COLORING MAPS OF PERIOD THREE

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We investigate the color number and genus for fixed-point free maps of order three. A result that has the flavor of the Ljusternik-Schnirelmann theorem for involutions is established. The $Y$-sphere, the combinatorial boundary of the product of tripods, is studied in detail. Problems of coloring non-invariant subspaces are touched upon.

Introduction.

All spaces are assumed to be separable metric and all mappings are assumed to be continuous.

Definition 1. Suppose that $f : X \to X$ is a map from $X$ to itself. An open subset $B$ of $X$ is called a color of $(X, f)$ if $f(B) \cap B = \emptyset$ or, equivalently, $f^{-1}(B) \cap B = \emptyset$. A coloring of $(X, f)$ is a finite cover $B$ of $X$ consisting of colors. The minimal cardinality of a coloring $B$ is called the color number $\text{col}(X, f)$ of $(X, f)$.

In the definition of color we could have used closed subsets as well. By shrinking an open coloring a closed coloring may be obtained and the colors of a closed coloring can be enlarged so as to obtain an open coloring. The situation is more delicate when considering non-invariant subspaces, as was shown in [8]. For a fixed-point free homeomorphism $f$ of an $n$-dimensional space $X$ we have $\text{col}(X, f) \leq n + 3$; if moreover the map $f$ is an involution then $\text{col}(X, f) \leq n + 2$, [3].

In this paper we study fixed-point free maps $\sigma : X \to X$ of order 3, i.e., $\sigma^3(x) = x$ for each $x \in X$. It is to be noted that for a color $B$ of $(X, \sigma)$, when $\sigma$ has period 3, the sets $B, \sigma(B)$ and $\sigma^2(B)$ are pairwise disjoint. One of the reasons for studying maps of period 3 is that they provide examples of pairs $(X, \sigma)$ with $\text{dim}(X) = n$ and $\text{col}(X, \sigma) = n + 3$. A second reason is that there is an intimate relation between the color number and the genus, which we now define.

Definition 2. Let $X$ be a space and $\sigma : X \to X$ a map of period 3 without fixed-points.

(1) A subset $B$ of $X$ is called a set of first type if there exists a color $C$ of $(X, \sigma)$ such that $B = C \cup \sigma(C) \cup \sigma^2(C)$; we also say that $B$ is generated by $C$. 

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(2) We say that the genus of the space $X$ is at most $k$ if $X$ can be written as a union of $k$ sets of first type. Notation: $\text{gen}(X,\sigma) \leq k$.

Recall that if $f: X \to X$ and $g: Y \to Y$ are mappings then a mapping $h: X \to Y$ is said to be equivariant if $h \circ f = g \circ h$. Note that for an equivariant $h: X \to Y$ and mappings $f$ and $g$ of order 3 we have $\text{col}(X,f) \leq \text{col}(Y,g)$ and $\text{gen}(X,f) \leq \text{gen}(Y,g)$.

We now formulate the main results, which are generalizations of the Ljusternik-Schnirelmann theorem for involutions on $S^n$.

**Theorem 1.** Let $X$ be a space and $\sigma: X \to X$ a map of period 3 without fixed-points. If $\text{gen}(X,\sigma) = n + 1$ and if $A_1, \ldots, A_{n+1}$ are colors of $X$ with $X = \bigcup_{i=1}^{n+1} [A_i \cup \sigma(A_i) \cup \sigma^2(A_i)]$ then
\[
\bigcap_{i=1}^{n+1} [A_i \cup \sigma(A_i)] \neq \emptyset.
\]

**Theorem 2.** Let $X$ be a space and $\sigma: X \to X$ a map of period 3 without fixed-points. If $\text{col}(X,\sigma) = n + 3$ and $\{A_1, \ldots, A_{n+1}, A_{n+2}, A_{n+3}\}$ is a coloring of $X$ then
\[
\bigcap_{i=1}^{n+1} [A_i \cup \sigma(A_i)] \neq \emptyset.
\]

We shall also obtain a bound on the color number of non-invariant subspaces, using the results of [3]. We shall use the results above to show that this bound is almost sharp. Theorem 1 can be extended to general periodic maps on paracompact spaces [1] by a different proof. The proof in the present paper concerns certain universal spaces, the $Y$-cube and the $Y$-sphere, which are interesting in their own right.

1. **The $Y$-cube $Y^n$ and the $Y$-sphere $S^n_Y$.**

The interval $I = [-1,1]$ with the antipodal-map is a standard space for the study of involutions and gives rise to the study of the $n$-cube $I^n$ and the $n$-sphere $S^n$. We introduce a similar space for the study of maps of order 3. A natural candidate is the tripod, a $Y$-shaped space.

Consider in the complex plane the rotation $\gamma$ of 120 degrees around 0 which is induced by the multiplication with $\zeta = \exp(\frac{2\pi i}{3})$. Let $I$ denote the closed segment between 0 and 1. The subspace $Y = I \cup \zeta(I) \cup \zeta^2(I)$ is called the **closed $Y$-interval**. The points 1, $\zeta$ and $\zeta^2$ are called the **end points** of $Y$ and $Y^o = Y \setminus \{1, \zeta, \zeta^2\}$ is referred to as the **open $Y$-interval**. Both $Y$ and $Y^o$ are invariant under $\gamma$. The space $Y^n$ is called the $n$-dimensional $Y$-cube. The product map $\gamma^n: Y^n \to Y^n$ is a map of order 3 with a unique fixed point. The subspace $S^n_{Y^o}$ of $Y^n$ is defined by
\[
S^n_{Y^o} = \{(x_1, \ldots, x_n) \in Y^n : x_i \in \{1, \zeta, \zeta^2\} \text{ for some } i\}.
\]
The space $S^{n-1}_Y$ is called the $(n - 1)$-dimensional $Y$-sphere. The 1-dimensional $Y$-sphere is the familiar bipartite cubic graph on six nodes $K(3, 3)$, which is a standard example of a non-planar graph.

Note that $S^{n-1}_Y$ is invariant under $\gamma^n$ and that $\gamma^n$ has no fixed points in $S^{n-1}_Y$. As $\dim(S^n_Y) = n$ we have

$$\text{col}(S^n_Y, \gamma^{n+1}) \leq n + 3.$$  

We will show later on that $\text{col}(S^n_Y, \gamma) = n + 3$. So $(S^n_Y, \gamma^{n+1})$ is a pair of an $n$-dimensional space and a mapping for which the color number is maximal.

The space $S^{n}_Y$ can be obtained from $S^{n-1}_Y$ as in the following way

$$S^n_Y = [S^{n-1}_Y \times Y] \cup \bigcup_{i=0}^{2} [Y^n \times \{\zeta^i\}] \subset Y^{n+1}$$  

and

$$S^n_Y = [S^{n-1}_Y \times Y^0] \cup \bigcup_{i=0}^{2} [Y^n \times \{\zeta^i\}] \subset Y^{n+1}.$$  

**Lemma 1.** Let $X$ be a space and $\sigma : X \to X$ a map of order 3 without fixed-points. Suppose $D$ is a closed subset such that $D \cap \sigma(D) = \emptyset$ and $E = D \cup \sigma(D) \cup \sigma^2(D)$, then there exist an equivariant map $f : X \to Y$ with $f^{-1}(1) = D$.

In the lemma, one may think of $D$ and $E$ as a closed color and closed set of the first kind.

**Proof.** By enlarging the $\sigma^i(D)$ for $i = 0, 1, 2$ we obtain open colors $U_i$. For $i = 0, 1, 2$ define real-valued Urysohn functions $f_i$ on $X$ such that $0 \leq f_i(x) \leq 1$ for all $x$, $f^{-1}_i(1) = \sigma^i(D)$ and $f^{-1}_i(0) = X \setminus U_i$. Let $f = f_0 + \zeta f_1 + \zeta^2 f_2$. $\square$

**Theorem 3.** Let $X$ be a space and $\sigma : X \to X$ a map of order 3 without fixed-points. The following statements are equivalent:

1. $\text{col}(X, \sigma) \leq n + 3$,
2. $\text{gen}(X, \sigma) \leq n + 1$,
3. there exist an equivariant map $f : (X, \sigma) \to (S^n_Y, \gamma^{n+1})$,
4. $X = \bigcup_{i=1}^{n+1} B_i$, where $B_i$ is a $\sigma$-invariant subspace with $\text{col}(B_i, \sigma) = 3$ for all $i$.

**Proof.** Suppose (1) holds. Let $A = \{A_1, \ldots, A_{n+1}, A_{n+2}, A_{n+3}\}$ be an open coloring of $X$. If $x \in A_{n+2} \cup A_{n+3}$ then at least one of the points $\sigma(x)$ and $\sigma^2(x)$ does not belong to $A_{n+2} \cup A_{n+3}$, as $A$ is a coloring of $X$. It follows that

$$X = \bigcup_{i=1}^{n+1} [A_i \cup \sigma(A_i) \cup \sigma^2(A_i)].$$
whence (2) holds. If \((X, \sigma)\) satisfies (2), then \(X = \bigcup_{i=1}^{n+1} B_i\) with \(B_i = C_i \cup \sigma(C_i) \cup \sigma^2(C_i)\) a set of first type. By shrinking the cover \(\{\sigma^j(C_i) : i = 1, \ldots, n; j = 0, 1, 2\}\) one can find closed subsets \(D_i \subset C_i\) such that \(\{\sigma^j(D_i) : i = 1, \ldots, n; j = 0, 1, 2\}\) is a cover of \(X\). By Lemma 1 there is for each \(i\) an equivariant map \(f_i : X \to Y\) with the property \(f_i^{-1}(1) = D_i\). The evaluation map \(f = (f_1, \ldots, f_{n+1})\) of \(X\) to \(Y^{n+1}\) is equivariant and sends \(X\) to \(S_Y^n\). So (3) holds. We have already observed that \(\text{col}(S^n_Y, \gamma^{n+1}) \leq n + 3\). Thus (1) follows from (3).

To complete the proof we show that (2) and (4) are equivalent. If \((X, \sigma)\) satisfies (2), then \(X = \bigcup_{i=1}^{n+1} B_i\) where each \(B_i = C_i \cup \sigma(C_i) \cup \sigma^2(C_i)\) is a set of first type. The subspaces \(B_i\) are invariant and have color number 3. Now, suppose that (4) holds. For each \(i\) let \(C_i\) be a color of the subspace \(B_i\) witnessing the fact that \(\text{col}(B_i, \sigma) = 3\). For each \(i\) the set \(C_i\) is open in the subspace \(B_i\) and the sets \(C_i, \sigma(C_i)\) and \(\sigma^2(C_i)\) are mutually disjoint subsets of \(B_i\). As the sets \(C_i, \sigma(C_i)\) and \(\sigma^2(C_i)\) are mutually separated in \(X\) there is an color \(U_i\) of \((X, \sigma)\) such that \(B_i \cap U_i = C_i\). The set \(V_i = U_i \cup \sigma(U_i) \cup \sigma^2(U_i)\) is of the first type and \(X = \bigcup_{i=1}^{n+1} V_i\), whence \(\text{gen}(X, \sigma) \leq n + 1\). □

From the equivalence of (2) and (4) in the previous theorem and the construction of \(S^n_Y\) out of \(S^{n-1}_Y\) one can obtain

\[
(1) \quad \text{gen}(S^n_Y, \gamma^{n+1}) \leq 1 + \text{gen}(S^{n-1}_Y, \gamma^n).
\]

To prove this formula let \(S^{n-1}_Y = \bigcup_{i=1}^k B_i\), where \(B_i\) is a \(\gamma^n\)-invariant subspace with \(\text{col}(B_i, \sigma) = 3\) for all \(i\) and \(k = \text{gen}(S^{n-1}_Y, \gamma^n)\). Then \(S^n_Y = [\bigcup_{i=1}^k B_i \times Y^n] \cup L^n\) and \(\text{gen}(S^n_Y, \gamma^{n+1}) \leq k + 1\), where \(L^n = Y^n \times \{1, 2, 3\}\).

**Theorem 4.** \(\text{col}(S^n_Y, \gamma^{n+1}) = n + 3\) and \(\text{gen}(S^n_Y, \gamma^{n+1}) = n + 1\).

*Proof.* It has already been observed that \(\text{col}(S^n_Y, \gamma^{n+1}) \leq n + 3\). It is known that for all odd \(n\) the standard sphere \(S^n\) with the standard map \(\sigma\) of period three has color number \(n + 3\). So there exists an equivariant map \(f : (S^n, \sigma) \to (S^n, \gamma^{n+1})\). It follows that \(\text{col}(S^n_Y, \gamma^{n+1}) \geq n + 3\) and by Theorem 3 one obtains \(\text{gen}(S^n_Y, \gamma^{n+1}) = n + 1\). Now suppose that \(n\) is even and \(\text{col}(S^n_Y, \gamma^{n+1}) < n + 3\). By Theorem 3 it follows that \(\text{gen}(S^n_Y, \gamma^{n+1}) < n + 1\). Then by the formula (1) we get \(\text{gen}(S^n_{Y+1}, \gamma^{n+2}) < n + 2\), which cannot be true, as \(n + 1\) is odd. □

2. Ljusternik-Schnirelmann for maps of order 3.

Theorem 2 is similar to the Ljusternik-Schnirelmann theorem which reads as follows. Suppose that \(\sigma\) is an involution on a space \(X\) and \(\text{col}(X, \sigma) = n + 2\). Suppose \(\{A_1, \ldots, A_{n+1}, A_{n+2}\}\) is a coloring of \(X\). Then \(\bigcap_{i=1}^{n+1} A_i \neq \emptyset\). The reason why this is true follows. If the two points \(x_1 = (1, \ldots, 1)\) and \(x_2 = (-1, \ldots, -1)\) are deleted from the standard sphere \(S^n \subset [-1, 1]^{n+1}\) one
obtains a space with a strictly smaller coloring number (for \( n \geq 1 \)). For maps of order 3 one can not make a similar claim, \( \bigcap_{i=1}^{n+1} A_i \neq \emptyset \), as easy examples on \( S^1 \) already show. If in the space \( S^n_Y \) the three points \( x_i = (\zeta_i, \ldots, \zeta_i) \), \( i = 0, 1, 2 \), are deleted one obtains a space with the same coloring number. The color number decreases only if more points are deleted.

Consider the \( n \)-dimensional \( Y \)-sphere \( S^n_Y \) and define the subset \( \Lambda^n_Y \) by

\[
x = (x_i)_i \in \Lambda^n_Y \text{ if and only if } x_i \in \{1, \zeta, \zeta^2\} \text{ for all } i.
\]

Note that the cardinality of \( \Lambda^n_Y \) is \( 3^n+1 \) and that \( \Lambda^n_Y \) is invariant under \( \gamma^{n+1} \). From Theorem 5 it follows that the color number can be reduced if we delete the subset \( \Lambda^n_Y \) from \( S^n_Y \).

We define a subspace \( \Sigma^n \subset \Lambda^n_Y \) in the following way. For \( i = 0, 1, 2 \), the subset \( \Sigma^n_i \) of \( \Lambda^n \) is defined by

\[
\Sigma^n_i = \{x = (x_j)_j : x_j = \zeta^{i-1} \text{ or } x_j = \zeta^{i+1}\} \setminus \Delta,
\]

where \( \Delta \) is the diagonal of \( Y^{n+1} \), i.e., \( \Delta \) is the set of points all whose coordinates are equal. For example, \( x = (\zeta, \zeta^2, \zeta) \in \Sigma^n_0 \), but \( y = (\zeta, \zeta, \zeta) \notin \Sigma^n_0 \). Note that the three sets \( \Sigma^n_i \), \( i = 0, 1, 2 \), are pairwise disjoint and \( \gamma^{n+1}(\Sigma^n_i) = \Sigma^n_{(i+1) \mod 3} \). It follows that the set \( \Sigma^n = \Sigma^n_0 \cup \Sigma^n_1 \cup \Sigma^n_2 \) is \( \gamma^{n+1} \)-invariant.

**Theorem 5.** For \( n \geq 1 \),

\[
\text{col}(S^n_Y \setminus \Sigma^n, \gamma^{n+1}) \leq n + 2 \quad \text{and} \quad \text{gen}(S^n_Y \setminus \Sigma^n, \gamma^{n+1}) \leq n.
\]

**Proof.** As the statements of the theorem are equivalent by Theorem 3, we prove the second. For \( n = 1 \) it is best to verify the statement by drawing a picture. There are six points in \( \Sigma^1 \). We mentioned already that \( S^1_Y \) is the graph \( K(3,3) \), which is cut by \( \Sigma^1 \) in six of its edges. The three edges that remain intact, connect vertices of two opposite parts of the graph. So \( S^1_Y \setminus \Sigma^1 \) consists of three components, that are permuted by \( \gamma^2 \). Thus \( S^1_Y \setminus \Sigma^1 \) is a set of the first type.

Now assume that the result holds for \( (S^n_{Y-1} \setminus \Sigma^{n-1}, \gamma^n) \). As \( S^n_Y = [S^n_{Y-1} \times Y^o] \cup \bigcup_{i=0}^2 [Y^n \times \{\zeta_i\}] \) and \( (S^n_{Y-1} \times Y^o) \cap \Sigma^n = \emptyset \), we have

\[
S^n_Y \setminus \Sigma^n = ((S^n_{Y-1} \setminus \Sigma^{n-1}) \times Y^o) \cup \left( (\Sigma^{n-1} \times Y^o) \cup \bigcup_{i=0}^2 \{(Y^n \times \{\zeta_i\}) \setminus \Sigma^n\} \right).
\]

The induction hypothesis implies that \( \text{gen}((S^n_{Y-1} \setminus \Sigma^{n-1}) \times Y^o, \gamma^{n+1}) \leq n - 1 \). So it suffices to show that the remaining two sets \( (\Sigma^{n-1} \times Y^o) \cup \bigcup_{i=0}^2 (Y^n \times \{\zeta_i\}) \setminus \Sigma^n \) form a set of genus 1. Define the sets \( A_i \) for \( i = 0, 1, 2 \) by

\[
A_i = (\Sigma_i^{n-1} \times Y^o) \cup (Y^n \times \{\zeta_i\}) \setminus \Sigma^n.
\]
One verifies that this set has the property \( \gamma^{n+1}(A_i) \cap A_i = \emptyset \) and that 
\[
(\Sigma^{n-1} \times Y^0) \cup \bigcup_{i=0}^{2} (Y^n \times \{ \zeta^i \}) \setminus \Sigma^n = A_0 \cup A_1 \cup A_2.
\]
So the proof is finished once we verify that \( A_0 \) is a closed subset of \( S^n_Y \setminus \Sigma^n \). It is obvious that 
\( Y^n \times \{ 1 \} \) is closed, so we have to verify that the closure of \( \Sigma^{n-1} \times Y^0 \) is contained in \( A_0 \). Observe that the set of density points of \( \Sigma^{n-1} \times Y^0 \) in the 
\( Y \)-sphere is \( \Sigma^{n-1}_0 \times \{ 1, \zeta, \zeta^2 \} \). So in \( S^n_Y \setminus \Sigma^n \) its set of density points is 
\( \Sigma^{n-1}_0 \times \{ 1 \} \), which is a subset of \( Y^n \times \{ 1 \} \). We conclude that \( A_0 \) is closed 
in \( S^n_Y \setminus \Sigma^n \).

The previous theorem has as a corollary Theorem 2.

Proof. Note first that the three statements
\[
(1) \bigcap_{i=1}^{n+1} [A_i \cup \sigma(A_i)] \neq \emptyset,
(2) \bigcap_{i=1}^{n+1} [\sigma(A_i) \cup \sigma^2(A_i)] \neq \emptyset,
(3) \bigcap_{i=1}^{n+1} [\sigma^2(A_i) \cup A_i] \neq \emptyset,
\]
are equivalent, as these sets are mapped onto each other by the map \( \sigma \).

Using the closed coloring \( \{ A_1, \ldots, A_{n+1}, A_{n+2}, A_{n+3} \} \), for \( i = 1, \ldots, n+1 \),
we can define by Lemma 1 equivariant maps \( f_i : X \to Y \) with \( f_i^{-1}(1) = A_i \).

The evaluation map \( F = (f_1, \ldots, f_{n+1}) : X \to Y^{n+1} \) is equivariant and 
\( F(X) \subset S^n_Y \). Since \( \text{col}(X, \sigma) = n + 3 \) and \( \text{col}(S^n_Y \setminus \Sigma_n, \gamma^{n+1}) \leq n + 2 \), it can not occur that \( F(X) \subset S^n_Y \setminus \Sigma^n \).

Choose \( x \in X \) with \( F(x) = (f_i(x))_{i \in \Sigma^n} \), say \( F(x) \in \Sigma^n \).

It follows that \( x \in \bigcap_{i=1}^{n+1} [A_i \cup \sigma^2(A_i)] \neq \emptyset \) and so \( \sigma(x) \in \bigcap_{i=1}^{n+1} [A_i \cup \sigma^2(A_i)] \neq \emptyset \).

Similar arguments can be used for the cases \( F(x) \in \Sigma^n_{2} \) and 
\( F(x) \in \Sigma^n_{0} \).

\section{3. Colorings of non-invariant subspaces.}

We have defined colors as special open subsets. It was already observed that in defining colors we could have used closed sets as well. However, when studying non-invariant subspaces we must stick to open colors.

\textbf{Definition 3.} Suppose that \( f : X \to X \) is a map from \( X \) to itself. Let \( A \) be 
a subset of \( X \). A \textit{coloring} of the subset \( A \) is a finite collection \( B \) consisting 
of colors of \( (X, f) \) such that \( A \subset \bigcup B \). We denote the minimal cardinality 
of such a collection by \( \text{col}(A, X, f) \).

With the technique of the proof of Theorem 3, the equivalence of (2) 
and (4) one can prove the following lemma. We use the notation \( \sigma_B \) to 
de note the restriction of the map \( \sigma \) to the subspace \( B \).

\textbf{Lemma 2 ([8])}. Let \( X \) be a space and \( \sigma : X \to X \) a map of period 3 without fixed-points. If \( A \subset B \subset X \) and \( B \) is \( \sigma \)-invariant, then 
\[
\text{col}(A, X, \sigma) = \text{col}(A, B, \sigma_B).
\]
It is a consequence of the lemma that to compute the color number of a subset $A$ of $X$, we may assume that $X = A \cup \sigma(A) \cup \sigma^2(A)$. The following theorem provides an upper bound for the color number of a subset related to its dimension.

**Theorem 6.** Suppose that $\sigma : X \to X$ is a map of period 3 without fixed-points. If $A$ is a subset of $X$ and $\dim(A) \leq n$ then $\col(A, X, \sigma) \leq 3n + 5$.

**Proof.** The $\sigma$-invariant subspace $T = A \cup \sigma(A) \cup \sigma^2(A)$ has dimension at most $3n + 2$ and contains no fixed points of $\sigma$. So $\col(T, \sigma_T) \leq \dim(T) + 3 = 3n + 5$. The result follows from the previous lemma.

We shall present an example of a map $\sigma$ of period 3 with $\col(A, X, \sigma) \geq 3n + 4$. We refer to [8] for a related example of an involution $\iota$ with color number $\col(A, X, \iota) = 2n + 3$. For the construction we need the following result, which is a consequence of Theorem 1.

**Theorem 7.** Let $\sigma : X \to X$ be a map of period 3 without fixed-points. Suppose $\col(X, \sigma) = n + 3$. If $A$ is a dense subset of $X$ with $A \cup \sigma(A) \cup \sigma^2(A) = X$ then $\col(A, \sigma) \geq n + 2$.

**Proof.** We argue by contradiction. Assume that $\{U_1, \ldots, U_{n+1}\}$ is a coloring of $(A, X, \sigma)$. Then $X = \bigcup_{i=1}^{n+1} (U_i \cup \sigma(U_i) \cup \sigma^2(U_i))$. As $\col(X, \sigma) = n + 3$, we have $\gen(X, \sigma) = n + 1$. From Theorem 1 it follows that $\bigcap_{i=1}^{n+1} [\sigma^{-1}(U_i) \cup \sigma(U_i)]$ is a nonempty open set, which by the density of $A$ contains an element $a \in A$. For each index $i$ we have either $\sigma(a) \in U_i$ or $\sigma^{-1}(a) \in U_i$ and therefore $a \notin U_i$. This contradicts that the $U_i$ cover $A$.

For the construction of our example we need the following lemmas. The first lemma is a special case of a result in [4].

**Lemma 3.** Let $X$ be a space of dimension 2 with a fixed-point free map $\sigma : X \to X$ of period 3. Then there exists a subspace $A$ of $X$ with the following properties

1. $A$ is dense in $X$,
2. $\dim A = 0$,
3. $A \cup \sigma(A) \cup \sigma^2(A) = X$.

The following lemma follows easily from the fact that a subset of $X$ is contained in a $G_\delta$-subset of the same dimension.

**Lemma 4.** Let $X$ be a space with a fixed-point free map $\sigma : X \to X$ of period 3. If $A$ is a $\sigma$-invariant subspace of $X$ with $\dim(A) = k$ then there exists a $\sigma$-invariant $G_\delta$ subset $A'$ of $X$ with $A \subset A'$ and $\dim A' = \dim A$.

**Lemma 5.** Let $X$ be a space with $\dim X = n$ and with a fixed-point free map $\sigma : X \to X$ of period 3. Then $X = X_0 \cup \ldots \cup X_n$, where $\dim X_i = 0$ and each $X_i$ is $\sigma$-invariant.
Proof. This is a version of the Decomposition Theorem [11]. We use induction on the dimension. If \( \dim X = 0 \) the result is trivial. Assume the result holds for all spaces of dimension \( \leq k - 1 \) and all maps of order 3 without fixed-point. Let \( X \) be a space with \( \dim X = k \) and let \( \sigma : X \to X \) be a map of period 3 without fixed-points. Let \( \{ U_n ; n \in \omega \} \) be a countable base with \( \dim(\text{cl} U_n \setminus U_n) \leq k - 1 \), for all \( n \). If we define

\[
X_k = X \setminus \left( \bigcup_n (\text{cl} U_n \setminus U_n) \cup \sigma(\text{cl} U_n \setminus U_n) \cup \sigma^2(\text{cl} U_n \setminus U_n) \right)
\]

then \( X_k \) is \( \sigma \)-invariant, \( \dim X_k = 0 \) and \( \dim(X \setminus X_k) \leq k - 1 \). So the induction argument applies. \( \square \)

Finally we construct the example of a complete \( n \)-dimensional subspace that cannot be colored with less than \( 3n + 4 \) colors.

**Example 1.** Consider the space \( S^{3n+2} \) with the standard map \( \gamma : S^{3n+2} \to S^{3n+2} \) of order 3. We know that \( \text{col}(S^{3n+2}, \gamma) = 3n + 5 \). Since the \( S^{3n+2} \) is \( (3n+2) \)-dimensional it can be written as the union of \( 3n+3 \) zero-dimensional subspaces, say \( S^{3n+2} = B_1 \cup \ldots \cup B_{3n+3} \) such that each \( B_i \) is zero-dimensional and \( \gamma \)-invariant. Let \( X_i = B_{3i-2} \cup B_{3i-1} \cup B_{3i} \) for \( i = 1, \ldots , n+1 \). Then each \( X_i \) is \( \leq 2 \)-dimensional, invariant under \( \gamma \). Lemma 3 implies that each \( X_i \) has a dense zero-dimensional subspace \( A_i \) such that \( A_i \cup \gamma(A_i) \cup \gamma^2(A_i) = X_i \). Write \( A' = A_1 \cup \ldots \cup A_{n+1} \). Then, \( A' \) is dense, \( A' \) has dimension \( \leq n \) and \( A' \) also has the property

\[
A' \cup \gamma(A') \cup \gamma^2(A') = S^{3n+2}.
\]

Note that Property (2) implies that \( \dim A' = n \). We enlarge \( A' \) to a dense \( G_\delta \) subset \( A \) of dimension \( n \). Then \( A \) satisfies Property (2). Finally, by Theorem 7 we have \( \text{col}(A, \gamma) \geq 3n + 4 \).

4. Some remarks on periodic maps.

For a general fixed-point free homeomorphism \( f : X \to X \) we were unable to obtain bounds on \( \text{col}(A, X, f) \) in terms of \( \dim A \). Indeed, we do not even know whether such a bound exists. However, we can find such a bound in the special case that the map is periodic.

We need the following result of Steinlein.

**Theorem 8 ([15]).** If \( f : S^{(m-1)(p-1)-1} \to S^{(m-1)(p-1)-1} \) is a fixed-point free map of prime-period \( p \), then

\[
\text{col}(S^{(m-1)(p-1)-1}, f) \leq 4m.
\]

To obtain our final result, one should know that any free periodic homeomorphism \( f : X \to X \) on a space of \( \dim X = n \) can be conjugated to
a free periodic homeomorphism on $S^n$ [15], so that it suffices to consider homeomorphisms on $S^n$.

**Theorem 9.** Let $\sigma : X \rightarrow X$ be a map of prime-period $p$, without fixed-points and suppose that $p - 1$ divides $n + 1$. If $A$ is a $n$-dimensional subset of $X$ then $\text{col}(A, X, \sigma) \leq 5n + 12$.

**Proof.** Suppose that $f : X \rightarrow X$ is a map of order $p$ and let $A$ be subspace of $X$ of dimension $n$. We can assume that $p \geq 5$, since the case $p = 2$ is done in [8] and $p = 3$ is done above. As we can assume that $X = A \cup f(A) \cup \cdots \cup f^{p-1}(A)$, we see that the maximal dimension of $X$ is $pn + p - 1$.

Choose a minimal $m$ such that $\dim(X) \leq (m - 1)(p - 1) - 1$ and so

$$pn + p - 1 \leq (m - 1)(p - 1) - 1.$$ 

This implies that we can take

$$m = \left\lceil \frac{pn + p}{p - 1} + 1 \right\rceil.$$ 

We conclude that we can color $X$, hence $A$, with $4m = 4\left\lceil \frac{pn + p}{p - 1} + 1 \right\rceil \leq 5n + 12$ colors. \hfill \square

**References**


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A CERTAIN QUOTIENT OF ETA-FUNCTIONS FOUND IN RAMANUJAN’S LOST NOTEBOOK

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In his lost notebook, Ramanujan defined a parameter $\lambda_n$ by

$$\lambda_n = \frac{e^{\pi/2\sqrt{n/3}}}{3\sqrt{3}} \left\{ \left( 1 + e^{-\pi\sqrt{n/3}} \right) \left( 1 - e^{-2\pi\sqrt{n/3}} \right) \left( 1 - e^{-4\pi\sqrt{n/3}} \right) \cdots \right\}^6,$$

and then devoted the remainder of the page to stating several elegant values of $\lambda_n$, for $n \equiv 1 \pmod{8}$, namely:

$$\lambda_1 = 1, \quad \lambda_9 = 3, \quad \lambda_{17} = 4 + \sqrt{17}, \quad \lambda_{25} = (2 + \sqrt{5})^2,$n

$$\lambda_{33} = 18 + 3\sqrt{33}, \quad \lambda_{41} = 32 + 5\sqrt{41}, \quad \lambda_{49} = 55 + 12\sqrt{21},$$

$$\lambda_{57} = , \quad \lambda_{65} = , \quad \lambda_{81} = , \quad \lambda_{89} = 500 + 53\sqrt{89},$$

$$\lambda_{73} = \left( \frac{\sqrt{11 + \sqrt{73}}}{8} + \frac{\sqrt{3 + \sqrt{73}}}{8} \right)^6,$$

$$\lambda_{97} = \left( \frac{\sqrt{17 + \sqrt{97}}}{8} + \frac{\sqrt{9 + \sqrt{97}}}{8} \right)^6,$$

$$\lambda_{121} = \left( \frac{3\sqrt{3} + \sqrt{11}}{4} + \frac{\sqrt{11 + 3\sqrt{33}}}{8} \right)^6,$$

$$\lambda_{169} = , \quad \lambda_{193} = , \quad \lambda_{217} = , \quad \lambda_{241} = ,$$

1. Introduction.

On the top of the page 212 in his lost notebook, Ramanujan defined the function $\lambda_n$ by

$$\lambda_n = \frac{e^{\pi/2\sqrt{n/3}}}{3\sqrt{3}} \left\{ \left( 1 + e^{-\pi\sqrt{n/3}} \right) \left( 1 - e^{-2\pi\sqrt{n/3}} \right) \left( 1 - e^{-4\pi\sqrt{n/3}} \right) \cdots \right\}^6,$$

and then devoted the remainder of the page to stating several elegant values of $\lambda_n$, for $n \equiv 1 \pmod{8}$, namely:
\[
\lambda_{265} = , \quad \lambda_{289} = , \quad \lambda_{361} = .
\]

Note that for several values of \(n\), Ramanujan did not record the corresponding value of \(\lambda_n\).

The purpose of this paper is to establish all the values of \(\lambda_n\) in (1.2), including the ones that are not explicitly stated by Ramanujan, by using the modular \(j\)-invariant, modular equations, Kronecker’s limit formula, and an empirical approach. Applications of values for \(\lambda_n\) will appear in papers by Chan, W.-C. Liaw, and V. Tan [16], Chan, A. Gee, and Tan [14], and by Berndt and Chan [6].

The function \(\lambda_n\) had been briefly introduced earlier in his third notebook [25, p. 393], where Ramanujan offered a formula for \(\lambda_n\) in terms of Klein’s \(j\)-invariant, first proved by Berndt and Chan [5], [3, p. 318, Entry 11.21] by using Ramanujan’s cubic theory of elliptic functions to alternative bases. As K.G. Ramanathan [23] pointed out, the formula in the third notebook is for evaluating \(\lambda_{n/3}\), especially for \(n = 11, 19, 43, 67, 163\). Observe that \(-11, -19, -43, -67, \) and \(-163\) are precisely the discriminants congruent to 5 modulo 8 of imaginary quadratic fields of class number one. (Ramanathan inadvertently inverted the roles of \(n\) and \(n/3\) in his corresponding remark.)

In Section 2, we discuss some of these results in Ramanujan’s third notebook and show how they can be used to calculate the values of \(\lambda_n\) when \(3|n\).

In this and the next two paragraphs, we offer some necessary definitions. Let \(\eta(\tau)\) denote the Dedekind eta-function, defined by

\[
\eta(\tau) := e^{2\pi i\tau/24} \prod_{n=1}^{\infty} (1 - e^{2\pi in\tau}) = q^{1/24} f(-q),
\]

where \(q = e^{2\pi i\tau}\) and \(\text{Im} \ \tau > 0\). Then (1.1) can be written in the alternative form

\[
\lambda_n = \frac{1}{3\sqrt{3}} \frac{f^6(q)}{\sqrt{q} f^6(q^3}) = \frac{1}{3\sqrt{3}} \left( \frac{\eta \left( \frac{1+i\sqrt{n/3}}{2} \right)}{\eta \left( \frac{1+i\sqrt{3n}}{2} \right)} \right)^6,
\]

where \(q = e^{-\pi\sqrt{n/3}}\).

Since much of this paper is devoted to the evaluation of \(\lambda_n\) by using modular equations, we now give a definition of a modular equation. Let \((a)_k = (a)(a+1)\cdots(a+k-1)\) and define the ordinary hypergeometric function \(2F_1(a, b; c; z)\) by

\[
2F_1(a, b; c; z) := \sum_{k=0}^{\infty} \frac{(a)_k(b)_k z^k}{(c)_k k!}, \quad |z| < 1.
\]
Suppose that
\[
\frac{2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; 1 - \beta\right)}{2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \beta\right)} = n \frac{2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; 1 - \alpha\right)}{2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \alpha\right)},
\]
for some positive integer \(n\). A relation between the moduli \(\sqrt{\alpha}\) and \(\sqrt{\beta}\) induced by (1.6) is called a modular equation of degree \(n\), and \(\beta\) is said to have degree \(n\) over \(\alpha\).

In Section 3, using a modular equation of degree 3, we derive a formula for \(\lambda_n\) in terms of the Ramanujan-Weber class invariant which is defined by
\[
(1.7) \quad G_n := 2^{-1/4}q^{-1/24}(-q; q^2)_\infty,
\]
where \(q = \exp(-\pi \sqrt{n})\) and
\[
(a; q)_\infty = \prod_{n=0}^{\infty} (1 - aq^n).
\]

In Sections 4 and 5, we establish all 8 values of \(\lambda_{n^2}\) in the table (1.2). Our proofs in Section 4 employ certain modular equations of degrees 3, 5, 7, and 11. The first three were claimed by Ramanujan [25], and the last one was newly discovered by Berndt, S. Bhargava, and F. G. Garvan [4] as a modular equation in the theory of signature 3 in [4]. In the theory of signature 3, we say that the modulus \(\sqrt{\beta}\) has degree \(n\) over the modulus \(\sqrt{\alpha}\) when
\[
(1.8) \quad \frac{2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; 1 - \beta\right)}{2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; \beta\right)} = n \frac{2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; 1 - \alpha\right)}{2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; \alpha\right)}.
\]
A modular equation of degree \(n\) in the theory of signature 3 is a relation between \(\alpha\) and \(\beta\) which is induced by (1.8). In Section 5, we employ recent discoveries of Chan and W.-C. Liaw [15], [20] on Russell-type modular equations of degrees 13, 17, and 19 in the theory of signature 3. In these two sections, we also determine the values of \(\lambda_5\), \(\lambda_7\), \(\lambda_{11}\), and \(\lambda_{17}\).

In two papers [22], [23], using Kronecker’s limit formula, Ramanathan determined several values of \(\lambda_n\). In [23], in order to determine two specific values of the Rogers–Ramanujan continued fraction, he evaluated \(\lambda_{25}\) by applying Kronecker’s limit formula to \(L\)-functions of orders of \(\mathbb{Q}(\sqrt{-3})\) with conductor 5. This method was also used to determine \(\lambda_{49}\). In the other paper [22], Ramanathan found a representation for \(\lambda_n\) in terms of fundamental units, where \(-3n\) is a fundamental discriminant of an imaginary quadratic field \(\mathbb{Q}(\sqrt{-3n})\) which has only one class in each genus of ideal classes. In particular, he calculated \(\lambda_{17}, \lambda_{41}, \lambda_{65}, \lambda_{89}\), and \(\lambda_{265}\). This
formula and all 14 values of such $\lambda_n$'s are given in Section 6. In the same section, we extend Ramanathan’s method to establish a similar result for $\lambda_n$ when $-3n \equiv 3 \pmod{4}$ and there is precisely one class per genus in each imaginary quadratic field $\mathbb{Q}(\sqrt{-3n})$.

Through Section 6, all values of $\lambda_n$ in (1.2) are calculated except for $n = 73, 97, 193, 217, 241$. In Section 7, we employ an empirical process, analogous to that employed by G. N. Watson [29], [30] in his calculations of class invariants, to determine $\lambda_n$ for these remaining values of $n$. This empirical method has been put on a firm foundation by Chan, A. Gee and V. Tan [14]. Their method works whenever $3 \nmid n$, $n$ is squarefree, and the class group of $\mathbb{Q}(\sqrt{-3n})$ takes the form $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \cdots \oplus \mathbb{Z}_{2k}$, with $1 \leq k \leq 4$.

The first representation of $\lambda_n$ in (1.4) suggests connections between $\lambda_n$ and Ramanujan’s alternative cubic theory. In fact, Berndt and Chan [5] have recently found such a relationship. In Section 8, applying one of their results, we establish an explicit formula for the $j$-invariant in terms of $\lambda_n$, and evaluate several values of the $j$-invariant. Values of the $j$-invariant play an important role in generating rapidly convergent series for $1/\pi$. For example, using the value of $\lambda_{1105}$, Berndt and Chan [6] established a series for $1/\pi$ which yields about 73 or 74 digits of $\pi$ per term. The previous record, which yields 50 digits per term, was given by the Borweins [9] in 1988.

The values of $\lambda_n$ are not only related to convergent series for $1/\pi$ via the $j$-invariants. In fact, Chan, Liaw, and Tan generated [16] a new class of series for $1/\pi$ depending on values of $\lambda_n$. Using the values of $\lambda_n$ proved in this paper, they derived many simple series for $1/\pi$ which are analogues of Ramanujan’s series “belonging to the theory of $q_2$”. For example, he proved that

$$\frac{4}{\pi \sqrt{3}} = \sum_{k=0}^{\infty} (5k + 1) \left( \frac{1}{3} \right)_k \left( \frac{2}{3} \right)_k \left( \frac{1}{2} \right)_k \left( -\frac{9}{16} \right)^k,$$

which follows from the value $\lambda_9 = 3$ and a certain Lambert-type series identity.

In the table (1.2), we observe that if $n$ is not divisible by 3, then $\lambda_n$ is a unit. In fact, in the final section, we show that $\lambda_n$ is a unit when $n$ is odd and $3 \nmid n$.

We conclude the introduction by summarizing in a table the values of $n$ for which $\lambda_n$ is determined in this paper, and the sections where the values can be located.
Recall that [17, p. 81] the invariants $J(\tau)$ and $j(\tau)$, for $\tau \in \mathbb{H} := \{ \tau : \text{Im} \tau > 0 \}$, are defined by

$$J(\tau) = \frac{g_2^3(\tau)}{\Delta(\tau)} \quad \text{and} \quad j(\tau) = 1728J(\tau),$$

where

$$\Delta(\tau) = g_2^3(\tau) - 27g_3^2(\tau),$$

$$g_2(\tau) = 60 \sum_{\substack{m,n=-\infty \atop (m,n) \neq (0,0)}}^{\infty} (m\tau + n)^{-4},$$

and

$$g_3(\tau) = 140 \sum_{\substack{m,n=-\infty \atop (m,n) \neq (0,0)}}^{\infty} (m\tau + n)^{-6}.$$

Furthermore, the function $\gamma_2(\tau)$ is defined by [18, p. 249]

$$\gamma_2(\tau) = \sqrt[3]{j(\tau)},$$

2. $\lambda_n$ and the modular $j$-invariant.

Recall that [17, p. 81] the invariants $J(\tau)$ and $j(\tau)$, for $\tau \in \mathbb{H} := \{ \tau : \text{Im} \tau > 0 \}$, are defined by

$$J(\tau) = \frac{g_2^3(\tau)}{\Delta(\tau)} \quad \text{and} \quad j(\tau) = 1728J(\tau),$$

where

$$\Delta(\tau) = g_2^3(\tau) - 27g_3^2(\tau),$$

$$g_2(\tau) = 60 \sum_{\substack{m,n=-\infty \atop (m,n) \neq (0,0)}}^{\infty} (m\tau + n)^{-4},$$

and

$$g_3(\tau) = 140 \sum_{\substack{m,n=-\infty \atop (m,n) \neq (0,0)}}^{\infty} (m\tau + n)^{-6}.$$
where that branch which is real when $\tau$ is purely imaginary is chosen.

In his third notebook, at the top of page 392 in the pagination of [26], Ramanujan defines a function $J_n$ by

$$J_n = -\frac{1}{32} \gamma^2 \left( 3 + \frac{\sqrt{-n}}{2} \right) = -\frac{1}{32} \sqrt{j} \left( 3 + \frac{\sqrt{-n}}{2} \right).$$

For 15 values of $n$, $n \equiv 3$ (mod 4), Ramanujan indicates the corresponding values for $J_n$. See [3, pp. 310-312] for proofs of these evaluations. In particular,

$$J_3 = 0, \quad J_{27} = 5 \cdot 3^{1/3}, \quad J_{51} = 3(\sqrt{17} + 4)^{2/3} \frac{5 + \sqrt{17}}{2},$$

$$J_{75} = 3 \cdot 5^{1/6} \frac{69 + 31\sqrt{5}}{2}, \quad J_{99} = (23 + 4\sqrt{33})^{2/3} \frac{77 + 15\sqrt{33}}{2}.$$

The first five values of $n$ for $\lambda_n$ in (1.2) are those for which $3n = 3, 27, 51, 75, 99$; the corresponding values of $J_n$ are given in (2.5). Then on the next page, which is the last page of his third notebook, Ramanujan gives a formula leading to a representation of $\lambda_n$.

**Theorem 2.1 (Ramanujan).** For $q = \exp(-\pi \sqrt{n})$, define

$$R := R_n := 3^{1/4} q^{1/36} \frac{f(q)}{f(q^{1/3})}.$$

Then

$$\frac{3\sqrt{3}}{R_n^6} = \sqrt{8J_n + 3 + \sqrt{2\sqrt{64J_n^2 - 24J_n} + 9 - 8J_n + 6}}.$$

Theorem 2.1 was first proved in [3, p. 318, Entry 11.21]. Since $\lambda_n = R_{3n}^{-6}$ by (1.4) and (2.6), (2.7) may be restated as

$$3\sqrt{3} \lambda_n^{1/3} = \sqrt{8J_n + 3 + \sqrt{2\sqrt{64J_n^2 - 24J_n} + 9 - 8J_n + 6}}.$$

By substituting $J_3 = 0$ into (2.8), we determine the first value of $\lambda_n$ in (1.2), and we state it as a corollary.

**Corollary 2.2.**

\[ \lambda_1 = 1. \]

Unfortunately, it is not so easy to find other values of $\lambda_n$ from Theorem 2.1. We have to struggle with complicated radicals even when $n = 9$ for which $\lambda_9 = 3$. It seems that Ramanujan used this formula to determine the values of $\lambda_{n/3}$ for rational integral values of $J_n$ as given in the following table, which constitutes the first part of the last page of the third notebook.
But we are going to use Ramanujan’s discoveries recorded between the table (2.9) and Theorem 2.1 on the last page in his third notebook. Ramanujan first sets, for \( q = \exp(-\pi \sqrt{n}) \),

\[
(2.10) \quad t_n := \sqrt{3} q^{1/18} \frac{f(q^{1/3}) f(q^3)}{f^2(q)}
\]

and

\[
(2.11) \quad u_n := \frac{1}{3} \sqrt{1 + \frac{8}{3} J_n}.
\]

(To avoid a conflict of notation, we have replaced Ramanujan’s second \( t_n \) by \( u_n \).) He then asserted that

\[
(2.12) \quad t_n = \left( 2 \sqrt{64 J_n^2 - 24 J_n + 9} - (16 J_n - 3) \right)^{1/6}
\]

and listed very simple polynomials satisfied by \( t_n \) and \( u_n \). The definition of \( u_n \) in (2.11) seems unmotivated, but by recalling from the proof of Theorem 2.1 in [3, p. 321, (11.33)] that

\[
(2.13) \quad 2 \sqrt{8 J_n + 3} = \frac{f^6(q^{1/3})}{q^{1/6} f^6(q)} - 27 \sqrt{q} \frac{f^6(q^3)}{f^6(q)},
\]

we find that

\[
(2.14) \quad 2 u_n = \frac{1}{3} \sqrt{3} q^{1/6} f^6(q) - 3 \sqrt{3} \sqrt{q} \frac{f^6(q^3)}{f^6(q)}.
\]

We summarize these results in the following two corollaries.

**Corollary 2.3.**

\[
\lambda_n/3 - \lambda_n^{-1} = \frac{2 \sqrt{8 J_n + 3}}{3 \sqrt{3}}.
\]

**Proof.** This is a restatement of either (2.13) or (2.14), with the definition of \( \lambda_n \) in (1.4).
Corollary 2.4. \[
\frac{\lambda_{3n}}{\lambda_{n/3}} = 2\sqrt{64J_n^2 - 24J_n} + 9 + (16J_n - 3).\]

Proof. By (1.4) and (2.10), \(\ell_n^6 = 27\lambda_{n/3}\lambda_{3n}^{-1}\). We obtain the result at once from (2.12).

Corollary 2.5. \(\lambda_9 = 3\).

Proof. Let \(n = 3\) in either Corollary 2.3 or Corollary 2.4. The result follows immediately from the fact that \(J_3 = 0\) and \(\lambda_1 = 1\).

Corollary 2.6.

(i) \(\lambda_{11/3} = \frac{2\sqrt{3} + \sqrt{11}}{3\sqrt{3}}\),

(ii) \(\lambda_{33} = (3\sqrt{3})(2\sqrt{3} + \sqrt{11}) = 18 + 3\sqrt{33}\).

Proof. (i) is an immediate consequence of Theorem 2.1 with \(n = 11\), since \(J_{11} = 1\). Using (i) and either Corollary 2.3 or Corollary 2.4 when \(n = 11\), we obtain (ii).

Corollary 2.7.

(i) \(\lambda_{19/3} = 3^{-3/4}\sqrt{2\sqrt{19} - 5\sqrt{3}} \left(\sqrt{46 + 6\sqrt{57}} + \sqrt{45 + 6\sqrt{57}}\right)\),

(ii) \(\lambda_{57} = 3^{3/4}\sqrt{2\sqrt{19} + 5\sqrt{3}} \left(\sqrt{46 + 6\sqrt{57}} + \sqrt{45 + 6\sqrt{57}}\right)\).

Proof. Using (2.8) with \(n = 19\), we find that \(\lambda_{19/3} = 3^{-3/4} \left(\sqrt{3\sqrt{3}} + \sqrt{2(\sqrt{19} - \sqrt{3})}\right)\).

Let us represent \(x = \sqrt{3\sqrt{3}} + \sqrt{2(\sqrt{19} - \sqrt{3})}\) as a product of units. If \(t = \sqrt{2(\sqrt{19} - \sqrt{3})}\), then \((x - t)^2 = 3\sqrt{3}\), or

(2.15) \[x^2 - 2tx + 2\sqrt{19} - 5\sqrt{3} = 0.\]

Let \(y = \frac{x}{\sqrt{2\sqrt{19} - 5\sqrt{3}}}\).

Then (2.15) becomes

(2.16) \[x\sqrt{2\sqrt{19} - 5\sqrt{3}} \left(y - \frac{2t}{\sqrt{2\sqrt{19} - 5\sqrt{3}}} + \frac{1}{y}\right) = 0.\]
Hence, by applying the quadratic formula to
\[ y + \frac{1}{y} = \frac{2\sqrt{2\sqrt{19} - 2\sqrt{3}}}{2\sqrt{19} - 5\sqrt{3}} = 2\sqrt{46 + 6\sqrt{57}}, \]
we find that
\[ y = \sqrt{46 + 6\sqrt{57}} + \sqrt{45 + 6\sqrt{57}}, \]
from which (i) follows. From (i) and either Corollary 2.3 or Corollary 2.4 with \( n = 19 \), we deduce (ii).

By similar methods, we can derive the following results.

**Corollary 2.8.**

(i) 
\[ \lambda_{43/3} = 3^{-3/4}\sqrt{14\sqrt{43} - 53\sqrt{3}} \left( \sqrt{4294 + 378\sqrt{129}} + \sqrt{4293 + 378\sqrt{129}} \right), \]

(ii) 
\[ \lambda_{129} = 3^{3/4}\sqrt{14\sqrt{43} + 53\sqrt{3}} \left( \sqrt{4294 + 378\sqrt{129}} + \sqrt{4293 + 378\sqrt{129}} \right). \]

**Corollary 2.9.**

(i) 
\[ \lambda_{67/3} = 3^{-3/4}\sqrt{62\sqrt{67} - 293\sqrt{3}} \cdot \left( \sqrt{129214 + 9114\sqrt{201}} + \sqrt{129213 + 9114\sqrt{201}} \right), \]

(ii) 
\[ \lambda_{201} = 3^{3/4}\sqrt{62\sqrt{67} + 293\sqrt{3}} \cdot \left( \sqrt{129214 + 9114\sqrt{201}} + \sqrt{129213 + 9114\sqrt{201}} \right). \]

**Corollary 2.10.**

(i) 
\[ \lambda_{163/3} = 3^{-3/4}\sqrt{4826\sqrt{163} - 35573\sqrt{3}} \cdot \left( \sqrt{1898210854 + 85840062\sqrt{489}} + \sqrt{1898210853 + 85840062\sqrt{489}} \right), \]

(ii) 
\[ \lambda_{489} = 3^{3/4}\sqrt{4826\sqrt{163} + 35573\sqrt{3}} \cdot \left( \sqrt{1898210854 + 85840062\sqrt{489}} + \sqrt{1898210853 + 85840062\sqrt{489}} \right). \]
3. $\lambda_n$ and the class invariant $G_n$.

In [23], Ramanathan introduced a new function $\mu_n$, defined by

$$\mu_n := \frac{1}{3\sqrt{3}} \left( \frac{\eta(i\sqrt{n/3})}{\eta(i\sqrt{3n})} \right)^6 = \frac{1}{3\sqrt{3}} \frac{f^6(-q^2)}{q f^6(-q^6)}, \quad q = e^{-\pi\sqrt{n/3}}. \quad (3.1)$$

Then by (1.4), (3.1), and Euler’s pentagonal number theorem, $f(-q) = (q;q)_\infty$,

$$\lambda_n \mu_n = q^{1/2} \left( \frac{f(q)f(-q^6)}{f(-q^2)f(q^2)} \right)^6 = q^{1/2} \left( \frac{(-q;q^2)_\infty}{(-q^3;q^6)_\infty} \right)^6.$$

Hence from (1.7), we deduce the following result:

**Theorem 3.1.**

$$\frac{\lambda_n}{\mu_n} = \left( \frac{G_{n/3}}{G_{3n}} \right)^6.$$ 

Let

$$P = \frac{f(-q)}{q^{1/12} f(-q^3)} \quad \text{and} \quad Q_p = \frac{f(-q^p)}{q^{p/12} f(-q^{3p})}. \quad (3.2)$$

Recall the modular equation [2, p. 204, Entry 51]

$$PQ^2 + \frac{9}{(PQ)^2} = \left( \frac{Q_2}{P} \right)^6 + \left( \frac{P}{Q_2} \right)^6. \quad (3.3)$$

By replacing $q$ by $-q$ in (3.3), we deduce from (1.4), (3.1), (3.2) and Theorem 3.1 that

$$3(\lambda_n \mu_n)^{1/3} - 3(\lambda_n \mu_n)^{-1/3} = \left( \frac{G_{3n}}{G_{n/3}} \right)^6 - \left( \frac{G_{n/3}}{G_{3n}} \right)^6. \quad (3.4)$$

Solving (3.4) for $\lambda_n \mu_n$, we find that

$$\lambda_n \mu_n = \left( \frac{c + \sqrt{c^2 + 9}}{3} \right)^3, \quad (3.5)$$

where $2c = \left( \frac{G_{3n}}{G_{n/3}} \right)^6 - \left( \frac{G_{n/3}}{G_{3n}} \right)^6$. Hence from Theorem 3.1 and (3.5), we derive the following theorem.

**Theorem 3.2.**

$$\lambda_n = \left( \frac{G_{n/3}}{G_{3n}} \right) \sqrt{\frac{c + \sqrt{c^2 + 9}}{3}}^3,$$

where $2c = \left( \frac{G_{3n}}{G_{n/3}} \right)^6 - \left( \frac{G_{n/3}}{G_{3n}} \right)^6$.
We give another proof of Corollary 2.2.

**Corollary 3.3.**

(i) \( \lambda_1 = 1 \),

(ii) \( \mu_1 = 1 \).

**Proof.** Since \( G_{1/n} = G_n [24] \), \((G_{1/3}/G_3)^6 = 1\). Substituting this value into Theorem 3.2, we have (i), and then using Theorem 3.1, we deduce (ii) at once.

**Corollary 3.4.**

\[ \lambda_3 = \frac{3^{3/4} \sqrt{3} - 1}{\sqrt{2}}. \]

**Proof.** Let \( n = 3 \) in Theorem 3.2 and use the values [3, p. 189],

\[ G_1 = 1 \quad \text{and} \quad G_9 = \left( \frac{1 + \sqrt{3}}{\sqrt{2}} \right)^{1/3}. \]

4. \( \lambda_n \) and modular equations.

We shall employ a certain type of modular equation of degree \( p \) in \( P \) and \( Q_p \) (defined in (3.2)) to calculate several values of \( \lambda_n \). First, recall the modular equation of degree 9 [1, p. 346, Entry 1(iv)],

\[
1 + 9q^{f^3(-q^9)} = \left( 1 + 27q^{f^{12}(-q^2)} \right)^{1/3}.
\]

After replacing \( q \) by \(-q\) in both sides, we deduce the following result from the definition of \( \lambda_n \) in (1.4).

**Theorem 4.1.**

\[ 1 - \frac{1}{\lambda_n^2} = \left( 1 - \sqrt{\frac{3}{\lambda_n \lambda_9}} \right)^3. \]

**Corollary 4.2.**

(i) \( \lambda_9 = 3 \),

(ii) \( \lambda_{81} = 3\sqrt{3}(52 + 36\sqrt{3} + 25\sqrt{3^2}). \)

**Proof.** Let \( n = 1 \) and \( n = 9 \) in Theorem 4.1 to obtain (i) and (ii), respectively.

At the end of Section 1, we remarked that \( \lambda_n \) is a unit for odd \( n \) not divisible by 3. With this as motivation, set

\[
\lambda_n = (\sqrt{a + 1} + \sqrt{a})^6,
\]
and let

\[(4.3) \quad \Lambda = \lambda_n^{1/3} + \lambda_n^{-1/3}. \]

Then

\[(4.4) \quad a = \frac{\Lambda - 2}{4}. \]

By determining \(\Lambda\) in (4.3) and then using (4.4) and (4.2), we next use modular equations to prove all the evaluations of \(\lambda_{p^2}\) given explicitly by Ramanujan in (1.2).

**Theorem 4.3.**

\[
(27\lambda_n\lambda_{25n})^{1/3} + \left(\frac{27}{\lambda_n\lambda_{25n}}\right)^{1/3} = \left(\frac{\lambda_{25n}}{\lambda_n}\right)^{1/2} - \left(\frac{\lambda_n}{\lambda_{25n}}\right)^{1/2} + 5.
\]

**Proof.** From [2, p. 221, Entry 62], we find that

\[(4.5) \quad (PQ_5)^2 + 5 + \frac{9}{(PQ_5)^2} = \left(\frac{Q_5}{P}\right)^3 - \left(\frac{P}{Q_5}\right)^3,
\]

where \(P\) and \(Q_5\) are defined by (3.2). We can deduce Theorem 4.3 from (4.5) and (1.4) immediately after replacing \(q\) by \(-q\).

**Corollary 4.4.**

\[
\lambda_{25} = \left(\frac{1 + \sqrt{5}}{2}\right)^6 = (2 + \sqrt{5})^2.
\]

**Proof.** For brevity, we set \(\lambda = \lambda_{25}\) in the proof. Let \(n = 1\) in Theorem 4.3. Then we have

\[(4.6) \quad 3(\lambda^{1/3} + \lambda^{-1/3}) = (\lambda^{1/2} - \lambda^{-1/2}) + 5.
\]

Set \(\Lambda = \lambda^{1/3} + \lambda^{-1/3}\). Since

\[
\lambda^{1/2} - \lambda^{-1/2} = (\lambda^{1/6} - \lambda^{-1/6})\left(\lambda^{1/3} + \lambda^{-1/3} + 1\right),
\]

(4.6) becomes

\[
3\Lambda - 5 = (\Lambda - 2)^{1/2}(\Lambda + 1),
\]

which can be simplified, after squaring both sides, to

\[
(\Lambda - 3)^3 = 0.
\]

Thus \(\Lambda = 3\) and \(a = 1/4\) by (4.4). Hence from (4.2),

\[
\lambda_{25} = \left(\sqrt{\frac{5}{4}} + \sqrt{\frac{1}{4}}\right)^6.
\]
Theorem 4.5.

\[
\left(27\lambda_n^49n\right)^{1/2} + \left(\frac{27}{\lambda_n^49n}\right)^{1/2} = \left(\frac{\lambda_n^49n}{\lambda_n}\right)^{2/3} + 7\left(\frac{\lambda_n^49n}{\lambda_n}\right)^{1/3} - 7\left(\frac{\lambda_n}{\lambda_n^49n}\right)^{1/3} - \left(\frac{\lambda_n}{\lambda_n^49n}\right)^{2/3}.
\]

Proof. By (3.2) and (1.4), this theorem can be deduced from [2, p. 236, Entry 69]

\[
(PQ_7)^3 + \frac{27}{(PQ_7)^3} = \left(\frac{Q_7}{P}\right)^4 - 7\left(\frac{Q_7}{P}\right)^2 + 7\left(\frac{P}{Q_7}\right)^2 - \left(\frac{P}{Q_7}\right)^4,
\]

with \(q\) replaced by \(-q\).

Corollary 4.6.

\[
\lambda_{49} = \left(\frac{\sqrt{7} + \sqrt{3}}{2}\right)^6 = 55 + 12\sqrt{21}.
\]

Proof. Let \(n = 1\) in Theorem 4.5 and set \(\lambda = \lambda_{49}\). Then

\[
(4.7) \quad 3\sqrt{3}(\lambda^{1/2} + \lambda^{-1/2}) = \lambda^{2/3} + 7\lambda^{1/3} - 7\lambda^{-1/3} - \lambda^{-2/3} = (\lambda^{1/3} - \lambda^{-1/3})(\lambda^{1/3} + \lambda^{-1/3} + 7),
\]

which can be simplified to

\[
(4.8) \quad 3\sqrt{3}(\lambda^{1/3} + \lambda^{-1/3} - 1) = (\lambda^{1/6} - \lambda^{-1/6})(\lambda^{1/3} + \lambda^{-1/3} + 7).
\]

Letting \(\Lambda = \lambda^{1/3} + \lambda^{-1/3}\) in (4.8), we have

\[
(4.9) \quad 3\sqrt{3}(\Lambda - 1) = \sqrt{\Lambda - 2}(\Lambda + 7).
\]

Squaring both sides of (4.9), we deduce that

\[
(\Lambda - 5)^3(\Lambda + 2) = 0.
\]

Hence \(\Lambda = 5\) and

\[
\lambda_{49} = \left(\frac{\sqrt{7}}{4} + \frac{\sqrt{3}}{4}\right)^6,
\]

by (4.2) and (4.4).
Theorem 4.7.

\[ 9\sqrt{3}\{(\lambda_n\lambda_{121n})^{5/6} + (\lambda_n\lambda_{121n})^{-5/6}\} - 99\{(\lambda_n\lambda_{121n})^{2/3} + (\lambda_n\lambda_{121n})^{-2/3}\} \]
\[ + 198\sqrt{3}\{(\lambda_n\lambda_{121n})^{1/2} + (\lambda_n\lambda_{121n})^{-1/2}\} \]
\[ - 759\{(\lambda_n\lambda_{121n})^{1/3} + (\lambda_n\lambda_{121n})^{-1/3}\} \]
\[ + 693\sqrt{3}\{(\lambda_n\lambda_{121n})^{1/6} + (\lambda_n\lambda_{121n})^{-1/6}\} - 1386 \]
\[ = \left(\frac{\lambda_n}{\lambda_{121n}}\right) + \left(\frac{\lambda_{121n}}{\lambda_n}\right). \]

Proof. A new modular equation of degree 11, which was not mentioned by Ramanujan, was proved by Berndt, Bhargava, and Garvan [4], and is given by

\[ (PQ_{11})^5 + \left(\frac{3}{PQ_{11}}\right)^5 + 11\left\{(PQ_{11})^4 + \left(\frac{3}{PQ_{11}}\right)^4\right\} + 66\left\{(PQ_{11})^3 \right\} \]
\[ + \left\{(PQ_{11})^3\right\} + 253\left\{(PQ_{11})^2 + \left(\frac{3}{PQ_{11}}\right)^2\right\} + 693\left\{(PQ_{11}) \right\} \]
\[ + \left\{(PQ_{11}) \right\} + 1368 = \left(\frac{P}{Q_{11}}\right)^6 + \left(\frac{Q_{11}}{P}\right)^6, \]

where \( P \) and \( Q_{11} \) are defined by (3.2). Replacing \( q \) by \(-q\) in the equation above yields Theorem 4.7.

Corollary 4.8.

\[ \lambda_{121} = \left(\sqrt[6]{\frac{19 + 3\sqrt{33}}{8}} + \sqrt[6]{\frac{11 + 3\sqrt{33}}{8}}\right)^6 \]
\[ = \left(3\sqrt{3} + \sqrt{11}\right)^6. \]

Proof. Letting \( n = 1 \) and \( \Lambda = \lambda_{121}^{1/3} + \lambda_{121}^{-1/3} \) in Theorem 4.7, we deduce that

\[ 9\sqrt{3}(\Lambda^2 - \Lambda - 1)(\Lambda + 2)^{1/2} - 99(\Lambda^2 - 2) + 198\sqrt{3}(\Lambda - 1)(\Lambda + 2)^{1/2} \]
\[ - 759\Lambda + 693\sqrt{3}(\Lambda + 2)^{1/2} - 1386 = \Lambda(\Lambda^2 - 3). \]

Rearranging (4.11) to

\[ 9\sqrt{3}(\Lambda + 2)^{1/2}(\Lambda^2 + 21\Lambda + 54) = \Lambda^3 + 99\Lambda^2 + 756\Lambda + 1188, \]
and then squaring both sides, we deduce the equation
\[(\Lambda^2 - 15\Lambda - 18)^3 = 0.\]
Thus
\[\Lambda = \frac{3(5 + \sqrt{33})}{2}.\]
We complete the proof by using (4.2) and (4.4).

We now establish \(\lambda_5, \lambda_7,\) and \(\lambda_{11}\) as well by using Theorems 4.3, 4.5, 4.7, and the following lemma.

**Lemma 4.9.**
\[
\lambda_{1/n} = \frac{1}{\lambda_n}.
\]

**Proof.** Recall that
\[
\eta(\tau + 1) = e^{\pi i/12}\eta(\tau) \quad \text{and} \quad \eta\left(-\frac{1}{\tau}\right) = (-i)^{1/2}\tau^{1/2}\eta(\tau).
\]
From these properties of the Dedekind eta-function or Entry 27(iv), Ch. 16 in [1, p. 43], we find that
\[
\eta^6\left(\frac{1 + i/\sqrt{3n}}{2}\right) = \left(3n\sqrt{3n}\right)\eta^6\left(\frac{1 + i\sqrt{3n}}{2}\right)
\]
and
\[
\eta^6\left(\frac{1 + i\sqrt{3/n}}{2}\right) = \left(\frac{n}{3}\sqrt{n/3}\right)\eta^6\left(\frac{1 + i\sqrt{n/3}}{2}\right).
\]
Hence the lemma follows from (1.4).

**Corollary 4.10.**
\[
\lambda_5 = \frac{1 + \sqrt{5}}{2}.
\]

**Proof.** Letting \(n = 1/5\) in Theorem 4.3 and using Lemma 4.9, we have
\[
1 = \lambda_5 - \lambda_5^{-1}.
\]
Solving this equation for \(\lambda_5\) completes the proof.

**Corollary 4.11.**
\[
\lambda_7 = (2 + \sqrt{3})^{3/2}\left(\frac{\sqrt{3} + \sqrt{7}}{2}\right)^{-3/2}.
\]
Proof. Letting $n = 1/7$ in Theorem 4.5 yields, by Lemma 4.9,

\[ 6\sqrt{3} = \lambda_7^{4/3} + 7\lambda_7^{2/3} - 7\lambda_7^{-2/3} - \lambda_7^{-4/3}. \]

If we set $x = \lambda_7^{2/3} - \lambda_7^{-2/3}$, then (4.13) is equivalent to

\[ 6\sqrt{3} = x\sqrt{x^2 + 4} + 7x, \]

which has the solution $x = -2\sqrt{3} + \sqrt{21}$. Hence by the quadratic formula,

\[ \lambda_7^{2/3} = \frac{-2\sqrt{3} + \sqrt{21} + \sqrt{37 - 12\sqrt{7}}}{2} = \frac{-2\sqrt{3} + \sqrt{21} + 2\sqrt{7} - 3}{2} \]

\[ = (2 + \sqrt{3}) \left( \frac{\sqrt{7} - \sqrt{3}}{2} \right) = (2 + \sqrt{3}) \left( \frac{\sqrt{7} + \sqrt{3}}{2} \right)^{-1}. \]

Then the value of $\lambda_7$ follows immediately.

**Corollary 4.12.**

\[ \lambda_{11} = (2\sqrt{3} + \sqrt{11})^{3/2}(10 + 3\sqrt{11})^{-1/2}. \]

**Proof.** If $n = 1/11$ in Theorem 4.7, then by Lemma 4.9,

\[ \lambda_{11}^2 + \lambda_{11}^{-2} = 2(900\sqrt{3} - 1551). \]

By the quadratic formula,

\[ \lambda_{11}^2 = 900\sqrt{3} - 1551 + 10\sqrt{48356 - 27918\sqrt{3}} \]

\[ = 900\sqrt{3} - 1551 + 470\sqrt{11} - 270\sqrt{3} \]

\[ = (90\sqrt{3} + 47\sqrt{11})(10 - 3\sqrt{11}) \]

\[ = (2\sqrt{3} + \sqrt{11})^3(10 + 3\sqrt{11})^{-1}. \]

Taking the positive square root above, we obtain the value of $\lambda_{11}$.

5. $\lambda_n$ and modular equations in the theory of signature 3.

