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K-GROUPS AND CLASSIFICATION OF SIMPLE QUOTIENTS OF GROUP C*-ALGEBRAS OF CERTAIN DISCRETE 5-DIMENSIONAL NILPOTENT GROUPS

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K-GROUPS AND CLASSIFICATION OF SIMPLE QUOTIENTS OF GROUP C*-ALGEBRAS OF CERTAIN DISCRETE 5-DIMENSIONAL NILPOTENT GROUPS

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The $K$-groups, the range of trace on $K_0$, and isomorphism classifications are obtained for simple infinite dimensional quotient C*-algebras of the group C*-algebras of six lattice subgroups, corresponding to each of the six non-isomorphic 5-dimensional connected, simply connected, nilpotent Lie groups. Connes’ non-commutative geometry involving cyclic cocycles and the Chern character play a key role in the proofs.

1. Introduction.

It is known that there are only six non-isomorphic 5-dimensional connected, simply connected, nilpotent Lie groups. These groups are denoted by $G_{5,j}(j = 1,\ldots,6)$, and were studied in great detail in Nielsen's paper [10]. In [9], Milnes and the author have studied a natural lattice subgroup $H_{5,j}$ of $G_{5,j}$. These subgroups are higher dimensional analogues of the well-known discrete Heisenberg group $H_3$, (but with more complicated multiplication rules inherited from $G_{5,j}$). The main result of [9] is an identification of all the simple infinite dimensional quotient C*-algebras of the group C*-algebra $C^*(H_{5,j})$ – more specifically, they consist, respectively, of the ‘primary’ algebras $A_{\theta}^{5,1}$, $A_{\theta,\varphi}^{5,2}$, $A_{\theta}^{5,3}$, $A_{\theta,\varphi}^{5,4}$, $A_{\theta}^{5,5}$, $A_{\theta,\varphi}^{5,6}$, (where $\theta, \varphi$ are irrational and are independent in the 5, 2 and 5, 4 cases), other simple C*-algebras isomorphic to matrix algebras over irrational rotation algebras (of any size and any irrational parameter), and a few more which are expressed as crossed products by the integers.

The objective of this paper is to find the $K$-groups, the range of the trace on $K_0$, and obtain a classification for the ‘primary’ simple quotient C*-algebras amongst themselves. Since each of these algebras is isomorphic to a crossed product by the integers, one uses the Pimsner-Voiculescu six term exact sequence [13] to compute their $K$-groups and Pimsner’s Theorem on the tracial range [12]. For the algebras $A_{\theta}^{5,1} \cong A_{\theta} \otimes A_{\theta}$, $A_{\theta,\varphi}^{5,2}$, $A_{\theta}^{5,3}$, $A_{\theta,\varphi}^{5,4}$, application of the Pimsner-Voiculescu exact sequence is not hard. This is done briefly in Section 2, and included for comparison and completion.
For the algebras $A^5_\theta$ and $A^6_\theta$, however, the application of the Pimsner-Voiculescu sequence is not so straightforward, as the action of the underlying automorphism (of the crossed product) on $K_*$ requires some careful work. More specifically, in order to calculate this action we make use of Connes’ non-commutative geometry involving cyclic cocycles and the Connes Chern character \[3\] in order to decipher $K$-group elements. This is dealt with in Sections 3 and 4, which are the main parts of the paper. In summary, we obtain the following result on the $K$-groups and range of the trace on $K_0$:

<table>
<thead>
<tr>
<th>Algebra</th>
<th>$A^3_\theta$</th>
<th>$A^4_\theta$</th>
<th>$A^{5,1}_\theta$</th>
<th>$A^{5,2}_\theta$</th>
<th>$A^{5,3}_\theta$</th>
<th>$A^{5,4}_\theta$</th>
<th>$A^{5,5}_\theta$</th>
<th>$A^{5,6}_\theta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$K_0 = K_1 = \mathbb{Z}$</td>
<td>$\mathbb{Z}^2$</td>
<td>$\mathbb{Z}^3$</td>
<td>$\mathbb{Z}^8$</td>
<td>$\mathbb{Z}^4$</td>
<td>$\mathbb{Z}^6$</td>
<td>$\mathbb{Z}^3$</td>
<td>$\mathbb{Z}^4$</td>
<td>$\mathbb{Z}^4$</td>
</tr>
<tr>
<td>$\tau_*K_0 = \mathbb{Z}^+$</td>
<td>$\mathbb{Z}\theta$</td>
<td>$\mathbb{Z}\theta$</td>
<td>$\mathbb{Z}\theta + \mathbb{Z}\theta^2$</td>
<td>$\mathbb{Z}\theta + \mathbb{Z}\varphi$</td>
<td>$\mathbb{Z}\theta + \mathbb{Z}\theta^2$</td>
<td>$\mathbb{Z}\theta + \mathbb{Z}\varphi$</td>
<td>$\mathbb{Z}\theta + \mathbb{Z}\theta^2$</td>
<td></td>
</tr>
</tbody>
</table>

Here, we have included the well-known result for the irrational rotation algebra $A_\theta = A^3_\theta$ (\[13\] and \[14\]), as well as for the Heisenberg C*-algebra $A^4_\theta$ studied by Packer in \[11\] (where it is referred to as class 2). (According to the convention adopted in \[8\] and \[9\], the superscripts on $A^3_\theta, A^4_\theta$ indicate the dimensions of the discrete nilpotent groups for which these are simple infinite dimensional quotients of the associate group C*-algebra – namely, the discrete Heisenberg group $H_3$ and the discrete group $H_4$ introduced in \[8\], respectively.)

The determination of the simple infinite dimensional quotients arising from 6-dimensional discrete nilpotent groups $H_{6,j}$ has been done by Milnes in \[6\] and \[7\] for $H_{6,4}$ and $H_{6,10}$ and by Junghenn and Milnes in \[5\] for $H_{6,7}$. It is known that there are exactly twenty-four non-isomorphic 6-dimensional connected, simply connected, nilpotent Lie groups $G_{6,j}$ ($j = 1, \ldots, 24$) (see Nielsen \[10\]), each of which contains a natural lattice subgroup $H_{6,j}$. In the 7-dimensional case, there are by contrast uncountably many non-isomorphic such Lie groups.

**Notation.** Throughout the paper we shall adopt Connes’ and Rieffel’s convention and write 

\[
e(t) := e^{2\pi it}.
\]

Briefly, recall Pimsner’s procedure \[12\] for finding the range of the trace in the case of crossed products by the integers. Let $A$ be a C*-algebra (for us unital) and $\sigma$ an automorphism of $A$. Let $\tau$ be a trace state on $A \rtimes_{\sigma} \mathbb{Z}$ and use it also to denote its restriction to $A$. Let $q : \mathbb{R} \to \mathbb{R}/\tau_*K_0(A)$ denote the quotient map. For an element $[u] \in \ker(\sigma_* - id) \subseteq K_1(A)$, where $u$ is in $U_n(A)$ (the $n \times n$ unitary matrices in $A$), consider the element of $\mathbb{R}/\tau_*K_0(A)$,
called the “determinant” of \([u]\), given by
\[
\Delta[u] = q \left( \frac{1}{2\pi i} \int_a^b (\tau \otimes \text{Tr})(\xi(t)\xi(t)^{-1}) dt \right)
\]
where \(\xi : [a, b] \to \mathbb{U}_n(A)\) is a piecewise continuously differentiable path such that \(\xi(a) = 1\) and \(\xi(b) = \sigma(u)u^{-1}\). Pimsner’s result \([12]\) (Theorem 3) is that the following is a short exact sequence:

\[
0 \longrightarrow \tau_*K_0(A) \overset{\iota}{\longrightarrow} \tau_*K_0(A \rtimes_{\sigma} \mathbb{Z}) \overset{q}{\longrightarrow} \Delta(\ker(\sigma_* - \text{id}_*) \longrightarrow 0
\]
where \(\iota\) is the canonical inclusion (as subgroups of \(\mathbb{R}\)) and \(q\) is the restriction of the canonical map \(q\).

