EGGERT’S CONJECTURE ON THE DIMENSIONS OF NILPOTENT ALGEBRAS

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In this paper we prove that for a finite dimensional commutative nilpotent algebra $A$ over a field of prime characteristic $p > 0$, $\dim A \geq p \dim A^{(p)}$, where $A^{(p)}$ is the subalgebra of $A$ generated by the elements $x^p$. In particular, this solves Eggert’s conjecture.

1. Introduction.

In 1971, Eggert [2] conjectured that for a finite commutative nilpotent algebra $A$ over a field $\mathbb{K}$ of prime characteristic $p > 0$, $\dim A \geq p \dim A^{(p)}$, where $A^{(p)}$ is the subalgebra of $A$ generated by all the elements $x^p$, $x \in A$ and $\dim A$, $\dim A^{(p)}$ denote the dimensions of $A$ and $A^{(p)}$ as vector spaces over $\mathbb{K}$.

In [3], Stack conjectures that $\dim A \geq p \dim A^{(p)}$ is true for every finite dimensional nilpotent algebra $A$ over $\mathbb{K}$. We point out that some particular cases of Eggert’s conjecture have been proved in [1, 2, 3, 4].

Here we prove the conjecture for finite dimensional commutative nilpotent algebras. This combined with the results of [2] completely describe the group of units of $A$ and the problem set in [1]: “When a finite abelian group is isomorphic to the group of units of some finite commutative nilpotent algebras?” is solved. Recall that the group of units of $A$ is the set $A$ with the following operation: $x \cdot y = x + y + xy$, $\forall x, y \in A$.

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2. Results.

Our main result is the following:

**Theorem.** Let $A$ be a finite dimensional commutative nilpotent algebra over a field $\mathbb{K}$ of characteristic $p > 0$ and let $A^{(p)}$ be the subalgebra of $A$ generated by all the elements $x^p$, $x \in A$. Then $\dim A \geq p \dim A^{(p)}$.

To prove the theorem we need an easy lemma on the partition of some sets in $\mathbb{Z}_{\geq 0}^d$ of $d$-tuples ($d > 0$) of nonnegative integers. Let $\alpha = (\alpha_1, \ldots, \alpha_d)$
and \( \beta = (\beta_1, \ldots, \beta_d) \) be in \( \mathbb{Z}_{\geq 0}^d \). Define \( \alpha > \beta \) if in the difference \( \alpha - \beta = (\alpha_1 - \beta_1, \ldots, \alpha_d - \beta_d) \), the left-most nonzero entry is positive and all other entries to the right are nonnegative. It is easy to prove that \( > \) is in fact a partial order on \( \mathbb{Z}_{\geq 0}^d \), which is compatible with the addition.

**Lemma 1.** Let \( (n_1, n_2, \ldots, n_d) = n \in \mathbb{Z}_{\geq 0}^d \) be a fixed \( d \)-tuple such that \((0, \ldots, 0, 0) \neq n \) and consider the following subsets of \( \mathbb{Z}_{\geq 0}^d \):

\[
\mathbb{Z}_{\geq 0}^d(n) = \{ \alpha, (0, \ldots, 0, 0) \neq \alpha \leq n \},
\]

\[
\mathbb{Z}_{\geq 0}^d(i_1, \ldots, i_{d-1}) = \{ (i_1, i_2, \ldots, i_{d-1}, j), 1 \leq j \leq n_d \}, \quad 0 \leq i_k \leq n_k, \ 1 \leq k \leq d-1,
\]

\[
\mathbb{Z}_{\geq 0}^d(0) = \{ (i_1, i_2, \ldots, i_{d-1}, 0), (i_1, i_2, \ldots, i_{d-1}, 0) \in \mathbb{Z}_{\geq 0}^d(n) \}.
\]

Then the sets \( \mathbb{Z}_{\geq 0}^d(i_1, \ldots, i_{d-1}) \), and \( \mathbb{Z}_{\geq 0}^d(0) \) form a partition of \( \mathbb{Z}_{\geq 0}^d(n) \).

The proof of the theorem requires also the following lemma due to Bautista [1, Proposition 2.1, p. 15]. For completeness, we will give a sketch of a proof of this result.