Suppose $\beta$ has degree $p$ over $\alpha$ in the theory of signature 3, and let

\[ z_1 := {}_2F_1 \left( \frac{1}{3}, \frac{2}{3}; \frac{1}{\alpha} \right) \quad \text{and} \quad z_p := {}_2F_1 \left( \frac{1}{3}, \frac{2}{3}; \frac{1}{\beta} \right). \]

For $\omega = \exp(2\pi i/3)$, if the cubic theta functions are defined by

\[ a(q) := \sum_{m,n=-\infty}^{\infty} q^{m^2+mn+n^2}, \]

(5.1)

\[ b(q) := \sum_{m,n=-\infty}^{\infty} \omega^{m-n}q^{m^2+mn+n^2}, \]

(5.2)
and

\begin{equation}
(5.3) \quad c(q) := \sum_{m,n=-\infty}^{\infty} q^{(m+1/3)^2 + (n+1/3)^2},
\end{equation}

then [4, Lemma 2.6 and Corollary 3.2]

\begin{align*}
a(q) &= z_1, \quad b(q) = (1 - \alpha)^{1/3} z_1, \quad c(q) = \alpha^{1/3} z_1, \\
a(q^p) &= z_p, \quad b(q^p) = (1 - \beta)^{1/3} z_p, \quad c(q^p) = \beta^{1/3} z_p.
\end{align*}

Thus it follows that

\begin{equation}
(5.4) \quad x := (\alpha \beta)^{1/6} = \left( \frac{c(q)c(q^p)}{a(q)a(q^p)} \right)^{1/2}
\end{equation}

and

\begin{equation}
(5.5) \quad y := \{(1 - \alpha)(1 - \beta)\}^{1/6} = \left( \frac{b(q)b(q^p)}{a(q)a(q^p)} \right)^{1/2}.
\end{equation}

But, since [4, Lemma 5.1]

\begin{equation}
(5.6) \quad b(q) = \frac{f^3(-q)}{f(-q^3)},
\end{equation}

\begin{equation}
(5.7) \quad c(q) = 3q^{1/3} \frac{f^3(-q^3)}{f(-q)},
\end{equation}

and [10]

\begin{equation}
(5.8) \quad a^3(q) = b^3(q) + c^3(q),
\end{equation}

we find that

\begin{equation}
(5.9) \quad a(q) = \left\{ \frac{f^{12}(-q) + 27q f^{12}(-q^3)}{f^3(-q) f^3(-q^3)} \right\}^{1/3}.
\end{equation}

(This was also proved by Ramanujan; see [1, p. 460, Entry 3(i)].) Hence, by (5.4)-(5.7) and (5.9),

\begin{equation}
(5.10) \quad x = \frac{3q(q+1)^{1/6} f^2(-q^3) f^2(-q^{3p})}{\{f^{12}(-q) + 27q f^{12}(-q^3)\}^{1/6} \{f^{12}(-q^p) + 27q^p f^{12}(-q^{3p})\}^{1/6}}
\end{equation}

and

\begin{equation}
(5.11) \quad y = \frac{f^2(-q) f^2(-q^p)}{\{f^{12}(-q) + 27q f^{12}(-q^3)\}^{1/6} \{f^{12}(-q^p) + 27q^p f^{12}(-q^{3p})\}^{1/6}}.
\end{equation}

Let

\begin{equation}
(5.12) \quad T := T(q) := \frac{f(-q)}{3^{1/4} q^{1/12} f(-q^3)} \quad \text{and} \quad U_p := U_p(q) := \frac{f(-q^p)}{3^{1/4} q^{p/12} f(-q^{3p})}.
\end{equation}
Then from (5.10) and (5.11), we find that

\[(5.13) \quad \frac{x}{y} = (TU_p)^{-2}\]

and

\[(5.14) \quad xy = (T^6 + T^{-6})^{-1/3}(U_p^6 + U_p^{-6})^{-1/3},\]

We now employ modular equations in \(x\) and \(y\) to calculate further values of \(\lambda_n\).

**Theorem 5.1.**

\[
32 \left\{ (\lambda_n \lambda_{289n})^{2/3} + (\lambda_n \lambda_{289n})^{-2/3} \right\} + 80 \left\{ (\lambda_n \lambda_{289n})^{1/3} + (\lambda_n \lambda_{289n})^{-1/3} \right\} \\
+ 118 + (\lambda_n - \lambda_n^{-1})^{1/3}(\lambda_{289n} - \lambda_{289n}^{-1})^{1/3}\left((\lambda_n \lambda_{289n})^{2/3} + (\lambda_n \lambda_{289n})^{-2/3}\right) \\
+ 35 \left\{ (\lambda_n \lambda_{289n})^{1/3} + (\lambda_n \lambda_{289n})^{-1/3} \right\} + 56) \\
- (\lambda_n - \lambda_n^{-1})^{2/3}(\lambda_{289n} - \lambda_{289n}^{-1})^{2/3} \\
\cdot \left((\lambda_n \lambda_{289n})^{1/3} + (\lambda_n \lambda_{289n})^{-1/3} - 14\right) = \frac{1}{3} \left\{ \frac{\lambda_n}{\lambda_{289n}} + \left(\frac{\lambda_n}{\lambda_{289n}}\right)^{-1} \right\}.
\]

**Proof.** The modular equation of degree 17 with which we start was proved by Chan and W.-C. Liaw [15] and is given by

\[(5.15) \quad x^6 + 96x^5y - 240x^4y^2 + 354x^3y^3 - 240x^2y^4 + 96xy^5 + y^6 \\
- 3x^4 + 105x^3y - 168x^2y^2 + 105xy^3 - 3y^4 + 3x^2 + 42xy + 3y^2 - 1 = 0,
\]

where \(x\) and \(y\) are defined by (5.4) and (5.5) with \(p = 17\). Dividing both sides of (5.15) by \(3x^3y^3\), we obtain

\[(5.16) \quad 32 \left(\frac{x^2}{y^3} + \frac{y^2}{x^3}\right) - 80 \left(\frac{x}{y} + \frac{y}{x}\right) + 118 \\
- \frac{1}{xy} \left\{ \left(\frac{x^2}{y^2} + \frac{y^2}{x^2}\right) - 35 \left(\frac{x}{y} + \frac{y}{x}\right) + 56 \right\} \\
+ \frac{1}{x^2y^2} \left\{ \left(\frac{x}{y} + \frac{y}{x}\right) + 14 \right\} = \frac{1}{3} \left\{ \frac{1}{x^3y^3} - \left(\frac{x^3}{y^3} + \frac{y^3}{x^3}\right) \right\},
\]

after a slight rearrangement.
By (5.13), (5.14), and (5.16),

\[
(5.17) \quad 32\{(TU_{17})^{-4} + (TU_{17})^{4}\} - 80\{(TU_{17})^{-2} + (TU_{17})^{2}\} + 118
\]

\[
- (T^6 + T^{-6})^{1/3}(U_{17}^6 + U_{17}^{-6})^{1/3} \left( (TU_{17})^{-4} + (TU_{17})^{4} - 35 \right)
\]

\[
\cdot \{(TU_{17})^{-2} + (TU_{17})^{2}\} + 56 \right)
\]

\[
+ (T^6 + T^{-6})^{2/3}(U_{17}^6 + U_{17}^{-6})^{2/3} \left( (TU_{17})^{-2} + (TU_{17})^{2} + 14 \right)
\]

\[
= \frac{1}{3} \left\{ \left( \frac{T}{U_{17}} \right)^6 + \left( \frac{T}{U_{17}} \right)^{-6} \right\}.
\]

Replacing \( q \) by \(-q\) in (5.17), setting \( q = e^{-\pi \sqrt{n/3}} \), and using (1.4), we complete the proof.

**Corollary 5.2.**

\[
\lambda_{289} = \left( \frac{\sqrt{34} + 13\sqrt{17} + 5\sqrt{17^2}}{2} + \frac{\sqrt{30} + 13\sqrt{17} + 5\sqrt{17^2}}{2} \right)^6.
\]

**Proof.** Set \( n = 1 \) and \( \lambda = \lambda_{289} \). It follows from Theorem 5.1 that

\[
(5.18) \quad 118 + 32(\lambda^{2/3} + \lambda^{-2/3}) + 80(\lambda^{1/3} + \lambda^{-1/3}) = \frac{\lambda + \lambda^{-1}}{3}.
\]

Let \( \Lambda := \lambda^{1/3} + \lambda^{-1/3} \). Then from (5.18), we have

\[
118 + 32(\Lambda^2 - 2) + 80\Lambda = \frac{\Lambda^3 - 3\Lambda}{3}.
\]

Hence \( \Lambda \) is a root of the equation

\[
\Lambda^3 - 96\Lambda^2 - 243\Lambda - 162 = 0.
\]

The proper solution for \( \Lambda \) is

\[
\Lambda = 32 + 13\sqrt{17} + 5\sqrt{17^2}.
\]

Corollary 5.2 now follows from (4.2) and (4.4).

**Corollary 5.3.**

\[
\lambda_{17} = 4 + \sqrt{17}.
\]

**Proof.** Let \( n = 1/17 \) in Theorem 5.1. Then by Lemma 4.9, we have

\[
171 - 64(\lambda_{17} - \lambda_{17}^{-1})^{2/3} + 6(\lambda_{17} - \lambda_{17}^{-1})^{4/3} = \frac{1}{6}(\lambda_{17}^2 + \lambda_{17}^{-2}).
\]

We complete the proof by solving this equation for \( \lambda_{17} \).
Theorem 5.4.

\[ \lambda_{169} = \left( \frac{2 + \sqrt{13}}{2} + \frac{\sqrt{13} + 4\sqrt{13}}{2} \right)^6. \]

Proof. Let \( x \) and \( y \) be defined by (5.4) and (5.5) with \( p = 13 \). In [20], W.-C. Liaw established a modular equation of degree 13, which, by (5.13) and (5.14), can be put in the abbreviated form

\[
5.19 \quad \{(TU_{13})^{-42} + (TU_{13})^{42}\} + 76142\{(TU_{13})^{-36} + (TU_{13})^{36}\}
\]

\[
+ 1932468187\{(TU_{13})^{-30} + (TU_{13})^{30}\} + 16346295812652
\]

\[
\times \{(TU_{13})^{-24} + (TU_{13})^{24}\} - 42859027901079\{(TU_{13})^{-18}\}
\]

\[
+ (TU_{13})^{18}\} + 30681672585330\{(TU_{13})^{-12} + (TU_{13})^{12}\}
\]

\[
+ 4443969755835\{(TU_{13})^{-6} + (TU_{13})^{6}\} - 90882188302360
\]

\[
+ R(x, y) = 0
\]

where \( R(x, y) \) contains a factor \( 1/(x^3y^3) \). Let \( q = e^{-\pi/\sqrt{3}} \), recall that \( \lambda_1 = 1 \), and set \( \lambda = \lambda_{169} \). Replacing \( q \) by \(-q\) in (5.19), by (1.4), we find that

\[
5.20 \quad -(\lambda^7 + \lambda^{-7}) + 76142(\lambda^6 + \lambda^{-6}) - 1932468187(\lambda^5 + \lambda^{-5})
\]

\[
+ 16346295812652(\lambda^4 + \lambda^{-4}) + 42859027901079(\lambda^3 + \lambda^{-3})
\]

\[
+ 30681672585330(\lambda^2 + \lambda^{-2}) - 4443969755835(\lambda + \lambda^{-1})
\]

\[
- 90882188302360 = 0,
\]

since \( R(x, y) \) equals 0 after \( q \) is replaced by \(-q\), for

\[ 1/x^3y^3 = (T^6 + T^{-6})(U_{13}^6 + U_{13}^{-6}) \]

is a factor of \( R(x, y) \), and

\[
\{T^6(-q) + T^{-6}(-q)\}\{U_{13}^6(-q) + U_{13}^{-6}(-q)\}
\]

\[
= -\{(\lambda_1 - 1^{-1})(\lambda_{169} - 1_{169}^{-1})\}.
\]

Set \( \Lambda = \lambda + \lambda^{-1} \). Then (5.20) has the equivalent form

\[ - (\Lambda^7 - 7\Lambda^5 + 14\Lambda^3 - 7\Lambda) + 76142(\Lambda^6 - 6\Lambda^4 + 9\Lambda^2 - 2)
\]

\[ - 1932468187(\Lambda^5 - 5\Lambda^3 + 5\Lambda) + 16346295812652(\Lambda^4 - 4\Lambda^2 + 2)
\]

\[ + 42859027901079(\Lambda^3 - 3\Lambda) + 30681672585330(\Lambda^2 - 2)
\]

\[ - 4443969755835\Lambda - 90882188302360 = 0,
\]

which simplifies to

\[ (\Lambda - 2)(\Lambda^2 - 25380\Lambda - 39100)^3 = 0. \]

Therefore,

\[ \lambda + \lambda^{-1} = \Lambda = 10(1269 + 352\sqrt{13}). \]
Solving the quadratic equation above, we find that
\[
\lambda_{169} = 6345 + 1760\sqrt{13} + 12\sqrt{559221 + 155100\sqrt{13}}.
\]
Using the formula [7, Eq. (3.1)]
\[
\left\{(32b^3 - 6b) + \sqrt{(32b^3 - 6b)^2 - 1}\right\}^{1/6} = \sqrt{b + \frac{1}{2}} + \sqrt{b - \frac{1}{2}},
\]
with \(b = \frac{15 + 4\sqrt{13}}{4}\), we deduce the value of \(\lambda_{169}\).

Equation (5.19) can also be used to deduce \(\lambda_{13}\).

**Theorem 5.5.**
\[
\lambda_{361} = \frac{1}{3} \left( 2928581 + 1097504(19)^{1/3} + 411296(19)^{2/3} \right.
\]
\[
+ 4\sqrt{1608109304409 + 602648894772(19)^{1/3} + 225846395748(19)^{2/3}} \right).
\]

**Proof.** The proof is similar to that for Theorem 5.4. Let \(x\) and \(y\) be defined by (5.4) and (5.5), respectively, and set \(u = x^3\), \(v = y^3\), and \(p = 19\). Liaw [20] found a modular equation of degree 19, which we give in the abbreviated form,
\[
\begin{align*}
&\left(\frac{u^{10}}{v^{10}}\right) + \left(\frac{v^{10}}{u^{10}}\right) - 17571484 \left\{\left(\frac{u^9}{v^9}\right) + \left(\frac{v^9}{u^9}\right)\right\} + 10291902724030 \\
&\cdot \left\{\left(\frac{u^8}{v^8}\right) + \left(\frac{v^8}{u^8}\right)\right\} - 2009378856109119740 \left\{\left(\frac{u^7}{v^7}\right) + \left(\frac{v^7}{u^7}\right)\right\} \\
&+ 363165905126589014509 \left\{\left(\frac{u^6}{v^6}\right) + \left(\frac{v^6}{u^6}\right)\right\} - 2745050674147219542832 \\
&\cdot \left\{\left(\frac{u^5}{v^5}\right) + \left(\frac{v^5}{u^5}\right)\right\} + 166925399271588508904 \left\{\left(\frac{u^4}{v^4}\right) + \left(\frac{v^4}{u^4}\right)\right\} \\
&- 9487507697742191502320 \left\{\left(\frac{u^3}{v^3}\right) + \left(\frac{v^3}{u^3}\right)\right\} - 7070474114231105014510 \\
&\cdot \left\{\left(\frac{u^2}{v^2}\right) + \left(\frac{v^2}{u^2}\right)\right\} - 7249503742499660191624 \left\{\left(\frac{u}{v}\right) + \left(\frac{v}{u}\right)\right\} \\
&- 29289891786172199497868 + R(u,v) = 0,
\end{align*}
\]
where \(R(u,v)\) is a sum of terms with a factor \(1/(uv)\), which equals 0 after setting \(q = e^{-\pi/\sqrt{3}}\) and replacing \(q\) by \(-q\). With \(\Lambda = \lambda_{361} + \lambda_{361}^{-1}\), we
eventually find that
\begin{align*}
&\left(\Lambda^{10} - 10\Lambda^8 + 35\Lambda^6 - 50\Lambda^4 + 25\Lambda^2 - 2\right) \\
&\quad - 17571484(\Lambda^9 - 9\Lambda^7 + 27\Lambda^5 - 30\lambda^3 + 9\Lambda) \\
&\quad + 102919027240030(\Lambda^8 - 8\Lambda^6 + 20\Lambda^4 - 16\Lambda^2 + 2) \\
&\quad - 20093785610911191740(\Lambda^7 - 7\Lambda^5 + 14\lambda^3 - 7\Lambda) \\
&\quad + 363165905126589014509(\Lambda^6 - 6\Lambda^4 + 9\Lambda^2 - 2) \\
&\quad - 2745050674147219542832(\Lambda^5 - 5\Lambda^3 + 5\Lambda) \\
&\quad + 166925399271588508904(\Lambda^4 - 4\Lambda^2 + 2) \\
&\quad - 9487507697742191502320(\Lambda^3 - 3\Lambda) \\
&\quad - 7070474114231105014510(\Lambda^2 - 2) \\
&\quad - 7249503742499660191624\Lambda - 29289891786172199497868 = 0,
\end{align*}
which simplifies to
\begin{align*}
(\Lambda + 2)(-18438200 + 7433420\Lambda - 5857162\Lambda^2 + \Lambda^3)^3 &= 0.
\end{align*}
Hence
\[\Lambda = \frac{1}{3}(5857162 + 2195008(19)^{1/3} + 822592(19)^{2/3}).\]
Solving the equation \(\Lambda = \lambda_{361} + \lambda_{361}^{-1}\) for \(\lambda_{361}\), we complete the proof.

The modular equation of degree 19 given above can also be used to calculate \(\lambda_{19}\); see also Corollary 6.5(iii).

6. \(\lambda_n\) and Kronecker’s limit formula.

Let \(m > 0\) be square-free and let \(K = \mathbb{Q}(\sqrt{-m})\), the imaginary quadratic field with discriminant \(d\), where
\begin{equation}
(6.1) \quad d = \begin{cases} 
-4m, & \text{if } -m \equiv 2, 3 \pmod{4}, \\
-m, & \text{if } -m \equiv 1 \pmod{4}.
\end{cases}
\end{equation}
Let \(d = d_1d_2\), where \(d_1 > 0\) and, for \(i = 1, 2\), \(d_i \equiv 0 \text{ or } 1 \pmod{4}\). If \(P\) denotes a prime ideal in \(K\), then the Gauss genus character \(\chi\) is defined by
\begin{equation}
(6.2) \quad \chi(P) = \begin{cases} 
\left( \frac{d_1}{N(P)} \right), & \text{if } N(P) \nmid d_1, \\
\left( \frac{d_2}{N(P)} \right), & \text{if } N(P) \mid d_1,
\end{cases}
\end{equation}
where \(N(P)\) is the norm of the ideal \(P\) and \((\cdot)\) denotes the Kronecker symbol.
Let
\[
\Omega = \begin{cases} 
\sqrt{-m}, & \text{if } -m \equiv 2, 3 \pmod{4}, \\
\frac{1 + \sqrt{-m}}{2}, & \text{if } -m \equiv 1 \pmod{4}.
\end{cases}
\]

It is known [21] that each ideal class in the class group \(C_K\) contains primitive ideals which are \(\mathbb{Z}\)-modules of the form \(A = [a, b + \Omega]\), where \(a\) and \(b\) are rational integers, \(a > 0, a|N(b + \Omega), |b| \leq a/2\); \(a\) is the smallest positive integer in \(A\) and \(N(A) = a\). Hence, Siegel’s Theorem [27, p. 72], obtained from Kronecker’s limit formula, can be stated as follows.

**Theorem 6.1 (Siegel).** Let \(\chi\) be a genus character arising from the decomposition \(d = d_1d_2\). Let \(h_1\) be the class number of the field \(\mathbb{Q}(\sqrt{d_1})\), \(\omega\) and \(\omega_2\) be the numbers of roots of unity in \(K\) and \(\mathbb{Q}(\sqrt{d_2})\), respectively, and \(\epsilon_1\) be the fundamental unit of \(\mathbb{Q}(\sqrt{d_1})\). Let
\[
F(A) = \frac{|\eta(z)|^2}{\sqrt{a}},
\]
where \(z = (b + \Omega)/a\) with \([a, b + \Omega] \in \mathcal{A}^{-1}\). Then
\[
\epsilon_1^{wh_1h_2/\omega_2} = \prod_{A \in C_K} F(A)^{-\chi(A)}.
\]

Ramanathan utilized Theorem 6.1 to compute \(\lambda_n\) and \(\mu_n\) [22, Theorem 4].

**Theorem 6.2 (Ramanathan).** Let \(3n\) be a positive square free integer and let \(K = \mathbb{Q}(\sqrt{-3n})\) be an imaginary quadratic field such that each genus contains only one ideal class. Then
\[
\prod_{\chi} \epsilon_1^{t_\chi} = \begin{cases} 
\lambda_n, & \text{if } n \equiv 1 \pmod{4}, \\
\mu_n, & \text{if } n \equiv 2, 3 \pmod{4},
\end{cases}
\]
where
\[
t_\chi = \frac{6\omega h_1h_2}{\omega_2 h},
\]
h, \(h_1, h_2\) are the class numbers of \(K, \mathbb{Q}(\sqrt{d_1})\), and \(\mathbb{Q}(\sqrt{d_2})\), respectively, \(\omega\) and \(\omega_2\) are the numbers of roots of unity in \(K\) and \(\mathbb{Q}(\sqrt{d_2})\), respectively, \(\epsilon_1\) is the fundamental unit in \(\mathbb{Q}(\sqrt{d_1})\), and \(\chi\) runs through all genus characters such that if \(\chi\) corresponds to the decomposition \(d_1d_2\), then either \((\frac{d_1}{3})\) or \((\frac{d_2}{3})\) = -1 and therefore \(d_1, d_2, h_1, h_2, \omega_2,\) and \(\epsilon_1\) are dependent on \(\chi\).

By Theorem 6.2, 14 values of \(\lambda_n\) and 19 of \(\mu_n\) can be evaluated. Among them, Ramanujan recorded only the values of \(\lambda_n\) for which the exponent \(t_\chi = 1\). We state all 14 values of such \(\lambda_n\)’s in the following corollary.
Corollary 6.3.

\[\lambda_5 = \frac{1 + \sqrt{5}}{2}, \quad \lambda_{17} = 4 + \sqrt{17},\]

\[\lambda_{41} = 32 + 5\sqrt{41}, \quad \lambda_{65} = \left(8 + \sqrt{65}\right) \left(\frac{1 + \sqrt{5}}{2}\right)^6,\]

\[\lambda_{89} = 500 + 53\sqrt{89}, \quad \lambda_{145} = \left(\frac{1 + \sqrt{5}}{2}\right)^9 \left(\frac{5 + \sqrt{29}}{2}\right)^3,\]

\[\lambda_{161} = \left(16\sqrt{23} + 29\sqrt{7}\right) \left(\frac{3\sqrt{3} + \sqrt{23}}{2}\right)^3,\]

\[\lambda_{185} = \left(68 + 5\sqrt{185}\right) \left(\frac{1 + \sqrt{5}}{2}\right)^{12},\]

\[\lambda_{209} = \left(46\sqrt{11} + 35\sqrt{19}\right) \left(2\sqrt{3} + \sqrt{11}\right)^3,\]

\[\lambda_{265} = \left(\frac{7 + \sqrt{53}}{2}\right)^3 \left(\frac{1 + \sqrt{5}}{2}\right)^{15},\]

\[\lambda_{385} = \left(\frac{1 + \sqrt{5}}{2}\right)^9 \left(\frac{5 + \sqrt{21}}{2}\right)^3 \left(\frac{\sqrt{7} + \sqrt{11}}{2}\right)^3 \left(\frac{\sqrt{15} + \sqrt{21}}{2}\right)^3,\]

\[\lambda_{665} = \left(14\sqrt{35} + 19\sqrt{19}\right) \left(2\sqrt{5} + \sqrt{21}\right)^3 \left(\frac{1 + \sqrt{5}}{2}\right)^{12} \left(\frac{\sqrt{15} + \sqrt{19}}{2}\right)^3,\]

\[\lambda_{1001} = \left(83\sqrt{77} + 202\sqrt{13}\right) \left(2\sqrt{3} + \sqrt{11}\right)^3 \left(\frac{9 + \sqrt{77}}{2}\right)^3 \left(\frac{7\sqrt{3} + \sqrt{143}}{2}\right)^3,\]

\[\lambda_{1105} = \left(4 + \sqrt{17}\right)^3 \left(8 + \sqrt{65}\right)^3 \left(\frac{1 + \sqrt{5}}{2}\right)^{12} \left(\frac{15 + \sqrt{221}}{2}\right)^3.\]

We next derive a theorem from Theorem 6.1 by which we can evaluate some \(\lambda_n\) for \(n \equiv 3 \mod 4\).

Theorem 6.4. Let \(n > 3\) be a positive square free integer, not divisible by 3, and \(n \equiv 3 \mod 4\). Let \(K = \mathbb{Q}(\sqrt{-3n})\) be an imaginary quadratic field such that each genus contains only one ideal class. Let \(C_0\) be the principal ideal class containing \([1, \Omega]\), where \(\Omega\) is defined by (6.3), and \(C_1\) and \(C_2\) be nonprincipal ideal classes containing \([2, 1 + \Omega]\) and \([6, 3 + \Omega]\), respectively.
Then
\[ \lambda_n = \left( \prod_{\chi(C_1) = -1} t_1^x \right)^{-1} \left( \prod_{\chi(C_2) = -1} t_1^x \right), \]
where \( t_1, d_1, d_2, h_1, h_2, \omega_1, \) and \( \epsilon_1 \) are defined in Theorem 6.2, and the products are over all characters \( \chi \) (the first with \( \chi(C_1) = -1 \) and the second with \( \chi(C_2) = -1 \), associated with the decomposition \( d = d_1d_2 \), and therefore \( d_1, d_2, h_1, h_2, \omega_1, \) and \( \epsilon_1 \) are dependent on \( \chi \).

**Proof.** If \( A \in C \) is any of the ideals \([1, \Omega], [2, 1 + \Omega], \) and \([6, 3 + \Omega]\), then \( A \sim A^{-1}, C = C^{-1}, \) and \( A \in C^{-1} \). For any ideal class \( C \), not \( C_0 \) nor \( C_1 \), Ramanathan \([22]\) showed that
\[ \sum_{\chi(C_1) = -1} \chi(C) = 0, \]
which implies that
\[ \prod_{\chi(C_1) = -1} F(C)^{-\chi(C)} = 1, \]
where \( F(C) \) is defined by (6.4). Therefore, by (6.5),
\[
(6.7) \prod_{\chi(C_1) = -1} e^{\omega h_1 h_2 / \omega_1} = \prod_{\chi(C_1) = -1} \prod_{A \in C_0 \cup C_1} F(A)^{-\chi(A)}
= \prod_{A \in C_0 \cup C_1} F(A)^{-\chi(A) h/2} = \left( \frac{F(C_1)}{F(C_0)} \right)^{h/2},
\]
since the number of genus characters is \( h \), and so the number of genus characters such that \( \chi(C_1) = -1 \) is \( h/2 \). With a similar argument, we find that
\[
(6.8) \prod_{\chi(C_2) = -1} e^{\omega h_1 h_2 / \omega_1} = \left( \frac{F(C_2)}{F(C_0)} \right)^{h/2}.
\]
Dividing (6.7) into (6.8), we have
\[
(6.9) \left( \frac{F(C_2)}{F(C_1)} \right)^{h/2} = \prod_{\chi(C_2) = -1} \epsilon_1 \frac{e^{\omega h_1 h_2 / \omega_1}}{\prod_{\chi(C_1) = -1} e^{\omega h_1 h_2 / \omega_1}}.
\]
But, by (6.4),
\[
(6.10) \frac{F(C_2)}{F(C_1)} = \frac{1}{\sqrt{3}} \left( \frac{\eta \left( \frac{1+i\sqrt{3n}}{2} \right)}{\eta \left( \frac{1+i\sqrt{3n}}{2} \right)} \right)^2.
\]
The theorem now follows from (6.9), (6.10), and (1.4).
We can apply Theorem 6.4 to evaluate \( \lambda_n \) when \( n = 7, 11, 19, 31, 35, 55, 59, 91, 115, 119, 455 \). We first determine all the values of such \( \lambda_n \) when \( \mathbb{Q}(\sqrt{-3n}) \) has class number 4 and each genus contains one ideal class.

Observe that \( \chi(C_1) = -1 \), where \( \chi \) has the decomposition \( d = d_1d_2 \) such that \( \left( \frac{d_1}{6} \right) = -1 \) or \( \left( \frac{d_2}{6} \right) = -1 \), i.e., \( d_1 \equiv \pm 3 \) or \( d_2 \equiv \pm 3 \) (mod 8), and \( \chi(C_2) = -1 \), where \( \left( \frac{d_1}{6} \right) = -1 \) or \( \left( \frac{d_2}{6} \right) = -1 \).

**Corollary 6.5.**

(i) \( \lambda_7 = \left( 2 + \sqrt{3} \right)^{3/2} \frac{4}{\sqrt{3} + \sqrt{7}} \),

(ii) \( \lambda_{11} = (2\sqrt{3} + \sqrt{11})^{3/2}(10 + 3\sqrt{11})^{-1/2} \),

(iii) \( \lambda_{19} = (151 + 20\sqrt{57})^{3/4}(2 + \sqrt{3})^{-3/2} \),

(iv) \( \lambda_{31} = (2 + \sqrt{3})^{9/2} \left( \frac{2\sqrt{7} + \sqrt{31}}{2} \right)^{-3/2} \),

(v) \( \lambda_{59} = (102\sqrt{3} + 23\sqrt{59})^{3/2}(530 + 69\sqrt{59})^{-1/2} \).

**Proof.** For all five values of \( n \) above, the corresponding imaginary quadratic field \( \mathbb{Q}(\sqrt{-3n}) \) has class number 4 and genus number 4. Since the values of \( \lambda_7 \) and \( \lambda_{11} \) are already determined in Corollaries 4.11 and 4.12, respectively, we will discuss only the last three values.

The following tables summarize the needed information about ideal classes and their characters.

Let \( K = \mathbb{Q}(\sqrt{-57}) \).

<table>
<thead>
<tr>
<th>( d_1 )</th>
<th>( d_2 )</th>
<th>( \chi ) ( C )</th>
<th>( \chi(C_0) ) ( \chi(C_1) ) ( \chi(C_2) ) ( \chi(C_3) )</th>
<th>( h_1 )</th>
<th>( h_2 )</th>
<th>( \omega_2 )</th>
<th>( \epsilon_1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-228</td>
<td>( \chi_0 ) ( [1, \Omega] )</td>
<td>( \frac{1}{1} ) ( \frac{1}{1} )</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>57</td>
<td>-4</td>
<td>( \chi_1 ) ( [2, 1 + \Omega] )</td>
<td>( \frac{1}{1} ) ( \frac{1}{1} )</td>
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<td>1</td>
<td>4</td>
<td>151 + 20\sqrt{57}</td>
</tr>
<tr>
<td>44</td>
<td>-3</td>
<td>( \chi_2 ) ( [6, 3 + \Omega] )</td>
<td>( \frac{1}{1} ) ( \frac{1}{1} )</td>
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<td>1</td>
<td>6</td>
<td></td>
</tr>
<tr>
<td>12</td>
<td>-11</td>
<td>( \chi_3 ) ( [3, \Omega] )</td>
<td>( \frac{1}{1} ) ( \frac{1}{1} )</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>2 + \sqrt{3}</td>
</tr>
</tbody>
</table>

By Theorem 6.4,

\[
\lambda_{19} = \frac{1}{\prod_{\chi_i} \epsilon_i^{\ell_1_{\chi_i}}} \cdot \frac{1}{\prod_{\chi_i \chi_j} \epsilon_i^{\ell_1_{\chi_j}}} = \frac{\epsilon_{\chi_1}^{\ell_{\chi_1}}}{\epsilon_{\chi_3}^{\ell_{\chi_3}}} = (151 + 20\sqrt{57})^{3/4}(2 + \sqrt{3})^{-3/2}.
\]

Let \( K = \mathbb{Q}(\sqrt{-93}) \).
<table>
<thead>
<tr>
<th>$d_1$</th>
<th>$d_2$</th>
<th>$\chi$</th>
<th>$C$</th>
<th>$\chi(C_0)$</th>
<th>$\chi(C_1)$</th>
<th>$\chi(C_2)$</th>
<th>$\chi(C_3)$</th>
<th>$h_1$</th>
<th>$h_2$</th>
<th>$\omega_2$</th>
<th>$\epsilon_1$</th>
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<td>$\frac{1}{1}$</td>
<td>$\frac{1}{1}$</td>
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<td></td>
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</tr>
<tr>
<td>93</td>
<td>-4</td>
<td>$\chi_1$</td>
<td>$[2, 1 + \Omega]$</td>
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<td>$\frac{-1}{1}$</td>
<td>$\frac{1}{1}$</td>
<td>$\frac{-1}{1}$</td>
<td>1</td>
<td>1</td>
<td>4</td>
<td>$\frac{29 + 3\sqrt{93}}{2}$</td>
</tr>
<tr>
<td>44</td>
<td>-3</td>
<td>$\chi_2$</td>
<td>$[6, 3 + \Omega]$</td>
<td>$\frac{1}{1}$</td>
<td>$\frac{-1}{1}$</td>
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<td>1</td>
<td>6</td>
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</tr>
<tr>
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<td>$[3, \Omega]$</td>
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<td>$\frac{-1}{1}$</td>
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<td>1</td>
<td>3</td>
<td>2</td>
<td>$2 + \sqrt{3}$</td>
</tr>
</tbody>
</table>

By Theorem 6.4,

$$
\lambda_{31} = \frac{\prod_{\chi_2, \chi_3} \epsilon_{t_1}^{t_x}}{\prod_{\chi_1, \chi_2} \epsilon_{t_1}^{t_x}} = \frac{\epsilon_{t_1}^{t_x}}{\epsilon_{t_1}^{t_x}} = (2 + \sqrt{3})^{9/2} \left( \frac{29 + 3\sqrt{93}}{2} \right)^{-3/4}.
$$

Let $K = \mathbb{Q}(\sqrt{-177})$.

<table>
<thead>
<tr>
<th>$d_1$</th>
<th>$d_2$</th>
<th>$\chi$</th>
<th>$C$</th>
<th>$\chi(C_0)$</th>
<th>$\chi(C_1)$</th>
<th>$\chi(C_2)$</th>
<th>$\chi(C_3)$</th>
<th>$h_1$</th>
<th>$h_2$</th>
<th>$\omega_2$</th>
<th>$\epsilon_1$</th>
</tr>
</thead>
<tbody>
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<td>$[1, \Omega]$</td>
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<td>$\frac{1}{1}$</td>
<td>$\frac{1}{1}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>177</td>
<td>-4</td>
<td>$\chi_1$</td>
<td>$[2, 1 + \Omega]$</td>
<td>$\frac{1}{1}$</td>
<td>$\frac{-1}{1}$</td>
<td>$\frac{1}{1}$</td>
<td>$\frac{-1}{1}$</td>
<td>1</td>
<td>1</td>
<td>4</td>
<td>$62423 + 4692\sqrt{177}$</td>
</tr>
<tr>
<td>236</td>
<td>-3</td>
<td>$\chi_2$</td>
<td>$[6, 3 + \Omega]$</td>
<td>$\frac{1}{1}$</td>
<td>$\frac{-1}{1}$</td>
<td>$\frac{1}{1}$</td>
<td>$\frac{-1}{1}$</td>
<td>1</td>
<td>1</td>
<td>6</td>
<td>$530 + 69\sqrt{59}$</td>
</tr>
<tr>
<td>12</td>
<td>-59</td>
<td>$\chi_3$</td>
<td>$[3, \Omega]$</td>
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<td>$\frac{-1}{1}$</td>
<td>1</td>
<td>3</td>
<td>2</td>
<td></td>
</tr>
</tbody>
</table>

By Theorem 6.4,

$$
\lambda_{59} = \frac{\prod_{\chi_1, \chi_3} \epsilon_{t_1}^{t_x}}{\prod_{\chi_2, \chi_3} \epsilon_{t_1}^{t_x}} = \frac{\epsilon_{t_1}^{t_x}}{\epsilon_{t_1}^{t_x}} = (62423 + 4692\sqrt{177})^{3/4} (530 + 69\sqrt{59})^{-1/2}
$$

$$
= (102\sqrt{3} + 23\sqrt{59})^{3/2} (530 + 69\sqrt{59})^{-1/2}.
$$

Next, we give examples of evaluations of $\lambda_n$ when $\mathbb{Q}(\sqrt{-3n})$ has class number 8 and each genus contains one ideal class.

**Corollary 6.6.**

(i)

$$
\lambda_{35} = (\sqrt{21} + 2\sqrt{5})^{3/2} (4 + \sqrt{15})^{3/2} (2 + \sqrt{5})^{-1} \left( \frac{\sqrt{5} + \sqrt{7}}{\sqrt{2}} \right)^{-1}.
$$
\( \lambda_{55} = (10 + 3\sqrt{11})^{3/2}(2 + \sqrt{3})^{3/(2 + \sqrt{5})} \left( \frac{13 + \sqrt{165}}{2} \right)^{-3/4}, \)

(ii)

\( \lambda_{91} = (25 + 4\sqrt{39})^{3/2}(727 + 44\sqrt{273})^{3/4}(2 + \sqrt{3})^{-3/2} \left( \frac{5 + \sqrt{21}}{2} \right)^{-3/2}, \)

(iii)

\( \lambda_{115} = (24 + 5\sqrt{23})^{3/2}(26\sqrt{5} + 7\sqrt{69})^{3/2}(2 + \sqrt{3})^{-3/2}, \)

(iv)

\( \lambda_{119} = (4 + \sqrt{17})^{3}(50 + 7\sqrt{51})^{3/2}(120 + 11\sqrt{119})^{-1/2} \left( \frac{19 + \sqrt{357}}{2} \right)^{-3}. \)

(v)

**Proof.** We give the calculation for only (i), as the calculations in the remaining four cases are similar. Let \( K = \mathbb{Q}(\sqrt{-105}) \). Then its discriminant \( d = -420 \).

<table>
<thead>
<tr>
<th>( d_1 )</th>
<th>( d_2 )</th>
<th>( \chi )</th>
<th>( C )</th>
<th>( \chi(C_0) )</th>
<th>( \chi(C_1) )</th>
<th>( \chi(C_2) )</th>
<th>( \chi(C_3) )</th>
<th>( h_1 )</th>
<th>( h_2 )</th>
<th>( \omega_2 )</th>
<th>( \epsilon_1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-420</td>
<td>( \chi_0 )</td>
<td>[1, ( \Omega )]</td>
<td>\begin{align*} 1 &amp; 1 \ 1 &amp; 1 \end{align*}</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>105</td>
<td>-4</td>
<td>( \chi_1 )</td>
<td>[2, 1 + ( \Omega )]</td>
<td>\begin{align*} 1 &amp; 1 \ -1 &amp; -1 \end{align*}</td>
<td>2</td>
<td>1</td>
<td>4</td>
<td>41 + 4\sqrt{105}</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>140</td>
<td>-3</td>
<td>( \chi_2 )</td>
<td>[6, 3 + ( \Omega )]</td>
<td>\begin{align*} 1 &amp; -1 \ 1 &amp; -1 \end{align*}</td>
<td>2</td>
<td>1</td>
<td>6</td>
<td>6 + 3\sqrt{5}</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>12</td>
<td>-35</td>
<td>( \chi_3 )</td>
<td>[3, ( \Omega )]</td>
<td>\begin{align*} 1 &amp; -1 \ -1 &amp; 1 \end{align*}</td>
<td></td>
<td></td>
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<td></td>
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</tr>
<tr>
<td>5</td>
<td>-84</td>
<td>( \chi_4 )</td>
<td></td>
<td>\begin{align*} 1 &amp; -1 \ 1 &amp; -1 \end{align*}</td>
<td>1</td>
<td>4</td>
<td>2</td>
<td>( \frac{1 + \sqrt{5}}{2} )</td>
<td></td>
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<td></td>
</tr>
<tr>
<td>21</td>
<td>-20</td>
<td>( \chi_5 )</td>
<td></td>
<td>\begin{align*} 1 &amp; -1 \ -1 &amp; 1 \end{align*}</td>
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<tr>
<td>60</td>
<td>-7</td>
<td>( \chi_6 )</td>
<td></td>
<td>\begin{align*} 1 &amp; 1 \ -1 &amp; -1 \end{align*}</td>
<td>2</td>
<td>1</td>
<td>2</td>
<td>4 + \sqrt{15}</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>28</td>
<td>-15</td>
<td>( \chi_7 )</td>
<td></td>
<td>\begin{align*} 1 &amp; 1 \ 1 &amp; 1 \end{align*}</td>
<td></td>
<td></td>
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<td></td>
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</tbody>
</table>

By Theorem 6.4,
\[ \lambda_{35} = \frac{\prod_{1,3,5,6} e_1^t}{\prod_{2,3,4,5} e_1^t} = \frac{e_1^t}{e_1^t} \]
\[ = (41 + 4\sqrt{105})^{3/4}(4 + \sqrt{15})^{3/2}(6 + \sqrt{35})^{-1/2} \left( \frac{1 + \sqrt{5}}{2} \right)^{-3}. \]

Lastly, by a similar calculation, we can determine \( \lambda_{455} \); here \( \mathbb{Q}(\sqrt{-1365}) \) has class number 16, and each genus contains one ideal class.

**Corollary 6.7.**
\[ \lambda_{455} = (14 + \sqrt{195})^{3/2}(8 + \sqrt{65})^{3/2}(6 + \sqrt{35})^{3/2}(41 + 4\sqrt{105})^{3/2} \times (2 + \sqrt{5})^{-1}(4 + \sqrt{15})^{-3/2}(64 + 3\sqrt{455})^{-1/2} \left( \frac{37 + \sqrt{1365}}{2} \right) ^{-3/4}. \]

**7. An empirical process for computing \( \lambda_n \).**

In this section, we describe an empirical process which can be used to derive values of \( \lambda_n \) for \( n = 73, 97, 193, 217, \) and \( 241 \). This empirical process is analogous to those used by G.N. Watson \([29], [30]\) in his computations of the Ramanujan–Weber class invariants \( G_n \) and \( g_n \).

**Theorem 7.1.** We have

(i) \[ \lambda_{73} = \left( \sqrt{\frac{11 + \sqrt{73}}{8}} + \sqrt{\frac{3 + \sqrt{73}}{8}} \right)^6, \]

(ii) \[ \lambda_{97} = \left( \sqrt{\frac{17 + \sqrt{97}}{8}} + \sqrt{\frac{9 + \sqrt{97}}{8}} \right)^6, \]

(iii) \[ \lambda_{241} = \left( 16 + \sqrt{241} + \sqrt{496 + 32\sqrt{241}} \right)^3, \]

(iv) \[ \lambda_{217} = \left( \sqrt{\frac{1901 + 129\sqrt{217}}{8}} + \sqrt{\frac{1893 + 129\sqrt{217}}{8}} \right)^{3/2} \times \left( \sqrt{\frac{1597 + 108\sqrt{217}}{4}} + \sqrt{\frac{1593 + 108\sqrt{217}}{4}} \right)^{3/2}, \]

(v) \[ \lambda_{193}^{1/3} + \frac{1}{\lambda_{193}^{1/3}} = \frac{1}{4} \left( 39 + 3\sqrt{193} + \sqrt{2690 + 194\sqrt{193}} \right). \]
Proof. By computing the numerical value of $\lambda_{73}$ using (1.1), we assume that

$$\lambda_{73}^{1/3} + \frac{1}{\lambda_{73}^{1/3}} = 7.77200187 \ldots = \frac{7 + \sqrt{73}}{2}.$$  

Solving this quadratic equation, we arrive at

$$\lambda_{73} = \left( \frac{7 + \sqrt{73}}{4} + \sqrt{\frac{53 + 7\sqrt{73}}{8}} \right)^3.$$  

Verifying that the value above is the same as Ramanujan’s value for $\lambda_{73}$ is straightforward.

The computations of $\lambda_{97}$ and $\lambda_{241}$ are similar. We assume that

$$\lambda_{97}^{1/3} + \frac{1}{\lambda_{97}^{1/3}} = 11.42442890089 \ldots = \frac{13 + \sqrt{97}}{2}$$

and

$$\lambda_{241}^{1/3} + \frac{1}{\lambda_{241}^{1/3}} = 63.04834939 \ldots = 32 + 2\sqrt{241},$$

which yield

$$\lambda_{97} = \left( \frac{13 + \sqrt{97}}{4} + \sqrt{\frac{125 + 13\sqrt{97}}{8}} \right)^3$$

and

$$\lambda_{241} = \left( 16 + \sqrt{241} + \sqrt{496 + 32\sqrt{241}} \right)^3.$$  

The computation of $\lambda_{217}$ is different from those of the previous three values. We compute $\lambda_{217}$ and $\lambda_{31/7}$ numerically and assume that

$$\left( \lambda_{217}\lambda_{31/7} \right)^{1/3} + \frac{1}{\left( \lambda_{217}\lambda_{31/7} \right)^{1/3}} = 56.4618397 \ldots = 27 + 2\sqrt{217}$$

and

$$\left( \frac{\lambda_{217}}{\lambda_{31/7}} \right)^{1/3} + \left( \frac{\lambda_{31/7}}{\lambda_{217}} \right)^{1/3} = 43.5963797 \ldots = \frac{43 + 3\sqrt{217}}{2},$$

which imply that

$$\lambda_{217} = \left( \sqrt{\frac{1901 + 129\sqrt{217}}{8}} + \sqrt{\frac{1893 + 129\sqrt{217}}{8}} \right)^{3/2} \times \left( \sqrt{\frac{1597 + 108\sqrt{217}}{4}} + \sqrt{\frac{1593 + 108\sqrt{217}}{4}} \right)^{3/2}.$$
In [14], this empirical method will be put on a firm foundation, and it will be shown that $\lambda_n$ can be explicitly determined when $n \equiv 1 \pmod{4}$, $3 \nmid n$ and the class group of $\mathbb{Q}(\sqrt{-3n})$ is of the type $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \cdots \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_4$. This result is analogous to that associated with the Ramanujan-Weber class invariant $G_n$ proved in [13]. It is clear that this method works for groups of the type $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \cdots \oplus \mathbb{Z}_2$ (which is the class group of imaginary quadratic fields having one class per genus). Hence, the empirical process illustrated here can also be used to determine the values of $\lambda_n$ given in Section 6.

It remains to determine the value of $\lambda_{193}$. The class group of $\mathbb{Q}(\sqrt{-579})$ is $\mathbb{Z}_8$, and so it does not belong to the set of $n$’s that we discuss here. However, we know that [14] $t := \lambda_{193} + \lambda_{193}^{-1}$ lies in a quartic extension of $\mathbb{Q}(\sqrt{-579})$ and so we may use MAPLE V’s “minpoly” function to guess a minimal polynomial of degree 4 satisfied by $t$. Our final result is

\[
\lambda_{193}^{1/3} + \frac{1}{\lambda_{193}^{1/3}} = \frac{1}{4} \left( 39 + 3\sqrt{193} + \sqrt{2690 + 194\sqrt{193}} \right).
\]

A rigorous proof of (7.1) can be found in [14].


The base $q$ in the alternative cubic theory is defined by

\[
q := q_3 := \exp \left( -\frac{2\pi}{\sqrt{3}} \frac{2F_1(\frac{1}{3}, \frac{2}{3}; 1; 1-\alpha)}{2F_1(\frac{1}{3}, \frac{2}{3}; 1; \alpha)} \right), \quad 0 < \alpha < 1,
\]

where $2F_1(a, b; c; z)$ is defined by (1.5). Recall the fundamental inversion formula for the cubic theta functions [4, Lemma 2.9],

\[
\alpha := \alpha(q) = \frac{c^3(q)}{a^3(q)},
\]

where the cubic theta functions $a(q)$, $b(q)$, and $c(q)$ are defined by (5.1)–(5.3). By (5.6)–(5.8), we find that

\[
\frac{1}{\alpha(q)} = \frac{f^{12}(-q)}{27qf^{12}(-q^3)} + 1.
\]

Let

\[
s := s(q) := \frac{f^{12}(-q)}{27qf^{12}(-q^3)}.
\]

From the relation between the $j$-invariant and the modulus $\sqrt{\alpha}$ in the cubic theory [5]

\[
j(3\tau) = 27 \frac{(1 + 8(1 - \alpha))^3}{(1 - \alpha)\alpha^3}, \quad q = e^{2\pi i \tau},
\]
and (8.3), we deduce that
\begin{equation}
(8.6) \quad j(3\tau) = 27(9s + 1)^3 \frac{s + 1}{s}.
\end{equation}
Thus by (2.3),
\begin{equation}
(8.7) \quad \gamma_2(3\tau) = \sqrt[3]{j(3\tau)} = 3(9s + 1)\sqrt[3]{\frac{s + 1}{s}}.
\end{equation}
Let $\tau = \frac{3 + \sqrt{-n}}{6}$. Then by (2.4), (1.3), and (1.4), we obtain the following
theorem after replacing $n$ by $3n$ and dividing both sides by $-1$.

**Theorem 8.1.**
\begin{equation}
32J_{3n} = -\gamma_2\left(\frac{3 + \sqrt{-3n}}{2}\right) = 3\lambda_n^2 - 1)\sqrt[3]{\lambda_n^2 - 1}.
\end{equation}

By Theorem 8.1, we can determine the corresponding values of $J_n$ for known values of $\lambda_n$. For example, the values of $J_n$ in the table (2.5) are easy consequences of Theorem 8.1. We give another example in the corollary below.

**Corollary 8.2.**
\begin{equation}
J_{123} = 15(8 + \sqrt{41})\sqrt[3]{32 + 5\sqrt{41}}.
\end{equation}

**Proof.** Let $n = 41$ in Theorem 8.1 and use the value of $\lambda_{41}$ in Table (1.2).

9. $\lambda_n$ is a unit when $n$ is odd and not divisible by 3.

Recall that an order $O_f$ with conductor $f$ in a quadratic field $K$ is a subset $O_f \subset K$ such that
(i) $O_f$ is a subring of $K$ containing 1,
(ii) $O_f$ is a finitely generated $\mathbb{Z}$-module,
(iii) $O_f$ contains a $\mathbb{Q}$-basis of $K$, and
(iv) $[O_K : O_f] = f$,
where $O_K = O_1$ is the ring of integers of $K$.

If $\alpha_1$ and $\alpha_2$ generate an $O_f$-ideal $A_{O_f}$ over $\mathbb{Z}$, we say that $[\alpha_1, \alpha_2]$ is a basis of $A_{O_f}$. An $O_K$ ideal $A_{O_f}$ is said to be proper if
\begin{equation}
O_f = \{\beta \in K : \beta A_{O_f} \subset A_{O_f}\}.
\end{equation}
Every ideal in $O_K$ prime to $f$ is proper.

If $\nu$ is an algebraic integer and $A$ is an $O_K$ ideal, we write $\nu \approx A$ to mean that $\nu O_L = A O_L$ in some larger number field $L$. Similarly, if $\nu_1$ and $\nu_2$ are algebraic integers, we write $\nu_1 \approx \nu_2$ to mean that $\nu_1/\nu_2$ is a unit.
Theorem 9.1 (Deuring, [19, p. 43]). Let \( f \) be a positive integer and \( p^t || f \), where \( p \) is a prime and \( t \) is a nonnegative integer. Let \( a, b, c, \) and \( d \) be integers, \( M := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) be a matrix with determinant \( p \), and \( \Delta(\tau) = q(q; q)^{24} \), where \( q = e^{2\pi \tau} \), with \( \text{Im} \tau > 0 \). Define

\[
\Phi_M(\tau) = p^{12} \frac{\Delta(M(\tau_1))}{\Delta(\tau_2)},
\]

where \( \tau = \tau_1/\tau_2 \) and \( \Delta(\tau_1) := \Delta((\tau_1)) := \tau_2^{-12} \Delta(\tau). \) Let \( [\alpha_1, \alpha_2] \) be a basis of a proper \( \mathcal{O}_f \)-ideal \( \mathcal{A}_{\mathcal{O}_f} \), and set \( \alpha = \alpha_1/\alpha_2. \) The action of \( M \) on the basis \( [a_1, a_2] \) is defined as \( M[a_1, a_2] = [a_1 a_1 + b a_2, a_1 a_2 + d a_2]. \)

I. When \( p \) splits completely in \( K \), namely \( p = \mathfrak{p} \mathfrak{p} \), then

- (I.a) \( \Phi_M(\alpha) \) is a unit if \( M[a_1, a_2] \) is a basis of a proper \( \mathcal{O}_{f \mathfrak{p}} \)-ideal,
- (I.b) \( \Phi_M(\alpha) \approx p^{12} \) if \( M[a_1, a_2] \) is a basis of a proper \( \mathcal{O}_{f \mathfrak{p}^{-1}} \)-ideal,
- (I.c) if \( p \nmid f \), then \( \Phi_M(\alpha) \approx \mathfrak{p}^{12} \) and \( \Phi_M(\alpha) \approx \mathfrak{p}^{12} \) when \( M[a_1, a_2] \) is a basis of \( \mathcal{A}_{\mathcal{O}_f} \mathfrak{p}_{\mathcal{O}_f} \) and \( M[a_1, a_2] \) is a basis of \( \mathcal{A}_{\mathcal{O}_f} \mathfrak{p}^{-1}_{\mathcal{O}_f} \), respectively.

II. When \( p \) ramifies in \( K \), namely \( p = \mathfrak{p}^2 \), then

- (II.a) \( \Phi_M(\alpha) \approx p^{6/p + 1} \) if \( M[a_1, a_2] \) is a basis of a proper \( \mathcal{O}_{f \mathfrak{p}} \)-ideal,
- (II.b) \( \Phi_M(\alpha) \approx p^{12 - 6/p} \) if \( M[a_1, a_2] \) is a basis of a proper \( \mathcal{O}_{f \mathfrak{p}^{-1}} \)-ideal,
- (II.c) \( \Phi_M(\alpha) \approx p^6 \) if \( M[a_1, a_2] \) is a basis of \( \mathcal{A}_{\mathcal{O}_f} \mathfrak{p}_{\mathcal{O}_f} \).

III. When \( p \) is inert in \( K \), then

- (III.a) \( \Phi_M(\alpha) \approx p^{12/p^t + 1} \) if \( M[a_1, a_2] \) is a basis of a proper \( \mathcal{O}_{f \mathfrak{p}} \)-ideal,
- (III.b) \( \Phi_M(\alpha) \approx p^{12(1 - 1/p^t - (p + 1))} \) if \( M[a_1, a_2] \) is a basis of a proper \( \mathcal{O}_{f \mathfrak{p}^{-1}} \)-ideal.

Corollary 9.2. \( \lambda_n \) is a unit if \( n \) is odd and not divisible by 3.

Proof. Let \( n = f^2 d \), where \( d \) is square-free, and let \( K = \mathbb{Q}(\sqrt{-3d}) \).

(i) First, suppose that \( n \equiv 1 \pmod{4} \). Then \( d \equiv 1 \pmod{4} \), and the ring of integers \( \mathcal{O}_K = \left[ 1 + \sqrt{-3d}/2, 1 \right] \), and \( (3) = \mathfrak{p}^2 \) ramifies in \( \mathcal{O}_K \). Consider the order \( \mathcal{O}_f \) of conductor \( f \). Then \( \mathcal{O}_f \) has a basis \( \mathcal{O}_f = \left[ \frac{f + \sqrt{-3f^2d}}{2}, 1 \right] \). Let

\[
M = \begin{pmatrix} 1 & f \\ 0 & 3 \end{pmatrix}
\]

Then \( M \left[ \frac{f + \sqrt{-3f^2d}}{2}, 1 \right] = \left[ \frac{3f + \sqrt{-3f^2d}}{2}, 3 \right] \) is a basis of \( \mathfrak{p} \mathcal{O}_f \).

Hence by (II.c) in Theorem 9.1, we have

\[
\Phi_M \left( \frac{f + \sqrt{-3f^2d}}{2} \right) \approx 3^6.
\]
On the other hand,

\[
\Phi_M \left( \frac{f + \sqrt{-3f^2d}}{2} \right) = 3^{12} \frac{\Delta \left( \frac{1}{2} f \left( f + \sqrt{-3f^2d} \right) \right)}{\Delta \left( \frac{f + \sqrt{-3f^2d}}{2} \right)}
\]

\[
= 3^{12} \frac{\Delta \left( \frac{3f + \sqrt{-3f^2d}}{6} \right)}{\Delta \left( \frac{f + \sqrt{-3f^2d}}{2} \right)} = 3^{12} \frac{6^{-12} \Delta \left( \frac{3f + \sqrt{-3f^2d}}{6} \right)}{2^{-12} \Delta \left( \frac{f + \sqrt{-3f^2d}}{2} \right)}
\]

\[
= \frac{\Delta \left( \frac{f + \sqrt{-f^2d/3}}{2} \right)}{\Delta \left( \frac{f + \sqrt{-3f^2d}}{2} \right)} = \frac{\Delta \left( \frac{1 + \sqrt{-n/3}}{2} \right)}{\Delta \left( \frac{1 + \sqrt{-3n}}{2} \right)}
\]

where the last equality follows from the facts that \( \Delta(\tau + 1) = \Delta(\tau) \) and \( f \) is odd. But, since \( \Delta(\tau) = \eta^24(\tau) \) by (1.3),

\[
\Phi_M \left( \frac{f + \sqrt{-3f^2d}}{2} \right) = \left( \frac{\eta \left( \frac{1 + \sqrt{-n/3}}{2} \right)}{\eta \left( \frac{1 + \sqrt{-3n}}{2} \right)} \right)^{24}.\]

Thus by (1.4),

\[
\Phi_M \left( \frac{f + \sqrt{-3f^2d}}{2} \right) = (3\sqrt{3}\lambda_n)^4 = 3^6\lambda_n^4.
\]

Combining (9.2) and (9.3), we have \( 3^6 \approx 3^6\lambda_n^4 \). Hence \( \lambda_n^4 \) is a unit, and so is \( \lambda_n \).

(ii) Suppose that \( n \equiv 3 \) (mod 4). Then \( d \equiv 3 \) (mod 4), \( \mathcal{O}_K = [\sqrt{-3d}, 1] \), and \( (3) = \mathcal{P}^2 \) ramifies in \( \mathcal{O}_K \). The order \( \mathcal{O}_f \) of conductor \( f \) has a basis \( \mathcal{O}_f = [\sqrt{-3f^2d}, 1] \). Let \( [f + \sqrt{-3f^2d}, 2] \) be a basis of a proper \( \mathcal{O}_f \)-ideal \( \mathcal{A}_{\mathcal{O}_f} \). Let \( M = \left( \begin{array}{cc} 1 & f \\ 0 & 3 \end{array} \right) \). Since \( M[f + \sqrt{-3f^2d}, 2] = [3f + \sqrt{-3f^2d}, 6] \) is a basis of \( \mathcal{A}_{\mathcal{O}_f} \mathcal{P}_{\mathcal{O}_f} \), we deduce from (II.c) in Theorem 9.1 that

\[
\Phi_M \left( \frac{f + \sqrt{-3f^2d}}{2} \right) \approx 3^6.
\]

As we have shown above,

\[
\Phi_M \left( \frac{f + \sqrt{-3f^2d}}{2} \right) = 3^6\lambda_n^4.
\]

Hence we can conclude that \( \lambda_n \) is a unit.
Corollary 9.3. If \( n \) is odd and congruent to \( 2 \) modulo \( 3 \), then \( 3^{-3/2} \lambda_{3n} \) is a unit.

Proof. Let \( n = f^2d \), where \( d \) is square-free, and let \( K = \mathbb{Q}(\sqrt{-d}) \).

(i) Suppose that \( 3n \equiv 1 \pmod{4} \). Then \( d \equiv 3 \pmod{4} \), \( \mathcal{O}_K = \left[ \frac{1 + \sqrt{-d}}{2}, 1 \right] \), and \( (3) = \mathcal{P}\bar{\mathcal{P}} \) splits completely in \( \mathcal{O}_K \), because \(-d \equiv 1 \pmod{3} \). Let \( \mathcal{O}_{3f} \) be an order of \( \mathcal{O}_K \) of conductor \( 3f \). Then \( \mathcal{O}_{3f} = \left[ \frac{3f + 3\sqrt{-f^2d}}{2}, 1 \right] \). Let \( M = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} \). Then \( M \left[ \frac{3f + 3\sqrt{-f^2d}}{2}, 1 \right] = \left[ \frac{3f + 3\sqrt{-f^2d}}{2}, 3 \right] \) is a basis of \( 3\mathcal{O}_f \). Hence, by (1.b) of Theorem 9.1,

\[
\Phi_M \left( \frac{3f + 3\sqrt{-f^2d}}{2} \right) \approx 3^{12}.
\]

On the other hand, by a calculation like that in the previous proof,

\[
\Phi_M \left( \frac{3f + 3\sqrt{-f^2d}}{2} \right) = 3^{12} \frac{\Delta \left( \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} \right) \left( \frac{3f + 3\sqrt{-f^2d}}{2} \right)}{\Delta \left( \frac{3f + 3\sqrt{-f^2d}}{2} \right)} = (3\sqrt{3}\lambda_{3n})^4 = 3^6\lambda_{3n}^4,
\]

by (1.3) and (1.4). Combining (9.4) and (9.5), we deduce that \( 3^{-3/2} \lambda_{3n} \) is a unit.

(ii) Suppose that \( 3n \equiv 3 \pmod{4} \). Then \( d \equiv 1 \pmod{4} \), \( \mathcal{O}_K = [\sqrt{-d}, 1] \), and \( (3) = \mathcal{P}\bar{\mathcal{P}} \) splits completely in \( \mathcal{O}_K \). Let \( \mathcal{O}_{3f} = [3\sqrt{-f^2d}, 1] \) be an order of \( \mathcal{O}_K \) with conductor \( 3f \). Let \([f + 3\sqrt{-f^2d}, 2]\) be a basis of a proper \( \mathcal{O}_{3f} \)-ideal \( \mathcal{A}_{3f} \). Let \( M = \begin{pmatrix} 1 & f \\ 0 & 3 \end{pmatrix} \). Then \( M[f + 3\sqrt{-f^2d}, 2] = [3f + 3\sqrt{-f^2d}, 6] \) is a basis of \( 3\mathcal{B}_\mathcal{O}_f \), where \( \mathcal{B}_\mathcal{O}_f = [f + \sqrt{-f^2d}, 2] \) is a proper ideal of \( \mathcal{O}_f \). Hence by (1.b) in Theorem 9.1,

\[
\Phi_M \left( \frac{f + 3\sqrt{-f^2d}}{2} \right) \approx 3^{12}.
\]

On the other hand, by calculations like those above,

\[
\Phi_M \left( \frac{f + 3\sqrt{-f^2d}}{2} \right) = 3^{12} \frac{\Delta \left( \begin{pmatrix} 1 & f \\ 0 & 3 \end{pmatrix} \right) \left( \frac{f + 3\sqrt{-f^2d}}{2} \right)}{\Delta \left( \frac{f + 3\sqrt{-f^2d}}{2} \right)} = 3^6\lambda_{3n}^4,
\]

where the last equality was deduced from (1.3) and (1.4). Hence, from (9.6) and (9.7), we can conclude that \( 3^{-3/2} \lambda_{3n} \) is a unit.
**Corollary 9.4.** If \( n \) is odd and congruent to 1 modulo 3, then \( 3^{-3/4} \lambda_{3n} \) is a unit.

*Proof.* Let \( n = f^2d \), where \( d \) is square-free, and let \( K = \mathbb{Q}(\sqrt{-d}) \).

(i) Suppose that \( 3n \equiv 1 \pmod{4} \). Then \( d \equiv 3 \pmod{4} \), \( \mathcal{O}_K = [1 + \sqrt{-d}, 1] \), and \( 3 \) is inert in \( \mathcal{O}_K \), because \( -d \equiv 2 \pmod{3} \). By an argument similar to that in (i) in the proof of Corollary 9.3 and by the use of (III.b) in Theorem 9.1, we can deduce that

\[
3^6 \lambda_{3n}^4 = \Phi_M \left( \frac{3f + 3\sqrt{-f^2d}}{2} \right) \approx 3^{12(1-1/4)} = 3^9.
\]

Hence \( 3^{-3/4} \lambda_{3n} \) is a unit.

(ii) We can deduce the same result when \( 3n \equiv 3 \pmod{4} \) by using a method similar to that in (ii) in the proof of Corollary 9.3 and by using (III.b) of Theorem 9.1.

**Corollary 9.5.** If \( n = 3^t \) for \( t \geq 1 \), then \( 3^{-\frac{3}{2}(1-3^{-t})} \lambda_n \) is a unit.

*Proof.* Suppose that \( n \equiv 1 \pmod{4} \). Let \( K = \mathbb{Q}(\sqrt{-3}) \). Then \( \mathcal{O}_K = [\frac{1+\sqrt{-3}}{2}, 1] \) and \( [3^{t}+\sqrt{-3^{2t+1}}, 1] \) is a basis of the order \( \mathcal{O}_{3^t} \) with conductor \( 3^t \).

Note that \( (3) = \mathcal{P}^2 \) ramifies in \( \mathcal{O}_K \). Let \( M = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} \). Then \( M[3^{t}+\sqrt{-3^{2t+1}}, 1] = [3^{t}+\sqrt{-3^{2t+1}}, 3] \) is a basis of the principal ideal \( 3\mathcal{O}_{3^{t-1}} \). By (II.b) in Theorem 9.1,

\[
\Phi_M \left( \frac{3^{t} + \sqrt{-3^{2t+1}}}{2} \right) \approx 3^{12-6-3^{-t}}.
\]

But, since, by (1.3) and (1.4),

\[
\Phi_M \left( \frac{3^{t} + \sqrt{-3^{2t+1}}}{2} \right) = 3^6 \lambda_n^4,
\]

\( 3^{-\frac{3}{2}(1-3^{-t})} \lambda_n \) is a unit.

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**References**


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LAZARD’S THEOREM FOR DIFFERENTIAL ALGEBRAIC GROUPS AND PROALGEBRAIC GROUPS

Marcin Chalupnik and Piotr Kowalski

We prove that a differential group whose underlying variety is an affine space is unipotent. The problem is reduced to an infinite-dimensional version of Lazard’s Theorem.

1. Introduction.

It is well-known that a connected unipotent algebraic group is isomorphic as a variety to an affine space. Lazard ([La]) proved the converse: An algebraic group isomorphic as a variety to an affine space is unipotent. So, the algebraic structure of a unipotent algebraic group is determined by its geometry. Buium and Cassidy ([BC]) asked if the same holds for differential algebraic groups, i.e., groups defined by differential polynomials in some differential field. It was proved for groups of small dimension (=1, 2) by Cassidy ([Ca]), and for arbitrary groups over a differentially closed field of characteristic 0 by Kowalski and Pillay [KP]. The principal result of the present paper is the proof of this theorem in the full generality:

Theorem 1. Suppose \((K, D)\) is a differential field. Let \(G\) be a differential algebraic group over \(K\), with underlying differential variety differentially isomorphic to \(A^n\). Then \(G\) is unipotent (i.e., \(G\) may be embedded into a unipotent algebraic group).

The strategy of the proof of this theorem is borrowed from [KP]. We deduce Theorem 1 from a purely algebro-geometric, and interesting for its own right result, which was proved in [KP] for \(K\) an algebraically closed field of characteristic 0:

Theorem 2. Let \(G\) be a group scheme over a field \(K\) whose underlying scheme is isomorphic to \(A^\infty\) (the projective limit of affine spaces). Then \(G\) is isomorphic in the category of group schemes over \(K\) to an inverse limit of unipotent algebraic groups.

The paper is organized as follows. In Section 2 we reduce Theorem 1 to Theorem 2 (replacing model theory of [KP] by a simple argument using Hopf algebras). Theorem 2 is proved in Section 4. The proof utilizes in an essential way étale cohomology of schemes and we collect necessary facts concerning it in Section 3. The reader may consult [BC] for background in
differential algebraic geometry, while our main reference on étale cohomology is [Mi] (for basics see also very readable [Ta]). The first author would like to thank Andrzej Weber for several enlightening comments concerning étale cohomology. The second author would like to thank Anand Pillay for suggesting this problem, and Ludomir Newelski for careful reading this paper and many helpful remarks.

2. The reduction.

Let us fix \((K, D)\) a differential field. \(K\{X\}\) denotes the ring of differential polynomials (in a set of variables \(X\)). As a ring \(K\{X\} = K[X, DX, D^2X, ...]\), where \(D^iX, i \geq 0\), are tuples of new variables, and derivation on \(K\{X\}\) extends \(D^iX \mapsto D^{i+1}X\) \((D^0X := X)\). A differential polynomial is an element of \(K\{X\}\), and we naturally regard differential polynomials as functions from \(K^n\) into \(K\). We recall some notions from differential algebraic geometry.

**Definition 2.1.**

i) A differential algebraic variety is a zero set of a finite number of differential polynomials.

ii) A morphism of differential algebraic varieties is a restriction of a differential polynomial function.

iii) A differential algebraic group is a group object in the category of differential algebraic varieties.

**Remark.** Usually (e.g., [Bu], [BC]) a morphism of differential algebraic varieties is defined as a function locally given by differential rational functions rather than a globally defined polynomial. However on an affine space these two notions coincide, so we may adopt the more convenient definition.

**Lemma 2.2.** If \(f : G \longrightarrow H\) is a morphism between affine reduced group schemes, then there exist inverse systems \((G_i), (H_i)\) of algebraic groups and a morphism \(F\) between them, such that \(G = \lim(G_i), H = \lim(H_i)\) and \(f = \lim(F)\).

**Proof.** The proof becomes straightforward when we turn to the category of Hopf algebras, which is dual to the category of affine group schemes. The morphism \(f\) corresponds to a morphism \(\phi : B \longrightarrow A\), where \(B\) is the Hopf algebra corresponding to \(H\) and \(A\) corresponds to \(G\). Since any Hopf algebra is a direct limit of a system of its finitely generated Hopf subalgebras [Wa, p. 24], there exist systems \((A_i)\), and \((B_i)\) of finitely generated reduced Hopf algebras, such that \(A = \text{colim}(A_i)\), and \(B = \text{colim}(B_i)\). Note that finitely generated reduced Hopf algebras correspond to algebraic groups. Denote by \(\phi^*_i\) the map \(\phi\) restricted to the \(B_i\). Then, since \(B_i\) is finitely generated, there exists \(n_i\) such that \(\phi_i^*\) can be factorized through the map \(\phi_i : B_i \longrightarrow A_{n_i}\). So \(\phi = \lim(\phi_i)\), where \((\phi_i)_{i \in N}\) is a map between direct systems \((B_i)_{i \in N}\) and
Going back to the category of affine group schemes we obtain the result.

**Remark.** A part of Lemma 2.2 (the existence of \((G_i)\)) was proved in [KP] by a model-theoretical argument (valid in any stable theory after a suitable reformulation) for algebraic groups over an algebraically closed field.

**Corollary 2.3.** Theorem 2 implies Theorem 1.

**Proof.** Let us consider the ring \(K\{X\}\) of differential polynomials in \(n\) variables. It is also a ring of differential regular functions (i.e., morphisms into \(A^1\)) on \(A^n\), so as in algebraic geometry, differential group structure on \(A^n\) gives us the Hopf algebra structure on \(K\{X\}\). We denote this Hopf algebra by \(A\) (it is also a differential algebra). It corresponds to an infinite-dimensional group scheme \(G^*\). The group \(G^*(K)\) may be thought of as the set \(K^\infty\) with the group operation given by the sequence \((\mu, D(\mu), D^2(\mu), \ldots)\), where \(D\) is a derivation in \(K\{X\}\), and \(K\{X\}\) acts on \(K^\infty\) as the ring of polynomials of infinitely many variables. Then the map \(\phi : a \mapsto (a, Da, D^2a, \ldots)\) is a homomorphism between \(G\) and \(G^*(K)\). From Lemma 2.2, we have \(G^* = \text{lim}(G_i)\), where \(G_i\)'s are algebraic groups. Denote by \(H_i\), the Hopf algebra of \(G_i\). The Hopf algebra \(A\) is a direct limit of the system \((H_i)\) and it is finitely generated as a differential algebra, so there exists \(n\) such that \(H_n\) differentially generates \(A\). Then the composition morphism \(G \rightarrow G_n\) is an embedding, since it induces epimorphism of algebras of differential regular functions. Using Theorem 2 and Lemma 2.2 we see that \(G_n\) is unipotent.