**2. The C*-algebra \(A_{\theta}^{5,k}\) for \(k = 1, 2, 3, 4\).**

2.1. The C*-algebra \(A_{\theta}^{5,1}\). Let us first look at the C*-algebra \(A_{\theta}^{5,1}\) generated by unitaries \(U, V, W, X\) satisfying

\[
\begin{align*}
UV &= \lambda VU, \quad WX = \lambda WX, \quad UW = WU, \\
UX &= XU, \quad VW = WV, \quad VX = XV,
\end{align*}
\]

where \(\lambda = e(\theta)\) and \(\theta\) is irrational (as in \([9]\), Section 1). It is clear that it is isomorphic to the simple C*-algebra \(A_{\theta} \otimes A_{\theta}\). We prefer to view it, however, as the crossed product \((A_{\theta} \otimes C(\mathbb{T})) \rtimes_{\sigma} \mathbb{Z}\) where \(A_{\theta}\) is generated by \(U, V, C(\mathbb{T})\) by \(W\), and \(\sigma = \text{Ad}_X\). So \(\sigma\) fixes \(U, V\) and \(\sigma(W) = \lambda W\). Since \(\sigma\) is homotopic to the identity automorphism (in the sense of \([1, 5.2.2]\), the Pimsner-Voiculescu sequence yields that \(K_j(A_{\theta}^{5,1})\) is isomorphic to \(K_j(A_{\theta} \otimes C(\mathbb{T}^2))\) \((j = 0, 1)\), which is isomorphic to \(\mathbb{Z}^8\). (One can also use the Küneth Theorem \([16]\) to get \(K_j(A_{\theta} \otimes A_{\theta}) = \mathbb{Z}^8\).) From Pimsner’s range of trace formula, one needs to know the generators of the kernel of \(id_* - \sigma_*\) in \(K_1(A_{\theta} \otimes C(\mathbb{T}))\). But \(id_* - \sigma_* = 0\). It is easy to show that a basis for \(K_1(A_{\theta} \otimes C(\mathbb{T}))\) consists of the following set \([\{V, [U, [W, [\xi]]\}\] where \(\xi = (1 - e) \otimes 1 + e \otimes W\) and \(e\) is a Powers-Rieffel projection in \(A_{\theta}\) of trace \(\theta\). This follows from the short exact sequence

\[
0 \longrightarrow A_{\theta} \otimes C_0(\mathbb{T}) \overset{i}{\longrightarrow} A_{\theta} \otimes C(\mathbb{T}) \overset{\epsilon}{\longrightarrow} A_{\theta} \longrightarrow 0
\]

where \(C_0(\mathbb{T})\) is the ideal of functions in \(C(\mathbb{T})\) vanishing at 1, \(\epsilon\) is evaluation at 1, and \(i\) is inclusion. Using the Bott periodicity isomorphism \(s^0 : K_0(A_{\theta}) \to K_1(A_{\theta} \otimes C_0(\mathbb{T}))\) (as given by Connes \([4]\)) one has \(s^0(e) = [1 \otimes 1 + e \otimes (W - 1)] = [\xi]\), giving us the fourth basis element. For the range of trace, and since we already know that \(\tau_*(K_0(A_{\theta})) = \mathbb{Z} + \mathbb{Z}\theta\), we need to compute the “determinant” of each basis element. From \(U, V, W\) we get determinants already in \(\mathbb{Z} + \mathbb{Z}\theta\), since \(\sigma(V)V^* = \sigma(U)U^* = 1\) and \(\sigma(W)W^* = \lambda = e(\theta)\). For \(\xi\) one has

\[
\sigma(\xi)\xi^* = ((1 - e) \otimes 1 + \lambda e \otimes W) \cdot ((1 - e) \otimes 1 + e \otimes W^*) = ((1 - e) + \lambda e) \otimes 1.
\]
A path of unitaries connecting this element to the identity is simply $\eta_t = ((1 - e) + e(t\theta)e) \otimes 1$ for $0 \leq t \leq 1$. Thus
\[
\frac{1}{2\pi i} \int_0^1 \tau(\eta_t\eta_t^*) \, dt = \frac{1}{2\pi i} \int_0^1 2\pi i \theta(e) \, dt = \theta^2
\]
since $\tau(e) = \theta$. From Pimsner’s trace formula one therefore obtains
\[
\tau_*(K_0(A_{\theta}^{5,1})) = \mathbb{Z} + \mathbb{Z}\theta + \mathbb{Z}\theta^2.
\]
One now has the isomorphism classification for the algebras $A_{\theta}^{5,1}$ for non-quartic irrationals $\theta$ (i.e., those that are not zeros of an integral polynomial of degree at most four). Therefore, for two such non-quartic irrationals $\theta, \theta'$, the algebras $A_{\theta}^{5,1}$ and $A_{\theta'}^{5,1}$ are isomorphic if and only if $\theta' = n \pm \theta$ for some integer $n$.

**The C*-algebra $A_{\theta,\phi}^{5,2}$.** The C*-algebra $A_{\theta,\phi}^{5,2}$ is generated by unitaries $U, V, W$ satisfying
\[
UV = \lambda VU, \quad UW = \mu WU, \quad VW = WV,
\]
where $\mu = e(\phi)$ and $\lambda = e(\theta)$ are assumed to be independent elements of the abelian group $\mathbb{T}$, so that in fact the algebra is simple. (See [9], Section 2.) This algebra can be realized as the crossed product $C(\mathbb{T}^2) \rtimes_\gamma \mathbb{Z}$ where $C(\mathbb{T}^2)$ is generated by $V, W$ and $\gamma(V) = \lambda V$, $\gamma(W) = \mu W$. Since this automorphism is homotopic to the identity, the Pimsner-Voiculescu sequence gives $K_j(A_{\theta,\phi}^{5,2}) = \mathbb{Z}^4$ since $K_j(C(\mathbb{T}^2)) = \mathbb{Z}^2$, for $j = 0, 1$. Since $K_1(C(\mathbb{T}^2))$ has basis $[V], [W]$ and since $\gamma(V)V^* = e(\theta)$ and $\gamma(W)W^* = e(\phi)$, one easily obtains the range of trace as
\[
\tau_*(K_0(A_{\theta,\phi}^{5,2})) = \mathbb{Z} + \mathbb{Z}\theta + \mathbb{Z}\phi.
\]
(This does in fact hold also for rational $\theta, \phi$, but one must use the canonical trace on the crossed product.) The classification for the algebras $A_{\theta,\phi}^{5,2}$ now follows:

**Proposition.** For independent irrationals $\theta, \phi$ the C*-algebras $A_{\theta,\phi}^{5,2}$ and $A_{\theta',\phi'}^{5,2}$ are isomorphic if, and only if there exists $X \in \text{GL}(2, \mathbb{Z})$ such that $[\theta' \phi'] = [\theta \phi]X$.

**Proof.** Given $X = [\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}]$ in $\text{GL}(2, \mathbb{Z})$ the substitutions $V' = V^a W^b$, $W' = V^c W^d$ satisfy the relations
\[
UV' = \lambda^a \mu^b V' U, \quad UW' = \lambda^c \mu^d W' U, \quad V' W' = W' V',
\]
so that $U, V', W'$, which already generate $A_{\theta,\phi}^{5,2}$, also generate $A_{\theta' + b\phi, \phi'}^{5,2}$, hence these algebras are isomorphic. Conversely, if $A_{\theta,\phi}^{5,2}$ and $A_{\theta',\phi'}^{5,2}$ are isomorphic then by the above one has $\mathbb{Z} + \mathbb{Z}\theta + \mathbb{Z}\phi = \mathbb{Z} + \mathbb{Z}\theta' + \mathbb{Z}\phi'$. Writing
each of \( \theta', \phi' \) in terms of \( \theta, \phi \) (modulo \( \mathbb{Z} \)) and vice versa, and using their rational independence, one easily obtains a matrix \( X \) in \( \text{GL}(2, \mathbb{Z}) \) such that \( [\theta' \phi'] = [\theta \phi]X \).

