COLORING MAPS OF PERIOD THREE

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We investigate the color number and genus for fixed-point free maps of order three. A result that has the flavor of the Ljusternik-Schnirelmann theorem for involutions is established. The $Y$-sphere, the combinatorial boundary of the product of tripods, is studied in detail. Problems of coloring non-invariant subspaces are touched upon.

Introduction.

All spaces are assumed to be separable metric and all mappings are assumed to be continuous.

Definition 1. Suppose that $f: X \to X$ is a map from $X$ to itself. An open subset $B$ of $X$ is called a color of $(X, f)$ if $f(B) \cap B = \emptyset$ or, equivalently, $f^{-1}(B) \cap B = \emptyset$. A coloring of $(X, f)$ is a finite cover $B$ of $X$ consisting of colors. The minimal cardinality of a coloring $B$ is called the color number $\text{col}(X, f)$ of $(X, f)$.

In the definition of color we could have used closed subsets as well. By shrinking an open coloring a closed coloring may be obtained and the colors of a closed coloring can be enlarged so as to obtain an open coloring. The situation is more delicate when considering non-invariant subspaces, as was shown in [8]. For a fixed-point free homeomorphism $f$ of an $n$-dimensional space $X$ we have $\text{col}(X, f) \leq n + 3$; if moreover the map $f$ is an involution then $\text{col}(X, f) \leq n + 2$, [3].

In this paper we study fixed-point free maps $\sigma: X \to X$ of order 3, i.e., $\sigma^3(x) = x$ for each $x \in X$. It is to be noted that for a color $B$ of $(X, \sigma)$, when $\sigma$ has period 3, the sets $B$, $\sigma(B)$ and $\sigma^2(B)$ are pairwise disjoint. One of the reasons for studying maps of period 3 is that they provide examples of pairs $(X, \sigma)$ with $\dim(X) = n$ and $\text{col}(X, \sigma) = n + 3$. A second reason is that there is an intimate relation between the color number and the genus, which we now define.

Definition 2. Let $X$ be a space and $\sigma: X \to X$ a map of period 3 without fixed-points.

(1) A subset $B$ of $X$ is called a set of first type if there exists a color $C$ of $(X, \sigma)$ such that $B = C \cup \sigma(C) \cup \sigma^2(C)$; we also say that $B$ is generated by $C$. 

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(2) We say that the genus of the space $X$ is at most $k$ if $X$ can be written as a union of $k$ sets of first type. Notation: $\text{gen}(X, \sigma) \leq k$.

Recall that if $f: X \to X$ and $g: Y \to Y$ are mappings then a mapping $h: X \to Y$ is said to be equivariant if $h \circ f = g \circ h$. Note that for an equivariant $h: X \to Y$ and mappings $f$ and $g$ of order 3 we have $\text{col}(X, f) \leq \text{col}(Y, g)$ and $\text{gen}(X, f) \leq \text{gen}(Y, g)$.

We now formulate the main results, which are generalizations of the Ljusternik-Schnirelmann theorem for involutions on $S^n$.

**Theorem 1.** Let $X$ be a space and $\sigma: X \to X$ a map of period 3 without fixed-points. If $\text{gen}(X, \sigma) = n + 1$ and if $A_1, \ldots, A_{n+1}$ are colors of $X$ with $X = \bigcup_{i=1}^{n+1} [A_i \cup \sigma(A_i) \cup \sigma^2(A_i)]$ then

$$\bigcap_{i=1}^{n+1} [A_i \cup \sigma(A_i)] \neq \emptyset.$$ 

**Theorem 2.** Let $X$ be a space and $\sigma: X \to X$ a map of period 3 without fixed-points. If $\text{col}(X, \sigma) = n + 3$ and $\{A_1, \ldots, A_{n+1}, A_{n+2}, A_{n+3}\}$ is a coloring of $X$ then

$$\bigcap_{i=1}^{n+1} [A_i \cup \sigma(A_i)] \neq \emptyset.$$ 

We shall also obtain a bound on the color number of non-invariant subspaces, using the results of [3]. We shall use the results above to show that this bound is almost sharp. Theorem 1 can be extended to general periodic maps on paracompact spaces [1] by a different proof. The proof in the present paper concerns certain universal spaces, the $Y$-cube and the $Y$-sphere, which are interesting in their own right.