**Remark.**

i) The construction of \(G^*\) is the same as in [KP]. However here we do not need our field to be differentially closed to check that we obtain a group.

ii) If \(G\) is an algebraic group, then \(G^*\) coincides with the Buium’s infinite prolongation, which is an inverse limit of so-called twisted jet spaces [Bu]. If \(D\) vanishes on the field of definition, then the infinite prolongation of an algebraic group is an inverse limit of the usual jet spaces.

### 3. Some étale cohomology of group schemes.

The aim of this section is to prove some facts about étale cohomology of group schemes which will be needed in the proof of Theorem 2. Throughout this section \(k\) is an algebraically closed field and all schemes are of finite type over \(k\). We are interested in étale cohomology groups with coefficients in the constant sheaf \(\mathbb{Z}/l\) where \(l\) is a prime distinct from the characteristic. We say that a scheme \(X\) is \(\mathbb{Z}/l\)-acyclic if it has the cohomology of a point i.e., \(H^i_{et}(X, \mathbb{Z}/l) = 0\) for \(i > 0\), and \(H^0_{et}(X, \mathbb{Z}/l) = \mathbb{Z}/l\); a morphism of schemes \((A_n)_{i \in N}\). Going back to the category of affine group schemes we obtain the result. □
is called $\mathbb{Z}/l$-acyclic if it induces an isomorphism on $H_{et}^*(\cdot, \mathbb{Z}/l)$. The main computational tool in étale cohomology theory is the Leray spectral sequence associated to a morphism of schemes $f: X \to Y$:

$$E_{2}^{ij} = H_{et}^i(X, R^j f_*(\mathbb{Z}/l)) \Rightarrow H_{et}^{i+j}(X, \mathbb{Z}/l).$$

The typical application of the Leray spectral sequence is the following criterion of the acyclicity of a morphism:

**Lemma 3.1.** Let $f: X \to Y$ be either proper or smooth. Suppose all fibres of $f$ are $\mathbb{Z}/l$-acyclic. Then $f$ is $\mathbb{Z}/l$-acyclic.

**Proof.** By the Proper (or Smooth) Base Change Theorem ([Mi], p. 224, p. 230) we may identify stalks of $R^j f_*(\mathbb{Z}/l)$ with $j$-th cohomology of fibres. Thus all rows except 0-th in the Leray spectral sequence disappear. Now it suffices to observe that since $f$ has connected fibre, then $R^0 f_*(\mathbb{Z}/l) = \mathbb{Z}/l$. □

If $f$ is locally trivial (i.e., locally a projection from a product) and $Y$ is smooth, then the Leray spectral sequence takes the form known from algebraic topology. We recall that a scheme is simply connected if it has trivial algebraic fundamental group (i.e., the group classifying étale coverings of a scheme).

**Lemma 3.2.** Suppose $f: X \to Y$ is locally trivial in the étale topology, and $Y$ is smooth, connected and simply connected. Denote by $F$ the fibre of $f$. Then the Leray spectral sequence associated with $f$ has the following form:

$$E_{2}^{ij} = H_{et}^i(X, H_{et}^j(F, \mathbb{Z}/l)) \Rightarrow H_{et}^{i+j}(X, \mathbb{Z}/l),$$

where $H_{et}^j(F, \mathbb{Z}/l)$ is a constant sheaf.

**Proof.** From the Smooth Base Change Theorem ([Mi], p. 230) we derive that if $f$ is a projection, then $R^j f_*(\mathbb{Z}/l)$ may be identified with a constant sheaf having a stalk $H_{et}^j(F, \mathbb{Z}/l)$, where $F$ is a fibre of $f$. So for $f$ locally trivial, the sheaf $R^j f_*(\mathbb{Z}/l)$ is locally trivial with a stalk $H_{et}^j(F, \mathbb{Z}/l)$. However, analogously to the classical context, a locally constant sheaf with finite stalks (we point out that $H_{et}^j(F, \mathbb{Z}/l)$ for smooth $F$ is always finite ([Mi], p. 244)) on a connected scheme $Y$ is determined by an action of algebraic fundamental group of $Y$ on a stalk ([Mi], p. 156). Thus for a simply connected scheme $Y$ a locally constant sheaf with finite stalks must be constant and the Leray spectral sequence has the required form. □
Another useful spectral sequence is the Hochschild-Serre spectral sequence corresponding to a Galois covering \( p : X \to Y \) with the structural group \( G \) ([Mi], p. 105):

\[
E_2^{ij} = H^i(G, H^j_{et}(X, \mathbb{Z}/l)) \Rightarrow H^{i+j}_{et}(Y, \mathbb{Z}/l),
\]

here \( H^i(G, -) \) denotes cohomology of the discrete group \( G \). Now we turn to the facts we need for the proof of Theorem 2.

**Proposition 3.3.** Let \( f : G \to H \) be an isogeny of connected affine algebraic groups. Then for almost all primes \( l \), \( f \) is \( \mathbb{Z}/l \)-acyclic.

**Proof.** Let \( K \) be the kernel of \( f \) regarded as a finite affine group scheme. Then it may be obtained as an extension: \( 1 \to K^0 \to K \to \Pi_0(K) \to 1 \), where \( K^0 \) is connected and \( \Pi_0(K) \) is étale ([Wa], p. 51). Hence we may factorize \( f \) as

\[
G \xrightarrow{f_c} G/K^0 \xrightarrow{f_{et}} (G/K^0)/\Pi_0(K) = H,
\]

where \( f_c \) has connected fibres and \( f_{et} \) is a Galois covering.

Since \( K^0 \) as a scheme is just a point (in general with multiplicity), fibres of \( f_c \) are \( \mathbb{Z}/l \)-acyclic. Moreover \( f_c \) being finite must be proper. Thus it satisfies the assumptions of Lemma 3.1, hence is \( \mathbb{Z}/l \)-acyclic for any \( l \).

Let us turn to \( f_{et} \) and take \( l \) prime to \( |\Pi_0(K)| \). Then the Hochschild-Serre spectral sequence corresponding to \( f_{et} \) degenerates and we get \( H^*_{et}(H, \mathbb{Z}/l) = (H^*_{et}((G/K^0), \mathbb{Z}/l))^{\Pi_0(K)} \), which gives us a monomorphism \( H^*_{et}(H, \mathbb{Z}/l) \to H^*_{et}((G/K^0), \mathbb{Z}/l) \). This monomorphism is compatible with \( f_{et}^* \) by the very construction of the spectral sequence. Now it remains to show that \( \Pi_0(K) \) acts trivially on \( H^*_{et}((G/K^0), \mathbb{Z}/l) \). But the action of \( \Pi_0(K) \) on \( G/K^0 \) extends to the action of the whole group \( G/K^0 \), which is connected. Thus the triviality of the action at the level of cohomology will follow if we show that any embedding \( \phi_g : G/K^0 \to G/K^0 \times G/K^0 \) defined by \( \phi_g(h) = (h, g) \) induces the same morphism on cohomology as \( \phi_g \). The last fact follows immediately from the Kunneth formula for \( G/K^0 \times G/K^0 \) ([SGA], p. 236), and the trivial observation that two constant morphisms into a connected scheme induce the same on étale cohomology. \( \square \)

**Proposition 3.4.** Let \( G \) be a simple group. Then for almost all primes \( l \), \( G \) is not \( \mathbb{Z}/l \)-acyclic.

**Proof.** First observe that \( G \) has finite algebraic fundamental group. Indeed, it suffices to show that there is only a finite number of étale coverings of \( G \). But this follows from the fact that any étale covering is étale (hence central) isogeny, while we have only a finite number of possibilities for centers of groups having root systems isomorphic to the root system of \( G \) ([Hu], p. 215), and all these centers are finite. Thus \( G \) has the universal covering being an étale isogeny. Therefore, according to Proposition 3.3, we may assume
that \( G \) is simply connected. Let \( B \subset G \) be a Borel subgroup. Then the sequence \( B \rightarrow G \rightarrow G/B \) is locally trivial in the étale topology \([\text{Se}]\), and \( G/B \) is simply connected (this follows from \( \Pi_1^{\text{alg}}(G) = 0 \) and connectivity of \( B \)). Then, by Lemma 3.2 we have:

\[
E_2^{ij} = H^i_{\text{et}}(G/B, H^j_{\text{et}}(B, \mathbb{Z}/l)) \Rightarrow H^{i+j}_{\text{et}}(G, \mathbb{Z}/l).
\]

Now take \( i_0 = \max\{i : H^i_{\text{et}}(G/B, \mathbb{Z}/l) \neq 0\} \). Since \( G/B \) is projective, it has the fundamental class ([Mi], pp. 247-252), so \( i_0 = 2 \cdot \dim(G/B) > 0 \). Similarly we take \( j_0 = \max\{j : H^j_{\text{et}}(B, \mathbb{Z}/l) \neq 0\} \). Then we have \( 0 \neq E_2^{i_0j_0} = E_\infty^{i_0j_0} \), hence \( H^{i_0+j_0}_{\text{et}}(G, \mathbb{Z}/l) \neq 0 \), so \( G \) is not \( \mathbb{Z}/l \)-acyclic.

\[ \square \]

4. The proof of Theorem 2.

The main ingredient in the proof is the following:

**Proposition 4.1.** Let \( f : G \rightarrow Q \) be an epimorphism of connected affine algebraic groups defined over an algebraically closed field \( k \), and assume that \( f \) factorizes through an affine space. Then \( Q \) is solvable.

**Proof.** Suppose \( Q \) is not solvable. Then we have an epimorphism \( p : Q \rightarrow S \) with simple \( S \), hence we may assume that \( Q \) is already simple. According to the acyclicity of \( \mathbb{A}^n \) ([Mi], p. 295) and Proposition 3.4 in order to obtain contradiction it suffices to show that \( f \) induces a monomorphism on étale cohomology with coefficients in \( \mathbb{Z}/l \) for almost all \( l \)’s.

Let \( R_u \) be the unipotent radical of \( G \). Then since \( R_u \subset \ker(f) \), we may factorize \( f \) as \( G \xrightarrow{g} G/R_u \xrightarrow{h} Q \). Let us first investigate \( g \). Observe that \( g \) is smooth and has acyclic fibres, since \( R_u \) as a variety is isomorphic to an affine space. Thus according to Lemma 3.1 it induces an isomorphism on étale cohomology for any \( l \) prime to the characteristic. So, we may focus on \( h \). Since \( G/R_u \) is reductive we have an isogeny \( i : T \times S_1 \times \cdots \times S_n \rightarrow G/R_u \), where \( T \) is a torus and each \( S_i \) is simple. We shall see that \((h \circ i)^* \) is a monomorphism on étale cohomology for some \( l \). Indeed, since \( Q \) is simple, there exists such \( j \) that \((h \circ i)|_{S_j} : S_j \rightarrow Q \) is also an isogeny. Hence 

\[ ((h \circ i)|_{S_j})^* \] 

is an isomorphism for almost all \( l \)’s, by Proposition 3.3. Thus 

\[ (h \circ i)^* \] (a fortiori \( h^* \)) is a monomorphism for these \( l \)’s. \[ \square \]

In fact we need the following, quite straightforward, generalization of the above proposition:

**Corollary 4.2.** Let \( K \) be any field, and \( f : G \rightarrow Q \) be an epimorphism of connected affine algebraic groups defined over \( K \). Suppose that \( f \) factors through an affine space. Then \( Q \) is unipotent.
Proof. Let $G_L = G \otimes L, Q_L = Q \otimes L, f_L = f \otimes L$, where $L$ is the algebraic closure of $K$. Then the epimorphism $f_L : G_L \rightarrow Q_L$ factors through an affine space, so by Proposition 4.1, $Q_L$ is solvable. Moreover, any connected solvable group is a semidirect product of its unipotent radical $R$ and some algebraic torus ([Hu], p. 123). But since $Q_L$ is an image of an affine space, it cannot be mapped onto a torus. Thus $Q_L$ is unipotent. It is also easy to see that $Q_L$ is unipotent iff $Q$ is (see e.g., [Wa], p. 64). This completes the proof. □

Now we are in a position to prove our main theorem. We recall that by Corollary 2.3 it suffices to prove Theorem 2.

Proof of Theorem 2. Let $G$ be a group scheme whose underlying scheme is $A^\infty$. By Lemma 2.2, $G$ is isomorphic to an inverse limit of a system $(G_i)$ of algebraic groups. It is enough to prove that $G_i$’s are unipotent. Fix a positive integer $n$. Then, like in the proof of Lemma 2.2, the isomorphism between $G$ and $\text{lim}(G_i)$ yields morphisms $f : G_N \rightarrow A^l$, and $g : A^l \rightarrow G_n$, for some $l$ and $N > n$. The composition $g \circ f$ is an epimorphism, since it comes from the identity map on $G$. From Corollary 4.2, $G_n$ is unipotent. □

References


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VERY AMPLE LINEAR SYSTEMS ON BLOWINGS-UP AT GENERAL POINTS OF SMOOTH PROJECTIVE VARIETIES

Marc Coppens

Let $X$ be a smooth projective variety, let $L$ be a very ample invertible sheaf on $X$ and assume $N + 1 = \dim(H^0(X, L))$, the dimension of the space of global sections of $L$. Let $P_1, \ldots, P_t$ be general points on $X$ and consider the blowing-up $\pi : Y \to X$ of $X$ at those points. Let $E_i = \pi^{-1}(P_i)$ be the exceptional divisors of this blowing-up. Consider the invertible sheaf $M := \pi^*(L) \otimes O_Y(-E_1 - \ldots - E_t)$ on $Y$. In case $t \leq N + 1$, the space of global section $H^0(Y, M)$ has dimension $N + 1 - t$. In case this dimension $N + 1 - t$ is at least equal to $2 \dim(X) + 2$, hence $t \leq N - 2 \dim(X) - 1$, it is natural to ask for conditions implying $M$ is very ample on $Y$ (this bound comes from the fact that “most” smooth varieties of dimension $n$ cannot be embedded in a projective space of dimension at most $2n$). For the projective plane $\mathbb{P}^2$ this problem is solved by J. d’Almeida and A. Hirschowitz. The main theorem of this paper is a generalization of their result to the case of arbitrary smooth projective varieties under the following condition. Assume $L = L' \otimes k$ for some $k \geq 3 \dim(X) + 1$ with $L'$ a very ample invertible sheaf on $X$: If $t \leq N - 2 \dim(X) - 1$ then $M$ is very ample on $Y$. Using the same method of proof we obtain very sharp result for $K3$-surface and let $L$ be a very ample invertible sheaf on $X$ satisfying $\text{Cliff}(L) \geq 3$ (“most” invertible sheaves on $X$ satisfy that property on the Clifford index), then $M$ is very ample if $t \leq N - 5$. Examples show that the condition on the Clifford index cannot be omitted.

0. Introduction.

0.1. Let $X \subset \mathbb{P}^N$ be a smooth projective variety of dimension $n$ defined over an algebraically closed field of characteristic 0 and assume $N \geq 2n + 2$. Let $\Lambda$ be a general linear subspace of dimension $N - (2n + 2)$ in $\mathbb{P}^N$. The projection with center $\Lambda$ induces an embedding $X \subset \mathbb{P}^{2n+1}$. In general this is the best one can hope for (i.e., projecting $X$ to $\mathbb{P}^{2n}$ in general one expects that the image of $X$ is not isomorphic to $X$). This embedding of $X$ in $\mathbb{P}^{2n+1}$ is described by a linear system associated to the same invertible
sheaf as the original embedding $X \subset \mathbb{P}^N$. In particular the linear system is not complete, i.e., the embedding $X \subset \mathbb{P}^{2n+1}$ is not linearly normal. An embedding is called linearly normal if its hyperplane sections give rise to a complete linear system.

0.2. Starting with a linearly normal embedding $X \subset \mathbb{P}^N$ one can try to find some linearly normal embedding in $\mathbb{P}^{2n+1}$ using projections from general points $P_1; \ldots ; P_{N-(2n+1)}$ on $X$. However, in order to define this morphism to $\mathbb{P}^{2n+1}$ one has to blow-up $X$ at $P_1; \ldots ; P_{N-(2n+1)}$. Let $Y$ be the blowing-up. One can hope to obtain an embedding $Y \subset \mathbb{P}^{2n+1}$ in this way. This would be linearly normal.

It is easy to find examples showing that this morphism $Y \rightarrow \mathbb{P}^{2n+1}$ is not always an embedding. As an example, let $X$ be a ruled subvariety of $\mathbb{P}^N$. At the end of the paper, using embeddings of some special types of $K3$-surfaces, we obtain some less trivial examples.

The obstruction for the morphism $Y \rightarrow \mathbb{P}^{2n+1}$ to be an embedding occurs in the following situation. A general set of $N - (2n + 1)$ points of $X$ is contained in a 0-dimensional subscheme $Z$ of $X$ of length $N - 2n + 1$ such that $Z$ is a subscheme of a linear subspace $V$ of $\mathbb{P}^N$ of dimension $N - 2n - 1$ (hence $Z$ imposes at most $N - 2n$ conditions on hyperplanes in $\mathbb{P}^N$). In that case we say that $V$ is a $(N - 2n + 1)$-secant $(N - 2n - 1)$-space for $X \subset \mathbb{P}^N$. So, the obstruction can be the existence of many such secant spaces. The main results of this paper (in particular their proofs) indicate that limited knowledge of postulation of points on curve sections of $X$ (i.e., 1-dimensional intersections of $X$ with hyperplanes of codimension $n - 1$ in $\mathbb{P}^N$) gives strong information on this question. (The meaning of “limited” is the following. In principle we need knowledge on 0-dimensional subschemes of length $N - 2n + 1$. It turns out that knowledge on 0-dimensional subschemes of length $3n + 2$ is sufficient.)

0.3. The problem considered in this paper can also be described as follows. Let $L$ be the invertible sheaf associated to some linearly normal embedding $X \subset \mathbb{P}^N$. Let $P_1; \ldots ; P_{N-(2n+1)}$ be general points on $X$. Let $\pi : Y \rightarrow X$ be the blowing-up of $X$ at those points $P_i$ and let $E_i = \pi^{-1}(P_i)$. We study very ampleness of $M := \pi^*(L) \otimes O_Y(-E_1 - \ldots - E_{N-(2n+1)})$ on $Y$.

0.4. The main result of this paper is the following.

**Theorem 1.** Let $X$ be a smooth projective variety of dimension $n$; let $L'$ be a very ample invertible sheaf on $X$ and let $L = L'^{\otimes t}$ for some $t \geq 3n + 1$. Let $N + 1 = \dim(\Gamma(X; L))$ and let $P_1; \ldots ; P_{N-(2n+1)}$ be general points on $X$. Let $\pi : Y \rightarrow X$ be the blowing-up at $P_1; \ldots ; P_{N-(2n+1)}$ and for $1 \leq i \leq N-(2n+1)$ let $E_i = \pi^{-1}(P_i)$. Then $M := \pi^*(L) \otimes O_Y(-E_1 - \ldots - E_{N-(2n+1)})$ is very ample on $Y$; i.e., it defines a linearly normal embedding $Y \subset \mathbb{P}^{2n+1}$. 
This theorem solves Conjecture 2 of my paper [7]. In case $X = \mathbb{P}^n$ and $L' = O_{\mathbb{P}^n}(1)$ it also gives information on Conjecture 1 of that paper. For sure, using more involved arguments, the lower bound on $t$ can be made better. Finding the best bound (or a better bound) on $t$ is interesting; in this paper I preferred to restrict to the development of a general method to prove very ampleness for the type of situation considered in this paper.

0.5. For special types of varieties the method of the proof of Theorem 1 can be used to give much better results. As an example we discuss the case of $K3$-surfaces. Let $X$ be a smooth $K3$-surface and let $L$ be a very ample invertible sheaf of $X$. As explained in the first part of part 2 of this paper we have a notion of the Clifford index $\text{Cliff}(L)$ of $L$. We prove the following theorem.

**Theorem 2.** Let $X$ be a smooth $K3$-surface and let $L$ be a very ample invertible sheaf of Clifford index $\text{Cliff}(L) \geq 3$. Let $g + 1 = \dim(\Gamma(X; L))$ and let $P_1; \ldots; P_9$ be general points on $X$; let $\pi : Y \to X$ be the blowing-up of $X$ at $P_1; \ldots; P_9$; let $E_i = \pi^{-1}(P_i)$ and let $M := \pi^*(L) \otimes O_Y(-E_1 - \ldots - E_9)$. Then $M$ is a very ample invertible sheaf on $Y$, in particular we obtain a linearly normal embedding $Y \subset \mathbb{P}^5$.

Examples show that the condition on $\text{Cliff}(L)$ in the statement cannot be omitted. Lazarsfeld proved that, in case $\text{Pic}(X) \cong \mathbb{Z}$ and $L$ is the ample generator of $\text{Pic}(X)$, then the curve $C$ corresponding to a general section of $L$ satisfies the results from Brill-Noether theory for linear systems on $C$ (see [15]). Hence in this case, if $g \geq 7$, one has $\text{Cliff}(L) \geq 3$ and Theorem 2 can be applied. For each value of $g$ there exist such $K3$-surfaces and they are “general”, this follows from the description of the moduli space of $K3$-surfaces using the Torelli map (see e.g., [3]).

0.6. Very ample linear systems on blowings-up of the projective plane are intensively studied, see e.g., [13]. In particular, the problem considered in this paper started with the paper of d’Almeida and Hirschowitz (see [9]) solving the problem for $\mathbb{P}^2$. Recently, De Volder and Chauvin considered similar problems on blowings-up of $\mathbb{P}^2$ admitting multiplicities for the exceptional divisors (see [4]). Also recently, T. Szemberg and H. Tutaj-Gasinski studied very ampleness of blowings-up of surfaces using Seshadri constants (see [20]).

**Notations and conventions.**

In this paper all varieties are defined over a fixed algebraically closed field of characteristic 0; see (1.4) for a discussion with respect to the proof of Theorem 1. A 0-dimensional subscheme $Z$ of a smooth variety $X$ is called *curvilinear* if there exists a smooth curve $C \subset X$ with $Z \subset C$. This is
equivalent to $\dim(T_P(Z)) \leq 1$ for all $P \in Z$. Let $X \subset \mathbb{P}^N$ be a smooth $n$-dimensional projective variety. The embedding is called linearly normal if its hyperplane sections give rise to a complete linear system. A curve section (resp. surface section) of $X$ is the scheme-theoretic intersection of $X$ with a linear subspace $\Lambda$ of codimension $n - 1$ (resp. $n - 2$) such that $X \cap \Lambda$ has dimension 1 (resp. 2).

$V_{e-f}^e(h)$: Set of $e$-secant $(e - f - 1)$-space divisors of a linear system $h$ on a smooth curve (see (1.1.2)).

$\mathbf{P}$: Set of general points on $X \subset \mathbb{P}^N$.

$P$: Linear span of $\mathbf{P}$.

$T_P(X); T_P(Z); \ldots$: Zariski tangent space.

$\langle Z \rangle$: Linear span of some subscheme $Z \subset \mathbb{P}^N$; this is the intersection of all hyperplanes in $\mathbb{P}^N$ containing $Z$ (and it is $\mathbb{P}^N$ if $Z$ is not contained in some hyperplane of $\mathbb{P}^N$).

Hilb$^k(X)$: Hilbert scheme of 0-dimensional subschemes of length $k$ on $X$.

$G(k, N)$: Grassmannian of $k$-planes in $\mathbb{P}^N$.

$(\mathbb{P}^N)^*$: The dual projective space (hence it is $G(N - 1, N)$).

$D_s$: Divisor on $Y$ associated to $s \in \Gamma(X; L \otimes I_P)$ (see (1.4)).

Cliff$(L)$ (resp. $g(L)$): Clifford index (resp. genus) of a very ample invertible sheaf $L$ on a K3-surface $X$ (see (2.1)).

$W^r_d(C)$: Subspace of the Jacobian of a smooth curve $C$ parametrizing invertible sheaves $L$ on $C$ of degree $d$ satisfying $h^0(L) \geq r + 1$.


1.1.1. Let $X$ be a smooth projective variety defined over an algebraically closed field of characteristic 0 and embedded in some projective space $\mathbb{P}^N$. Let $P_1, \ldots, P_k$ be $k$ general points on $X$. In case there exist two more different points $Q_1, Q_2$ on $X$ such that the $k + 2$ points $P_1, \ldots, P_k, Q_1, Q_2$ belong to some $(k + 2)$-secant $k$-space for $X \subset \mathbb{P}^N$ then the projection to $\mathbb{P}^{N-k}$ with center $P_1, \ldots, P_k$ does not give rise to an embedding into $\mathbb{P}^{N-k}$ of the blowing-up of $X$ at those points. In that case there exists a family of $(k+2)$-secant $k$-spaces for $X \subset \mathbb{P}^N$ of large dimension. Classifying varieties according to the existence of such family of secant spaces is a very hard problem in general. Some papers related to similar types of questions are [18] and [19]. The benefit of using the $t$-th Veronese embeddings of some original embedding of $X$ is as follows: For the new embedding of $X$ every 0-dimensional subscheme of $X$ of length $t + 1$ imposes independent conditions on hyperplanes (this can be easily seen using hyperplanes in the
original embedding). However the dimension of the ambient projective space grows by taking such a Veronese embedding, hence for our purpose this does not look very suitable. The main ingredient in the proof of Theorem 1 solves this problem. This ingredient is Proposition 1.2 in my paper [7], which is an application of the results from my paper [8]. Before restating that proposition (see (1.1.3)) let me recall some notations and definitions.

1.1.2. . Let $C$ be a smooth complete connected curve of genus $g$. Let $h$ be a linear system on $C$, let $E$ be an effective divisor of degree $e$ on $C$. We say that $E$ is an $e$-secant $(e - f - 1)$-space divisor for $C$ if $\dim(\{D \in h | D \text{ contains } E\}) \geq \dim(h) - e + f$. In order to understand this notion, assume $h$ is a very ample linear system on $C$, hence it corresponds to some embedding of $C$ in $\mathbb{P}^r$. Then the linear span of $E$, i.e., the intersection of the hyperplanes in $\mathbb{P}^r$ containing $E$ (as a scheme) has dimension at most $e - f - 1$.

We use the following notation: $V_e^{e-f}(h)$ is the set of $e$-secant $(e - f - 1)$-space divisors for $h$. It is a closed subset of the $e$-th symmetric product of $C$.

1.1.3. . Proposition (see [7], Proposition 1.2). Let $e \leq \dim(h)$ and let $V$ be an irreducible component of $V_e^{e-f}(h)$. Assume that for a general point $E$ of $V$, if $Q \in E$, then $E - Q \not\in V_{e-1}^{e-f-1}(h)$. Then:

a) $3 \dim(V) \leq 2e - 1$ if $2 \dim(V) \leq \dim(h) + 1$,
b) $\dim(V) \leq 2e - 2 - \dim(h)$ if $2 \dim(V) \geq \dim(h) + 1$.

This proposition will be used in the proof as follows: In case there exists, for some large value of $k$, a lot of $(k + 2)$-secant $k$-planes such that the projections do not give rise to an embedding of the blowing-up of $X$ (as explained in (1.1.1)), then this already occurs for small values of $k$. Then the benefit of the Veronese embedding as explained in (1.1.1) becomes clear.

1.2. . Main lemma. Consider $X \subset \mathbb{P}^N$ as in the statement of Theorem 1. Let $a \geq 0$ be an integer at most $N - (2n + 1)$. For general points $P_1, \ldots, P_a$ on $X$ there exists no curvilinear subscheme $Z$ of $X$ of length $a + 2$ containing $P_1, \ldots, P_a$ such that $\dim(\Gamma(X, L \otimes I_Z)) \geq N - a$. (The inequality $\dim(\Gamma(X, L \otimes I_Z)) \geq N - a$ would imply that $Z$ is contained in some $(a + 2)$-secant $a$-space.)

For the reader’s convenience first I give a small survey of the proof of the main lemma. We use induction on $a$. Assume for general points $P_1, \ldots, P_a$ on $X$ there exists a curvilinear subscheme $Z$ of length $a + 2$ containing those points with $\dim(Z) = a$. We consider the intersection $(Z) \cap X$. 
First in (1.2.1) we prove that, a general curve section containing \( \langle Z \rangle \cap X \) is not a smooth curve on \( X \). Then in (1.2.3) we prove that this implies that \( \langle Z \rangle \cap X \) contains a curve \( \Lambda \) on \( X \). We explain that \( \Lambda \) has to be a rational normal curve, hence \( X \) contains many rational normal curves. In case \( X \) is a surface then the arguments in (1.2.4.2) give a contradiction. In (1.2.4.1) we prove that there exist suitable surface sections for \( X \), reducing the general case to the surface case.

**Proof.** We are going to use induction on \( a \). Since \( L \) is very ample on \( X \) the case \( a = 0 \) is trivial. During the proof, we also point out that the case \( a = 1 \) is proved without using the induction hypothesis. We write \( P \) to denote the set of points \( \{ P_1, \ldots, P_a \} \) (we also consider it as a reduced scheme). Assume that there exists a curvilinear subscheme \( Z \) of \( X \) of degree \( a + 2 \) containing \( P \) such that \( \dim(\Gamma(X, L \otimes I_Z)) \geq N - a \). This means, \( \dim(\langle Z \rangle) \leq a \). Let \( P \) be the linear span of \( P \). Since \( a \leq N - (2n + 1) \) and the points \( P_1, \ldots, P_a \) are general on \( X \), the dimension of \( P \) is \( a - 1 \) and \( P \cap X = P \) as a scheme. This can be seen using the general position lemma (see e.g., [1], p. 109) using a general curve section of \( X \) containing \( P \) (we need \( a \leq \deg(X) \) but this inequality holds because \( \deg(X) \geq N - n + 1 \) (see e.g., [11], Proposition 0) and we assumed \( a \leq N - 2n - 1 \). In particular \( \dim(\langle Z \rangle) \) cannot be less than \( a \), hence \( \dim(\langle Z \rangle) = a \).

Let \( T' \subset \text{Hilb}^{a+2}(X) \times X^a \) be the closure of the set of points \( (Z; P_1, \ldots, P_a) \) with \( Z \) a curvilinear subscheme of \( X \) of degree \( a + 2 \) satisfying \( \dim(\Gamma(X, L \otimes I_Z)) \geq N - a \) and containing the points \( P_1, \ldots, P_a \) with \( P_i \neq P_j \) for \( i \neq j \). Let \( T \) be an irreducible component of \( T' \) dominating \( X^a \) (such a component exists, that’s the assumption). Then \( \dim(T) \geq an \). Consider \( I \subset T \times G(N - n + 1, N) \) with \( (Z; P_1, \ldots, P_a; \Lambda) ) \in I \) if and only if \( \langle Z \rangle \subset \Lambda \). The fibers of the projection \( I \rightarrow T \) have dimension \( (n - 1)(N - n + 1 - a) \), hence \( \dim(I) \geq an + (n-1)(N-n+1-a) \). Consider the projection \( \tau: I \rightarrow G(N-n+1,N) \).

1.2.1. .

**Subclaim.** For \( \Lambda \in \tau(I) \) general the intersection \( \Lambda \cap X \) (as a scheme) is not a smooth curve. (Notice that \( \dim(\Lambda \cap X) \geq 1 \).)

Assume for some \( \Lambda \in \tau(I) \) the intersection \( \Lambda \cap X \) is a smooth curve (call it \( C \)). The embedding \( C \subset \Lambda \) corresponds to a linear system \( g \) on \( C \) of dimension \( N - n + 1 \). Elements of \( \tau^{-1}(\Lambda) \) correspond to effective divisors \( Z \) on \( C \) of degree \( a + 2 \) such that \( \{ D \in g : D - Z \geq 0 \} \) has dimension at least \( N - n - a \), i.e., \( Z \in V_{a+2}^a(g) \).

Since \( \dim[G(N-n+1,N)] = (N-n+2)(n-1) \) one has \( \dim(\tau^{-1}(\Lambda)) \geq an + (n-1)(N-n+1-a) - (n-1)(N-n+2) = (a+2) - n - 1 \). Let \( e \leq a+2 \leq N-2n+1 \) be the integer defined as follows. For a general element of \( \tau^{-1}(\Lambda) \) the divisor \( Z \) contains a subdivisor \( E \) of degree \( e \) not imposing
independent conditions on the linear system $g$ (hence $E \in V^e_{-1}(g)$) while each subdivisor $E'$ of degree $e - 1$ imposes independent conditions on $g$. This implies $Z = E + P_1 + \cdots + P_{a+2-e}$ for some points $P_1, \ldots, P_{a+2-e}$ on $C$. Since $Z$ is general in a subvariety of dimension at least $(a + 2) - n - 1$ of $V^e_{a+2}(g)$, the subdivisor $E$ is general in a subvariety of $V^e_{e-1}(g)$ of dimension at least $e - n - 1$. So, for some $e \leq a + 2 \leq N - 2n + 1$ there exists some component $V$ of $V^e_{e-1}(g)$ with $\dim(V) = e - n - 1 + t$ for some integer $t \geq 0$ and for $E \in V$ general and $P \in E$ one has $E - P \notin V^e_{e-2}(g)$.

Now we use Proposition (1.1.3). The condition $e \leq \dim(g) = N - n + 1$ holds. In case $2 \dim(V) \geq N - n + 2$ we find $e - n - 1 + t \leq 2e - 2 - (N - n + 1)$ hence $e \geq N - 2n + 2$, a contradiction. (Notice: Here we use the condition $a \leq N - (2n + 1)$; we should use that condition because in general the statement of Theorem 1 is sharp with respect to the upper bound on $a$.) So we are in case $2 \dim(V) \leq N - n + 2$ and we find $3(e - n - 1 + t) \leq 2e - 1$, i.e., $e \leq 3n + 2 - 3t \leq 3n + 2$, hence $\dim(V^e_{3n+1}(g)) \geq 3n + 2 - n - 1 = 2n + 1$. But, because $L = L^{\otimes t}$ we find that $g$ contains all sums of $t$ divisors from another very ample linear system on $C$. Since $t \geq 3n + 1$, all effective divisors on $C$ of degree $3n + 2$ impose independent conditions on $g$. This gives a contradiction, proving the subclaim.

1.2.2. .

**Remark.** We only used the assumption $L = L^{\otimes t}$ at the end of the proof of Subclaim (1.2.1). In general we proved the following lemma.

**Lemma.** Let $X \subset \mathbb{P}^N$ be a smooth $n$-dimensional variety. Let $a \leq N - 2n - 1$ be an integer. Assume for $P = \{P_1; \ldots; P_a\}$ a set of general points on $X$ there exists a curvilinear subscheme $Z \subset X$ of length $a + 2$ containing $P$ such that $\dim(\langle Z \rangle) = a$. Let $\Lambda \subset \mathbb{P}^N$ be a general linear subspace of dimension $N - n + 1$ containing $\langle Z \rangle$ and assume $\Lambda \cap X$ is a smooth curve $C$ on $X$. Let $g$ be the linear system on $C$ corresponding to the embedding $C \subset \Lambda$. Then $V^e_{3n+1}(g)$ has an irreducible component of dimension at least $2n + 1$.

1.2.3. .

**Subclaim.** For $(Z; P_1, \ldots, P_a) \in T$ general the intersection $\langle Z \rangle \cap X$ is not finite.

Take $(Z; P_1, \ldots, P_a) \in T$ general and assume $\langle Z \rangle \cap X$ is finite. We know that $Z$ contains $P$, a set of a general points on $X$. Since $P \cap X = \mathbb{P}$ as a scheme and $\dim(\langle Z \rangle) = a$, we know that $P$ is a hyperplane in $\langle Z \rangle$.

1.2.3.1. Assume $\langle Z \rangle \cap X$ would be curvilinear. Consider a general $\Lambda \in G(N - n + 1, N)$ containing $\langle Z \rangle$. Because of Bertini’s Theorem, singular points of $\Lambda \cap X$ belong to $\langle Z \rangle \cap X$. For all $z \in \langle Z \rangle \cap X$ and $\Lambda \supset \langle Z \rangle$ general
one has \( \dim(\Lambda \cap T_q(X)) \leq 1 \). Indeed: \( \dim(T_q(X)(\langle Z \rangle)) \leq 1 \) because we assume \( X \cap \langle Z \rangle \) is curvilinear and \( \dim(T_q(X)) = n \), hence \( \{ \Lambda \in G(N - n + 1, N) : \Lambda \supset \langle Z \rangle \} \) has dimension at most \( (N - n + 1 - a)(n - 1) - 1 \), while \( \{ \Lambda \in G(N - n + 1, N) : \Lambda \supset \langle Z \rangle \} \) has dimension \( (N - n + 1 - a)(n - 1) \). Since \( \langle Z \rangle \cap X \) is finite it follows that \( \Lambda \cap X \) is smooth for \( \Lambda \in G(N - n + 1, N) \) general and \( \Lambda \supset \langle Z \rangle \). This contradicts Subclaim (1.2.1).

1.2.3.2. So there exists some point \( Q \in \langle Z \rangle \cap X \) such that \( \dim(\langle Z \rangle \cap T_Q(X)) \geq 2 \). Notice that this cannot happen if \( a = 1 \). Since \( P \) is a hyperplane in \( \langle Z \rangle \) also \( \dim(P \cap T_Q(X)) \geq 1 \). In particular since \( P \cap X \) is reduced, one finds \( Q \notin P \). Let \( P' = \langle P_1; \ldots; P_{a-1} \rangle \). Since \( P' \) is a hyperplane in \( P \) one also finds \( P' \cap T_Q(X) \neq \emptyset \). Let \( W = \{ P_1; \ldots; P_{a-1}; Q \} \), then \( \dim((W) \cap T_Q(X)) \geq 1 \), hence \( (W) \cap X \) contains some length 2 subscheme \( Z_Q \) with support at \( Q \); its union with \( \{ P_1; \ldots; P_{a-1} \} \) is a curvilinear subscheme \( Z' \) of \( X \) of length \( a + 1 \). Since \( \langle Z' \rangle = \langle W \rangle \) we find that \( \dim(\langle Z' \rangle) = a - 1 \). Now we use the induction hypothesis of the main lemma, this gives a contradiction. This finishes the proof of Subclaim (1.2.3).

1.2.4. So we find \( \dim(\langle Z \rangle \cap X) \geq 1 \). Since \( P \) is a hyperplane in \( \langle Z \rangle \) and \( \dim(P \cap X) = 0 \) we find that, for \( \langle Z; P_1, \ldots, P_a \rangle \in T \) general, \( \dim(\langle Z \rangle \cap X) = 1 \). Let \( \Gamma \) be a 1-dimensional irreducible component of \( \langle Z \rangle \cap X \). Then \( \Gamma \cap P \) is a hyperplane section of \( \Gamma \subset \langle \Gamma \rangle \) and it consists of \( b \) points of \( \{ P_1; \ldots; P_a \} \). Those points are independent (say they are \( P_1; \ldots; P_b \)) hence \( \dim((\Gamma \cap P)) = b - 1 \), so \( \dim(\langle \Gamma \rangle) = b \) and \( \Gamma \) has degree \( b \). So \( \Gamma \) is a rational normal curve of degree \( b \). But \( \Gamma \) is embedded by a linear system that contains all sums of \( t \) divisors from another linear system on \( \Gamma \), hence \( b \geq t \geq 3n + 1 \). Since \( a \geq b \) we find a contradiction if \( a = 1 \); hence the main lemma is proved in case \( a = 1 \). So we find: For \( b \geq 3n + 1 \) general points \( P_1; \ldots; P_b \) on \( X \) there is a rational normal curve \( \Gamma \) of degree \( b \) on \( X \subset P \) containing those points. In case \( b < a \) then for any \( b \) general points \( P_1; \ldots; P_b \) on \( X \) we find a curvilinear subscheme \( Z' \) of length \( b + 2 \) containing \( \{ P_1; \ldots; P_b \} \) with \( \dim(\langle Z' \rangle) = b \), a contradiction to the induction hypothesis of the main lemma, hence \( b = a \). Let \( C \) be the space parametrizing such curves \( \Gamma \) and let \( I_1 \subset C \times X^a \) be the set of points \( (\Gamma; P_1, \ldots, P_a) \) with \( P_i \in \Gamma \). Let \( I_1 \) be an irreducible component of \( I_1 \) dominating \( X^a \). We obtain \( \dim(I_1) \geq na \). Also the projection \( I_1 \to C \) has fibers of dimension \( a \). Let \( U \subset C \) be the image of \( I_1 \) then \( \dim(U) \geq a(n - 1) \).

1.2.4.1. Subclaim. Let \( \Gamma \) be a curve corresponding to a general element of \( U \). There exists \( \Lambda \in G(N - n + 2; N) \) such that \( X \cap \Lambda \) is a smooth surface containing \( \Gamma \).

Proof. Of course, in case \( n = 2 \), there is nothing to prove, so assume \( n > 2 \). Let \( G(\Gamma) = \{ \Lambda \in G(N - n + 2, N) : \Lambda \subset \langle \Gamma \rangle \} \). Since \( \dim(\langle \Gamma \rangle) = a \), we find
dim(G(\Gamma)) = (N - n + 2 - a)(n - 2). Assume there exists Q \in \langle \Gamma \rangle \cap X with dim(T_Q(X) \cap \langle \Gamma \rangle) \geq 2. Then one finds a contradiction to the induction hypothesis of the main lemma as in (1.2.3.2). So for each Q \in \langle \Gamma \rangle \cap X we find dim(T_Q(X) \cap \langle \Gamma \rangle) \leq 1. Because of Bertini’s Theorem, for \Lambda \in G(\Gamma) general we find Sing(\Lambda \cap X) \subset \langle \Gamma \rangle \cap X. If \Lambda \cap X would be singular at Q \in \langle \Gamma \rangle \cap X then dim(\Lambda \cap T_Q(X)) \geq 3. Consider G(\Gamma; Q) = \{\Lambda \in G(\Gamma) : dim(\Lambda \cap T_Q(X)) \geq 3\}. Then dim[G(\Gamma; Q)] \leq (n - 2)(N - n - a + 2) - 2 since dim(\langle \Gamma \rangle \cap T_Q(X)) \leq 1. It follows that the union of G(\Gamma; Q) for all \Lambda \in G(\Gamma) general \Lambda \cap X = S is a smooth surface. This finishes the proof of Subclaim (1.2.4.1).

1.2.4.2. Now, consider I_2 \subset G(N - n + 2, N) \times U defined by (\Lambda; \Gamma) \in I_2 if and only if \Lambda \supseteq \langle \Gamma \rangle. The fiber of the projection morphism I_2 \to U for a general \Gamma \in U is G(\Gamma), hence dim(I_2) \geq a(n - 1) + (n - 2)(N - n + 2 - a). Consider the projection morphism \nu : I_2 \to G(N - n + 2; N). Take \Lambda general in the image of \nu. We just proved that \Lambda \cap X is a smooth surface S. We find dim(\nu^{-1}(\Lambda)) \geq a(n - 1) + (n - 2)(N - n + 2 - a) - (N - n + 3)(n - 2) = a - n + 2 \geq 2n + 3. Hence S is smooth surface containing a family of rational curves of dimension at least 2n + 3, in particular each 2 points on S contain a rational curve. This implies h^1(O_S) = 0, hence those rational curves are linearly equivalent.

Take a general hyperplane section C of S \subset \Lambda; it is a smooth curve not containing any of those rational curves. Hence those rational curves induce a linear system h on C of dimension at least a - n + 2. On the other hand the degree of that linear system is a (a general hyperplane section intersects such rational normal curve at a points). Since a - 2(a - n + 2) = 2n - a - 4 \leq 2n - 3n - 1 - 4 = -n - 5 < 0, we find that the linear system h is non-special on C (Clifford’s Theorem, see e.g., [1], p. 107). So g(C) \leq n - 2. The linear system on S defining the embedding S \subset \Lambda contains the sum of t divisors from another very ample linear system on S, hence C is linearly equivalent to the sum of t mutually intersecting smooth curves on S. Such a sum has arithmetic genus at least (t - 1)(t - 2)/2, hence g(C) \geq (t - 1)(t - 2)/2 \geq 3n(3n - 1)/2. This can be proved using the adjunction formula: \( (C + K_S.C) = g(C) - 2 \) while C is linearly equivalent to \( C_1 + \cdots + C_t \) with \( (C_j.C_j) > 0 \) and \( (C_i + K_S.C_i) \geq -2 \). So, we find \( 2n - 4 \geq 9n^2 - 3n \), hence \( 0 \geq 9n^2 - 5n + 4 \). This is a contradiction. This finally finishes the proof of the main lemma.

1.3. Proof of Theorem 1. We continue to use the notation P and P introduced in the proof of the Main Lemma with \( a = N - (2n + 1) \). A section \( s \in \Gamma(X, L \otimes I_P) \) will be identified with \( s \in \Gamma(Y, M) \). We write \( D_s \) to denote the divisor on Y. We write E for the union of the exceptional divisors \( E_1, \ldots, E_{N-(2n+1)} \) and we identify a point Q on \( Y \setminus E \) with the corresponding point Q on \( X \setminus P \).
1.3.1. First we prove base-point freeness. Let \( Q \in Y \). If \( Q \notin E \), choose \( H \in (\mathbf{P}^N)^* \) with \( H \supset P \) and \( Q \notin H \). (Remember \( P \cap X = \mathbf{P} \) as a scheme.) This hyperplane \( H \) defines \( s \in \Gamma(X, L \otimes I_P) \) with \( Q \notin D_s \).

Assume \( Q \in E_1 \). This corresponds to a tangent line \( T_Q \) to \( X \) at \( P_1 \) in \( \mathbf{P}^N \). Since \( T_Q \) is not contained in \( P \) we can find \( H \in (\mathbf{P}^N)^* \) with \( P \subset H \) but \( T_Q \not\subset H \). Then \( H \) corresponds to \( s \in \Gamma(X, L \otimes I_P) \) with \( Q \notin D_s \).

1.3.2. Next we prove separation of points. Take \( Q_1; Q_2 \), two different points on \( Y \). First assume \( Q_1 \) and \( Q_2 \) are outside of \( E \). Then \( Z = \mathbf{P} \cup \{Q_1; Q_2\} \) is a curvilinear subscheme of length \( N - 2n + 1 \) containing \( \mathbf{P} \).

Because of the main lemma we find \( \dim((Z)) = \dim P + 2 \), hence there exists \( H \in (\mathbf{P}^N)^* \) with \( P \cup \{Q_1\} \subset H \) but \( Q_2 \notin H \). Then \( H \) defines \( s \in \Gamma(X, L \otimes I_P) \) with \( Q_1 \in D_s \) but \( Q_2 \notin D_s \).

Assume \( Q_1 \in E_1 \), but \( Q_2 \notin E \). The point \( Q_1 \) defines a tangent direction to \( X \) at \( P_1 \), hence it defines a 0-dimensional subscheme \( Z_{Q_1} \) of length 2 with support \( P_1 \). Let \( Z = \mathbf{P} \cup Z_{Q_1} \cup \{Q_2\} \). It is a curvilinear subscheme of length \( N - 2n + 1 \) containing \( \mathbf{P} \), hence \( \dim((Z)) = \dim(P) + 2 \). This implies that there exists \( H \in (\mathbf{P}^N)^* \) with \( H \supset P \cup Z_{Q_1} \) but \( Q_2 \notin H \). Then \( H \) defines \( s \in \Gamma(X, L \otimes I_P) \) with \( Q_1 \in D_s \) but \( Q_2 \notin D_s \).

Next assume \( Q_1 \) and \( Q_2 \) both belong to \( E_1 \). Let \( T_1 \) and \( T_2 \) be the corresponding tangent lines to \( X \subset \mathbf{P}^N \). Since \( P \cap X = \mathbf{P} \) as a scheme, one finds \( \dim((P \cup T_1 \cup T_2)) = \dim P + 2 \). Hence there exists \( H \in (\mathbf{P}^N)^* \) such that \( H \supset P \) with \( T_1 \subset H \) and \( T_2 \not\subset H \). Then \( H \) defines \( s \in \Gamma(X, L \otimes I_P) \) with \( Q_1 \in D_s \) but \( Q_2 \notin D_s \).

Finally assume \( Q_1 \in E_1 \) and \( Q_2 \in E_2 \). Let \( Z_1 \) and \( Z_2 \) be the curvilinear subschemes of length 2 on \( X \) with support at \( P_1 \) and \( P_2 \) corresponding to those points. Consider \( Z = Z_1 \cup Z_2 \cup \mathbf{P} \). It is a curvilinear subscheme of length \( N - 2n + 1 \) containing \( \mathbf{P} \) hence the main lemma implies \( \dim((Z)) = \dim(P) + 2 \). So there exists \( H \in (\mathbf{P}^N)^* \) with \( P \cup Z_1 \subset H \) but \( Z_2 \not\subset H \). Then \( H \) defines \( s \in \Gamma(X, L \otimes I_P) \) with \( Q_1 \in D_s \) but \( Q_2 \notin D_s \).

1.3.3. Finally we prove separation of tangent directions. Let \( Q \in Y \) and let \( v \in T_Q(Y) \). First assume \( Q \notin E \). Then \( v \) corresponds to a subscheme \( Z_v \) of length 2 of \( X \) with support \( Q \). Let \( Z = Z_v \cup \mathbf{P} \). It is a curvilinear subscheme of \( X \) of length \( N - 2n + 1 \) containing \( \mathbf{P} \). The main lemma implies that \( \dim((Z)) = \dim P + 2 \). Hence there exists \( H \in (\mathbf{P}^N)^* \) with \( P \cup \{Q\} \subset H \) but \( Z_v \not\subset H \). Then \( H \) defines \( s \in \Gamma(X, L \otimes I_P) \) with \( Q \in D_s \) but \( v \notin T_Q(D_s) \).

Assume that \( Q \in E_1 \). First assume \( v \in T_Q(E_1) \). Identifying \( E_1 \) with \( \mathbf{P}^{n-1} \), the direction defined by \( v \) corresponds to a line \( L_v \) containing \( Q \). Let \( Q' \) be another point on that line. We already know that there exists \( s \in \Gamma(X, L \otimes I_P) \) with \( Q \in D_s \) but \( v \notin T_Q(D_s) \). In particular \( L_v \not\subset D_s \), hence under the identification of \( E_1 \) and \( \mathbf{P}^{n-1} \) the intersection of \( D_s \) and \( E_1 \) is a hyperplane intersecting \( L_v \) transversally at \( Q \). It follows that \( v \notin T_Q(D_s) \).
EMBEDDINGS OF BLOWINGS-UP

So assume \( v \notin T_Q(E_1) \). Then \( Q \) corresponds to a subscheme \( Z_Q \) of length 2 of \( X \) and \( v \) corresponds to a curvilinear subscheme \( Z_v \) of length 3 of \( X \) with support \( P \) and containing \( Z_Q \). Then \( Z = Z_v \cup P \) is a subscheme of length \( N - 2n + 1 \) containing \( P \) hence the main lemma implies \( \dim(\langle Z \rangle) = \dim P + 2 \). Hence we find \( H \in (\mathbb{P}^N)^* \) with \( Z_Q \cup P \subset H \) but \( Z_v \notin H \). Then \( H \) defines \( s \in \Gamma(X, L \otimes I_P) \) with \( Q \in D_s \) but \( v \notin T_Q(D_s) \).

1.3.4. . In this proof we only used the fact that \( P \) is a set of \( a \) points on \( X \) such that for each curvilinear subscheme \( Z \subset X \) of length \( a + 2 \) containing \( P \) one has \( \dim(\langle Z \rangle) = \dim(P) + 2 \). So we obtain:

Proposition. Let \( X \subset \mathbb{P}^N \) be a smooth projective variety and let \( P \) be a set of \( a \) points on \( X \). Let \( P \) be the span of \( P \). Assume for all curvilinear subschemes \( Z \subset X \) of length \( a + 2 \) and containing \( P \) one has \( \dim(\langle Z \rangle) = \dim(P) + 2 \). Let \( Y \) be the blowing-up of \( X \) at \( P \). Then the projection of \( X \) with center \( P \) induces an embedding of \( Y \).

1.4. . In the proof of Theorem 1 we use the characteristic zero assumption in two arguments. First of all there is the use of Bertini’s Theorem. There exist Bertini Theorems for positive characteristic. Maybe they can be used causing a more involved proof and maybe a worse assumption on \( t \). Next there is the use of Proposition (1.1.3). See [8], Remark 1.7 for a discussion of it. The main problem is: The linear system \( g \) on the curve \( C \) in the proof of Subclaim (1.2.1) need not be complete.


2.1. . For the convenience of the reader we recall the definition of the Clifford index of a smooth curve \( C \). Let \( L \) be an invertible sheaf on \( C \). The Clifford index of \( L \) is \( \text{Cliff}(L) = \deg(L) - 2h^0(L) + 2 \). Let \( K_C \) be the canonical sheaf on \( C \). The invertible sheaf \( L \) is very special if \( h^0(L) > 1 \) and \( h^0(K_C \otimes L^{-1}) > 1 \). The Clifford index of \( C \) is \( \text{Cliff}(C) = \min(\{\text{Cliff}(L) : L \) is a very special invertible sheaf on \( C \}) \). From the Riemann-Roch Theorem it follows that in the definition of \( \text{Cliff}(C) \) we can restrict to line bundles with \( \deg(L) \leq g - 1 \). In the proof of Theorem 2 we will use that \( \text{Cliff}(C) \leq 2 \) if and only if there exist integers \( r \geq 1 ; d \leq g - 1 \) with \( d - 2r \leq 2 \) such that \( W^r_d(C) \) is not empty.

Let \( X \) be a \( K3 \)-surface and let \( L \) be a very ample invertible sheaf on \( X \). Let \( C \) be a smooth curve on \( X \) associated to a global section of \( L \). It is proved in [12] that \( \text{Cliff}(C) \) is independent of the curve, so we call it the Clifford index of \( L \), denoted by \( \text{Cliff}(L) \). Also remember that the dimension of the complete linear system defined by \( L \) is equal to the genus of \( C \). We denote it by \( g = g(L) \) and we consider \( X \subset \mathbb{P}^g \) defined by \( L \).
2.2. Proof of Theorem 2. In case $g \leq 6$ it follows from the existence of special divisors (see e.g., [1], p. 206 (1.1)) that $W_d^1(C)$ is not empty for a smooth curve section of $X$, hence $\text{Cliff}(L) \leq 2$. So, we can assume $g \geq 7$.

2.2.1. Using (1.3.4), it is enough to prove the main lemma (1.2) in this situation. Let $t \leq g - 5$ and assume for general points $P_1; \ldots ; P_t$ on $X$ there exists a curvilinear subscheme $Z$ on $X$ of length $t + 2$ containing those points such that $\dim(\langle Z \rangle) \leq t$ and so $\dim(\langle Z \rangle) = t$ follows from the general position lemma as explained in the first lines of the proof of (1.2). Since $L$ is very ample on $X$ this is not possible for $t = 0$. So we use induction on $t$ and we assume $t > 0$ and it is not possible to find such a subscheme $Z$ for a smaller number of general points on $X$.

2.2.2. Let $T \subset X^t \times \text{Hilb}^{t+2}(X)$ be the closure of the set of points $(P_1, \ldots , P_t; Z)$ with $P_i \neq P_j$ for $i \neq j$ and $Z$ a curvilinear subscheme of length $t + 2$ containing the points $P_1, \ldots , P_t$ and imposing at most $t + 1$ independent conditions on $\Gamma (X; L)$. The assumption implies that $T$ dominates $X^t$, let $T'$ be a component dominating $X^t$. Let $I \subset T' \times (\mathbb{P}^g)^*$ be the closure of the set of points $(P_1, \ldots , P_t; Z; H)$ such that $\langle Z \rangle \subset H$. Since $\dim(\langle Z \rangle) \leq t$, the fiber of $I$ over $(P_1, \ldots , P_t; Z)$ has dimension at least $g - 1 - t$, hence $\dim(I) \geq t + g - 1$. Assume $H \in (\mathbb{P}^g)^*$ belongs to the image of $I$, then the fiber of $H$ over $I$ has dimension at least $t - 1$. Take $(P_1, \ldots , P_t; Z; H)$ general on $I$.

2.2.3. Assume $H \cap X$ is a smooth curve $C$. The curve section $C \subset H$ is embedded by means of the canonical linear system on $C$. In that case secant space divisors correspond to special divisors. So, in this part of the proof, we are going to make use of the notations $W_d^1(C)$ for the subsets of the Jacobian $J(C)$ parametrizing special invertible sheaves on $C$ (see [1], Chapter IV). We find $\dim(V_{t+2}^1([K_C])) \geq t - 1$. (Here we use $|K_C|$ to denote the canonical linear system on $C$, i.e., the complete linear system associated to the canonical sheaf $K_C$.) Let $E$ be a general element of a component of $V_{t+2}^1([K_C])$. Define the integer $\varepsilon \geq 0$ such that $E \in V_{t+2}^1([K_C])$ but $E \notin V_{t+2}^{-\varepsilon}([K_C])$. Since $t + 2 \leq g$ it follows from the geometric Riemann-Roch Theorem (see e.g., [1], p. 12) that $\dim(|E|) = \varepsilon$. Choose a subdivisor $P_1 + \cdots + P_{\varepsilon - 1}$ of $E$ such that $\dim(|E - P_1 - \cdots - P_{\varepsilon - 1}|) = 1$. For $Q_1; \ldots ; Q_{\varepsilon - 1}$ general on $C$ one has $\dim(|E - P_1 - \cdots - P_{\varepsilon - 1} + Q_1 + \cdots + Q_{\varepsilon - 1}|) = 1$ and using the geometric Riemann-Roch Theorem one finds $E - P_1 - \cdots - P_{\varepsilon - 1} + Q_1 + \cdots + Q_{\varepsilon - 1} \notin V_{t+2}^1([K_C])$ but $E - P_1 - \cdots - P_{\varepsilon - 1} + Q_1 + \cdots + Q_{\varepsilon - 1} \notin V_{t+2}^1([K_C])$. Since $E$ is a specialisation of $E - P_1 - \cdots - P_{\varepsilon - 1} + Q_1 + \cdots + Q_{\varepsilon - 1}$ one finds a contradiction unless $\varepsilon = 0$. So, $\dim(|E|) = 1$ and $|E|$ is a one-dimensional subspace of $V_{t+2}^1([K_C])$. Mapping divisors to their associated invertible sheaf we find $O_C(E) \in W_{t+2}^1(C)$, hence $\dim(W_{t+2}^1(C)) \geq t - 2$. In
case \( g = 7 \) we have \( t \leq 2 \) and we find that \( C \) has a linear system \( g_1^t \), hence \( \text{Cliff}(C) \leq 2 \). Now, assume \( g \geq 8 \). We are going to use some dimension theorems on the varieties \( W_g^t \).

First of all, if \( g \geq 11 \), the following is proved in [14], Theorem 2.1. Let \( d \) and \( r \) be integers satisfying \( d \leq g + r - 4 \) and \( r \geq 1 \) and assume \( \dim(W_g^r(C)) \geq d - 2r - 2 \geq 0 \). Then \( W_g^1(C) \) is not empty, hence \( \text{Cliff}(C) \leq 2 \) (in [14] there is a list of possibilities for those curves, in all cases it is easy to find elements in \( W_g^1(C) \)). In our case, we find \( \dim(W_{t+2}^1(C)) \geq (t+2) - 2 - 2 \) and \( t + 2 \leq g - 3 = g - 1 - 2 \), hence we find \( \text{Cliff}(C) \leq 2 \) if \( g \geq 11 \). In case \( g = 8 \) we find \( \dim(W_5^1(C)) \geq 1 \). In case \( g = 9 \) (resp. \( g = 10 \)) we can use [6], Proposition 12: From \( \dim(W_5^1(C)) \geq 2 \) (resp. \( \dim(W_5^1(C)) \geq 3 \)) it follows that \( \dim(W_5^1(C)) \geq 1 \). Hence in the cases \( g = 8; 9; 10 \) we find the existence of a component of \( W_5^1(C) \) of dimension at least 1. In case a general element of it is of the type \( L = L'(P) \) for some \( P \in C \) with \( h^0(L') \geq 1 \), we find \( L' \in W_5^1(C) \) hence \( \text{Cliff}(C) \leq 2 \). So we assume for a general such \( L \) such a point \( P \) does not exist (in terms of linear systems: We obtain a base point free linear system \( g_1^2 \)). In those cases \( C \) is birationally equivalent to a plane curve of degree 6 (for \( g = 10 \) see [16]; for \( g = 11 \) see [5]; for \( g = 8 \) see [2]). The associated map from \( C \) to \( \mathbb{P}^2 \) with image that plane curve of degree 6 defines an invertible sheaf \( L \) belonging to \( W_6^5(C) \) and again we find \( \text{Cliff}(C) \leq 2 \). So we conclude that \( H \cap X \) is not a smooth curve \( C \).

2.2.4. Assume \( \langle Z \rangle \cap X \) is a 0-dimensional subscheme of \( X \). In case it is curvilinear then for a general hyperplane \( H \) containing \( \langle Z \rangle \) the intersection \( H \cap X \) is a smooth curve. This can be proved as in (1.2.3.1) using Bertini’s Theorem. So \( \langle Z \rangle \cap X \) can not be curvilinear. As explained in (1.2.3.2) we obtain a contradiction to the induction hypothesis on \( t \).

2.2.5. So we conclude that \( \dim(\langle Z \rangle \cap X) \geq 1 \). As in (1.2.4) this implies the intersection contains a rational normal curve \( \Gamma \). Since \( \Gamma \subset \langle Z \rangle \) and \( P \) is a hyperplane in \( \langle Z \rangle \), it follows that \( \Gamma \) intersects \( P \). Hence \( \Gamma \) contains at least one of the points \( P_1; \ldots; P_t \). Since those points are general points on \( X \) it follows that a general point on \( X \) is contained in a smooth rational curve on \( X \). But \( X \) is a \( K3 \)-surface, hence each smooth rational curve on \( X \) is a linear system on its own. Also \( \text{Pic}(X) \) is discrete and finitely generated, hence \( X \) cannot have a one-dimensional family of smooth rational curves. This implies a contradiction.

2.3. The condition \( \text{Cliff}(L) \geq 3 \) cannot be omitted in general.

2.3.1. In case a general curve \( C \) in the linear system associated to \( L \) is trigonal (hence \( \text{Cliff}(L) = 1 \)) then for a general point \( P \) on \( X \) and a general curve \( C \) through \( P \) we find two more points \( P_1 \) and \( P_2 \) on \( C \), such that \( P + P_1 + P_2 \) belongs to the \( g_1^3 \) on \( C \). Hence \( P; P_1; P_2 \) are three points on a line. Projecting with center \( P \) does not give an embedding of the blowing-up \( Y \) of \( X \) at \( P \) in \( \mathbb{P}^{s-1} \).
2.3.2. As a second example consider the so-called Donagi-example (see [10], 2.2). Let \( f : X \to \mathbb{P}^2 \) be the double covering branched along a smooth plane sextic. Let \( L = f^*(O_{\mathbb{P}^2}(3)) \), then \( \dim(\Gamma(X; L)) = 11 \) and for \( X \subset \mathbb{P}^{10} \) using \( L \) there exists \( Q \in \mathbb{P}^{10} \setminus X \) such that the projection \( \pi : X \to \mathbb{P}^9 \) with center \( Q \) is the composition of \( f \) and the 3-Veronese embedding of \( \mathbb{P}^2 \). Taking \( P_1; P_2 \) general on \( X \) one finds \( P'_1; P'_2 \) with \( P_i + P'_i \) a fiber of \( f \) and \( \langle P_i; P'_i \rangle \subset \mathbb{P}^{10} \) a line containing \( Q \), hence \( \langle P_1; P'_1 \rangle \) and \( \langle P_2; P'_2 \rangle \) intersect. So projection with center \( \langle P_1; P_2 \rangle \) does not give rise to an embedding of the blowing-up of \( X \) at \( P_1 \) and \( P_2 \) in \( \mathbb{P}^8 \). A general section of \( X \subset \mathbb{P}^{10} \) is isomorphic to a smooth plane curve of degree 6 (see [10]), hence \( \text{Cliff}(C) = 2 \).

2.3.3. One more example can be found in [17], 4.2. Let \( f : X \to \mathbb{P}^1 \times \mathbb{P}^1 \) be a double covering branched along a smooth curve of bidegree \((4;4)\) on \( \mathbb{P}^1 \times \mathbb{P}^1 \). Let \( L = f^*(O(2;2)) \), then \( \dim(\Gamma(X; L)) = 10 \) and one obtains an embedding \( X \subset \mathbb{P}^9 \) using \( L \). There exists \( Q \in \mathbb{P}^9 \setminus X \) such that projection \( \pi : X \to \mathbb{P}^8 \) with center \( Q \) is the composition of \( f \) and the embedding of \( \mathbb{P}^1 \times \mathbb{P}^1 \) using \( O(2;2) \). We conclude as in the previous example finding no embedding for the blowing-up of \( X \) at two general points using a projection in \( \mathbb{P}^7 \). In this example a general section of \( X \subset \mathbb{P}^9 \) has gonality 4 (see [17]); hence \( \text{Cliff}(L) = 2 \).

References


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ON BUCHSBAUM SIMPLICIAL AFFINE SEMIGROUPS

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We give an arithmetic characterization which allow us to determine algorithmically when the semigroup ring associated to a simplicial affine semigroup is Buchsbaum. This characterization is based on a test performed on the Apéry sets of the extremal rays of the semigroup. We use this method to obtain the cardinality of minimal presentations for semigroups with minimal Apéry set.

Introduction.

Let \( S = \langle n_1, \ldots, n_r, n_{r+1}, \ldots, n_{r+m} \rangle \subseteq \mathbb{N}^r \) be a simplicial affine semigroup, that is \( L_{\mathbb{Q}^+}(S) = L_{\mathbb{Q}^+}^+(\{n_1, \ldots, n_r\}) \), where \( L_{\mathbb{Q}^+}^+(A) = \{ \sum q_i a_i \mid q_i \in \mathbb{Q}^+ \text{ and } a_i \in A \} \). We assume that the elements \( n_1, \ldots, n_r \) are linearly independent (otherwise \( S \) can be embedded in \( \mathbb{N}^s \) with \( s < r \)). This enables us to suppose that, up to isomorphism, \( n_i = \alpha_i e_i \) with \( \alpha_i \in \mathbb{N} \setminus \{0\} \) (as usual, \( e_i \) denotes the element in \( \mathbb{N}^r \) all of whose coordinates are equal to zero except the \( i \)-th which is equal to one). We will refer to \( n_1, \ldots, n_r \) as the extremal rays of \( S \).

Let \( K[S] = \bigoplus_{s \in S} Ky_s \) be the semigroup ring associated to \( S \). We say that \( S \) is Cohen-Macaulay if the ring \( K[S] \) is Cohen-Macaulay. The same stands for the notions of Gorenstein, Buchsbaum and complete intersection semigroup. In \([10]\) the authors gave a characterization of the Cohen-Macaulay and Gorenstein property for simplicial affine semigroups in terms of the Apéry sets of its extremal rays. In that paper we also studied the form and cardinality of a minimal system of generators of the defining ideals of this type of semigroup rings. The mentioned paper was inspired mostly in the characterization given by Goto, Suzuki and Watanabe in \([5]\) and in the generalization given in \([16]\) by Trung and Hoa. Here we focus our attention on Buchsbaum semigroups. There are a lot of papers devoted to the study of the structure of arithmetically Buchsbaum monomial curves (see for instance \([1, 7, 12, 15]\)). Using as a starting point the characterizations given by Trung in \([14]\) and by Kamoi in \([8]\), we present an alternative characterization of the Buchsbaum property in Theorem 5 (compare with Theorem 1.1, page 230, in \([13]\)). This result is used later to achieve Theorem 9 which is the main result of this paper and presents an arithmetical characterization of the Buchsbaum property for simplicial affine semigroups.
in terms of the Apéry sets of their extremal rays. This main theorem provides us with a procedure for deciding whether or not a given simplicial affine semigroup is Buchsbaum. Finally these results are also used to give the exact cardinality of a minimal presentation of a Buchsbaum simplicial affine semigroup with minimal Apéry sets (using the notation in [4], these are Buchsbaum simplicial affine semigroups with maximal embedding dimension, and what we count here is the number of elements in a minimal system of generators of the defining ideal of the semigroup ring associated to the given monoid; see this reference for an explicit expression of the Hilbert polynomial for this semigroup ring). The number of elements of a minimal presentation for this kind of monoids is obtained from the Apéry sets of its extremal rays and in this way this result generalizes the bound given for Cohen-Macaulay simplicial affine semigroups with maximal codimension presented in [10].

1. A characterization of Buchsbaum semigroups.

For every $k \in \mathbb{N}$, define

$$S_k = \{ x \in \mathbb{N}^r \mid \text{there exists } 1 \leq i < j \leq r \text{ such that } x + kn_i \in S, x + kn_j \in S \}.$$ 

The characterizations given here for Buchsbaum semigroups are based on the following result.

**Proposition 1.** The following conditions are equivalent.

(i) $S$ is a Buchsbaum semigroup.

(ii) $S_2 + (S \setminus \{0\}) \subseteq S$.

(iii) For every $1 \leq i < j \leq r$ and $u, v \in S$, if $v + 2n_j = u + 2n_i$ then $v + (S \setminus \{0\}) \subseteq 2n_i + S$.

(iv) For every $x \in S$, if $x - 2n_i, x - 2n_j \in S$, for some $i \neq j \in \{1, \ldots, r\}$, then $x + n_k - (2n_i + 2n_j) \in S$ for all $k \in \{1, \ldots, r + m\}$.

**Proof.** The equivalence between (i) and (ii) appears in Lemma 3 of [14]. The conditions (i) and (iii) are equivalent by Proposition 2.3 of [8]. Finally (iii) if and only if (iv) follows easily taking $x = v + 2n_j = u + 2n_i$. □

There are several characterizations of the Cohen-Macaulay property similar to Proposition 1. Next we give one of them.

**Proposition 2.** The semigroup $S$ is a Cohen-Macaulay semigroup if and only if $S_k = S$ for every $k \in \mathbb{N} \setminus \{0\}$.

**Proof.** The fact that $S_1 = S$ is equivalent to the Cohen-Macaulay property for $S$ is part of Corollary 4.4 in [16]. Besides, once $S_1 = S$, one can prove that $S_k = S$ for all $k \geq 2$. □
In order to reformulate the Buchsbaum property for simplicial affine semigroups, we have to introduce some notation. The Apéry set of an element \( n \) of \( S \) is the set
\[
S(n) = \{ s \in S \mid s - n \notin S \}.
\]
The subgroup of \( \mathbb{Z}^r \) generated by \( \{n_1, \ldots, n_r\} \) is denoted by \( G(\{n_1, \ldots, n_r\}) \).