\[
\text{\textbf{The C*-algebra} } A^{5,3}_\theta. \text{ The C*-algebra } A^{5,3}_\theta \text{ is generated by unitaries } U, V, W, X \text{ satisfying }
\begin{align*}
UV &= XVU, \quad UX = \lambda XU, \quad VX = XV, \\
VW &= \lambda WV, \quad UW = WU, \quad WX = WX.
\end{align*}
\]

where \( \lambda = e(\theta) \) and \( \theta \) is irrational. (See [9], Section 3.) One can view this algebra as the crossed product \( A^1_\theta \rtimes_{\nu} \mathbb{Z} \), where \( A^1_\theta \) is the Heisenberg C*-algebra generated by the unitaries \( U, V, X \) satisfying the three relations in the first line of (2.3), and \( \nu(X) = X, \, \nu(U) = U, \, \nu(V) = \lambda V \). Since this automorphism is also homotopic to the identity, and since \( K_j(A^1_\theta) = \mathbb{Z}^3 \), the Pimsner-Voiculescu exact sequence immediately gives \( K_j(A^{5,3}_\theta) = \mathbb{Z}^6 \) for \( j = 0, 1 \). To find the range of the trace on \( K_0 \) using Pimsner’s Theorem we will need to do the following. The Pimsner-Voiculescu exact sequence applied to \( A^4_\theta \), viewed as the crossed product \( A^1_\theta \rtimes_{\sigma} \mathbb{Z} \) (where \( A^1_\theta \) is generated by \( U, X \) and \( \sigma = \text{Ad}V \)) is

\[
\begin{array}{cccc}
& & K_0(A^1_\theta) & \\
& \overset{i_{\sigma}-\sigma_*}{\longrightarrow} & K_0(A^1_\theta) & \overset{i_*}{\longrightarrow} & K_0(A^1_\theta) \\
\delta_1 & & & \downarrow & \delta_0 \\
K_1(A^1_\theta) & \overset{i_*}{\longrightarrow} & K_1(A^1_\theta) & \overset{i_{\sigma}-\sigma_*}{\longrightarrow} & K_1(A^1_\theta).
\end{array}
\]

Recall from Lemma 1.2 of [13] that the group \( K_1(A^1_\theta \rtimes_{\sigma} \mathbb{Z}) \) is generated by classes of unitaries of the form

\[
(1 \otimes I_n - F) + Fx(V^{-1} \otimes I_n)F,
\]

where \( F \) is a projection in \( M_n(A) \) and \( x \in M_n(A) \). (Here, \( V \) is the canonical unitary of the crossed product: \( \sigma(a) = VaV^{-1} \).) In addition, from page 102 of [13], the connecting homomorphism \( \delta_1 : K_1(A \rtimes_{\sigma} \mathbb{Z}) \to K_0(A) \) is given on classes of such unitaries by

\[
(2.4) \quad \delta_1[(1 \otimes I_n - F) + Fx(V^{-1} \otimes I_n)F] = [F].
\]

\textbf{Lemma.} A basis for \( K_1(A^1_\theta) \) is \( \{[U], \, [V], \, [\xi]\} \) where

\[
\xi := (1 - e) + ew^{-1}V^{-1}e
\]

and \( e \in A_\theta = C^*(X, U) \) is a Powers-Rieffel projection of trace \( \theta \) and \( w \) is a unitary in \( A_\theta \) such that \( we^{-1}V^{-1}V = V^{-1}eV \).

\textbf{Proof.} From the above exact sequence we see that \( \delta_1 \) is surjective and thus \( K_1(A^1_\theta) \) contains elements that are mapped by \( \delta_1 \) to \([1]\) and \([e]\). Applying (2.4) with \( F = 1 \) one has \( \delta_1[V^{-1}] = [1] \). To find an element that \( \delta_1 \) maps to
[\varepsilon], note that $V^{-1}eV$ is a projection in $A_\theta$ whose trace is $\theta$, so by Rieffel’s Cancellation Theorem [15] there exists a unitary $w$ in $A_\theta$ such that $wew^{-1} = V^{-1}eV$. Now it is straightforward to see that $(1-e) + ew^{-1}V^{-1}e$ is a unitary in $A_\theta^1$ (with inverse $(1-e) + eVwe$). Therefore, one has

$$\delta_1[(1-e) + ew^{-1}V^{-1}e] = \varepsilon.$$  

Finally, on $K_1(A_\theta)$, $\text{id}_\ast - \alpha_\ast$ maps $[X]$ to zero and $[U]$ to $[X]$, hence $[U]$ is the third basis element. 

First, since $\nu(V)V^{-1} = \lambda = e(-\theta)$, Pimsner’s Theorem gives us $\theta$ in the range (which is already contained in the range of the trace on $K_0(A_\theta^1)$). Since $e$ is in $A_\theta = C^\ast(X,U)$, which is fixed by $\nu$, one obtains for $\xi$ (since $Vw$ commutes with $e$)

$$\nu(\xi)\xi^{-1} = ((1-e) + \lambda w^{-1}V^{-1}e) \cdot ((1-e) + Vwe) = (1-e) + \lambda e.$$  

This is exactly the same situation we had for the algebra $A^{5,1}_\theta$ which yielded $\theta^2$ in the trace range. Therefore one concludes in the same manner that $\tau_*([K_0(A^{5,3}_\theta)]) = \mathbb{Z} + \mathbb{Z}\theta + \mathbb{Z}\theta^2$ which also yields the same type of classification statement as for the case of the algebra $A^{5,1}_\theta$ above (namely, for non-quartic irrationals $\theta$).

**The C*-algebra $A^{5,4}_{\theta,\phi}$.** The C*-algebra $A^{5,4}_{\theta,\phi}$ is generated by unitaries $U, V, W$ satisfying

$$WV = UVW, \quad WU = \lambda UW, \quad VU = \mu UV,$$

where $\mu = e(\phi)$ and $\lambda = e(\theta)$ are assumed to be independent. This algebra is Packer’s Heisenberg C*-algebra of class 3 [11]. As shown in [9], Section 4, this algebra is simple with a unique trace state. As in [9], we can view $A^{5,4}_{\theta,\phi}$ as the crossed product $A_\phi \rtimes_{\sigma} \mathbb{Z}$, where $A_\phi$ is generated by $U, V$, and $\sigma$ is the “Anzai” automorphism $\sigma(U) = \lambda U, \quad \sigma(V) = UV$. Since $\sigma_\ast$ induces the identity map on $K_0(A_\phi)$, and since on $K_1(A_\phi) = \mathbb{Z}[U] + \mathbb{Z}[V]$ one has $(id_\ast - \alpha_\ast)[U] = 0, \quad (id_\ast - \alpha_\ast)[V] = -[U]$, the Pimsner-Voiculescu exact sequence gives $K_0(A^{5,4}_{\theta,\phi}) = K_1(A^{5,4}_{\theta,\phi}) = \mathbb{Z}^3$. Now Pimsner’s machine states that the range of trace is obtained from that of $\tau_*K_0(A_\phi) = \mathbb{Z} + \mathbb{Z}\phi$ and from the class $[U]$. But $\sigma(U)U^* = \lambda = e(\theta)$ so that one has

$$\tau_*K_0(A^{5,4}_{\theta,\phi}) = \mathbb{Z} + \mathbb{Z}\phi + \mathbb{Z}\theta.$$  

The classification for underlying algebras is the same as for the algebras $A^{5,2}_{\theta,\phi}$ above.

**Proposition.** For independent irrationals $\theta, \phi$ the C*-algebras $A^{5,4}_{\theta,\phi}$ and $A^{5,4}_{\theta',\phi'}$ are isomorphic if, and only if there exists $X \in \text{GL}(2, \mathbb{Z})$ such that $[\theta', \phi'] = [\theta, \phi]X$.  