**Lemma 2.** Let \( A \) be a commutative nilpotent algebra over a field \( \mathbb{K} \) generated by \( X_1, \ldots, X_d \). Let \( (\alpha_1, \ldots, \alpha_d) \) be an element of \( \mathbb{Z}_{\geq 0}^d \) such that \( X_1^{\alpha_1} \cdots X_d^{\alpha_d} \neq 0 \) but \( \forall (\beta_1, \ldots, \beta_d) \in \mathbb{Z}_{\geq 0}^d, (\beta_1, \ldots, \beta_d) > (\alpha_1, \ldots, \alpha_d), X_1^{\beta_1} \cdots X_d^{\beta_d} = 0 \). Then for the set of ordered \( d \)-tuples

\[
S = \left\{ (i_1, \ldots, i_d) \in \mathbb{Z}_{\geq 0}^d; (\alpha_1, \ldots, \alpha_d) - (i_1, \ldots, i_d) \in \mathbb{Z}_{\geq 0}^d \right\},
\]

\[
\{ X_1^{i_1} \cdots X_d^{i_d}; (i_1, \ldots, i_d) \in S \} \text{ is linearly independent.}
\]

**Sketch of Proof.** Suppose that the family

\[
\{ X_1^{i_1} \cdots X_d^{i_d}; (i_1, \ldots, i_d) \in \mathbb{Z}_{\geq 0}^d; (\alpha_1, \ldots, \alpha_d) - (i_1, \ldots, i_d) \in \mathbb{Z}_{\geq 0}^d \}
\]

is linearly dependent. Then there exists a set of nonzero elements \( \lambda_{i_1, \ldots, i_d} \in \mathbb{K} \) such that \( \sum_{\alpha - I \in \mathbb{Z}_{\geq 0}^d} \lambda_{i_1, \ldots, i_d} X_1^{i_1} \cdots X_d^{i_d} = 0, \ \alpha = (\alpha_1, \ldots, \alpha_d), I = (i_1, \ldots, i_d) \).

Let \( L = (l_1, \ldots, l_d) \) be a minimal element such that \( \lambda_{l_1, \ldots, l_d} \neq 0 \). Then

\[
\lambda_{l_1, \ldots, l_d} X_1^{l_1} \cdots X_d^{l_d} + \sum_{I > L} \lambda_{i_1, \ldots, i_d} X_1^{i_1} \cdots X_d^{i_d} = 0.
\]

By multiplying on the right by \( X_1^{(\alpha_1 - l_1)} \cdots X_d^{(\alpha_d - l_d)} \) and using the commutativity of \( A \), we obtain:

\[
\lambda_{l_1, \ldots, l_d} X_1^{\alpha_1 - l_1} \cdots X_d^{\alpha_d - l_d} + \sum_{I > L} \lambda_{i_1, \ldots, i_d} X_1^{i_1 + (\alpha_1 - l_1)} \cdots X_d^{i_d + (\alpha_d - l_d)} = 0.
\]

However, it is easy to see that \( (i_1 + \alpha_1 - l_1, \ldots, i_d + \alpha_d - l_d) > (\alpha_1, \ldots, \alpha_d) \).
Thus,
\[ \sum_{l > L} \lambda_{i_1, \ldots, i_d} X^{i_1 + (\alpha_1 - l_1)} \cdots X^{i_d + (\alpha_d - l_d)} = 0. \]

So, \( \lambda_{i_1, \ldots, i_d} X^{\alpha_1} \cdots X^{\alpha_d} = 0 \). But, \( \lambda_{i_1, \ldots, i_d} \neq 0 \). Thus, \( X^{\alpha_1} \cdots X^{\alpha_d} = 0 \). This contradicts our hypothesis and proves the lemma.

**Lemma 3.** Let \( A \) be a commutative nilpotent algebra over a field \( \mathbb{K} \) generated by \( d \) elements \( X_1, \ldots, X_d \). Suppose that \( A \) cannot be generated by \( d - 1 \) elements. Let \( B = \{ X_1^{i_1} \cdots X_d^{i_d}, (i_1, i_2, \ldots, i_d) \in \mathbb{Z}_{\geq 0}^d, \text{ with the convention } X_k^0 = 1, 1 \leq k \leq d \} \) be a basis of \( A \) as a vector space over \( \mathbb{K} \). Then \( X_d \in B \) and some of the basis \( B \) are such that, if for some \( (j_1, \ldots, j_d) \), \( j_d \geq 2, X_1^{j_1} \cdots X_d^{j_d} \in B \) then \( X_1^{j_1} \cdots X_{d-1}^{j_{d-1}} X_d^{j_d-1} \in B \).

**Proof.** Suppose that \( X_d \notin B \) and let us write it as a linear combination of elements of \( B \), \( X_d = \sum_{i_1, \ldots, i_d} \lambda_{i_1, \ldots, i_d} X_1^{i_1} \cdots X_d^{i_d}, \lambda_{i_1, \ldots, i_d} \in \mathbb{K} \). Since \( A \) is not generated by \( d - 1 \) elements, for some \( i_d \) we have \( i_d \geq 1 \). So, one can write

\[ X_d = \left( \sum_{i_1, \ldots, i_d} \lambda_{i_1, \ldots, i_d} X_1^{i_1} \cdots X_d^{i_d-1} \right) \left( \sum_{i_1, \ldots, i_d} \lambda_{i_1, \ldots, i_d} X_1^{i_1} \cdots X_d^{i_d} \right). \]

Since \( A \) is commutative and nilpotent, by repeating the above process we can write \( X_d \) as a linear combination of monomials in \( X_1, \ldots, X_{d-1} \). Thus \( A \) is generated by \( d - 1 \) elements. This contradiction proves our assertion, \( X_d \in B \).