1. **The $Y$-cube $Y^n$ and the $Y$-sphere $S^n_Y$.**

The interval $I = [-1, 1]$ with the antipodal-map is a standard space for the study of involutions and gives rise to the study of the $n$-cube $I^n$ and the $n$-sphere $S^n$. We introduce a similar space for the study of maps of order 3. A natural candidate is the tripod, a $Y$-shaped space.

Consider in the complex plane the rotation $\gamma$ of 120 degrees around 0 which is induced by the multiplication with $\zeta = \exp(\frac{2\pi i}{3})$. Let $I$ denote the closed segment between 0 and 1. The subspace $Y = I \cup \zeta(I) \cup \zeta^2(I)$ is called the closed $Y$-interval. The points 1, $\zeta$, and $\zeta^2$ are called the end points of $Y$ and $Y^o = Y \setminus \{1, \zeta, \zeta^2\}$ is referred to as the open $Y$-interval. Both $Y$ and $Y^o$ are invariant under $\gamma$. The space $Y^n$ is called the $n$-dimensional $Y$-cube. The product map $\gamma^n: Y^n \to Y^n$ is a map of order 3 with a unique fixed point. The subspace $S^n_Y$ of $Y^n$ is defined by

$$S^n_Y = \{(x_1, \ldots, x_n) \in Y^n: x_i \in \{1, \zeta, \zeta^2\} \text{ for some } i\}.$$
The space $S^n_{Y}$ is called the \((n-1)\)-dimensional \(Y\)-sphere. The 1-dimensional \(Y\)-sphere is the familiar bipartite cubic graph on six nodes \(K(3, 3)\), which is a standard example of a non-planar graph.

Note that $S^n_{Y}$ is invariant under $\gamma^n$ and that $\gamma^n$ has no fixed points in $S^n_{Y}$. As $\dim(S^n_{Y}) = n$ we have

$$\text{col}(S^n_{Y}, \gamma^{n+1}) \leq n + 3.$$  

We will show later on that $\text{col}(S^n_{Y}, \gamma) = n + 3$. So $(S^n_{Y}, \gamma^{n+1})$ is a pair of an \(n\)-dimensional space and a mapping for which the color number is maximal.

The space $S^n_{Y}$ can be obtained from $S^{n-1}_{Y}$ as in the following way

$$S^n_{Y} = [S^{n-1}_{Y} \times Y] \cup \bigcup_{i=0}^{2}[Y^n \times \{\zeta^i\}] \subset Y^{n+1}$$

and

$$S^n_{Y} = [S^{n-1}_{Y} \times Y^0] \cup \bigcup_{i=0}^{2}[Y^n \times \{\zeta^i\}] \subset Y^{n+1}.$$

**Lemma 1.** Let $X$ be a space and $\sigma : X \to X$ a map of order 3 without fixed-points. Suppose $D$ is a closed subset such that $D \cap \sigma(D) = \emptyset$ and $E = D \cup \sigma(D) \cup \sigma^2(D)$, then there exist an equivariant map $f : X \to Y$ with $f^{-1}(1) = D$.

In the lemma, one may think of $D$ and $E$ as a closed color and closed set of the first kind.