Let \( \mathcal{S} \) be the set of elements \( x \in \mathbb{Z}^r \) such that \( x + n_i \in S \) for all \( i \in \{1, \ldots, r + m\} \). Note that this set is a semigroup that contains \( S \) and that
\[
\mathcal{S} = \{ x \in \mathbb{N}^r \mid x + n_i \in S \text{ for all } i \in \{1, \ldots, r + m\} \}.
\]
Since \( \mathcal{S} \subseteq \mathbb{N}^r \) and \( L_{\mathbb{Q}_0^+}(n_1, \ldots, n_r) = (\mathbb{Q}_0^+)^r \), then \( \mathcal{S} \) is simplicial as well.

As a consequence of the following result, \( \mathcal{S} \) is finitely generated.

**Lemma 3.** Every submonoid \( T \) of \( \mathbb{N}^r \) containing \( \{n_1, \ldots, n_r\} \) is finitely generated.

**Proof.** It is easy to see that since \( T \) is simplicial, every element \( t \) in \( T \) can be written as \( t = \sum_{i=1}^r a_in_i + w \) for some \( a_i \in \mathbb{N} \) and \( w \in \bigcap_{i=1}^r T(n_i) \). If we want to demonstrate that \( T \) is finitely generated, it suffices to prove that the set \( \bigcap_{i=1}^r T(n_i) \) has a finite number of elements. For proving this, define in \( \bigcap_{i=1}^r T(n_i) \) the following equivalence relation:
\[
x \sim y \text{ if } x - y \in G(\{n_1, \ldots, n_r\}).
\]
Since there are at most \( \alpha_1 \cdots \alpha_r \) elements in \( \mathbb{N}^r \) modulo \( G(\{n_1, \ldots, n_r\}) \), there is a finite number of \( \sim \)-classes in \( \bigcap_{i=1}^r T(n_i) \). If we show that for every \( x = (x_1, \ldots, x_r) \in \bigcap_{i=1}^r T(n_i) \), its \( \sim \)-class \( [x] \) is finite, then we conclude the proof. Set \( m = (x_1 \mod \alpha_1, \ldots, x_r \mod \alpha_r) \). Clearly, for every element \( y \in [x] \), there exists \( a_1^y, \ldots, a_r^y \in \mathbb{N} \) such that \( y = \sum_{i=1}^r a_i^yn_i + m \). If there exists \( y, z \in [x] \) for which \( (a_1^y, \ldots, a_r^y) < (a_1^z, \ldots, a_r^z) \), then \( z = y + \sum_{i=1}^r (a_i^z - a_i^y)n_i \notin \bigcap_{i=1}^r T(n_i) \), which contradicts \( z \in \bigcap_{i=1}^r T(n_i) \). Hence the set of elements \( A = \{(a_1^y, \ldots, a_r^y) \mid y \in [x]\} \) is a set of incomparable elements with respect to the usual partial order in \( \mathbb{N}^r \) (product order). Using Dickson’s lemma, it follows that there exists a finite number of elements in \( A \) and thus a finite number of elements in \( [x] \). \( \square \)

The following result indicates a connection between \( S_k \) and \( \mathcal{S} \).

**Lemma 4.**
\[
S \subseteq \mathcal{S} \subseteq S_1 \subseteq (\mathcal{S})_1 \subseteq S_2 \subseteq (\mathcal{S})_2.
\]

**Proof.** Follows easily from the definitions of \( S_k \) and \( \mathcal{S} \). \( \square \)

The next statement justifies the definition of the semigroup \( \mathcal{S} \).

**Theorem 5.** The semigroup \( S \) is Buchsbaum if and only if \( \mathcal{S} \) is Cohen-Macaulay.
Proof. Necessity. By Proposition 1, we have $S_2 + (S \setminus \{0\}) \subseteq S$, whence $S_2 \subseteq \overline{S}$. Since the opposite inclusion always holds, $S_2 = \overline{S}$ and by Lemma 4, it follows that $\overline{S} = (\overline{S})_1$, which by Proposition 2 implies that $\overline{S}$ is Cohen-Macaulay.

Sufficiency. By Proposition 2, $\overline{S} = (\overline{S})_2$. From Lemma 4 it follows that $S_2 = \overline{S}$, which by the definition of $\overline{S}$ leads to $S_2 + (S \setminus \{0\}) \subseteq S$. Proposition 1 asserts that $S$ is Buchsbaum. □

From the characterization of the Cohen-Macaulay property, if $S$ is Cohen-Macaulay and $x + n_i, x + n_j$ are in $S$, for $i \neq j \in \{1, \ldots, r\}$, then is $x \in S$. Thus if $S$ is Cohen-Macaulay and $r \geq 2$, then $S = \overline{S}$. (The case $r = 1$ is the numerical case and all numerical semigroups are Cohen-Macaulay and Buchsbaum.)

As an easy consequence of Theorem 5 we obtain the following remark (the proof is left to the reader).

**Corollary 6.** Let $T$ be a simplicial affine Cohen-Macaulay semigroup minimally generated by $\{n_1, \ldots, n_r, n_{r+1}, \ldots, n_{r+m}\}$ and $A$ be a nonempty subset of $\{n_{r+1}, \ldots, n_{r+m}\}$. Then $T \setminus A$ is Buchsbaum but not Cohen-Macaulay.

We illustrate this with an example.

**Example 7.** Let $S$ be the semigroup finitely generated by
\[(2,0), (0,2), (1,1)\].
Clearly $S$ is Cohen-Macaulay (it is even a complete intersection). The semigroup $S \setminus \{(1,1)\}$ can be generated by
\[
\{(2,0), (0,2), 2(1,1), 3(1,1), (2,0) + (1,1), (0,2) + (1,1)\},
\]
whence $S \setminus \{(1,1)\} = \langle (2,0), (0,2), (3,1), (1,3) \rangle$ is Buchsbaum but not Cohen-Macaulay.

2. How to determine whether a simplicial affine semigroup is Buchsbaum.

As we did in the previous section, we define in $\bigcap_{i=1}^r S(n_i)$ the following binary relation:
\[x \sim y \text{ if } x - y \in G(\{n_1, \ldots, n_r\}).\]
The next result is used in [10] for giving a procedure for determining whether a simplicial affine semigroup is Cohen-Macaulay.

**Proposition 8.** The following statements are equivalent.
(i) $S$ is a Cohen-Macaulay semigroup.
(ii) For any $s \in S$ and $i \neq j \in \{1, \ldots, r\}$, if $s - n_i$ and $s - n_j$ are in $S$ then $s - (n_i + n_j)$ also belongs to $S$. 
(iii) For every element \( s \in S \) there exists a unique element \( (a_1, \ldots, a_r) \in \mathbb{N}^r \) and a unique element \( w \) in \( \bigcap_{i=1}^r S(n_i) \) such that \( s = \sum_{i=1}^r a_i n_i + w \).

(iv) For every \( x, y \in \bigcap_{i=1}^r S(n_i) \), if \( x - y \in \mathcal{G}(\{n_1, \ldots, n_r\}) \), then \( x = y \).

(v) For every \( x \in \bigcap_{i=1}^r S(n_i) \), \( [x] = \{x\} \).

Proof. The equivalence between (i)-(iv) appears in [10]. Condition (v) is a reformulation of (iv).

The next proposition shows what happens in the Buchsbaum case.

**Theorem 9.** The affine semigroup \( S \) is Buchsbaum if and only if, for every \( x \in \bigcap_{i=1}^r S(n_i) \), if \( \#[x] \geq 2 \), then there exists \( m \in S \) such that \([x] = \{m + n_1, \ldots, m + n_r\}\).

**Proof.** Necessity. Let us assume that \( \#[x] \geq 2 \). By Theorem 5, \( \overline{S} \) is Cohen-Macaulay (recall that \( \overline{S} \) is a simplicial affine semigroup whose extremal rays are the extremal rays of \( S \)). Proposition 8 ensures that there exist unique \( m \in \bigcap_{i=1}^r \overline{S}(n_i) \) and \( (a_1, \ldots, a_r) \in \mathbb{N}^r \) such that \( x = \sum_{i=1}^r a_i n_i + m \). In particular, this implies that \( x - m \in \mathcal{G}(\{n_1, \ldots, n_r\}) \), whence \( y - m \in \mathcal{G}(\{n_1, \ldots, n_r\}) \) for all \( y \in [x] \). We show that in this case \( m \) cannot be in \( S \). If this were not the case, then \( a_1 = \cdots = a_r = 0 \), since \( x \in \bigcap_{i=1}^r S(n_i) \). Thus \( m = x \). Recall that \( \#[x] \geq 2 \) and hence there exists \( y \in [x] \setminus \{x\} \). Using once more Proposition 8, there exist \( m' \in \bigcap_{i=1}^r \overline{S}(n_i) \) and \( (b_1, \ldots, b_r) \in \mathbb{N}^r \) such that \( y = \sum_{i=1}^r b_i n_i + m' \). It follows that \( m' - m = (x - m) + (y - x) + (m' - y) \in \mathcal{G}(\{n_1, \ldots, n_r\}) \). Condition (v) of Proposition 8 ensures that \( [m] = \{m\} \), which leads to \( m = m' \). Therefore \( b_1 = \cdots = b_r = 0 \), since \( y \in [x] \subseteq \bigcap_{i=1}^r S(n_i) \). This means that \( y = m = x \), which contradicts \( y \neq x \).

We show next that \( \{m + n_1, \ldots, m + n_r\} \subseteq [x] \). Since \( m \in S \), we get that \( m + n_k \in S \) for all \( 1 \leq k \leq r \). The affine semigroup \( S \) is simplicial and for this reason there exists \( (c_1, \ldots, c_r) \in \mathbb{N}^r \) and \( w \in \bigcap_{i=1}^r S(n_i) \) such that \( m + n_1 = \sum_{i=1}^r c_i n_i + w \) (observe that this forces \( w \) to be in \([x]\)). In addition, \( w \in S \) and \( \overline{S} \) is Cohen-Macaulay, which by Proposition 8 implies that there exist \( (d_1, \ldots, d_r) \in \mathbb{N}^r \) and \( m' \in \bigcap_{i=1}^r \overline{S}(n_i) \) such that \( w = \sum_{i=1}^r d_i n_i + m' \).

As before, we can deduce that \( m = m' \). It follows \( m + n_1 = (c_1 + d_1)n_1 + \cdots + (c_r + d_r)n_r + m \). From Proposition 8 we get that \( c_1 + d_1 = 1 \) and that \( c_2 + d_2 = \cdots = c_r + d_r = 0 \). This leads to \( w = m + n_1 \in [x] \). Similarly it is shown that \( m + n_i \in [x] \) for all \( i \in \{2, \ldots, r\} \).

For the opposite inclusion, take \( y \in [x] \). Then \( y \in S \subseteq \overline{S} \). By the same argument used above, there exists \( (a_1, \ldots, a_r) \in \mathbb{N}^r \) for which \( y = \sum_{i=1}^r a_i n_i + m \). The fact that \( y \in S \) implies that \( \sum_{i=1}^r a_i \geq 1 \) and \( y \in \bigcap_{i=1}^r S(n_i) \) forces \( \sum_{i=1}^r a_i = 1 \). Hence \( y = m + n_i \) for some \( i \in \{1, \ldots, r\} \).

**Sufficiency.** Define 
\[
A = \{ m_{[x]} \mid [x] \in \bigcap_{i=1}^r S(n_i) / \sim \},
\]
where \( m_{[x]} = \begin{cases} x & \text{if } \# [x] = 1, \\ m & \text{if } [x] = \{m + n_1, \ldots, m + n_r\}. \end{cases} \)

From the definition of \( A \), any two of its elements are incongruent modulo \( G(\{n_1, \ldots, n_r\}) \). If we prove that \( \bigcap_{i=1}^r S(n_i) \subseteq A \), we get that \( \bigcap_{i=1}^r S(n_i) \) fulfills the same condition, which by Proposition 8 means that \( S \) is Cohen-Macaulay and by Theorem 5 that \( S \) is Buchsbaum. Thus it suffices to show that \( \bigcap_{i=1}^r S(n_i) \subseteq A \). Take \( x \in \bigcap_{i=1}^r S(n_i) \). Then \( x \in S \), whence \( x + n_1, x + n_2 \in S \). Since \( S \) is simplicial, there exist \( (c_1, \ldots, c_r), (d_1, \ldots, d_r) \in \mathbb{N}^r \) and \( w, w' \in \bigcap_{i=1}^r S(n_i) \) such that \( x + n_1 = \sum_{i=1}^r c_i n_i + w \) and \( x + n_2 = \sum_{i=1}^r d_i n_i + w' \). It follows that \( w' \in [w] \). By the definition of \( A \), there exists \( m \in A \) for which \( w = m \) or \( w = m + n_i \) for some \( i \in \{1, \ldots, r\} \) and \( m' \in A \) such that \( w' = m' \) or \( w' = m' + n_j \) for some \( j \in \{1, \ldots, r\} \). In any case, since \( w' \in [w] \), we have \( m = m' \). Thus both \( x + n_1 \) and \( x + n_2 \) can be written as \( x + n_1 = \sum_{i=1}^r a_i n_i + m \) and \( x + n_2 = \sum_{i=1}^r b_i n_i + m \) for some \( (a_1, \ldots, a_r), (b_1, \ldots, b_r) \in \mathbb{N}^r \), which this leads to

\[
a_1 n_1 + (a_2 + 1)n_2 + a_3 n_3 + \cdots + a_r n_r = (b_1 + 1)n_1 + b_2 n_2 + \cdots + b_r n_r.
\]

Since \( \{n_1, \ldots, n_r\} \) is a basis of \( \mathbb{Q}^r \), we get that \( a_1 = b_1 + 1 \), which implies that \( a_1 \geq 1 \). Hence \( x = (a_1 - 1)n_1 + a_2 n_2 + \cdots + a_r n_r + m \). In addition, \( m \in S \) and \( x \in \bigcap_{i=1}^r S(n_i) \), which forces \( x \) to be equal to \( m \). \( \square \)

If we know \( \bigcap_{i=1}^r S(n_i) \), then we can check for every \( x \in \bigcap_{i=1}^r S(n_i) \), whether \( \# [x] = 1 \) or \( [x] = \{m + n_1, m + n_2, \ldots, m + n_r\} \) for some \( m \in S \). If this is not the case, then \( S \) is not Buchsbaum. In [3, 10] an algorithm for computing the set \( \bigcap_{i=1}^r S(n_i) \) is presented. This idea is based on the fact that

\[
\bigcap_{i=1}^r S(n_i) \subseteq \{ \sum_{i=1}^m \gamma_{r+i} n_{r+i} \mid \gamma_{r+i} < c_{r+i} \text{ for all } i \in \{1, \ldots, m\} \},
\]

where \( c_{r+i} = \min \{ k \in \mathbb{N} - \{0\} : kn_{r+i} \in \langle n_1, n_2, \ldots, n_r \rangle \} \leq \alpha_1 \cdots \alpha_r \).

Thus Theorem 9, together with the algorithm for computing \( \bigcap_{i=1}^r S(n_i) \), constitutes a method for deciding whether a simplicial affine semigroup is Buchsbaum.

**Example 10.** Let \( S = \langle (2,0), (0,1), (1,2), (3,1) \rangle \). We compute \( S((2,0)) \cap S((0,1)) \) as explained in [10] and obtain

\[
S((2,0)) \cap S((0,1)) = \{(0,0), (1,2), (3,1)\}.
\]

Observe that

\[
[(0,0)] = \{(0,0)\}, \quad [(1,2)] = \{(1,2), (3,1)\} = \{(1,1) + (2,0), (1,1) + (0,1)\}.
\]

Taking \( m = (1,1), m + (2,0), m + (0,1), m + (1,2), m + (3,1) \in S \) and hence \( m \in S \). By Theorem 9, \( S \) is Buchsbaum but not Cohen-Macaulay, since \( \# [(1,2)] \neq 1 \) (Proposition 8).
The condition \( \#x \in \{1, r\} \) is not sufficient for \( S \) to be Buchsbaum, as the following example shows.

**Example 11.** Let \( S = \langle (2, 0), (0, 2), (3, 1), (1, 3), (1, 2) \rangle \). Using the procedure presented in [10] to compute \( \bigcap_{i=1}^{r} S(n_i) = S((2, 0)) \cap S((0, 2)) \), we get 
\[ S((2, 0)) \cap S((0, 2)) = \{ (0, 0), (3, 1), (1, 3), (1, 2), (4, 3), (2, 5) \} \]
It follows that 
\[ [(0, 0)] = \{ (0, 0) \}, \quad [(3, 1)] = \{ (3, 1), (1, 3) \}, \]
\[ [(1, 2)] = \{ (1, 2) \}, \quad [(4, 3)] = \{ (4, 3), (2, 5) \}. \]
By looking at \([3, 1] \), the only possible candidate to be \( m \) is \( (1, 1) \). However, 
\[ m + (1, 2) = (2, 3) \not\in S, \] which by Theorem 9 implies that \( S \) is not Buchsbaum, since \( m \not\in S \).

### 3. Buchsbaum semigroups with minimal Apéry set.

In the sequel we assume that \( \{ n_1, \ldots, n_r, n_{r+1}, \ldots, n_{r+m} \} \) is a minimal system of generators of \( S \). By the definition of \( \bigcap_{i=1}^{r} S(n_i) \), this implies that \( \{ n_{r+1}, \ldots, n_{r+m} \} \) is included in \( \bigcap_{i=1}^{r} S(n_i) \) \{0\}. We say that \( S \) has **minimal Apéry set** if
\[ \{ 0, n_{r+1}, \ldots, n_{r+m} \} = \bigcap_{i=1}^{r} S(n_i). \]
Here we transfer a result known for Cohen-Macaulay simplicial affine semigroups fulfilling this condition to the Buchsbaum case. To this end, we need to recall some basic concepts in order to fix notation.

Let \( \varphi \) be the map defined by
\[ \varphi : \mathbb{N}^{r+m} \to S, \quad \varphi(a_1, \ldots, a_{r+m}) = \sum_{i=1}^{r+m} a_i n_i, \]
and denote its kernel congruence by \( \sigma \). Then \( S \) is isomorphic to \( \mathbb{N}^{r+m}/\sigma \).

We say that \( \rho \) is a **minimal system of generators** of \( \sigma \) if \( \rho \) generates \( \sigma \) and its cardinal is minimal among the cardinal of the sets generating \( \sigma \). In this case we also say that \( \rho \) is a **minimal presentation** of \( S \). It can be shown that \( \#\rho \geq r + m - r = m \) (see [6]).

Let \( n \in S - \{0\} \). Define the graph \( G_n \) as the graph whose vertices are
\[ V(G_n) = \{ n_i \mid n - n_i \in S, i \in \{1, \ldots, r + m\} \} \]
and whose edges are
\[ E(G_n) = \{ n_i n_j \mid n - (n_i + n_j) \in S, i, j \in \{1, \ldots, r + m\}, i \neq j \}. \]
Define \( \rho_n \) as follows.

1) If \( G_n \) is connected, then \( \rho_n = \emptyset \).
2) If $G_n$ is not connected and $G_{n_1}^1, \ldots, G_{n_t}^t$ are the connected components of $G_n$, then choose a vertex $n_j \in V(G_{n_i}^i)$ and an element $\alpha_i^n = (a_1^i, \ldots, a_{r+m}^i) \in \mathbb{N}^{r+m}$ such that $\varphi(\alpha_i^n) = n$ and $a_{j_i}^i \neq 0$; define

$$\rho_n = \{(\alpha_1^n, \alpha_2^n), \ldots, (\alpha_t^n, \alpha_1^n)\}.$$ 

Take $\rho = \bigcup_{n \in S} \rho_n$. Then $\rho$ is a minimal system of generators of $\sigma$ (this follows from a straightforward generalization presented in [3, 11] of the results given in [9]). Furthermore, every minimal system of generators of $\sigma$ has the same cardinality.

**Example 12.** Let

$$S = \langle (2, 0), (0, 1), (1, 2), (3, 1) \rangle \subseteq \mathbb{N}^2.$$ 

The elements $n \in S$ for which $G_n$ is not connected are $(3, 2), (6, 2), (4, 3)$ and $(2, 4)$.

<table>
<thead>
<tr>
<th>Graph</th>
<th>Connected components</th>
<th>Relators</th>
</tr>
</thead>
<tbody>
<tr>
<td>$G_{(3,2)}$</td>
<td>${(2,0), (1,2)}, {(0,1), (3,1)}$</td>
<td>$e_1 + e_3 = e_2 + e_4$</td>
</tr>
<tr>
<td>$G_{(6,2)}$</td>
<td>${(2,0), (0,1)}, {(3,1)}$</td>
<td>$3e_1 + 2e_2 = 2e_4$</td>
</tr>
<tr>
<td>$G_{(4,3)}$</td>
<td>${(2,0), (0,1)}, {(1,2), (3,1)}$</td>
<td>$2e_1 + 3e_2 = e_3 + e_4$</td>
</tr>
<tr>
<td>$G_{(2,4)}$</td>
<td>${(2,0), (0,1)}, {(1,2)}$</td>
<td>$e_1 + 4e_2 = 2e_3$</td>
</tr>
</tbody>
</table>

Hence

$$\rho = \{((1,0,1,0),(0,1,0,1)), ((3,2,0,0),(0,0,0,2)), ((2,3,0,0),(0,0,1,1)), ((1,4,0,0),(0,0,2,0))\}$$

is a minimal presentation of $S$.

In [10] the authors show that if $S$ is a Cohen-Macaulay simplicial affine semigroup with minimal Apéry set (there called with maximal codimension), then $\#\rho = m(m+1)/2$. Moreover, this property characterizes Cohen-Macaulay simplicial affine semigroup with minimal Apéry set. Let us see what happens in the Buchsbaum case.

**Theorem 13.** Let $S$ be a Buchsbaum simplicial affine semigroup with minimal Apéry set. Let $\sim$ be the equivalence relation defined over $\bigcap_{i=1}^r S(n_i)$ as before and $\lambda = \# \{x \in \bigcap_{i=1}^r S(n_i)/\sim : \#x = r\}$. For every minimal system of presentation $\rho$ of $S$,

$$\#\rho = \frac{m(m+1)}{2} + \lambda \frac{r(r-1)}{2}. $$
Proof. As we have indicated before, every minimal system of generators has the same cardinality. Hence it suffices to count the elements belonging to \( \rho = \bigcup_{n \in S} \rho_n \). For doing this, we must know which are the elements in \( S \) fulfilling that \( G_n \) is not connected. If \( n \in S \) and \( G_n \) is not connected, then this graph must contain a connected component with some of its vertices lying in \( \{n_1, \ldots, n_r\} \), otherwise \( n - n_i \not\in S \) for all \( i \in \{1, \ldots, r\} \) and thus \( n \in \bigcap_{i=1}^r S(n_i) = \{0, n_{r+1}, \ldots, n_{r+m}\} \), contradicting that \( \{n_1, \ldots, n_{r+m}\} \) is a minimal system of generators of \( S \). For the rest of the proof and for a given \( n \in S \) such that \( G_n \) is not connected, we fix \( G_n^1 \) (defined in the description of \( \rho \) given above) as one of these connected components of \( G_n \) fulfilling that some of its vertices are contained in \( \{n_1, \ldots, n_r\} \). From the construction of \( \rho \), for every \( n \in S \) and every component of \( G_n \) other than \( G_n^1 \), we get a new element in \( \rho \). It follows that in order to count the cardinality of \( \rho \), we only have to decide how many connected components different from the fixed \( G_n^1 \)’s are in all the possible non-connected graphs \( G_n \)’s. We first count those connected components in all the possible non-connected graphs not having vertices in \( \{n_1, \ldots, n_r\} \) (these are of course different from any \( G_n^1 \)) and then we will count those connected components having some vertices in \( \{n_1, \ldots, n_r\} \) and different from the fixed \( G_n^1 \)'s.

Take \( n \in S \) such that \( G_n \) is not connected and contains a connected component \( C \) whose vertices belong to \( \{n_{r+1}, \ldots, n_{r+m}\} \). Then \( n \) can be expressed as \( n = \sum_{i=1}^m a_in_{r+i} \) with \( (a_1, \ldots, a_m) \in \mathbb{N}^m \). Observe that \( \sum_{i=1}^m a_i \geq 2 \), since otherwise \( n \in \bigcap_{i=1}^r S(n_i) \). We claim that \( \sum a_i = 2 \). If this were not the case, then there would exist \( i, j, k \in \{1, \ldots, m\} \) (maybe not different) such that \( n = n_{r+i} + n_{r+j} + n_{r+k} + s \), for some \( s \in \{n_{r+1}, \ldots, n_{r+m}\} \). Since \( n_{r+i} + n_{r+j} \not\in \bigcap_{i=1}^r S(n_i) \), there exists \( l \in \{1, \ldots, r\} \) such that \( n_{r+i} + n_{r+j} - n_l \in S \). However, this leads to \( n = \sum_{i=1}^m a_in_{r+i} \) \( \not\in S \), which implies that \( n_l \) is a vertex of \( C \), contradicting \( V(C) \subseteq \{n_{r+1}, \ldots, n_{r+m}\} \). Hence each \( n \) must be of the form \( n = n_{r+i} + n_{r+j} \) with \( i, j \in \{1, \ldots, m\} \). Conversely, since \( n_{r+i} + n_{r+j} \not\in \bigcap_{i=1}^r S(n_i) \) for all \( i, j \in \{1, \ldots, m\} \), each element of this form yields an element in \( \rho \). In this way we collect \( m(m+1)/2 \) elements in \( \rho \).

Now we determine for which \( n \in S \) the graph \( G_n \) has at least two connected components containing vertices belonging to \( \{n_1, \ldots, n_r\} \) (recall that one of these was taken to be \( G_n^1 \)). If \( n \) fulfills this condition, then there must exist \( i, j \in \{1, \ldots, r\} \) such that \( n - n_i, n - n_j \in S \) and \( n - (n_i + n_j) \not\in S \) (\( n_i \) and \( n_j \) are in different connected components of \( G_n \)). Since \( S \) is Buchsbaum, Theorem 5 ensures that \( \overline{S} \) is Cohen-Macaulay. The elements \( n - n_i, n - n_j \) belong to \( S \), which implies that they belong to \( \overline{S} \) and by Proposition 8, we obtain that \( n - (n_i + n_j) \in \overline{S} \). This leads to \( n - (n_i + n_j) = m \in \overline{S} \setminus S \). As we did in the proof of Theorem 9, it is easy to show that \( \{m + n_1, \ldots, m + n_r\} \subseteq \bigcap_{i=1}^r S(n_i) = \{0, n_{r+1}, \ldots, n_{r+m}\} \). Since \( n - n_i = m + n_j \) and \( n - n_j = m + n_i \), there exists \( s, t \in \{1, \ldots, m\} \) such that
\( n = n_{r+t} + n_j = n_{r+s} + n_i = (m+n_i) + n_j = (m+n_j) + n_i \), which implies that \( n_{r+s} \in [n_{r+t}] \) and thus \( \# [n_{r+t}] = r = \# [n - n_i] \). In addition, \( m - n_k \not\in \mathcal{S} \) for all \( k \in \{1, \ldots, r\} \), since otherwise \( (m - n_k) + n_k \) should belong to \( S \). Hence \( m \in \bigcap_{i=1}^r \mathcal{S}(n_i) \). By Proposition 8, if there exists \( m' \in \mathcal{S} \setminus S \) and \( i', j' \in \{1, \ldots, r\} \) such that \( n = (m + n_i) + n_j = (m' + n_i') + n_j' \), then \( m \) must be equal to \( m' \) and \( \{i, i\} = \{i', j'\} \). This implies that in this case there are exactly two connected components of \( G_n \) with some of its vertices in \( \{n_1, \ldots, n_r\} \) (and this yields a new element in \( \rho \)). Thus for a fixed \( m \), we get as many new elements in \( \rho \) as elements of the form \( (m + n_i) + n_j \) we can write with \( i < j \). This makes \( r(r - 1)/2 \) new elements in \( \rho \). Moreover, for each element \( x \in \bigcap_{i=1}^r S(n_i) \) such that \( \# [x] = r \), we get an element \( m \) as before. It follows that we obtain \( r \lambda(r - 1)/2 \) elements in \( \rho \) corresponding to the graphs having at least two connected components (and therefore exactly two) with some of its vertices lying in \( \{n_1, \ldots, n_r\} \).

We conclude that \( \# \rho = m(m+1)/2 + r \lambda(r - 1)/2 \).

\[ \square \]

References


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CHEBYSHEV PROPERTY
OF COMPLETE ELLIPTIC INTEGRALS
AND ITS APPLICATION TO ABELIAN INTEGRALS

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This paper has two parts. In the first one we study the maximum number of zeros of a function of the form \( f(k)K(k) + g(k)E(k) \), where \( k \in (-1, 1) \), \( f \) and \( g \) are polynomials, and

\[
K(k) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} \quad \text{and} \quad E(k) = \int_0^{\pi/2} \sqrt{1-k^2\sin^2\theta}d\theta
\]

are the complete normal elliptic integrals of the first and second kinds, respectively. In the second part we apply the first one to obtain an upper bound for the number of limit cycles which appear from a small polynomial perturbation of the planar isochronous differential equation \( \dot{z} = iz + z^3 \), where \( z = x + iy \in \mathbb{C} \).

1. Introduction and statement of the main results.

In the qualitative theory of real planar differential systems the main open problem is the determination of limit cycles. A classical way to obtain limit cycles is perturbing the periodic orbits of a center. There are several methods for studying the bifurcated limit cycles from a center. The major part of the methods are based either on the Poincaré return map, or on the Poincaré-Melnikov integral or Abelian integral which are equivalent in the plane (see for instance [1]). Recently some other methods are presented, ones based on the inverse integrating factor (see [7]), others are based in the reduction of the problem to a one dimensional differential equation (see [10] and [13]). In general these methods are difficult to apply for studying the limit cycles that bifurcate from the periodic orbits of a center when the system is integrable but not Hamiltonian. As far as we know few papers study the non–Hamiltonian centers, see for instance [3], [5], [8], [9], [10], [11] and [13].

By definition a polynomial system is a differential system of the form

\[
\frac{dx}{dt} = P(x, y), \quad \frac{dy}{dt} = Q(x, y),
\]

where \( P \) and \( Q \) are polynomials with real coefficients. We say that \( n = \max\{\deg P, \deg Q\} \) is the degree of the polynomial system.
In [11] we studied the limit cycles that bifurcate from the periodic orbits of quadratic isochronous centers when we perturb these centers inside the class of all polynomial systems of degree \( n \). The case \( n = 2 \) was studied by Chicone and Jacobs [3].

The number of zeros of Abelian integrals for reversible isochronous cubic centers whose all orbits are conic were studied in [12].

The technique for studying the limit cycles that bifurcate from the periodic orbits of a nonintegrable non-Hamiltonian center when we perturb the center is classical, see for instance [17]. If the perturbed system is

\[
\begin{array}{l}
\dot{x} = f(x, y) + \varepsilon P(x, y), \\
\dot{y} = g(x, y) + \varepsilon Q(x, y),
\end{array}
\]

and \( 1/R(x, y) \) is an integrating factor when \( \varepsilon = 0 \), then the number of zeros of the following Abelian integral

\[
M(h) := \int_{\{H(x,y) = h\}} \frac{P(x,y)dx + Q(x,y)dy}{R(x,y)},
\]

where \( H(x, y) \) is such that \( \frac{\partial H(x,y)}{\partial x} = \frac{g(x,y)}{R(x,y)} \), \( \frac{\partial H(x,y)}{\partial y} = \frac{g(x,y)}{R(x,y)} \), and \( \{ H(x, y) = h \} \) are periodic orbits of the unperturbed system, controls the number of limit cycles of the perturbed system for \( \varepsilon \) small enough.

When we calculate an Abelian integral we often meet elliptic integrals which can be expressed as the following complete normal elliptic integrals of the first and second kinds:

\[
K(k) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}}, \quad E(k) = \int_0^{\pi/2} \sqrt{1 - k^2 \sin^2 \theta} d\theta.
\]

Therefore to estimate the number of zeros of a function of the form

\[
f(k)K(k) + g(k)E(k),
\]

where \( f, g \) are real polynomials of \( k \), is important. In order to get a good estimation we need to prove a kind of Chebyshev property for (3). This is the goal of the first part of this paper.

We denote by \( \mathcal{P}_n \) the set of all real polynomials in one variable of degree at most \( n \). The next two theorems are the main results of this paper.

**Theorem 1.** For \( f \in \mathcal{P}_n \), \( g \in \mathcal{P}_m \) and \( k \in (-1, 1) \) an upper bound for the number of zeros of the function \( M(k) = f(k)K(k) + g(k)E(k) \), taking into account their multiplicities, is \( n + m + 2 \). Moreover:

(a) For arbitrary \( k_i \in (-1, 1) \) with \( i = 1, 2, \ldots, n + m + 1 \) there exist \( f \in \mathcal{P}_n \) and \( g \in \mathcal{P}_m \) such that \( M(k) \neq 0 \) and \( M(k_i) = 0 \) for \( i = 1, 2, \ldots, n + m + 1 \).

(b) There exist values of \( n \) and \( m \) (for instance, both values even) for which the upper bound \( n + m + 2 \) is attained.
There exist values of \( n \) and \( m \) (for instance, \( n < m + 2 \) and \( n + m \) odd) for which the upper bound is \( n + m + 1 \).

Theorem 1 will be proved in Section 2 by using the Argument Principle. As far as we know this method to estimate the number of zeros of Abelian integrals was introduced by Petrov in [14, 15, 16].

As an application of Theorem 1, in the second part of this paper we study the number of limit cycles which bifurcate from the closed curves surrounding the origin of the planar holomorphic isochronous center, \( \dot{z} = iz + z^3 \). In other words, we study the number of zeros of the Abelian integral associated to the system

\[
\dot{z} = iz + z^3 + \varepsilon P(z, \bar{z}),
\]

where \( P \) is a polynomial of degree \( n \).

**Theorem 2.** An upper bound for the number of zeros (taking into account their multiplicities) of the Abelian integral associated to system (4) for \( n \geq 9 \) is \( 3n - 1 \).

Theorem 2 will be proved in Section 3. The cases \( n < 9 \) also would follow from a more accurate analysis of the computations made in this paper, but we are just interested into obtaining an asymptotic bound.

Note that, as in most known examples, the upper bound for the number of zeros of the Abelian integrals associated to the perturbed system depends linearly on the degree of the polynomial perturbation.

### 2. Elliptic functions.

Before starting the proof we need several preliminary results on the elliptic functions \( K(k) \) and \( E(k) \) and their extensions to the complex plane.

**Lemma 3 ([2, formulas 118.02-710.00]).** The functions \( K \) and \( E \) satisfy the following Picard-Fuchs equation:

\[
\frac{dK}{dk} = \frac{E - (1 - k^2)K}{k(1 - k^2)}, \quad \frac{dE}{dk} = \frac{E - K}{k},
\]

and the following linear differential equations of order two:

\[
k(1 - k^2)\frac{d^2K}{dk^2} + (1 - 3k^2)\frac{dK}{dk} - kK = 0,
\]

\[
k(1 - k^2)\frac{d^2E}{dk^2} + (1 - k^2)\frac{dE}{dk} + kE = 0.
\]

From (5) and since \( K \) and \( E \) are clearly analytic at \( k = 0 \), the elliptic integrals \( E \) and \( K \) can be continuously extended to single-valued analytic functions on the region

\[
D = \mathbb{C} \setminus \{ z \in \mathbb{R}, \ |z| \geq 1 \}.
\]
Denote the upper and lower banks of the cut \( \{ z \in \mathbb{R}, z \geq 1 \} \) by \( L_1^+ \) and \( L_2^+ \) respectively; and the upper and lower banks of the cut \( \{ z \in \mathbb{R}, z \leq -1 \} \) by \( L_1^- \) and \( L_2^- \), respectively. Next lemma collects several properties of these extended functions.

**Lemma 4.** Consider the extensions of \( K \) and \( E \) to \( D \subset \mathbb{C} \). As usual, we denote by \( \log \) the principal determination of the logarithm function. They satisfy the following properties:

(a) The asymptotic expansions of \( K \) and \( E \) near \( \pm 1 \) are given by
\[
K = \log 4 - \frac{1}{2} \log(1 - k^2) + O \left( \left| \frac{\log(1 - k^2)}{(1 - k^2)} \right| \right),
E = 1 + \frac{1}{2} \left[ \log 4 - \frac{1}{2} \log(1 - k^2) - \frac{1}{2} \right] (1 - k^2)
+ O \left( \left| \frac{\log(1 - k^2)}{(1 - k^2)^2} \right| \right).
\]

(b) The asymptotic expansions of \( K \) and \( E \) near \( \infty \) are given by
\[
K \sim k^{-1} \log k, \quad E \sim k.
\]

(c) For \( k \in L_i^\pm, i = 1, 2 \) the following hold
\[
KE \neq 0, \quad \text{Im} \left( \frac{K}{E} \right) \neq 0, \quad \text{Im} \left( \frac{E}{K} \right) \neq 0.
\]

(d) For \( k \in L_i^\pm, i = 1, 2 \) the following holds
\[
(\text{Im} K)(\text{Im} E) \neq 0.
\]

**Proof.** (a). These expressions are given in formulas 900.05 and 900.10 of [2].

(b). By introducing the new variable \( t = 1/k \), the differential equation for \( K \) in (6) is changed into
\[
t^2 \frac{d^2 K}{dt^2} + t \frac{t^2 + 1}{t^2 - 1} \frac{dK}{dt} + \frac{1}{1 - t^2} K = 0.
\]

The indicial equation associated with (7) is \( \lambda(\lambda - 1) - \lambda + 1 = 0 \), which has the double root \( \lambda = 1 \). By applying the Frobenius method (see for instance [4, pp. 132-135]), we know that Equation (7) has two independent solutions of the form
\[
\varphi_1(t) = t \sum_{j=0}^{\infty} c_j t^j, \quad c_0 = 1,
\]

and
\[
\varphi_2(t) = \varphi_1(t) \log t + \sum_{j=2}^{\infty} d_j t^j.
\]
Therefore
\[ K(k) = a_1 \varphi_1(k^{-1}) + a_2 \left[ -\varphi_1(k^{-1}) \log k + \sum_{j=2}^{\infty} d_j k^{-j} \right], \]
for some constants \( a_1 \) and \( a_2 \). Now we prove that \( a_2 \neq 0 \).

Consider \( k = is, s \in \mathbb{R}^+ \). Then note that
\[
|kK(k)| = \left| i \int_0^{\pi/2} \frac{s}{\sqrt{1 + s^2 \sin^2 \theta}} d\theta \right|
\]
\[
= \int_0^{\pi/2} \frac{s}{\sqrt{1 + s^2 \sin^2 \theta}} d\theta \geq \int_0^{\pi/2} \frac{s}{\sqrt{1 + s^2 \theta^2}} d\theta
\]
\[
= \int_0^{s\pi/2} \frac{dt}{\sqrt{1 + t^2}} \xrightarrow{s \to \infty} \infty.
\]
Hence \( a_2 \neq 0 \) and we have proved that \( K(k) \sim k^{-1} \log k \).

The proof that \( E(k) \sim k \) follows the same steps.

(c). All the results of this statement will follow if we prove that
\[
f(k) := (\text{Re } K)(\text{Im } E) - (\text{Re } E)(\text{Im } K)
\]
does not vanish on \( k \in L_{i \pm}^i, i = 1, 2 \). In fact we will prove that
\[
f(k) = \pm (-1)^i \frac{\pi}{2} \neq 0, \quad k \in L_{i \pm}^i, \quad i = 1, 2.
\]

Note that for \( k \in L_{i \pm}^i \), the vector \((\text{Re } K, \text{Re } E)\) and \((\text{Im } K, \text{Im } E)\) are the solutions of \((5)\). Hence by Liouville’s Formula
\[
\frac{df}{dk} = \text{tr} \left( -\frac{1}{k} \frac{k(1-k^2)}{1-k^2} \right) f = 0,
\]
which implies that \( f \) is a constant. On the other hand, by statement (a), we have
\[
f(k) = \left( \log 4 - \frac{1}{2} \log |1 - k^2| + o\left( |1 - k^2|^\frac{1}{2} \right) \right) (O(|1 - k^2|))
\]
\[
- \left( 1 + o\left( |1 - k^2|^\frac{1}{2} \right) \right) \left( \mp (-1)^i \frac{\pi}{2} + o(1 - k^2) \right), \quad k \in L_{i \pm}^i.
\]
Let \( k \to \pm 1 \), we get \( f = \pm (-1)^i \frac{\pi}{2} \), \( k \in L_{i \pm}^i, i = 1, 2 \), so statement (c) is proved.
(d). Remember that on the banks $L_{i}^{i}$, the vector $(\text{Im } K, \text{Im } E)$ is a solution of \eqref{eq:1}. Therefore the function $P = \text{Im } E / \text{Im } K$ satisfies the following Ricatti equation:

$$\frac{dP}{dk} = -\frac{P^2 + 2(1 - k^2)P + k^2 - 1}{k(1 - k^2)},$$

or its equivalent system

$$(8) \quad \frac{dk}{dt} = k(1 - k^2),$$

$$(8) \quad \frac{dP}{dt} = -P^2 + 2(1 - k^2)P + k^2 - 1.$$  

We will only prove the result when $k \in L_{i}^{i}$, the case $k \in L_{i}^{-}$ can be proved in a similar way.

The phase portrait of \eqref{eq:8} is given in Figure 1. Note that by Lemma 4.(a), $\lim_{k \to 1} P(k) = 0$, so $P(k)$ belongs to the stable set of the saddle-node $O_{+} = (1, 0)$. All orbits except the stable separatrix $\gamma$ in the stable set of $O_{+}$ tend to $O_{+}$ in the tangential direction along the half line $k = 1, P \geq 0$. On the other hand, again by Lemma 4.(a), $|\text{Im } E / \text{Im } K| \sim 1 - k$, at $k = 1$. Therefore $P(k)$ must be the stable manifold of $O_{+}$, which is located between the horizontal isocline $\Gamma$ and $x$-axis, as shown in Figure 1. Hence $(\text{Im } E)(\text{Im } K)$ never vanishes and the proof of statement (d) is complete. \qed
Let $G = G_{R,\varepsilon} \subset D$ be a simple connected region with $\partial G = C$, where $C = C_{R,\varepsilon} := C^1_{\varepsilon} \cup C^2_{\varepsilon} \cup C_R \cup L^1_\pm(R, \varepsilon) \cup L^2_\pm(R, \varepsilon)$; $C^1_{\varepsilon} := \{|k - 1| = \varepsilon \ll 1\}; C^2_{\varepsilon} := \{|k + 1| = \varepsilon \ll 1\}; C_R := \{|k| = R \gg 1\};$ and $L^1_\pm(R, \varepsilon) = L^1_\pm \cap \{\varepsilon \leq |k| \leq R\}$, see Figure 2.

**Figure 2.** Domain $G_{R,\varepsilon} = G \subset D$.

**Lemma 5.** The elliptic functions $K$ and $E$ have no zeros in the domain $D \subset \mathbb{C}$.

**Proof.** In order to see that $E$ and $K$ do not vanish in $D$ we apply the Argument Principle to $G = G_{R,\varepsilon}$ for $R$ and $1/\varepsilon$ positive and big enough.

We shall prove that the rotation number of $E$, when $k$ turns around the boundary of $G$ is less than 4.

By Lemma 4.(b) the number of complete turns around $C_R$ is at most $1 + \mu(R)$, where $\mu(R)$ tends to zero as $R$ goes to infinity. By Lemma 4.(c), since $\text{Im } E \neq 0$ on $L^1_\pm$, the number of complete turns on $L^1_\pm(R, \varepsilon) \cup L^2_\pm(R, \varepsilon)$ is less than 2 (in fact less than one half turn in each bank). Finally, by Lemma 4.(a), the number of complete turns of $E$ on $C^1_{\varepsilon} \cup C^2_{\varepsilon}$ when $\varepsilon$ goes to 0 tends to zero. Therefore, all together gives that the rotation number of $E$ when $k$ turns one time around the boundary of $G$ is less than 4.

The study of $K$ is similar and gives the same result.

On the other hand,

$E(ik), K(ik) > 0, \text{ for } k \in \mathbb{R}, E(k), K(k) > 0, \text{ for } k \in (-1, 1)$,
and
\[ E(-k) = E(k), \quad E(\overline{k}) = \overline{E(k)}, \quad K(-k) = K(k), \quad K(\overline{k}) = \overline{K(k)}. \]
So if \( E \) (or \( K \)) has a zero \( z \) in \( D \) it must have at least 4 zeros \( \pm z, \pm \overline{z} \) in \( D \), which is in contradiction with the Argument Principle. Hence the proof of this lemma is ended. \( \square \)

In the sequel we will use the notation #\{ \( k \in A \mid f(k) = 0 \) \} = #\{ \( k \in A \mid f = 0 \) \} to indicate the number of zeros of the function \( f \) in the set \( A \) taking into account their multiplicities.

**Proof of Theorem 1.** We begin by proving that \( n + m + 2 \) is an upper bound for the number of zeros of \( fK + gE \). As a first step we show how the case in which \( f \) and \( g \) have a non-constant common factor can be reduced to the case \( \gcd(f, g) = 1 \). Assume that \( h = \gcd(f, g) \), and that it has degree \( d \). Then the equality
\[ fK + gE = h \left( \frac{f}{h} K + \frac{g}{h} E \right) \]
implies that
\[ \#\{ k \in D \mid fK + gE = 0 \} \leq \#\{ k \in D \mid h(k) = 0 \} + \# \left\{ k \in D \mid \frac{f}{h} K + \frac{g}{h} E = 0 \right\}. \]
Since \( \gcd(f/h, g/h) = 1 \), by assuming the theorem to be true in this case, we have
\[ \#\{ k \in D \mid fK + gE = 0 \} \leq d + ((n - d) + (m - d) + 2) \leq n + m + 2, \]
as we wanted to prove.

So from now on we just consider the case \( \gcd(f, g) = 1 \).

In our proof we consider two cases: \( n \geq m + 2 \) and \( n < m + 2 \).

Case \( n \geq m + 2 \). By Lemma 5,
\[ \#\{ k \in D \mid M(k) = 0 \} = \# \left\{ k \in D \mid \overline{P}(k) = f + g \frac{E}{K} = 0 \right\}. \]
As in the proof of Lemma 5 we apply the Argument Principle to \( \overline{P} \) in \( G = G_{R, \varepsilon} \) for \( R \) and \( 1/\varepsilon \) big enough.

By Lemma 4.(b) the number of complete turns around \( C_R \) is at most \( n + \mu(R) \), where \( \mu(R) \) tends to 0 as \( R \) goes to infinity.

By Lemma 4.(c)
\[ \#\{ k \in L^i_{\pm} \mid \text{Im} \overline{P}(k) = 0 \} = \# \left\{ k \in L^i_{\pm} \mid g \text{Im} \frac{E}{K} = 0 \right\} = \#\{ k \in L^i_{\pm} \mid g = 0 \}, \]
and hence the number of complete turns on $L^1_\pm(R,\varepsilon) \cup L^2(R,\varepsilon)$ is less than $m + 2$. Note that we have used that $\gcd(f, g) = 1$ to ensure that $\overline{P}$ is not zero on the banks.

Finally, by Lemma 4.(a) the number of complete turns of $\overline{P}$ on $C^1_\varepsilon \cup C^2_\varepsilon$ when $\varepsilon$ goes to 0 tends to zero.

Therefore the rotation number of $\overline{P}(k)$ on $\partial G$ is at most $n + m + 2$ and by the Argument Principle,

$$\# \{ k \in D \mid \overline{P}(k) = 0 \} \leq m + n + 2,$$

as we wanted to see.

The case $n < m + 2$ follows by similar considerations by taking $g + f K^E$ instead of $f + g E$. Hence the first part of Theorem 1 is proved.

Next we prove the remaining three statements in the theorem.

(a). For any given $k_i \in (-1, 1), i = 1, 2, \ldots, n + m + 1$, consider the system of linear equations

$$\sum_{j=0}^{n} a_j k^j K(k_i) + \sum_{j=0}^{m} b_j k^j E(k_i) = 0, \quad i = 1, 2, \ldots, n + m + 1. \quad (9)$$

Since the number of unknown variables $\{a_j\}, \{b_j\}$ is greater than the number of equations, there exists a solution of (9)

$$\{a_j\}_{j=0,1,\ldots,n}, \quad \{b_j\}_{j=0,1,\ldots,m} \text{ with } \sum_{j=0}^{n} a_j^2 + \sum_{j=0}^{m} b_j^2 > 0.$$

On the other hand, since by Lemma 4.(a), $K'/E$ is not a rational function, we have that

$$M(k) = \left(\sum_{i=0}^{n} a_i k^i\right) K + \left(\sum_{i=0}^{m} b_i k^i\right) E \not\equiv 0,$$

and

$$M(k_i) = 0, \quad i = 1, 2, \ldots, n + m + 1,$$

as we wanted to see.

(b). By arguing as in statement (a) but with the function $f(k)K(\sqrt{k}) + g(k)E(\sqrt{k})$ and taking $k_i$ values in $(0, 1), i = 1, 2, \ldots, n + m + 1$, we have a function with $f$ and $g$ of degrees $n$ and $m$ respectively, and $n + m + 1$ positive zeros. Hence the function $f(k^2)K(k) + g(k^2)E(k)$ is an even function with $2n + 2m + 2$ zeros (the values $\pm \sqrt{k_i}$) and $f(k^2)$ and $g(k^2)$ polynomials of degrees $2n$ and $2m$, respectively. Therefore the upper bound $n + m + 2$ is attained as we wanted to see.

(c). Our proof is divided in two cases:
(c.1) If \( f \) has zeros in \([-1, 1] \), then arguing as in the proof of the general upper bound we have that the complete turns of \( \overline{P} \) on \( L_1^\pm(R, \varepsilon) \cup L_2^\pm(R, \varepsilon) \) is at most \( n + 1 \) (one less than if we have no information about \( f \)) and the result follows.

(c.2) If \( f \) has no zeros in \([-1, 1] \), then when \( k \to \pm 1 \) with \( k \in (-1, 1) \), the function \( g + f \overline{K} \) is real and tends to infinity with the same sign in both cases. Therefore the difference between the argument of \( g + f \overline{K} \) for \(|k| < 1 \) near \(+1\) and near \(-1\) tends to be \( 2L\pi \) for some integer number \( L \). On the other hand arguing also as in the proof of the general upper bound but just taking the upper half part of the boundary of \( G \) we obtain that this difference is smaller or equal than \((n + m + 2)\pi\). Since \( n + m \) is odd, in fact this difference has to be smaller than \((n + m + 1)\pi\). By applying the same reasoning to the lower half boundary the result follows.

3. Perturbation of an isochronous center.

We need some preliminary results. In all this section \( P_i, Q_i \) denote polynomials of degree \( i \).

Lemma 6. Let \( f \) be a continuous function and let \( i, j \geq 0 \) be integers. Then the following hold:

(a) If \( i + j \) is odd, then
\[
\int_0^{2\pi} f(\sin 2\theta) \cos^i \theta \sin^j \theta \, d\theta = 0.
\]

(b) If \( i + j = 2N \) even, then there exist real constants \( C_0, C_1, \ldots, C_N \), such that
\[
\int_0^{2\pi} f(\sin 2\theta) \cos^i \theta \sin^j \theta \, d\theta = \sum_{s=0}^{N} C_s \int_{-\pi}^{\pi} f(\cos \theta) \cos^s \theta \, d\theta
\]
\[
= \sum_{s=0}^{N} C_s \int_{-\pi}^{\pi} f(\sin \theta) \sin^s \theta \, d\theta.
\]

Proof. (a) Suppose that \( i + j \) is odd. Then
\[
I = \int_0^{2\pi} f(\sin 2\theta) \cos^i \theta \sin^j \theta \, d\theta \quad (\theta = \pi + \varphi)
\]
\[
= -\int_0^{2\pi} f(\sin 2\varphi) \cos^i \varphi \sin^j \varphi \, d\varphi
\]
\[
= -I,
\]
which implies \( I = 0 \).
(b) Assume \(i + j = 2N\) even. Then

\[
I = \int_0^{2\pi} f(\sin 2\theta) \cos^i \theta \sin^j \theta d\theta \quad \left( \theta = \frac{\pi}{4} - \varphi \right)
\]

\[
= \int_0^{2\pi} f(\cos 2\varphi) \left( \frac{1}{\sqrt{2}} \cos \varphi + \frac{1}{\sqrt{2}} \sin \varphi \right)^i \left( \frac{1}{\sqrt{2}} \cos \varphi - \frac{1}{\sqrt{2}} \sin \varphi \right)^j d\varphi
\]

\[
= \sum_{s=0}^{i+j} d_s \int_{-\pi}^\pi f(\cos 2\varphi)(\cos \varphi)^{i+j-s}(\sin \varphi)^s d\varphi
\]

\[
= \sum_{s=0}^N d_{2s} \int_{-\pi}^\pi f(\cos 2\varphi)(\cos \varphi)^{2N-2s}(\sin \varphi)^{2s} d\varphi
\]

\[
= \sum_{s=0}^N d_{2s} \int_{-\pi}^\pi f(\cos 2\varphi) \left( \frac{1 + \cos 2\varphi}{2} \right)^{N-s} \left( \frac{1 - \cos 2\varphi}{2} \right)^s d\varphi
\]

\[
= \sum_{s=0}^N C_s \int_{-\pi}^\pi f(\cos 2\varphi)(\cos 2\varphi)^s d\varphi \quad (2\varphi = \theta)
\]

\[
= \sum_{s=0}^N C_s \int_{-\pi}^\pi f(\cos \theta)(\cos \theta)^s d\theta \quad \left( \theta = \frac{\pi}{2} - \varphi \right)
\]

\[
= \sum_{s=0}^N C_s \int_{-\pi}^\pi f(\sin \varphi)(\sin \varphi)^s d\varphi,
\]

where the value of the constants might vary from one expression to the other. Hence the proof of the lemma is complete. \(\square\)

**Lemma 7.** Consider

\[
J_m = J_m(h) := \int_{-\pi}^\pi (\sin \theta)^{2m} \sqrt{h^2 \sin^2 \theta + h} \ d\theta,
\]

where \(m\) is zero, or a natural number. Then there exist polynomials \(P_m\) and \(Q_m\) of degree \(m\), such that

\[
J_m = \frac{k^{1-2m}}{1-k^2} \left( P_m(k^2)K + Q_m(k^2)E \right),
\]

where \(k^2 = h/(1 + h)\).
Proof. We consider

\[ J_m = \int_{-\pi}^{\pi} (\sin \theta)^2 \sqrt{h^2 \sin^2 \theta + h} \, d\theta \]
\[ = \int_{-\pi}^{\pi} (\cos \theta)^2 \sqrt{h^2 \cos^2 \theta + h} \, d\theta \]
\[ = \int_{-\pi}^{\pi} (1 - \sin^2 \theta)^m \sqrt{h^2 + h - h^2 \sin^2 \theta} \, d\theta \]
\[ = \sqrt{h^2 + h} \int_{-\pi}^{\pi} (1 - \sin^2 \theta)^m \sqrt{1 - \frac{h}{h+1} \sin^2 \theta} \, d\theta \quad \left( k^2 = \frac{h}{1+h} \right) \]
\[ = \frac{k}{1-k^2} \int_{-\pi}^{\pi} \sum_{i=0}^{m} (-1)^i C_m^{m-i} (\sin \theta)^{2i} \sqrt{1 - k^2 \sin^2 \theta} \, d\theta \]
\[ = \frac{k}{1-k^2} \sum_{i=0}^{m} (-1)^i C_m^{m-i} B_i, \]

where

\[ B_i = \int_{-\pi}^{\pi} (\sin \theta)^{2i} \sqrt{1 - k^2 \sin^2 \theta} \, d\theta. \]

If \( \sin \theta = t \), then
\[ B_i = 4 \int_{0}^{1} t^{2i} \sqrt{\frac{1-k^2 t^2}{1-t^2}} \, dt \]
\[ = 4(O_{2i} - k^2 O_{2i+2}), \]

where
\[ O_{2i} = \int_{0}^{1} \frac{t^{2i}}{\sqrt{(1-t^2)(1-k^2 t^2)}} \, dt. \]

By formula 320.05 of [2], \( O_{2i} \) satisfy the following recurrence formula:
\[ O_{2i+2} = \frac{2i(1+k^2)O_{2i} + (1-2i)O_{2i-2}}{(2i+1)k^2} \]

and
\[ O_0 = K, \quad O_2 = \frac{1}{k^2}(K - E). \]

Thus, by induction we get

\[ B_i = P_i \left( \frac{1}{k^2} \right) K + Q_i \left( \frac{1}{k^2} \right) E, \]

(11)
and hence substituting (11) into (10), we get
\[
J_m = \frac{k}{1-k^2} \left( P_m \left( \frac{1}{k^2} \right) K + Q_m \left( \frac{1}{k^2} \right) E \right).
\]

From the above expression the lemma follows easily. \(\square\)

We also need the following result which has a straightforward proof.

**Lemma 8.** Consider
\[
W_s = W_s(h) := \int_{-\pi}^{\pi} \log(h \sin \theta + \sqrt{h^2 \sin^2 \theta + h})(\sin \theta)^s \ d\theta,
\]
where \(s\) is zero, or a natural number. Then
\[
\frac{\partial W_s(h)}{\partial h} = \begin{cases} 
\frac{1}{2h} \int_{-\pi}^{\pi} \sin^s(\theta) \ d\theta, & \text{if } s \text{ is even,} \\
\frac{1}{2h} \left( J_{s+1}(h) - h \frac{\partial J_{s+1}(h)}{\partial h} \right), & \text{if } s \text{ is odd,}
\end{cases}
\]
where \(J_m\) are defined in Lemma 7.

**Lemma 9.** Let \(f(x), g(x)\) be analytic functions on \((a, b) \subset \mathbb{R}\), then
\[
\# \{x \in (a, b) \mid f(x) + g(x) = 0\} \leq \# \{x \in (a, b) \mid f(x) = 0\} + \# \{x \in (a, b) \mid fg' - gf' = 0\} + 1.
\]

**Proof.** Set \(F = f + g\). Then
\[
F'f - Ff' = fg' - gf'.
\]
We just make the proof for the case of simple zeros of \(F\). The case of multiple zeros follows in a similar way. Let \(x_1 < x_2\) be two consecutive simple zeros of \(F\). If \(f\) does not vanish in \([x_1, x_2]\) then \(F'f\) has different signs in \(x_1\) and \(x_2\). Therefore the above expression implies that \(fg' - gf'\) has a zero in \((x_1, x_2)\). Hence the lemma follows. \(\square\)

**Proof of Theorem 2.** Consider the polynomial perturbation of the isochronous system \(\dot{z} = iz + z^3\):
\[
\begin{align*}
\dot{x} &= -y + x^3 - 3xy^2 + \varepsilon P(x, y), \\
\dot{y} &= x + 3x^2y - y^3 + \varepsilon Q(x, y),
\end{align*}
\]
where \(P, Q\) are real polynomials of degree \(n\). For \(\varepsilon = 0\), (13) has a first integral \((1 + 4xy)(x^2 + y^2)^{-2}\) with integrating factor \((x^2 + y^2)^{-3}/4\).

Denote by \(\Gamma_h: H = h^{-1} (h > 0)\) all periodic orbits surrounding the center \((0, 0)\). In polar coordinates \(x = r \cos \theta, y = r \sin \theta,\)
\[
\Gamma_h: r = r_h(\theta) = r(\theta) = \sqrt{h \sin 2\theta + \sqrt{h^2 \sin^2 2\theta + h}}.
\]
By using (2) we know that the Abelian integral associated to (13) is defined as

\[ M(h) = \int_{\Gamma_h} \frac{P}{4(x^2 + y^2)^3} dy - \frac{Q}{4(x^2 + y^2)^3} dx. \]

Denote by \( D_h \) the simple connected region enclosed by \( \Gamma_h \) and \( D_{h,\delta} = D_h \setminus \{ r \leq \delta \} \).

By Green’s formula

\[ M(h) = \iint_{D_{h,\delta}} \left[ \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right] \frac{4}{x^2 + y^2} \left[ \frac{P}{4(x^2 + y^2)^3} - \frac{3xP + 3yQ}{2(x^2 + y^2)^4} \right] dxdy - T_\delta, \]

where

\[ T_\delta = \int_{r=\delta} \frac{P}{(x^2 + y^2)^3} dy - \frac{Q}{(x^2 + y^2)^3} dx. \]

Let \( \frac{1}{4} [(x^2 + y^2)(\partial P/\partial x + \partial Q/\partial y) - 6xP - 6yQ] = \sum_{1 \leq i + j \leq n + 1} C_{i,j} x^i y^j \).

Then

\[ M(h) = \sum_{1 \leq i + j \leq n + 1} C_{i,j} \int \int_{D_{h,\delta}} \frac{x^i y^j}{(x^2 + y^2)^4} dx dy - T_\delta \]

\[ (x = r \cos \theta, \ y = r \sin \theta) \]

\[ = \sum_{1 \leq i + j \leq n + 1} C_{i,j} \int_0^{2\pi} \int_\delta r^{i+j-7} \cos^i \theta \sin^j \theta dr d\theta - T_\delta \]

\[ = \sum_{i+j \neq 6} C_{i,j} \int_0^{2\pi} \frac{1}{i+j-6} r^{i+j-6} \cos^i \theta \sin^j \theta d\theta \]

\[ + \sum_{i+j=6} C_{i,j} \int_0^{2\pi} \log(r(\theta)) \cos^i \theta \sin^j \theta d\theta - C_\delta, \]

where

\[ C_\delta = \sum_{i+j \neq 6} C_{i,j} \delta^{i+j-6} \int_0^{2\pi} \cos^i \theta \sin^j \theta d\theta \]

\[ + \sum_{i+j=6} C_{i,j} \log \delta \int_0^{2\pi} \cos^i \theta \sin^j \theta d\theta + T_\delta. \]

We want to control the number of positive zeros of \( M(h) \). In fact, in the final step we will study \( \partial M(h)/\partial h \). Since \( C_\delta \) does not depend on \( h \) we do not take care of this constant.
From the above formulas

\begin{equation}
M(h) = \sum_{1 \leq i+j \leq n+1} C_{i,j} I_{i,j} - C_\delta,
\end{equation}

where

\begin{align*}
I_{i,j} &= \frac{1}{i+j-6} \int_0^{2\pi} r(\theta)^{i+j-6} \cos^i \theta \sin^j \theta \, d\theta, \quad i+j \neq 6, \\
I_{i,j} &= \int_0^{2\pi} \log(r(\theta)) \cos^i \theta \sin^j \theta \, d\theta, \quad i+j = 6.
\end{align*}

By Lemma 6, we have that

\begin{equation}
I_{i,j} = 0, \quad \text{if} \quad i+j \quad \text{odd},
\end{equation}

and that there exist constants $C_s = C_s(i,j)$, $s = 0, 1, \ldots, N$, such that $I_{i,j}$ is equal to

\begin{equation}
\begin{cases}
\sum_{s=0}^{N} C_s \int_{-\pi}^{\pi} (\sin \theta)^s \left( h \sin \theta + \sqrt{h^2 \sin^2 \theta + h} \right)^{N-3} d\theta, & \text{if } i+j = 2N \neq 6, \\
\sum_{s=0}^{3} C_s \int_{-\pi}^{\pi} (\sin \theta)^s \log \left( h \sin \theta + \sqrt{h^2 \sin^2 \theta + h} \right) d\theta, & \text{if } i+j = 6.
\end{cases}
\end{equation}

Thus,

\begin{equation*}
M(h) = \sum_{N=1}^{n+1} I_N - C_\delta,
\end{equation*}

where

\begin{equation*}
I_N = \sum_{i+j=2N} C_{i,j} I_{i,j}.
\end{equation*}

We calculate some compact expressions for $I_N$, for $N < 3, N = 3$ and $N > 3$, separately.
For $i + j = 2N < 6$, by Lemma 6, we have

$$I_{i,j} = \sum_{s=0}^{N} C_s \int_{-\pi}^{\pi} (\sin \theta)^s \left( \frac{1}{\sqrt{h^2 \sin^2 \theta + h + h \sin \theta}} \right)^{3-N} d\theta$$

$$= \sum_{s=0}^{N} C_s \int_{-\pi}^{\pi} (\sin \theta)^s \left( \frac{\sqrt{h^2 \sin^2 \theta + h - h \sin \theta}}{h} \right)^{3-N} d\theta$$

$$= h^{N-3} \sum_{s=0}^{N} C_s \int_{-\pi}^{\pi} (\sin \theta)^s \sum_{m=0}^{3-N} C_m^m (-h)^m (\sin \theta)^m$$

$$\cdot (h^2 \sin^2 \theta + h)^{3-N-m} d\theta$$

$$= \begin{cases} 
2\pi C_0 (1 + \frac{1}{h}) - 2C_1 J_1, & \text{if } N = 1, \\
-C_1 \pi + \frac{C_0}{\pi} J_0 + \frac{C_2}{h} J_1, & \text{if } N = 2. 
\end{cases}$$

Next we study $I_{i,j}$ for $i + j = 6$. By (16),

$$I_{i,j} = \sum_{s=0}^{3} C_s W_s, \quad W_s = \int_{-\pi}^{\pi} \log(h \sin \theta + \sqrt{h^2 \sin^2 \theta + h})(\sin \theta)^s d\theta.$$

By Lemma 8,

$$\frac{\partial W_0}{\partial h} = \frac{\pi}{h}, \quad \frac{\partial W_1}{\partial h} = \frac{1}{h} \left( J_1 - h \frac{\partial J_1}{\partial h} \right),$$

$$\frac{\partial W_2}{\partial h} = \frac{\pi}{2h}, \quad \frac{\partial W_3}{\partial h} = \frac{1}{h} \left( J_2 - h \frac{\partial J_2}{\partial h} \right).$$

Hence, we have

$$\frac{\partial I_3}{\partial h} = \frac{d_0}{h} + d_1 \frac{J_1}{h} + d_2 \frac{J_2}{h} + d_3 \frac{\partial J_1}{\partial h} + d_4 \frac{\partial J_2}{\partial h},$$

where $d_i$ are constants.
For $i + j = 2N > 6$,

\begin{align*}
I_{i,j} &= \sum_{s=0}^{N} C_s \int_{-\pi}^{\pi} (\sin \theta)^s \sum_{m=0}^{N-3} C_{N-3}^m h^m (\sin \theta)^m (h^2 \sin^2 \theta + h)^{\frac{N-3-m}{2}} d\theta \\
&= \begin{cases} 
    h^{N-3} \sum_{s=0}^{N} C_s \int_{-\pi}^{\pi} (\sin \theta)^s \left( \sum_{m=0}^{N-3} p_m (\sin \theta)^{2m} h^m 
    + \sum_{m=0}^{N-5} q_m (\sin \theta)^{2m+1} h^m \sqrt{h^2 \sin^2 \theta + h} \right) d\theta, & \text{if } N \text{ odd}, \\
    h^{N-2} \sum_{s=0}^{N} C_s \int_{-\pi}^{\pi} (\sin \theta)^s \left( \sum_{m=0}^{N-4} p_m (\sin \theta)^{2m+1} h^{m+1} 
    + \sum_{m=0}^{N-2} q_m (\sin \theta)^{2m} h^m \sqrt{h^2 \sin^2 \theta + h} \right) d\theta, & \text{if } N \text{ even},
\end{cases}
\end{align*}

where $p_m, q_m$ are constants and we just have separated the odd and even values of $m$.