We prove now our second assertion. It is easy to see that \( X_1^{j_1} \cdots X_d^{j_d} \in B \) implies that there exists \( (\alpha_1, \ldots, \alpha_d) \in \mathbb{Z}_{\geq 0}^d \) satisfying the hypothesis of Lemma 2 such that

\( (\alpha_1, \ldots, \alpha_d) > (j_1, \ldots, j_d) \) and \( (\alpha_1 - j_1, \ldots, \alpha_d - j_d) \in \mathbb{Z}_{\geq 0}^d \).

But \( (j_1, \ldots, j_d) > (j_1, \ldots, j_{d-1}, j_d - 1) \). So, \( (\alpha_1 - j_1, \ldots, \alpha_{d-1} - j_{d-1}, \alpha_d - j_d - 1) \in \mathbb{Z}_{\geq 0}^d \). Thus, Lemma 2 applies here.

Suppose now that \( X_1^{j_1} \cdots X_{d-1}^{j_{d-1}} X_d^{j_d-1} \notin B \). Then \( \{ X_1^{j_1} \cdots X_{d-1}^{j_{d-1}} X_d^{j_d-1}, B \} \) is linearly dependent which contradicts the preceding lemma.

**Proof of the Theorem.** We prove our theorem by induction on the number \( l \) of generators of the algebra \( A \).

We first prove the conjecture for \( l = 1 \). Let \( X \) be a generator of \( A \) and \( m+1 \) be the degree of nilpotency of \( X \). Then \( \{ X, X^2, \ldots, X^m \} \) is a basis for the vector space \( A \) and since \( A \) is commutative over a field of characteristic \( p \), \( \{ X^p, \ldots, X^{pk} \} \) is a basis of \( A^{(p)} \). But the fact that \( m+1 \) is the degree of nilpotency of \( X \) yields to \( m \geq pk \). So, \( \dim A \geq pk = p \dim A^{(p)} \).

Suppose that the theorem is proved for every algebra generated by \( l \) elements, \( l \leq d-1 \) and consider a finite dimensional commutative nilpotent
algebra $A$ over $\mathbb{K}$ generated by $d$ elements, $X_1, \ldots, X_d$. Since $A$ is nilpotent, there exists a $d$-tuple $(n_1, n_2, \ldots, n_d) = n \in \mathbb{Z}^d_{\geq 0}$ such that $n_1 + 1, \ldots, n_d + 1$ are the degrees of nilpotency of $X_1, \ldots, X_d$ respectively. Since $A$ is commutative over a field of characteristic $p$, as vector spaces over $\mathbb{K}$, $A$ and $A^{(p)}$ are generated by the monomials of the form $X_1^{\beta_1} \cdots X_d^{\beta_d}, (\beta_1, \ldots, \beta_d) \in \mathbb{Z}^d_{\geq 0}$, where $X_1^0 = 1$ and $X_1^{p\beta_1} \cdots X_d^{p\beta_d}$ respectively. So, one can extract a basis $\mathcal{B}$ of $A^{(p)}$ from the last cited monomials. Let $\overline{\mathcal{B}}$ be a basis of $A$ obtained by completing $\mathcal{B}$. Let $\mathbb{Z}^d_{\geq 0}(\overline{\mathcal{B}})$ be the set of all $d$-tuples $(\alpha_1, \ldots, \alpha_d) \in \mathbb{Z}^d_{\geq 0}$ such that $X_1^{\alpha_1} \cdots X_d^{\alpha_d} \in \overline{\mathcal{B}}$ and denote by $\mathbb{Z}^d_{\geq 0}(\mathcal{B})$ the set of all $d$-tuples $(\alpha_1, \ldots, \alpha_d) \in \mathbb{Z}^d_{\geq 0}$ such that $X_1^{\alpha_1} \cdots X_d^{\alpha_d} \in \mathcal{B}$.

With these notations, $\dim A \geq p \dim A^{(p)}$ is the same as $\# \mathbb{Z}^d_{\geq 0}(\overline{\mathcal{B}}) \geq p \# \mathbb{Z}^d_{\geq 0}(\mathcal{B})$, where $\# Y$ is the number of the elements of the set $Y$.

