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Lakhdar Hammoudi

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In this paper we prove that for a finite dimensional commutative nilpotent algebra A over a field of prime characteristic $p > 0$, dim$A \geq p$ dim$A^{(p)}$, where $A^{(p)}$ is the subalgebra of A generated by the elements x^p. In particular, this solves Eggert's conjecture.

1. Introduction.

In 1971, Eggert [**2**] conjectured that for a finite commutative nilpotent algebra A over a field $\mathbb{K}$ of prime characteristic $p > 0$, dim$A \geq p$ dim$A^{(p)}$, where $A^{(p)}$ is the subalgebra of A generated by all the elements x^p, $x \in A$ and dim$A$, dim$A^{(p)}$ denote the dimensions of A and $A^{(p)}$ as vector spaces over $\mathbb{K}$.

In [**3**], Stack conjectures that dim$A \geq p$ dim$A^{(p)}$ is true for every finite dimensional nilpotent algebra A over $\mathbb{K}$. We point out that some particular cases of Eggert's conjecture have been proved in [**1**, **2**, **3**, **4**].

Here we prove the conjecture for finite dimensional commutative nilpotent algebras. This combined with the results of [**2**] completely describe the group of units of A and the problem set in [**1**]: "When a finite abelian group is isomorphic to the group of units of some finite commutative nilpotent algebras?" is solved. Recall that the group of units of A is the set A with the following operation: $x \cdot y = x + y + xy$, $\forall x, y \in A$.

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2. Results.

Our main result is the following:

Theorem. *Let A be a finite dimensional commutative nilpotent algebra over a field $\mathbb{K}$ of characteristic $p > 0$ and let $A^{(p)}$ be the subalgebra of A generated by all the elements x^p, $x \in A$. Then* $\dim A \geq p \dim A^{(p)}$.

To prove the theorem we need an easy lemma on the partition of some sets in $\mathbb{Z}^d_{\geq 0}$ of d-tuples ($d > 0$) of nonnegative integers. Let $\alpha = (\alpha_1, \dots, \alpha_d)$

and $\beta = (\beta_1, \dots, \beta_d)$ be in $\mathbb{Z}^d_{\geq 0}$. Define $\alpha > \beta$ if in the difference $\alpha - \beta = (\alpha_1 - \beta_1, \dots, \alpha_d - \beta_d)$, the left-most nonzero entry is positive and all other entries to the right are nonnegative. It is easy to prove that $>$ is in fact a partial order on $\mathbb{Z}^d_{\geq 0}$, which is compatible with the addition.

Lemma 1. *Let* $(n_1, n_2, \dots, n_d) = n \in \mathbb{Z}^d_{\geq 0}$ *be a fixed d-tuple such that* $(0, \dots, 0, 0) \neq n$ *and consider the following subsets of* $\mathbb{Z}^d_{\geq 0}$*:*

$$\mathbb{Z}^d_{\geq 0}(n) = \{\alpha,\ (0, \dots, 0, 0) \neq \alpha \leq n\},$$

$\mathbb{Z}^d_{\geq 0}(i_1, \dots, i_{d-1}) = \{(i_1, i_2, \dots, i_{d-1}, j),\ 1 \leq j \leq n_d\},\quad 0 \leq i_k \leq n_k,\ 1 \leq k \leq d-1,$

$$\mathbb{Z}^d_{\geq 0}(0) = \{(i_1, i_2, \dots, i_{d-1}, 0),\ (i_1, i_2, \dots, i_{d-1}, 0) \in \mathbb{Z}^d_{\geq 0}(n)\}.$$

Then the sets $\mathbb{Z}^d_{\geq 0}(i_1, \dots, i_{d-1})$*, and* $\mathbb{Z}^d_{\geq 0}(0)$ *form a partition of* $\mathbb{Z}^d_{\geq 0}(n)$*.*

The proof of the theorem requires also the following lemma due to Bautista [**1**, Proposition 2.1, p. 15]. For completness, we will give a sketch of a proof of this result.

Lemma 2. *Let A be a commutative nilpotent algebra over a field $\mathbb{K}$ generated by $X_1, \dots, X_d$. Let $(\alpha_1, \dots, \alpha_d)$ be an element of $\mathbb{Z}^d_{\geq 0}$ such that $X_1^{\alpha_1} \cdots X_d^{\alpha_d} \neq 0$ but $\forall (\beta_1, \dots, \beta_d) \in \mathbb{Z}^d_{\geq 0}$, $(\beta_1, \dots, \beta_d) > (\alpha_1, \dots, \alpha_d)$, $X_1^{\beta_1} \cdots X_d^{\beta_d} = 0$. Then for the set of ordered d-tuples*

$$S = \left\{(i_1, \dots, i_d) \in \mathbb{Z}^d_{\geq 0};\ (\alpha_1, \dots, \alpha_d) - (i_1, \dots, i_d) \in \mathbb{Z}^d_{\geq 0}\right\},$$