Remember that

\[ J_m = \int_{-\pi}^{\pi} (\sin \theta)^{2m} \sqrt{h^2 \sin^2 \theta + h} d\theta. \]

From (19), we have

\begin{align*}
I_{i,j} &= \begin{cases} 
    h^{N-3} \left( P_{N-3}^0 (h) + \sum_{m=1}^{N-2} U_m(h) J_m \right), & \text{if } N \text{ odd}, \\
    h^{N-2} \left( P_{N-2}^0 (h) + \sum_{m=0}^{N-2} V_m(h) J_m \right), & \text{if } N \text{ even},
\end{cases}
\end{align*}

where $P_{N-3}^0, P_{N-2}^0, U_m, V_m$ are polynomials of $h$ with

\begin{align*}
\deg P_{N-3}^0 &\leq \frac{N-3}{2}, & \deg P_{N-2}^0 &\leq \frac{N-2}{2}, \\
\deg U_m &\leq \min \left\{ m - 1, \frac{N-5}{2} \right\}, & \deg V_m &\leq \min \left\{ m, \frac{N-4}{2} \right\}.
\end{align*}

From now on we introduce the variable $k$, as $k^2 = h/(1+h)$. By Lemma 7, equality (17) writes as

\begin{align*}
I_1 &= \frac{1}{k^2} \left( P_0 + kP_1 \left( \frac{1}{k^2} \right) K + kQ_1 \left( \frac{1}{k^2} \right) E \right), \\
I_2 &= P_0 + \frac{1}{k} \left( P_1 \left( \frac{1}{k^2} \right) K + Q_1 \left( \frac{1}{k^2} \right) E \right).
\end{align*}
By using again Lemma 7, equality (20) writes as

\[
I_N = \begin{cases} 
\frac{P_{N-3}(k^2)}{(1-k^2)^{N-3}} + \frac{1}{k^{N-2}(1-k^2)^{N-3}} \left[ P_{\frac{3}{2}N-\frac{7}{2}}(k^2)K + Q_{\frac{3}{2}N-\frac{9}{2}}(k^2)E \right] 
& \text{if } N \geq 5 \text{ odd,} \\
\frac{P_{N-3}(k^2)}{(1-k^2)^{N-3}} + \frac{1}{k^{N-1}(1-k^2)^{N-3}} \left[ P_{\frac{3}{2}N-4}(k^2)K + Q_{\frac{3}{2}N-2}(k^2)E \right] 
& \text{if } N \geq 4 \text{ even.}
\end{cases}
\]

Let \( N = \left[ \frac{n+1}{2} \right] \), then for \( n \geq 7 \), we have that

\[
M(h) = \sum_{i=1}^{N} I_i - C_\delta
\]

\[
= \begin{cases} 
\frac{P_{1}(k^2)}{k^2} + I_3 - C_\delta + \frac{P_{N-3}(k^2)}{(1-k^2)^{N-3}} \\
& \text{if } N \geq 5 \text{ odd,} \\
\frac{P_{1}(k^2)}{k^2} + I_3 - C_\delta + \frac{P_{N-3}(k^2)}{(1-k^2)^{N-3}} \\
& \text{if } N \geq 4 \text{ even,}
\end{cases}
\]

By (5), (18), taking into account that \( dI_3/dk = 2k(1-k^2)^{-2}(dI_3/dh) \) and direct computations give that

\[
dM/dk = \begin{cases} 
k^{1-N}(1-k^2)^{2-N}(R_0 + R_1), 
& \text{if } N \geq 5 \text{ odd,} \\
k^{-N}(1-k^2)^{2-N}(R_0 + R_1), 
& \text{if } N \geq 4 \text{ even,}
\end{cases}
\]

where

\[
R_0 = \begin{cases} 
k^{N-4}P_{N-1}(k^2), 
& \text{if } N \geq 5 \text{ odd,} \\
k^{N-3}P_{N-1}(k^2), 
& \text{if } N \geq 4 \text{ even,}
\end{cases}
\]

\[
R_1 = \begin{cases} 
P_{\frac{3}{2}N-\frac{7}{2}}(k^2)K + Q_{\frac{3}{2}N-\frac{9}{2}}(k^2)E, 
& \text{if } N \geq 5 \text{ odd,} \\
P_{\frac{3}{2}N-3}(k^2)K + Q_{\frac{3}{2}N-3}(k^2)E, 
& \text{if } N \geq 4 \text{ even.}
\end{cases}
\]
Next we estimate the number of zeros of \( dM/dk \) by applying Lemma 9 to \( R_0 + R_1 \). Straightforward computations show that

\[
R_0 R'_1 - R'_0 R_1 = \begin{cases} 
\frac{k^{N-5}}{1-k^2} R, & \text{if } N \geq 5 \text{ odd}, \\
\frac{k^{N-4}}{1-k^2} R, & \text{if } N \geq 4 \text{ even}, 
\end{cases}
\]

where

\[
R = \begin{cases} 
P_{\frac{N-7}{2}}(k^2)K + Q_{\frac{N-7}{2}}(k^2)E, & \text{if } N \geq 5 \text{ odd}, \\
P_{\frac{N-3}{2}}(k^2)K + Q_{\frac{N-3}{2}}(k^2)E, & \text{if } N \geq 4 \text{ even}. 
\end{cases}
\]

By Theorem 1, we obtain

\[
\# \{-1 < k < 1 \mid R = 0\} \leq \begin{cases} 
2(5N - 7) + 2 = 10N - 12, & \text{if } N \geq 5 \text{ odd}, \\
2(5N - 6) + 2 = 10N - 10, & \text{if } N \geq 4 \text{ even}. 
\end{cases}
\]

Note that \( R \) is an even function. Therefore we have

\[
\# \{0 < k < 1 \mid R = 0\} \leq \begin{cases} 
5N - 6, & \text{if } N \geq 5 \text{ odd}, \\
5N - 5, & \text{if } N \geq 4 \text{ even}. 
\end{cases}
\]

By Lemma 9 we obtain

\[
\# \left\{ 0 < k < 1 \mid \frac{dM}{dk} = 0 \right\} \leq \begin{cases} 
N - 1 + 5N - 6 + 1 = 6N - 6, & \text{if } N \geq 5 \text{ odd}, \\
N - 1 + 5N - 5 + 1 = 6N - 5, & \text{if } N \geq 4 \text{ even}. 
\end{cases}
\]

From Rolle’s Theorem, it follows that

\[
\# \{h > 0 \mid M(h) = 0\} = \# \{0 < k < 1 \mid M(k) = 0\} \leq \begin{cases} 
6N - 5 = 6 \left\lceil \frac{n+1}{2} \right\rceil - 5, & \text{if } \left\lceil \frac{n+1}{2} \right\rceil \geq 5 \text{ odd}, \\
6N - 4 = 6 \left\lceil \frac{n+1}{2} \right\rceil - 4, & \text{if } \left\lceil \frac{n+1}{2} \right\rceil \geq 4 \text{ even}, 
\end{cases}
\]

\[
\leq 3n - 1.
\]

From the above inequality the theorem follows. \( \square \)

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References


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NON-COMMUTATIVE CLARKSON INEQUALITIES FOR UNITARILY INVARIANT NORMS

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It is shown that if $A$ and $B$ are operators on a separable complex Hilbert space and if $||| \cdot |||$ is any unitarily invariant norm, then

\[
2 ||| |A|^p + |B|^p ||| \leq ||| |A + B|^p + |A - B|^p ||| \\
\leq 2^{p-1} ||| |A|^p + |B|^p |||
\]

for $2 \leq p < \infty$, and

\[
2^{p-1} ||| |A|^p + |B|^p ||| \leq ||| |A + B|^p + |A - B|^p ||| \\
\leq 2 ||| |A|^p + |B|^p |||
\]

for $0 < p \leq 2$. These inequalities are natural generalizations of some of the classical Clarkson inequalities for the Schatten $p$-norms. Generalizations of these inequalities to larger classes of functions including the power functions are also obtained.

1. Introduction.

The classical Clarkson inequalities for the Schatten $p$-norms of Hilbert space operators assert that

(1) \[
2 (||A||_p^p + ||B||_p^p) \leq ||A + B||_p^p + ||A - B||_p^p \leq 2^{p-1} (||A||_p^p + ||B||_p^p)
\]

for $2 \leq p < \infty$,

(2) \[
2^{p-1} (||A||_p^p + ||B||_p^p) \leq ||A + B||_p^p + ||A - B||_p^p \leq 2 (||A||_p^p + ||B||_p^p)
\]

for $0 < p \leq 2$,

(3) \[
2 (||A||_p^p + ||B||_p^p)^{q/p} \leq ||A + B||_p^q + ||A - B||_p^q
\]

for $2 \leq p < \infty$; $\frac{1}{p} + \frac{1}{q} = 1$, and

(4) \[
||A + B||_p^q + ||A - B||_p^q \leq 2 (||A||_p^p + ||B||_p^p)^{q/p}
\]

for $1 < p \leq 2$; $\frac{1}{p} + \frac{1}{q} = 1$.

These inequalities, which can be found in [11], are non-commutative versions of the celebrated Clarkson inequalities for the classical sequence spaces. These inequalities have useful applications in operator theory and in mathematical physics (see, e.g., [2], [5], [7], [10], [12], and references therein). In
particular, the uniform convexity of the Schatten \( p \)-classes, for \( 1 < p < \infty \), is an immediate consequence of the inequalities (1) and (4). For a comprehensive account of the Clarkson inequalities, the reader is referred to [8].

Extensions, with proof simplification, of the inequalities (1) and (2) (for \( 1 \leq p \leq 2 \)) to wider classes of unitarily invariant norms including the Schatten \( p \)-norms have been given in [4]. This has been achieved by formulating these inequalities in terms of direct sums of operators.

In this paper we give pretty natural generalizations of the inequalities (1) and (2) to all unitarily invariant norms. In fact, our new inequalities seem natural enough and applicable to be widely useful.

Let \( B(H) \) denote the \( C^* \)-algebra of all bounded linear operators on a separable complex Hilbert space \( H \). If \( A \) is a compact operator in \( B(H) \), then the singular values of \( A \) are, by definition, the eigenvalues of the positive operator \( |A| = (A^*A)^{1/2} \) enumerated as \( s_1(A) \geq s_2(A) \geq \cdots \geq 0 \).

Recall that, with the exception of the usual operator norm, which is defined on all of \( B(H) \), each unitarily invariant norm is a symmetric gauge function of the singular values and is defined on a norm ideal contained in the ideal of compact operators. For the sake of brevity, we will make no explicit mention of this norm ideal. Thus, when we talk of \( |||A||| \), we are assuming that \( A \) belongs to the norm ideal associated with \( ||| \cdot ||| \).

If \( A \) is a compact operator in \( B(H) \), let

\[
|||A|||_p = \left( \sum_{j=1}^{\infty} s_j(A)^p \right)^{1/p} = \left( \text{tr} |A|^p \right)^{1/p}
\]

for \( 0 < p \leq \infty \), where \( \text{tr} \) is the usual trace functional. This defines the Schatten \( p \)-norm (quasinorm) for \( 1 \leq p \leq \infty \) (\( 0 < p < 1 \)), where by convention \( |||A|||_{\infty} = s_1(A) \) is the usual operator norm of \( A \).

Since \( |||A|||_p \geq |||A|||_1 = \text{tr} |A|^p \) for \( 0 < p < \infty \), our generalizations of the inequalities (1) and (2) will be much appreciated if we rewrite them as

\[
(5) \quad 2|||A|||_p + |||B|||_p \leq |||A + B|||_1 \leq |||A + B|||_p + |||A - B|||_1 \leq 2^{p-1} |||A|||_p + |||B|||_p \leq |||A|||_1
\]

for \( 2 \leq p < \infty \), and

\[
(6) \quad 2^{p-1} |||A|||_p + |||B|||_p \leq |||A + B|||_1 \leq |||A + B|||_p + |||A - B|||_1 \leq 2|||A|||_p + |||B|||_p \leq |||A|||_1
\]

for \( 0 < p \leq 2 \).

In Section 2 of this paper, we will show that the trace norm \( ||| \cdot |||_1 \) in (5) and (6) can be replaced by any unitarily invariant norm \( ||| \cdot ||| \), and that the power functions \( f(t) = t^p \) can be replaced by more general classes of functions.
2. Main results.

To achieve our goal of generalizing the inequalities (1) and (2), we need the following two lemmas. The first lemma is a well-known result that can be proved by using the spectral theorem and Jensen’s inequality. The inequalities in this lemma are of the Peierls-Bogoliubov type (see, e.g., [3, p. 281] or [12, pp. 101-102]).

**Lemma 1.** Let $A$ be a positive operator in $B(H)$.

(a) If $g$ is a convex function on $[0, \infty)$, then

$$g(\langle Ax, x \rangle) \leq \langle g(A)x, x \rangle$$

for every unit vector $x$ in $H$.

(b) If $h$ is a concave function on $[0, \infty)$, then

$$\langle h(A)x, x \rangle \leq h(\langle Ax, x \rangle)$$

for every unit vector $x$ in $H$.

The second lemma, which is due to Ando and Zhan [1], contains norm inequalities comparing $f(A + B)$ and $f(A) + f(B)$ for certain functions $f$ (see, also [6]).

In this lemma and in the sequel, $||| \cdot |||$ designates any unitarily invariant norm.

**Lemma 2.** Let $A$ and $B$ be positive operators in $B(H)$.

(a) If $g$ is an increasing function on $[0, \infty)$ such that $g(0) = 0$, $\lim_{t \to \infty} g(t) = \infty$, and $g^{-1}$ is an operator monotone function, then

$$|||g(A) + g(B)||| \leq |||g(A + B)|||.$$  \hspace{1cm} (9)

(b) If $h$ is a nonnegative operator monotone function on $[0, \infty)$, then

$$|||h(A + B)||| \leq |||h(A) + h(B)|||.$$  \hspace{1cm} (10)

Now we are in a position to present our main results. The first result is a considerable generalization of the inequalities (1).

**Theorem 1.** Let $A$ and $B$ be operators in $B(H)$ and let $f$ be an increasing function on $[0, \infty)$ such that $f(0) = 0$, $\lim_{t \to \infty} f(t) = \infty$, and the inverse function of $g(t) = f(\sqrt{t})$ is operator monotone. Then

$$2 |||f(|A|) + f(|B|)||| \leq |||f(|A + B|) + f(|A - B|)||| \leq \frac{1}{2} |||f(2|A|) + f(2|B|)|||.$$  \hspace{1cm} (11)

**Proof.** Since $g^{-1}$ is operator monotone, it follows that it is concave (see, e.g., [3, p. 120]). Since $g$ is increasing, the concavity of $g^{-1}$ implies that $g$
is convex. Now for any unit vector $x$ in $H$, we have

\[
\langle f(|A + B|) + f(|A - B|) \rangle x, x \rangle \\
= \langle g(|A + B|^2) x, x \rangle + \langle g(|A - B|^2) x, x \rangle \\
\geq g(\langle |A + B|^2 x, x \rangle) + g(\langle |A - B|^2 x, x \rangle) \\
\geq 2g\left(\frac{\langle |A + B|^2 x, x \rangle + \langle |A - B|^2 x, x \rangle}{2}\right) \\
= 2g(\langle |A|^2 + |B|^2 \rangle x, x).
\]

Using the min-max principle (see, e.g., [3, p. 58] or [9, p. 25]) and the fact that $g$ is increasing, we see that

\[
s_j(f(|A + B|) + f(|A - B|)) \geq 2g(s_j(|A|^2 + |B|^2)) \\
= 2s_j(g(|A|^2 + |B|^2))
\]

for $j = 1, 2, \ldots$. Since unitarily invariant norms are increasing with respect to singular values (see, e.g., [3, p. 52] or [9, p. 71]), it follows that

\[
\|\|f(|A + B|) + f(|A - B|)\|\| \\
\geq 2\|\|g(|A|^2 + |B|^2)\|\| \\
\geq 2\|\|g(|A|^2) + g(|B|^2)\|\| \\
= 2\|\|f(|A|) + f(|B|)\|\|,
\]

which proves the first inequality in (11). The second inequality in (11) follows from the first one by replacing $A$ and $B$ by $A + B$ and $A - B$, respectively.

Based on Lemmas 1(b) and 2(b), one can employ an argument similar to that used in the proof of Theorem 1 to derive the following generalization of the inequality (2).

**Theorem 2.** Let $A$ and $B$ be operators in $B(H)$ and let $f$ be a nonnegative function on $[0, \infty)$ such that $h(t) = f(\sqrt{t})$ is operator monotone. Then

\[
\frac{1}{2}\|\|f(2|A|) + f(2|B|)\|\| \leq \|\|f(|A + B|) + f(|A - B|)\|\| \\
\leq 2\|\|f(|A|) + f(|B|)\|\|.
\]

Specializing Theorems 1 and 2 to the functions $f(t) = t^p$ $(2 \leq p < \infty)$ and $f(t) = t^p$ $(0 < p \leq 2)$, respectively, we obtain our promised natural generalizations of the inequalities (1) and (2).

**Corollary 1.** Let $A$ and $B$ be operators in $B(H)$. Then

\[
2\|\||A|^p + |B|^p || \leq \|\||A + B|^p + |A - B|^p || \leq 2^{p-1} \|\||A|^p + |B|^p ||
\]
for $2 \leq p < \infty$, and
\begin{align}
2^{p-1} \left\| |A|^p + |B|^p \right\| &\leq \left\| |A + B|^p + |A - B|^p \right\| \leq 2 \left\| |A|^p + |B|^p \right\| \\
&\leq 2 \left\| |A + B|^p + |A - B|^p \right\|
\end{align}
for $0 < p \leq 2$.

It should be observed here that the inequalities (5) and (6) (and so the inequalities (1) and (2)) follow from the inequalities (13) and (14) specialized to the trace norm.

It has been remarked in [4] that, although the inequalities (1) and (2) are usually proved separately, they follow from (3) and (4), respectively, simply by the convexity of the function $f(t) = t^{p/q}$ ($2 \leq p < \infty; \frac{1}{p} + \frac{1}{q} = 1$) and the concavity of the function $f(t) = t^{p/q}$ ($1 < p \leq 2; \frac{1}{p} + \frac{1}{q} = 1$). Thus, it would be desirable to find natural generalizations (perhaps along the lines of our generalizations of (1) and (2)) of the inequalities (3) and (4) to all unitarily invariant norms.

Let $f(t) = e^t - 1$. Then $f$ is increasing on $[0, \infty)$, $f(0) = 0$, $\lim_{t \to \infty} f(t) = \infty$, and the inverse of $g(t) = f(\sqrt{t}) = e^t - 1$ is the operator monotone function $g^{-1}(t) = \log(t + 1)$.

Applying Theorem 1 to this special function, we have the following corollary.

**Corollary 2.** Let $A$ and $B$ be operators in $B(H)$. Then
\begin{align}
2 \left\| e^{|A|^2} + e^{|B|^2} - 2I \right\| &\leq \left\| e^{|A + B|^2} + e^{|A - B|^2} - 2I \right\| \\
&\leq \frac{1}{2} \left\| e^{|A|^2} + e^{|B|^2} - 2I \right\|.
\end{align}

Now let $f(t) = \log(t + 1)$. Then $h(t) = f(\sqrt{t}) = \log(\sqrt{t} + 1)$ is operator monotone on $[0, \infty)$. So applying Theorem 2 to this function, we have the following corollary.

**Corollary 3.** Let $A$ and $B$ be operators in $B(H)$. Then
\begin{align}
\frac{1}{2} \left\| \log(2|A| + I) + \log(2|B| + I) \right\| &\leq \left\| \log(|A + B| + I) + \log(|A - B| + I) \right\| \\
&\leq 2 \left\| \log(|A| + I) + \log(|B| + I) \right\|.
\end{align}

The most basic unitarily invariant norms are the Ky Fan norms $\| \cdot \|_{(k)}$ defined as
\[ \|A\|_{(k)} = \sum_{j=1}^{k} s_j(A) \]
for $k = 1, 2, \ldots$. The Ky Fan dominance principle says that $\|A\| \leq \|B\|$ for all unitarily invariant norms if and only if $\|A\|_{(k)} \leq \|B\|_{(k)}$ for all $k = 1, 2, \ldots$ (see, e.g., [3, p. 93] or [9, p. 72]).
Utilizing the Ky Fan dominance principle, enables us to conclude the following finite-dimensional consequence of Corollary 2.

**Corollary 4.** Let $A$ and $B$ be operators in $B(H)$, where $H$ is an $n$-dimensional Hilbert space. Then

\begin{equation}
2 \left\| e^{|A|^2} + e^{|B|^2} \right\| \leq \frac{1}{2} \left\| e^{4|A|^2} + e^{4|B|^2} + 6I \right\|.
\end{equation}

**Proof.** Applying Corollary 2 to the Ky Fan norms, we have

\begin{equation*}
2 \left\| e^{|A|^2} + e^{|B|^2} - 2I \right\|_{(k)} \leq \left\| e^{|A+B|^2} + e^{|A-B|^2} - 2I \right\|_{(k)} \leq \frac{1}{2} \left\| e^{4|A|^2} + e^{4|B|^2} - 2I \right\|_{(k)}
\end{equation*}

for $k = 1, 2, \ldots, n$. Thus,

\begin{equation*}
2 \left\| e^{|A|^2} + e^{|B|^2} \right\|_{(k)} - 4k \leq \left\| e^{|A+B|^2} + e^{|A-B|^2} + 2I \right\|_{(k)} - 4k \leq \frac{1}{2} \left\| e^{4|A|^2} + e^{4|B|^2} + 6I \right\|_{(k)} - 4k
\end{equation*}

for $k = 1, 2, \ldots, n$, from which we get

\begin{equation*}
2 \left\| e^{|A|^2} + e^{|B|^2} \right\|_{(k)} \leq \left\| e^{|A+B|^2} + e^{|A-B|^2} + 2I \right\|_{(k)} \leq \frac{1}{2} \left\| e^{4|A|^2} + e^{4|B|^2} + 6I \right\|_{(k)}
\end{equation*}

for $k = 1, 2, \ldots, n$. Now the desired inequalities (17) follow by the Ky Fan dominance principle.

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**References**


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SOME EXAMPLES IN COHOMOLOGICAL DIMENSION THEORY

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We construct examples having remarkable properties of cohomological dimension.

1. Introduction.

It is well-known that $\dim X \leq n$ if and only if every map of a closed subspace of $X$ into the $n$-dimensional sphere $S^n$ can be extended over $X$. It is also well-known that for the cohomological dimension $\dim_G X$ of $X$ with respect to an abelian coefficient group $G$, $\dim_G X \leq n$ if and only if every map of a closed subspace of $X$ into the Eilenberg-Mac Lane complex $K(G,n)$ extends over $X$. These properties give rise to the notion of extensional dimension 

\[ e\dim_X \leq K \]

where $e\dim_X$ is the extensional dimension of $X$. Let $K$ be a CW complex. The extensional dimension of $X$ does not exceed $K$, written $e\dim X \leq K$, if every map of a closed subset of $X$ into $K$ extends over $X$. Here $e\dim X > K$ means that $e\dim X \leq K$ does not hold. We write $e\dim X > n$ if $e\dim X > K$ for every CW-complex $K$ which is not $n$-connected. Thus $e\dim > n$ implies both $\dim > n$ and $\dim_G > n$ for every group $G \neq 0$.

Below are listed some remarkable examples in cohomological dimension.

Theorem 1.1 (Dranishnikov \[1\]). There is a locally compact separable metric space $X$ such that $\dim_{\mathbb{Z}} X \leq 4$ and $\dim_{\mathbb{Z}} \beta X = \infty$ where $\beta X$ is the Stone-Cech compactification of $X$.

Theorem 1.2 (Dydak \[5\], cf. \[6\]). For each abelian group $G$ there is a separable metric space $X$ such that $\dim_G X \leq 3$ and every Hausdorff compactification of $X$ is of $\dim_G > 3$.

Theorem 1.3 (Dranishnikov-Repovš \[4\], cf. \[11\]). There is a compactum $X$ such that $\dim_{\mathbb{Z}_2} X \leq 1$ and $e\dim X > \mathbb{R}P^m$ for all integers $m > 0$.

The goal of this note is to improve these results with a simpler construction. Namely we will prove the following theorems.

Theorem 1.4. There is a locally compact separable metric space $X$ such that for every abelian group $G$ and every non-contractible CW-complex $P$, $\dim_G X \leq 2$ and $e\dim \beta X > P$. 

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Theorem 1.5. There is a separable metric space $X$ such that for every abelian group $G$ and every Hausdorff compactification $X'$ of $X$, $\dim_G X \leq 2$ and $e\dim X' > 2$.

Theorem 1.6. There is a compactum $X$ such that for every cyclic finite CW-complex $P$ and every abelian group $G$, $e\dim X > P$, $\dim_G X \leq 2$ and $\dim_G X \leq 1$ if $G$ is finite.

A space is called cyclic if at least one of its (reduced integral) homology groups does not vanish. We call a map homologically essential if it induces a nontrivial homomorphism of at least one of the homology groups.

The main tool for constructing our examples is the following theorem which was proved in [8]. We will formulate this theorem without using notations of truncated cohomology (note that no algebraic properties of truncated cohomology were used in [8]).

For a CW-complex $K$ and a space $L$, $[K,L]$ denotes the set of pointed homotopy classes of maps from $K$ to $L$. Let $\text{map}(K,L)$ stand for the space of pointed maps from $K$ to $L$. By $\text{map}(K,L) \cong 0$ we mean that $\text{map}(K,L)$ is weakly homotopy equivalent to a point, that is $\pi_n(\text{map}(K,L)) = [\Sigma^n K, L] = [K, \Omega^n L] = 0$ for every $n \geq 0$. Clearly $\text{map}(K,L) \cong 0$ implies both $\text{map}(\Sigma^n K, L) \cong 0$ and $\text{map}(K, \Omega^n L) \cong 0$ for all $n \geq 0$. A space $L$ and CW-complexes in Theorems 1.7-1.10 are assumed to be pointed. Maps between pointed spaces are also assumed to be pointed.

Theorem 1.7 ([8]). Let $K$ and $P$ be countable CW-complexes and let a space $L$ have finite homotopy groups. If $\text{map}(K,L) \cong 0$ and $[P,L] \neq 0$ then there exists a compactum $X$ such that $P < e - \dim X \leq K$.

Theorem 1.7 was formulated in [8] in a slightly different form. First, it was assumed in [8] that $K$ and $P$ are countable simplicial complexes. Since each countable CW-complex is homotopy equivalent to a countable simplicial complex we can replace simplicial complexes by CW-complexes. Secondly, it was assumed in [8] that $\text{map}(K,L) \cong 0$ and $|\pi_i(L)| < \infty$, $i = 0, 1, \ldots$ for any base point in $L$. This restriction can be omitted. Indeed, this is obvious if $L$ is path connected. Let a pointed map $f : P \longrightarrow L$ be essential. If $P$ is mapped by $f$ into the path component of the base point of $L$ then replace $L$ by the path component of the base point and we are done. So the only case one needs to consider is when both $P$ and $L$ are not path connected. Then the condition $[K,L] = 0$ (derived from $\text{map}(K,L) \cong 0$) implies that $K$ is connected. In this case one can define $X = [0,1]$ which obviously satisfies $P < e - \dim X \leq K$.

We will need a more precise version of Theorem 1.7 (which was actually proved in [8]).
Theorem 1.8. Let $K$, $P = P_0$ be countable CW-complexes, let a space $L$ have finite homotopy groups and let map $(K, L) \cong 0$. Let $P_1, \ldots, P_n$ be CW-complexes and let maps $f_i : P_i \to P_{i+1}, i = 0, 1, \ldots, n-1$ and $f_n : P_n \to L$ be such that $f = f_n \circ \cdots \circ f_0 : P \to L$ is essential. Then there are a compactum $X$, a closed subset $X'$ of $X$ and a map $g : X' \to P$ such that $e$-dim $X \leq K$ and the maps $g_0 = g$ and $g_i = f_{i-1} \circ \cdots \circ f_0 \circ g : X' \to P_i, i = 1, 2, \ldots, n$ do not extend over $X$. In particular $e$-dim $X > P_i$ for every $i = 0, 1, \ldots, n$.

Theorem 1.8 was proved in [8] for the case $n = 0$, see the proof of Theorem 1.2(b) in [8]. The general case can easily be derived from the case $n = 0$. We recall that $X$ and $X'$ were constructed in [8] as the inverse limit $(X, X') = \lim_{\leftarrow}((M_j, N_j), p^j_{j-1})$ of a sequence of pairs of finite complexes $(M_j, N_j), j = 0, 1, \ldots$ with bonding maps $p^j_{j-1} : (M_j, N_j) \to (M_{j-1}, N_{j-1})$ such that $N_0$ is a finite subcomplex of $P$, $N_j = (p^j_{j-1})^{-1}(N_{j-1})$ and the map $p^j_0 = p^j_0 \circ \cdots \circ p^j_{j-2} \circ p^j_{j-1} : (M_j, N_j) \to (M_0, N_0)$ has the property that $f \circ p^j_0|_{N_j} : N_j \to L$ does not extend over $M_j$ where $p^j_0|_{N_j} : N_j \to N_0$ is considered as a map to $P$. Let $p : (X, X') \to (M_0, N_0)$ be the projection. Consider $g_0 = p|_{X'} : X' \to N_0$ as a map to $P$ and let $g_i = f_{i-1} \circ \cdots \circ f_0 \circ g_0 : X' \to P_i, i = 1, 2, \ldots, n$. Then for every $i$ the map $g_i$ does not extend over $X$ since otherwise for a sufficiently large $j$ the map $(f_{i-1} \circ \cdots \circ f_0) \circ p^j_0|_{N_j} : N_j \to P_i$ if $i \geq 1$ or the map $p^j_0|_{N_j} : N_j \to P_i$ if $i = 0$ would extend over $M_j$ and this would imply that $f \circ p^j_0|_{N_j} : N_j \to L$ also extends over $M_j$. This contradiction proves Theorem 1.8.

Note that if $L$ is a CW-complex (or a space homotopy equivalent to a CW-complex) then we can assume $P_{n+1} = L$ and get that $g_{n+1} = f_n \circ g_n : X' \to L$ does not extend over $X$ and hence $e$-dim $X > L$ (cf. the remark at the end of [8]).

The following two theorems provide us with a very important class of CW-complexes to which Theorems 1.7 and 1.8 apply.

Theorem 1.9 (Miller’s theorem (the Sullivan conjecture) [10]). Let $G$ be a finite group and $L$ a finite CW-complex. Then map $(K(G, 1), L) \cong 0$.

Theorem 1.10 (Dydak-Walsh [7]). Let $L$ have finite homotopy groups. Then map $(K(\mathbb{Q}, 1), L) \cong 0$.

The Dranishnikov-Repovš example (Theorem 1.3) can be obtained as an application of Theorem 1.7 and Miller’s theorem. Indeed, fix $m > 0$ and let $k \geq m$ be even. The homology groups of $\mathbb{R}^k$ are finite and hence so are the homology groups of $\Sigma \mathbb{R}^k$. Since $\Sigma \mathbb{R}^k$ is simply connected the Hurewicz isomorphism theorem modulo the class of finite abelian groups ([12], Sec. 9.6) implies that the homotopy groups of $\Sigma \mathbb{R}^k$ are finite and hence so are the homotopy groups of $\Omega \Sigma \mathbb{R}^k$. The inclusion $i : \mathbb{R}^m \to \mathbb{R}^k$ induces
the nontrivial homomorphism $i_* : H_1(\mathbb{R}P^m) \to H_1(\mathbb{R}P^k)$ and hence $\Sigma i : \Sigma \mathbb{R}P^m \to \Sigma \mathbb{R}P^k$ is essential. Thus $[\mathbb{R}P^m, \Omega \Sigma \mathbb{R}P^k] = [\Sigma \mathbb{R}P^m, \Sigma \mathbb{R}P^k] \neq 0$. By the Sullivan conjecture (Theorem 1.9) map $(K(\mathbb{Z}_2, 1), \Sigma \mathbb{R}P^k) \cong 0$ and hence map $(K(\mathbb{Z}_2, 1), \Omega \Sigma \mathbb{R}P^k) \cong 0$. Then Theorem 1.7 applied to $K = K(\mathbb{Z}_2, 1), P = \mathbb{R}P^m$ and $L = \Omega \Sigma \mathbb{R}P^k$ produces a compactum $X_m$ with $\dim_{\mathbb{R}} X_m \leq 1$ and $e\dim X_m > \mathbb{R}P^m$. Let $X$ be the one point compactification of the disjoint union of $X_m, m = 1, 2, \ldots$ and we have constructed the Dranishnikov-Repovš example (Theorem 1.3).

More or less the same strategy is applied for proving Theorems 1.4, 1.5 and 1.6 but this time instead of a specific structure of the real projective spaces $\mathbb{R}P^m$ we need Lemma 2.1 which plays a key role in our proofs.

2. Proofs.

**Lemma 2.1.** Let $m \geq 2$ and let $A$ be a finite CW-complex with $H_m(A) \neq 0$. Then there exists a finite CW-complex $B$ with finite homotopy groups such that $[A, B] \neq 0$. Moreover, if $0 \neq \alpha \in H_m(A)$ then $B$ can be constructed such that there is a map $\phi : A \rightarrow B$ with $\phi_*(\alpha) \neq 0$.

**Proof.** By adjoining to $A$ finitely many cells of $\dim \leq m$ we can kill the homotopy groups $\pi_i(A)$ for $i = 0, 1, \ldots, m-1$. Clearly $\alpha$ will remain nonzero in this enlarged complex and hence without loss of generality we may assume that $\pi_i(A) = 0$ for $i = 0, 1, \ldots, m-1$.

Assume that $\alpha$ is of infinite order. By Hurewicz’s isomorphism theorem we can adjoin an $(m+1)$-cell to $A$ to kill the element $2\alpha$, leaving $\alpha \neq 0$. Thus we may assume that $\alpha$ is of finite order.

Let $z_1, \ldots, z_k \in H_m(A)$ be a maximal collection of elements of infinite order such that $t_1z_1 + \cdots + t_kz_k$, $t_i \in \mathbb{Z}$ is of finite order if and only if $t_i = 0$ for all $1 \leq i \leq k$. By Hurewicz’s isomorphism theorem attach to $A$ $k$ cells of $\dim = m + 1$ to kill $z_i$, $1 \leq i \leq k$. Then the elements of $H_m(A)$ of finite order, and $\alpha$ in particular, will remain untouched and $H_m(A)$ will become a finite group. Thus we may assume that $H_m(A)$ is finite.

Let $n = \dim A > m$. If $m + 1 < n$ adjoin to $A$ finitely many cells of $m + 2 \leq \dim \leq n$ to kill the homotopy groups $\pi_i(A)$ for $m + 1 \leq i \leq n - 1$. Then $H_m(A)$ remains unchanged and by the Hurewicz isomorphism theorem modulo the class of finite abelian groups we may assume that $H_i(A)$ are finite for $i < n$.

Now once again take a maximal collection $z_1, \ldots, z_k \in H_n(A)$ of elements of infinite order such that $t_1z_1 + \cdots + t_kz_k$, $t_i \in \mathbb{Z}$ is of finite order only if $t_i = 0$ for all $1 \leq i \leq k$. Let $\psi : \pi_n(A) \rightarrow H_n(A)$ be the Hurewicz homomorphism. By the Hurewicz isomorphism theorem modulo the class of finite abelian groups, coker$\psi$ is finite and hence there are $t_i \neq 0, 1 \leq i \leq k$ such that $t_iz_i \in \psi(\pi_n(A))$. Attach to $A$ cells $C_1, \ldots, C_k$ of $\dim = n + 1$ to kill $t_1z_1, \ldots, t_kz_k$ respectively. Then $H_n(A)$ will become a finite group.
Let $C = n_1C_1 + \cdots + n_kC_k$, $n_i \in \mathbb{Z}$ be an $(n+1)$-dimensional chain. Then $\partial C = t_1n_1z_1 + \cdots + t_kn_kz_k$ if $z_i$'s are considered as cycles. Therefore $\partial C = 0$ only if $C = 0$ and hence $H_{n+1}(A) = 0$.

Thus after all the enlargements of $A$ we get a simply connected finite CW-complex $B$ with finite homotopy groups such that for the inclusion $\phi : A \longrightarrow B$, $\phi_*(\alpha) \neq 0$. Then the homotopy groups of $B$ are finite and the lemma follows. \hfill \qed

**Proof of Theorem 1.6.** Fix a cyclic finite CW-complex $P$. Then $\Sigma^2P$ is simply connected and cyclic and hence by Lemma 2.1 there exists a finite CW-complex $B$ with finite homotopy groups such that $[\Sigma^2P,B] \neq 0$. Then $[P,\Omega^2B] = [\Sigma^2P,B] \neq 0$ and $\Omega^2B$ also has finite homotopy groups. Let $K = K(\mathbb{Q},1)\bigvee(\bigvee\{K(G,1) : G \text{ is finite}\})$ be the wedge of $K(\mathbb{Q},1)$ and $K(G,1)$'s over all possible (up to isomorphism) finite abelian groups $G$. Since there are only countably many non-isomorphic finite groups, $K$ is a countable CW-complex. By the Sullivan conjecture and Theorem 1.10 map $(K,\Omega^2B) \cong 0$. Apply Theorem 1.7 to $K,P$ and $L = \Omega^2B$ and construct a compactum $X_P$ such that $e\text{-dim}X_P > P$, $\dim_{\mathbb{Q}}X_P \leq 1$ and $\dim_GX_P \leq 1$ for every finite abelian group $G$. $e\text{-dim} \leq K$ implies $\dim_{\mathbb{Q}} \leq 1$ and $\dim_G \leq 1$ for every finite abelian group $G$. Hence by Bockstein's theorem and inequalities $\dim_GX_P \leq 2$ for every abelian group $G$. Since there are only countably many finite CW-complexes of different homotopy types define $X$ as the one point compactification of the disjoint union of $X_P$'s over all possible (up to homotopy equivalence) cyclic finite CW-complexes $P$. Then $X$ is the desired compactum. \hfill \qed

**Proof of Theorem 1.5.** By a couple $(f,F)$ we mean a finite CW-complex $F$ and a map $f : S^2 \longrightarrow F$ such that $f_* : H_2(S^2) \longrightarrow H_2(F)$ is nontrivial. Two couples $(f,F)$ and $(f',F')$ are said to be of the same homotopy type or homotopy equivalent if there is a homotopy equivalence $h : F \longrightarrow F'$ such that $h \circ f \cong f'$.

Let $K = K(\mathbb{Q},1)\bigvee(\bigvee\{K(Z_p,1) : p \text{ prime }\})$ and let $T = (f,F)$ be a couple. By Lemma 2.1 and Theorems 1.8, 1.9, 1.10 there are a compactum $X_1^T$, a closed subset $X_1^T$ of $X_1^T$ and a map $g_1^T : X_1^T \longrightarrow S^2$ such that $e\text{-dim}X_1^T \leq K$ and $g_1^T = f \circ g_1^T : X_1^T \longrightarrow F$ does not extend over $X_1^T$. Let $X_T$ be the quotient space of $X_1^T$ obtained by replacing $X_1^T$ by $S^2$ according to the map $g_1^T$. Then $S^2$ can be considered as a subspace $S^2 \subset X_T$ of $X_T$ such that the map $f : S^2 \longrightarrow F$ does not extend over $X_T$. $e\text{-dim} \leq K$ implies $\dim_{\mathbb{Q}} \leq 1$ and $\dim_{\mathbb{Q}} \leq 1$ for every prime $p$. Hence by Bockstein's theorem and inequalities $\dim_GX_1^T \leq 2$ for every $G$ and clearly the latter property also holds for $X_T$.

Let $T$ be a countable family of couples which includes all possible homotopy types of couples. Let $X$ be the set obtained from the disjoint union of $X_T,T \in T$ by identifying all the spheres $S^2$, that is $X = \bigcup\{X_T : T \in T\}$.
with \( S^2 = \cap\{X^T : T \in \mathcal{T}\} \). Endow \( X \) with a separable metric topology which agrees with the topology of \( X^T \) for each \( X^T \). Then \( \dim_G X \leq 2 \) for every \( G \).

We are going to show that \( X \) has the required properties. Let \( X' \) be a Hausdorff compactification of \( X \). Since \( S^2 \subset X \subset X' \), \( \text{e-dim} X' > P \) for every CW-complex \( P \) which is not simply connected. Now assume that \( P \) is simply connected but not 2-connected. Take \( f' : S^2 \to P \) such that \( f'_* (\pi_2 (S^2)) \neq 0 \). Assume that \( f' \) extends to \( f'' : X' \to P \) and take a finite subcomplex \( F' \) of \( P \) such that \( f'' (X') \subset F' \). Consider \( f' \) and \( f'' \) as maps to \( F' \). Then \( T' = (f', F') \) is a couple and hence there is a couple \( T = (f, F) \in \mathcal{T} \) which is homotopy equivalent to \( T' \), that is there is a homotopy equivalence \( h : F' \to F \) such that \( f \cong h \circ f' \). By our construction \( f \) does not extend over \( X^T \) and hence neither does \( h \circ f' \). On the other hand \( h \circ f'|_{X^T} \) is an extension of \( h \circ f' \). This contradiction proves the theorem. \( \Box \)

**Proof of Theorem 1.4.** By a pair \( T = (P_1, P_0) \) we mean a pair of finite CW-complexes \( P_0 \subset P_1 \) such that such that the inclusion \( f_0 : P_0 \to P_1 \) is homologically essential. By Lemma 2.1 there are a finite CW-complex \( B \) with finite homotopy groups and a map \( \phi : \Sigma^2 P_1 \to B \) such that \( \phi \circ (\Sigma^2 f_0) : \Sigma^2 P_0 \to B \) is essential. Let \( f_1 : P_1 \to \Omega^2 B \) be the adjoint of \( \phi \). Then \( f = f_1 \circ f_0 : P_0 \to \Omega^2 B \) is the adjoint of \( \phi \circ (\Sigma^2 f_0) \) and hence \( f \) is also essential. Define \( K = K(\mathbb{Q}, 1) \vee (\bigvee \{K(Z_p, 1) : p \text{ prime }\}) \), \( L = \Omega^2 B \) and apply Theorems 1.8, 1.9 and 1.10 to construct a compactum \( X^T \), a closed subset \( X^T_0 \) of \( X^T \) and a map \( g^T : X^T_0 \to P_0 \) such that \( \text{e-dim} X^T \leq K \) and \( g^T \) does not extend over \( X^T \) as a map to \( P_1 \).

Let \( T = (P_1, P_0) \) be a pair. One can find a countable collection \( Q^T \) of maps from \( P_0 \) to \( P_0 \) such that each map from \( P_0 \) to \( P_0 \) is homotopic to some element of \( Q^T \). Consider \( Q^T \) as a discrete space. Let \( T \) be a countable collection of pairs which includes all possible homotopy types of pairs and define \( X \) as the disjoint union of \( X^T \times Q^T, T \in \mathcal{T} \). Clearly \( X \) is separable metrizable and locally compact and \( \text{e-dim} X \leq K \). By the Bockstein theorem and inequalities \( \dim_G X \leq 2 \) for every abelian \( G \).

Let us show that \( \text{e-dim} \beta X > P \) for every non-contractible simply connected CW-complex \( P \). Take a finite subcomplex \( P' \) of \( P \) supporting a nontrivial homology cycle in \( P \). Then for any finite subcomplex \( P'' \) of \( P \) containing \( P' \) the inclusion of \( P' \) into \( P'' \) is homologically essential. Let \( T' = \{T : T = (P_0, P_1) \in \mathcal{T} \text{ such that } P' \cong P_0 \} \) and let \( X' = \bigcup \{X^T_0 \times Q^T : T \in \mathcal{T}'\} \). Then \( X' \) is a closed subset of \( X \). For each \( T = (P_1, P_0) \in \mathcal{T}' \) fix a homotopy equivalence \( q^T : P_0 \to P' \).

Define \( f' : X' \to P' \) by \( f'(x, q) = q^T (q(g^T(x))) \) for \( (x, q) \in X^T_0 \times Q^T, T \in T' \).

Consider \( \beta X' \) as a closed subset of \( \beta X \) and let \( \beta f' : \beta X' \to P' \) be the extension of \( f' \). Let us show that \( \beta f' \) considered as a map to \( P \) does not
extend over $\beta X$. Assume that there is an extension $h : \beta X \to P$ of $\beta f'$ and let $P''$ be a finite subcomplex of $P$ containing both $h(\beta X)$ and $P'$. Take $T = (P_1, P_0) \in T'$ such that $(P_1, P_0) \cong (P'', P')$ and let $q'' : (P'', P') \to (P_1, P_0)$ be a homotopy equivalence. Let $q \in Q^T$ be a homotopy inverse of $q'' \circ q' : P_0 \to P_0$, that is $q'' \circ q' \circ q : P_0 \to P_0$ is homotopic to the identity map.

From now we identify $X^T_0 \times \{q\}$ and $X^T \times \{q\}$ with $X^T_0$ and $X^T$ respectively. Then the map $r = q'' \circ f'|_{X^T_0} = q'' \circ q' \circ q \circ g^T : X^T_0 \to P_0$ is homotopic to $g^T$ and hence by our construction $r$ does not extend over $X^T$ as a map to $P_1$. On the other hand $q'' \circ h|_{X^T} : X^T \to P_1$ is an extension of $r$ where $h$ is considered as a map to $P''$. This contradiction shows that $e\text{-dim}\beta X > P$.

Now, by adding a 2-dimensional disk to $X$, we get that $e\text{-dim}\beta X > K$ for every non-simply connected CW-complex $K$. Clearly all the cohomological dimensions of $X$ remain $\leq 2$ and the theorem follows. $\square$

Remarks. An interesting property of Theorems 1.4 and 1.5 is that the CW-complexes are not required to be countable and fixed in advance. This was achieved by using the so-called Rubin-Schapiro trick [9]. Another interesting point is that the space $X$ constructed in the proof of Theorem 1.6 has the property $\dim_G X^n \leq n + 1$ for every $G$ and $n$. And finally let us note that it would be interesting to find out if Theorem 1.6 holds for non-contractible (not necessarily cyclic) finite complexes $P$.

References


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IRREDUCIBILITY OF TENSOR SQUARES, SYMMETRIC SQUARES AND ALTERNATING SQUARES

KAY MAGAARD, GUNTER MALLE, AND PHAM HUU TIEP

We investigate the question when the tensor square, the alternating square, or the symmetric square of an absolutely irreducible projective representation $V$ of an almost simple group $G$ is again irreducible. The knowledge of such representations is of importance in the description of the maximal subgroups of simple classical groups of Lie type. We show that if $G$ is of Lie type in odd characteristic, either $V$ is a Weil representation of a symplectic or unitary group, or $G$ is one of a finite number of exceptions. For $G$ in even characteristic, we derive upper bounds for the dimension of $V$ which are close to the minimal possible dimension of nontrivial irreducible representations. Our results are complete in the case of complex representations. We will also answer a question of B. H. Gross about finite subgroups of complex Lie groups $G$ that act irreducibly on all fundamental representations of $G$.

1. Introduction.

Let $R = R(ℓ^f)$ be a finite classical group of Lie type. Let $G < R$ be a quasi-simple subgroup acting absolutely irreducibly on the natural module of $R$, not of Lie type in characteristic $ℓ$. In continuation of [18] we study those cases where $G$ has the same number of composition factors on the adjoint module for $R$ as $R$ itself. These embeddings are of importance in the determination of maximal subgroups of the finite classical groups of Lie type.

Let $V$ be the natural module for $R$. We will write $\tilde{\Lambda}^2(V), \tilde{\Sigma}^2(V)$ respectively $\tilde{A}(V)$ for the largest irreducible $R$-sub-quotient of $\Lambda^2(V), \text{Sym}^2(V), V \otimes V^*$. In Table 1.1 we recall the dimension of $X(V)$ for certain choices $(R, X)$ with $X \in \{\tilde{\Lambda}^2, \tilde{\Sigma}^2, \tilde{A}\}$.

In this paper we study quasi-simple subgroups $G$ of classical groups $R$ which act irreducibly on $V$ as well as on $X(V)$ with $X$ as in Table 1.1. It is known that the following families of examples do occur:

(1) $V$ is the heart of the natural permutation module of $G = \mathfrak{A}_n$ (see [18]),
(2) $V$ is a Weil module of $G = \text{Sp}_{2n}(q)$, $q \in \{3, 5, 9\}$, (see [19, 20]),
Table 1.1. Choices for $R$ and $X$.

<table>
<thead>
<tr>
<th>$R$</th>
<th>$X$</th>
<th>$\text{dim}(X(V))$</th>
<th>condition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\text{SL}_m$</td>
<td>$A$</td>
<td>$m^2 - 1$</td>
<td>$\ell \nmid m$</td>
</tr>
<tr>
<td></td>
<td>$\bar{A}$</td>
<td>$m^2 - 2$</td>
<td>$\ell</td>
</tr>
<tr>
<td>$\text{Sp}_m$</td>
<td>$\tilde{\Sigma}^2$</td>
<td>$\frac{1}{2} m(m + 1)$</td>
<td>$\ell$ odd</td>
</tr>
<tr>
<td></td>
<td>$\Sigma^2$</td>
<td>$\frac{1}{2} m(m - 1) - 1$</td>
<td>$\ell = 2$, $m \equiv 2 \pmod{4}$</td>
</tr>
<tr>
<td></td>
<td>$\bar{\Sigma}^2$</td>
<td>$\frac{1}{2} m(m - 1) - 2$</td>
<td>$\ell = 2$, $m \equiv 0 \pmod{4}$</td>
</tr>
<tr>
<td>$\text{SO}_m$</td>
<td>$\bar{\Lambda}^2$</td>
<td>$\frac{1}{2} m(m - 1)$</td>
<td>$\ell$ odd</td>
</tr>
<tr>
<td></td>
<td>$\Lambda^2$</td>
<td>$\frac{1}{2} m(m - 1) - 1$</td>
<td>$\ell = 2$, $m \equiv 2 \pmod{4}$</td>
</tr>
<tr>
<td></td>
<td>$\bar{\Lambda}^2$</td>
<td>$\frac{1}{2} m(m - 1) - 2$</td>
<td>$\ell = 2$, $m \equiv 0 \pmod{4}$</td>
</tr>
<tr>
<td></td>
<td>$\tilde{\Sigma}^2$</td>
<td>$\frac{1}{2} m(m + 1) - 1$</td>
<td>$\ell \nmid m$</td>
</tr>
<tr>
<td></td>
<td>$\bar{\Sigma}^2$</td>
<td>$\frac{1}{2} m(m + 1) - 2$</td>
<td>$\ell</td>
</tr>
</tbody>
</table>

(3) $V$ is a Weil module of $G = \text{SU}_n(q)$, $q \in \{2, 3\}$ (see [20, 14]),
(4) $V$ is a module of dimension $(2^n - 1)(2^{n-1} - 1)/3, (2^n + 1)(2^{n-1} + 1)/3$
or $(2^{2n} - 1)/3$ for $G = \text{Sp}_{2n}(2)$ and $X = \bar{\Lambda}^2$ (see Prop. 7.4).

We expect that the above are the only infinite series of examples. Our
main result is somewhat weaker; in order to formulate it denote by $l(G)$ the
(known lower bounds for the) minimal dimensions of nontrivial representa-
tions (Landázuri-Seitz-Zalesskii bound):

**Theorem 1.2.** Let $G$ be quasi-simple and $V$ a nontrivial absolutely irre-
ducible representation in characteristic $\ell \geq 0$ (which is different from the
defining characteristic if $G$ is of Lie type). Let $X = \bar{A}$ if $V$ is not self-dual,
and $X = \Sigma^2$ or $\Lambda^2$ otherwise. Then one of

(i) $X(V)$ is reducible, or
(ii) $(G, V)$ are as in (1)-(4) above, or
(iii) $G = G(q)$ is classical, $q \in \{2, 4, 8\}$ and $\dim(V)$ is at most $c \cdot l(G)^2$, or
(iv) $G = G(q)$ is exceptional, $q$ is even and $\dim(V)$ is at most $4l(G)$, or
(v) $G$ is on a known finite list of groups.

More precise formulations can be found in Theorem 3.1 and the Proposi-
tions in Section 5.

Clearly this result is true for the finitely many sporadic groups, see also
Section 6. Complete results for alternating and special linear groups were
obtained by the first two authors [18] (see also the references given there).
The case of complex representations is completely solved in Theorem 7.14.
Observe that the classification of complex modules $V$ with irreducible $\Sigma^2(V)$
has interesting applications in the theory of integral Euclidean lattices, cf. [14].

Our analysis of the remaining groups of Lie type splits into two cases, the second of which only occurs for groups defined over fields of characteristic 2. The first case leads to the examples in (2) and (3) above. In the second case we either show that \( X(V) \) is reducible or at least derive upper bounds for \( \dim(V) \) which are very close to \( l(G) \), the worst case being that of groups in characteristic 2 over fields of characteristic 3.

Also we improve the Landázuri-Seitz-Zalesskii bounds for the twisted exceptional groups \( ^3D_4(q) \) and \( ^2E_6(q) \), which might be of independent interest (see Section 4). To our knowledge, this only leaves the groups of types \( ^2F_4 \) and \( F_4 \) for which no sharp lower bound for the dimension of nontrivial representations in cross characteristic has been proved.

The finite irreducible complex reflection groups \( G \) are known to have the property that all exterior powers of their reflection representations remain irreducible. This can be rephrased by saying that \( G \) is a subgroup of \( \mathfrak{G} = \text{SL}_d(\mathbb{C}) \) acting irreducibly in all fundamental representations of \( \mathfrak{G} \). In the final section of our paper we determine all finite subgroups of complex simple simply-connected Lie groups with this property, thus answering a question asked by B.H. Gross:

**Theorem 1.3.** Let \( G \) be a finite subgroup of a complex simple simply-connected Lie group \( \mathfrak{G} \) which is irreducible in all fundamental representations of \( \mathfrak{G} \). Assume that the dimension \( d \) of the natural module \( V \) for \( \mathfrak{G} \) is at least 5 and \( \mathfrak{G} \neq \text{Spin}_5(\mathbb{C}) \cong \text{Sp}_4(\mathbb{C}), \mathfrak{G} \neq \text{Spin}_6(\mathbb{C}) \cong \text{SL}_4(\mathbb{C}) \). Then up to a finite subgroup of the center \( Z(\mathfrak{G}) \) one of the following holds, where \( \overline{G} \) denotes the image of \( G \) in its action on the natural module for \( \mathfrak{G} \):

(i) \( G = k^m \cdot H \) is monomial on the natural module with \( k \geq 2 \), \( d - 1 \leq m \leq d \), \( \mathfrak{G} = \text{SL}_d(\mathbb{C}) \), and \( \mathfrak{A}_d \leq H \leq \mathfrak{S}_d \) or \( H \) is as in Table 7.18.

(ii) \( \overline{G} = 2^m \cdot H \) is monomial on the natural module with \( d - 1 \leq m \leq d \), \( \mathfrak{G} = \text{Spin}_d(\mathbb{C}) \) and \( \mathfrak{A}_d \leq H \leq \mathfrak{S}_d \), or \( H \) is as in Table 7.18.

(iii) \( G = 2^3 \cdot \text{SL}_3(2) \) and \( \mathfrak{G} = G_2(\mathbb{C}) \).

(iv) \( G \leq 5^{1+2} : \text{SL}_2(5) \) and \( \mathfrak{G} = \text{SL}_5(\mathbb{C}) \).

(v) \( \overline{G} \leq 2^{1+6} : \mathfrak{S}_8 \) and \( \mathfrak{G} = \text{Spin}_8(\mathbb{C}) \).

(vi) \( G \) is almost quasi-simple and \( (\mathfrak{G}, G) \) are as in Table 7.22.

In particular, except for finitely many cases, \( G \) contains the derived group of an irreducible complex reflection group.

2. **Generalities.**

Let \( G \) be a finite group and \( \mathbb{F} \) an algebraically closed field of characteristic \( \ell \).
Let $V$ be any irreducible $FG$-module. We say that $V$ is of type $+\,$ if it carries a nondegenerate $G$-invariant quadratic form, and that $V$ is of type $-$ if it carries a bilinear alternating but no quadratic form. We would like to define some modules arising from $V$. If $V$ is not of type $+$ (resp. not of type $-$), then $\Sigma^2(V)$ (resp. $\Lambda^2(V)$) denotes $\text{Sym}^2(V)$ (resp. $\Lambda^2(V)$). Next, let $Y$ be $V \otimes V^*$ if $V$ is not self-dual, $\text{Sym}^2(V)$ if $V$ is of type $+$, $\Lambda^2(V)$ if $V$ is of type $-$. Then $Y$ is self-dual and $\dim \text{Hom}_G(Y, 1_G) = \dim \text{Hom}_G(1_G, Y) = 1$. Let $T$ be the (unique) submodule of $Y$ such that $Y/T \cong 1_G$, and let $I$ be the unique trivial submodule of $Y$. Then we will denote $T/(T \cap I)$ by $\tilde{A}(V)$, resp. $\tilde{\Sigma}^2(V)$, $\tilde{\Lambda}^2(V)$. If $V$ is a composition factor of an $FG$-module $U$, we will write $V \hookrightarrow U$.

According to Table 1.1, $\tilde{A}(V)$, $\tilde{\Sigma}^2(V)$, $\tilde{\Lambda}^2(V)$ respectively is irreducible for the ambient classical group.

**Lemma 2.1.** Let $U$ be a uniserial $FG$-module with $\text{soc}(U) = I$ being the trivial module and $U/I = V$ being irreducible, and $V \neq 1$. Let $D$ be a subgroup of $G$, $L$ a one-dimensional $FD$-module, and let $W = L \uparrow^G$. Suppose that $\text{Hom}_G(U, W)$ has dimension $t \geq 1$. Then $V$ is a composition factor of $W$ of multiplicity at least $t$.

**Proof.** Pick a nonzero map $f \in \text{Hom}_G(U, W)$. Since $U$ is uniserial, $V \hookrightarrow f(U)$, hence $V \hookrightarrow W$ and we get the statement for the case $t = 1$.

Next suppose that $t \geq 2$ and fix a basis $(f_1, \ldots, f_t)$ of $\text{Hom}_G(U, W)$. Let $v$ be a generator of $I$. If $f_i(v) = 0$ for all $i$, then $f_i \in \text{Hom}_G(V, W)$ for all $i$, whence $\dim \text{Hom}_G(V, W) = t$ and the multiplicity of $V$ in $W$ is at least $t$, as stated. Suppose $f_1(v) \neq 0$. Observe that $\dim \text{Hom}_G(I, W) = \dim \text{Hom}_G(1_D, L) \leq 1$, since $\dim L = 1$. Hence $f_i(v) = \lambda_i f_1(v)$ for some $\lambda_i \in F$. Replacing $f_i$ by $f_i - \lambda_i f_1$, we may assume that $f_i(v) = 0$ for all $i \geq 2$. Let $V' = \sum_{i=2}^t f_i(U)$. Then $V'$ is a sum of some copies of $V$. But $f_2, \ldots, f_t$ are linearly independent elements of $\text{Hom}_G(V, V')$. Therefore $V'$ is a direct sum of $t - 1$ copies of $V$. Finally, let $W' = f_1(I) + V'$. Then $W' \cong I \oplus (t - 1)V$. Since $U$ is uniserial, $f_1(U) \not\subset W'$. Thus

$$V \simeq f_1(U)/f_1(I) = f_1(U)/(f_1(U) \cap W') \cong (f_1(U) + W')/W' \subseteq W/W',$$

whence $W/W'$ has $V$ as a composition factor, and so the multiplicity of $V$ in $W$ is at least $t$.

**Corollary 2.2.** Let $U$ be a uniserial $FG$-module with $\text{soc}(U) = V$ being irreducible, $U/V \cong I$ being the trivial module, and $V \neq 1$. Let $D$ be a subgroup of $G$, $L$ a one-dimensional $FD$-module, and let $W = L \uparrow^G$. Suppose that $\text{Hom}_G(W, U)$ has dimension $t \geq 1$. Then $V$ is a composition factor of $W$ of multiplicity at least $t$.

**Proof.** Dualize Lemma 2.1. □
A key ingredient of our arguments is the following proposition, in which $X(V)$ means either $\tilde{\Sigma}^2(V)$ or $\tilde{A}^2(V)$ if $V$ is self-dual, and $\tilde{A}(V)$ if $V$ is not self-dual.

**Proposition 2.3.** Let $V$ be an irreducible $FG$-module such that $X(V)$ is irreducible. Let $Z$ be an abelian $t'$-subgroup of $G$ and denote $C = C_G(Z)$, $N = N_G(Z)$. Suppose that $V|_Z$ affords at least $t \geq 2$ linear characters $\alpha_i$, $1 \leq i \leq t$, no two of which are dual to each other.

(i) If $(\text{type}(V), X) \in \{(\alpha, \tilde{A}), (+, \tilde{\Sigma}^2), (-, \tilde{A}^2)\}$, then $X(V)$ is a composition factor of $(1_C)^G$ of multiplicity at least $t - 1$. If in addition at least one $\alpha_i$ is $N$-invariant, then $X(V) \subseteq (1_N)^G$ and so $\dim X(V) \leq (G : N)$. If $N/C$ is abelian then we also have $\dim X(V) \leq (G : N)$.

(ii) If $(\text{type}(V), X) \in \{(-, \tilde{\Sigma}^2), (+, \tilde{A}^2)\}$ and $|Z|$ is odd, then $X(V)$ is a composition factor of $(1_C)^G$ of multiplicity at least $t - 1$. If in addition $N/C$ is abelian, then $\dim X(V) \leq (G : N)$.

**Proof.** 1) Observe that the $FZ$-module $V$ is semisimple. In Cases (i) and (ii) we put $D = C$ and write $V|_Z = A_1 \oplus \ldots \oplus A_s$, where $s = t$, $A_i$, $1 \leq i \leq s - 1$, can afford only the $Z$-characters $\alpha_i$ and $\overline{\alpha}_i$, and $A_s$ affords all the rest of $Z$-characters. In Case (i), if $\alpha_i$ is $N$-invariant, we can also put $s = 2$ and $D = N$. Assuming for definiteness that $\alpha_1$ is $N$-stable, we write $V|_Z = A_1 \oplus A_2$, where $A_1$ can afford only $\alpha_1$ and $\overline{\alpha}_1$, and $A_2$ affords all the rest of $Z$-characters. Clearly, each $A_i$ is $D$-stable, hence we can view $A_i$ as an $FD$-module. The construction of $A_i$ ensures that

\[
(2.4) \quad \text{Hom}_Z(A_i, A_j) = \text{Hom}_Z(A_i, A_j^*) = 0
\]

whenever $i \neq j$. In particular, $\text{Hom}_D(A_i, A_j) = \text{Hom}_D(A_i, A_j^*) = 0$ if $i \neq j$.

We will let $W = (1_D)^G$.

2) Here we consider the subcase $X(V) = \tilde{A}(V)$ of (i).

2a) Take $D = C$, $s = t$. Then

\[
(V \otimes V^*)|_D = \sum_{i=1}^s A_i \otimes A_i^* \oplus \sum_{1 \leq i \neq j \leq s} A_i \otimes A_j^*.
\]

As mentioned above, $\text{Hom}_D((1_D, A_i \otimes A_j^*)) = 0$ if $i \neq j$. Hence the trivial $FG$-module $I$ inside $V \otimes V^*$ has to be contained (as a subspace) in $\sum_i (A_i \otimes A_i^*)$. Let $J$ be the sum of the $s$ (nonzero) $D$-fixed point subspaces inside $A_i \otimes A_i^*$, $1 \leq i \leq s$. Then $I \subseteq J$ and

\[
(2.5) \quad \dim \text{Hom}_D(1_D, J/I) \geq s - 1,
\]

whence

\[
s - 1 \leq \dim \text{Hom}_D(1_D, Y(V)/I) = \dim \text{Hom}_G(W, Y(V)/I),
\]
(recall $Y(V) = V \otimes V^*$ in this case). If $T \cap I = 0$, then $\tilde{A}(V) \cong Y(V)/I$, and $\tilde{A}(V)$ is irreducible by assumption, so $\tilde{A}(V)$ has multiplicity at least $s-1$ in $W$. Suppose $T \supseteq I$. Then $Y(V)/I$ is a uniserial $FG$-module with socle equal $\tilde{A}(V)$. By Corollary 2.2, the multiplicity of $\tilde{A}(V)$ in $W$ is again at least $s-1$ as stated.

2b) The same argument yields the statement for the subcase $X(V) = \tilde{A}(V)$ and $\alpha_i$ is $N$-invariant: We just need to put $D = N$ and $s = 2$.

2c) Now suppose that $N/C$ is abelian. According to (2.5), $\text{Hom}_C(1_C, Y(V)/I) \neq 0$. Thus the subspace $F$ of $C$-fixed points in $Y(V)/I$ is nonzero, and $F$ is acted on by $N$. But $N/C$ is abelian, hence the $N/C$-module $F$ contains a one-dimensional $FN$-module $L$. In this case $0 \neq \text{Hom}_N(L, Y(V)/I) \cong \text{Hom}_G(L^{|G|}, Y(V)/I)$. Arguing as in 2a) and using Corollary 2.2 if $T \supseteq I$, we get $\tilde{A}(V) \hookleftarrow L^{|G|};$ in particular, $\dim \tilde{A}(V) \leq (G : N)$.

3) It remains to consider the subcases $X(V) = \Sigma^2(V)$ and $\tilde{A}^2(V)$. Take $D = C$, $s = t$. Then $V$ supports a nondegenerate $G$-invariant bilinear form $b$. We claim that $b|A_i$ is also nondegenerate. For, let $A_i^\perp$ be the orthogonal complement to $A_i$ in $V$ (with respect to $b$) and $B = \sum_{j \neq i} A_j$. Then $B/(B \cap A_i^\perp) \cong (B + A_i^\perp)/A_i^\perp \subseteq V/A_i^\perp \simeq A_i$. But due to (2.4), the $Z$-modules $B$ and $A_i^\perp$ have no common composition factors. Hence $B \subseteq A_i^\perp$.

Comparing the dimension we get $B = A_i^\perp$ and so $A_i \cap A_j^\perp = 0$, as stated. Now for $Y(V) = \Sigma^2(V)$ or $\Lambda^2(V)$ we have

$$Y(V)|_D = \sum_{i=1}^s Y(A_i) \oplus \sum_{1 \leq i < j \leq s} A_i \otimes A_j.$$  

4a) In Case (i), let $J$ be the sum of the $D$-fixed point subspaces inside $Y(A_i)$, $1 \leq i \leq s$; all of them are nonzero because of the nondegeneracy of $b|A_i$. Then $\dim J \geq s$. By (2.4) above, $\text{Hom}_D(1_D, A_i \otimes A_j) = 0$ if $i \neq j$. Hence the trivial $FG$-module $I$ inside $Y(V)$ has to be contained (as a subspace) in $J$. We again have

$$\dim \text{Hom}_D(1_D, J/I) \geq s - 1$$

and so $\dim \text{Hom}_G(W, Y(V)/I) \geq s - 1$. At this point we can repeat the arguments of 2a) to show that the multiplicity of $X(V)$ in $W$ is at least $s-1$.

If in addition $\alpha_1$ is $N$-invariant, then we can use the same argument, with changing $D$ to $N$ and $s$ to $2$.

Suppose $N/C$ is abelian. According to (2.6), $\text{Hom}_C(1_C, Y(V)/I) \neq 0$. At this point we can repeat the argument of 2c).

4b) Finally, we consider Case (ii). Note that, if $\ell = 2$, we may argue as above. Thus we may now assume that $Y(V) = X(V)$ is irreducible. By 3), the restriction to $A_i$ of the nondegenerate $G$-invariant bilinear form $b$ is nondegenerate. But unlike Case 4a), now we cannot conclude that $Y(A_i)$
has nonzero $D$-fixed points for every $i$ (the form $b$ is not of the right type!). We may assume that $\alpha_i \neq 1_Z$ for $i \geq 2$. We claim that for each $i \geq 2$, $Y(A_i)$ has nonzero $D$-fixed points. For, since $V$ is self-dual, $A_i|_Z = B_i \oplus B'_i$, where $B_i$ affords only $\alpha_i$ (with some multiplicity) and $B'_i$ affords only $\overline{\alpha}_i$. Since $|Z|$ is odd, $\alpha_i \neq \overline{\alpha}_i$. Clearly, $B_i$ and $B'_i$ are $D$-stable. Again by the self-duality of $V$, $B'_i \cong B^*_i$ as $D$-modules. Now $B_i \otimes B^*_i$ is a submodule of $Y(A_i)$, and $B_i \otimes B^*_i$ has nonzero $D$-fixed points, hence the claim follows.

Let $J$ be the sum of the $D$-fixed point subspaces inside $Y(A_i)$, $1 \leq i \leq s$. Then $\dim J \geq s - 1$. Therefore

\begin{equation}
\dim \text{Hom}_D(1_D, J) \geq s - 1
\end{equation}

and so $\dim \text{Hom}_G(W, Y(V)) \geq s - 1$. Since $X(V) = Y(V)$ in (ii), we conclude that the multiplicity of $X(V)$ in $W$ is at least $s - 1$.

Now suppose in addition that $N/C$ is abelian. According to (2.7) we have that $\dim \text{Hom}_C(1_C, X(V)) \neq 0$. It remains to repeat the argument of 2c. \hfill \Box

In what follows, we will apply Proposition 2.3 in the following set-up: $G$ is a finite group of Lie type, defined over a field $\mathbb{F}_q$ in characteristic $p$ and of universal type, and $Z$ is a long-root subgroup $\{x_\alpha(t) \mid t \in \mathbb{F}_q\}$, $\alpha$ a long root. A visual description of $Z$ is given for instance in [19].

To apply Proposition 2.3 efficiently, one therefore needs to know the spectrum $\text{Spec}(Z, V)$ of abelian subgroups $Z$ in any irreducible representation $V$, that is, the set of (distinct) linear characters of $Z$ occurring in $V$. If $Z = \langle g \rangle$, we denote $\text{Spec}(Z, V)$ by $\text{Spec}(g, V)$. The following theorem is the main result of [27]:

**Theorem 2.8 ([27]).** Let $G$ be a finite group of Lie type, defined in characteristic $p$ and of universal type, and $g \in G$ a non-central element of order $p$. Suppose $G$ has an irreducible representation $V$ over an algebraically closed field $\mathbb{F}$ of characteristic $\ell \neq p$ such that $1 < |\text{Spec}(g, V)| < p$. Then $p > 2$ and one of the following holds.

(i) $G = \text{SU}_3(p)$ or $\text{Sp}_{2n}(p)$, and $g$ is a transvection.

(ii) $G = \text{SL}_2(p^2)$ or $\text{Sp}_4(p)$.

Moreover, if $1 \notin \text{Spec}(g, V)$ and $G = [G, G]$ then $p > 2$ and

$G \in \{\text{SL}_2(p), \text{SL}_2(p^2), \text{SU}_3(p), \text{Sp}_4(p)\}$.

The group $Z$ is elementary abelian of order $q$ and may be identified with the additive group $\{t \mid t \in \mathbb{F}_q\}$. Fix a $p^{th}$ primitive root $\epsilon$ of unity in $\mathbb{C}$. Then any irreducible Brauer character of $Z$ in characteristic $\ell \neq p$ is of the form

$\lambda_c : t \mapsto \epsilon^{tr_{\mathbb{F}_q/\mathbb{F}_p}(tc)}$

for some $c \in \mathbb{F}_q$. Let $\Omega^+$, resp. $\Omega^-$, be the set of all $\lambda_c$, where $c$ is any square, resp. non-square, in $\mathbb{F}_q^\times$.

We will need the following supplement to Theorem 2.8:
Lemma 2.9. Let $G$ be a universal-type quasi-simple finite group of Lie type defined over $\mathbb{F}_q$, $q = p^r$, and $Z$ a long-root subgroup as above. Let $\mathbb{F}$ be an algebraically closed field of characteristic $\ell \neq p$ and $V$ a nontrivial irreducible $\mathbb{F}G$-module. Suppose $\text{Spec}(Z,V) \neq \text{IBr}(Z)$. Then one of the following holds.

(i) $q > p$, and $\text{Spec}(Z,V) = \Omega^+ \cup \Omega^-.$
(ii) $G = \text{SU}_3(p)$, $p > 2$, and $\text{Spec}(Z,V) = \Omega^+ \cup \Omega^-.$
(iii) $G = \text{Sp}_{2n}(q)$, $n \geq 3$, $p > 2$, and $\text{Spec}(Z,V) = \Omega^+ \cup \{1_Z\}$, or $\Omega^- \cup \{1_Z\}.$
(iv) $G = \text{Sp}_4(q)$, $p > 2$, and $\text{Spec}(Z,V) = \Omega^+ \cup \{1_Z\}$, $\Omega^- \cup \{1_Z\}$, or $\Omega^+ \cup \Omega^-.$
(v) $G = \text{SL}_2(q)$, $p > 2$, and $\text{Spec}(Z,V) = \Omega^+, \Omega^-, \Omega^+ \cup \{1_Z\}, \Omega^- \cup \{1_Z\}$, or $\Omega^+ \cup \Omega^-.$

Proof. Let $P = N_G(Z)$ and $C = C_G(Z)$. Assume that $\text{Spec}(Z,V) \neq \text{IBr}(Z)$.

First suppose $G \neq \text{Sp}_{2n}(q)$, or $G = \text{Sp}_{2n}(q)$ but $q$ is even. Then $P/C \simeq Z_{q-1}$ and $P$ acts transitively on $\Omega^+ \cup \Omega^- = \text{IBr}(Z) \setminus \{1_Z\}$. Since $\text{Spec}(Z,V)$ contains at least one nontrivial linear character of $Z$, $\text{Spec}(Z,V) = \Omega^+ \cup \Omega^-.$ Moreover, if $q = p$, then $G = \text{SU}_3(p)$ by Theorem 2.8. Thus we arrive at (ii) or (i).