**Proof.** By enlarging the $\sigma^i(D)$ for $i = 0, 1, 2$ we obtain open colors $U_i$. For $i = 0, 1, 2$ define real-valued Urysohn functions $f_i$ on $X$ such that $0 \leq f_i(x) \leq 1$ for all $x$, $f_i^{-1}(1) = \sigma^i(D)$ and $f_i^{-1}(0) = X \setminus U_i$. Let $f = f_0 + \zeta f_1 + \zeta^2 f_2$. \(\square\)

**Theorem 3.** Let $X$ be a space and $\sigma : X \to X$ a map of order 3 without fixed-points. The following statements are equivalent:

1. $\text{col}(X, \sigma) \leq n + 3$,
2. $\text{gen}(X, \sigma) \leq n + 1$,
3. there exist an equivariant map $f : (X, \sigma) \to (S^n_{Y}, \gamma^{n+1})$,
4. $X = \bigcup_{i=1}^{n+1} B_i$, where $B_i$ is a $\sigma$-invariant subspace with $\text{col}(B_i, \sigma) = 3$ for all $i$.

**Proof.** Suppose (1) holds. Let $A = \{ A_1, \ldots, A_{n+1}, A_{n+2}, A_{n+3} \}$ be an open coloring of $X$. If $x \in A_{n+2} \cup A_{n+3}$ then at least one of the points $\sigma(x)$ and $\sigma^2(x)$ does not belong to $A_{n+2} \cup A_{n+3}$, as $A$ is a coloring of $X$. It follows that

$$X = \bigcup_{i=1}^{n+1} [A_i \cup \sigma(A_i) \cup \sigma^2(A_i)],$$
whence (2) holds. If \((X, \sigma)\) satisfies (2), then \(X = \bigcup_{i=1}^{n+1} B_i\) with \(B_i = C_i \cup \sigma(C_i) \cup \sigma^2(C_i)\) a set of first type. By shrinking the cover \(\{ \sigma^j(C_i) : i = 1, \ldots, n; j = 0, 1, 2 \}\) one can find closed subsets \(D_i \subset C_i\) such that \(\{ \sigma^j(D_i) : i = 1, \ldots, n; j = 0, 1, 2 \}\) is a cover of \(X\). By Lemma 1 there is for each \(i\) an equivariant map \(f_i : X \to Y\) with the property \(f_i^{-1}(1) = D_i\). The evaluation map \(f = (f_1, \ldots, f_{n+1})\) of \(X\) to \(Y^{n+1}\) is equivariant and sends \(X\) to \(S^0_Y\). So (3) holds. We have already observed that \(\text{col}(S^n_Y, \gamma^{n+1}) \leq n + 3\). Thus (1) follows from (3).

To complete the proof we show that (2) and (4) are equivalent. If \((X, \sigma)\) satisfies (2), then \(X = \bigcup_{i=1}^{n+1} B_i\) where each \(B_i = C_i \cup \sigma(C_i) \cup \sigma^2(C_i)\) is a set of first type. The subspaces \(B_i\) are invariant and have color number 3. Now, suppose that (4) holds. For each \(i\) let \(C_i\) be a color of the subspace \(B_i\) witnessing the fact that \(\text{col}(B_i, \sigma) = 3\). For each \(i\) the set \(C_i\) is open in the subspace \(B_i\) and the sets \(C_i, \sigma(C_i)\) and \(\sigma^2(C_i)\) are mutually disjoint subsets of \(B_i\). As the sets \(C_i, \sigma(C_i)\) and \(\sigma^2(C_i)\) are mutually separated in \(X\) there is an color \(U_i\) of \((X, \sigma)\) such that \(B_i \cap U_i = C_i\). The set \(V_i = U_i \cup \sigma(U_i) \cup \sigma^2(U_i)\) is of the first type and \(X = \bigcup_{i=1}^{n+1} V_i\), whence \(\text{gen}(X, \sigma) \leq n + 1\). \(\square\)

From the equivalence of (2) and (4) in the previous theorem and the construction of \(S^n_Y\) out of \(S^0_Y\), one can obtain
\[
(1) \quad \text{gen}(S^n_Y, \gamma^{n+1}) \leq 1 + \text{gen}(S^n_Y, \gamma^n).
\]
To prove this formula let \(S^0_Y = \bigcup_{i=1}^{n+1} B_i\), where \(B_i\) is a \(\gamma^n\)-invariant subspace with \(\text{col}(B_i, \sigma) = 3\) for all \(i\) and \(k = \text{gen}(S^n_Y, \gamma^n)\). Then \(S^n_Y = [\bigcup_{i=1}^{n+1} B_i \times Y^n] \cup L^n\) and \(\text{gen}(S^n_Y, \gamma^{n+1}) \leq k + 1\), where \(L^n = Y^n \times \{1, 2, 3\}\).