Let $R$ be the subalgebra of $A$ generated by $\{X_1, \ldots, X_{d-1}\}$. Then by the hypothesis of induction, $\dim R \geq p \dim R^{(p)}$. But, $\dim R = \# (\mathbb{Z}^d_{\geq 0}(\overline{\mathcal{B}}) \cap \mathbb{Z}^d_{\geq 0}(0))$ and $\dim R^{(p)} = \# (\mathbb{Z}^d_{\geq 0}(\mathcal{B}) \cap \mathbb{Z}^d_{\geq 0}(0))$. On the other hand, since $\mathbb{Z}^d_{\geq 0}(\overline{\mathcal{B}})$ and $\mathbb{Z}^d_{\geq 0}(\mathcal{B})$ are included in $\mathbb{Z}^d_{\geq 0}(n)$, by Lemma 1 we have:

$$\mathbb{Z}^d_{\geq 0}(\overline{\mathcal{B}}) = \left( \bigcup_{i_1, \ldots, i_{d-1}} \mathbb{Z}^d_{\geq 0}(\overline{\mathcal{B}}) \cap \mathbb{Z}^d_{\geq 0}(i_1, \ldots, i_{d-1}) \right) \bigcup \left( \mathbb{Z}^d_{\geq 0}(\overline{\mathcal{B}}) \cap \mathbb{Z}^d_{\geq 0}(0) \right)$$

$$\mathbb{Z}^d_{\geq 0}(\mathcal{B}) = \left( \bigcup_{i_1, \ldots, i_{d-1}} \mathbb{Z}^d_{\geq 0}(\mathcal{B}) \cap \mathbb{Z}^d_{\geq 0}(i_1, \ldots, i_{d-1}) \right) \bigcup \left( \mathbb{Z}^d_{\geq 0}(\mathcal{B}) \cap \mathbb{Z}^d_{\geq 0}(0) \right).$$

Also, by Lemma 1 we have partitions of $\mathbb{Z}^d_{\geq 0}(\overline{\mathcal{B}})$ and $\mathbb{Z}^d_{\geq 0}(\mathcal{B})$. Thus, we only need to prove that

$$\# \bigcup_{i_1, \ldots, i_{d-1}} \left( \mathbb{Z}^d_{\geq 0}(\overline{\mathcal{B}}) \cap \mathbb{Z}^d_{\geq 0}(i_1, \ldots, i_{d-1}) \right) \geq p \# \bigcup_{i_1, \ldots, i_{d-1}} \left( \mathbb{Z}^d_{\geq 0}(\mathcal{B}) \cap \mathbb{Z}^d_{\geq 0}(i_1, \ldots, i_{d-1}) \right).$$

Moreover, since we have a disjoint union of sets, we prove that

$$\# \left( \mathbb{Z}^d_{\geq 0}(\overline{\mathcal{B}}) \cap \mathbb{Z}^d_{\geq 0}(i_1, \ldots, i_{d-1}) \right) \geq p \# \left( \mathbb{Z}^d_{\geq 0}(\mathcal{B}) \cap \mathbb{Z}^d_{\geq 0}(i_1, \ldots, i_{d-1}) \right).$$

Fix $(i_1, \ldots, i_{d-1})$ and let $j$ be the greatest integer such that:

$$X_1^{i_1} \cdots X_{d-1}^{i_{d-1}} X_d^j \in \mathcal{B} \quad \text{(i.e., } (i_1, \ldots, i_{d-1}, j) \in \mathbb{Z}^d_{\geq 0}(\mathcal{B})).$$

If $j = 0$ or $j = 1$ then $\mathbb{Z}^d_{\geq 0}(\overline{\mathcal{B}}) \cap \mathbb{Z}^d_{\geq 0}(i_1, \ldots, i_{d-1}) = \emptyset$ and our claim is obvious.
If \( j \geq 2 \) then by Lemma 3, \((i_1, \ldots, i_{d-1}, k) \in \mathbb{Z}^d_{\geq 0}(B), \forall k, 1 \leq k \leq j\) and so, by the choice of the integer \( j \),

\[
\# \left( \mathbb{Z}^d_{\geq 0}(B) \cap \mathbb{Z}^d_{\geq 0}(i_1, \ldots, i_{d-1}) \right) = j.
\]

On the other hand

\[
\mathbb{Z}^d_{\geq 0}(B) \cap \mathbb{Z}^d_{\geq 0}(i_1, \ldots, i_{d-1}) = \begin{cases} 
\emptyset \\
\{(i_1, \ldots, i_{d-1}, pk), 1 \leq pk \leq j\}
\end{cases}
\]

The first case is obvious and in the second as for an algebra generated by one element, we have

\[
p\# \left( \mathbb{Z}^d_{\geq 0}(B) \cap \mathbb{Z}^d_{\geq 0}(i_1, \ldots, i_{d-1}) \right) = pt \leq j.
\]

This ends the proof of the theorem.

References


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