sum_{i} \left( \phi \otimes \psi|_{C \otimes B_k}, \sum_{m=1}^{r} \phi_m \otimes \left( \sum_{i_k=i} \psi_{m,i_1,\ldots,i_n} \right)|_{C \otimes B_k} \right)
\]

\[
= \sum_{k=1}^{n} \sum_{i} \sum_{m=1}^{r} \phi(p_m) S \left( \psi|_{B_k}, \sum_{i_k=i} \psi_{m,i_1,\ldots,i_n}|_{B_k} \right).
\]

If \( a_i \geq 0 \) then \( \eta \left( \sum_{i} a_i \right) \leq \sum_{i} \eta(a_i) \). Hence we have

\[
\sum_{i_1,\ldots,i_n} \eta\omega_{i_1,\ldots,i_n}(1)
\]

\[
\leq \sum_{m=1}^{r} \sum_{i_1,\ldots,i_n} \eta(\phi_m \otimes \psi_{m,i_1,\ldots,i_n})(1)
\]

\[
= \sum_{m=1}^{r} \eta \phi(p_m) \sum_{i_1,\ldots,i_n} \psi_{m,i_1,\ldots,i_n}(1) + \sum_{m=1}^{r} \phi(p_m) \sum_{i_1,\ldots,i_n} \eta \psi_{m,i_1,\ldots,i_n}(1)
\]

\[
= \sum_{m=1}^{r} \eta \phi(p_m) + \sum_{m=1}^{r} \phi(p_m) \sum_{i_1,\ldots,i_n} \eta \psi_{m,i_1,\ldots,i_n}(1).
\]
Thus
\[ H_{\{\phi \otimes \psi = \sum_{i_1 \ldots i_n} \omega_1 \cdots \omega_n \}}(A_1 \otimes B_1, \ldots, A_n \otimes B_n) \leq \sum_{m=1}^{r} \eta \phi(p_m) + \sum_{m=1}^{r} \phi(p_m) H_{\{\psi = \sum_{i_1 \ldots i_n} \psi_{m,i_1} \cdots \psi_{m,i_n} \}}(B_1, \ldots, B_n). \]

Since \( \sum_{m=1}^{r} \eta \phi(p_m) = H_\phi(C) = H_\phi(A_1, \ldots, A_n) \), we conclude that
\[ H_{\phi \otimes \psi}(A_1 \otimes B_1, \ldots, A_n \otimes B_n) \leq H_\phi(A_1, \ldots, A_n) + H_\psi(B_1, \ldots, B_n), \]
completing the proof of the Theorem. \( \square \)

References


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