**Theorem 4.** \(\text{col}(S^n_Y, \gamma^{n+1}) = n + 3\) and \(\text{gen}(S^n_Y, \gamma^{n+1}) = n + 1\).

**Proof.** It has already been observed that \(\text{col}(S^n_Y, \gamma^{n+1}) \leq n + 3\). It is known that for all odd \(n\) the standard sphere \(S^n\) with the standard map \(\sigma\) of period three has color number \(n + 3\). So there exists an equivariant map \(f : (S^n, \sigma) \to (S^n, \gamma^{n+1})\). It follows that \(\text{col}(S^n_Y, \gamma^{n+1}) \geq n + 3\) and by Theorem 3 one obtains \(\text{gen}(S^n_Y, \gamma^{n+1}) = n + 1\). Now suppose that \(n\) is even and \(\text{col}(S^n_Y, \gamma^{n+1}) < n + 3\). By Theorem 3 it follows that \(\text{gen}(S^n_Y, \gamma^{n+1}) < n + 1\). Then by the formula (1) we get \(\text{gen}(S^n_Y, \gamma^{n+1}) < n + 2\), which cannot be true, as \(n + 1\) is odd. \(\square\)

**2. Ljusternik-Schnirelmann for maps of order 3.**

Theorem 2 is similar to the Ljusternik-Schnirelmann theorem which reads as follows. Suppose that \(\sigma\) is an involution on a space \(X\) and \(\text{col}(X, \sigma) = n + 2\). Suppose \(\{ A_1, \ldots, A_{n+1}, A_{n+2} \}\) is a coloring of \(X\). Then \(\bigcap_{i=1}^{n+1} A_i \neq \emptyset\). The reason why this is true follows. If the two points \(x_1 = (1, \ldots, 1)\) and \(x_2 = (-1, \ldots, -1)\) are deleted from the standard sphere \(S^n \subset [-1, 1]^{n+1}\) one
obtains a space with a strictly smaller coloring number (for \( n \geq 1 \)). For maps of order 3 one can not make a similar claim, \( \bigcap_{i=1}^{n+1} A_i \neq \emptyset \), as easy examples on \( S^1 \) already show. If in the space \( S^n_Y \) the three points \( x_i = (\zeta^i, \ldots, \zeta^3) \), \( i = 0, 1, 2 \), are deleted one obtains a space with the same coloring number. The color number decreases only if more points are deleted.

Consider the \( n \)-dimensional \( Y \)-sphere \( S^n_Y \) and define the subset \( \Lambda^n_Y \) by

\[
x = (x_i) \in \Lambda^n_Y \quad \text{if and only if} \quad x_i \in \{1, \zeta, \zeta^2\} \quad \text{for all} \quad i.
\]

Note that the cardinality of \( \Lambda^n_Y \) is \( 3^{n+1} \) and that \( \Lambda^n_Y \) is invariant under \( \gamma^{n+1} \). From Theorem 5 it follows that the color number can be reduced if we delete the subset \( \Lambda^n_Y \) from \( S^n_Y \).

We define a subspace \( \Sigma^n \subset \Lambda^n_Y \) in the following way. For \( i = 0, 1, 2 \), the subset \( \Sigma^n_i \) of \( \Lambda^n \) is defined by

\[
\Sigma^n_i = \{ x = (x_j) : x_j = \zeta^{i-1} \quad \text{or} \quad x_j = \zeta^{i+1} \} \setminus \Delta,
\]

where \( \Delta \) is the diagonal of \( Y^{n+1} \), i.e., \( \Delta \) is the set of points all whose coordinates are equal. For example, \( x = (\zeta, \zeta^2, \zeta) \in \Sigma^n_2 \), but \( y = (\zeta, \zeta, \zeta) \notin \Sigma^n_2 \).

