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We investigate the color number and genus for fixed-point free maps of order three. A result that has the flavor of the Ljusternik-Schnirelmann theorem for involutions is established. The Y-sphere, the combinatorial boundary of the product of tripods, is studied in detail. Problems of coloring non-invariant subspaces are touched upon.

Introduction.

All spaces are assumed to be separable metric and all mappings are assumed to be continuous.

Definition 1. Suppose that $f\colon X \to X$ is a map from X to itself. An open subset B of X is called a *color* of (X, f) if $f(B) \cap B = \emptyset$ or, equivalently, $f^{-1}(B) \cap B = \emptyset$. A *coloring* of (X, f) is a finite cover $\mathcal{B}$ of X consisting of colors. The minimal cardinality of a coloring $\mathcal{B}$ is called the *color number* $\mathrm{col}(X, f)$ of (X, f).

In the definition of color we could have used closed subsets as well. By shrinking an open coloring a closed coloring may be obtained and the colors of a closed coloring can be enlarged so as to obtain an open coloring. The situation is more delicate when considering non-invariant subspaces, as was shown in [**8**]. For a fixed-point free homeomorphism f of an n-dimensional space X we have $\mathrm{col}(X, f) \leq n + 3$; if moreover the map f is an involution then $\mathrm{col}(X, f) \leq n + 2$, [**3**].

In this paper we study fixed-point free maps $\sigma\colon X \to X$ of order 3, i.e., $\sigma^3(x) = x$ for each $x \in X$. It is to be noted that for a color B of (X, σ), when σ has period 3, the sets B, $\sigma(B)$ and $\sigma^2(B)$ are pairwise disjoint. One of the reasons for studying maps of period 3 is that they provide examples of pairs (X, σ) with $\dim(X) = n$ and $\mathrm{col}(X, \sigma) = n + 3$. A second reason is that there is an intimate relation between the color number and the genus, which we now define.

Definition 2. Let X be a space and $\sigma\colon X \to X$ a map of period 3 without fixed-points.

(1) A subset B of X is called a *set of first type* if there exists a color C of (X, σ) such that $B = C \cup \sigma(C) \cup \sigma^2(C)$; we also say that B is *generated* by C.

(2) We say that the *genus* of the space X is at most k if X can be written as a union of k sets of first type. Notation: $\mathrm{gen}(X,\sigma) \le k$.

Recall that if $f\colon X \to X$ and $g\colon Y \to Y$ are mappings then a mapping $h\colon X \to Y$ is said to be *equivariant* if $h \circ f = g \circ h$. Note that for an equivariant $h\colon X \to Y$ and mappings f and g of order 3 we have $\mathrm{col}(X,f) \le \mathrm{col}(Y,g)$ and $\mathrm{gen}(X,f) \le \mathrm{gen}(Y,g)$.

We now formulate the main results, which are generalizations of the Ljusternik-Schnirelmann theorem for involutions on S^n.

Theorem 1. *Let X be a space and $\sigma\colon X \to X$ a map of period 3 without fixed-points. If $\mathrm{gen}(X,\sigma) = n+1$ and if $A_1, \dots, A_{n+1}$ are colors of X with $X = \bigcup_{i=1}^{n+1}[A_i \cup \sigma(A_i) \cup \sigma^2(A_i)]$ then*

$$\bigcap_{i=1}^{n+1}[A_i \cup \sigma(A_i)] \neq \emptyset.$$

Theorem 2. *Let X be a space and $\sigma\colon X \to X$ a map of period 3 without fixed-points. If $\mathrm{col}(X,\sigma) = n+3$ and $\{A_1, \dots, A_{n+1}, A_{n+2}, A_{n+3}\}$ is a coloring of X then*

$$\bigcap_{i=1}^{n+1}[A_i \cup \sigma(A_i)] \neq \emptyset.$$

We shall also obtain a bound on the color number of non-invariant subspaces, using the results of [**3**]. We shall use the results above to show that this bound is almost sharp. Theorem 1 can be extended to general periodic maps on paracompact spaces [**1**] by a different proof. The proof in the present paper concerns certain universal spaces, the Y-cube and the Y-sphere, which are interesting in their own right.