Proof. Since the group $GL(2,\mathbb{Z})$ is generated by the matrices $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, it suffices to check that $A_{\phi,\theta}^{5,4}$ and $A_{\theta,\phi}^{5,4}$ are isomorphic to $A_{\theta,\phi}^{5,4}$. To get the algebra $A_{\phi,\theta}^{5,4}$, one lets $W'=V^*$, $V'=W^*$, $U' = \mu \lambda U^*$ so that the universal relations (2.5) are checked for $W', V', U'$ in place of $W, V, U$, respectively, and with $\mu$ and $\lambda$ switched. To get the algebra $A_{\theta,\phi}^{5,4}$, in the same manner one lets $W'=W$, $V'=WV$, $U' = \lambda U$ which satisfy (2.5) with $\lambda$ remaining the same and $\mu$ replaced by $\lambda \mu$. Conversely, if $A_{\phi,\theta}^{5,4}$ and $A_{\phi',\theta'}^{5,4}$ are isomorphic then exactly as in the proof of previous proposition one shows that there is a matrix $X$ in $GL(2,\mathbb{Z})$ such that $\begin{bmatrix} \theta' \\ \phi' \end{bmatrix} = \begin{bmatrix} \theta \\ \phi \end{bmatrix} X$. $\square$

3. The C*-algebra $A_{\theta}^{5,5}$.

Let us view the commutative 3-torus $C(T^3)$ as generated by its three canonical unitaries $X, W, V$, where $X(r, s, t) = e(r)$, $W(r, s, t) = e(s)$, $V(r, s, t) = e(t)$. The C*-algebra $A_{\theta}^{5,5}$ can be viewed as the crossed product $C(T^3)\rtimes_{\sigma} \mathbb{Z}$ where

\begin{equation}
\sigma(X) = \lambda X, \quad \sigma(W) = XW, \quad \sigma(V) = WV.
\end{equation}

(3.1)

(where $\lambda = e(\theta)$) which was introduced in [9] (Section 5). When $\theta$ is irrational, $A_{\theta}^{5,5}$ is the unique C*-algebra generated by unitaries $X, W, V, U$ satisfying the relations

\begin{align*}
UV &= WVU, \\
UW &= XWU, \\
UX &= \lambda XU, \\
VW &= WV, \\
VX &= XV, \\
WX &= WX.
\end{align*}

(3.2)

We shall prove the following result.

**Theorem 3.1.** For any $\theta$ (rational or irrational) one has

\[ K_0(A_{\theta}^{5,5}) = \mathbb{Z}^4, \quad K_1(A_{\theta}^{5,5}) = \mathbb{Z}^4. \]

If $\theta$ is irrational, and if $\tau$ is the unique trace state on $A_{\theta}^{5,5}$, then $\tau K_0(A_{\theta}^{5,5}) = \mathbb{Z} + \mathbb{Z} \theta$. (This yields the usual isomorphism classification for irrational $\theta$ upon noting that $A_{\theta}^{5,5} \cong A_{1-\theta}^{5,5}$)

The Pimsner-Voiculescu exact sequence corresponding to the above crossed product is

\[ K_0(C(T^3)) \xrightarrow{id_* - \sigma_*} K_0(C(T^3)) \xrightarrow{i_*} K_0(A_{\theta}^{5,5}) \]

(3.3)

\[ \delta_1 \leftarrow K_1(A_{\theta}^{5,5}) \xrightarrow{i_*} K_1(C(T^3)) \xrightarrow{id_* - \sigma_*} K_1(C(T^3)) \]

and our goal is to compute $id_* - \sigma_*$ at the $K_0$ and $K_1$ levels.
**The Connes Chern character on** \( K_* (C(T^3)) \). First we need to find concrete bases for the K-groups of \( C(T^3) \). It is already known that \( K_0 (C(T^3)) = \mathbb{Z}^4 \) and \( K_1 (C(T^3)) = \mathbb{Z}^4 \). Let \( B \) denote the Bott projection in \( M_2 (C(T^2)) \) given by

\[
B = \begin{bmatrix} 1 - f & g \\ g & f \end{bmatrix}
\]

where \( f, g \in C(T^2) \) are smooth functions satisfying

\[
(\phi \# \text{Tr})(B, B, B) = -6 \phi(f, g, g) = -\frac{6}{2\pi i} \int \int f [g_x g_y - g_y g_x] \, dx \, dy = 1,
\]

where \( g_x := \frac{\partial g}{\partial x} \), which is just the Connes pairing of \([B]\) with \([\phi]\) (this number being often called the ‘twist’ of \(B\) in the C*-literature), where \(\phi\) is the fundamental cyclic cocycle on \(T^2\):

\[
\phi(f^0, f^1, f^2) = \frac{1}{2\pi i} \int \int f^0 [f^1_j f^2_i - f^1_i f^2_j] \, dx_1 \, dx_2 \, dx_3.
\]

For \( 1 \leq i < j \leq 3 \), let \( P_{ij} \) denote the Bott projection in \( M_2 (C(T^3)) \) in the variables \( i, j \). More specifically,

\[
P_{12}(r, s, t) = B(r, s), \quad P_{13}(r, s, t) = B(r, t), \quad P_{23}(r, s, t) = B(s, t).
\]

Putting \( b_{ij} = [P_{ij}] - [1] \) (the Bott elements), it is not hard to check that \( \{[1], b_{12}, b_{13}, b_{23}\} \) is a basis for \( K_0 (C(T^3)) \). Now the (numerical) Connes Chern character \( ch_0 \) is the homomorphism

\[
ch_0 : K_0 (C(T^3)) \rightarrow \mathbb{Z}^4
\]

given by

\[
ch_0(x) = (\tau(x), (x, \phi_{12}), (x, \phi_{13}), (x, \phi_{23}))
\]

where

\[
\phi_{ij}(f^0, f^1, f^2) = \frac{1}{2\pi i} \int \int \int_{T^3} f^0 [f^1_j f^2_i - f^1_i f^2_j] \, dx_1 \, dx_2 \, dx_3
\]

is a cyclic 2-cocycle on \(C(T^3)\) and \( f_k := \partial f / \partial x_k \). (Henceforth, all triple integrals are over the 3-torus.) From (3.4) one gets

\[
\langle [P_{ij}], [\phi_{kl}] \rangle = \delta_{i,k} \delta_{j,l}
\]

which gives

\[
ch_0[1] = (1, 0, 0, 0), \quad ch_0[b_{12}] = (0, 1, 0, 0),
\]

\[
ch_0[b_{13}] = (0, 0, 1, 0), \quad ch_0[b_{23}] = (0, 0, 0, 1),
\]

so that \( ch_0 \) is injective on \( K_0 (C(T^3)) \).
Lemma 3.2. One has the following action of $\sigma_*$ on $K_0(C(T^3))$: 

$$
\sigma_*(1) = [1], \quad \sigma_*(b_{12}) = b_{12}, \\
\sigma_*(b_{13}) = b_{12} + b_{13}, \quad \sigma_*(b_{23}) = b_{12} + b_{13} + b_{23}.
$$

Proof. For simplicity consider the change of variables $(u, v, w) = (r + \theta, r + s, s + t)$, and note that by the chain rule one has 

$$
\frac{\partial}{\partial r} h(u, v, w) = h_1(u, v, w) + h_2(u, v, w), \\
\frac{\partial}{\partial s} h(u, v, w) = h_2(u, v, w) + h_3(u, v, w), \\
\frac{\partial}{\partial t} h(u, v, w) = h_3(u, v, w),
$$

which can be simplified by writing 

$$
\frac{\partial}{\partial x_i} h(u, v, w) = h_i(u, v, w) + h_{i+1}(u, v, w)
$$

where $h_4 = 0$, and $x_1 = r, x_2 = s, x_3 = t$. From this one gets 

$$
\frac{\partial}{\partial x_i} g(u, v, w) \frac{\partial}{\partial x_j} \bar{g}(u, v, w) = (g_i + g_{i+1})(\bar{g}_j + \bar{g}_{j+1})(u, v, w)
$$

and 

$$
\frac{\partial}{\partial x_i} g(u, v, w) \frac{\partial}{\partial x_j} \bar{g}(u, v, w) - \frac{\partial}{\partial x_j} g(u, v, w) \frac{\partial}{\partial x_i} \bar{g}(u, v, w)
$$

$$
= [(g_i + g_{i+1})(\bar{g}_j + \bar{g}_{j+1}) - (g_j + g_{j+1})(\bar{g}_i + \bar{g}_{i+1})](u, v, w).
$$