Note that the three sets \( \Sigma^n_i \), \( i = 0, 1, 2 \), are pairwise disjoint and \( \gamma^{n+1}(\Sigma^n_i) = \Sigma^n_{i+1} \mod 3 \). It follows that the set \( \Sigma^n = \Sigma^n_0 \cup \Sigma^n_1 \cup \Sigma^n_2 \) is \( \gamma^{n+1} \)-invariant.

**Theorem 5.** For \( n \geq 1 \),

\[
\text{col}(S^n_Y \setminus \Sigma^n, \gamma^{n+1}) \leq n + 2 \quad \text{and} \quad \text{gen}(S^n_Y \setminus \Sigma^n, \gamma^{n+1}) \leq n.
\]

**Proof.** As the statements of the theorem are equivalent by Theorem 3, we prove the second. For \( n = 1 \) it is best to verify the statement by drawing a picture. There are six points in \( \Sigma^1 \). We mentioned already that \( S^1_Y \) is the graph \( K(3, 3) \), which is cut by \( \Sigma^1 \) in six of its edges. The three edges that remain intact, connect vertices of two opposite parts of the graph. So \( S^1_Y \setminus \Sigma^1 \) consists of three components, that are permuted by \( \gamma^2 \). Thus \( S^1_Y \setminus \Sigma^1 \) is a set of the first type.

Now assume that the result holds for \( (S^n_{Y-1} \setminus \Sigma^{n-1}, \gamma^n) \). As \( S^n_Y = [S^n_{Y-1} \times Y^o] \cup \bigcup_{i=0}^{2} [Y^n \times \{\zeta^i\}] \) and \( (S^n_{Y-1} \times Y^o) \cap \Sigma^n = \emptyset \), we have

\[
S^n_Y \setminus \Sigma^n = [(S^n_{Y-1} \setminus \Sigma^{n-1}) \times Y^o] \cup \left( \Sigma^{n-1} \times Y^o \right) \cup \bigcup_{i=0}^{2} \left( (Y^n \times \{\zeta^i\}) \setminus \Sigma^n \right).
\]

The induction hypothesis implies that \( \text{gen}((S^n_{Y-1} \setminus \Sigma^{n-1}) \times Y^o, \gamma^{n+1}) \leq n-1 \), so it suffices to show that the remaining two sets \( (\Sigma^{n-1} \times Y^o) \cup \bigcup_{i=0}^{2} (Y^n \times \{\zeta^i\}) \setminus \Sigma^n \) form a set of genus 1. Define the sets \( A_i \) for \( i = 0, 1, 2 \) by

\[
A_i = (\Sigma^{n-1} \times Y^o) \cup (Y^n \times \{\zeta^i\}) \setminus \Sigma^n.
\]
One verifies that this set has the property \( \gamma^{n+1}(A_i) \cap A_i = \emptyset \) and that 
\( (\Sigma^{n-1} \times Y^o) \cup \bigcup_{i=0}^{2^n} (Y^n \times \{\zeta^i\}) \setminus \Sigma^n = A_0 \cup A_1 \cup A_2 \). So the proof is finished 
once we verify that \( A_0 \) is a closed subset of \( S_Y^n \setminus \Sigma^n \). It is obvious that 
\( Y^n \times \{1\} \) is closed, so we have to verify that the closure of \( \Sigma^{n-1} \times Y^o \) is 
contained in \( A_0 \). Observe that the set of density points of \( \Sigma^{n-1} \times Y^o \) in 
the \( Y \)-sphere is \( \Sigma^1 \). So in \( S_Y^n \setminus \Sigma^n \) its set of density points is 
\( \Sigma^1 \), which is a subset of \( Y^n \times \{1\} \). We conclude that \( A_0 \) is closed 
in \( S_Y^n \setminus \Sigma^n \). \( \square \)

The previous theorem has as a corollary Theorem 2.