1. The Y-cube Y^n and the Y-sphere S^n_Y.

The interval $I = [-1,1]$ with the antipodal-map is a standard space for the study of involutions and gives rise to the study of the n-cube I^n and the n-sphere S^n. We introduce a similar space for the study of maps of order 3. A natural candidate is the tripod, a Y-shaped space.

Consider in the complex plane the rotation γ of 120 degrees around 0 which is induced by the multiplication with $\zeta = \exp(\frac{2\pi i}{3})$. Let I denote the closed segment between 0 and 1. The subspace $Y = I \cup \zeta(I) \cup \zeta^2(I)$ is called the *closed Y-interval.* The points 1, ζ and ζ^2 are called the *end points* of Y and $Y^\circ = Y \setminus \{1,\zeta,\zeta^2\}$ is referred to as the *open Y-interval.* Both Y and Y° are invariant under γ. The space Y^n is called the *n-dimensional Y-cube.* The product map $\gamma^n : Y^n \to Y^n$ is a map of order 3 with a unique fixed point. The subspace S_Y^{n-1} of Y^n is defined by

$$S_Y^{n-1} = \{(x_1, \dots, x_n) \in Y^n : \ x_i \in \{1,\zeta,\zeta^2\} \text{ for some } \ i\}.$$

The space S_Y^{n-1} is called the $(n-1)$-*dimensional* Y-*sphere*. The 1-dimensional Y-sphere is the familiar bipartite cubic graph on six nodes $K(3,3)$, which is a standard example of a non-planar graph.

Note that S_Y^{n-1} is invariant under γ^n and that γ^n has no fixed points in S_Y^{n-1}. As $\dim(S_Y^n) = n$ we have

$$\mathrm{col}(S_Y^n, \gamma^{n+1}) \leq n+3.$$

We will show later on that $\mathrm{col}(S_Y^n, \gamma) = n+3$. So $\left(S_Y^n, \gamma^{n+1}\right)$ is a pair of an n-dimensional space and a mapping for which the color number is maximal. The space S_Y^n can be obtained from S_Y^{n-1} as in the following way

$$S_Y^n = [S_Y^{n-1} \times Y] \cup \bigcup_{i=0}^{2} [Y^n \times \{\zeta^i\}] \subset Y^{n+1}$$

and

$$S_Y^n = [S_Y^{n-1} \times Y^\circ] \cup \bigcup_{i=0}^{2} [Y^n \times \{\zeta^i\}] \subset Y^{n+1}.$$

Lemma 1. *Let X be a space and $\sigma\colon X \to X$ a map of order 3 without fixed-points. Suppose D is a closed subset such that $D \cap \sigma(D) = \emptyset$ and $E = D \cup \sigma(D) \cup \sigma^2(D)$, then there exist an equivariant map $f : X \to Y$ with $f^{-1}(1) = D$.*

In the lemma, one may think of D and E as a closed color and closed set of the first kind.

Proof. By enlarging the $\sigma^i(D)$ for $i = 0, 1, 2$ we obtain open colors U_i. For $i = 0, 1, 2$ define real-valued Urysohn functions f_i on X such that $0 \leq f_i(x) \leq 1$ for all x, $f_i^{-1}(1) = \sigma^i(D)$ and $f_i^{-1}(0) = X \setminus U_i$. Let $f = f_0 + \zeta f_1 + \zeta^2 f_2$. □

Theorem 3. *Let X be a space and $\sigma\colon X \to X$ a map of order 3 without fixed-points. The following statements are equivalent:*

1. $\mathrm{col}(X, \sigma) \leq n+3$,
2. $\mathrm{gen}(X, \sigma) \leq n+1$,
3. *there exist an equivariant map* $f : (X, \sigma) \to (S_Y^n, \gamma^{n+1})$,
4. $X = \bigcup_{i=1}^{n+1} B_i$, *where B_i is a σ-invariant subspace with* $\mathrm{col}(B_i, \sigma) = 3$ *for all i.*