If $G = \text{SL}_2(q)$, then we can check (v) directly.

So suppose that $G = \text{Sp}_{2n}(q)$, $q$ is odd, and $n \geq 2$. In this case, any nontrivial element of $Z$ is a transvection, and $P$-orbits on $\Omega^+ \cup \Omega^-$ are $\Omega^+$ and $\Omega^-$. Moreover, $P$ contains a subgroup $P' = QL$, where $Q$ is a normal subgroup of extraspecial type of order $q^{2n-1}$, with $Z = Z(Q)$, and $L = \text{Sp}_{2n-2}(q)$.

Fix a nontrivial linear character $\lambda$ occurring in $V$ and consider the $\lambda$-eigenspace $V'$ for $Z$ in $V$. Then, as an $FP'$-module, $V' \simeq W \otimes U$, where $W$ is an irreducible representation of degree $q^{n-1}$ (extending an irreducible representation of $Q$), and $U$ is a representation of $L = P'/Q$ inflated to $P'$. Moreover, $W|_L$ is the sum of two Weil representations $W^\pm$ of degree $(q^{n-1} \pm 1)/2$ if $r \neq 2$, and has three composition factors, one of dimension 1 and two, say $W_1, W_2$, of dimension $(q^{n-1} - 1)/2$ if $r = 2$.

Since $n \geq 2$, $L$ contains a long-root subgroup $Z'$ which is $G$-conjugate to $Z$. Let $n = 2$. The above discussion shows that $|\text{Spec}(Z',W)| \geq (q+1)/2$.

(For, if $r = 2$ then $1_{Z'} \in \text{Spec}(Z',W)$, hence the claim follows. If $r \neq 2$ then $|\text{Spec}(Z',W^+)| = (q+1)/2$ and we are again done.) Thus $|\text{Spec}(Z',W \otimes U)| \geq (q+1)/2$, whence

$$\frac{q+1}{2} \leq |\text{Spec}(Z',W \otimes U)| \leq |\text{Spec}(Z',V)| = |\text{Spec}(Z,V)|.$$

Therefore we arrive at (iv).

Finally, suppose $n \geq 3$. It suffices to show that $1_{Z'} \in \text{Spec}(Z',W \otimes U)$.

Assume the contrary. If $r = 2$, then clearly $1_{Z'} \in \text{Spec}(Z',W)$.

If $r \neq 2$, then we can embed $Z'$ into a standard subgroup $L'$ of type $\text{Sp}_2(q)$, and $W^+|_{L'}$
has a composition factor of dimension \((q + 1)/2\), whence \(1_{Z'} \in \text{Spec}(Z', W)\). Therefore \(1_{Z'} \not\in \text{Spec}(Z', U)\). Due to (iv), \(\text{Spec}(Z', U) = \Omega^+ \cup \Omega^-\). In this case, pick a character \(\lambda_c\) with \(c \in \mathbb{F}_q^\times\) occurring in \(\text{Spec}(Z', W')\). Then \(\lambda_c \in \text{Spec}(Z', U)\), whence \(1_{Z'} = \lambda_c^{-1} \in \text{Spec}(Z', W \otimes U)\), a contradiction. \(\square\)

**Corollary 2.10.** Let \(G\) be a universal-type quasi-simple finite group of Lie type defined over \(\mathbb{F}_q\), \(q = p^f\), and \(Z\) a long-root subgroup as above. Let \(\mathbb{F}\) be an algebraically closed field of characteristic \(\ell \neq p\) and \(V\) a nontrivial irreducible \(\mathbb{F}G\)-module. Then either

(i) \(\text{Spec}(Z, V)\) contains at least two distinct characters which are not dual to each other, or

(ii) \(G \in \{\text{SL}_2(5), \text{SU}_3(3), \text{Sp}_4(3)\}\).

**Proof.** First assume that \(q = 2\). Then \(Z\) is of order 2 and \(Z\) is not central, hence \(V|_Z\) affords both linear characters of \(Z\), and they are real and distinct, i.e., we are in (i). Next assume that \(q = 3\) and \(G \neq \text{SU}_3(3), \text{Sp}_4(3)\). By Lemma 2.9, \(\text{Spec}(Z, V)\) contains \(1_Z\) and at least one more character, so we arrive at (i). Assume that \(q = 5\) and \(G \neq \text{SL}_2(5)\). By Lemma 2.9, \(|\text{Spec}(Z, V)| \geq 3\), which implies (i). Finally, if \(q = 4\) or \(q \geq 7\), then \(|\text{Spec}(Z, V)| \geq 3\) by Lemma 2.9, and we again arrive at (i). \(\square\)

### 3. The good case.

We first treat the **good** cases where Proposition 2.3 applies, that is, if either \(p \neq 2\) or \((\text{type}(V), X) \not\in \{(-, \Sigma^2), (+, \Lambda^2)\}\), where we obtain reasonably complete results.

Let \(\mathbb{F}\) be an algebraically closed field of characteristic \(\ell\). The main result of this section is the following:

**Theorem 3.1.** Let \(G\) be a quasi-simple group with \(S := G/Z(G)\) being a finite group of Lie type in characteristic \(p\). Suppose that \(G\) has a nontrivial irreducible \(\mathbb{F}G\)-module \(V\) such that \(X(V)\) is irreducible, where \(X = \Lambda\) if \(V\) is not self-dual, and \(X = \Sigma^2\) or \(\Lambda^2\) otherwise. Then one of the following holds.

(i) \(S = \text{S}_2n(q), q = 3, 5, 9, V\) is a Weil module of \(\text{Sp}_{2n}(q)\) of degree \((q^n \pm 1)/2\).

(ii) \(S = U_n(q), q = 2, 3, V\) is a Weil module of \(\text{SU}_n(q)\) of degree \((q^n + q(-1)^n)/(q + 1)\) or \((q^n - (-1)^n)/(q + 1)\).

(iii) \(S = \text{E}_4(3), F_4(2), F_4(3), E_6(3), E_7(2)\) or \(E_8(2)\).

(iv) “Small groups”: \(S\) is \(\text{L}_n(q)\) as in Table 3.1 of [18], or as in Table 3.2.

(v) \(p = 2\) and \((\text{type}(V), X) \in \{(-, \Sigma^2), (+, \Lambda^2)\}\).

**Remark 3.3.** The Weil modules of \(\text{Sp}_{2n}(q), q = 3, 5, 9,\) and of \(\text{SU}_n(q), q = 2, 3,\) do indeed give irreducible examples. In characteristic \(\ell = 0\), this question has been resolved in [19] for \(\text{Sp}_{2n}(q)\), and in [20, 14] for \(\text{SU}_n(q)\).
Table 3.2. Non-generic examples.

<table>
<thead>
<tr>
<th>$G$</th>
<th>$\ell$</th>
<th>$\dim(V)$</th>
<th>$\Lambda^2$</th>
<th>$\Sigma^2$</th>
<th>$A$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$U_3(4)$</td>
<td>$\neq 2$</td>
<td>12</td>
<td>$\times$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$3_1.U_4(3)$</td>
<td>2</td>
<td>6</td>
<td>$\times$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$6_1.U_4(3)$</td>
<td>$\neq 2,3$</td>
<td>6</td>
<td>$\times$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$2.U_6(2)$</td>
<td>$\neq 2$</td>
<td>56</td>
<td>$\times$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$S_4(4)$</td>
<td>$\neq 2$</td>
<td>18</td>
<td>$\times$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$2.S_6(2)$</td>
<td>$\neq 2$</td>
<td>8</td>
<td>$\times$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$2.O_8^+(2)$</td>
<td>$\neq 2$</td>
<td>8</td>
<td>$\times$</td>
<td>$\times$</td>
<td></td>
</tr>
<tr>
<td>$2.2B_2(8)$</td>
<td>5</td>
<td>8</td>
<td>$\times$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$G_2(3)$</td>
<td>2</td>
<td>14</td>
<td>$\times$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$G_2(3)$</td>
<td>$\neq 2,3$</td>
<td>14</td>
<td>$\times$</td>
<td>$\times$</td>
<td></td>
</tr>
<tr>
<td>$2.G_2(4)$</td>
<td>$\neq 2$</td>
<td>12</td>
<td>$\times$</td>
<td>$\times$</td>
<td></td>
</tr>
<tr>
<td>$^2F_4(2)'$</td>
<td>$\neq 2$</td>
<td>26</td>
<td>$\times$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$^3D_4(2)$</td>
<td>3</td>
<td>25</td>
<td>$\times$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Proof. We start by making some obvious reductions. The corresponding universal-type group of Lie type is the universal cover for $S$, with a few exceptions. These exceptions as well as those groups emerging in Corollary 2.10(ii) can be handled directly using [12], and the arising examples are recorded in (iii) or (iv). Throughout the proof we will therefore assume that $V$ is a nontrivial (could be non-faithful) irreducible module of a universal-type group $G$ of Lie type defined over a field $\mathbb{F}_q$ of characteristic $p$ and satisfying Corollary 2.10(i). Moreover, we can and will assume that $G$ is not a special linear group, since that case has been treated in [18].

1) We will apply Proposition 2.3 to a long-root subgroup $Z$ of $G$. If $C := C_G(Z)$ and $N := N_G(Z)$, then $N/C$ is a cyclic group, of order $(q - 1)/2$ if $G = Sp_{2n}(q)$ with $q$ odd, $q^2 - 1$ if $G = ^2B_2(q^2)$, $^2G_2(q^2)$, $^2F_4(q^2)$, and $q - 1$ otherwise.

By our initial reductions we may assume by Corollary 2.10 that $(V, Z)$ satisfies the assumptions of Proposition 2.3. According to Proposition 2.3(i) and (ii), $\dim X(V) \leq (G : N)$. Since $\dim X(V) \geq \dim(V)(\dim(V) - 1)/2 - 2$, this implies that

$$\dim(V) \leq \frac{1}{2} + \sqrt{2(G : N) + \frac{17}{4}}. \quad (3.4)$$

2) We assume that $G$ is one of the groups in the following table, where moreover $(n, q) \notin \{(3, 3), (3, 4), (4, 2), (4, 3), (6, 2)\}$ for $G = SU_n(q)$, $(n, q) \notin \{(3, 3), (3, 4), (4, 2), (4, 3), (6, 2)\}$.
{(2, 2), (2, 3), (3, 2)} for \( G = \text{Sp}_{2n}(q) \). Then \( N \) is as indicated, cf. [13]. Here, for an integer \( m \), \([m]\) is a certain group of that order.

Table 3.5. The group \( N \).

<table>
<thead>
<tr>
<th>( G )</th>
<th>condition</th>
<th>( N )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \text{SU}_n(q) )</td>
<td>( n \geq 3 )</td>
<td>([q^{2n-3}] \cdot (\text{SU}<em>{n-2}(q) \cdot Z</em>{q^2-1}))</td>
</tr>
<tr>
<td>( \text{Sp}_{2n}(q) )</td>
<td></td>
<td>([q^{2n-1}] \cdot (\text{Sp}<em>{2n-2}(q) \cdot Z</em>{q-1}))</td>
</tr>
<tr>
<td>( \text{Spin}_{2n+1}(q) )</td>
<td>( n \geq 3 ), ((n, q) \neq (3, 3))</td>
<td>([q^{4n-5}] \cdot (\text{SL}<em>2(q) \times \text{Spin}</em>{2n-3}(q)) \cdot Z_{q-1})</td>
</tr>
<tr>
<td>( \text{Spin}^5_{2n}(q) )</td>
<td>( n \geq 4 ), ((n, q) \neq (4, 2))</td>
<td>([q^{4n-7}] \cdot (\text{SL}<em>2(q) \times \text{Spin}^5</em>{2n-4}(q)) \cdot Z_{q-1})</td>
</tr>
<tr>
<td>( ^2B_2(q^2) )</td>
<td>( q^2 \geq 8 )</td>
<td>([q^6] \cdot Z_{q^2-1})</td>
</tr>
<tr>
<td>( ^2G_2(q^2) )</td>
<td>( q^2 \geq 27 )</td>
<td>([q^6] \cdot Z_{q^2-1})</td>
</tr>
<tr>
<td>( G_2(q) )</td>
<td>( q \geq 5 )</td>
<td>([q^3] \cdot (\text{SL}<em>2(q) \cdot Z</em>{q-1}))</td>
</tr>
<tr>
<td>( ^3D_4(q) )</td>
<td>( q \geq 3 )</td>
<td>([q^9] \cdot (\text{SL}<em>2(q^3) \cdot Z</em>{q-1}))</td>
</tr>
<tr>
<td>( F_4(q) )</td>
<td>( q \geq 3 )</td>
<td>([q^{15}] \cdot (\text{Sp}<em>6(q) \cdot Z</em>{q-1}))</td>
</tr>
<tr>
<td>( ^2F_4(q^2) )</td>
<td>( q^2 \geq 2 )</td>
<td>([q^{22}] \cdot (^2B_2(q^2) \cdot Z_{q^2-1}))</td>
</tr>
<tr>
<td>( E_6(q) )</td>
<td></td>
<td>([q^{21}] \cdot (\text{SL}<em>6(q) \cdot Z</em>{q-1}))</td>
</tr>
<tr>
<td>( ^2E_6(q) )</td>
<td>( q \geq 3 )</td>
<td>([q^{21}] \cdot (\text{SU}<em>6(q) \cdot Z</em>{q-1}))</td>
</tr>
<tr>
<td>( E_7(q) )</td>
<td></td>
<td>([q^{33}] \cdot (\text{Spin}^1_{12}(q) \cdot Z_{q-1}))</td>
</tr>
<tr>
<td>( E_8(q) )</td>
<td></td>
<td>([q^{57}] \cdot (E_7(q) \cdot Z_{q-1}))</td>
</tr>
</tbody>
</table>

Let \( d \) be the smallest degree of nontrivial irreducible projective representations of \( G \) in cross-characteristics. Lower bounds on \( d \) were given in [13, 23, 6, 9, 10]. The obvious lower bound \( \dim(V) \geq d \) and the upper bound (3.4) imply \( \sqrt{2(G : N) + 17/4} + 1/2 \geq d \). The last inequality can hold only when \( G \) is one of the following groups:

(3.6) \( \text{SU}_n(2) \), \( \text{SU}_n(3) \), \( \text{Sp}_{2n}(3) \), \( \text{Sp}_{2n}(5) \), \( \text{Sp}_{2n}(7) \), \( \text{Sp}_{2n}(9) \), \( O_{2n}^+(2) \), \( \text{Spin}^+_{8}(3) \), \( ^3D_4(3) \), \( F_4(3) \), \( E_6(2) \), \( ^2E_6(3) \), \( E_7(2) \), \( E_8(2) \).

(A typical calculation is given in the case of \( G = E_8(q) \) as follows: \( (G : N) = (q^{30} - 1)(q^{12} + 1)(q^{10} + 1)(q^6 + 1)/(q - 1) \) and \( d \geq q(q^{12} + 1)(q^{10} + 1)(q^6 + 1) - 3 \) (cf. [10]), hence \( \sqrt{2(G : N) + 17/4} + 1/2 < d \) if \( q \geq 3 \).

3) To complete the proof of the theorem, we have to analyze the groups singled out in (3.6).

Assume \( G = \text{SU}_n(q) \) and \( q = 2, 3 \). Then (3.4) implies that \( \dim V < 2d \).

By [7], Th. 16, \( V \) is one of the Weil modules of \( G \).
Assume \( G = \text{Sp}_{2n}(q) \), \( n \geq 2 \), \( q \) odd and \( q \leq 9 \). If \( q \geq 5 \), then
\[
\dim V \leq \sqrt{2(G : N)} + 17/4 + 1/2
= \sqrt{2(q^{2n} - 1)/(q - 1)} + 17/4 + 1/2 < q^n - 1.
\]

Suppose \( q = 3 \). Since \( n \geq 3 \), \( \text{Spec}(Z, V) \ni 1 \) by Corollary 2.10. By Proposition 2.3(i), \( X(V) \) is a constituent of a reduction modulo \( \ell \) of the complex character \( \rho := (1_N)^\dagger \). Observe that \( \rho \) is the sum of the trivial character, a character of degree \((q^n + 1)(q^n - q)/(2(q - 1))\), and a character of degree \((q^n - 1)(q^n + q)/(2(q - 1))\), whence \( \dim V < q^n - 1 \). Thus in all cases \( \dim V < q^n - 1 \). By [6], Thm. 9.9.2, \( \dim V = (q^n + 1)/2 \). This implies by the main result of [5] that \( V \) is a Weil module. Now apply Prop. 5.5 in [19] and use that a Weil module is self-dual if and only if \( q \equiv 1 \) (mod 4).

4) Assume that \( G = \text{O}_{2n}^e(2) \) and \( n \geq 4 \). Consider the natural module \( \mathbb{F}_{2^n}^2 \) for \( G \) and the stabilizer \( P \) of an isotropic vector in this module. Then \( P = U \cdot L \), where \( U \) is a normal elementary abelian subgroup of order \( 2^{2n-2} \) and \( L = \text{O}_2^{2n-2}(2) \). The group \( L \) acts on \( \text{IBr}_\ell(U) \) with three orbits, of length 1, \( a := (2^{n-2} + \epsilon 1)(2^{n-1} - \epsilon 1) \), and \( b := 2^{n-2}(2^{n-1} - \epsilon 1) \), respectively. Observe that
\[
\max\{a, b\} < d \leq \dim V \leq \sqrt{2(G : N)} + 17/4 + 1/2 < 2 \min\{a, b\}.
\]
This shows that \( L \) has at least two different orbits on \( \text{Spec}(U, V) \). Each orbit gives rise to a direct summand in the \( P \)-module \( V \). Thus we can write \( V|_P = V_1 \oplus V_2 \) with \( V_1 \) and \( V_2 \) having no common composition factors. Here we have \( p = 2 \), so due to our assumption \( V \) is of type \( + \) if \( X = \Sigma^2 \) and \( V \) is of type \( - \) if \( X = \Lambda^2 \). Arguing as in the proof of Proposition 2.3(i) (with \( P \) in place of \( C \)), we see that \( X(V) \ni (1_P)^\dagger \), in particular, \( \dim X(V) \leq (G : P) \). Thus
\[
\dim V \leq \sqrt{2(G : P) + 17/4 + 1/2} = \sqrt{2(2^n - \epsilon 1)(2^{n-1} + \epsilon 1) + 17/4 + 1/2},
\]
whence \( \dim V \leq 2^n + 1 < d \), a contradiction.

5) Let \( G = \text{Spin}^+_8(3) \). In this case (3.4) implies that \( \dim V \leq 270 \). We can embed \( \text{Spin}^+_7(3) \) in \( G \). According to [12], the dimension of any faithful representation of \( \text{Spin}^+_7(3) \) in characteristic \( \neq 2, 3 \) is at least 520. From this it follows that \( V \) is actually a representation of \( G/Z(G) = \text{O}_8^+(3) \). The modular character tables of this group are known. \( \square \)

For the groups in Case (iii) of Theorem 3.1, (3.4) implies at least the upper bounds for \( \dim(V) \) given in the following table:
Here, \( l(G) \) denotes the lower bound for nontrivial irreducible representations in cross-characteristic from [13, 23, 10]. If \( \ell = 0 \), then each \( G \) in the table has exactly one nontrivial irreducible module \( V \) of the indicated dimension, cf. [17], and one can check that \( \dim(X(V)) \) does not divide \( |G| \). The complex characters of \( 2.F_4(2) \) and \( 2.E_6(2) \) can be checked by inspecting the character table and using GAP. Finally, let \( G = 3.E_6(2) \) or \( 6.E_6(2) \), and \( V \) a faithful irreducible \( \mathbb{C}G \)-module. Then \( V \) is not self-dual, so we need to consider only \( \bar{A}(V) \). Let \( z \) be an inverse image in \( G \) of a long-root element and let \( Z = \langle z \rangle \). In order to apply Proposition 2.3, we need to show that \( z \) has at least two non-conjugate eigenvalues in \( V \). Suppose not, then \( z \) has exactly two eigenvalues say \( \alpha \) and \( \alpha^{-1} \). Write \( V = U \oplus W \) for the corresponding eigenspaces. Then \( A(V) = A(U) \oplus A(W) \oplus \ldots \) contains at least two copies of the trivial \( C \)-module (with \( C := C_G(Z) \)), so \( \bar{A}(V) \) contains \( 1_C \). Thus in any case we have \( \dim(V) \leq \sqrt{|G:C|} + 1 \), i.e., \( \dim(V) \leq 1991 \). According to [17], there is no such faithful \( G \)-module.

Thus we have completed the good case for \( \ell = 0 \).

4. Lower bounds for representations of \( 3^2D_4(q) \) and \( 2^2E_6(q) \).

In this section we improve the Landázuiri-Seitz-Zalesskii bounds for the smallest degree of a nontrivial representation in non-defining characteristic for the twisted exceptional groups \( 3^2D_4(q) \) and \( 2^2E_6(q) \). Our method is a direct extension of the one used by Hoffman [10] for the non-twisted groups of type \( E_n \).

**Theorem 4.1.** Let \( V \) be a nontrivial irreducible representation of \( 3^2D_4(q) \) in characteristic \( \ell \mid q \). Then

\[
\dim(V) \geq q(q^4 - q^2 + 1) - 1.
\]

**Proof.** Let \( N = Q.SL_2(q^3).Z_{q-1} \) be a maximal parabolic subgroup of \( 3^2D_4(q) \) with special unipotent radical \( Q \) of order \( q^{1+8} \). The restriction of \( V \) to \( Q \) splits into \( V_1 \oplus V_2 \oplus V_3 \) where \( V_1 = C_V(Q), V_2 \) is the part on which \( Q \) acts by nontrivial linear characters, and \( V_3 = [Z(Q), V] \). Any non-linear irreducible
representation of \( Q \) has degree \( q^4 \), and all these are conjugate by an element of order \( q - 1 \) in \( L \). Thus \( \dim(V_3) = mq^4(q - 1) \) for some positive \( m \).

Let \( g \) be a long root element in \( Z(Q) \). Then the Brauer character on \( V_3 \) takes value \( -mq^4 \) on \( g \). On the other hand, it vanishes on any long root element in \( Q \setminus Z(Q) \) by [13], Lemma 2.3. Since all long root elements are conjugate in \( 3D_4(q) \) and \( Q \) acts trivially on \( V_1 \) we conclude that \( V_2 \neq 0 \).

The space \( U \) of linear characters of \( Q \) is isomorphic to \( M \otimes M^F \otimes M^{F^2} \) as \( SL_2(q^3) \)-module, where \( M \) is the natural \( SL_2 \)-module and \( F \) is the Frobenius map of \( F_q^3/F_q \). Clearly \( SL_2(q^3) \) has no fixed points on \( U \), nor has the maximal subgroup \( SL_2(q) \). The Borel subgroup stabilizes a line, hence a subgroup of index \( q - 1 \) fixes a vector, giving an orbit of length \( (q^3 + 1)(q - 1) \). All other subgroups of \( SL_2(q^3) \) have index at least \( \frac{1}{2}q^2 \) times that large. Hence \( \dim(V_2) \geq (q^3 + 1)(q - 1) \), and we obtain

\[
\dim(V) \geq \dim(V_2) + \dim(V_3) \geq q^4(q - 1) + (q^3 + 1)(q - 1) = q(q^4 - q^2 + 1) - 1.
\]

Note that, by the results of Lusztig, \( 3D_4(q) \) has a complex irreducible unipotent character of degree \( q(q^4 - q^2 + 1) \). Moreover, Harish-Chandra induction of projectives from the Levi subgroup \( L = SL_2(q^3).Z_{q-1} \) shows that in characteristic \( \ell \mid (q + 1) \) this splits off a trivial composition factor, hence the above result is best possible. In particular our bound is better than the bound \( q^3(q^2 - 1) \) given in [13]. In the case \( q \) odd and \( \ell \neq 2 \) an alternative proof of Theorem 4.1 using generalized Gelfand-Graev characters was given in [17], 4.4.

**Theorem 4.2.** Let \( V \) be a nontrivial irreducible representation of \( ^2E_6(q)_{sc} \) in characteristic \( \ell \parallel q \). Then

\[
\dim(V) \geq q(q^4 + 1)(q^6 - q^3 + 1) - 2.
\]

**Proof.** Let \( N = Q.SU_6(q).Z_{q-1} \) be a maximal parabolic subgroup of \( ^2E_6(q) \) with special unipotent radical \( Q \) of order \( q^{1+20} \). We proceed as in the previous proof. Now \( V_3 := [Z(Q), V] \) has dimension \( mq^{10}(q - 1) \) for some positive \( m \), and consideration of values of Brauer characters on long root elements shows that the linear part \( V_2 \) is nontrivial.

In order to determine the orbits of the Levi factor \( L := SU_6(q).Z_{q-1} \) on the linear characters of \( Q \) we first look at the case of the algebraic group of type \( E_6 \). By the result of Hoffman in the case of the untwisted group [10], Sect. 2, the Levi subgroup of type \( SL_6 \) has five nontrivial orbits on the linear characters of the corresponding unipotent radical. Representatives for these are known explicitly. Taking fixed points under the twisted Frobenius
morphism of $E_6$ then yields that in our case the nontrivial orbits have lengths 
\[(q^2 - 1)(q^3 + 1)(q^5 + 1), \quad q^2(q^3 + 1)(q^5 + 1)(q^6 - 1)/(q + 1), \]
\[q^4(q^4 - 1)(q^5 + 1)(q^6 - 1), \quad q^6(q - 1)(q^3 + 1)(q^6 + 1)(q^5 + 1)/2.\]

Thus we have $\dim(V_2) \geq (q^2 - 1)(q^3 + 1)(q^5 + 1)$, the length of the shortest orbit.

By Lusztig’s classification there exists a unipotent complex module $\tilde{V}$ of $G$ of dimension $q(q^4 + 1)(q^6 - q^3 + 1)$. The preceding argument shows that this satisfies

\[
\dim(C_{\tilde{V}}(Q)) = \dim(\tilde{V}) - (q^2 - 1)(q^3 + 1)(q^5 - 1) - q^{10}(q - 1) = (q^6 + 2q + 1)/(q + 1).
\]

The permutation character of the Weyl group of $E_6$ on the cosets of $S_6$ has five constituents, hence (by Harish-Chandra theory) $L$ has at most five trivial composition factors on $C_{\tilde{V}}(Q)$. Any nontrivial $L$-composition factor of $C_{\tilde{V}}(Q)$ has dimension at least $(q^6 - 1)/(q + 1)$ (see for example [7]).

Comparison of the Brauer characters of $V$ and $\tilde{V}$ on long root elements [10] shows that hence $L$ must also act nontrivially on $V_1 = C_{\tilde{V}}(Q)$, whence $\dim(V_1) \geq (q^6 - 1)/(q + 1)$. In conclusion we obtain

\[
\dim(V) = \dim(V_1) + \dim(V_2) + \dim(V_3) \\
\geq (q^6 - 1)/(q + 1) + (q^2 - 1)(q^3 + 1)(q^5 + 1) + q^{10}(q - 1) = q(q^4 + 1)(q^6 - q^3 + 1) - 2.
\]

As stated in the proof, there exists a complex unipotent representation of $E_6(q)$ of degree $q(q^4 + 1)(q^6 - q^3 + 1)$. The lower bound given in [13] was $q^9(q^2 - 1)$.

### 5. The bad case.

In this section we deal exclusively with the bad case, that is, where $p = 2$ (so $\ell$ is odd) and $(\text{type}(V), X) \in \{(-, \tilde{\Sigma}^2), (+, \tilde{\Lambda}^2)\}$. In particular we may assume here that $V$ carries a nondegenerate bilinear form and $X(V) = Y(V)$.

We are able to eliminate all classical groups over fields $\mathbb{F}_q$ with $q \geq 16$ ($q \geq 8$ if $G$ is not symplectic) and to derive upper bounds for $\dim(V)$ for the remaining $q$. These bounds are very close to the known lower bounds $l(G)$ for the dimension of nontrivial representations of $G$ in cross characteristic, the worst case occurring for $(q, \ell) = (2, 3)$. For exceptional groups not of type $F_4$ we show that necessarily $X = \tilde{\Lambda}^2$ and $\dim(V) < 2l(G)$. Finally, for $^2F_4$ and $F_4$ we obtain that $\dim(V) \leq 4ql(G)$.
5A. Groups of small rank.

We first show that certain small rank groups do not lead to examples. For

\[ G \in \{ \text{SU}_3(q), \text{Sp}_4(q), \text{^2B}_2(q^2) \}, \]

q even, we may argue as follows. A lower bound for the dimension of an irreducible nontrivial \( \mathbb{F}G \)-module is \( \frac{(q^3 - q)}{(q+1)}, \frac{q(q-1)^2}{2}, \frac{q\sqrt{2}(q^2-1)}{q/q^2}, \frac{2}{2}, \frac{q/q^2}{\sqrt{2}}(q^2 - 1) \) (for \( q^2 > 8 \)) by [23]. On the other hand, the largest degree of an irreducible complex representation for \( G \) is given by \( (q^2 - 1)(q + 1), (q + 1)^2(q^2 + 1), (q^2 - 1)(q^2 + \sqrt{2}q + 1) \) respectively (for example by Deligne-Lusztig theory).

The assumption that \( X(V) \) is irreducible now leads to a contradiction for \( q \geq 8, q \geq 8, q^2 > 8 \) respectively. The tables of Brauer characters of the remaining groups are contained in [12] and allow to verify that no examples arise.

Lemma 5.1. Let \( G = G_2(q), q \geq 8 \) even, \( \ell \neq 2 \), and \( V \) a self-dual absolutely irreducible faithful \( \mathbb{F}G \)-module. Then \( X(V) \) is reducible for \( X \in \{ ^\Lambda^2, ^\Sigma^2 \} \).

Proof. The largest ordinary character degree of \( G \) equals \( (q + 1)(q^2 + q + 1)(q^2 + 1) \), thus \( \dim(V) \) is bounded above by \( 2q^3 \). By the known decomposition numbers for \( G_2(q) \) [8] this implies that \( V \) is the largest irreducible constituent of the ordinary irreducible \( \hat{V} \) of degree \( q^3 + \epsilon \) (where \( q \equiv \epsilon \) (mod 3)). Moreover \( \hat{V} \) remains irreducible in positive characteristic unless \( \ell = 3 \) and \( \epsilon = 1 \), and in the latter case it splits off one trivial composition factor. The ordinary character table of \( G_2(q) \) is known, and it can be verified with Chevie [2] that both the symmetric and the alternating square of the character \( \chi \) of \( \hat{V} \) contain several irreducible constituents in characteristic 0, hence in characteristic \( \ell \neq 3 \).

If \( \ell = 3 \), one checks again in characteristic 0 that the tensor product of \( \chi-1 \) with itself decomposes positively, and neither alternating nor symmetric square can be irreducible. \( \square \)

5B. Unitary groups.

Recall that \( \text{SU}_3(q) \) was handled in part A. For unitary groups in dimension at least 4 we first assume that \( q \neq 2 \).

Proposition 5.2. Let \( G \) be a covering group of \( \text{U}_n(q), n \geq 4, 2 \neq q \) even, \( \ell \neq 2 \) and \( V \) a self-dual absolutely irreducible faithful \( \mathbb{F}G \)-module. Then \( X(V) \) is reducible for \( X \in \{ ^\Lambda^2, ^\Sigma^2 \} \).

Proof. We first claim that \( V \) must be the \( \ell \)-modular reduction of a Weil representation by applying gap results for low-dimensional irreducible representations and a recognition theorem for Weil representations proved in
[7]. Thus assume that \( V \) can not be obtained in this way. Then we have

\[
\dim(V) \geq \begin{cases} 
(q^2 - q + 1)(q^2 + 1) - 1 & \text{if } n = 4, \\
q^{n-2}(q - 1)(q^{n-2} - q)/(q + 1) & \text{if } n \geq 5 \text{ is odd}, \\
q^{n-2}(q - 1)(q^{n-2} - 1)/(q + 1) & \text{if } n \geq 6 \text{ is even},
\end{cases}
\]

by [7] Th. 16.

We now derive an upper bound for \( \dim(X(V)) \). If \( n = 4, 5 \) we just take the largest degree \( q(q+1)^3(q^2+1) \), resp. \( (q+1)^3(q^2+1)(q^5+1) \) of an ordinary irreducible character. This contradicts the lower bound. For \( n \geq 6 \) let \( N \) be the maximal parabolic subgroup considered in Section 3, the stabilizer of an isotropic 1-space, with Levi subgroup \( L \) such that \( L' = SU_{n-2}(q) \) and special unipotent radical \( Q \) of type \( q^{1+2(n-2)} \). The restriction of \( V \) to \( Q \) thus splits into a direct sum of \( C_V(Q) = V_1, V_3 = [Z(Q), V] \) and the part \( V_2 \) on which \( Q \) acts by nontrivial linear characters. Furthermore, \( V_3 \) splits into the isotypic components \( V_\psi \) for the \( q - 1 \) non-linear irreducible characters \( \psi \) of \( Q \) of degree \( q^{n-2} \). Let \( M \) be an isotypic \( Q \)-component on \( V_3 \) of dimension \( dq^{n-2} \). By Lemma 2.3 and Proposition 2.4 in [19] \( M \otimes M \) contains an \( N \)-submodule of dimension \( d^2 \). Since \( V \) is self-dual and \( 2q \) we deduce that \( X(V) \) also contains an \( N \)-submodule of dimension \( d(d \pm 1)/2 \) (the sign depending on \( X \) and the type of \( Q \)). If \( d > 1 \), this gives the upper bound \( \dim(X(V)) \leq [G : N]d(d \pm 1)/2 \), while on the other hand clearly \( \dim(V) \geq dq^{n-2}(q - 1) \), which leads to a contradiction for \( q \geq 4 \). Thus all \( Q \)-isotypic parts of \( V_3 \) are irreducible, and \( \dim(V_3) = q^{n-2}(q - 1) \).

We next estimate the dimension of \( V_2 \). Let \( \mu \) denote the Brauer character of \( V \) and \( \mu_i \) the Brauer character of \( V_i \), \( i = 2, 3 \). Let \( g \in Z(Q) \) be a central involution of \( Q \). Since \( V_2 \) contains all non-linear characters of \( Q \) exactly once, \( \mu_3(g) = -q^{n-2} \) and \( \mu(g) = \dim(V) - \dim(V_3) - q^{n-2} = \dim(V) - q^{n-1} \) since \( Z(Q) \) acts trivially on \( V_1 \oplus V_2 \). By the main result of [3], \( \mu(g)/\dim(V) \leq 3/4 \), hence \( \dim(V) \leq 4q^{n-1} \). But this contradicts the lower bound given above.

Thus \( V \) is the \( \ell \)-modular reduction of a complex Weil representation \( \bar{V} \) and has dimension \( (q^n + (-1)^n q)/(q + 1) \) or \( (q^n - (-1)^n)/(q + 1) \) (see [7], Th. 16). In particular \( X(V) \) is reducible unless the complex representation \( X(\bar{V}) \) is irreducible. The latter cannot happen according to [14], Prop. 3.8.

\[ \Box \]

**Proposition 5.3.** Let \( G \) be a covering group of \( U_n(2) \), \( n \geq 5 \), \( \ell \neq 2 \) and \( V \) a self-dual absolutely irreducible faithful \( FG \)-module. Then \( X(V) \) is reducible unless possibly if

\[
\dim(X(V)) \leq \begin{cases} 
2^{n-4}(2^n - (-1)^n)(2^{n-1} + (-1)^n)/9 & \text{if } \ell \neq 3, \\
2^n(2^n - (-1)^n)(2^{n-1} + (-1)^n)(2^{n-3} + 1)/3 & \text{if } \ell = 3.
\end{cases}
\]

**Proof.** The cases \( n \leq 6 \) can be checked directly. Hence we may assume \( n \geq 7 \) and consider \( V \) as a \( G \)-module with \( G = SU_n(2) \).
For \( \ell \neq 3 \) consider the subgroup \( H = \text{SU}_2(2) \times \text{SU}_{n-2}(2) \). The first factor is isomorphic to the symmetric group \( \mathfrak{S}_3 \). The nontrivial eigenspaces for an element \( g \) of order 3 in this factor are dual to each other as \( C \)-modules, where \( C := C_G(g) \). Thus \( X(V) \) contains a trivial \( C \)-module and so a 1-dimensional module for \( M := N_G(\langle g \rangle) \). Observe that \( |M| = 3|H| \). This leads to the estimate

\[
\dim(X(V)) \leq (G : M) = 2^{2n-4} \frac{(2^n - (1)^n)(2^{n-1} + (-1)^n)}{9}.
\]

Now assume that \( \ell = 3 \). We first prove a crude upper bound for \( \dim(X(V)) \) as follows. Let \( H = \text{SU}_3(2) \times \text{SU}_{n-3}(2) \). We may now restrict to the eigenspaces of order-4 elements in the quaternion group contained in the first factor to obtain a trivial composition factor for \( \text{SU}_{n-3}(2) \) in the socle of \( X(V) \). The 3-modular Brauer characters of \( \text{SU}_3(2) \) have degree at most 2, and we conclude

\[
(5.4) \quad \dim(X(V)) \leq 2^{3n-8} \frac{(2^n - (1)^n)(2^{n-1} + (-1)^n)(2^{n-2} + (-1)^n)}{27}.
\]

To improve this bound, as above let \( N = Q.L \) be a maximal parabolic subgroup of \( G \) with \( L' = \text{SU}_{n-2}(2) \). Assume first that \( C_V(Q) =: V_1 \neq 0 \) and let \( S \) be an \( L \)-composition factor of \( V_1 \). By [5], Lemma 4.2(iii), this occurs again as an \( L \)-composition factor of \( [V, Q] \). So both the symmetric and the alternating square of \( V \) contain a trivial \( L \)-composition factor and

\[
\dim(X(V)) \leq [G : L] = 2^{2n-3} \frac{(2^n - (1)^n)(2^{n-1} + (-1)^n)}{3}.
\]

Otherwise, as \( V \) is faithful for \( G \), the center of the extraspecial group \( Q \) doesn’t act by scalars. So there exists a nontrivial linear character \( \lambda \) of \( Q \) such that the corresponding isotypic component \( V_\lambda \) of \( V \) is nonzero. Denote by \( I_\lambda \) the stabilizer of \( \lambda \) in \( L \). Then \( I_\lambda \) stabilizes two further characters \( \lambda', \lambda'' \) of \( Q \), and \( \lambda, \lambda', \lambda'' \) are conjugate by an element of order three in the normalizer of \( I_\lambda \). Write \( \bar{Q} \) for the intersection of the kernels of \( \lambda, \lambda', \lambda'' \), a subgroup of \( Q \) of index 4. We distinguish two cases according to the type of \( \lambda \).

If \( \lambda \) is anisotropic, then \( I_\lambda = \text{SU}_{n-3}(2) \). If \( 3 \nmid n \) then \( \lambda, \lambda', \lambda'' \) are already conjugate in the centralizer of \( I_\lambda \). In particular the selfdual \( \bar{Q}.I_\lambda \)-modules \( V_\lambda, V'_\lambda \) are equivalent, so \( X(V) \) contains a trivial \( \bar{Q}.I_\lambda \)-composition factor in the socle. This gives the upper bound

\[
\dim(X(V)) \leq [G : \bar{Q}.I_\lambda] \leq 4.2^{n-3}(2^{n-2} + 1)(2^n - (1)^n)(2^{n-1} + (-1)^n)/3.
\]

When \( 3 \mid n \) the trivial and the (at most two) Weil representations of \( I_\lambda \) are invariant under the diagonal outer automorphism of order 3. Thus if the socle of \( V_\lambda \) contains one of these representation, then the same is true for \( V_{\lambda'} \), and we conclude as in the previous case. Otherwise, by [7], Th. 16, we...
have \( \dim(V_\lambda) \geq 2^{n-4}(2^{n-6} - 1)/3 \), so

\[
\dim(V) \geq [L : I_\lambda] \dim(V_\lambda) \geq 2^{n-3}(2^{n-2} - (-1)^n)2^{n-4}(2^{n-6} - 1)/3,
\]
violating the upper bound (5.4) if \( n \geq 10 \), respectively larger than the square root of the largest character degree if \( n = 9 \) (note that \( 3|n \)).

It remains to consider the case of isotropic \( \lambda \), with

\[
I_\lambda = 2^{1+2(n-4)}GU_{n-4}(2).
\]

If the unipotent radical \( R \) of \( I_\lambda \) acts trivially on \( V_\lambda \), we may argue as in the previous case, either obtaining the upper bound

\[
\dim(X(V)) \leq [G : \widetilde{Q}.I_\lambda] \leq 4.2^{n-3}(2^{(n-3) + 1})(2^n - (-1)^n)(2^{n-1} + (-1)^n)/3
\]
or a contradiction to (5.4) respectively to the largest degree of an ordinary irreducible character when \( n = 9 \).

On the other hand, if \( V_\lambda \) contains a nontrivial linear character \( \mu \) of \( R \), then

\[
\dim(V) \geq [L : I_\lambda] \dim(V_\lambda)
\]

\[
\geq [L : I_\lambda][I_\lambda : I_{\lambda\mu}] = 2^{n-2}(2^{n-3} + 1)2^{n-4}(2^{n-5} + 1),
\]
with the stabilizer \( I_{\lambda\mu} \) in \( GU_{n-4}(2) \) of \( \mu \), too large compared to (5.4) for \( n \geq 7 \).

If finally \( V_\lambda \) contains the faithful character of \( R \), then at least

\[
\dim(V) \geq [G : \widetilde{Q}.I_\lambda]2^{n-4}\dim(D) = 2^n(2^{n-3} + 1)2^{n-4}\dim(D)
\]
with \( D \) an irreducible representation of \( SU_{n-4}(2) \). If \( D \) is nontrivial, then \( \dim(D) \geq (2^{n-4} - 2)/3 \), too large for \( n \geq 7 \). Thus \( SU_{n-4}(2) \) has a trivial composition factor in the socle of \( V_\lambda \), and the (unique) faithful representation of \( R \) occurs in the socle of \( V_\lambda \). But then the same is true for \( V_\lambda \) and we find a trivial \( \bar{U}.I_\lambda \)-composition factor in the socle of \( X(V) \). Arguing as before we obtain the desired bound.

\[ \square \]

5C. Symplectic groups.

We next deal with the symplectic groups \( S_{2n}(q) \), \( n \geq 3 \):

**Proposition 5.5.** Let \( G = S_{2n}(q) \), \( q \) even, \( n \geq 3 \), \( (n,q) \neq (3,2), (\ell \neq 2, (q,\ell) \neq (2,3) \), and \( V \) a self-dual absolutely irreducible faithful \( FG \)-module. Then \( X(V) \) is reducible for \( X \in \{\Lambda^2, \text{Sym}^2\} \) unless possibly if \( X = \Lambda^2 \) and

\[
\frac{(q^n - 1)(q^n - q)}{2(q + 1)} \leq \dim(V) \leq \begin{cases} 2(q^{2n-1} - 1) & \text{if } q = 2, \\ q^n(q^{n-1} - 1) & \text{if } q = 4, \\ \frac{1}{2}(q^n - q)^2/(q - 1) & \text{if } q = 8. \end{cases}
\]

**Proof.** The case \( G = S_8(2) \) can be checked from [12] so we may also assume \( (n,q) \neq (4,2) \). Let \( G = S_{2n}(q) \) and \( V \) a self-dual absolutely irreducible faithful \( FG \)-module such that \( X(V) \) is irreducible. Let \( H := S_2(q) \times S_{2n-2}(q) \leq G \)
be the stabilizer of a 2-dimensional subspace of the natural module. Upon restriction to the Sylow 2-subgroup $U$ of the first factor of $H$ the module $V$ decomposes into a direct sum $\oplus \lambda V_\lambda$, $\lambda \in \text{Hom}(U, \mathbb{F}^\times)$. The $V_\lambda$ for $\lambda \neq 1$ are permuted by the normalizer of $U$ in the first factor of $H$, so they are isomorphic $S_{2n-2}(q)$-modules. For $q \geq 4$ we thus obtain at least 3 trivial $S_{2n-2}(q)$-composition factors in the socle of $X(V)$. Hence $X(V)$ is a constituent of $1^G_{S_{2n-2}(q)}$. If $q = 2$, $\ell \neq 3$, we consider instead the eigenspaces $V_\lambda$ of the element of order 3 in the first factor $S_2(2)$ of $H$ and reach the same conclusion. But the irreducible complex characters of $S_2(q)$ have degree at most $q + 1$, so we obtain the estimate

$$\dim(X(V)) \leq (q + 1)[G:H] = \frac{q^{2n-2}(q^{2n} - 1)}{q - 1}.$$  

On the other hand by [23] any faithful $\mathbb{F}G$-module $V$ satisfies

$$\dim(V) \geq \frac{(q^n - 1)(q^n - q)}{2(q + 1)}.$$  

This leads to a contradiction unless $q \leq 8$.

Next we restrict $V$ to the unipotent radical $U$ of the maximal parabolic subgroup $N$ in Table 3.5, and decompose $V$ as

$$V = \bigoplus_{\lambda \in \text{Hom}(U, \mathbb{F}^\times)} V_\lambda$$  

into $U$-isotypic components. Now $|U| = q^{2n-1}$, and the Levi factor $L = \text{CSp}_{2n-2}(q)$ has orbits $O_1, O_2, O_3, O_4$ of lengths

$$1, q^{2n-2} - 1, \frac{1}{2}q^{n-1}(q^{n-1} - 1)(q - 1), \frac{1}{2}q^{n-1}(q^{n-1} + 1)(q - 1)$$  

on $\text{Hom}(U, \mathbb{F}^\times)$. All elements in $O_2$ have the invariant 1-dimensional subspace of $U$ in their kernel, so since $V$ is faithful, some $V_\lambda$ for $\lambda$ in $O_3$ or $O_4$ has to be nontrivial. Writing $d_\lambda$ for its dimension we get

$$\dim(V) \geq \frac{1}{2}q^{n-1}(q^{n-1} - 1)(q - 1)d_\lambda$$  

and comparison with (5.6) yields $d_\lambda = 1$ if $q = 8$, $d_\lambda \leq 2$ if $q = 4$, $d_\lambda \leq 6$ if $q = 2$ and $n \geq 5$, respectively. Thus the stabilizer $I_\lambda$ in $L$ of $\lambda$ acts trivially on $V_\lambda$. Hence if $X = \text{Sym}^2$ or if $d_\lambda > 1$ then for each $\lambda$ in a fixed orbit the module $X(V)$ has a trivial $I_\lambda$-composition factor in the socle. But $L$ permutes these $\lambda$, thus we obtain a trivial constituent in $X(V)$ for the derived group of $L$. This forces $\dim(X(V)) \leq q^{2n} - 1$ which gives a contradiction to the Seitz-Zalesskii bound.

So we have $q \leq 8$, $d_\lambda \leq 1$ for $\lambda \in O_3 \cup O_4$ and $X = \Lambda^2$. For $q = 4, 8$ the upper bound stated in the Proposition now follows from (5.6). For $q = 2$ we obtain at least $\dim(V) \leq 2(|U| - 1)$. Indeed, each nontrivial orbit can occur at most once, and the dimension of the fixed point space cannot be
larger than the commutator, for otherwise the parabolic subgroup \( N \) and its opposite (which together generate \( G \)) would have a common fixed vector. \( \square \)

**Proposition 5.7.** Let \( G = S_{2n}(2) \), \( n \geq 4 \), \( \ell = 3 \), and \( V \) a self-dual absolutely irreducible faithful \( \mathbb{F}G \)-module. Then \( X(V) \) is reducible for \( X \in \{ \Lambda^2, \text{Sym}^2 \} \) unless possibly if

\[
\dim(X(V)) \leq 2^{4n-8} \frac{(2^{2n} - 1)(2^{2n-2} - 1)}{5}.
\]

**Proof.** Let \( H = S_4(2) \times S_{2n-4}(2) \) and note that \( S_4(2) \) is isomorphic to the symmetric group \( S_6 \). We restrict \( V \) to a subgroup of order 5 of the first factor. Since elements of order 5 are rational in \( S_6 \), the eigenspaces for the nontrivial eigenvalues are permuted transitively and hence are isomorphic as \( S_{2n-4}(2) \)-modules. Thus in \( X(V) \) we find a trivial \( S_{2n-4}(2) \)-module in the socle. The largest degree of a 3-modular irreducible for \( S_6 \) is 9, hence we find an \( H \)-module of dimension at most 9 in the socle of \( X(V) \). This shows

\[
\dim(X(V)) \leq 2^{4n-4} \frac{(2^{2n} - 1)(2^{2n-2} - 1)}{80}
\]

as claimed. \( \square \)

In \( \S 7 \) we will show that \( G = S_{2n}(2) \) does in fact lead to examples of irreducible tensor products and irreducible alternating squares.

**5D. Orthogonal groups.**

The case of orthogonal groups is the least pleasant:

**Proposition 5.8.** Let \( G = O^\delta_{2n}(q) \), \( n \geq 4 \), \( q \) even, \( \ell \neq 2 \), \( (q, \ell) \neq (2, 3) \), and \( V \) a self-dual absolutely irreducible faithful \( \mathbb{F}G \)-module. Then \( X(V) \) is reducible for \( X \in \{ \Lambda^2, \text{Sym}^2 \} \) unless possibly if \( q = 2, 4 \) and

\[
\dim(X(V)) \leq \begin{cases} 
\frac{1}{2} q^{2n-2}(q^n - \epsilon_1)(q^{n-1} + \epsilon_1)/(q - 1) & \text{if } q = 4, \ell = 5, \\
\frac{1}{2} q^{2n-2}(q^n - \epsilon_1)(q^{n-1} - \epsilon_1)/(q + 1) & \text{if } q = 2 \text{ or } \ell \neq 5.
\end{cases}
\]

**Proof.** Let \( G = O^\delta_{2n}(q) \) and let \( \delta \in \{ \pm \} \) be such that \( \ell, |(q - \delta 1) \) and \( q - \delta 1 > 1 \) (which is possible since \( (q, \ell) \neq (2, 3) \)). We restrict \( V \) to a natural subgroup \( O^\delta_2(q) \times O^\delta_{2n-2}(q) \). Let \( V_\lambda, V_{\lambda^{-1}} \) denote eigenspaces for the element of order \( q - \delta 1 \) in the first factor of a nontrivial \( (q - \delta 1) \)-th root of unity \( \lambda \) and its inverse. Since \( V \) is self-dual, both \( V_\lambda, V_{\lambda^{-1}} \) are singular with respect to the nondegenerate form on \( V \) and dual to each other. Thus \( V_\lambda \otimes V_{\lambda^{-1}} \leq X(V) \) has a trivial \( O^\delta_{2n-2}(q) \)-composition factor in the socle. If \( X(V) \) is irreducible this yields the upper bound

\[
\dim(X(V)) \leq \frac{q^{2n-2}(q^n - \epsilon_1)(q^{n-1} + \epsilon_1)}{2(q - 1)}.
\]

\[5.9\]
Observe that if \( \gcd(\ell, q + 1) = 1 \) then we may choose \( \delta = - \), which gives the better bound

\[
\dim(X(V)) \leq \frac{q^{2n-2}(q^n - \epsilon 1)(q^{n-1} - \epsilon 1)}{2(q+1)}.
\]

On the other hand, let \( P = U.L \) be the parabolic subgroup with Levi complement \( L \) of type \( O_{2n-2}'(q) \). The restriction of \( V \) to the unipotent radical \( U \) of \( P \) decomposes as \( V = \oplus_{\lambda} V_{\lambda} \) for \( \lambda \in \text{Hom}(U, \mathbb{F}^\times) \). The Levi factor \( L \) acts on \( U \) and hence on \( \text{Hom}(U, \mathbb{F}^\times) \) as on its natural module. It thus has two nontrivial orbits on \( \text{Hom}(U, \mathbb{F}^\times) \) of lengths \( (q^{n-1} - \epsilon 1)(q^{n-2} + \epsilon 1) \) and \( q^{n-2}(q^{n-1} - \epsilon 1)(q-1) \), consisting of isotropic respectively anisotropic elements. We first claim that \( d_\lambda := \dim(V_{\lambda}) = 0 \) for anisotropic \( \lambda \) unless possibly \( q = 4 \), \( d_\lambda = 1 \), or \( q = 2, d_\lambda \leq 2 \).

So assume that \( d_\lambda \neq 0 \). Then

\[
\dim(V) \geq q^{n-2}(q^n - \epsilon 1)(q-1)d_\lambda.
\]

Comparison with (5.9) yields \( d_\lambda \leq 1 \) if \( q \geq 4 \) respectively \( d_\lambda \leq 3 \) if \( q = 2 \). Thus \( V_{\lambda} \) has to be the trivial module for the stabilizer \( I_\lambda = O_{2n-3}(q) \) of \( \lambda \) in \( L \). If \( q \geq 4 \) the group \( I_\lambda \) stabilizes at least three different elements of \( \text{Hom}(U_1, \mathbb{F}^\times) \). Thus if \( d_\lambda > 1 \) or \( q \geq 4 \) we get a trivial \( U_{\lambda}' : I_\lambda \)-constituent in the socle of \( X(V) \), where \( U_{\lambda}' = \ker(\lambda) \), hence a linear constituent for \( U_1'I_{\lambda} \) extended by an element of order \( q - 1 \). So we obtain the better upper bound

\[
\dim(X(V)) \leq \frac{q^{2n-2}(q^n - \epsilon 1)(q^{n-1} - \epsilon 1)}{q - 1}
\]

which violates (5.11) unless \( q = 4 \), \( d_\lambda = 1 \), or \( q = 2, d_\lambda \leq 2 \).

So next assume that \( d_\lambda \neq 0 \) for an isotropic \( \lambda \). Then

\[
\dim(V) \geq (q^n - \epsilon 1)(q^{n-2} + \epsilon 1)d_\lambda,
\]

and comparison with (5.9) shows that \( d_\lambda < 2(q-2) \). But the smallest degree of a non-linear representation of \( I_{\lambda} \) is at least \( 2(q-1) \) unless \( n = 4 \), \( q \in \{2, 4\} \), \( \epsilon = + \). (Note that for \( n = 4 \), \( \epsilon = + \), we have \( I_{\lambda} = L_2(q) \).) Thus either \( G = O_8^+(2) \), \( G = O_8^-(4) \), or \( I_{\lambda} \) has to act by a linear character of order at most \( 2 \) on \( V_{\lambda} \) (in fact, trivially if \( (n, \epsilon) \neq (4, +) \)). So each \( X(V_{\lambda}) \) contains a trivial \( I_{\lambda}' \)-composition factor in the socle. The group \( I_{\lambda}' \) permutes the isotropic \( \lambda \), so we get a trivial constituent for \( P' \) in \( X(V) \). This yields

\[
\dim(X(V)) \leq (q^n - \epsilon 1)(q^{n-1} + \epsilon 1),
\]

which is a contradiction to (5.12).

Thus if \( X(V) \) is irreducible, then \( q = 2, 4 \) and \( \dim(X(V)) \) is bounded as either in (5.10) or (5.9).

\[ \square \]

**Proposition 5.13.** Let \( G = O_{2n}^{\epsilon}(2) \), \( n \geq 5 \), \( \ell = 3 \), and \( V \) a self-dual absolutely irreducible faithful \( \mathbb{F}G \)-module. Then \( X(V) \) is reducible unless possibly if

\[
\dim(X(V)) \leq 2^{2n-6}(2^n - \epsilon 1)(2^{n-2} - 1)(2^{n-2} - \epsilon 1)/15.
\]
4.1. \( Q \) trivial linear characters of the center \( C \).

Proposition 2.4 in \( V \) isotypic part of \( X \) in the socle of \( \mathfrak{g} \).

Proof. Let \( H = O_{2n}^e(2) \times O_{2n-4}^e(2) \). The first factor is isomorphic to \( L_2(4) \).

The nontrivial eigenspaces of elements of order 5 in this factor yield isomorphic \( O_{2n-4}^e(2) \)-modules, and hence we force a trivial constituent for \( O_{2n-4}^e(2) \) in the socle of \( X(V) \). Since the largest 3-modular degree of \( L_2(4) \) is 4 the claim follows. \( \square \)

5E. Large exceptional groups.

For the following statement we collect the lower bounds for cross-characteric representations of certain exceptional groups from Propositions 4.1 and 4.2 respectively from [10].

\[
\begin{align*}
\text{l}(G) = \begin{cases} 
q(q^4 - q^2 + 1) - 1 & \text{if } G = 3D_4(q) \\
q(q^4 + 1)(q^6 + q^3 + 1) - 1 & \text{if } G = E_6(q) \\
q(q^4 + 1)(q^6 - q^3 + 1) - 2 & \text{if } G = 2E_6(q) \\
q(q^4 - q^2 + 1)(q^{12} + q^{10} + q^8 + q^6 + q^4 + q^2 + 1) - 2 & \text{if } G = E_7(q) \\
q(q^6 + 1)(q^{10} + 1)(q^{12} + 1) - 3 & \text{if } G = E_8(q).
\end{cases}
\end{align*}
\]

Proposition 5.14. Let \( G = 3D_4(q), E_6(q), 2E_6(q), E_7(q) \) or \( E_8(q) \), \( q = 2^n > 2 \), \( \ell \neq 2 \), and \( V \) a self-dual absolutely irreducible faithful \( \mathbb{F}G \)-module. Then \( X(V) \) is reducible unless possibly if \( X = \Lambda^2 \), and

\[
\text{l}(G) \leq \dim(V) < 2\text{l}(G).
\]

Proof. 1) Let \( N = Q.L \) be the maximal parabolic subgroup of \( G \) from Table 3.5, with special unipotent radical \( Q \) of type \( q^{1+2k} \) where \( k = 4, 10, 16, 28 \) respectively. The restriction of \( V \) to \( Q \) splits into the centralizer \( V_1 := C_V(Q) \), the part \( V_2 \) on which \( Q \) acts nontrivial linearly, and \( V_3 := [Z(Q), V] \). The non-linear characters of \( Q \) (of degree \( q^k \)) are indexed by the \( q-1 \) nontrivial characters of the center \( Z(Q) \). Since \( Q \) is normal in \( N \), each isotypic part of \( V|_Q \) is an \( L' \)-module.

Let \( M \) be an isotypic part of \( V_3 \) of dimension \( dq^k \). By Lemma 2.3 and Proposition 2.4 in [19] the tensor square of \( M \) contains an \( N \)-submodule of dimension \( d^2 \). Since \( V \) is self-dual and \( 2|q \) we deduce that \( X(V) \) also contains an \( N \)-submodule of dimension \( d(d+1)/2 \) (the sign depending on \( X \) and the type of \( Q \)). If \( d > 1 \), this gives the upper bound \( \dim(X(V)) \leq [G : N]d(d+1)/2 \), while on the other hand clearly \( \dim(V) \geq dq^k(q-1) \). Using the values

\[
[G : N] = \begin{cases} 
\frac{(q^6-1)(q^4-q^2+1)}{q^2-1} & \text{if } G = 3D_4(q), \\
\frac{(q^4+1)(q^6+q^3+1)(q^{12}+1)}{q^4-1} & \text{if } G = E_6(q), \\
\frac{(q^4+1)(q^6-q^3+1)(q^{12}+1)}{q^4-1} & \text{if } G = 2E_6(q), \\
\frac{(q^4+q^2+1)(q^{14}+1)(q^{18}+1)}{q^4-1} & \text{if } G = E_7(q), \\
\frac{(q^{10}+1)(q^{24}+1)(q^{30}+1)}{(q^4-1)(q^6-1)} & \text{if } G = E_8(q),
\end{cases}
\]

this leads to a contradiction for \( q \geq 4 \). Thus all \( Q \)-isotypic parts of \( V_3 \) are irreducible, and \( \dim(V_3) = q^k(q - 1) \).

If \( Z' \) is a subgroup of index 2 in \( Z(Q) \), then \( Q/Z' \) is an extraspecial 2-group of type \( \epsilon \) for some \( \epsilon = \pm \). Then any non-linear irreducible character of \( Q \) has Schur-Frobenius indicator \( \epsilon \). Since \( d = 1 \) and since we are in the bad case, we see that \( (\epsilon, X) = (+, \Lambda^2) \) or \( (-, \Sym^2) \).

We next estimate the dimension of \( V_2 \). Let \( \mu \) denote the Brauer character of \( V \) and \( \mu_i \) the Brauer character of \( V_i, i = 2, 3 \). Let \( g \in Z(Q) \) be a central involution of \( Q \). Since \( V_3 \) contains all non-linear characters of \( Q \) exactly once, \( \mu_3(g) = -q^k \) and \( \mu(g) = \dim(V) - \dim(V_3) - q^k = \dim(V) - q^{k+1} \) since \( Z(Q) \) acts trivially on \( V_1 \oplus V_2 \). We now use the main result of [3] which states that the value of any nontrivial irreducible Brauer character on any non-identity unipotent element is equal to at most 3/4 of its degree. Plugging this into our above computations we get

\[
\dim(V) \leq 4q^{k+1}.
\]

2) Let first \( G \neq 3D_4(q), E_6(q) \). The linear characters of \( Q \) are just the irreducible characters of \( U := Q/Z(Q) \). The orbits of \( L \) on \( \Hom(U, F^\times) \) are known for \( G = E_7(q), E_8(q) \) [10] and given in the proof of Proposition 4.2 for \( 2E_6(q) \).

Comparison with the upper bound \( \dim(V_2) \leq 3q^{k+1} + q^k \) shows that in all cases only the shortest nontrivial orbit \( O_1 \) can occur. For \( \lambda \in O_1 \) let \( V_\lambda \) denote the \( \lambda \)-isotypic component of \( V_2 \) and \( I_\lambda \) the stabilizer of \( \lambda \) in \( L \) (of semisimple type \( \SL_3(q^2), \SL_6(q), E_6(q) \)). The lower bounds for nontrivial representations of \( I'_\lambda \) compared with the upper bound for \( \dim(V_2) \) above implies that \( I'_\lambda \) acts trivially on \( V_\lambda \). Assume that \( d_\lambda := \dim(V_\lambda) \geq 2 \). Then \( X(V_\lambda) \) contains a trivial composition factor for \( I'_\lambda \), but clearly also for \( U \) (since \( 2|q \)). The representation of \( L' \) on the \( L' \)-orbit of that trivial submodule is the permutation module of \( L' \) on the cosets of \( I'_\lambda \), thus it contains a trivial \( L' \)-composition factor. We hence obtain that \( \dim(X(V)) \leq [G : N'] \) which gives a contradiction to the lower bound for \( \dim(V) \) from [23]. Thus \( d_\lambda \leq 1 \) and we find

\[
\dim(V) \leq 2(q^k(q - 1) + |O_1|) < 2l(G).
\]

3) Let now \( G = 3D_4(q) \). We first claim that \( G \) is generated by four long root elements. Indeed, the normalizer of a Coxeter torus is maximal in \( G \), and d not contain long root elements. Thus any four long root elements whose product is a generator of a Coxeter torus must generate \( G \). Using the character table in [2] it can be verified that the structure constant for the corresponding 5-tuples is nonzero.

Thus in any irreducible representation \( V \) the \( \pm 1 \)-eigenspaces of a long root element \( g \) can have dimension at most 3/4 \( \dim(V) \). If \( \mu \) denotes the corresponding Brauer character, it follows that \( |\mu(g)/\mu(1)| \leq 1/2 \). This yields \( \dim(V) \leq 2q^5 \). Thus again only the shortest orbit \( O_1 \) of \( L \) on the
nontrivial linear characters of \( Q/Z(Q) \) can occur. In this case the stabilizer \( I_\lambda \) of \( \lambda \in O_1 \) is a subgroup of order \( q^3(q^2 + q + 1) \) of the Borel subgroup. Its non-linear irreducible characters have degree \( q^2 + q + 1 \). Comparison with the upper bound for \( \text{dim}(V) \) shows that \( I_\lambda \) has to act linearly. Since \( V \) is self-dual, \( V_{\lambda} \) is also self-dual. If \( \text{dim}(V_{\lambda}) > 1 \) then \( X(V) \) contains a trivial composition factor for \( I_\lambda \). Arguing as in 2) above this leads to a contradiction, forcing \( d_\lambda \leq 1 \).

4) Let \( G = E_6(q) \). Let \( P = U.\text{Spin}^+_10(q).Z_{q-1} \) be the \( D_5 \)-parabolic subgroup of \( G \). Its unipotent radical \( U \) is elementary abelian of order \( q^{16} \), and the Levi factor \( L \) has two nontrivial orbits \( O_1, O_2 \) on \( \text{Hom}(U, \mathbb{F}^\times) \), of lengths

\[
(q^3 + 1)(q^8 - 1), \quad q^3(q^5 - 1)(q^8 - 1)
\]

(see [10]). For \( \lambda \in \text{Hom}(U, \mathbb{F}^\times) \) let \( V_{\lambda} \) denote the \( \lambda \)-isotypic component of the restriction of \( V \) to \( U \) and \( d_\lambda \) its dimension. By part 1), \( d_\lambda = 0 \) if \( \lambda \in O_2 \) and \( d_\lambda \leq 5 \) if \( \lambda \in O_1 \). Let \( \lambda \in O_1 \) and denote by \( I_\lambda \) the stabilizer of \( \lambda \) in \( L_2 \), with semisimple part \( \text{SL}_5(q) \). Since any nontrivial representation of \( I_\lambda' \) has dimension at least \( (q^5 - 1)/(q - 1) - 5 \) by [23], \( I_\lambda' \) has to act trivially on \( V_\lambda \). Assume that \( d_\lambda \geq 2 \). Then again \( X(V_{\lambda}) \) contains a trivial composition factor for \( I_\lambda \). We may proceed as before to obtain that \( \text{dim}(X(V)) \leq [G: P'] \), again contradicting the lower bound for \( \text{dim}(V) \) from [13]. Thus \( d_\lambda = 1 \) for \( \lambda \in O_1 \). In particular we find

\[
\text{dim}(V) \leq 2|O_1| = 2(q^3 + 1)(q^8 - 1) < 2l(G)
\]
as claimed.

5) Finally, observe that \( (\epsilon, X) = (+, A^2) \). For, \( V|_Q \) contains linear characters of \( Q \) with multiplicity 1, and clearly those characters are of type +, whence \( V \) itself is of type +. \( \square \)

**Proposition 5.15.** Let \( G = 2F_4(q^2) \) with \( q^2 > 2 \), or \( G = F_4(q) \) with \( 2 \neq q \) even, and let \( \ell \neq 2 \), and \( V \) a self-dual absolutely irreducible faithful \( \mathbb{F}G \)-module. Then \( X(V) \) is reducible unless possibly if

\[
\text{dim}(V) \leq 2q^{11}(q + 1).
\]

**Proof.** The largest degree of an irreducible complex character of \( G \) is \( (q^2 - 1)(q^6 + 1)(q^8 - 1)(q^{12} + 1)/(q^2 - \sqrt{2}q + 1)^2 \), respectively \( (q^2 - 1)(q^6 - 1)(q^8 - 1)(q^{12} - 1)/(q - 1)^4 \), thus we get the trivial upper bound \( \text{dim}(V) \leq 2q^{11}(q + 1) \). \( \square \)

6. The sporadic groups.

**Proposition 6.1.** Let \( G \) be a covering group of a sporadic simple group in

\[\{M_{11}, M_{12}, J_1, M_{22}, J_2, M_{23}, HS, J_3, M_{24}, McL, He, Suz, Co_3, Co_2\}\],

\( \ell \geq 0 \) and \( V \) an absolutely irreducible faithful \( \mathbb{F}G \)-module. Then \( X(V) \) is reducible for \( X \in \{A^2, S^2, A\} \) unless \( (G, \ell, \text{dim}(V), X) \) are as in Table 6.2.
Table 6.2. Small sporadic groups.

| $G$ |
|---|---|---|---|---|
| $M_{11}$ |
| $\ell$ |
| 3 |
| 11 |
| $\neq 11$ |
| $\neq 2,3$ |
| 2.M$_{12}$ |
| 3 |
| 2 |
| $\neq 2$ |
| $\neq 2,3$ |
| 2.M$_{12}$ |
| 3 |
| 2 |
| $\neq 2$ |
| $\neq 2,3$ |
| J$_1$ |
| 11 |
| 2 |
| 2 |
| $\neq 2,7$ |
| M$_{22}$ |
| 11 |
| 2 |
| $\neq 2,11$ |
| 2.J$_2$ |
| 2 |
| $\neq 2$ |
| 2.J$_2$ |
| 3 |
| $\neq 2$ |
| M$_{23}$ |
| 2 |
| 23 |
| $\neq 2,3$ |
| M$_{23}$ |
| 7 |
| 23 |
| $\neq 2,5,7$ |

$\ell$ denotes the smallest degree of an irreducible character of $G$, $\dim(V)$ is the dimension of the corresponding irreducible representation, and $\lambda^2$, $\Sigma^2$, and $\Lambda$ are the corresponding Brauer characters.

Proof. This can be checked from the known tables of Brauer characters [12].
Table 6.2. Small sporadic groups (continued).

<table>
<thead>
<tr>
<th>$G$</th>
<th>$\ell$</th>
<th>dim($V$)</th>
<th>$\Lambda^2$</th>
<th>$\Sigma^2$</th>
<th>$A$</th>
</tr>
</thead>
<tbody>
<tr>
<td>HS</td>
<td>5</td>
<td>21</td>
<td>$\times$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>HS</td>
<td>$\neq$ 2, 5</td>
<td>22</td>
<td>$\times$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3.J3</td>
<td>2</td>
<td>9</td>
<td>$\times$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>J3</td>
<td>3</td>
<td>18</td>
<td>$\times$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3.J3</td>
<td>$\neq$ 3</td>
<td>18</td>
<td>$\times$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>M24</td>
<td>2</td>
<td>11</td>
<td>$\times$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>M24</td>
<td>3</td>
<td>22</td>
<td>$\times$</td>
<td>$\times$</td>
<td></td>
</tr>
<tr>
<td>M24</td>
<td>$\neq$ 2, 3</td>
<td>23</td>
<td>$\times$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>M24</td>
<td>7</td>
<td>45</td>
<td>$\times$</td>
<td>$\times$</td>
<td></td>
</tr>
<tr>
<td>M24</td>
<td>$\neq$ 2, 3, 7</td>
<td>45</td>
<td>$\times$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>McL</td>
<td>3</td>
<td>21</td>
<td>$\times$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>McL</td>
<td>5</td>
<td>21</td>
<td>$\times$</td>
<td>$\times$</td>
<td></td>
</tr>
<tr>
<td>McL</td>
<td>2</td>
<td>22</td>
<td>$\times$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>McL</td>
<td>$\neq$ 2, 3, 5</td>
<td>22</td>
<td>$\times$</td>
<td>$\times$</td>
<td></td>
</tr>
<tr>
<td>Ru</td>
<td>2</td>
<td>28</td>
<td>$\times$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2.Ru</td>
<td>$\neq$ 2</td>
<td>28</td>
<td>$\times$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3.Suz</td>
<td>2</td>
<td>12</td>
<td>$\times$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2.Suz</td>
<td>3</td>
<td>12</td>
<td>$\times$</td>
<td>$\times$</td>
<td></td>
</tr>
<tr>
<td>6.Suz</td>
<td>$\neq$ 2, 3</td>
<td>12</td>
<td>$\times$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Co3</td>
<td>2, 3</td>
<td>22</td>
<td>$\times$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Co3</td>
<td>$\neq$ 2, 3</td>
<td>23</td>
<td>$\times$</td>
<td>$\times$</td>
<td></td>
</tr>
<tr>
<td>Co3</td>
<td>5</td>
<td>230</td>
<td>$\times$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Co3</td>
<td>$\neq$ 2, 3, 5</td>
<td>253</td>
<td>$\times$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Co2</td>
<td>2</td>
<td>22</td>
<td>$\times$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Co2</td>
<td>$\neq$ 2</td>
<td>23</td>
<td>$\times$</td>
<td>$\times$</td>
<td></td>
</tr>
</tbody>
</table>

Complete results for sporadic groups in characteristic 0 are given in the next section.
7. Results in characteristic 0.