**Proof.** Note first that the three statements

\[
(1) \quad \bigcap_{i=1}^{n+1} [A_i \cup \sigma(A_i)] \neq \emptyset,
\]
\[
(2) \quad \bigcap_{i=1}^{n+1} [\sigma(A_i) \cup \sigma^2(A_i)] \neq \emptyset,
\]
\[
(3) \quad \bigcap_{i=1}^{n+1} [\sigma^2(A_i) \cup A_i] \neq \emptyset,
\]

are equivalent, as these sets are mapped onto each other by the map \( \sigma \).

Using the closed coloring \( \{A_1, \ldots, A_{n+1}, A_{n+2}, A_{n+3}\} \), for \( i = 1, \ldots, n+1 \),
we can define by Lemma 1 equivariant maps \( f_i : X \to Y \) with \( f_i^{-1}(1) = A_i \).

The evaluation map \( F = (f_1, \ldots, f_{n+1}) : X \to Y^{n+1} \) is equivariant and 
\( F(X) \subset S_Y^n \). Since \( \text{col}(X, \sigma) = n + 3 \) and \( \text{col}(S_Y^n \setminus \Sigma, \gamma^{n+1}) \leq n + 2 \), it can 
not occur that \( F(X) \subset S_Y^n \setminus \Sigma^n \). Choose \( x \in X \) with \( F(x) = (f_i(x))_i \in \Sigma^n \),
say \( F(x) \in \Sigma^n \).

It follows that \( x \in \bigcap_{i=1}^{n+1} [A_i \cup \sigma(A_i)] \neq \emptyset \) and so \( \sigma(x) \in \bigcap_{i=1}^{n+1} [A_i \cup \sigma(A_i)] \neq \emptyset \). Similar arguments can be used for the cases 
\( F(x) \in \Sigma^n \setminus \Sigma^n \) and \( F(x) \in \Sigma^n \setminus \Sigma_0^n \). \( \square \)

3. **Colorings of non-invariant subspaces.**

We have defined colors as special open subsets. It was already observed 
that in defining colors we could have used closed sets as well. However, 
when studying non-invariant subspaces we must stick to open colors.

**Definition 3.** Suppose that \( f : X \to X \) is a map from \( X \) to itself. Let \( A \) be 
a subset of \( X \). A **coloring** of the subset \( A \) is a finite collection \( B \) consisting 
of colors of \( (X, f) \) such that \( A \subset \bigcup B \). We denote the minimal cardinality 
of such a collection by \( \text{col}(A, X, f) \).

With the technique of the proof of Theorem 3, the equivalence of (2) 
and (4) one can prove the following lemma. We use the notation \( \sigma_B \) to 
denote the restriction of the map \( \sigma \) to the subspace \( B \).

**Lemma 2 ([8]).** Let \( X \) be a space and \( \sigma : X \to X \) a map of period 3 without 
fixed-points. If \( A \subset B \subset X \) and \( B \) is \( \sigma \)-invariant, then 
\[
\text{col}(A, X, \sigma) = \text{col}(A, B, \sigma_B).
\]
It is a consequence of the lemma that to compute the color number of a subset \( A \) of \( X \), we may assume that \( X = A \cup \sigma(A) \cup \sigma^2(A) \). The following theorem provides an upper bound for the color number of a subset related to its dimension.

**Theorem 6.** Suppose that \( \sigma : X \to X \) is a map of period 3 without fixed-points. If \( A \) is a subset of \( X \) and \( \dim(A) \leq n \) then \( \text{col}(A, X, \sigma) \leq 3n + 5 \).

**Proof.** The \( \sigma \)-invariant subspace \( T = A \cup \sigma(A) \cup \sigma^2(A) \) has dimension at most \( 3n + 2 \) and contains no fixed points of \( \sigma \). So \( \text{col}(T, \sigma_T) \leq \dim(T) + 3 = 3n + 5 \). The result follows from the previous lemma.