Proof. Suppose (1) holds. Let $\mathcal{A} = \{ A_1, \dots, A_{n+1}, A_{n+2}, A_{n+3} \}$ be an open coloring of X. If $x \in A_{n+2} \cup A_{n+3}$ then at least one of the points $\sigma(x)$ and $\sigma^2(x)$ does not belong to $A_{n+2} \cup A_{n+3}$, as $\mathcal{A}$ is a coloring of X. It follows that

$$X = \bigcup_{i=1}^{n+1} [A_i \cup \sigma(A_i) \cup \sigma^2(A_i)],$$

whence (2) holds. If (X,σ) satisfies (2), then $X=\bigcup_{i=1}^{n+1}B_i$ with $B_i=C_i\cup\sigma(C_i)\cup\sigma^2(C_i)$ a set of first type. By shrinking the cover $\{\,\sigma^j(C_i):i=1,\ldots,n;\ j=0,1,2\,\}$ one can find closed subsets $D_i\subset C_i$ such that $\{\,\sigma^j(D_i):i=1,\ldots,n;\ j=0,1,2\,\}$ is a cover of X. By Lemma 1 there is for each i an equivariant map $f_i:X\to Y$ with the property $f_i^{-1}(1)=D_i$. The evaluation map $f=(f_1,\ldots,f_{n+1})$ of X to Y^{n+1} is equivariant and sends X to S^n_Y. So (3) holds. We have already observed that $\mathrm{col}(S^n_Y,\gamma^{n+1})\le n+3$. Thus (1) follows from (3).

To complete the proof we show that (2) and (4) are equivalent. If (X,σ) satisfies (2), then $X=\bigcup_{i=1}^{n+1}B_i$ where each $B_i=C_i\cup\sigma(C_i)\cup\sigma^2(C_i)$ is a set of first type. The subspaces B_i are invariant and have color number 3. Now, suppose that (4) holds. For each i let C_i be a color of the subspace B_i witnessing the fact that $\mathrm{col}(B_i,\sigma)=3$. For each i the set C_i is open in the subspace B_i and the sets C_i, $\sigma(C_i)$ and $\sigma^2(C_i)$ are mutually disjoint subsets of B_i. As the sets C_i, $\sigma(C_i)$ and $\sigma^2(C_i)$ are mutually separated in X there is an color U_i of (X,σ) such that $B_i\cap U_i=C_i$. The set $V_i=U_i\cup\sigma(U_i)\cup\sigma^2(U_i)$ is of the first type and $X=\bigcup_{i=1}^{n+1}V_i$, whence $\mathrm{gen}(X,\sigma)\le n+1$. □

From the equivalence of (2) and (4) in the previous theorem and the construction of S^n_Y out of S^{n-1}_Y one can obtain

$$\mathrm{gen}(S^n_Y,\gamma^{n+1})\le 1+\mathrm{gen}(S^{n-1}_Y,\gamma^n). \tag{1}$$

To prove this formula let $S^{n-1}_Y=\bigcup_{i=1}^k B_i$, where B_i is a γ^n-invariant subspace with $\mathrm{col}(B_i,\sigma)=3$ for all i and $k=\mathrm{gen}(S^{n-1}_Y,\gamma^n)$. Then $S^n_Y=[\bigcup_{i=1}^n B_i\times Y^\circ]\cup L^n$ and $\mathrm{gen}(S^n_Y,\gamma^{n+1})\le k+1$, where $L^n=Y^n\times\{1,2,3\}$.

Theorem 4. $\mathrm{col}(S^n_Y,\gamma^{n+1})=n+3$ *and* $\mathrm{gen}(S^n_Y,\gamma^{n+1})=n+1$.

Proof. It has already been observed that $\mathrm{col}(S^n_Y,\gamma^{n+1})\le n+3$. It is known that for all odd n the standard sphere S^n with the standard map σ of period three has color number $n+3$. So there exists an equivariant map $f:(S^n,\sigma)\to(S^n_Y,\gamma^{n+1})$. It follows that $\mathrm{col}(S^n_Y,\gamma^{n+1})\ge n+3$ and by Theorem 3 one obtains $\mathrm{gen}(S^n_Y,\gamma^{n+1})=n+1$. Now suppose that n is even and $\mathrm{col}(S^n_Y,\gamma^{n+1})<n+3$. By Theorem 3 it follows that $\mathrm{gen}(S^n_Y,\gamma^{n+1})<n+1$. Then by the formula (1) we get $\mathrm{gen}(S^{n+1}_Y,\gamma^{n+2})<n+2$, which cannot be true, as $n+1$ is odd. □