$\{X_1^{i_1} \cdots X_d^{i_d};\ (i_1, \dots, i_d) \in S\}$ *is linearly independent.*

Sketch of Proof. Suppose that the family

$$\left\{X_1^{i_1} \cdots X_d^{i_d};\ (i_1, \dots, i_d) \in \mathbb{Z}^d_{\geq 0};\ (\alpha_1, \dots, \alpha_d) - (i_1, \dots, i_d) \in \mathbb{Z}^d_{\geq 0}\right\}$$

is linearly dependent. Then there exists a set of nonzero elements $\lambda_{i_1, \dots, i_d} \in \mathbb{K}$ such that $\sum_{\alpha - I \in \mathbb{Z}^d_{\geq 0}} \lambda_{i_1, \dots, i_d} X_1^{i_1} \cdots X_d^{i_d} = 0$, $\alpha = (\alpha_1, \dots, \alpha_d)$, $I = (i_1, \dots, i_d)$.

Let $L = (l_1, \dots, l_d)$ be a minimal element such that $\lambda_{l_1, \dots, l_d} \neq 0$. Then

$$\lambda_{l_1, \dots, l_d} X_1^{l_1} \cdots X_d^{l_d} + \sum_{I > L} \lambda_{i_1, \dots, i_d} X_1^{i_1} \cdots X_d^{i_d} = 0.$$

By multiplying on the right by $X_1^{(\alpha_1 - l_1)} \cdots X_d^{(\alpha_d - l_d)}$ and using the commutativity of A, we obtain:

$$\lambda_{l_1, \dots, l_d} X_1^{\alpha_1} \cdots X_d^{\alpha_d} + \sum_{I > L} \lambda_{i_1, \dots, i_d} X_1^{i_1 + (\alpha_1 - l_1)} \cdots X_d^{i_d + (\alpha_d - l_d)} = 0.$$

However, it is easy to see that $(i_1 + \alpha_1 - l_1, \dots, i_d + \alpha_d - l_d) > (\alpha_1, \dots, \alpha_d)$.

Thus,

$$\sum_{I>L} \lambda_{i_1,\dots,i_d} X_1^{i_1+(\alpha_1-l_1)} \cdots X_d^{i_d+(\alpha_d-l_d)} = 0.$$

So, $\lambda_{l_1,\dots,l_d} X_1^{\alpha_1} \cdots X_d^{\alpha_d} = 0$. But, $\lambda_{l_1,\dots,l_d} \neq 0$. Thus, $X_1^{\alpha_1} \cdots X_d^{\alpha_d} = 0$. This contradicts our hypothesis and proves the lemma.

Lemma 3. *Let A be a commutative nilpotent algebra over a field $\mathbb{K}$ generated by d elements $X_1, \dots, X_d$. Suppose that A cannot be generated by $d-1$ elements. Let $\mathcal{B} = \{X_1^{i_1} \cdots X_d^{i_d}$, $(i_1, i_2, \dots, i_d) \in \mathbb{Z}_{\geq 0}^d$, with the convention $X_k^0 = 1$, $1 \leq k \leq d\}$ be a basis of A as a vector space over $\mathbb{K}$. Then $X_d \in \mathcal{B}$ and some of the basis $\mathcal{B}$ are such that, if for some $(j_1, \dots, j_d)$, $j_d \geq 2$, $X_1^{j_1} \cdots X_d^{j_d} \in \mathcal{B}$ then $X_1^{j_1} \cdots X_{d-1}^{j_{d-1}} X_d^{j_d-1} \in \mathcal{B}$.*

Proof. Suppose that $X_d \notin \mathcal{B}$ and let us write it as a linear combination of elements of $\mathcal{B}$, $X_d = \sum_{i_1,\dots,i_d} \lambda_{i_1,\dots,i_d} X_1^{i_1} \cdots X_d^{i_d}$, $\lambda_{i_1,\dots,i_d} \in \mathbb{K}$. Since A is not generated by $d-1$ elements, for some i_d we have $i_d \geq 1$. So, one can write

$$X_d = \left(\sum_{i_1,\dots,i_d} \lambda_{i_1,\dots,i_d} X_1^{i_1} \cdots X_d^{i_d-1} \right) \left(\sum_{i_1,\dots,i_d} \lambda_{i_1,\dots,i_d} X_1^{i_1} \cdots X_d^{i_d} \right).$$

Since A is commutative and nilpotent, by repeating the above process we can write X_d as a linear combination of monomials in $X_1, \dots, X_{d-1}$. Thus A is generated by $d-1$ elements. This contradiction proves our assertion, $X_d \in \mathcal{B}$.