Now if we write 

$$
P_{ij} = \begin{bmatrix} 1 - f & g \\ \bar{g} & f \end{bmatrix}
$$

where $f, g$ depend only on the $i, j$ coordinates ($i < j$), then 

$$
\sigma(P_{ij}) = \begin{bmatrix} 1 - f(u, v, w) & g(u, v, w) \\ \bar{g}(u, v, w) & f(u, v, w) \end{bmatrix}
$$

for which one has 

$$
\langle [\sigma(P_{ij})], [\phi_{kl}] \rangle = (\phi_{kl} \# \text{Tr})(\sigma(P_{ij}), \sigma(P_{ij}), \sigma(P_{ij}), \sigma(P_{ij}))
$$

$$
= -6 \phi_{kl}(f(u, v, w), g(u, v, w), \bar{g}(u, v, w))
$$

$$
= -\frac{6}{2\pi i} \iiint f(u, v, w) \left[ \frac{\partial}{\partial x_k} g(u, v, w) \frac{\partial}{\partial x_l} \bar{g}(u, v, w) - \frac{\partial}{\partial x_l} g(u, v, w) \frac{\partial}{\partial x_k} \bar{g}(u, v, w) \right] drdsdt
$$

$$
= -\frac{6}{2\pi i} \iiint f(u, v, w) \left[ (g_k + g_{k+1})(\bar{g}_l + \bar{g}_{l+1}) - (g_l + g_{l+1})(\bar{g}_k + \bar{g}_{k+1}) \right](u, v, w) drdsdt.
$$
Now since the transformation \((u, v, w) = (r + \theta, r + s, s + t)\) has Jacobian
determinant 1, the change of variables formula gives
\[
\langle [\sigma(P_{ij})], [\phi_{k\ell}] \rangle = -\frac{6}{2\pi i} \iiint f(r, s, t) \cdot \left( (g_k + g_{k+1})(\bar{g}_\ell + \bar{g}_{\ell+1}) \\
- (g_\ell + g_{\ell+1})(\bar{g}_k + \bar{g}_{k+1}) \right) (r, s, t) \, drdsdt
\]
which yields
\[
\langle [\sigma(P_{12})], [\phi_{12}] \rangle = 1, \quad \langle [\sigma(P_{13})], [\phi_{13}] \rangle = 0, \quad \langle [\sigma(P_{13})], [\phi_{23}] \rangle = 0,
\]
\[
\langle [\sigma(P_{23})], [\phi_{23}] \rangle = 1, \quad \langle [\sigma(P_{23})], [\phi_{13}] \rangle = 1, \quad \langle [\sigma(P_{23})], [\phi_{12}] \rangle = 1,
\]
and the injectivity of \(\text{ch}_0\) thus yields the following equalities in \(K_0(C(\mathbb{T}^3))\)
\[
\sigma_*(b_{12}) = b_{12}, \quad \sigma_*(b_{13}) = b_{12} + b_{13}, \quad \sigma_*(b_{23}) = b_{12} + b_{13} + b_{23}.
\]
These give the desired result.

We now turn our attention to \(K_1\).

**Lemma 3.3.** A basis for \(K_1(C(\mathbb{T}^3))\) is \(\{[X], [W], [V], [\xi]\}\), where \(\xi = I_2 + (V - 1) \otimes P_{12}\) is a unitary in \(M_2(C(\mathbb{T}^3))\) and \(P_{12}\) is the Bott projection in the variables \(X, W\).

**Proof.** This immediately follows from the Künneth Theorem applied to the tensor product expansion of \(K_1(C(\mathbb{T}^3)) = K_1(C(\mathbb{T}^2) \otimes C(\mathbb{T}))\) and using the individual generators of each factor.

**Lemma 3.4.** The action of \(\sigma\) on \(K_1(C(\mathbb{T}^3))\) is given by
\[
\sigma_*[X] = [X], \quad \sigma_*[W] = [X] + [W], \quad \sigma_*[V] = [W] + [V], \quad \sigma_*[\xi] = [\xi] + [W].
\]

**Proof.** The only nontrivial part is to show \(\sigma_*[\xi] = [\xi] + [W]\) (the rest follow trivially from the definition of \(\sigma\)). From Lemma 3.2 one has \([\sigma(P_{12})] = [P_{12}]\), and since \(C(\mathbb{T}^2)\) has the cancellation property, there is a unitary \(R\) in \(M_2(C(\mathbb{T}^3))\) (which depends only on the first two variables) such that
\[ \sigma(P_{12}) = RP_{12}R^*. \] Hence
\[ \sigma_*[\xi] = [I_2 + (WV - 1) \otimes \sigma(P_{12})] = [I_2 + (WV - 1) \otimes RP_{12}R^*] = [R(I_2 + (WV - 1) \otimes P_{12})R^*] = [I_2 + (WV - 1) \otimes P_{12}] = [I_2 + (W - 1) \otimes P_{12}] + [I_2 + (V - 1) \otimes P_{12}] = [\xi + [I_2 + (W - 1) \otimes P_{12}]] \]

and now we claim that \([I_2 + (W - 1) \otimes P_{12}] = [W].\) It is enough to show that this equality holds in \(K_1(C(T^2))\) (since all concerned variables here are the first two – involving \(X,W\)). This is shown in the following remark. □

**Remark.** Let us view the 2-torus \(T^2\) as \(T \times [0,1]\) with the endpoints of the interval identified. Recall that the Bott projection in \(M_2(C(T^2))\) can be given by \(P(x,s) = M(x,s)e_0M(x,s)^*\) for \(0 \leq s \leq 1,\) where
\[ M(x,s) = E^s \begin{bmatrix} \pi & 0 \\ 0 & 1 \end{bmatrix} E^{-s}, \]
\[ E^s = \begin{bmatrix} \cos(\pi s/2) & -\sin(\pi s/2) \\ \sin(\pi s/2) & \cos(\pi s/2) \end{bmatrix}, \quad e_0 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \]

where \(M : T \times [0,1] \to U_2(C)\) is smooth (but clearly \(M\) is not in \(M_2(C(T^2))\)), and \(E^s\) satisfies the usual exponential property. In our case above, the unitary \(V\) corresponds to \(x\) and \(W(x,s) = e(s).\) The unitary \(\eta := I_2 + (W - 1) \otimes P\) can now be written as
\[ \eta = I_2 + (W - 1) \otimes M e_0 M^* = M(I_2 + (W - 1) \otimes e_0)M^* = M \begin{bmatrix} W & 0 \\ 0 & 1 \end{bmatrix} M^*. \]

Now an explicit path of unitaries \(t \mapsto \eta_t\) in \(M_2(C(T^2))\) connecting \(\eta\) to \(\begin{bmatrix} W & 0 \\ 0 & 1 \end{bmatrix}\) can be given by
\[ (3.5) \quad \eta_t(x,s) = M(x,ts) \begin{bmatrix} e(s) & 0 \\ 0 & 1 \end{bmatrix} M(x,ts)^*. \]

(For each \(t\) one has \(\eta_t(x,0) = \eta_t(x,1) = I_2\) so that \(\eta_t \in M_2(C(T^2)).\) It follows, in particular, that \(\eta\) and \(W\) give the same class in \(K_1(C(T^2))\)). The explicit form of the unitary path (3.5) is used in the trace computation below.