2. Ljusternik-Schnirelmann for maps of order 3.

Theorem 2 is similar to the Ljusternik-Schnirelmann theorem which reads as follows. Suppose that σ is an involution on a space X and $\mathrm{col}(X,\sigma)=n+2$. Suppose $\{\,A_1,\ldots,A_{n+1},A_{n+2}\,\}$ is a coloring of X. Then $\bigcap_{i=1}^{n+1}A_i\ne\emptyset$. The reason why this is true follows. If the two points $\mathbf{x_1}=(1,\ldots,1)$ and $\mathbf{x_2}=(-1,\ldots,-1)$ are deleted from the standard sphere $S^n\subset[-1,1]^{n+1}$ one obtains a space with a strictly smaller coloring number (for $n \geq 1$). For maps of order 3 one can not make a similar claim, $\bigcap_{i=1}^{n+1} A_i \neq \emptyset$, as easy examples on S^1 already show. If in the space S_Y^n the three points $\mathbf{x_i} = (\zeta^i, \dots, \zeta^i)$, $i = 0, 1, 2$, are deleted one obtains a space with the same coloring number. The color number decreases only if more points are deleted.

Consider the n-dimensional Y-sphere S_Y^n and define the subset Λ_Y^n by

$$\mathbf{x} = (x_i)_i \in \Lambda_Y^n \text{ if and only if } x_i \in \{1, \zeta, \zeta^2\} \text{ for all } i.$$

Note that the cardinality of Λ_Y^n is 3^{n+1} and that Λ_Y^n is invariant under γ^{n+1}. From Theorem 5 it follows that the color number can be reduced if we delete the subset Λ_Y^n from S_Y^n.

We define a subspace $\Sigma^n \subset \Lambda_Y^n$ in the folowing way. For $i = 0, 1, 2$, the subset Σ_i^n of Λ^n is defined by

$$\Sigma_i^n = \{ \mathbf{x} = (x_j)_j : x_j = \zeta^{i-1} \text{ or } x_j = \zeta^{i+1} \} \setminus \Delta,$$

where Δ is the diagonal of Y^{n+1}, i.e., Δ is the set of points all whose coordinates are equal. For example, $\mathbf{x} = (\zeta, \zeta^2, \zeta) \in \Sigma_0^2$, but $\mathbf{y} = (\zeta, \zeta, \zeta) \notin \Sigma_0^2$. Note that the three sets Σ_i^n, $i = 0, 1, 2$, are pairwise disjoint and $\gamma^{n+1}(\Sigma_i^n) = \Sigma_{(i+1) \bmod 3}^n$. It follows that the set $\Sigma^n = \Sigma_0^n \cup \Sigma_1^n \cup \Sigma_2^n$ is γ^{n+1}-invariant.

Theorem 5. *For $n \geq 1$,*

$$\operatorname{col}(S_Y^n \setminus \Sigma^n, \gamma^{n+1}) \leq n + 2 \quad \textit{and} \quad \operatorname{gen}(S_Y^n \setminus \Sigma^n, \gamma^{n+1}) \leq n.$$

Proof. As the statements of the theorem are equivalent by Theorem 3, we prove the second. For $n = 1$ it is best to verify the statement by drawing a picture. There are six points in Σ^1. We mentioned already that S_Y^1 is the graph $K(3, 3)$, which is cut by Σ^1 in six of its edges. The three edges that remain intact, connect vertices of two opposite parts of the graph. So $S_Y^1 \setminus \Sigma^1$ consists of three components, that are permuted by γ^2. Thus $S_Y^1 \setminus \Sigma^1$ is a set of the first type.