We prove now our second assertion. It is easy to see that $X_1^{j_1} \cdots X_d^{j_d} \in \mathcal{B}$ implies that there exists $(\alpha_1, \dots, \alpha_d) \in \mathbb{Z}_{\geq 0}^d$ satisfying the hypothesis of Lemma 2 such that

$$(\alpha_1, \dots, \alpha_d) > (j_1, \dots, j_d) \text{ and } (\alpha_1 - j_1, \dots, \alpha_d - j_d) \in \mathbb{Z}_{\geq 0}^d.$$

But $(j_1, \dots, j_d) > (j_1, \dots, j_{d-1}, j_d - 1)$. So, $(\alpha_1 - j_1, \dots, \alpha_{d-1} - j_{d-1}, \alpha_d - j_d - 1) \in \mathbb{Z}_{\geq 0}^d$. Thus, Lemma 2 applies here.

Suppose now that $X_1^{j_1} \cdots X_{d-1}^{j_{d-1}} X_d^{j_d-1} \notin \mathcal{B}$. Then $\{X_1^{j_1} \cdots X_{d-1}^{j_{d-1}} X_d^{j_d-1}, \mathcal{B}\}$ is linearly dependent which contradicts the preceeding lemma.

Proof of the Theorem. We prove our theorem by induction on the number l of generators of the algebra A.

We first prove the conjecture for $l = 1$. Let X be a generator of A and $m+1$ be the degree of nilpotency of X. Then $\{X, X^2, \dots, X^m\}$ is a basis for the vector space A and since A is commutative over a field of characteristic p, $\{X^p, \dots, X^{pk}\}$ is a basis of $A^{(p)}$. But the fact that $m+1$ is the degree of nilpotency of X yields to $m \geq pk$. So, $\dim A = m \geq pk = p \dim A^{(p)}$.

Suppose that the theorem is proved for every algebra generated by l elements, $l \leq d-1$ and consider a finite dimensional commutative nilpotent algebra A over $\mathbb{K}$ generated by d elements, $X_1, \ldots, X_d$. Since A is nilpotent, there exists a d-tuple $(n_1, n_2, \ldots, n_d) = n \in \mathbb{Z}^d_{\geq 0}$ such that $n_1+1, \ldots, n_d+1$ are the degrees of nilpotency of $X_1, \ldots, X_d$ respectively. Since A is commutative over a field of characteristic p, as vector spaces over $\mathbb{K}$, A and $A^{(p)}$ are generated by the monomials of the form $\{X_1^{\beta_1} \cdots X_d^{\beta_d}$, $(\beta_1, \ldots, \beta_d) \in \mathbb{Z}^d_{\geq 0}$, where $X_i^0 = 1\}$ and $X_1^{p\beta_1} \cdots X_d^{p\beta_d}$ respectively. So, one can extract a basis $\mathcal{B}$ of $A^{(p)}$ from the last cited monomials. Let $\bar{\mathcal{B}}$ be a basis of A obtained by completing $\mathcal{B}$. Let $\mathbb{Z}^d_{\geq 0}(\bar{\mathcal{B}})$ be the set of all d-tuples $(\alpha_1, \ldots, \alpha_d) \in \mathbb{Z}^d_{\geq 0}$ such that $X_1^{\alpha_1} \cdots X_d^{\alpha_d} \in \bar{\mathcal{B}}$ and denote by $\mathbb{Z}^d_{\geq 0}(\mathcal{B})$ the set of all d-tuples $(\alpha_1, \ldots, \alpha_d) \in \mathbb{Z}^d_{\geq 0}$ such that $X_1^{\alpha_1} \cdots X_d^{\alpha_d} \in \mathcal{B}$.

With these notations, $\dim A \geq p \dim A^{(p)}$ is the same as $\#\mathbb{Z}^d_{\geq 0}(\bar{\mathcal{B}}) \geq p\#\mathbb{Z}^d_{\geq 0}(\mathcal{B})$, where $\#Y$ is the number of the elements of the set Y.