In this section we complete the answer in the case of characteristic 0 (or more generally, in the case of coprime characteristic). In order to achieve this, by the remark following the proof of Theorem 3.1, we have to consider those groups in the bad case not handled in Section 5. Here our proof relies very much on Lusztig’s classification of the (degrees of) irreducible characters of groups of Lie type, more precisely, on the Jordan decomposition of characters and the determination of the degrees of unipotent characters. We refer to [25] for a short survey of the phenomenology of the necessary results.

A. The classical groups in characteristic $\ell = 0$.

First we handle the unitary groups.

Proposition 7.1. Let $G$ be a cover of $U_n(q)$ with $n \geq 4$ and $V$ a nontrivial irreducible $\mathbb{C}G$-module. Then $X(V)$ is reducible unless $q = 2, 3$ and $V$ is a Weil module of $G$.

Proof. The results of §§3.5 allow us to assume that we are in the bad case and that $q = 2$. The cases $n \leq 6$ can be checked directly, hence we assume that $n \geq 7$, $V$ is a module for $G = SU_n(2)$, $V$ is not a Weil module, and $X(V)$ is irreducible.

Suppose that $V$ is extendible to $H := GU_n(2)$. But $V$ is self-dual, so $V^H$ is the sum of three irreducible $H$-modules, and at least one of them, which we denote by the same letter $V$, is self-dual. Consider a pseudoreflection $g$ of order 3 in $H$. Then the nontrivial eigenspaces of $g$ in $V$ are dual to each other as $\mathbb{C}$-modules, where $C = GU_1(2) \times GU_{n-1}(2)$. Thus $X(V)$ contains the trivial $C$-module, and so $\dim(X(V)) \leq (H:C) = (2^n - (-1)^n)2^{n-1}/3$, which implies by [25] that $V$ is a Weil module. Consequently, $V$ cannot be extended to $H$. In particular, we are done if $\gcd(n, 3) = 1$.

So we may assume that $n \geq 9$. The bound in Proposition 5.3 implies that $\dim(V) < 2^{n-1}(2^{n-1} + 1)/3$. Carefully following the proof of Theorem 4.1 of [25], one can show that $G$ has exactly 9 nontrivial irreducible modules satisfying this bound; namely three Weil modules, one of dimension $(2^n - (-1)^n)(2^{n-1} + 4(-1)^n)/9$, two of dimension $(2^n - (-1)^n)(2^{n-1} - 2(-1)^n)/9$, and three in the dimension $(2^n - (-1)^n)(2^{n-1} + (-1)^n)/9$. From Lusztig’s parametrization of irreducible characters of $G$ and $H$, it follows that all these modules extend to $H$, which completes the proof. \[\square\]

The irreducible $X(V)$ for complex Weil modules $V$ of $SU_n(2)$ and $SU_n(3)$ are determined in [20] and [14].

To handle the symplectic and orthogonal groups, we need the following observation, which follows from Lusztig’s classification of unipotent characters.

Lemma 7.2. Let $\chi$ be a complex irreducible unipotent character of a finite group of Lie type $G$ in characteristic $p$.
(i) Let $G$ be $\text{GL}_n(q)$ or $\text{GU}_n(q)$. Then the $p$-part $\chi(1)_p$ of $\chi(1)$ is a power of $q$, and this $p$-part is 1 if and only if $\chi$ is trivial.

(ii) Let $G = S_{2n}(q)$ and $p = 2$. Then either $\chi$ is trivial, or $\chi$ is labeled by one of the symbols

$$
\begin{pmatrix}
0 & 1 & n \\
- & 1 \\
0 & n
\end{pmatrix},
\begin{pmatrix}
0 & n \\
1
\end{pmatrix},
\begin{pmatrix}
0 & 1 \\
n
\end{pmatrix},
\begin{pmatrix}
1 & n \\
0
\end{pmatrix}
$$

and $\chi(1)_2 = q/2$, or $\chi(1)_2 \geq q$.

(iii) Let $G = \text{O}_{2n}(q)$ with $p = 2$ and $n \geq 4$. Then either $\chi$ is trivial, or $\chi$ is the smallest unipotent character of degree $(q^n - c)(q^{n-1} + c)/q^2 - 1)$, or $\chi(1)_2 \geq q^2/2$.

Proof. We refer to [25], for example, for explicit formulae giving the degree polynomials of unipotent characters. In Case (i), assume $\chi$ is labeled by the partition $(\alpha_1, \alpha_2, \ldots, \alpha_m)$ of $n$ (with $1 \leq \alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_m$) and let $\lambda_i = \alpha_i + i - 1 \geq i$. Then $\chi(1)_p$ is $q^d$, where

$$
d = \sum_{i=1}^{m-1} (m - i)\lambda_i - \sum_{i=2}^{m-1} \binom{i}{2} \geq \sum_{i=1}^{m-1} (m - i)i - \sum_{i=2}^{m-1} \binom{i}{2} = \binom{m}{2}.
$$

In particular, if $d = 0$ then $m = 1$ and $\chi$ is the trivial character.

In Cases (ii), (iii), unipotent characters $\chi$ are labeled by symbols

$$
\begin{pmatrix}
\lambda \\
\mu
\end{pmatrix} = \begin{pmatrix}
\lambda_1 & \lambda_2 & \ldots & \lambda_a \\
\mu_1 & \mu_2 & \ldots & \mu_b
\end{pmatrix}
$$

of rank $n$, where $0 \leq \lambda_1 < \lambda_2 < \ldots < \lambda_a$, $0 \leq \mu_1 < \mu_2 < \ldots < \mu_b$, $(\lambda_1, \mu_1) \neq (0, 0)$, with $a - b$ odd in Case (ii) and even in (iii). While estimating the 2-powers dividing $\chi(1)$, we will also occasionally change the rank of the symbols.

Assume $a, b \geq 1$ and $(a, b) \neq (1, 1)$. Then we can consider the unipotent character $\chi'$ corresponding to

$$
\begin{pmatrix}
\lambda_1 & \lambda_2 & \ldots & \lambda_{a-1} \\
\mu_1 & \mu_2 & \ldots & \mu_{b-1}
\end{pmatrix}.
$$

It follows from the explicit degree formulae that $\chi(1)_2/\chi'(1)_2$ is at least $q^d/2$, where

$$
d = \sum_{i=1}^{a-1} \lambda_i + \sum_{j=1}^{b-1} \mu_j + \sum_{j=1}^{b} \min(\lambda_a, \mu_j) + \sum_{i=1}^{a-1} \min(\lambda_i, \mu_b) - \binom{a + b - 2}{2},
$$

and $d$ will attain its smallest value when

$$
\begin{pmatrix}
\lambda \\
\mu
\end{pmatrix} = \begin{pmatrix}
0 & 1 & \ldots & a - 1 \\
1 & 2 & \ldots & b
\end{pmatrix}, \begin{pmatrix}
1 & 2 & \ldots & a \\
0 & 1 & \ldots & b - 1
\end{pmatrix},
$$

respectively.
for which \( d \) is exactly \( a + b - 2 \). It follows that \( \chi(1)_2 \geq q^{a+b-2}/2 \). In Case (ii) we get \( \chi(1)_2 \geq q/2 \). Moreover, if \( \chi(1)_2 = q/2 \) then \( a + b = 3 \), i.e., \( (a, b) = (2, 1) \). Direct calculation then shows that \( \chi \) is labeled by one of the symbols given in the statement.

In Case (iii) \( a + b \) is even, so \( a + b \geq 4 \) and \( \chi(1)_2 \geq q^2/2 \). If \( (a, b) = (1, 1) \), then \( G = O_{2n}^+(q) \). Here either \( \chi \) is trivial, or \( \begin{pmatrix} \lambda \\ \mu \end{pmatrix} = \begin{pmatrix} 1 \\ n-1 \end{pmatrix} \) and \( \chi(1) = (q^n - 1)(q^{n-1} + q)/(q^2 - 1) \), or \( \chi(1)_2 \geq q^2/2 \).

Suppose \( b = 0 \) and \( a \geq 3 \). Then we can consider the unipotent character \( \chi'' \) corresponding to \( \begin{pmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_{a-2} \\ -1 \end{pmatrix} \). One can show that \( \chi(1)_2/\chi''(1)_2 \) is at least \( q^{d''}/2 \), where

\[
d' = 2 \sum_{i=1}^{a-2} \lambda_i + \lambda_{a-1} - \binom{a-2}{2},
\]

and \( d' \) will attain its smallest value when \( \lambda = (0, 1, \ldots, a-2, \lambda_a) \), for which \( d \) is exactly \( (a-1)(a-2)/2 \). It follows that \( \chi(1)_2 \geq q^2/2 \). Moreover, if \( \chi(1)_2 = q/2 \) then \( a = 3 \), \( G = S_{2n}(q) \), and \( \lambda = (0, 1, n) \). In Case (iii) \( a \) is even, so \( a \geq 4 \) and \( \chi(1)_2 \geq q^3/2 \).

Finally, let \( b = 0 \) and \( a \leq 2 \). If \( a = 2 \), then \( G = O_{2n}^+(q) \), and either \( \chi \) is trivial, or \( \lambda = (1, n-1) \) and \( \chi(1) = (q^n + 1)(q^{n-1} - q)/(q^2 - 1) \), or \( \chi(1)_2 \geq q^2 \). If \( a = 1 \), then \( G = S_{2n}(q) \) and \( \chi \) is trivial. \( \square \)

Next we handle the bad case for symplectic groups \( G = S_{2n}(q) \), \( q \) even. Let \( W = F_{q}^{2n} \) be the natural module for \( G \), and we consider the permutation character \( \omega \) of \( G \) on \( W \). Then \( \omega(g) = q^\dim \text{Ker}(g-1) \) for any \( g \in G \). We will also consider the class function

\[
\widetilde{\omega}_n : g \mapsto (-q)^{\dim \text{Ker}(g-1)}.
\]

Using certain dual pairs in characteristic 2, it was shown in [24] that the permutation action of \( G \) on the 1-spaces of \( W \) affords the character \( 1_G + \rho_1 + \rho_2 \) where \( \rho_1 \) and \( \rho_2 \) are irreducible characters of degree \((q^n+1)(q^n-q)/2(q-1)\) and \((q^n-1)(q^n+q)/2(q-1)\) respectively, and that \( \widetilde{\omega}_n \) is actually the restriction of the (reducible) Weil character \( \sum_{i=0}^{q-1} \zeta_n^i \) of SU\(_{2n}(q)\) to \( G \), when \( G \) is naturally embedded in SU\(_{2n}(q)\). Moreover, \( \zeta_n^0|_G = \alpha_n + \beta_n \), where \( \alpha_n \) and \( \beta_n \) are irreducible characters of degree \((q^n+1)(q^n+q)/2(q+1)\) and \((q^n-1)(q^n-q)/2(q+1)\) respectively, and \( \zeta_n^i|_G = \zeta_n^{q^n-1-i}|_G = \gamma_n^i \) is an irreducible character of degree \((q^{2n}-1)/(q+1)\) when \( 1 \leq i \leq q/2 \). If \( q = 2 \), then we will use the notation \( \gamma_n \) instead of \( \gamma_n^1 \). If \( \xi \), resp. \( \delta \), is a primitive
On the other hand, by looking at the character degree and using the equality
\[ (G \leq 1) \]
Proposition 7.4. Let \( G = S_{2n}(2), \ n \geq 3 \). Then the characters \( \Lambda^2(\alpha_n), \Lambda^2(\beta_n), \Lambda^2(\gamma_n) \), and \( \alpha_n \beta_n \) are irreducible. Moreover, if \( n \geq 4 \) then all the other \( X(\chi) \) with \( \chi \in \{\alpha_n, \beta_n, \gamma_n\} \) and \( X \in \{A, \bar{A}, \bar{2}\} \), and \( \alpha_n \gamma_n, \beta_n \gamma_n \), are reducible.

Proof. The cases \( n = 3, 4 \) are easy to check, so we will assume \( n \geq 5 \). We begin with the obvious observation that \( \omega = \tilde{\omega} \). Next, \( \bar{G} = \omega - 1 \bar{G} = 1G + \rho_1 + \rho_2 \) is the permutation character of \( G \) on \( W^\times \). Hence, \( (\omega - 1G)^2, \omega - 1G \), resp. \( (\omega - 1G)^2, (\omega - 1G)^2 \) is the number of \( G \)-orbits on \( W^\times \times W^\times \times W^\times \times W^\times \), resp. on \( W^\times \times W^\times \times W^\times \times W^\times \), which is 17, resp. 179, as can be seen by direct counting. Thus
\[ (\tilde{\omega}^2 - 3 \cdot 1G - 2\rho_1 - 2\rho_2, \mu) = 17, \]
\[ (\tilde{\omega}^2 - 3 \cdot 1G - 2\rho_1 - 2\rho_2, \tilde{\omega}^2 - 3 \cdot 1G - 2\rho_1 - 2\rho_2) = 179. \]

It is known that \( \zeta_0^2 \) has Schur-Frobenius indicator 1, and so do \( \alpha_n \) and \( \beta_n \). We write
\[ \text{Sym}^2(\alpha_n) = 1G + \chi_1 + a_1 \rho_1 + b_1 \rho_2, \quad \Lambda^2(\alpha_n) = \chi_2 + a_2 \rho_1 + b_2 \rho_2, \]
\[ \text{Sym}^2(\beta_n) = 1G + \chi_3 + a_3 \rho_1 + b_3 \rho_2, \quad \Lambda^2(\beta_n) = \chi_4 + a_4 \rho_1 + b_4 \rho_2, \]
\[ \text{Sym}^2(\gamma_n) = c_1 \cdot 1G + \chi_5 + a_5 \rho_1 + b_5 \rho_2, \quad \Lambda^2(\gamma_n) = c_2 \cdot 1G + \chi_6 + a_6 \rho_1 + b_6 \rho_2, \]
\[ \alpha_n \beta_n = \chi_7 + a_7 \rho_1 + b_7 \rho_2, \quad \alpha_n \gamma_n = \chi_8 + a_8 \rho_1 + b_8 \rho_2, \quad \beta_n \gamma_n = \chi_9 + a_9 \rho_1 + b_9 \rho_2 \]
for some non-negative integers \( a_i, b_i, c_i \); furthermore, each \( \chi_i \) is either 0 or a \( G \)-character not involving \( 1G, \rho_1, \rho_2 \). Clearly \( c_1 + c_2 = 1 \). From the decomposition
\[ \tilde{\omega}^2 = \alpha_n^2 + \beta_n^2 + 4\gamma_n^2 + 2\alpha_n \beta_n + 4\alpha_n \gamma_n + 4\beta_n \gamma_n \]
and (7.5) it follows that \( a + b = 18 \), where \( a = \sum_{i=0}^{9} t_i \alpha_i, \ b = \sum_{i=0}^{9} t_i \beta_i, \) with \( t_i = 1 \) for \( 1 \leq i \leq 4, 2 \) for \( i = 7, \) and \( 4 \) otherwise. In this case \( (a - 2)^2 + (b - 2)^2 \geq 98. \) Together with (7.6), this implies that
\[ \left( \sum_{i=1}^{9} t_i \chi_i ; \sum_{i=1}^{9} t_i \chi_i \right)_{G} \leq 179 - 9 - 98 = 72 = \sum_{i=1}^{9} t_i^2. \]
On the other hand, by looking at the character degree and using the equality \( a + b = 18 \) we see that all \( \chi_i \) are nonzero. Thus (7.7) means that the \( \chi_i, 1 \leq i \leq 9, \) are distinct irreducible characters of \( G \).

Next we restrict various characters to the first parabolic subgroup \( P \) of \( G \). Recall that \( P = U.L, \) where \( U \) is elementary abelian of order \( 2^{2n-1} \) and \( L \simeq S_{2n-2}(2) \). We can define the characters \( \alpha_{n-1}, \beta_{n-1}, \gamma_{n-1} \) of \( L \) in a
similar manner. Also, $L$ acts on the set of linear characters of $U$ with four orbits $O_j$, $1 \leq j \leq 4$, see the proof of Prop. 5.5. Restricting to $L$ and using the explicit formula (7.3), we see that the restriction of $\alpha_n$, $\beta_n$, and $\gamma_n$ to $L$ involves only the characters $\alpha_{n-1}$, $\beta_{n-1}$, $\gamma_{n-1}$. Knowing this information and the length of each $O_j$, we can show that

\begin{equation}
\beta_n|_P = \beta' + \beta_{n-1},
\end{equation}

where $\beta_{n-1}$ is inflated from $L$ to $P$ and $\beta'|_U = \sum_{\lambda \in O_3} \lambda$, in particular, $\beta'$ is irreducible over $P$. Thus

\[ 2 = (\beta_n|_P, \beta_n|_P)_P = (\beta^2_n, 1^G_P)_G = (\beta^2_n, \mu)_G, \]

i.e., $a_3 + b_3 + a_4 + b_4 = 1$. By Theorem 3.1 $\text{Sym}^2(\beta_n) - 1_G$ is reducible. Consequently, $a_3 + b_3 = 1, a_4 = b_4 = 0$, i.e., $\Lambda^2(\beta_n)$ is irreducible.

Similarly,

\begin{equation}
\alpha_n|_P = \alpha' + \alpha_{n-1},
\end{equation}

where $\alpha'|_U = \sum_{\lambda \in O_4} \lambda$, in particular, $\alpha'$ is irreducible over $P$. Arguing as above, we see that $a_1 + b_1 = 1, a_2 = b_2 = 0$, i.e., $\text{Sym}^2(\alpha_n) - 1_G$ is reducible and $\Lambda^2(\alpha_n)$ is irreducible. Also, (7.8) and (7.9) imply that $4 = ((\alpha_n + \beta_n)|_P, (\alpha_n + \beta_n)|_P)_P = ((\alpha_n + \beta_n)^2, \mu)_G$. But $(\alpha^2_n, \mu)_G = (\beta^2_n, \mu)_G = 2$, hence $(\alpha_n + \beta_n, \mu)_G = 0$. Thus $a_7 = b_7 = 0$, i.e., $\alpha_n \beta_n$ is irreducible.

Now let $g$ be the central involution in $P$. Since $\gamma_n(g) = -(2^{2n-1} + 1)/3$, the $g$-fixed point subspace $V_+$ in the representation space $V$ for $\gamma_n$ has dimension equal to $\gamma_{n-1}(1)$ and less than $|O_j|$ for any $j > 1$. Again, every constituent of $\gamma_n|_L$ is either $\alpha_{n-1}$, $\beta_{n-1}$, or $\gamma_{n-1}$. Comparing the character degrees we see that $U$ acts trivially on $V_+$ and $V_+$ affords the $L$-character $\gamma_{n-1}$. Also, the $-1$-eigenspace for $g$ on $V$ has to afford the $P$-characters $\gamma'$ and $\gamma''$, where $\gamma'|_U = \sum_{\lambda \in O_4} \lambda$, and $\gamma''|_U = \sum_{\lambda \in O_4} \lambda$, in particular, $\gamma'$ and $\gamma''$ are distinct irreducible $P$-characters. Thus

\begin{equation}
\gamma_n|_P = \gamma' + \gamma'' + \gamma_{n-1}.
\end{equation}

Arguing inductively on $n$, we see that $\gamma_n$ has Schur-Frobenius indicator $1$, i.e., $c_1 = 1$ and $c_2 = 0$. From (7.10) it follows that $3 = (\gamma_n|_P, \gamma_n|_P)_P = (\gamma^2_n, \mu)_G$, and so $a_5 + b_5 + a_6 + b_6 = 2$. From (7.9) and (7.10) one obtains $1 \geq (\alpha_n|_P, \alpha_n|_P)_P = (\alpha_n \alpha_n, \mu)_G$, i.e., $a_8 + b_8 \leq 1$. Similarly, (7.8) and (7.10) imply that $a_9 + b_9 \leq 1$. But we know that $a + b = 18$, so in fact we have equality in all three previous inequalities. Thus $a_8 + b_8 = a_9 + b_9 = 1$ (which means $\alpha_n \gamma_n$ and $\beta_n \gamma_n$ are reducible), $\alpha' = \gamma''$ and $\beta' = \gamma'$. Since $a_2 = b_2 = 0$, we see that $\Lambda^2(\gamma'')|_P$ does not involve $1_P$. Similarly, $a_4 = b_4 = 0$ implies that $\Lambda^2(\gamma')|_P$ does not involve $1_P$. Recall that $\gamma_{n-1}$ is of type $+$ and $\gamma_{n-1}$, $\gamma'$, and $\gamma''$ are distinct irreducibles. Together with (7.10), this implies that $0 = (\Lambda^2(\gamma_n)|_P, 1_P)_P = (\Lambda^2(\gamma_n), \mu)_G$, i.e., $a_6 + b_6 = 0$ and $\Lambda^2(\gamma_n)$ is irreducible. Finally, we see that $a_5 + b_5 = 2$ and $\text{Sym}^2(\gamma_n)$ is reducible. □
Proposition 7.11. Let $G = S_{2n}(q)$, $q$ even, $n \geq 3$, $(n, q) \neq (3, 2)$, and $V$ a self-dual nontrivial irreducible $\mathbb{C}G$-module. Then $X(V)$ is reducible for $X \in \{\Lambda^2, \text{Sym}^2\}$ unless $q = 2$, $X = \Lambda^2$, and $V$ affords one of the characters $\alpha_n$, $\beta_n$, $\gamma_n$.

Proof. The case $G = S_8(2)$ can be checked from [1] so we may also assume $(n, q) \neq (4, 2)$. Let $G = S_{2n}(q)$ and $V$ a self-dual nontrivial irreducible $\mathbb{C}G$-module such that $X(V)$ is irreducible. By Proposition 5.5, $X = \Lambda^2$, $q \leq 8$, and $\dim(V) < (q^{2n} - 1)/(q - 1)$. Denote $d = \dim(V)$, $e = \dim(X(V)) = d(d - 1)/2$, and let $\rho$ and $\chi$ be the character of $V$ and $X(V)$, respectively.

1) Recall that we are assuming $n \geq 3$ if $q \geq 4$ and $n \geq 5$ if $q = 2$. Following the proof of Proposition 5.1 and Theorem 5.5 of [25], one can show that $G$ has exactly $4 + q/2$ nontrivial irreducible characters of degree $< (q^{2n} - 1)/(q - 1)$, namely $4$ unipotent characters $\alpha_n$, $\beta_n$, $\rho_1$, $\rho_2$, labeled by

$$
\begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}, \begin{pmatrix} 0 & 1 \\
1 & -1
\end{pmatrix}, \begin{pmatrix} 1 & n \\
0 & 1
\end{pmatrix}, \begin{pmatrix} 0 & n \\
1 & 1
\end{pmatrix}
$$

respectively, and $q/2$ semisimple characters $\gamma_n^i$, $1 \leq i \leq q/2$.

2) Let $q = 8$. By Proposition 5.5, $\dim(V) \leq (q^2 - q^2)/2(q - 1)$. According to 1), $\rho$ is either $\alpha_n$ or $\beta_n$. In particular, $d \in 8\mathbb{Z} + 4$, whence $e \in 4\mathbb{Z} + 2$. Suppose that $\chi$ corresponds to a semisimple class $(s)$ in the dual group $G^* \simeq G$ and a unipotent character $\psi$ of $C := C_{G^*}(s)$ in Lusztig’s Jordan decomposition of characters. One can show that $C$ is a direct product of subgroups of the form $\mathrm{Gl}_m(q^k)$, $\mathrm{Gu}_m(q^k)$, and $S_{2m}(q)$. By Lemma 7.2, if $\psi(1) > 1$ then $\psi(1)$ and so $e$ is divisible by $4$, a contradiction. Hence $\psi(1) = 1$ and so $e = (G^* : C)_2$ is odd, again a contradiction.

Henceforth we may assume that $q \leq 4$. By Zsigmondy’s Theorem [28], if $m \geq 3$ and $(q, m) \neq (2, 6)$, then $q^m - 1$ has a prime divisor which is coprime to $\prod_{i=1}^{m-1} (q^i - 1)$. We will denote such a prime by $\ell_{m,q}$.

3) Assume that $q = 4$ and $\rho$ is either $\rho_1$ or $\rho_2$. Then $d - 1$ is divisible by $\ell_{4n-2,2}$ or $\ell_{2n-1,2}$ respectively. None of these primes divides $|G|$, a contradiction. Thus $d = (4^{2n} - 1)/5, (4^n + 1)(4^n + 4)/10$ or $(4^n - 1)(4^n - 4)/10$. In particular, $e$ is odd. Using Lemma 7.2 and arguing as in 2), we see that $\chi$ is a semisimple character and $e = (G : C_G(s))_2$ for some semisimple element $1 \neq s \in G$. First suppose that $3 \leq n \leq 5$. The condition $e$ divides $|G|$ implies that $(n, d, e) = (3, 442, 97461)$ or $(4, 6426, 20643525)$. In neither case has $G$ a semisimple element $s$ such that $e = (G : C_G(s))_2$. Hence we may assume that $n \geq 6$. Then $e$ is coprime to $\ell_{2n-4,4}$.

In the case $q = 2$, the characters $\alpha_n$, $\beta_n$, and $\gamma_n$ have already been treated in Proposition 7.4, hence we may assume that $\rho$ is either $\rho_1$ or $\rho_2$. Since $e$ does not divide $|G|$ when $n = 5$, we must have $n \geq 6$, in which case $e$ is odd and coprime to $\ell_{2n-4,2}$. Now Lemma 7.2 implies that $\chi$ is not unipotent, and so $\chi$ corresponds to a semisimple element $s \neq 1$ in $G \simeq G^*$. 

Thus in either case we obtain a semisimple element $1 \neq s \in G$ such that the $2'$-part of the index of $C := C_G(s)$ in $G$ is coprime to $\ell_{2n-4,q}$ and smaller than $q^{4n-2}$. Recall that $C$ is a direct product of subgroups of the form $\text{GL}_m(q^k)$ or $\text{GU}_m(q^k)$ with $mk \leq n$, or $S_{2m}(q)$ with $m \leq n - 1$. Since $|C|$ is divisible by $\ell_{2n-4,q}$, it follows that $C$ has a subgroup $D = \text{GU}_m(q^k)$ with $mk \geq n - 2$ or $S_{2m}(q)$ with $m = n - 2, n - 1$. Also, we are assuming that $n \geq 6$.

Assume $D = \text{GU}_m(q^k)$. If $mk = n$ and $n \geq 7$ or $(n, k) \neq (6, 1)$ then $(G : C)_{2'} \geq q^{4n-2}$, a contradiction. If $m = n = 6$ then $(G : C)$ is divisible by $\ell_{2n-4,q}$, again a contradiction. If $mk = n - j$ with $j = 1, 2$, then $C \leq S_{2j}(q) \times \text{GU}_m(q^k)$, whence $(G : C)_{2'} \geq q^{4n-2}$, a contradiction.

Thus $D = S_{2n-2j}(q)$ with $j = 1$ or $2$. Therefore $C \leq S_{2j}(q) \times S_{2n-2j}(q)$. If $q = 4$, then, as shown above, $e = (G : C)_{2'}$, whence $(4^{2n} - 1)(4^{2n-2} - 1)$ is divisible by $e = (4^{2n} - 1)(4^{2n} - 6)/50$, $(4^{2n} - 1)(4^n + 4)(4^n + 6)/200$, or $(4^{2n} - 1)(4^n - 4)(4^n - 6)/200$, a contradiction. If $q = 2$, then $45 \cdot (G : C)_{2'}$ is divisible by $2^{2n} - 1$, which implies that $2^{2n} - 1$ divides $45e$, with $e = (2^n + 1)(2^{n-1} - 1)(2^{n-2}(2^n - 1) - 1)$ or $e = (2^n - 1)(2^{n-1} + 1)(2^n - 2)(2^n + 1) - 1$, again a contradiction. \hfill \qed

**Proposition 7.12.** Let $G = O^\pm_{2n}(q)$, $n \geq 4$, $q$ even, and $V$ a self-dual non-trivial irreducible $\mathbb{C}$-module. Then $X(V)$ is reducible for $X \in \{\Lambda^2, \text{Sym}^2\}$.

**Proof.** 1) The case $O^+_8(2)$ can be checked directly, hence we may assume that $(n, q) \neq (4, 2)$. Assume that $X(V)$ is irreducible. Denote $d = \dim(V)$, $e = \dim(X(V)) = d(d + 1)/2$, and let $\rho$ and $\chi$ be the character of $V$ and $X(V)$, respectively. By Proposition 5.8, $q = 2, 4$ and $d < q^{2n-2}$. Applying Propositions 7.1 and 7.2 of [25] in the case $\rho$ is unipotent, and following the proof of Theorem 7.6 of [25] in the non-unipotent case, we conclude that either $\rho$ is the smallest unipotent character $\rho_n$ of degree $(q^n - \epsilon)(q^{n-1} + \epsilon q)/(q^2 - 1)$, or it is one of $q/2$ semisimple characters $\vartheta_i^\rho$, $1 \leq i \leq q/2$, of degree $(q^n - \epsilon)(q^{n-1} - \epsilon)/(q + 1)$, or $G = O^+_8(4)$ and $\rho(1) = 3213$.

2) Suppose $q = 4$. First we consider the case $G = O^+_8(4)$. Then either $d = 1156$ or $d = 3213$, and in either case neither $d - 1$ nor $d + 1$ divides $|G|$, a contradiction. Hence we may assume that $(n, \epsilon) \neq (4, +)$. Under this assumption, the proof of Proposition 5.8 shows that the restriction of $\rho$ to the parabolic subgroup $P = U.L$ (in the notation of that proof) contains a linear character of the elementary abelian 2-group $U$ with multiplicity 1. Since this linear character obviously is of type $+$, $\rho$ itself is of type $+$, and so $X = \Lambda^2$, $e = d(d - 1)/2$. Since $\rho$ is either $\rho_n$ or $\vartheta_i^\rho$, we see that $e \in 4\mathbb{Z} + 2$.

Assume that under the Jordan decomposition $\chi$ corresponds to a semisimple class $(s)$ in $G^* \simeq G$ and a unipotent character $\psi$ of $C := C_G(s)$. One can show that $C$ is a direct product of subgroups of the form $\text{GL}_m(q^k)$, $\text{GU}_m(q^k)$ or $O^\pm_{2m}(q)$. Hence by Lemma 7.2 $\psi(1)$ is either 1 or divisible by 4. Thus $e = (G^* : C)_{2'} \psi(1)$ cannot belong to $4\mathbb{Z} + 2$, again a contradiction.
3) We are reduced to consider \( q = 2 \). The case \( G = \text{O}^{\pm}_m(2) \) can be checked directly. Therefore we are left with the case \( q = 2, n \geq 6 \), and \( \rho \) is either \( \rho_n \) or \( \vartheta_n := \vartheta_n^1 \). Recall that, in the notation of the proof of Proposition 5.8, the Levi subgroup \( L \) of \( P \) acts on the nontrivial linear characters of \( U \) with two orbits say \( O_l \) and \( O_a \) of lengths \((2^{n-1} - e)(2^{n-2} + e)\) and \( 2^{n-2}(2^{n-1} - e) \), and \( O_l \) does not occur on \( V \). Observe that \( d < 2|O_a| \) and also \( U \) cannot act trivially on \( V \). Thus each character from \( O_a \) occurs on \( V \) with multiplicity 1. It follows that \( \rho|_P = \lambda + \beta \), where \( \lambda \) is an irreducible \( P \)-character whose restriction to \( U \) is \( \sum_{\alpha \in O_a} \alpha \), and \( \beta \) is an \( L \)-character inflated to \( P \). Also, the type of \( \rho \) is +, whence \( X = \Lambda^2 \) and \( e = d(d - 1)/2 \).

First we suppose that \( \rho = \rho_n \). Then \( \beta \) is an \( L \)-character of degree \( (2^{2n-3} + 9e2^{n-2} - 2)/3 < 2^{2n-4} \) as \( n \geq 6 \). According to 1) applied to \( L = \text{O}^\pm_{2n-2}(2) \), \( \beta = a \cdot 1_L + b \cdot \rho_{n-1} + c \cdot \vartheta_{n-1} \) for some non-negative integers \( a, b, c \). Thus
\[
\frac{2^{2n-3} + 9e2^{n-2} - 2}{3} = a + b \cdot \frac{2^{2n-3} + 3e2^{n-2} - 2}{3} + c \cdot \frac{2^{2n-3} - 3e2^{n-2} + 1}{3}.
\]
Since \( a, b, c \) are non-negative integers, we come to the conclusion that \( a \geq 2^{n-1} \). Hence \( 2^{n-1} \leq (\rho|_P, 1_P) = (\rho, 1^n_P)_G \), and so \( \rho(1) \leq (G : P)/2^{n-1} \), a contradiction.

Finally, let \( \rho = \vartheta_n \). Since \( e \), and so \( |G| \), is divisible by \( (2^{2n-1} - 1)/3 \), we must have that \( \epsilon = -1 \) and \( n \geq 6 \) is even. Since \( e \) is odd, \( \chi \) cannot be unipotent by Lemma 7.2. Suppose that \( \chi \) corresponds to a semisimple class (s) in \( G \) and a unipotent character \( \psi \) of \( C := C_G(s) \). One can show that either \( C \leq \text{GU}(2) \times \text{O}^\pm_{2n-4}(2) \), or \( C \) is a direct product of subgroups of the forms \( \text{GL}_m(q^k) \), \( \text{GU}_m(q^k) \), \( \text{O}^\pm_{2m}(q) \). In the former case \( G : C \) is divisible by \( \ell_{n-1,2} \) if \( n \neq 7 \), and by 31 if \( n = 7 \), meanwhile \( e \) is not, a contradiction. In the latter case, the oddness of \( e \) and Lemma 7.2 imply that \( e = (G : C)_{2^e} \).

In the case \( n = 6 \), one can show directly that \( G = \text{O}^{\pm}_6(2) \) has no such \( C \) with \( e = (G : C)_{2^e} = 255255 \). Therefore we assume that \( n \geq 8 \) is even. In this case, \( e \) is not divisible by \( \ell_{2n-6,2} \), whence \( C \) must contain a subgroup \( D \) of the form \( \text{GU}_m(q^k) \) with \( mk \geq n - 3 \) or \( \text{O}^\pm_{2m}(q) \) with \( n - 3 \leq m - 1 \). Assume \( D = \text{GU}_m(q^k) \). If \( mk = n \), then \( |C| \) is divisible by \( 2^n \pm 1 \), and so \( (G : C)_{2^e} \) is not divisible by \( \ell_{2n,2} \) or \( \ell_{n,2} \), while \( e \) is divisible by that prime, a contradiction. If \( mk = n - 1 \), then \( e \), but not \( (G : C)_{2^e} \), is divisible by \( \ell_{2n-2,2} \). If \( mk = n - 2 \) or \( n - 3 \), then \( (G : C)_{2^e} \), but not \( e \), is divisible by \( \ell_{n-1,2} \). Assume \( D = \text{O}^\pm_{2m}(q) \). If \( m = n - 3 \) or \( n - 2 \), then \( (G : C)_{2^e} \), but not \( e \), is divisible by \( \ell_{n-1,2} \). If \( m = n - 1 \), then \( e \), but not \( (G : C)_{2^e} \), is divisible by \( \ell_{2n-4,2} \). \( \square \)

7B. The exceptional groups in characteristic \( \ell = 0 \).

For the exceptional groups we can obtain a complete answer in characteristic 0 thanks to the tables of low-dimensional irreducible representations compiled by Frank Lübeck [17]. These in turn again rely on Lusztig’s classification of irreducible characters of finite reductive groups.
Proposition 7.13. Let $G$ be a quasi-simple exceptional group of Lie type and $V$ a nontrivial irreducible $\mathbb{C}G$-module. Then $X(V)$ is reducible unless possibly if $(G, \dim(V), X)$ are as in Table 7.15.

Proof. By Theorem 3.1 and the remark after its proof we may assume that $q$ is even and we are in the bad case. The cases $^2B_2(q^2)$ and $G_2(q)$ were treated in 5A. Now assume that $G = ^3D_4(q)$. By [17] the smallest degree of a nontrivial complex irreducible representation of $G$ is $q(q^4 - q^2 + 1)$, while the next largest is $d_2 = q^3(q - 1)^2(q^4 - q^2 + 1)/2$ (since $q$ is even). Using [1] we may moreover assume that $q > 2$. Then $d_2(d_2 - 1)/2$ is already larger than the largest degree of an irreducible complex character of $G$, hence only the smallest nontrivial character of $G$ remains. But $\dim(X(V))$ involves a factor $q^4 + q^3 + 1$ which does not divide the order of $G$.

For $G = ^2F_4(q^2)$ or $G = F_4(q)$ the cases $q^2 = 2$ respectively $q = 2$ can be dealt with by [1]. Otherwise Proposition 5.15 gives $\dim(V) \leq 2q^{11}(q + 1)$. By the tables in [17] this forces $\dim(V) = \sqrt{2}(q^3 - 1)(q^6 + 1)/2$ respectively

$$\dim(V) \in \left\{ \frac{1}{2}q(q^3 - 1)^2(q^4 + 1), \frac{1}{2}q(q^6 + 1)(q^6 + 1), \frac{1}{2}q(q^3 + 1)^2(q^4 + 1) \right\}.$$

In the first case the precise power of 2 dividing $\dim(X(V))$ is $\sqrt{2}q/4$, but $^2F_4(q^2)$ does not have an irreducible character with this property unless $q^2 = 8$. For $q^2 = 8$ it is readily checked that $\dim(X(V))$ does not divide $|^2F_4(8)|$. In the second case, the precise power of $q$ dividing $\dim(X(V))$ is $q/4$. If $q > 4$ then $F_4(q)$ does not have such a character. For $q = 4$ the dimension $\dim(X(V))$ does not divide $|G|$.

Now assume that $G = ^2E_6(q)$ or $G = E_6(q)$ with $n = 6, 7, 8$. We may treat the covering groups of $^2E_6(2)$ as follows: The faithful characters of coverings with center of order divisible by 3 cannot be self-dual. The remaining characters are printed in [1]. By Proposition 5.14 and [17] we are reduced to the case where the character $\chi$ of $V$ is the smallest nontrivial one of $G$, that is, $\chi = \phi_{2,4}, \phi_{6,1}, \phi_{7,1}, \phi_{8,1}$ respectively, or $q = 2$.

In the first case, $\dim(X(V))$ does not divide $|G|$. For $G = E_6(2)$ the tables in [17] show that only $\dim(V) \in \{2482, 137020, 443548\}$ are below $\sqrt{2\sqrt{|G|}}$, and for any of them $\dim(X(V))$ does not divide $|G|$. Similarly, for $G = E_7(2)$ only the first five characters, of degrees 141986, 86507701, 95420052, 181785768, 2422215628 can occur, but for none of these $\dim(X(V))$ divides $|G|$. For $G = E_8(2)$ only the first seven characters, of degrees 545925250, 76321227908420, 46453389380074796, 51320060161363500, 97697128859455125, 144074197011621500, 148940867792910204 can occur, but for none of these $\dim(X(V))$ is the degree of an irreducible character of $G$. (The authors are thankful to Lübeck for kindly providing them with the table of character degrees for $E_8(2)$.)

□

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Thus, together with the results from [18] we have completed the case \( \ell = 0 \) (see also [20] for the case \( X = \bar{A} \)):

**Theorem 7.14.** Let \( G \) be a quasi-simple group and let \( S := G/Z(G) \). Suppose that \( G \) has a nontrivial irreducible \( \mathbb{C}G \)-module \( V \) such that \( X(V) \) is irreducible, where \( X = \bar{A} \) if \( V \) is not self-dual and \( X = \bar{A}^2 \) or \( \bar{A}^2 \) otherwise. Then one of the following holds.

(i) \( S = \mathfrak{A}_n \) and \( V \) is the heart of the natural permutation module.
(ii) \( S = S_{2n}(q) \), \( q = 3, 5, 9 \), \( V \) is a Weil module of \( \text{Sp}_{2n}(q) \) of degree \( (q^n \pm 1)/2 \).
(iii) \( S = S_{2n}(2) \), \( X = \Lambda^2 \), and the character of \( V \) is one of the unipotent characters \( \alpha_n, \beta_n \) labeled by \( \begin{pmatrix} 0 & 1 & n \\ - & 0 & 1 \end{pmatrix} \), \( \begin{pmatrix} 0 & 1 \\ n \end{pmatrix} \), of degree \( (2^n + \epsilon)(2^n - 1)/3 \) with \( \epsilon = \pm 1 \), or a (unique) semisimple character \( \gamma_n \) of degree \( (2^{2n} - 1)/3 \).
(iv) \( S = U_n(q) \), \( q = 2, 3 \), \( V \) is a Weil module of \( \text{SU}_n(q) \) of degree \( (q^n + q(-1)^n)/(q + 1) \), \( (q^n - (-1)^n)/(q + 1) \).
(v) “Small groups”: \( (G, \dim(V), X) \) is as in Table 7.15.

**7C. A question of Gross.**

B.H. Gross asked the question which finite subgroups \( G \) of complex simple simply-connected Lie groups \( \mathfrak{G} \) have the property that they act irreducibly in all fundamental representations of \( \mathfrak{G} \). A well-known family of examples is provided by the finite irreducible complex reflection groups: All exterior powers of their reflection representations remain irreducible, thus they give examples where \( \mathfrak{G} = \text{SL}_n(\mathbb{C}) \) (see for example \( \mathfrak{S}_{n+1} < \text{SL}_n(\mathbb{C}) \)).

Clearly, if \( G \) has the above mentioned property, then so has the product of \( G \) with any subgroup \( Z \) of the centre of \( \mathfrak{G} \). We adopt the following notation:

If \( G < \mathfrak{G} = \text{Spin}_d(\mathbb{C}) \) then we write \( \overline{G} \) for the image of \( G \) in \( \text{SO}_d(\mathbb{C}) \), and otherwise set \( \overline{G} = G \).

We start our investigation by reducing the general case of Gross’ question to the monomial and the almost quasi-simple case, which will then be treated subsequently.

**Theorem 7.16.** Let \( G \) be a finite subgroup of the simple simply-connected complex Lie group \( \mathfrak{G} \) which is irreducible in all fundamental representations of \( \mathfrak{G} \). Assume that the dimension \( d \) of the natural module \( V \) for \( \mathfrak{G} \) is at least 5. Then up to a finite subgroup of \( Z(\mathfrak{G}) \) one of the following holds.

(i) \( \overline{G} \) is an irreducible monomial group in \( \text{GL}(V) \).
(ii) \( G = 2^3 \cdot \text{SL}_3(2) \) and \( \mathfrak{G} = G_2(\mathbb{C}) \).
(iii) \( G \leq 5_1^{1+2} : \text{SL}_2(5) \) and \( \mathfrak{G} = \text{SL}_5(\mathbb{C}) \).
Table 7.15. Non-generic examples for $\ell = 0$.

<table>
<thead>
<tr>
<th>$G$</th>
<th>$\dim(V)$</th>
<th>$A^2$</th>
<th>$\Xi^2$</th>
<th>$A$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.$A_6$</td>
<td>3</td>
<td>$\times$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2.$A_7$</td>
<td>4</td>
<td>$\times$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>6.$A_7$</td>
<td>6</td>
<td>$\times$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2.$A_8$</td>
<td>8</td>
<td>$\times$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2.$A_9$</td>
<td>8</td>
<td>$\times$</td>
<td>$\times$</td>
<td></td>
</tr>
<tr>
<td>L$_2$(7)</td>
<td>3</td>
<td>$\times$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>6.L$_3$(4)</td>
<td>6</td>
<td>$\times$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4$_1$.L$_3$(4)</td>
<td>8</td>
<td>$\times$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>U$_3$(4)</td>
<td>12</td>
<td>$\times$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>6$_1$.U$_4$(3)</td>
<td>6</td>
<td>$\times$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2.U$_6$(2)</td>
<td>56</td>
<td>$\times$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>S$_4$(4)</td>
<td>18</td>
<td>$\times$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>S$_6$(2)</td>
<td>7</td>
<td>$\times$</td>
<td>$\times$</td>
<td></td>
</tr>
<tr>
<td>2.S$_6$(2)</td>
<td>8</td>
<td>$\times$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2.O$_7^+$ (2)</td>
<td>8</td>
<td>$\times$</td>
<td>$\times$</td>
<td></td>
</tr>
<tr>
<td>G$_2$(3)</td>
<td>14</td>
<td>$\times$</td>
<td>$\times$</td>
<td></td>
</tr>
<tr>
<td>2.G$_2$(4)</td>
<td>12</td>
<td>$\times$</td>
<td>$\times$</td>
<td></td>
</tr>
<tr>
<td>2$^{F_4(2)'}$</td>
<td>26</td>
<td>$\times$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2.F$_4(2)$</td>
<td>52</td>
<td>$\times$</td>
<td>$\times$</td>
<td></td>
</tr>
<tr>
<td>2.2$^E_6(2)$</td>
<td>2432</td>
<td>$\times$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>M$_{11}$</td>
<td>10</td>
<td>$\times$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>M$_{11}$</td>
<td>11</td>
<td>$\times$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2.M$_{12}$</td>
<td>10</td>
<td>$\times$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>M$_{12}$</td>
<td>11</td>
<td>$\times$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

(iv) $G \leq 2^{1+6+6} \cdot S_8$ and $\mathfrak{g} = \text{Spin}_8(\mathbb{C})$.

(v) $G$ is almost quasi-simple.

Proof. 1) First we assume that $G$ is an exceptional group. Suppose $G$ is not Lie primitive (in the sense of [4]). Then there is a proper closed subgroup $H$ of $G$ which contains $G$, whence $H$ and $G$ are not irreducible on the adjoint fundamental representation. So $G$ is Lie primitive. By Theorem 1.7 of [4], either we are in Case (v) or (the image of) $G$ in $\mathfrak{g}_{ad}$ is contained in the
Table 7.15. Non-generic examples for $\ell = 0$ (continued).

<table>
<thead>
<tr>
<th>$G$</th>
<th>$\text{dim}(V)$</th>
<th>$\Lambda^2$</th>
<th>$\Sigma^2$</th>
<th>$A$</th>
</tr>
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<tbody>
<tr>
<td>2.M$_{12}$</td>
<td>12</td>
<td>$\times$</td>
<td></td>
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</tr>
<tr>
<td>2.M$_{22}$</td>
<td>10</td>
<td></td>
<td>$\times$</td>
<td></td>
</tr>
<tr>
<td>M$_{22}$</td>
<td>21</td>
<td>$\times$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2.J$_2$</td>
<td>6</td>
<td>$\times$</td>
<td>$\times$</td>
<td></td>
</tr>
<tr>
<td>2.J$_2$</td>
<td>14</td>
<td>$\times$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>M$_{23}$</td>
<td>22</td>
<td>$\times$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>M$_{23}$</td>
<td>45</td>
<td>$\times$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>HS</td>
<td>22</td>
<td>$\times$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3.J$_3$</td>
<td>18</td>
<td>$\times$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>M$_{24}$</td>
<td>23</td>
<td>$\times$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>M$_{24}$</td>
<td>45</td>
<td>$\times$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>McL</td>
<td>22</td>
<td>$\times$</td>
<td>$\times$</td>
<td></td>
</tr>
<tr>
<td>2.Ru</td>
<td>28</td>
<td>$\times$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>6.Suz</td>
<td>12</td>
<td>$\times$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Co$_3$</td>
<td>23</td>
<td>$\times$</td>
<td>$\times$</td>
<td></td>
</tr>
<tr>
<td>Co$_3$</td>
<td>253</td>
<td>$\times$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Co$_2$</td>
<td>23</td>
<td>$\times$</td>
<td>$\times$</td>
<td></td>
</tr>
<tr>
<td>3.ON</td>
<td>342</td>
<td>$\times$</td>
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<td></td>
</tr>
<tr>
<td>Fi$_{22}$</td>
<td>78</td>
<td>$\times$</td>
<td>$\times$</td>
<td></td>
</tr>
<tr>
<td>HN</td>
<td>133</td>
<td>$\times$</td>
<td>$\times$</td>
<td></td>
</tr>
<tr>
<td>Th</td>
<td>248</td>
<td>$\times$</td>
<td>$\times$</td>
<td></td>
</tr>
<tr>
<td>2.Co$_1$</td>
<td>24</td>
<td>$\times$</td>
<td>$\times$</td>
<td></td>
</tr>
<tr>
<td>$J_4$</td>
<td>1333</td>
<td></td>
<td></td>
<td>$\times$</td>
</tr>
<tr>
<td>B</td>
<td>4371</td>
<td>$\times$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

normalizer $N = N_{\mathfrak{g}}(J)$ of a so-called Jordan subgroup $J$. If $\mathfrak{g} = G_2(\mathbb{C})$, then $J = 2^3$ and $N = 2^3 \cdot \text{SL}_3(2)$. One can check that $N$ is irreducible on both fundamental representations of $\mathfrak{g}$, and no proper subgroup of $N$ has this property, so $G = N$. In the other cases, $|N|$ is not divisible by 7, 5, 19, respectively, whence $N$ and $G$ cannot act irreducibly on the fundamental representation of degree 273, resp. 2925, 147250, of $\mathfrak{g} = F_4(\mathbb{C})$, resp. $E_6(\mathbb{C})$, $E_8(\mathbb{C})$. 
From now on we assume that \( \mathfrak{G} \) is classical. We apply the main result of [15] to \( \mathcal{G} \) and see that we are in case (v) or \( \mathcal{G} \) is in one of the families \( \mathcal{C}_j \) with \( 1 \leq j \leq 6 \) defined in [15].

Suppose that \( \mathcal{G} \) is in \( \mathcal{C}_6 \). Then \( \mathfrak{G} = \text{SL}(V), \mathcal{G} = G \leq N_{\mathfrak{G}}(\mathcal{H}) = Z \ast \mathcal{H}, \) where \( Z = Z(\mathfrak{G}) \) and \( \mathcal{H} = \text{Sp}(V) \) or \( \text{SO}(V) \). Set \( H = ZG \cap \mathcal{H} \). Clearly \( ZG = Z \ast H \), and the finite subgroups \( ZG \) and \( H \) also act irreducibly on all fundamental representations of \( \mathfrak{G} \). At this point we may apply [15] again to the subgroup \( H \) of \( \mathcal{H} \). Thus we may assume that \( \mathcal{G} \in \mathcal{C}_j \) with \( j \leq 5 \). Moreover, \( j > 1 \) since \( G \) is irreducible on the natural module \( V \) of \( \mathfrak{G} \).

2) Assume \( \mathcal{G} \) is in \( \mathcal{C}_5 \), that is, \( \mathcal{G} \leq N := N_{\mathfrak{G}}(E) \) where either \( E \) is an extraspecial \( r \)-group of order \( r^{2m+1} \), or \( Z^2 \ast 2^m+2m \). In particular, \( d = r^m \), resp. \( 2^m \).

Suppose \( d = 5 \), so \( r = 5 \). Then \( \mathfrak{G} = \text{SL}_5(\mathbb{C}) \). Clearly \( N \) acts irreducibly on \( V \) and \( \Lambda^4(V) \). We claim that \( N \) is also irreducible on \( \Lambda^2(V) \) and so on \( \Lambda^3(V) \) as well. Assume \( N \) is reducible on \( \Lambda^2(V) \). Then the character \( \chi \) of \( N \) on this module is a sum of two faithful irreducible characters of degree 5. Each of them restricted to \( \text{SL}_5(5) \) is a reducible Weil character of degree 5 and so it takes value 1 on the central involution \( z \) of \( \text{SL}_2(5) \). Thus \( \chi(z) = 2 \). On the other hand, \( z \) acts on \( \Lambda^2(V) \) with trace \(-2\), a contradiction. Consequently, \( N \) is irreducible on all fundamental representations of \( \mathfrak{G} \). Now \( G \) can be any subgroup of \( N \) of the form \( E : H \), where \( H \) is a subgroup of \( \text{SL}_2(5) \) which acts irreducibly on a 2-dimensional complex representation of \( \text{SL}_2(5) \).

Suppose \( d = 7 \). Then \( \mathfrak{G} = \text{SL}_7(\mathbb{C}), G \leq 7^{1+2} : \text{SL}_2(7), \) and 5 does not divide \( |G| \), so \( G \) is not irreducible on \( \Lambda^3(V) \).

Suppose \( d = 8 \). Then \( \mathfrak{G} = \text{Sp}_8(\mathbb{C}), \text{SL}_8(\mathbb{C}) \) or \( \text{Spin}_8(\mathbb{C}) \). In the first case, the order of \( N = 2^{1+6} \cdot O^-_6(2) \) is not divisible by 7, so \( N \) is reducible on the fourth fundamental representation of \( \text{Sp}_8(\mathbb{C}) \). In the second case, the character of \( E \) on \( \Lambda^4(V) \) is a sum of some linear characters of \( E \), at least two of which are distinct. Since \( N \) is irreducible on \( \Lambda^4(V) \), this sum is \( 70/s \) times the sum of \( s \geq 2 \) distinct linear characters of \( E \). On the other hand, \( N/E \cong \text{Sp}_6(2) \) does not have a subgroup of such an index \( s \), a contradiction. So we are in the third case: \( N = 2^{1+6} \cdot \mathfrak{S}_8 \). Clearly, \( N \) acts irreducibly on \( V \). \( N \) is also irreducible on \( \Lambda^2(V) \), since \( N \) acts transitively on 28 linear characters of \( E \) occurring on \( \Lambda^2(V) \). The third and the fourth fundamental representations of \( \text{Spin}_8(\mathbb{C}) \) have kernel \( Z_2 \) and dimension 8.
So $E$ acts nontrivially on them. If $Z(E)$ is nontrivial on any of them, then $E$ is irreducible on that representation. On the other hand, if $Z(E)$ acts trivially on one of them, then $N$ is irreducible on it since $N$ is transitive on the linear characters of $E$ occurring on it. Thus $N$ is indeed irreducible on all fundamental representations. A similar argument shows that $G$ has this property if $G = E \cdot H$ where $H$ is any 2-transitive subgroup of $S_8$.

Suppose $d \geq 9$. If $d \neq 11$ then by Lemma 7.19 there is a prime $p$ such that $\lceil d/2 \rceil + 3 \leq p \leq d - 2$ and $p$ divides the degree of a fundamental representation of $G$, see the proof of Proposition 7.21. If $d = 11$ then $G = SL_{11}(C)$, and 7 divides the degree of the fifth fundamental representation of $G$, so we may choose $p = 7$. By assumption $p$ divides $|G|$, and clearly $p \neq r$. So $p$ divides the order of $G/E \leq Sp_{2m}(r)$. Such a prime is either $\leq r^{m-1} + 1 \leq d/2 + 1$, or $\leq (r^m + 1)/2 = (d + 1)/2$, or equal to $r^m \pm 1 = d \pm 1$. Any of these (in)equalities contradicts the choice of $p$.

4) Let $G$ be in $C_4$ (i), i.e., $G$ preserves a tensor product decomposition $V = V_1 \otimes V_2$, with $\dim(V_i) = e_i > 1$ and $e_2 \geq e_1$.

First assume that $d > 6$ and $d \neq 9$. Then $1 < e_1 \leq d/2 - 2$. If $G$ is not of type $C$, then the fixed point subspace $F$ for $SL(V_1)$ on the fundamental representation $\Lambda^{e_1}(V)$ is clearly a nonzero proper $GL(V_1) \otimes GL(V_2)$-submodule, a contradiction. If $G$ is of type $C$, then we may assume $G \leq O(V_1) \otimes Sp(V_2)$.

Fix a nonzero singular vector $e \in V_2$. Then $V' = V_1 \otimes e$ is totally singular, whence $0 \neq U := \Lambda^{e_1}(V')$ is contained in the kernel of the contraction map $\Lambda^{e_1}(V) \to \Lambda^{e_1-2}(V)$. Thus $U$ is contained in the $e_{1}$th fundamental representation of $G$, and clearly $U$ is fixed by $SO(V_1)$ pointwise. Now we can repeat the above argument with $SO(V_1)$ instead of $SL(V_1)$.

Observe that this argument also works if $d = 9$, or if $d = 6$ and $G \neq SP_{16}(C)$, as $\Lambda^{e_1}(V)$ is a fundamental representation for $G$. The case $SP_{16}(C)$ can be viewed as $SL_4(C)$, in which case we apply the same argument to $\Lambda^2(V)$.

5) Finally, let $G$ be in $C_4$ (ii), i.e., $G$ preserves a tensor power decomposition $V = U_1 \otimes U_2 \otimes \ldots \otimes U_m$, with $U_i \simeq U$ and $\dim(U) = e > 1$. Fix a Borel subgroup $B_1$ in $GL(U_1)$ and a nonzero singular $B_1$-invariant subspace $\langle v \rangle_C$. Set $V' = v \otimes U_2 \otimes \ldots \otimes U_m$. Then $V'$ is totally singular, so as in 4), $T := \Lambda^s(V')$ is contained in the $s$th fundamental representation $W$ of $G$, where $s = e^{m-1}$. Also, $T$ is one-dimensional and invariant under $B := B_1 \otimes GL(U_2) \otimes \ldots \otimes GL(U_m)$. Thus $T$ is a highest weight subspace for $B$ (affording the highest weight $sw_1$ for $B_1$), so $W$ contains a nonzero $H$-submodule say $W_1$ of dimension at most $e^s$, where $H = GL(U_1) \otimes \ldots \otimes GL(U_m)$. Repeating the same argument for $i$ instead of 1 we get a nonzero $H$-submodule $W_i$, $i = 1, \ldots , m$. Observe that $G_m$ permutes the $W_i$'s, so $W' := \sum_i W_i$ is a $G$-submodule of dimension at most $me^s$, as $G \leq H : S_m$. Now if $d = e^m \neq 8$ then $me^s < \dim(W)$, a contradiction. If $d = 8$ then $m = 3$, $e = 2$, and instead of $\dim(W_i) \leq 16$ we have
the better bound \( \dim(W_i) \leq 11 \) (as \( W_1 \) is a \( \text{GL}(U_1) \)-submodule of \( U_1^{\otimes 4} \) and \( \dim(\text{Sym}^4(U_1)) = 5 \)), whence \( \dim(W') \leq 33 < 42 \leq \dim(W) \) as well. \( \square \)

One can say more about the monomial case in this theorem:

**Lemma 7.17.** In Case (i) of Theorem 7.16 we have \( G = E \cdot H \), where \( E \) is a normal abelian homocyclic subgroup and either \( A_d \leq H \leq \mathfrak{S}_d \), or \( d \leq 11 \) and \((\mathfrak{S}, H)\) are as in Table 7.18, where moreover \( |E| = 2^s \) with \( d - 2 \leq s \leq d \) when \( \mathfrak{S} = \text{Spin}_d(\mathbb{C}) \).

**Proof.** By the proof of Thm. 7.16 we have that \( \mathfrak{S} \) is a classical group on \( V \), and \( G \) permutes a basis of \( V \) and is of the form \( E \cdot H \) with \( E \) a normal abelian subgroup and \( H \leq \mathfrak{S}_d \).

The set of weight spaces of \( E \) is a \( G \)-invariant decomposition of \( V \), so as in part 2) of the proof of Thm. 7.16, each one is one dimensional (else we violate irreducibility), and \( H \) acts transitively on the \( E \)-weight spaces of \( V \). By transitivity of \( H \) either all weight spaces are singular or non-singular. In the latter case \( \mathfrak{S} \) is \( \text{Spin} \) and \( E \) is an elementary abelian 2-group and the weight spaces form an orthonormal basis of \( V \).

Now we assume that all \( E \)-weight spaces \( V_{\chi_i} \) of \( V \) are singular and one dimensional. If \( f \) is the \( \mathfrak{S} \)-invariant bilinear form and \( v_i \in V_{\chi_i} \), then for all \( g \) in \( E \)

\[
f(v_i, v_j) = f(gv_i, gv_j) = \chi_i(g)\chi_j(g)f(v_i, v_j).
\]

So \( f(v_i, v_j) = 0 \) unless \( \chi_i(g) = \chi_j(g)^{-1} \). As \( f \) is nondegenerate there exists for every weight space \( V_{\chi_i} \) at least one weight space \( V_{\chi_j} \) outside of the orthogonal complement of \( V_{\chi_i} \). By the computation above \( V_{\chi_j} = V_{-\chi_i} \).

Thus the set of \( E \)-weight spaces is a \( G \) invariant hyperbolic basis of \( V \). As \( \mathfrak{S} \) is transitive on hyperbolic bases, and the normalizer of a hyperbolic basis is the normalizer of a split torus, we get the embedding of \( G \) into \( N_\mathfrak{S}(T) \).

But in this case we observe that when \( \mathfrak{S} \) is not a linear group, then the zero weight space of the heart of \( \Lambda^2(V) \) is a proper nontrivial \( N_\mathfrak{S}(T) \) submodule, by [11], Ex. 13.13. But we assumed that \( G \) is conjugate to a subgroup of \( N_\mathfrak{S}(T) \), so \( \mathfrak{S} \) must be linear in the second case.

Since \( \Lambda^2(V) \) is a fundamental representation for \( \mathfrak{S} \), it is irreducible, so \( H \) is 2-transitive. Note that if \( \Lambda^k(V) \) is irreducible then \( H \) must be a \( k \)-homogeneous subgroup of \( \mathfrak{S}_d \). We first deal with the linear case. Since \( \Lambda^r(V) \) is a fundamental representation of \( \text{SL}_d(\mathbb{C}) \) for any \( r \leq d - 2 \), the group \( H \) must act \( r \)-homogeneously for those \( r \). According to [16], a 6-homogeneous group is 6-transitive if \( d \geq 12 \), hence contains the alternating group. We conclude that in the linear case either \( H \geq \mathfrak{A}_d \) or \( d \leq 11 \). Among the transitive groups of degree \( 5 \leq d \leq 11 \) not containing \( \mathfrak{A}_d \) only \( H = 5:4 \) with \( d = 5 \), \( H = \text{PGL}_2(5) \) with \( d = 6 \), and \( H = \text{L}_2(8), H = \text{L}_2(8):3 \) with \( d = 9 \) are \( r \)-homogeneous for all \( r \leq d - 2 \). These give rise to examples.
Now assume that $G = \text{Spin}_d(\mathbb{C})$. Here $\Lambda_r$ is irreducible for $r \leq \lfloor (d-3)/2 \rfloor$, thus $H$ is $\lfloor (d-3)/2 \rfloor$-homogeneous. Arguing as before we conclude that either $d \leq 14$ or $H \geq \mathfrak{A}_d$. Among the transitive groups of degree $d \leq 14$ only the following ones are $r$-homogeneous for all $r \leq \lfloor (d-3)/2 \rfloor$:

- $d = 7$ : $7:3$, $7:6$, $L_3(2)$
- $d = 8$ : $2^3:7$, $2^3:7:3$, $L_2(7)$, $\text{PGL}_2(7)$, $2^3:L_3(2)$
- $d = 9$ : $L_2(8)$, $L_2(8):3$
- $d = 10$ : $\text{PGL}_2(9)$, $M_{10}$, $\text{Aut}(\mathfrak{A}_6)$
- $d = 11$ : $M_{11}$
- $d = 12$ : $M_{12}$.

Now note that, whenever $|E| \geq 2^{d-1}$ if $d$ is odd, respectively $|E| \geq 2^{d-2}$ if $d$ is even, then $E$ lifts to an extraspecial group $E$ in $\text{Spin}_d(\mathbb{C})$. This $E$ has a faithful representation of degree equal to the degree of the spin representation, which moreover lifts to the extension $EH$. Hence, for any of the groups $H$ above, whenever $|E|$ is large enough we get an example. □

Note that the extensions $G = E \cdot H$ in Lemma 7.17 need not necessarily split, as is shown by the example of Weyl groups in $\text{SL}_n(\mathbb{C})$.

**Table 7.18.** Monomial subgroups $G = E.H$ of complex simple Lie groups $G$ irreducible in all fundamental representations.

<table>
<thead>
<tr>
<th>$G$</th>
<th>$H$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\text{SL}_5$</td>
<td>$5:4$</td>
</tr>
<tr>
<td>$\text{SL}_6$</td>
<td>$\text{PGL}_2(5)$</td>
</tr>
<tr>
<td>$\text{SL}_9$</td>
<td>$L_2(8)$, $L_2(8):3$</td>
</tr>
<tr>
<td>$\text{Spin}_7$</td>
<td>$7:3$, $7:6$, $L_3(2)$</td>
</tr>
<tr>
<td>$\text{Spin}_8$</td>
<td>$2^3:7$, $2^3:7:3$, $L_2(7)$, $\text{PGL}_2(7)$, $2^3:L_3(2)$</td>
</tr>
<tr>
<td>$\text{Spin}_9$</td>
<td>$L_2(8)$, $L_2(8):3$</td>
</tr>
<tr>
<td>$\text{Spin}_{10}$</td>
<td>$\text{PGL}<em>2(9)$, $M</em>{10}$, $\text{Aut}(\mathfrak{A}_6)$</td>
</tr>
<tr>
<td>$\text{Spin}_{11}$</td>
<td>$M_{11}$</td>
</tr>
<tr>
<td>$\text{Spin}_{12}$</td>
<td>$M_{12}$</td>
</tr>
</tbody>
</table>

We now turn to the almost quasi-simple case. For this we first need the following lemma:
Lemma 7.19. Let $n \geq 13$ be an integer. Then the interval $(\frac{n}{2}, n]$ contains at least two different primes.

Proof. The statement can be checked directly if $13 \leq n \leq 37$. Let $n \geq 38$. Then $x = 2n/3 > 25$, hence the intervals $(x, 6x/5)$ and $(6x/5, 36x/25)$ both contain at least one prime, cf. [22], whence the result follows. □

For a finite group $G$, let $d(G)$ denote the smallest degree of a faithful complex projective representation of $G$.

Lemma 7.20. Let $S$ be a finite simple group with $d(S) \geq 13$. Suppose that

(i) $S$ has a faithful projective complex representation $\Phi$ of degree $e$, and

(ii) $|S|$ has at least two prime divisors $p, p'$ with $p > p' > \frac{2}{3}e$.

Then either $S = S_n$ and $\Phi$ is the smallest representation of degree $n - 1$ of $S$, or $S = L_2(q)$ with $q = p$, $p' = (q \pm 1)/2$ and $e = (q \pm 1)/2$.

Proof. First suppose that $S = S_n$. Then $e \geq d(S) \geq 13$, whence $n \geq 14$. Next, $n \geq p > 2e/3 \geq 2d(S)/3$, whence $13 \leq d(S) < 3p/2$. This condition excludes all finite groups of Lie type, except possibly $S = S_{2n}(q)$ with $n \geq 2$ and $q$ odd, $L_n(q)$ with $n \geq 2$, $U_n(q)$ with $n \geq 3$. Assume $S = S_{2n}(q)$, $n \geq 2$, and $q$ odd. Since $d(S) \geq 13$, we have $q^n \geq 27$ and $d(S) = (q^n - 1)/2$. Now

\[ p > 2d(S)/3 = (q^n - 1)/3 > \max\{ (q^{n-1}/2), (q^n + 1)/4, q, 2\} \]

But $p$ is a prime divisor of $|S|$, hence $p = (q^n \pm 1)/2$. The same holds for $p'$. Thus both $(q^n - 1)/2$ and $(q^n + 1)/2$ are primes, a contradiction. The cases $L_n(q)$, $n \geq 3$, and $U_n(q)$ can be excluded similarly. □

Proposition 7.21. Let $S$ be a finite simple group with $d(S) \geq 13$ and $G$ a finite group such that $S \leq G/Z(G) \leq \text{Aut}(S)$. Suppose that $G$ can be embedded in a simple simply-connected complex classical group $\mathfrak{G}$ in such a way that $G$ acts irreducibly on all fundamental representations of $\mathfrak{G}$. Then $S = S_n$, $Z(G) \times S_n \leq G \leq Z(G) \rtimes S_n$, $\mathfrak{G} = \text{SL}_{n-1}(\mathbb{C})$.

Proof. Let $K = G^{(\infty)}$, $C = C_G(K) = Z(G)$, $L = K \rtimes C$. Then $K$ is a finite quasi-simple group and $K/Z(K) \simeq S$. Let $V$ be the natural module for $\mathfrak{G}$ and let $d = \text{dim}(V)$. By Clifford’s Theorem, $V|_K$ is a direct sum of irreducible $K$-modules of dimension $e$, each of which is a faithful projective representation of $S$. In particular, $d \geq e \geq d(S) \geq 13$. By Lemma 7.19, one can find primes $p > p'$ in the interval $(\frac{d}{2}, d]$. Since $d \geq 13$, we have $p > p' \geq [d/2] + 3$. Now the $(d - p + 1)^{th}$-fundamental representation
W of $\mathfrak{G}$ has dimension
\[
\binom{d}{d-p+1} = \frac{p(p+1)\ldots d}{(d-p+1)!}
\]
if $\mathfrak{G}$ is of type $A$, $B$ or $D$. If $\mathfrak{G}$ is of type $C$, then the $(d-p+3)^{th}$-fundamental representation $W$ of $\mathfrak{G}$ has dimension
\[
\binom{d}{d-p+3} - \binom{d}{d-p+1} = \frac{(p-2)(p-1)p\ldots d}{(d-p+3)!} - \frac{p(p+1)\ldots d}{(d-p+1)!}.
\]
In either case, $\dim(W)$ is divisible by $p$.