Now assume that the result holds for $(S_Y^{n-1} \setminus \Sigma^{n-1}, \gamma^n)$. As $S_Y^n = [S_Y^{n-1} \times Y^\circ] \cup \bigcup_{i=0}^2 [Y^n \times \{\zeta^i\}]$ and $(S_Y^{n-1} \times Y^\circ) \cap \Sigma^n = \emptyset$, we have

$$S_Y^n \setminus \Sigma^n = [(S_Y^{n-1} \setminus \Sigma^{n-1}) \times Y^\circ] \ \cup \left[(\Sigma^{n-1} \times Y^\circ) \cup \bigcup_{i=0}^{2} \{(Y^n \times \{\zeta^i\}) \setminus \Sigma^n\} \right].$$

The induction hypothesis implies that $\operatorname{gen}\big((S_Y^{n-1} \setminus \Sigma^{n-1}) \times Y^\circ, \gamma^{n+1}\big) \leq n-1$. So it suffices to show that the remaining two sets $(\Sigma^{n-1} \times Y^\circ) \cup \bigcup_{i=0}^2 (Y^n \times \{\zeta^i\}) \setminus \Sigma^n$ form a set of genus 1. Define the sets A_i for $i = 0, 1, 2$ by

$$A_i = (\Sigma_i^{n-1} \times Y^\circ) \cup (Y^n \times \{\zeta^i\}) \setminus \Sigma^n.$$

One verifies that this set has the property $\gamma^{n+1}(A_i) \cap A_i = \emptyset$ and that $(\Sigma^{n-1} \times Y^\circ) \cup \bigcup_{i=0}^{2} (Y^n \times \{ \zeta^i \}) \backslash \Sigma^n = A_0 \cup A_1 \cup A_2$. So the proof is finished once we verify that A_0 is a closed subset of $S_Y^n \backslash \Sigma^n$. It is obvious that $Y^n \times \{1\}$ is closed, so we have to verify that the closure of $\Sigma^{n-1} \times Y^\circ$ is contained in A_0. Observe that the set of density points of $\Sigma_0^{n-1} \times Y^\circ$ in the Y-sphere is $\Sigma_0^{n-1} \times \{1, \zeta, \zeta^2\}$. So in $S_Y^n \backslash \Sigma^n$ its set of density points is $\Sigma_0^{n-1} \times \{1\}$, which is a subset of $Y^n \times \{1\}$. We conclude that A_0 is closed in $S_Y^n \backslash \Sigma^n$. □

The previous theorem has as a corollary Theorem 2.

Proof. Note first that the three statements

(1) $\bigcap_{i=1}^{n+1} [A_i \cup \sigma(A_i)] \neq \emptyset$,
(2) $\bigcap_{i=1}^{n+1} [\sigma(A_i) \cup \sigma^2(A_i)] \neq \emptyset$,
(3) $\bigcap_{i=1}^{n+1} [\sigma^2(A_i) \cup A_i] \neq \emptyset$,

are equivalent, as these sets are mapped onto each other by the map σ. Using the closed coloring $\{ A_1, \dots , A_{n+1}, A_{n+2}, A_{n+3} \}$, for $i = 1, \dots , n+1$, we can define by Lemma 1 equivariant maps $f_i \colon X \to Y$ with $f_i^{-1}(1) = A_i$. The evaluation map $F = (f_1, \dots , f_{n+1}) \colon X \to Y^{n+1}$ is equivariant and $F(X) \subset S_Y^n$. Since $\operatorname{col}(X, \sigma) = n + 3$ and $\operatorname{col}(S_Y^n \backslash \Sigma_n, \gamma^{n+1}) \leq n + 2$, it can not occur that $F(X) \subset S_Y^n \backslash \Sigma^n$. Choose $x \in X$ with $F(x) = \big(f_i(x)\big)_i \in \Sigma^n$, say $F(x) \in \Sigma_1^n$.

It follows that $x \in \bigcap_{i=1}^{n+1} [A_i \cup \sigma^2(A_i)] \neq \emptyset$ and so $\sigma(x) \in \bigcap_{i=1}^{n+1} [A_i \cup \sigma(A_i)] \neq \emptyset$. Similar arguments can be used for the cases $F(x) \in \Sigma_2^n$ and $F(x) \in \Sigma_0^n$. □

3. Colorings of non-invariant subspaces.

We have defined colors as special open subsets. It was already observed that in defining colors we could have used closed sets as well. However, when studying non-invariant subspaces we must stick to open colors.