Let R be the subalgebra of A generated by $\{X_1, \ldots, X_{d-1}\}$. Then by the hypothesis of induction, $\dim R \geq p \dim R^{(p)}$. But, $\dim R = \#(\mathbb{Z}^d_{\geq 0}(\bar{\mathcal{B}}) \cap \mathbb{Z}^d_{\geq 0}(0))$ and $\dim R^{(p)} = \#(\mathbb{Z}^d_{\geq 0}(\mathcal{B}) \cap \mathbb{Z}^d_{\geq 0}(0))$. On the other hand, since $\mathbb{Z}^d_{\geq 0}(\bar{\mathcal{B}})$ and $\mathbb{Z}^d_{\geq 0}(\mathcal{B})$ are included in $\mathbb{Z}^d_{\geq 0}(n)$, by Lemma 1 we have:

$$\mathbb{Z}^d_{\geq 0}(\bar{\mathcal{B}}) = \left(\bigcup_{i_1,\cdots,i_{d-1}} \mathbb{Z}^d_{\geq 0}(\bar{\mathcal{B}}) \cap \mathbb{Z}^d_{\geq 0}(i_1, \ldots, i_{d-1})\right) \bigcup \left(\mathbb{Z}^d_{\geq 0}(\bar{\mathcal{B}}) \cap \mathbb{Z}^d_{\geq 0}(0)\right)$$

$$\mathbb{Z}^d_{\geq 0}(\mathcal{B}) = \left(\bigcup_{i_1,\cdots,i_{d-1}} \mathbb{Z}^d_{\geq 0}(\mathcal{B}) \cap \mathbb{Z}^d_{\geq 0}(i_1, \ldots, i_{d-1})\right) \bigcup \left(\mathbb{Z}^d_{\geq 0}(\mathcal{B}) \cap \mathbb{Z}^d_{\geq 0}(0)\right).$$

Also, by Lemma 1 we have partitions of $\mathbb{Z}^d_{\geq 0}(\bar{\mathcal{B}})$ and $\mathbb{Z}^d_{\geq 0}(\mathcal{B})$. Thus, we only need to prove that

$$\begin{aligned} &\# \bigcup_{i_1,\cdots,i_{d-1}} \left(\mathbb{Z}^d_{\geq 0}(\bar{\mathcal{B}}) \cap \mathbb{Z}^d_{\geq 0}(i_1, \ldots, i_{d-1})\right) \\ &\geq p\# \bigcup_{i_1,\cdots,i_{d-1}} \left(\mathbb{Z}^d_{\geq 0}(\mathcal{B}) \cap \mathbb{Z}^d_{\geq 0}(i_1, \ldots, i_{d-1})\right). \end{aligned}$$

Moreover, since we have a disjoint union of sets, we prove that

$$\#\left(\mathbb{Z}^d_{\geq 0}(\bar{\mathcal{B}}) \cap \mathbb{Z}^d_{\geq 0}(i_1, \ldots, i_{d-1})\right) \geq p\#\left(\mathbb{Z}^d_{\geq 0}(\mathcal{B}) \cap \mathbb{Z}^d_{\geq 0}(i_1, \ldots, i_{d-1})\right).$$

Fix $(i_1, \ldots, i_{d-1})$ and let j be the greatest integer such that: $X_1^{i_1} \cdots X_{d-1}^{i_{d-1}} X_d^j \in \bar{\mathcal{B}}$ (i.e., $(i_1, \ldots, i_{d-1}, j) \in \mathbb{Z}^d_{\geq 0}(\bar{\mathcal{B}})$).

If $j = 0$ or $j = 1$ then $\mathbb{Z}^d_{\geq 0}(\mathcal{B}) \cap \mathbb{Z}^d_{\geq 0}(i_1, \ldots, i_{d-1}) = \emptyset$ and our claim is obvious.

If $j \geq 2$ then by Lemma 3, $(i_1, \dots, i_{d-1}, k) \in \mathbb{Z}^d_{\geq 0}(\overline{\mathcal{B}})$, $\forall k$, $1 \leq k \leq j$ and so, by the choice of the integer j,

$$\# \left(\mathbb{Z}^d_{\geq 0}(\overline{\mathcal{B}}) \cap \mathbb{Z}^d_{\geq 0}(i_1, \dots, i_{d-1}) \right) = j.$$

On the other hand

$$\mathbb{Z}^d_{\geq 0}(\mathcal{B}) \cap \mathbb{Z}^d_{\geq 0}(i_1, \dots, i_{d-1}) = \begin{cases} \emptyset \\ \text{or} \\ \{(i_1, \dots, i_{d-1}, pk),\ 1 \leq pk \leq j\}. \end{cases}$$

The first case is obvious and in the second as for an algebra generated by one element, we have

$$p\# \left(\mathbb{Z}^d_{\geq 0}(\mathcal{B}) \cap \mathbb{Z}^d_{\geq 0}(i_1, \dots, i_{d-1}) \right) = pt \leq j.$$

This ends the proof of the theorem.

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DEPARTMENT OF MATHEMATICS AND STATISTICS MIAMI UNIVERSITY
OXFORD, OHIO 45056
E-mail address: hammoul@muohio.edu