We shall present an example of a map \( \sigma \) of period 3 with \( \text{col}(A, X, \sigma) \geq 3n + 4 \). We refer to [8] for a related example of an involution \( \iota \) with color number \( \text{col}(A, X, \iota) = 2n + 3 \). For the construction we need the following result, which is a consequence of Theorem 1.

**Theorem 7.** Let \( \sigma : X \to X \) be a map of period 3 without fixed-points. Suppose \( \text{col}(X, \sigma) = n + 3 \). If \( A \) is a dense subset of \( X \) with \( A \cup \sigma(A) \cup \sigma^2(A) = X \) then \( \text{col}(A, \sigma) \geq n + 2 \).

**Proof.** We argue by contradiction. Assume that \( \{U_1, \ldots, U_{n+1}\} \) is a coloring of \( (A, X, \sigma) \). Then \( X = \bigcup_{i=1}^{n+1} (U_i \cup \sigma(U_i) \cup \sigma^2(U_i)) \). As \( \text{col}(X, \sigma) = n + 3 \), we have \( \text{gen}(X, \sigma) = n + 1 \). From Theorem 1 it follows that \( \bigcap_{i=1}^{n+1} [\sigma^{-1}(U_i) \cup \sigma(U_i)] \) is a nonempty open set, which by the density of \( A \) contains an element \( a \in A \). For each index \( i \) we have either \( \sigma(a) \in U_i \) or \( \sigma^{-1}(a) \in U_i \) and therefore \( a \notin U_i \). This contradicts that the \( U_i \) cover \( A \).

For the construction of our example we need the following lemmas. The first lemma is a special case of a result in [4].

**Lemma 3.** Let \( X \) be a space of dimension 2 with a fixed-point free map \( \sigma : X \to X \) of period 3. Then there exists a subspace \( A \) of \( X \) with the following properties

1. \( A \) is dense in \( X \),
2. \( \dim A = 0 \),
3. \( A \cup \sigma(A) \cup \sigma^2(A) = X \).

The following lemma follows easily from the fact that a subset of \( X \) is contained in a \( G_\delta \)-subset of the same dimension.

**Lemma 4.** Let \( X \) be a space with a fixed-point free map \( \sigma : X \to X \) of period 3. If \( A \) is a \( \sigma \)-invariant subspace of \( X \) with \( \dim(A) = k \) then there exists a \( \sigma \)-invariant \( G_\delta \) subset \( A' \) of \( X \) with \( A \subset A' \) and \( \dim A' = \dim A \).

**Lemma 5.** Let \( X \) be a space with \( \dim X = n \) and with a fixed-point free map \( \sigma : X \to X \) of period 3. Then \( X = X_0 \cup \ldots \cup X_n \), where \( \dim X_i = 0 \) and each \( X_i \) is \( \sigma \)-invariant.
Proof. This is a version of the Decomposition Theorem [11]. We use induction on the dimension. If \( \dim X = 0 \) the result is trivial. Assume the result holds for all spaces of dimension \( \leq k - 1 \) and all maps of order 3 without fixed-point. Let \( X \) be a space with \( \dim X = k \) and let \( \sigma : X \to X \) be a map of period 3 without fixed-points. Let \( \{ U_n ; n \in \omega \} \) be a countable base with \( \dim (\text{cl} \ U_n \setminus U_n) \leq k - 1 \), for all \( n \). If we define

\[
X_k = X \setminus \left[ \bigcup_n (\text{cl} \ U_n \setminus U_n) \cup \bigcup_n \sigma(\text{cl} \ U_n \setminus U_n) \cup \bigcup_n \sigma^2(\text{cl} \ U_n \setminus U_n) \right]
\]

then \( X_k \) is \( \sigma \)-invariant, \( \dim X_k = 0 \) and \( \dim (X \setminus X_k) \leq k - 1 \). So the induction argument applies.