Definition 3. Suppose that $f \colon X \to X$ is a map from X to itself. Let A be a subset of X. A *coloring* of the subset A is a finite collection $\mathcal{B}$ consisting of colors of (X, f) such that $A \subset \bigcup \mathcal{B}$. We denote the minimal cardinality of such a collection by $\operatorname{col}(A, X, f)$.

With the technique of the proof of Theorem 3, the equivalence of (2) and (4) one can prove the following lemma. We use the notation σ_B to denote the restriction of the map σ to the subspace B.

Lemma 2 ([8])**.** *Let X be a space and $\sigma : X \to X$ a map of period 3 without fixed-points. If $A \subset B \subset X$ and B is σ-invariant, then*

$$\operatorname{col}(A, X, \sigma) = \operatorname{col}(A, B, \sigma_B).$$

It is a consequence of the lemma that to compute the color number of a subset A of X, we may assume that $X = A \cup \sigma(A) \cup \sigma^2(A)$. The following theorem provides an upper bound for the color number of a subset related to its dimension.

Theorem 6. *Suppose that $\sigma\colon X \to X$ is a map of period 3 without fixed-points. If A is a subset of X and $\dim(A) \leq n$ then $\mathrm{col}(A, X, \sigma) \leq 3n + 5$.*

Proof. The σ-invariant subspace $T = A \cup \sigma(A) \cup \sigma^2(A)$ has dimension at most $3n+2$ and contains no fixed points of σ. So $\mathrm{col}(T, \sigma_T) \leq \dim(T) + 3 = 3n+5$. The result follows from the previous lemma. □

We shall present an example of a map σ of period 3 with $\mathrm{col}(A, X, \sigma) \geq 3n + 4$. We refer to [8] for a related example of an involution ι with color number $\mathrm{col}(A, X, \iota) = 2n + 3$. For the construction we need the following result, which is a consequence of Theorem 1.

Theorem 7. *Let $\sigma : X \to X$ be a map of period 3 without fixed-points. Suppose $\mathrm{col}(X, \sigma) = n + 3$. If A is a dense subset of X with $A \cup \sigma(A) \cup \sigma^2(A) = X$ then $\mathrm{col}(A, \sigma) \geq n + 2$.*

Proof. We argue by contradiction. Assume that $\{ U_1, \dots , U_{n+1} \}$ is a coloring of (A, X, σ). Then $X = \bigcup_{i=1}^{n+1} (U_i \cup \sigma(U_i) \cup \sigma^2(U_i))$. As $\mathrm{col}(X, \sigma) = n+3$, we have $\mathrm{gen}(X, \sigma) = n+1$. From Theorem 1 it follows that $\bigcap_{i=1}^{n+1} [\sigma^{-1}(U_i) \cup \sigma(U_i)]$ is a nonempty open set, which by the density of A contains an element $a \in A$. For each index i we have either $\sigma(a) \in U_i$ or $\sigma^{-1}(a) \in U_i$ and therefore $a \notin U_i$. This contradicts that the U_i cover A. □

For the construction of our example we need the following lemmas. The first lemma is a special case of a result in [4].

Lemma 3. *Let X be a space of dimension 2 with a fixed-point free map $\sigma\colon X \to X$ of period 3. Then there exists a subspace A of X with the following properties*

(1) *A is dense in X,*
(2) *$\dim A = 0$,*
(3) *$A \cup \sigma(A) \cup \sigma^2(A) = X$.*

The following lemma follows easily from the fact that a subset of X is contained in a G_δ-subset of the same dimension.

Lemma 4. *Let X be a space with a fixed-point free map $\sigma\colon X \to X$ of period 3. If A is a σ-invariant subspace of X with $\dim(A) = k$ then there exists a σ-invariant G_δ subset A' of X with $A \subset A'$ and $\dim A' = \dim A$.*

Lemma 5. *Let X be a space with $\dim X = n$ and with a fixed-point free map $\sigma\colon X \to X$ of period 3. Then $X = X_0 \cup \ldots \cup X_n$, where $\dim X_i = 0$ and each X_i is σ-invariant.*