In view of Lemmas 3.2 and 3.4 one obtains, from the Pimsner-Voiculescu exact sequence (3.3), the \(K_0\) and \(K_1\) groups of \(A_g^{5,5}\) as stated in Theorem 3.1.
Tracial Range on $K_0(A^{5,5}_\theta)$. To complete the proof of Theorem 3.1 we now use Pimsner’s Theorem. In the present case, the quotient map is $q : \mathbb{R} \to \mathbb{R}/\tau_{k}(K_0(C(\mathbb{T}^3))) = \mathbb{R}/\mathbb{Z}$, since the range of the canonical trace state $\tau$ on $K_0(C(\mathbb{T}^3))$ is $\mathbb{Z}$. From Lemma 3.4 the kernel of $id_{\mathcal{A}} - \sigma_{\mathcal{A}}$ in $K_1(C(\mathbb{T}^3))$ is generated by the classes $[X]$ and $[\xi] - [V]$. For $[X]$, since $\sigma(X) = \lambda X$, one clearly has $\Delta[X] = q(\theta)$. For $[\xi] - [V]$, it suffices to show that $\Delta([\xi] - [V]) = 0$, and this will complete the proof that $\tau_{k}K_0(A^{5,5}_\theta) = \mathbb{Z} + \mathbb{Z}\theta$. As in the proof of Lemma 3.4, we noted that

$$\sigma(\xi) = R\xi(I + (W - 1) \otimes P_{12})R^* = R\xi MW_1 M^* R^*$$

where $R$ is a unitary in $M_2(C(\mathbb{T}^2))$ such that $\sigma(P_{12}) = RP_{12}R^*$, and $I + (W - 1) \otimes P_{12} = MW_1 M^*$ (in the notation of the above remark). Writing $V_1 = [\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}]$ and similarly for $W_1$, we have $[\xi] - [V] = [\xi V_1^*]$. Thus one has

$$\sigma \left( \left[ \begin{array}{cc} \xi V_1^* & I \\ \end{array} \right] \right)^{-1} \sigma \left( \left[ \begin{array}{cc} \xi V_1^* & I \\ \end{array} \right] \right) = \left[ \begin{array}{cc} \sigma(\xi) & I \\ \end{array} \right] \left[ \begin{array}{cc} V_1^* W_1^* & I \\ \end{array} \right] \left[ \begin{array}{cc} V_1 & I \\ \end{array} \right] \left[ \begin{array}{cc} \xi^* & I \\ \end{array} \right] = \left[ \begin{array}{cc} R & R^* \\ \end{array} \right] \left[ \begin{array}{cc} \xi & I \\ \end{array} \right] \left[ \begin{array}{cc} MW_1 M^* & I \\ \end{array} \right] \left[ \begin{array}{cc} R^* & R \\ \end{array} \right] \left[ \begin{array}{cc} W_1^* & I \\ \end{array} \right] \left[ \begin{array}{cc} \xi^* & I \\ \end{array} \right] .$$

Letting $t \mapsto \mathcal{R}_t$ be the standard unitary path such that $\mathcal{R}_0 = I$ and $\mathcal{R}_1 = \left[ \begin{array}{cc} R & R^* \end{array} \right]$ one considers the following path of unitaries in $M_4(C(\mathbb{T}^2))$

$$\gamma_t = \mathcal{R}_t \left[ \begin{array}{cc} \xi & I \\ \end{array} \right] \left[ \begin{array}{cc} \eta_t & I \\ \end{array} \right] \mathcal{R}_t^* \left[ \begin{array}{cc} W_1^* & I \\ \end{array} \right] \left[ \begin{array}{cc} \xi^* & I \\ \end{array} \right]$$

where $\eta_t$ is the path defined by (3.5) such that $\eta_0 = W_1$, $\eta_1 = MW_1 M^*$. The path $\gamma_t$ clearly connects the above element to the identity. Now it is straightforward to see that $(\mathbf{r} \otimes \text{Tr}_4) (\gamma_t \gamma_t^*) = (\mathbf{r} \otimes \text{Tr}_2) (\eta_t \eta_t^*)$, since the fact that both $\mathcal{R}_t$ and $\mathcal{R}_t^*$ appear in $\gamma_t$ leads to their cancellation under the trace. Since $\eta_t$ has the form (3.5), one similarly obtains $(\mathbf{r} \otimes \text{Tr}_2) (\eta_t \eta_t^*) = 0$. Therefore, $\Delta([\xi V_1^*]) = 0$ which completes the proof of Theorem 3.1.

4. The $C^*$-algebra $A^{5,6}_\theta$.

The $C^*$-algebra $A^{5,6}_\theta$ can be characterized as the unique $C^*$-algebra (when $\theta$ is irrational) generated by unitaries $U, V, W, Z$ such that

$$ZV = \lambda V Z, \quad ZU = V^{-1} U Z, \quad ZW = W Z, \quad UV = W V U, \quad UW = \lambda W U, \quad VW = W V .$$

(As in [9].) It will be convenient to present $A^{5,6}_\theta$ as the crossed product $(C(\mathbb{T}) \otimes A_\theta) \rtimes_{\gamma} \mathbb{Z}$, where $C(\mathbb{T})$ is generated by $W$, $A_\theta$ is generated by $V, Z$,.
and ν = Ad_U := U( )U* is the automorphism given by
\[ \nu(W) = \lambda W, \quad \nu(Z) = VZ, \quad \nu(V) = W \otimes V = WV. \]

The aim of this section is to prove the following.

**Theorem 4.1.** For any θ (rational or irrational) one has
\[ K_0(A_{θ}^{5,6}) = \mathbb{Z}^4, \quad K_1(A_{θ}^{5,6}) = \mathbb{Z}^4. \]

If θ is irrational, and if τ is the unique trace state on A_{θ}^{5,6}, then \( τ_{*}K_0(A_{θ}^{5,6}) = \mathbb{Z} + \mathbb{Z}θ + \mathbb{Z}θ^2 \). (This yields the isomorphism classification for non-quartic irrationals θ upon noting that \( A_{θ}^{5,6} \cong A_{1-θ}^{5,6} \)).

Since \( ZV = \lambda VZ \), let \( p = V^*g + f + gV \) be a Powers-Rieffel projection of trace θ, where \( f = f(Z) \), \( g = g(Z) \) are \( C^\infty \) real functions of \( Z \) satisfying the usual properties that would make \( p \) a projection. Among these, to be used below, are (after interpreting \( f, g \) as functions of period 1 on \( \mathbb{R} \))
\begin{equation}
(4.2) \quad f(t + \theta) = 1 - f(t) \text{ for } 0 \leq t \leq 1 - \theta,
\end{equation}
\[ g(t) - f(t)g(t) = g(t)f(t + \theta) \text{ for all } t, \]
\[ \int_{0}^{1} f(g^2)^* = -\int_{0}^{1} g^2 fdt = \frac{1}{6}, \]
where we may assume, with no loss of generality, that \( \frac{1}{2} < \theta < 1 \), and where the dot indicates usual differentiation of a real function. Recall that \( g \) is supported on \([0,1]\) on which it is given by \((f - f^2)^{1/2}\) and \( f = 1 \) on \([1 - \theta, \theta]\).

**Lemma 4.2.** One has the following equalities in \( K_0(C(\mathbb{T}) \otimes A_θ) \) for irrational θ:
\[ \nu_*(p) = [p] + ([P_{W,V}] - [1]) + ([P_{W,Z}] - [1]) \]
\[ \nu_*[P_{W,Z}] = [P_{W,Z}] + [P_{W,V}] - [1]. \]

**Proof.** We first compute the Connes Chern character for the algebra \( C(\mathbb{T}) \otimes A_θ \). Consider the canonical cyclic 1-cocycles \( ϕ_0 \) and \( ϕ_j, j = 1, 2, \) of \( C(\mathbb{T}) \) and \( A_θ \), respectively, given by
\[ ϕ_0(f^0, f^1) = \frac{1}{2\pi i} \int_{0}^{1} \frac{df^0}{dt}(f^1)dt, \]
\[ ϕ_1(x^0, x^1) = \frac{1}{2\pi i} τ(x^0δ_V(x^1)), \quad ϕ_2(x^0, x^1) = \frac{1}{2\pi i} τ(x^0δ_Z(x^1)) \]
where \( δ_V, δ_Z \) are the canonical derivations of \( A_θ \), and \( τ \) is the canonical trace. Let \( ρ \) denote Connes’ canonical cyclic 2-cocycle of \( A_θ \)
\[ ρ(x^0, x^1, x^2) = \frac{1}{2\pi i} τ(x^0[δ_V(x^1)δ_Z(x^2) - δ_Z(x^1)δ_V(x^2)]). \]
The Connes Chern character now takes the form of the group homomorphism
\[ ch_0 : K_0(C(\mathbb{T}) \otimes A_θ) \rightarrow (\mathbb{Z} + \mathbb{Z}θ) \oplus \mathbb{Z}^3 \]
given by taking the Connes pairings with the various cup products as

\[ \chi_0(x) := (\tau(x), \langle x, \varphi_0 \# \varphi_1 \rangle, \langle x, \varphi_0 \# \varphi_2 \rangle, \langle x, \tau_0 \# \rho \rangle) \]

where \( \tau_0 \) is the canonical trace of \( C(T) \). It is straightforward to check that it assumes the following values on the basis for \( K_0(C(T) \otimes A_\theta) \) given by the classes \{[1], [1 \otimes p], [P_{W,V}], [P_{W,Z}]\}:

\[
\begin{align*}
\chi_0[1] &= (1, 0, 0, 0) \\
\chi_0[1 \otimes p] &= (\theta, 0, 0, 1) \\
\chi_0[P_{W,V}] &= (1, 1, 0, 0) \\
\chi_0[P_{W,Z}] &= (1, 0, 1, 0).
\end{align*}
\]

(This follows immediately from the multiplicative property of Connes’ canonical pairing with respect to tensor products of algebras, see [3, III.3].) It is immediate that \( \chi_0 \) is injective on \( K_0 \) (for any \( \theta \)).