Finally we construct the example of a complete \( n \)-dimensional subspace that cannot be colored with less than \( 3n + 4 \) colors.

**Example 1.** Consider the space \( S^{3n+2} \) with the standard map \( \gamma : S^{3n+2} \to S^{3n+2} \) of order 3. We know that \( \text{col}(S^{3n+2}, \gamma) = 3n + 5 \). Since the \( S^{3n+2} \) is \( (3n+2) \)-dimensional it can be written as the union of \( 3n+3 \) zero-dimensional subspaces, say \( S^{3n+2} = B_1 \cup \ldots \cup B_{3n+3} \) such that each \( B_i \) is zero-dimensional and \( \gamma \)-invariant. Let \( X_i = B_{3i-2} \cup B_{3i-1} \cup B_{3i} \) for \( i = 1, \ldots, n+1 \). Then each \( X_i \) is \( \leq 2 \)-dimensional, invariant under \( \gamma \). Lemma 3 implies that each \( X_i \) has a dense zero-dimensional subspace \( A_i \) such that \( A_i \cup \gamma(A_i) \cup \gamma^2(A_i) = X_i \).

Write \( A' = A_1 \cup \ldots \cup A_{n+1} \). Then, \( A' \) is dense, \( A' \) has dimension \( \leq n \) and \( A' \) also has the property

\[
A' \cup \gamma(A') \cup \gamma^2(A') = S^{3n+2}.
\]

Note that Property (2) implies that \( \dim A' = n \). We enlarge \( A' \) to a dense \( G_\delta \) subset \( A \) of dimension \( n \). Then \( A \) satisfies Property (2). Finally, by Theorem 7 we have \( \text{col}(A, X, \gamma) \geq 3n + 4 \).

4. Some remarks on periodic maps.

For a general fixed-point free homeomorphism \( f : X \to X \) we were unable to obtain bounds on \( \text{col}(A, X, f) \) in terms of \( \dim A \). Indeed, we do not even know whether such a bound exists. However, we can find such a bound in the special case that the map is periodic.

We need the following result of Steinlein.

**Theorem 8 ([15]).** If \( f : S^{(m-1)(p-1)-1} \to S^{(m-1)(p-1)-1} \) is a fixed-point free map of prime-period \( p \), then

\[
\text{col}(S^{(m-1)(p-1)-1}, f) \leq 4m.
\]

To obtain our final result, one should know that any free periodic homeomorphism \( f : X \to X \) on a space of \( \dim X = n \) can be conjugated to
a free periodic homeomorphism on \( S^n \) \cite{15}, so that it suffices to consider homeomorphisms on \( S^n \).

**Theorem 9.** Let \( \sigma : X \to X \) be a map of prime-period \( p \), without fixed-points and suppose that \( p - 1 \) divides \( n + 1 \). If \( A \) is a \( n \)-dimensional subset of \( X \) then \( \text{col}(A, X, \sigma) \leq 5n + 12 \).

*Proof.* Suppose that \( f : X \to X \) is a map of order \( p \) and let \( A \) be subspace of \( X \) of dimension \( n \). We can assume that \( p \geq 5 \), since the case \( p = 2 \) is done in \cite{8} and \( p = 3 \) is done above. As we can assume that \( X = A \cup f(A) \cup \cdots \cup f^{p-1}(A) \), we see that the maximal dimension of \( X \) is \( pn + p - 1 \). Choose a minimal \( m \) such that \( \dim(X) \leq (m - 1)(p - 1) - 1 \) and so

\[
\begin{align*}
 pn + p - 1 &\leq (m - 1)(p - 1) - 1. 
\end{align*}
\]

This implies that we can take

\[
 m = \left\lceil \frac{pn + p}{p - 1} + 1 \right\rceil .
\]

We conclude that we can color \( X \), hence \( A \), with \( 4m = 4\lceil \frac{pn + p}{p - 1} + 1 \rceil \leq 5n + 12 \) colors. \( \square \)

**References**


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