Proof. This is a version of the Decomposition Theorem [**11**]. We use induction on the dimension. If $\dim X = 0$ the result is trivial. Assume the result holds for all spaces of dimension $\leq k-1$ and all maps of order 3 without fixed-point. Let X be a space with $\dim X = k$ and let $\sigma\colon X \to X$ be a map of period 3 without fixed-points. Let $\{U_n; n \in \omega_0\}$ be a countable base with $\dim(\operatorname{cl} U_n \backslash U_n) \leq k-1$, for all n. If we define

$$X_k = X \Bigg\backslash \left[\bigcup_n (\operatorname{cl} U_n \backslash U_n) \cup \bigcup_n \sigma(\operatorname{cl} U_n \backslash U_n) \cup \bigcup_n \sigma^2(\operatorname{cl} U_n \backslash U_n)\right]$$

then X_k is σ-invariant, $\dim X_k = 0$ and $\dim(X \backslash X_k) \leq k-1$. So the induction argument applies. □

Finally we construct the example of a complete n-dimensional subspace that cannot be colored with less than $3n+4$ colors.

Example 1. Consider the space S_Y^{3n+2} with the standard map $\gamma : S_Y^{3n+2} \to S_Y^{3n+2}$ of order 3. We know that $\operatorname{col}(S_Y^{3n+2}, \gamma) = 3n+5$. Since the S_Y^{3n+2} is $(3n+2)$-dimensional it can be written as the union of $3n+3$ zero-dimensional subspaces, say $S_Y^{3n+2} = B_1 \cup \ldots \cup B_{3n+3}$ such that each B_i is zero-dimensional and γ-invariant. Let $X_i = B_{3i-2} \cup B_{3i-1} \cup B_{3i}$ for $i = 1, \ldots, n+1$. Then each X_i is ≤ 2-dimensional, invariant under γ. Lemma 3 implies that each X_i has a dense zero-dimensional subspace A_i such that $A_i \cup \gamma(A_i) \cup \gamma^2(A_i) = X_i$. Write $A' = A_1 \cup \ldots \cup A_{n+1}$. Then, A' is dense, A' has dimension $\leq n$ and A' also has the property

$$A' \cup \gamma(A') \cup \gamma 2(A') = S_Y^{3n+2}. \tag{2}$$

Note that Property (2) implies that $\dim A' = n$. We enlarge A' to a dense G_δ subset A of dimension n. Then A satisfies Property (2). Finally, by Theorem 7 we have $\operatorname{col}(A, X, \gamma) \geq 3n+4$.

4. Some remarks on periodic maps.

For a general fixed-point free homeomorphism $f\colon X \to X$ we were unable to obtain bounds on $\operatorname{col}(A, X, f)$ in terms of $\dim A$. Indeed, we do not even know whether such a bound exists. However, we can find such a bound in the special case that the map is periodic.

We need the following result of Steinlein.

Theorem 8 ([**15**])**.** *If $f\colon S^{(m-1)(p-1)-1} \to S^{(m-1)(p-1)-1}$ is a fixed-point free map of prime-period p, then*

$$\operatorname{col}(S^{(m-1)(p-1)-1}, f) \leq 4m.$$

To obtain our final result, one should know that any free periodic homeomorphism $f\colon X \to X$ on a space of $\dim X = n$ can be conjugated to a free periodic homeomorphism on S^n [**15**], so that it suffices to consider homeomorphisms on S^n.

Theorem 9. *Let $\sigma : X \to X$ be a map of prime-period p, without fixed-points and suppose that $p-1$ divides $n+1$. If A is a n-dimensional subset of X then* $\mathrm{col}(A, X, \sigma) \le 5n + 12$*.*

Proof. Suppose that $f \colon X \to X$ is a map of order p and let A be subspace of X of dimension n. We can assume that $p \ge 5$, since the case $p = 2$ is done in [**8**] and $p = 3$ is done above. As we can assume that $X = A \cup f(A) \cup \cdots \cup f^{p-1}(A)$, we see that the maximal dimension of X is $pn+p-1$. Choose a minimal m such that $\dim(X) \le (m-1)(p-1) - 1$ and so

$$pn + p - 1 \le (m-1)(p-1) - 1.$$

This implies that we can take

$$m = \left\lceil \frac{pn+p}{p-1} + 1 \right\rceil.$$

We conclude that we can color X, hence A, with $4m = 4\lceil \frac{pn+p}{p-1} + 1 \rceil \le 5n + 12$ colors. □

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