It is clear that \( \tau(\nu(p)) = \theta \). So to compute \( \chi_0(\nu(p)) \), we will have to calculate the above three 2-cocycles on \( \nu(p) \). First, we show that \( \langle \nu(p) \rangle, \tau_0 \# \rho \rangle = 1 \). Since \( p = V^*g + f + gV \), where \( f = f(Z), g = g(Z) \), one has \( \nu(p) = V^*W^*G + F + GWV \), where \( F = \nu(f) = f(VZ), G = \nu(g) = g(VZ) \), and hence

\[
(\tau_0 \# \rho)(\nu(p), \nu(p)) \equiv (\tau_0 \# \rho)(W^*V^*G + F + GWV, W^*V^*G + F + GWV)
= \rho(F, F, F) + 3 \rho(F, GV, V^*G) + 3 \rho(F, V^*G, GV).
\]

(In the expansion, the only possibly nonzero terms are ones of the form \((\tau_0 \# \rho)(W^a \ldots, W^b \ldots, W^c \ldots)\) where \( a + b + c = 0 \).) First, it is easy to verify that

\[
\nu^{-1} \delta_Z \nu = \delta_Z, \quad \nu^{-1} \delta_V \nu = \delta_V + \delta_Z.
\]

We thus see that \( \rho(F, F, F) = \rho(f, f, f) = 0 \) since \( \delta_V(f) = 0 \). Next, we have

\[
2\pi i \rho(F, GV, V^*G)
= \tau(F[\delta_V(GV)\delta_Z(V^*G) - \delta_Z(GV)\delta_V(V^*G)])
= \tau(F[(\delta_V(G) + 2\pi iGV)\delta_Z(G) - \delta_Z(G)\delta_V(G)] - 2\pi iGV + \tau_V(G))
= \tau(F[\delta_V(G) + 2\pi iGV)\delta_Z(G) - \delta_Z(G)[(-2\pi iGV + \tau_V(G)])]
= \tau(f[\delta_V(g) + \delta_Z(g) + 2\pi iG] - \delta_Z(g)[(-2\pi iG + \delta_V(g) + \delta_Z(g)])
= 2\pi i \tau(f[\delta_Z(g) + \delta_Z(g)g]) = 2\pi i \tau(f \delta_Z(g^2)).
\]

Similarly, one checks that

\[
\rho(F, V^*G, GV) = -\tau(VfV^*\delta_Z(g^2)).
\]
Therefore, using the properties (4.2) one gets
\[(\tau_0 \# \rho)(\nu(p), \nu(p), \nu(p)) = 3\tau((f - VfV^*)\delta Z(g^2)) = 1.\]

Fix \( j = 1, 2 \) and for simplicity let \( \psi = \varphi_0 \# \varphi_j \). From the definition of the cup product it can easily be shown that
\[
\psi(a^0 \otimes b^0, a^1 \otimes b^1, a^2 \otimes b^2) = \varphi_0(a^2a^0, a^1)\varphi_j(b^0b^1, b^2) - \varphi_0(a^0a^1, a^2)\varphi_j(b^2b^0, b^1)
\]
for \( a^k \in C(T) \) and \( b^k \in A_0 \). We want to calculate \( \psi(\nu(p), \nu(p), \nu(p)) \). From \( p = V^*g + f + gV \) and \( \nu(p) = V^* \otimes V^*G + 1 \otimes F + W \otimes GV \) and upon expanding the expression \( \psi(\nu(p), \nu(p), \nu(p)) \) we note that the only possibly nonzero terms are of the form \( \psi(W^a, W^b, W^c) \) for \( a + b + c = 0 \). Hence using (4.4) and the cyclicity of \( \psi \) we get
\[
\psi(W^* \otimes V^*G + 1 \otimes F + W \otimes GV, V^* \otimes V^*G + 1 \otimes F + W \otimes GV,
\]
\[V^* \otimes V^*G + 1 \otimes F + W \otimes GV) = \psi(W^* \otimes V^*G, 1 \otimes F, W \otimes GV) + \psi(W^* \otimes V^*G, W \otimes GV, 1 \otimes F) \]
+ \( \psi(1 \otimes F, W^* \otimes V^*G, W \otimes GV) + \psi(1 \otimes F, 1 \otimes F, 1 \otimes F) \)
+ \( \psi(1 \otimes F, W \otimes GV, W^* \otimes V^*G) \)
+ \( \psi(W \otimes GV, W^* \otimes V^*G, 1 \otimes F) + \psi(W \otimes GV, 1 \otimes F, W^* \otimes V^*G) \)
= \( 3\psi(W^* \otimes V^*G, 1 \otimes F, W \otimes GV) \)
+ \( 3\psi(W^* \otimes V^*G, W \otimes GV, 1 \otimes F) + \psi(1 \otimes F, 1 \otimes F, 1 \otimes F) \).

Note that \( \psi(1 \otimes F, 1 \otimes F, 1 \otimes F) = 0 \) since \( \varphi_0(x, 1) = 0 \). Also, since \( \varphi_0(W^*, W) = 1 \) one has
\[
\psi(W^* \otimes V^*G, 1 \otimes F, W \otimes GV) = -\varphi_0(W^*, W)\varphi_j(G^2, F) = -\varphi_j(G^2, F),
\]
\[
\psi(W^* \otimes V^*G, W \otimes GV, 1 \otimes F) = \varphi_0(W^*, W)\varphi_j(V^*G^2V, F) = \varphi_j(V^*G^2V, F)
\]
and hence
\[
\langle \nu(p), \varphi_0 \# \varphi_j \rangle = -3\varphi_j(G^2, F) + 3\varphi_j(V^*G^2V, F).
\]

First, for \( j = 1 \), one has
\[
2\pi i \varphi_1(G^2, F) = \tau(\nu(g^2)\delta_V(\nu(f))) = \tau(g^2\delta_Z(f)) = -\tau(f\delta_Z(g^2))
\]
and
\[
2\pi i \varphi_1(V^*G^2V, F) = \tau(V^*\nu(g^2)V\delta_V(\nu(f))) = \tau(V^*g^2V\delta_Z(f)) = \tau(g^2\delta_Z(VfV^*)) = -\tau(VfV^*\delta_Z(g^2))
\]
hence by (4.2)
\[
\langle \nu(p), \varphi_0 \# \varphi_1 \rangle = \psi(\nu(p), \nu(p), \nu(p)) = \frac{3}{2\pi i} \tau((f - VfV^*)\delta Z(g^2)) = 1.
\]
When \( j = 2 \), one similarly gets
\[
\varphi_2(C, F) = -\frac{1}{2\pi i} \tau(f\delta Z(g^2)), \quad \varphi_2(V^*G^2V, F) = -\frac{1}{2\pi i} \tau(VfV^*\delta Z(g^2))
\]
and thus \( \langle \nu(p), \varphi_0 \# \varphi_2 \rangle = 1 \). Therefore, \( \text{ch}_0(\nu(p)) = (\theta, 1, 1, 1) \) from which one concludes the equality in the lemma. The proof of the second equality in the lemma follows in a similar way (in fact more like the proof of the third equality in Lemma 3.2 except with \( A_\theta \) in place of \( C(\mathbb{T}^2) \)). \( \square \)

Since
\[
K_1(C(\mathbb{T}) \otimes A_\theta) = [K_1(C(\mathbb{T})) \otimes K_0(A_\theta)] \oplus [K_0(C(\mathbb{T})) \otimes K_1(A_\theta)] = \mathbb{Z}^2 \oplus \mathbb{Z}^2
\]
it is easily seen that it has as basis the four elements \([W], [Z], [V], [\zeta] \), where \( \zeta := W \otimes p + (1 - p) \). From Lemma 4.2 one has
\[
\nu(p) + [p_0] + [p_0] = [p] + [P] + [Q]
\]
where \( P = P_{W,V}, Q = P_{W,Z} \), and \( p_0 = [1_{0,0}] \). Therefore, there exist integers \( m, n \) and an invertible matrix \( w \) over \( C(\mathbb{T}) \otimes A_\theta \) such that
\[
\nu(p) + p_0 + p_0 + e_0 = w(p \oplus P \oplus Q \oplus e_0)w^{-1}
\]
where \( e_0 = I_n \oplus O_m \). By suitably enlarging \( m \) one could assume that \( w \) is connectable to the identity by a smooth path of invertibles (upon replacing \( w \) by \( w \oplus w^{-1} \)). So let \( t \mapsto w_t \) be such a path with \( w_0 = I, w_1 = w \). Let
\[
p' = p \oplus p_0 \oplus p_0 + e_0, \quad \text{and} \quad \zeta' = W \otimes p' + (I - p')
\]
so that \( \nu(p') = w(p \oplus P \oplus Q \oplus e_0)w^{-1} \) and it is easily seen that \( [\nu(\zeta')] = [\zeta'] \) in \( K_1 \) of \( C(\mathbb{T}) \otimes A_\theta \). Now since \( [\zeta'] = [\zeta] + (n + 2)[W] \), one gets \( \nu_*[\zeta] = [\zeta] \).

It now follows that on \( K_1(C(\mathbb{T}) \otimes A_\theta) \) one has
\[
\ker(\nu_* - \text{id}_*) = \mathbb{Z}[W] + \mathbb{Z}[\zeta], \quad \text{Im}(\nu_* - \text{id}_*) = \mathbb{Z}[W] + \mathbb{Z}[V].
\]
In view of the basis in (4.3) and the second equality in Lemma 4.2, on \( K_0(C(\mathbb{T}) \otimes A_\theta) \) one has
\[
\ker(\nu_* - \text{id}_*) = \mathbb{Z}[1] + \mathbb{Z}([P_{W,V}] - [1]),
\]
\[
\text{Im}(\nu_* - \text{id}_*) = \mathbb{Z}([P_{W,V}] - [1]) + \mathbb{Z}([P_{W,Z}] - [1]).
\]
The Pimsner-Voiculescu exact sequence for \( A_{\theta}^{5,6} = C \rtimes_{\nu} \mathbb{Z} \), where \( C := C(\mathbb{T}) \otimes A_\theta \):
\[
\begin{array}{ccc}
K_0(C) & \xrightarrow{\text{id}_* - \nu_*} & K_0(C) \\
\uparrow & & \downarrow \\
K_0(A_{\theta}^{5,6}) & \xrightarrow{\nu_*} & K_0(C) \\
& & \downarrow \\
K_1(A_{\theta}^{5,6}) & \xleftarrow{\nu_*} & K_1(C) & \xrightarrow{\text{id}_* - \nu_*} & K_1(C)
\end{array}
\]
now immediately yields $K_0(A_{θ}^{5,6}) = K_1(A_{θ}^{5,6}) = \mathbb{Z}^4$, as stated in Theorem 4.1.

It remains to obtain the range of the trace on $K_0$. For convenience, let us use the notation

$$
\begin{bmatrix}
X \\
Y \\
\vdots
\end{bmatrix} = X \oplus Y \oplus \cdots
$$

for block diagonal matrices. One then has (since $W$ is central in $C(\mathbb{T}) \otimes A_θ$)

(4.5)

$$
\nu(\zeta')\zeta'^{-1} = (\lambda W \otimes w(p \oplus P \oplus Q \oplus e_0)w^{-1} + (I - w(p \oplus P \oplus Q \oplus e_0)w^{-1})) \\
\cdot (W^{-1} \otimes (p \oplus p_0 \oplus p_0 \oplus e_0) + (I - p \oplus p_0 \oplus p_0 \oplus e_0))
$$

$$
= w
\begin{bmatrix}
\lambda W \otimes p + (1 - p) \\
\lambda W \otimes P + (I - P) \\
\lambda W \otimes Q + (I - Q) \\
\lambda W \otimes e_0 + (I - e_0)
\end{bmatrix}
\begin{bmatrix}
W^{-1} \otimes p + (1 - p) \\
W^{-1} \otimes P + (I - P) \\
W^{-1} \otimes Q + (I - Q) \\
W^{-1} \otimes e_0 + (I - e_0)
\end{bmatrix}
$$

For $0 \leq t \leq 1$, let $t \mapsto a_t$ be a smooth path of invertibles in $C^*(W; V) \cong C(\mathbb{T}^2)$ such that $a_0 = \begin{bmatrix} W & 0 \\
0 & 1 \end{bmatrix}$ and $a_1 = \lambda W \otimes P + (I - P)$, let $t \mapsto b_t$ be a smooth path of invertibles in $C^*(W, Z) \cong C(\mathbb{T}^2)$ such that $b_0 = \begin{bmatrix} W & 0 \\
0 & 1 \end{bmatrix}$ and $b_1 = \lambda W \otimes Q + (I - Q)$, and let $t \mapsto c_t$ be a smooth path of invertibles in $C^*(W) \cong C(\mathbb{T})$ such that $c_0 = \begin{bmatrix} W & 0 \\
0 & 1 \end{bmatrix}$ and $c_1 = \lambda W \otimes e_0 + (I - e_0)$. Let

$$
\eta_t := \begin{bmatrix}
e(\theta t)W \otimes p + (1 - p) \\
a_t \\
b_t \\
c_t
\end{bmatrix}
$$

and consider the smooth path

$$
\gamma_t := (w_t \eta_t w_t^{-1} \eta_t^{-1}) \cdot \eta_t \xi
$$

where $\xi$ is the right-most matrix in (4.5). Clearly, $\gamma_0 = I$ and $\gamma_1 = \nu(\zeta')\zeta'^{-1}$. Now $v_t := w_t \eta_t \eta_t^{-1} \eta_t^{-1} \xi$ being a commutator, one obtains (under the trace)

$$
(r \otimes \text{Tr})(\dot{\gamma_t} v_t^{-1}) = 0.
$$

Hence

$$
(r \otimes \text{Tr})(\dot{\gamma_t} v_t^{-1}) = (r \otimes \text{Tr})(\dot{\eta_t} v_t^{-1})
$$

$$
= 2\pi i \theta r(p) + (r \otimes \text{Tr})(\dot{a_t} a_t^{-1}) + (r \otimes \text{Tr})(\dot{b_t} b_t^{-1})
$$

$$
+ (r \otimes \text{Tr})(\dot{c_t} c_t^{-1})
$$
and since $\tau(p) = \theta$, one gets

$$\Delta[\zeta'] = q\left(\frac{1}{2\pi i} \int_0^1 (\tau \otimes \text{Tr})(\dot{\gamma}_t \gamma_t^{-1}) dt\right)$$

$$= q\left(\frac{1}{2\pi i} \int_0^1 \left[2\pi i\theta^2 + (\tau \otimes \text{Tr})(\dot{a}_t a_t^{-1}) + (\tau \otimes \text{Tr})(\dot{b}_t b_t^{-1})
\right.ight.$$

$$\left.\left.\quad + (\tau \otimes \text{Tr})(\dot{c}_t c_t^{-1})\right]\right) dt$$

$$= q(\theta^2)$$

since the last three integrals are integers (as $a_t, b_t, c_t$ are paths of invertibles in matrix algebras over $C(\mathbb{T}^2)$). Now as $q : \mathbb{R} \to \mathbb{R}/(\mathbb{Z} + \mathbb{Z}\theta)$, one deduces that $\tau_\ast K_0(A_\theta^{5,0}) = \mathbb{Z} + \mathbb{Z}\theta + \mathbb{Z}\theta^2$.

References


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