By assumption, $W|_G$ is irreducible. Hence by Clifford’s Theorem, $W|_L$ is a direct sum of $t$ irreducible $L$-modules, each of dimension $l$, and $t|(G:L)$. Clearly, $p$ divides $tl$. Assume that $p$ does not divide $|S|$. Since $p > 2d(S)/3$, we can conclude that $p$ does not divide $|\text{Aut}(S)|$ and $|\text{Mult}(S)|$. Thus $p$ does not divide $(G:L)$, and so $p|t$. But $l$ is an irreducible degree of $L = K * C$ and $C = Z(G)$ is abelian. Therefore, $p$ divides $|K|$, a contradiction. Consequently, $p$ divides $|S|$. Similarly, $p'$ divides $|S|$. Now we can apply Lemma 7.20. Observe that the case $S = L_2(q)$ with $q = p$ is impossible: $(G:L) \leq 2$ and $d \geq e \geq (q-1)/2$, and so already the third fundamental representation has degree too big to be a sum of $\leq 2$ irreducible representations of $L$.

The simple groups with a nontrivial projective complex representation of degree at most twelve are $\mathfrak{A}_5, \mathfrak{A}_6, \mathfrak{A}_7, \mathfrak{A}_8, \mathfrak{A}_9, \mathfrak{A}_{10}, \mathfrak{A}_{11}, \mathfrak{A}_{12}, \mathfrak{A}_{13}, \mathfrak{A}_{14}, \mathfrak{A}_{15}, \mathfrak{A}_{16}, \mathfrak{A}_{17}, \mathfrak{A}_{18}, \mathfrak{A}_{19}, \mathfrak{A}_{20}, \mathfrak{A}_{21}, \mathfrak{A}_{22}$, $\mathfrak{L}_2(7), \mathfrak{L}_2(8), \mathfrak{L}_2(11), \mathfrak{L}_2(13), \mathfrak{L}_2(17), \mathfrak{L}_2(19), \mathfrak{L}_2(23), \mathfrak{L}_2(25), \mathfrak{L}_3(3), \mathfrak{L}_3(4), \mathfrak{U}_3(2), \mathfrak{U}_3(3), \mathfrak{U}_3(2), \mathfrak{S}_4(2), \mathfrak{S}_6(2), \mathfrak{O}_8^+(2), \mathfrak{G}_2(4), \mathfrak{M}_{11}, \mathfrak{M}_{12}, \mathfrak{J}_2, \mathfrak{M}_{22}$ and $\mathfrak{Suz}$. The character tables of all these groups are contained in the Atlas [1] and can be checked directly.

The quasi-simple subgroups of simple complex exceptional groups have been classified, see for example [4] for references. Apart from the groups listed above, these are $\mathfrak{A}_{14}, \mathfrak{A}_{15}, \mathfrak{A}_{16}, \mathfrak{A}_{17}, \mathfrak{A}_{18}, \mathfrak{A}_{19}, \mathfrak{A}_{20}, \mathfrak{A}_{21}, \mathfrak{A}_{22}, \mathfrak{L}_2(16), \mathfrak{L}_2(17), \mathfrak{L}_2(19), \mathfrak{L}_2(23), \mathfrak{L}_2(25), \mathfrak{L}_3(5), \mathfrak{U}_3(8), \mathfrak{G}_2(3), \mathfrak{3D}_4(2), \mathfrak{2B}_2(8)$ and $\mathfrak{2F}_4(2)'$. The groups $E_7(C)$ and $E_8(C)$ both have fundamental representations of degrees divisible by 13 and by 19. None of the quasi-simple groups from the above lists has this property. The only groups with an irreducible complex representation of degree 27 are $\mathfrak{A}_8, \mathfrak{L}_2(53), \mathfrak{U}_3(3), \mathfrak{S}_6(2), \mathfrak{3O}_7(3), \mathfrak{3G}_2(3)$, and $\mathfrak{2F}_4(2)'$. None of these has a character of degree 2925. The only groups with an irreducible complex representation of degree 26 are $\mathfrak{L}_2(25), \mathfrak{L}_2(27), \mathfrak{L}_2(53), \mathfrak{L}_4(3), \mathfrak{3D}_4(2)$ and $\mathfrak{2F}_4(2)'$. Of these, only $\mathfrak{3D}_4(2)$ has characters of degree 52, 273, 1274 as well. The only groups with an irreducible complex representation of degree 7 are $\mathfrak{A}_8, \mathfrak{L}_2(8), \mathfrak{L}_2(13), \mathfrak{U}_3(3)$ and $\mathfrak{S}_6(2)$. Of these, only $\mathfrak{A}_8, \mathfrak{L}_2(13)$ and $\mathfrak{U}_3(3)$ also have a projective character of degree 14. But $2\mathfrak{A}_8$ is known not to be contained in $G_2(C)$. 


### Table 7.22.
Almost quasi-simple subgroups $G$ of complex simple Lie groups $\mathfrak{g}$ irreducible in all fundamental representations.

<table>
<thead>
<tr>
<th>$\mathfrak{g}$</th>
<th>$G$</th>
<th>ext.</th>
<th>remarks</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathrm{SL}_n$</td>
<td>$\mathfrak{S}_{n+1}$</td>
<td></td>
<td>$\mathrm{Weyl}(A_n)$</td>
</tr>
<tr>
<td>$\mathrm{SL}_3$</td>
<td>$\mathfrak{A}_5$</td>
<td></td>
<td>$\mathrm{Weyl}(H_3)$</td>
</tr>
<tr>
<td>$\mathrm{SL}_3$</td>
<td>$\Lambda_3(2)$</td>
<td></td>
<td>$\mathrm{CRG}(24)$</td>
</tr>
<tr>
<td>$\mathrm{SL}_3$</td>
<td>$3,\mathfrak{A}_6$</td>
<td></td>
<td>$\mathrm{CRG}(27)$</td>
</tr>
<tr>
<td>$\mathrm{SL}_4$</td>
<td>$2,\Lambda_3(2)$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\mathrm{SL}_4$</td>
<td>$2,\Lambda_4(2)$</td>
<td></td>
<td>$\mathrm{CRG}(32)$</td>
</tr>
<tr>
<td>$\mathrm{SL}_4$</td>
<td>$2,\mathfrak{A}_7$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\mathrm{SL}_5$</td>
<td>$L_2(11)$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\mathrm{SL}_5$</td>
<td>$U_4(2)$</td>
<td></td>
<td>$\mathrm{CRG}(33)$</td>
</tr>
<tr>
<td>$\mathrm{SL}_6$</td>
<td>$6,\Lambda_3(4),2_1$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\mathrm{SL}_6$</td>
<td>$U_4(2),2$</td>
<td></td>
<td>$\mathrm{Weyl}(E_6)$</td>
</tr>
<tr>
<td>$\mathrm{SL}_6$</td>
<td>$6_1,\Lambda_4(3)$</td>
<td></td>
<td>$2_2$ $\mathrm{CRG}(34)$</td>
</tr>
<tr>
<td>$\mathrm{SL}_6$</td>
<td>$6,\mathfrak{A}_7$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\mathrm{SL}_7$</td>
<td>$S_6(2)$</td>
<td></td>
<td>$\mathrm{Weyl}(E_7)$</td>
</tr>
<tr>
<td>$\mathrm{SL}_8$</td>
<td>$4_1,\Lambda_3(4),2_3$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\mathrm{SL}_8$</td>
<td>$2,\Lambda_5(2),2$</td>
<td></td>
<td>$\mathrm{Weyl}(E_8)$</td>
</tr>
<tr>
<td>$\mathrm{SL}_{10}$</td>
<td>$2,\Lambda_4(2),2$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

In Table 7.22, the examples in Lie groups of type $A$ or $C$ can be seen to occur (using GAP for instance). Gross observed (cf. for instance [14]) that $U_3(3)$, resp. $3D_4(2)$, give rise to examples for type $G_2$, resp. $F_4$. Next, let $G = L_2(13)$. It is known that $G$ embeds in $\mathfrak{g} = G_2(\mathbb{C})$, cf. [4]. If $\omega$ is the fundamental representation of degree 7 of $\mathfrak{g}$, then $\omega' := \Lambda^2(\omega) - \omega$ is the fundamental representation of degree 14 of $\mathfrak{g}$. Now it is easy to check that the restrictions of $\omega$ and $\omega'$ to $G$ are irreducible.

Finally, let $\mathfrak{g} = \mathrm{Spin}_n(\mathbb{C})$. If $n = 8$ then clearly $G = 2,\Lambda_5(2)$ or $2,\mathfrak{A}_9$ embeds in $\mathfrak{g}$, and since $d(G) = 8$ one sees that each $G$ gives rise to an example in $\mathfrak{g}$. Let $n = 7$ and let $\omega_1$ be the fundamental representation of $\mathfrak{g}$ (on a 7-dimensional module $V$). Let $G = L_2(8),3$ or $2,\mathfrak{S}_6(2)$. Then $G$ embeds in $O_7(\mathbb{C}) = O(V)$, and the restrictions of $\omega_1$ and $\omega_2 = \Lambda^2(\omega_1)$ to $G$ are irreducible. It is known that the square of the third fundamental
Table 7.22. Almost quasi-simple subgroups $G$ of complex simple Lie groups $\mathfrak{g}$ irreducible in all fundamental representations (continued).

<table>
<thead>
<tr>
<th>$\mathfrak{g}$</th>
<th>$G$</th>
<th>ext.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\text{Sp}_4$</td>
<td>$2A_5$</td>
<td>21</td>
</tr>
<tr>
<td>$\text{Sp}_4$</td>
<td>$2A_6$</td>
<td>2</td>
</tr>
<tr>
<td>$\text{Sp}_6$</td>
<td>$2L_2(13)$</td>
<td>2</td>
</tr>
<tr>
<td>$\text{Sp}_6$</td>
<td>$2J_2$</td>
<td>2</td>
</tr>
<tr>
<td>$\text{Spin}_7$</td>
<td>$L_2(8).3$</td>
<td>2</td>
</tr>
<tr>
<td>$\text{Spin}_7$</td>
<td>$2S_6(2)$</td>
<td>2</td>
</tr>
<tr>
<td>$\text{Spin}_8$</td>
<td>$2O^+(2)$</td>
<td>2</td>
</tr>
<tr>
<td>$\text{Spin}_8$</td>
<td>$2A_9$</td>
<td>2</td>
</tr>
<tr>
<td>$G_2$</td>
<td>$L_2(13)$</td>
<td>3</td>
</tr>
<tr>
<td>$G_2$</td>
<td>$U_3(3)$</td>
<td>3</td>
</tr>
<tr>
<td>$F_4$</td>
<td>$3D_4(2)$</td>
<td>3</td>
</tr>
</tbody>
</table>

representation, $\omega_3$, of $\mathfrak{g}$ is just the representation of $\mathfrak{g}$ on the even part of the Clifford algebra $C(V)$. Hence

$$\omega_3^2 = 1 + \Lambda^2(\omega_1) + \Lambda^4(\omega_1) + \Lambda^6(\omega_1) = \sum_{i=0}^{3} \Lambda^i(\omega_1).$$

In particular, if $g \in G$ is of order 9, then $\omega_3(g)^2 = 1$. Since $\omega_3|_G$ is a complex representation of degree 8, this implies that $\omega_3|_G$ is irreducible.

Thus all examples in Table 7.22 do indeed occur. This completes the proof of Theorem 1.3.

References


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HYPERBOLIC 2-FOLD BRANCHED COVERINGS OF LINKS AND THEIR QUOTIENTS

MATTIA MECCHIA AND MARCO RENI

Many 3-manifolds can be represented as 2-fold branched coverings of links, but this representation is, in general, not unique. In the Seifert fibered case the problem is usually local: For example, if $K$ is a Montesinos knot its 2-fold branched covering is Seifert fibered and there exists a complete system of local geometric modifications on $K$ by which we can get every other Montesinos knot with the same 2-fold branched covering. On the other hand, if the 2-fold covering $M$ of a knot is hyperbolic, the situation is globally determined by the structure of the isometry group of $M$. In this paper we develop a global approach for the case that $M$ is hyperbolic and we study the orbifolds which are quotients of $M$ by the action of a 2-group of isometries. This leads to a complete description of the geometry of the possible configurations of knots with the same 2-fold branched coverings. Moreover we are also able to settle the 2-component link case, which was still open, by finding an explicit bound on the number of inequivalent 2-component links which have the same hyperbolic 2-fold branched coverings.

1. Introduction.

Many 3-manifolds can be represented as 2-fold coverings of the 3-sphere $S^3$ branched over links (“2-fold branched coverings of links”), but this representation is, in general, not unique. Examples of non-uniqueness of the representation have been known for a long time ([1] and [19]); however a complete description of the general situation is not yet available (see Problems 3.25 and 1.22 in Kirby’s list [8]).

As usual the two basic cases of the theory are the Seifert and the hyperbolic one.

The Seifert case is well understood and the problem is usually local.

The representation is unique for spherical Seifert fibered manifolds, because they have (up to conjugation) a unique involution with orbit space the 3-sphere $S^3$: This is proved in [21] for $S^3$, in [6] for lens spaces (which are 2-fold branched coverings of two-bridge links) and in [7], [10] for the other spherical manifolds.
In the nonspherical case the representation is highly not unique. By [3], [9] and [18] any involution on a Seifert manifold is standard, that is it is equivalent to a fiber-preserving one. If $M$ is a Seifert fibered 2-fold branched covering of a link $L$, there are two possible situations: If the covering involution respects the orientation of the fibers, the link $L$ is a Seifert link, that is its complement in $S^3$ admits a Seifert fibration by circles [4]; on the other hand if the covering involution reverses the orientation of the fibers, $L$ is a Montesinos link, that is $S^3$ admits a Seifert fibration by circles and intervals such that $L$ consists of all boundary points of the intervals [11], [5]. Both situations may occur simultaneously. Moreover the number of inequivalent Montesinos links which have the same 2-fold branched coverings may be arbitrarily large because a Seifert space does not change if we change the order of its exceptional fibers, but this permutation may affect the corresponding Montesinos branch sets. This phenomenon is local and well understood: If $M$ is a Seifert fibered 2-fold branched covering of a Montesinos link $L$ every other Montesinos link with the same 2-fold branched covering can be obtained by a sequence of elementary geometric modifications of $L$ (mutations along Conway spheres [11], [20]).

The case that $M$ is hyperbolic is quite different. By Thurston’s Orbifold Theorem [2] any involution with nonempty fixed point set on a hyperbolic manifold $M$ is standard, that is it is equivalent to an isometry. This implies that any link with 2-fold branched covering $M$ is $\pi$-hyperbolic, that is $S^3$ admits a Riemannian metric of constant negative curvature which becomes singular folding with an angle $\pi$ around the link.

The first difference with the Seifert case is that the number of inequivalent links which have the same hyperbolic 2-fold branched covering $M$ is bounded by a constant $C$ not depending on $M$. The estimate for $C$ depends on the number of components of the link (by homological reasons two links with the same 2-fold branched coverings have the same number of components). It has been proved that $C \leq 9$ for knots [13] and that $C \leq 5$ for links which have at least three components [15]; for the most difficult case of 2-component links no explicit bound was known before.

A second major difference between the Seifert and the hyperbolic case is how inequivalent links with the same 2-fold branched coverings are related. We have recalled above that, if $M$ is a Seifert fibered 2-fold covering of a Montesinos knot $K$, there exists a complete system of local geometric modifications on $K$ by which we can get every other Montesinos knot with the same 2-fold branched coverings. But if $M$ is hyperbolic there is no analogous system of local geometric modifications: Indeed the arguments of [13] and [15] make clear that the hyperbolic situation is globally determined by the structure of the isometry group of $M$. 
In this paper we develop a global approach for the case that $M$ is hyperbolic and we study the orbifolds which are quotients of $M$ by the action of a 2-group of isometries. This leads to a new proof of the main Theorem of [13] and also to a complete description of the geometry of the possible configurations of knots with the same 2-fold branched coverings. More important we are able to settle the 2-component link case, which was still open, by finding the explicit bound *nine* on the number of inequivalent 2-component links which have the same hyperbolic 2-fold branched coverings.

The key result of the paper is (for 2-fold branched covering of a link we mean that every meridian of the link corresponds to a generator of the covering group):

**Theorem 1.** Let $M$ be the hyperbolic 2-fold branched covering of a link with one or two components. For any (finite) 2-group $S$ of orientation-preserving isometries of $M$ which contains the covering involution of the link, the singularity graph of the quotient orbifold $M/S$ is combinatorially equivalent to one of the twelve graphs $\text{IA}, \ldots, \text{IID}$ (Figure 1).

By Mostow’s Rigidity Theorem the number of inequivalent $\pi$-hyperbolic knots, respectively 2-component links, with the same 2-fold branched covering $M$ is bounded by the number of the conjugacy classes of non-free involutions in the orientation-preserving isometry group of $M$. So, as a consequence of Theorem 1, by simply counting the number of edges and loops (at most nine) of the twelve graphs $\text{IA}, \ldots, \text{IID}$, we get the following:

**Theorem 2.** There are at most nine different $\pi$-hyperbolic knots with the same 2-fold branched coverings.

The word “different” in Theorem 2 must be made precise. In this paper we shall always work in the category of oriented manifolds and orientation-preserving diffeomorphisms. So two knots $K$ and $K'$ in $S^3$ are equivalent if and only if there is an orientation-preserving diffeomorphism of $S^3$ which carries $K$ onto $K'$. This is equivalent to say that $K$ and $K'$ are ambient isotopic.

Thurston’s Orbifold Theorem [2] and Theorem 2 imply the purely topological result that there are at most nine different simple Conway-irreducible knots (that is: Knots with no pairwise incompressible embedded 2-spheres and such that every embedded incompressible 2-torus is boundary parallel) with the same 2-fold branched coverings.

It is not completely clear if the bound ‘nine’ in Theorem 2 is best possible. In [14] explicit examples of four different $\pi$-hyperbolic knots in $S^3$ with the same 2-fold branched coverings are constructed. Recently the author has obtained an example of six different $\pi$-hyperbolic knots in $S^3$ with the same 2-fold branched coverings (unpublished). There is some evidence that this
last construction can be possibly generalized to give an example with nine different knots, but computations are still in progress.

The estimate in Theorem 2 had already been obtained in [13] by abstract group-theoretical methods; the advantage here is that Theorem 1 describes also the possible configurations of knots with the same 2-fold branched coverings. Moreover if we turn to the 2-component link case, the algebraic methods of [13] and [15] become too involved; but, from a geometrical point of view, the situation is analogous to the knot case. Indeed Theorem 1 applies simultaneously also to the 2-component link case:
**Theorem 3.** There are at most nine different $\pi$-hyperbolic 2-component links with the same 2-fold branched coverings.

Finally a Corollary of Theorem 2, Theorem 3 and [15, Theorem 1] is the following explicit bound which does not depend on the number of components of the link:

**Corollary.** There are at most nine different $\pi$-hyperbolic links with the same 2-fold branched coverings.

2. **Proof of Theorems 2 and 3.**

By Mostow’s Rigidity Theorem, the number of inequivalent $\pi$-hyperbolic knots, respectively 2-component links, with the same 2-fold branched covering $M$ is bounded above by the number of the conjugacy classes of non-free involutions in the orientation-preserving isometry group of $M$. So it is also bounded above by the number of conjugacy classes of non-free involutions in a Sylow 2-subgroup $S$ of the orientation-preserving isometry group of $M$.

The projection of the fixed point set of an involution of $S$ to the quotient orbifold $M/S$ contains one edge or one loop of the singularity graph of $M/S$. Moreover if two involutions of $S$ are not conjugate, the projections of their fixed point sets have no common interior points. The thesis follows from Theorem 1 by counting the number of edges and loops (at most nine) of the twelve graphs IA, . . . , IID.

3. **The family $\mathcal{F}$ of quotient orbifolds.**

In this section we associate a family $\mathcal{F}$ of 3-orbifolds to each pair $(M, S)$ where $M$ is a hyperbolic 2-fold branched covering of a link $L$ with one or two components and $S$ a (finite) 2-group of orientation-preserving isometries of $M$ containing the covering involution. Each orbifold of the family $\mathcal{F}$ will be a quotient $M/H$ of $M$ for some subgroup $H$ of $S$; however we will not include in $\mathcal{F}$ all the quotient orbifolds of $M$ (we are now concerned only with elements of $S$ which have nonempty fixed point sets). We conclude the section by proving the most important properties of $\mathcal{F}$.

The following elementary algebraic result on 2-groups is crucial in the construction of $\mathcal{F}$:

**Proposition 1 ([17, page 88, Theorem 1.6]).** If $H$ is a proper subgroup of a finite 2-group $S$, then the normalizer $N_S H$ is strictly larger than $H$.

**Construction of $\mathcal{F}$.**

We define $\mathcal{F}$ as a disjoint union of subfamilies $\mathcal{F}_1, \ldots, \mathcal{F}_n$. 
The first subfamily $\mathcal{F}_1$ of $\mathcal{F}$ is the set $\{O(L)\}$ where $O(L)$ is the orbifold with underlying topological space $S^3$ and singular set $L$ with singular index two.

To construct $\mathcal{F}_2$, let $S(L)$ be the group of isometries of $O(L)$ such that their lifts to $M$ are elements of $S$ (we briefly say: “Isometries which lift to $S$”). By classical Smith theory for finite group actions on $S^3$, if the fixed point set of an involution of $S(L)$ is nonempty, then it is connected. The subfamily $\mathcal{F}_2$ is the set of all the 3-orbifolds (up to isometry) which are quotients $O(L)/u$ for some involution $u$ of $S(L)$ which has nonempty fixed point set (in $O(L)$). In particular, if $S(L)$ acts freely on $M$, the subfamily $\mathcal{F}_2$ is empty. By the positive solution of the Smith conjecture \cite{12} the underlying topological space of any orbifold of $\mathcal{F}_2$ is $S^3$. Note that any orbifold of $\mathcal{F}_2$ is also a quotient $M/H$ for some subgroup $H$ of $S$ which contains $h$; moreover $H$ is generated by elements which have nonempty fixed point sets in $M$ (if $v$ is an involution of $S(L)$ with nonempty fixed point set, then at least one lift of $v$ to $M$ has also nonempty fixed point set).

The construction of $\mathcal{F}_3$ is analogous to $\mathcal{F}_2$. For any orbifold $O \in F_2$, let $S(O)$ be the group of the isometries of $O$ which lift to $S$. The subfamily $\mathcal{F}_3$ is the set of all the 3-orbifolds which are quotients $O/u'$ of an orbifold $O \in \mathcal{F}_2$ for some involution $u'$ of $S(O)$ with nonempty fixed point set in $O$ (if any). By the positive solution of the Smith conjecture the underlying topological space of any orbifold of $\mathcal{F}_3$ is $S^3$. Again any orbifold of $\mathcal{F}_3$ is also a quotient $M/H'$ for some subgroup $H'$ of $S$ which contains $h$ and it is generated by elements with nonempty fixed point sets (in $M$).

Iteratively, the subfamily $\mathcal{F}_4$ is the set of quotients of orbifolds of the subfamily $\mathcal{F}_3$ by involutions which have nonempty fixed point sets and lift to $S$.

Since $S$ is a finite group, after a finite number of steps the construction ends. We denote by $\mathcal{F}_n$ the last nonempty subfamily we get, by $\mathcal{F}$ the disjoint union $\mathcal{F}_1 \cup \ldots \cup \mathcal{F}_n$.

We now turn to describe some properties of the family $\mathcal{F}$ (Propositions 2 and 3) which we need for the proof of Theorem 1.

We recall what is already evident from the construction above:

- the underlying topological space of each orbifold of $\mathcal{F}$ is $S^3$ (positive solution of the Smith conjecture);
- each orbifold of $\mathcal{F}_i$ for $1 \leq i \leq n$ is a quotient $M/H$ of $M$ for some subgroup $H$ of $S$ of order $2^i$ which contains $h$ and is generated by elements with nonempty fixed point sets (in $M$).

We first characterize the last subfamily $\mathcal{F}_n$ of $\mathcal{F}$ by the following Proposition.
Proposition 2. The last subfamily $F_n$ contains exactly one orbifold: This orbifold is the quotient $M/H$ of $M$ where $H$ is the subgroup of $S$ which is generated by all the elements with nonempty fixed point sets.

Proof. Let $O \in F_n$ be any orbifold. By construction $O$ is a quotient $= M/H$ of $M$ for some subgroup $H$ of $S$ which has order $2^n$ and is generated by elements with nonempty fixed point sets. To prove the thesis it is enough to show that any element of $S - H$ acts freely on $M$: It follows that $H$ is the unique subgroup of $S$ of order $2^n$ which is generated by elements with nonempty fixed point sets.

Suppose, ad absurdum, that there exists an element $g$ of $S - H$ which has nonempty fixed point set in $M$. In particular $H$ is a proper subgroup of $S$. By Proposition 1 the normalizer $N_S H$ of $H$ in $S$ is larger than $H$ and the factor group $N_S H/H$ projects to a 2-group of isometries of $O$. Hence there exists at least one isometry $u$ of $O$ which has order two and lifts to $S$. Since $F_n$ is the last subfamily of $F$, the involution $u$ acts freely on $O$. The quotient $O_1 := O/u$ is a hyperbolic 3-orbifold and it is also the quotient $M/H_1$ of $M$ for a subgroup $H_1$ of $S$ containing $H$ as a normal subgroup of index two.

Note that $g$ is not an element of $H_1$ because if $g \in H_1$ the fixed point set of $g$ in $M$ projects to (a subset of) the fixed point set of $u$ in $O$ which is impossible, since $u$ acts freely. So again $H_1$, which does not contain $g$, is a proper subgroup of $S$ and, by Proposition 1, there exists one isometry of $O_1$ which is an involution and lifts to $S$. If every involution of $O_1$ which lifts to $S$ acts freely on $O_1$ we can construct a further quotient $O_2$ of $O_1$ by any involution which lifts to $S$.

More generally, for some $m \geq 1$, we can construct iteratively a hierarchy $O_1, \ldots, O_m$ of orbifolds and a sequence $H_1, \ldots, H_m$ of corresponding subgroups of $S$ such that every orbifold $O_i$ is the quotient of $O_{i-1}$ by an involution acting freely on $O_{i-1}$; every group $H_i$ contains $H_{i-1}$ as a normal subgroup of index two. Since we know that $S - H$ contains at least one element $g$ which has nonempty fixed point set, after finitely many steps we must find an orbifold, say $O_m$, which admits an involution $v$ which has nonempty fixed point set (in $O_m$) and lifts to $S$. By construction $O_m$ is the quotient $M/H_m$ for the corresponding subgroup $H_m$ of $S$. A lift $\tilde{v}$ of $v$ to $S$ has nonempty fixed point set and it lies in the normalizer $N_S H_m$ of $H_m$ in $S$; so $\tilde{v}$ also normalizes the subgroup $H$ of $H_m$ because $H$ is the subgroup of $H_m$ generated by all the elements of $H_m$ with nonempty fixed point set and thus $\tilde{v}$ descends to an involution of $O$ with nonempty fixed point set. This is impossible because $F_n$ is the last subfamily of $F$.

This finishes the proof.
We conclude this section by showing that the procedure described at the beginning of the paragraph can be reversed. Indeed we have constructed $\mathcal{F}$ ‘up-bottom’ starting from $\mathcal{F}_1$ and taking quotients by involutions until the last set $\mathcal{F}_n$ has been reached; but it is also possible to go back ‘bottom-up’ from $\mathcal{F}_n$ to $\mathcal{F}_1$, taking 2-fold branched coverings at each step. This reverse construction is made precise in Proposition 3. Before that we need to introduce the notion of first homology group and 2-fold branched covering of an orbifold.

The first homology group of an orbifold.

The first homology group $H_1(\mathcal{O})$ of a 3-orbifold $\mathcal{O}$ is the abelianization of the orbifold fundamental group $\pi_1(\mathcal{O})$ of $\mathcal{O}$. In our case the underlying topological space of $\mathcal{O}$ is $S^3$, each component of its singularity graph $\Gamma$ is a knot or a trivalent graph and the singularity order at each point of the edges of $\Gamma$ is a power of two (all vertices are of dihedral type).

Starting with a planar projection of the graph, one sees that $\pi_1(\mathcal{O})$ admits a (Wirtinger) presentation of the form:

$$\pi_1(\mathcal{O}) = \langle x_1, \ldots, x_n | r_1 = 1, \ldots, r_m = 1; x_1^{i_1} = \cdots = x_m^{i_m} = 1 \rangle$$

where each $x_j$ represents a loop around an arc contained either in some edge or in some knot of $\Gamma$ and $i_j$ its singularity order (a power of two). There are two possible types of relations $r_j$. The first type corresponds to the vertices of $\Gamma$ and has the form $x_jx_k^{-1}x_k^{-1}$ with $e = +1$ or $-1$ and $d = +1$ or $-1$; since each vertex is of dihedral type at least two among the three elements $x_j, x_k$ and $x_s$ have order two. This type of relations involve only loops around edges of $\Gamma$ and not around components which are knots. The second type of relations corresponds to the double points of $\Gamma$ in the Wirtinger projection and has the form $x_jx_k^{-1}x_s^{-1}$ with $e = +1$ or $-1$. When abelianizing, relations of the first type imply that $x_j^2 = x_k^2 = x_s^2 = 1$ and get the form $x_jx_kx_s$. Relations of the second type get the form $x_jx_s^{-1}$ where $x_j$ and $x_s$ correspond to two loops around the same edge or the same knot of $\Gamma$.

This shows that, if the singularity graph of $\mathcal{O}$ has $q$ edges, $p$ vertices and $s$ knots, its first homology group $H_1(\mathcal{O})$ is the abelianization of a group which admits a presentation of the form:

$$\langle m_1, \ldots, m_q, n_1, \ldots, n_s | R_1 = 1, \ldots, R_p = 1; m_1^2 = \cdots = m_q^2 = n_1^{k_1} = \cdots = n_s^{k_s} = 1 \rangle$$

where $m_i$ is a small meridian around the $i$-th edge, $n_j$ is a small meridian around the $j$-th knot with singularity order $k_j$ and $R_s$ is the abelianized relation at the $s$th-vertex.
2-fold branched covering of \( O \) along a cycle.

Let \( O \) be an orbifold of \( \mathcal{F} \), \( \Gamma \) its singularity graph and \( c \) a subgraph of \( \Gamma \) which is either a cycle of edges or a knot. If there exists a map \( \psi : H_1(O) \rightarrow \mathbb{Z}_2 \) which sends a small meridian around each edge contained in \( c \), respectively around the knot \( c \), to the generator of \( \mathbb{Z}_2 \) and all the other small meridians around edges of \( \Gamma \) to the trivial element, we call 2-fold branched covering of \( O \) along \( c \) the covering of \( O \) defined by \( \psi \).

We are now ready to state Proposition 3 (recall that \( h \) is the covering involution of the covering \( M \rightarrow O(L) \)):

**Proposition 3.** Let \( O \) be an orbifold of the subfamily \( \mathcal{F}_i \) for \( i \geq 2 \) and \( \tilde{O} \) the 2-fold branched covering of \( O \) along a cycle of edges or a loop \( c \). If \( c \) contains no interior points of the projection to \( O \) of the fixed point set of \( h \), then \( \tilde{O} \) is an orbifold of the subfamily \( \mathcal{F}_{i-1} \).

**Proof.** We first prove that the covering \( O(L) \rightarrow O \) factors through \( \tilde{O} \). Up to identification of \( \pi_1O(L) \) and \( \pi_1\tilde{O} \) with subgroups of \( \pi_1O \), this is equivalent to prove that \( \pi_1\tilde{O} \) is a subgroup of \( \pi_1O \). Since \( \pi_1\tilde{O} \) has index two in \( \pi_1O \) the intersection \( \pi_1\tilde{O} \cap \pi_1O(L) \) has index at most two in \( \pi_1O(L) \). So it is enough to prove that the case that \( \pi_1\tilde{O} \cap \pi_1O(L) \) has index two in \( \pi_1O(L) \) is impossible.

Suppose, ad absurdum, that \( \pi_1\tilde{O} \cap \pi_1O(L) \) is the fundamental group \( \pi_1N \) of an orbifold \( N \) which is a 2-fold covering of \( O(L) \). The 2-fold coverings of \( O(L) \) are branched and classified by epimorphisms of \( H_1O(L) \) onto \( \mathbb{Z}_2 \). If \( L \) is connected, \( H_1O(L) \cong \mathbb{Z}_2 \) and \( O(L) \) has a unique 2-fold covering, \( M_0 := M \), with covering involution \( h_0 := h \). If \( L \) has two components, \( H_1O(L) \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \) and \( O(L) \) has three distinct 2-fold coverings, namely: The 3-manifold \( M_0 := M \) with covering involution \( h_0 = h \), the 2-fold covering \( M_1 \) branched along the first component of \( L \) with covering involution, say \( h_1 \), the 2-fold covering \( M_2 \) branched along the second component of \( L \) with covering involution, say \( h_2 \). So for some \( i \), \( N \) is homeomorphic to \( M_i \) and the involution \( h_i \) projects to a nontrivial involution of \( \pi_1\tilde{O} \). Thus the quotient orbifold \( \tilde{O}/h_i \) is homeomorphic to \( O \) and the projection of the fixed point set of \( h_i \) to \( O \) is a subset of the branching set of the covering \( \tilde{O} \rightarrow O \). But this is impossible, because we have supposed that the branching set \( c \) contains no interior points of the projection of the fixed point set of \( h \) to \( O \).

We have thus proved that the covering \( O(L) \rightarrow O \) factors through \( \tilde{O} \).

Finally we have to show that \( \tilde{O} \) is an element of \( \mathcal{F} \) (note that what we have proved above implies that \( \tilde{O} \) is a quotient of \( O(L) \), but it is not yet clear at this point if we can get \( \tilde{O} \) by only quotienting by involutions with
nonempty fixed point sets as it is the case when we construct the orbifolds of $\mathcal{F}$.

To fix notation let $\tilde{O} = M/\tilde{H}$ for some subgroup $\tilde{H}$ of $S$ containing $h$. To the pair $(M, \tilde{H})$ we can associate a family $\mathcal{E}$ of hyperbolic 3-orbifolds which is the disjoint union of subfamilies $\mathcal{E}_1, \ldots, \mathcal{E}_m$. Each orbifold of $\mathcal{E}_j$ is the quotient of $M$ by a subgroup of $\tilde{H}$ of order $2^j$ and it is also an element of $\mathcal{F}_i$ (indeed now we are using only involutions with nonempty fixed point sets).

So by Proposition 2, the last subfamily $\mathcal{E}_m$ of $\mathcal{E}$ contains a unique orbifold, say $\tilde{O}' = M/H'$ where $H'$ is the subgroup of $\tilde{H}$ generated by all the elements with nonempty fixed point sets. To conclude the proof, it is enough to show that $\tilde{O} = \tilde{O}'$, because $\tilde{O}'$, hence $\tilde{O}$ is an element of $\mathcal{F}$ (in particular of $\mathcal{F}_{i-1}$ because it is the quotient of $M$ by a subgroup of $S$ of order $2^{i-1}$).

If $H' = \tilde{H}$, then $\tilde{O} = \tilde{O}'$ and the proof is complete. We shall show that the case that $H'$ is a proper subgroup of $\tilde{H}$ is impossible. If $H'$ is a proper subgroup of $\tilde{H}$, the normalizer $N_\tilde{H}H'$ is larger than $H'$ by Proposition 1 and the factor group $N_\tilde{H}H'/H'$ projects to a 2-group of isometries of $\tilde{O}'$. Hence there exists at least one involution $u$ of $\tilde{O}'$ which lifts to $\tilde{H}$. The involution $u$ acts freely on $\tilde{O}'$, because $H'$ is the subgroup of $\tilde{H}$ generated by all the elements with nonempty fixed point sets. The quotient $O_1$ of $\tilde{O}'$ by $u$ is a hyperbolic 3-orbifold which is also the quotient of $M$ by a 2-subgroup $H_1$ of $\tilde{H}$ containing $H'$ as a subgroup of index two.

If $H_1$ is a proper subgroup of $\tilde{H}$, again by Proposition 1, there exists at least one involution $v$ of $O_1$ which lifts to $\tilde{H}$. So we can construct a quotient $O_2 := O_1/v$ of $O_1$ by $v$ and $v$ acts freely, because $H'$ contains all the elements of $\tilde{H}$ with nonempty fixed point sets.

Iteratively, for some $m \geq 1$ we can construct a hierarchy $O_1, \ldots, O_m$ of orbifolds and a sequence $H_1, \ldots, H_m$ of corresponding subgroups of $\tilde{H}$ such that each orbifold $O_i$ is the quotient of $O_{i-1}$ by an involution acting freely on $O_i$ and each $H_i$ contains $H_{i-1}$ as a subgroup of index two. After finitely many steps our procedure stops because we have got the quotient orbifold $\tilde{O} = M/\tilde{H}$ of $M$ by the full group of isometries $\tilde{H}$.

Since $m \geq 1$, the orbifold $\tilde{O}$ admits a free regular 2-fold covering, which is impossible because the underlying topological space of $\tilde{O}$, which is the 2-fold branched covering of $S^3$ branched along a knot, is a $\mathbb{Z}_2$-homology 3-sphere [16, Sublemma 15.4].

This finishes the proof of Proposition 3.

4. The singularity graphs.

In this section we explain which combinatorial modifications may take place on the singularity graph of an orbifold $O$ of $\mathcal{F}_i$ when passing to a quotient of
\[ \mathcal{O} \text{ in } \mathcal{F}_{i+1}. \] Indeed this section may be skipped without affecting the proof of Theorem 1. We have included it in the paper just to give some intuition why one considers graphs of Type IA, \ldots , IID as natural candidates for the singularity graphs of the orbifolds of \( \mathcal{F} \).

First of all we need a generalized definition of graph which includes loops, because, in general, the singularity graph of an orbifold \( \mathcal{O} \in \mathcal{F} \) is a union of trivalent graphs and disjoint knots (we use the term ‘loop’ in the combinatorial setting, ‘knot’ in the topological one). A graph \( \Gamma \) is a set \( (V(\Gamma), E(\Gamma), c_1, \ldots, c_r) \) for some nonnegative integer \( r \), where the vertex-set \( V(\Gamma) \) is a finite set of elements called vertices, the edge-set \( E(\Gamma) \) is a finite set of ordered pairs of distinct elements of \( V(\Gamma) \) called edges and \( c_1, \ldots, c_r \) is a finite set of disjoint loops.

A graph is called admissible if it is one of the twelve graphs IA, \ldots , IID, inadmissible in any other case. So our notation for the twelve admissible graphs (see Figure 1) is:

**IA** one loop:
\[ (V(\Gamma) = \emptyset, E(\Gamma) = \emptyset, c_1) \]

**IIA** two loops:
\[ (V(\Gamma) = \emptyset, E(\Gamma) = \emptyset, c_1, c_2) \]

**IIIA** three loops:
\[ (V(\Gamma) = \emptyset, E(\Gamma) = \emptyset, c_1, c_2, c_3) \]

**IB** theta-graph:
\[ (V(\Gamma) = \{v_1, v_2\}, E(\Gamma) = \{(v_1, v_2), (v_1, v_2)', (v_1, v_2)'' \text{ and three reverse edges}\}) \]

**IIB** tetrahedral graph:
\[ (V(\Gamma) = \{v_1, v_2, v_3, v_4\}, E(\Gamma) = \{(v_1, v_2), (v_1, v_3), (v_1, v_4), (v_2, v_3), (v_2, v_4), (v_3, v_4) \text{ and six reverse edges}\}) \]

**IIIB** Kuratowski graph:
\[ (V(\Gamma) = \{v_1, v_2, v_3, v_4, v_5, v_6\}, E(\Gamma) = \{(v_1, v_2), (v_1, v_4), (v_1, v_6), (v_2, v_3), (v_2, v_5), (v_3, v_4), (v_3, v_6), (v_4, v_5), (v_5, v_6) \text{ and nine reverse edges}\}) \]

**IC** pince-nez graph:
\[ (V(\Gamma) = \{v_1, v_2\}, E(\Gamma) = \{(v_1, v_1), (v_1, v_2), (v_2, v_2) \text{ and } (v_2, v_1)\}) \]

**IIC** \( (V(\Gamma) = \{v_1, v_2, v_3, v_4\}, E(\Gamma) = \{(v_1, v_3), (v_1, v_3)', (v_1, v_2), (v_2, v_4), (v_2, v_4)', (v_3, v_4) \text{ and six reverse edges}\}) \)
IIC  \( (V(\Gamma) = \{v_1, v_2, v_3, v_4, v_5, v_6\}, E(\Gamma) = \{(v_1, v_2), (v_1, v_3), (v_1, v_5), (v_2, v_4), (v_2, v_6), (v_3, v_4), (v_3, v_5), (v_4, v_6), (v_5, v_6) \text{ and nine reverse}
edges\})

ID a theta-graph and a loop:  
\( (V(\Gamma) = \{v_1, v_2\}, E(\Gamma) = \{(v_1, v_2), (v_1, v_2)', (v_1, v_2)'' \text{ and three reverse}
edges\}, c_1) \)

IID a pince-nez and a loop:  
\( (V(\Gamma) = \{v_1, v_2\}, E(\Gamma) = \{(v_1, v_1), (v_1, v_2), (v_2, v_2) \text{ and (v_2, v_1)}\}, c_1) \)

IIID  \( (V(\Gamma) = \{v_1, v_2, v_3, v_4\}, E(\Gamma) = \{(v_1, v_1), (v_1, v_2), (v_2, v_3), (v_2, v_4), (v_3, v_4), (v_3, v_4)', (v_2, v_1), (v_3, v_2), (v_4, v_2), (v_4, v_3) \text{ and (v_4, v_3)'}\}). \)

Our point here is to understand which combinatorial modifications occur on the singularity graph of an orbifold of \( F_i \) when passing to \( F_{i+1} \). We are concerned only with the combinatorial structure of the graph, so we forget singularity orders at the various points.

As a warming up example, start with an orbifold \( O(K) \) of \( F_1 \), with \( K \) a knot, and let \( u \) be an involution of \( O(K) \) with connected fixed point set. Either the fixed point set of \( u \) intersects \( K \) into two points and \( u \) acts as a reflection on \( K \) or the fixed point set of \( u \) is disjoint from \( K \) and \( u \) acts as a rotation on \( K \) (the fixed point set of \( u \) is distinct from \( K \) by the positive solution of the Smith conjecture). Correspondingly, the graph of the quotient orbifold \( O(K)/u \) is of Type IB or IIA. Thus, in the case of a knot, the singular set of any orbifold of the subfamily \( F_2 \) is combinatorially a theta-graph or a set of two disjoint loops.

Passing to \( F_3 \) we construct the quotients of the orbifolds of \( F_2 \). For example, consider an orbifold \( O \) of \( F_2 \) with singularity graph of Type IB. An involution \( v \) of \( O \) with connected fixed point set may act in the following ways (up to combinatorial equivalence):

i) \( v_1 \to v_1 \quad v_2 \to v_2 \)

\( (v_1, v_2) \to (v_1, v_2) \quad (v_2, v_1) \to (v_2, v_1) \)

\( (v_1, v_2)', (v_1, v_2)' \to (v_1, v_2)', (v_2, v_1)' \to (v_2, v_1)' \)

\( (v_1, v_2)', (v_1, v_2)', (v_2, v_1)', (v_2, v_1)' \to (v_2, v_1)', (v_2, v_1)' \)
The singularity graph of the corresponding quotient orbifold is, respectively, of Type IB, IIB or IC.

By a routine, not unpleasant, exercise in combinatorial theory, it is easy to show that the twelve graphs IA, . . . , IID appear after a few steps. This is a purely combinatorial operation, which gives many graphs which are inadmissible. But not all graphs we get combinatorially are singularity graphs of some orbifolds in some family F. The proof of Theorem 1 in Section 5 will make clear that topological obstructions exclude graphs which are inadmissible.

5. Proof of Theorem 1.

Let M be a hyperbolic 2-fold branched covering of a link with one or two components, S a 2-group of orientation-preserving isometries of M containing the covering involution and F the family of orbifolds associated to M and S (see Section 3). To prove Theorem 1 we show that inadmissible graphs can not occur as singularity graphs of the orbifolds of F.

This implies the thesis that the singularity graph of the quotient orbifold M/S is of Type IA, . . . , IID. In fact, by Proposition 2, the last subfamily F_n of F contains the quotient orbifold M/H where H is the subgroup of S which is generated by all the elements with nonempty fixed point sets. By Proposition 1 the quotient orbifold M/S is the final output of a hierarchy of quotients, starting with M/H and quotienting by involutions at each step. Since any element of S − H acts freely on M the quotienting involutions act also freely at each step and it is easy to check that, since the singularity graph of M/H is admissible, the singularity graph we get at each step is also admissible.
To prove that the singularity graphs of the orbifolds of \( F \) are admissible, we proceed by contradiction: Let \( i \) be the minimal index such that \( F_i \) contains an orbifold \( O \) with inadmissible singularity graph. Minimality of \( i \) implies that \( O \) is a quotient of an orbifold of \( F_{i-1} \) which has admissible singularity graph. Combinatorially the graph of \( O \) is obtained, when passing from \( F_{i-1} \) to \( F_i \), by a modification of one of the graphs IA, \ldots, IIID as explained in Section 4.

Again it is a routine exercise to check that the graphs we get combinatorially, when passing from \( F_{i-1} \) to \( F_i \), from the graphs IA, \ldots, IIID are the following:

\[
\begin{align*}
IA: & \rightarrow IA, \ II, IB \\
IIB: & \rightarrow II, IA, \ III, IIB, IIC, ID \\
III: & \rightarrow III, IIA, IIB, IIC, ID, X1, X2, X3, X4, X5, X6, X7 \\
IB: & \rightarrow IB, IIA, IC \\
IIC: & \rightarrow IIB, IIC, IID \\
IIC: & \rightarrow IIB, IIC, IID, X2, X4 \\
IID: & \rightarrow IIB, IIC, IID, X2, X3, X4, X5, X7 \\
IID: & \rightarrow IIC, IIID, X2.
\end{align*}
\]

The seven inadmissible graphs \( X_j \) are (Figure 2):

\[
\begin{align*}
X1 & \text{ four loops: } (V(\Gamma) = \emptyset, E(\Gamma) = \emptyset, c_1, c_2, c_3, c_4 ) \\
X2 & \text{ } (V(\Gamma) = \{v_1, v_2, v_3, v_4, v_5, v_6\}, E(\Gamma) = \{(v_1, v_2), (v_1, v_2)', (v_1, v_6), (v_2, v_3), (v_3, v_4), (v_3, v_5), (v_4, v_5), (v_4, v_6), (v_5, v_6) \text{ and nine reverse edges}\}) \\
X3 & \text{ } (V(\Gamma) = \{v_1, v_2, v_3, v_4, v_5, v_6\}, E(\Gamma) = \{(v_1, v_2), (v_1, v_2)', (v_1, v_6), (v_2, v_3), (v_3, v_4), (v_3, v_4)', (v_4, v_5), (v_5, v_6) \text{ and nine reverse edges}\}) \\
X4 & \text{ a tetrahedral graph and a loop: } (V(\Gamma) = \{v_1, v_2, v_3, v_4\}, E(\Gamma) = \{(v_1, v_2), (v_1, v_3), (v_1, v_4), (v_2, v_3), (v_2, v_4), (v_3, v_4) \text{ and six reverse edges}\}, c_1)
\end{align*}
\]
Figure 2

\[ (V(\Gamma) = \{v_1, v_2, v_3, v_4\}, E(\Gamma) = \{(v_1, v_3), (v_1, v_3)', (v_1, v_2), (v_2, v_4), (v_2, v_4)', (v_3, v_4)\} \text{ and six reverse edges}, c_1) \]

\[ \text{X6 a theta-graph and two loops:}\]
\[ (V(\Gamma) = \{v_1, v_2\}, E(\Gamma) = \{(v_1, v_2), (v_1, v_2)', (v_1, v_2)'' \text{ and three reverse edges}\}, c_1, c_2) \]

\[ \text{X7}\]
\[ (V(\Gamma) = \{v_1, v_2, v_3, v_4, v_5, v_6\}, E(\Gamma) = \{(v_1, v_2), (v_2, v_3), (v_2, v_3)', (v_3, v_4), (v_1, v_4), (v_4, v_5), (v_5, v_6), (v_5, v_6)', (v_6, v_1) \text{ and nine reverse edges}\}). \]

Thus the singularity graph of $\mathcal{O}$ is combinatorially one of the seven graphs $X_j$.

The thesis now follows from the following Claim which contradicts minimality of $i$. The proof of the Claim occupies the rest of the section.

**Claim.** If $\mathcal{F}_i$ contains an orbifold with singularity graph of Type $X_j$, there exists an index $k$, $k < i$ and $k \geq 1$, such that $\mathcal{F}_k$ contains an orbifold with inadmissible singularity graph.
Proof of the Claim.

We first prove that if $O \in \mathcal{F}_i$ has singularity graph $X_j$, there exists an orbifold $\tilde{O}$ of $\mathcal{F}_{i-1}$ such that its singularity graph is either inadmissible or of Type IIIA.

The orbifold $\tilde{O}$ is a 2-fold branched covering of $O$ along a cycle which satisfies the hypotheses of Proposition 3. More precisely, we can always find a cycle of edges $c$ or a loop in the singularity graph of $O$ such that there exists the 2-fold branched covering $\tilde{O}$ of $O$ along $c$ and $c$ contains no interior points of the projection $F(h)$ of the fixed point set of $h$ to $O$. Here is the cycle to choose for the various graphs $X_j$ (remember that $F(h)$ has at most two disjoint components and each component is an edge or a loop of the singularity graph of $O$):

X1 Set $c = c_1$ (up to renaming loops). The singularity graph of $\tilde{O}$ is a set of at least three disjoint loops, so inadmissible or of Type IIIA.

X2 Whatever is the projection $F(h)$, we can always make one of the following choices: $c = (v_1, v_2) \cup (v_2, v_1)'$; $c = (v_4, v_5) \cup (v_6, v_6) \cup (v_6, v_4)$; $c = (v_3, v_4) \cup (v_4, v_6) \cup (v_6, v_5) \cup (v_5, v_3)$. In all cases the singularity graph of $\tilde{O}$ is inadmissible.

X3 Set $c = (v_1, v_2) \cup (v_2, v_1)'$ (up to renaming vertices). The singularity graph of $\tilde{O}$ is inadmissible.

X4 Whatever is the projection $F(h)$, we can make one of the following choices: $c = c_1'$; $c = (v_1, v_2) \cup (v_2, v_3) \cup (v_3, v_4) \cup (v_4, v_5)$; if the singularity order of, say $(v_1, v_3)$, is greater than two, set $c = (v_1, v_3) \cup (v_3, v_4) \cup (v_4, v_1)$. In the first case the singularity graph of $\tilde{O}$ is inadmissible, in the second case it is of Type IIIA, in the third case is inadmissible.

X5 Either we can choose $c = c_1$ or $c = (v_1, v_3) \cup (v_3, v_1)'$ (up to renaming vertices). In all cases the singularity graph of $\tilde{O}$ is inadmissible.

X6 Either we can choose $c = c_1$ or $c = (v_1, v_2) \cup (v_2, v_1)'$. In the first case the singularity graph of $\tilde{O}$ is inadmissible, in the second case it is of Type IIIA or inadmissible.

X7 Either we can choose $c = (v_2, v_3) \cup (v_3, v_2)'$ or $c = (v_1, v_2) \cup (v_2, v_3) \cup (v_3, v_4) \cup (v_4, v_1)$. In all cases the singularity graph of $\tilde{O}$ is inadmissible.

To complete the proof we finally show that in the remaining case that $O$ has a 2-fold branched covering $\tilde{O} \in \mathcal{F}_{i-1}$ with singularity graph of Type IIIA, there also exists an index $k$, $k < i$ and $k \geq 1$, such that $\mathcal{F}_k$ contains an orbifold with inadmissible singularity graph, again contradicting minimalitiy of $i$. The rest of the section describes how to construct such an orbifold in the various cases.
In most cases the orbifold with inadmissible singularity graph is again a 2-fold branched covering of \( \hat{\mathcal{O}} \) along a cycle which satisfies the hypotheses of Proposition 3. More precisely, we can always find a loop \( c_1 \) in the singularity graph \( (V(\Gamma) = \emptyset, E(\Gamma) = \emptyset, c_1, c_2, c_3) \) of \( \hat{\mathcal{O}} \) which contains no interior points of the projection of the fixed point set of \( h \) to \( \hat{\mathcal{O}} \). Thus, by Proposition 3, the 2-fold branched covering \( \mathcal{O}' \) of \( \hat{\mathcal{O}} \) along \( c_1 \) is an orbifold of \( \mathcal{F}_{i-2} \).

Before going on we fix some notation. We denote by \( u \), respectively by \( v \), the covering involution of the covering \( \hat{\mathcal{O}} \to \mathcal{O} \), respectively \( \mathcal{O}' \to \hat{\mathcal{O}} \). Since we assume that the singularity graph of \( \mathcal{O} \) is of Type \( X_j \), it can be easily checked that \( u \) fixes setwise each component \( c_i \) of the singularity graph of \( \hat{\mathcal{O}} \). In particular \( u \) lifts to \( \mathcal{O}' \) and its two lifts \( u_1 \) and \( u_2 := (u_1)v \) generate a dihedral group \( D \) of order four.

The fixed point set \( \mathcal{O}(u) \) of \( u \) in \( \hat{\mathcal{O}} \) may intersect the singularity graph of \( \hat{\mathcal{O}} \) in three different ways. We have to distinguish these three cases.

a) \( \mathcal{O}(u) \cap c_2 = \mathcal{O}(u) \cap c_3 = \emptyset \).

In this case the singularity graph of \( \mathcal{O}' \) is inadmissible. Indeed the singularity graph of \( \mathcal{O}' \) contains the preimages \( \tilde{c}_2 \) and \( \tilde{c}_3 \) in \( \mathcal{O}' \) of \( c_2 \) and \( c_3 \). The preimage \( \tilde{c}_2 \) can not be connected because, if \( \tilde{c}_2 \) is connected, the group \( D \) would contain three distinct rotations of order two of \( \tilde{c}_2 \) such that their fixed point sets do not intersect \( \tilde{c}_2 \), which is impossible. So \( \tilde{c}_2 \) has two components. An analogous argument holds for the preimage \( \tilde{c}_3 \) of \( c_3 \) in \( \mathcal{O}' \) which also has two components.

b) \( \mathcal{O}(u) \cap c_2 \neq \emptyset \); \( \mathcal{O}(u) \cap c_3 \neq \emptyset \).

In this case either the singularity graph of \( \mathcal{O}' \) is inadmissible or there exists in \( \mathcal{F}_{i-1} \) a quotient \( \mathcal{O}'/u_1 \) of \( \mathcal{O}' \) with inadmissible singularity graph. By the same argument as in a), \( \tilde{c}_2 \) has two components which are interchanged by the action of the covering involution \( v \).

If \( \tilde{c}_3 \) has also two components, the singularity graph of \( \mathcal{O}' \) is inadmissible.

If \( \tilde{c}_3 \) is connected we show that the quotient \( \mathcal{O}'/u_1 \) of \( \mathcal{O}' \) by a lift \( u_1 \) of \( u \) is an orbifold of \( \mathcal{F}_{i-1} \) and its singularity graph is inadmissible. To prove this it is enough to look at the action induced by \( u_1 \) on \( \tilde{c}_2 \) and \( \tilde{c}_3 \). First of all, if \( \tilde{c}_3 \) is connected, both \( u_1 \) and \( u_2 \) act as reflections on \( \tilde{c}_3 \). Indeed an involution of \( \mathcal{O}' \) either acts freely or it has connected fixed point set because its underlying topological space is \( S^3 \). This implies that each of \( u_1 \) and \( u_2 \) has a connected fixed point set intersecting \( \tilde{c}_3 \) into two distinct points (the preimage of the two intersection points \( \mathcal{O}(u) \cap c_3 \) in \( \hat{\mathcal{O}} \) consists of four distinct points of \( \tilde{c}_3 \)).

On the other hand \( u_2 \) is the product \( u_2 = (u_1)v \); so one between \( u_1 \) and \( u_2 \), say \( u_2 \), interchanges the two components of \( \tilde{c}_2 \) as \( v \), while \( u_1 \) fixes setwise each of the two components of \( \tilde{c}_2 \) acting as a rotation on them. It follows
now that the singularity graph of the quotient orbifold $\tilde{O}'/u_1$, which is an orbifold of $F_{i-1}$, is inadmissible.

c) $F(u) \cap c_2 \neq \emptyset; F(u) \cap c_3 \neq \emptyset$.

In this case either the singularity graph of $\tilde{O}'$ is inadmissible or there exists in $\mathcal{O}_{i-1}$ a quotient $\tilde{O}'/u_1$ of $\tilde{O}'$ with inadmissible singularity graph.

If both $\tilde{c}_2$ and $\tilde{c}_3$ have two components, the singularity graph of $\tilde{O}'$ is inadmissible.

So assume for the following that $\tilde{c}_3$ is connected; in this case arguing as in b), we find that both lifts $u_1$ and $u_2$ of $u$ to $\tilde{O}'$ act as reflections on $\tilde{c}_3$.

If $\tilde{c}_2$ has two components, one between $u_1$ and $u_2$, say $u_1$, fixes setwise each of the two components acting as a reflection on them. Factoring by $u_1$ we find an orbifold in $\mathcal{O}_{i-1}$ with singularity graph which is either of Type IIIB or IIIC or inadmissible. The case that the singularity graph is of Type IIIB or IIIC is impossible because this orbifold is a 2-fold covering of $\tilde{O}$ and it can not have a inadmissible singularity graph (see the list at the beginning of this section).

The only possible left case is that both $\tilde{c}_2$ and $\tilde{c}_3$ are connected. So the singularity graph of $\tilde{O}'$ has two components and both $u_1$ and $u_2$ acts as reflections on each component. This case is impossible because, in this case, the quotient $\tilde{O}'/D$ which is homeomorphic to $\tilde{O}$ has a singularity graph of Type IIIB or IIIC and not of Type X, a contradiction.

This finishes the proof of the Claim.

References


HYPERBOLIC 2-FOLD BRANCHED COVERINGS OF LINKS


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ON THE EXISTENCE OF INFINITELY MANY ESSENTIAL SURFACES OF BOUNDED GENUS

ULRICH OERTEL

A theorem of William Jaco and Eric Sedgwick states that if \( M \) is an irreducible, \( \partial \)-irreducible 3-manifold with boundary a single torus, and if \( M \) contains no genus one essential (incompressible and \( \partial \)-incompressible) surfaces, then \( M \) cannot contain infinitely many distinct isotopy classes of essential surfaces of uniformly bounded genus. The main result in this paper is a generalization: If \( M \) is an irreducible \( \partial \)-irreducible 3-manifold with boundary, and \( M \) contains no genus one or genus zero essential surfaces, then \( M \) cannot contain infinitely many isotopy classes of essential surfaces of uniformly bounded genus.

1. Introduction.

In this paper, a Haken manifold is an orientable, irreducible, \( \partial \)-irreducible 3-manifold containing a (2-sided) incompressible surface. Even if a 3-manifold is not Haken, we shall always, for simplicity, assume that it is orientable. An irreducible, \( \partial \)-irreducible manifold \( M \) is simple if it contains no incompressible, \( \partial \)-incompressible tori or annuli. We say a properly embedded 2-sided surface \( S \hookrightarrow M \) is essential if it is incompressible and \( \partial \)-incompressible and not a sphere or disc. In this paper all essential surfaces will be 2-sided and embedded, and it will be understood that “essential surface” means “isotopy class of an essential surface.” The following is a well-known result attributed to Wolfgang Haken, see [7].

**Theorem 1.1 (W. Haken).** Let \( M \) be a simple Haken manifold, then \( M \) cannot contain infinitely many essential surfaces of uniformly bounded Euler characteristic \( \chi \).

Here, in the introduction, we give sketchy explanation of the proof. For simplicity, we assume \( M \) is closed. The proof uses normal surface theory. There are many sources: The original sources are [9] and [4]; an early source recommended for readability is [14]. For a more modern version in terms of triangulations, see for example [8] or [5]. To any given essential surface \( S \) in a closed Haken 3-manifold \( M \) equipped with a triangulation, one can associate an Euler ratio \(|\chi(S)|/\text{Area}(S) = \mathcal{R}(S)|. To measure area here,
we first note that the surface $S$ can be isotoped to make it a normal surface relative to the triangulation; the area is then the area of the least area normal representative in the isotopy class of $S$, where the area is the usual the combinatorial measure of area defined in normal surface theory, i.e., the number of intersections of the surface with the 1-skeleton of the triangulation. The Euler ratio then measures the average “curvature per unit area” in the surface.

Now, given infinitely many distinct normal surfaces $S_i$, their areas must be unbounded, since there are just finitely many normal surfaces of a given area. If the Euler characteristics of these surfaces are uniformly bounded, $|\chi(S_i)| \leq U$ for all $i$, then the Euler ratios $R(S_i)$ are not bounded away from 0. After passing to a subsequence, $R(S_i) \to 0$. There is a way of taking limits of (normalized) subsequences of the sequence $\{S_i\}$, which we shall describe in a later section. The limit is an embedded, measured, zero-Euler-characteristic lamination, which can be approximated by an embedded incompressible torus. The method of taking limits is well-known, see for example [10], [13], [11]. This completes the sketch of the proof of the Haken result stated above.

We observe that the Euler ratio is closely related to the “isoperimetric ratio,” see for example [11]. For a null homotopic curve $\gamma$ in $M$, the isoperimetric ratio is defined as $\text{Area}(D)/\text{Length}(\gamma)$ where $D$ is a least area null homotopy for $\gamma$ and length is also measured combinatorially. Spaces for which the isoperimetric ratios are uniformly bounded have negatively curved fundamental groups in the sense of Gromov, [3].

William Jaco and Eric Sedgwick have proved an interesting variation on the Haken finiteness theorem:

**Theorem 1.2** (Jaco-Sedgwick). Let $M$ be an irreducible, $\partial$-irreducible 3-manifold with boundary a single torus, and suppose $M$ contains no essential surfaces of genus 1, then $M$ cannot contain infinitely many essential surfaces of uniformly bounded genus.

We shall give an easy proof of this result using branched surface techniques and the limiting argument mentioned above.

We give a further generalization to arbitrary Haken manifolds with boundary:

**Theorem 1.3.** Let $M$ be an irreducible, $\partial$-irreducible 3-manifold, and suppose $M$ contains no essential surfaces of genus 1 or genus 0, then $M$ cannot contain infinitely many essential surfaces of uniformly bounded genus.

Returning to the original Haken finiteness theorem, we shall prove, as we have already mentioned, that in the case of a simple 3-manifold $M$ there is a constant $K$ such that any essential surface $S \hookrightarrow M$ satisfies $|\chi(S)| \geq K\text{Area}(S)$. “Large area surfaces have large topology.” This statement implies Theorem 1.1 and is only slightly stronger. When we measure
the area of an essential surface, we assume, as usual, that we have chosen a
triangulation for $M$ and that $\text{Area}(S)$ means the area of a least area normal
surface in the isotopy class of $S$.

In view of the Jaco-Sedgwick result, one might then ask whether a similar
result is true in a manifold $M$ with boundary one torus and no genus one
essential surfaces, whether “large area connected essential surfaces have large
genus.” This is true. We will prove the following stronger version of the Jaco-
Sedgwick theorem. Given an essential surface $S \hookrightarrow M$, a natural modified
Euler ratio is $\overline{R} = |\chi(S)|/\text{Area}(S)$, where $\overline{S}$ is obtained from $S$ by capping
all boundary components with discs.

**Theorem 1.4.** Let $M$ be a Haken 3-manifold with boundary a single torus
and with a given triangulation. Suppose $M$ contains no genus one essential
surfaces. Then there is a uniform lower bound $K > 0$ for the Euler ratio
$\overline{R}(S)$ on all essential surfaces $S$.

We also have a slightly stronger version of Theorem 1.3:

**Theorem 1.5.** Let $M$ be a Haken 3-manifold and with a given triangula-
tion. Suppose $M$ contains no genus one or genus zero essential surfaces.
Then there is a uniform lower bound $K > 0$ for the Euler ratio $\overline{R}(S)$ on all
essential surfaces $S \hookrightarrow M$.

2. Preliminaries.

We give here a brief informal overview of normal surfaces and incompressible
or essential branched surfaces. There are numerous sources for these related
subjects. We note that the earlier versions of normal surface theory were
described in terms of handle decompositions, whereas the currently used
normal surface theory is in terms of triangulations. In fact, the latter can
be regarded as a special case of the former. For normal surface theory, see
for example [4], [12], [5]; for branched surfaces as applied to incompressible
surfaces and essential laminations see [1], [12], [2].

The basic result in normal surface theory says that any essential surface
$S$ in a Haken 3-manifold $M$ can be isotoped to be normal relative to a tri-
angulation. The same result applies to other essential surfaces, for example
to essential spheres in reducible 3-manifolds, but we will not need these re-
results. To say that an embedded surface is normal means that it intersects
each 3-simplex of the triangulation in arbitrarily many discs, each combina-
torially isomorphic to one shown in Figure 1. For each 3-simplex there are
seven types of discs as in Figure 1 which are known as disc types, so that
for a triangulation with $n$ 3-simplices, there are $7n$ disc types. Typically
one goes further and chooses a normal representative of least area for the
isotopy class of $S$, where $\text{Area}(S)$ is the number of intersections with the
1-skeleton of the triangulation.
The transition to branched surfaces is made as follows. Given an essential normal surface of least area in its isotopy class, we push together adjacent discs of the same disc type and identify them. At the boundary of the region where several discs are identified, branching may occur, and we may suppose that the identification space, a branched surface $B \hookrightarrow M$, has a smooth structure which pulls back to a smooth structure on the surface $S$. The locus of points where $B$ is not a manifold is called the branch locus of $B$. The completion of a component of the complement in $B$ of the branch locus is called a sector of $B$. The branched surface just described is called a normal branched surface, since it intersects 3-simplices in normal discs. It has non-generic branch locus. One can perturb the branch locus by increasing slightly, in a generic manner, the area of identification, and the result is a branched surface with generic branch locus. A branched surface with generic branch locus is locally modelled as shown in Figure 2. It has a
fibered neighborhood $N(B)$ modelled as shown in Figure 2, which is foliated by interval fibers. The frontier of $N(B)$ is partitioned into the horizontal boundary $\partial_h N(B)$ and the vertical boundary $\partial_v N(B)$ as shown. There is a similar fibered neighborhood for a branched surface with non-generic branch locus. Clearly, there is a projection map $\pi : N(B) \to B$ which collapses each fiber. A surface is carried by $B$ if it can be embedded in $N(B)$ transverse to fibers, and it is fully carried if it also intersects every fiber of $N(B)$.

Figure 3.

If $S$ is carried by $B$, it induces a weight vector on $B$. To each sector $Z_i$ of $B$ the weight vector assigns the weight $w_i \geq 0$ equal to the number of intersections of $S$ with a fiber $\pi^{-1}(p)$, where $p$ is any point in $Z_i$. Furthermore, the weights satisfy branch equations as indicated in Figure 3. Any weight vector on $B$ satisfying all the branch equations for a branched surface $B$ yields an invariant weight vector on $B$. If the entries of an invariant weight vector $w$ are integers, it is not hard to show that the weight vector determines a surface which we denote $B(w)$. Invariant weight vectors with rationally related weights give weighted surfaces. In general, an invariant weight vector $w$ determines a measured lamination $B(w)$, which we will not describe here in detail. One can think of a measured lamination simply as branched surface with an invariant weight vector. There is an Euler characteristic associated to a measured lamination $B(w)$. Assuming the branched surface is generic, we first assign an Euler characteristic $z_i$ to each sector $Z_i$. This is not the Euler characteristic of $Z_i$ as a compact surface; instead, letting $|Z_i|$ denote the underlying topological surface, $z_i = \chi(Z_i) = \chi(|Z_i|) - (1/4)k_i$, where $k_i$ is the number of “corners” of the sector. The corners can be seen in the local model of Figure 2. Now, we define the Euler characteristic of a measured lamination $B(w)$ as $\chi(B(w)) = \sum_i z_i w_i$. This coincides with the usual Euler characteristic of a surface when $w$ is an invariant weight vector with integer entries, see [11].

Given a branched surface $B \hookrightarrow M$ with $s$ sectors, the set of all invariant weight vectors gives a weight cone in $\mathbb{R}^s$ which we denote $C(B)$. It is a cone in the first orthant bounded by finitely many hyperplanes. If we normalize
invariant weight vectors, requiring for example the sum of entries to equal 1, then we obtain the weight cell which we denote $\mathcal{M}(B)$. Later in the paper, we will often normalize a weight vector $w$ with integer entries on a normal branched surface by dividing by the area $\text{Area}(B(w))$. The normalized weight vectors obtained in this way also lie in a compact set homeomorphic to $\mathcal{M}(B)$. This is because $\sum w_i$ differs from $\text{Area}(B(w))$ only by positive scalar multiples, i.e., there exists a constant $C$ such that

$$\frac{1}{C} \sum w_i \leq \text{Area}(B(w)) \leq C \sum w_i.$$ 

It turns out that the branched surfaces obtained by the construction described above satisfy certain conditions which ensure that any surface carried by the branched surface is essential. We do not need to describe here the exact technical conditions the branched surface satisfies, but we will call a branched surface satisfying these conditions essential. The conditions are those given in [2] to define essential branched surfaces, though essential branched surfaces are there defined for a larger class of 3-manifolds than the class of Haken manifolds. They are also the conditions given in [12] to define Reebless incompressible branched surfaces.

**Theorem 2.1** ([12]). Let $M$ be Haken 3-manifold. Then there is a finite collection of essential normal branched surfaces such that every essential surface is fully carried by one of the branched surfaces of the collection as a least area surface.

**Theorem 2.2** ([12]). Let $M$ be an irreducible, $\partial$-irreducible, 3-manifold and let $B \hookrightarrow M$ be an essential branched surface in $M$. Then any surface carried by $B$ is essential.

### 3. Proofs.

We shall begin with a proof of the following, which easily implies Theorem 1.1.

**Theorem 3.1.** Let $M$ be a Haken 3-manifold, equipped with a triangulation. Suppose $M$ contains no essential annulus or torus. Then there is a uniform lower bound $K > 0$ for the Euler ratio $R(S)$ on all essential surfaces $S$.

**Proof.** Suppose there is no uniform lower bound for $R(S)$, and let $\{S_i\}$ be a sequence of surfaces such that $R(S_i) \to 0$. Without loss of generality in view of Theorem 2.1, we may pass to a subsequence and assume that all surfaces $S_i$ are fully carried by one essential branched surface $B \hookrightarrow M$. The proof of the Theorem 2.1, see [12], shows that $B$ can be taken to be a normal branched surface, and we can assume that each $S_i$ carried by $B$ is a least area normal representative of the isotopy class of $S_i$. Each surface $S_i$ determines a weight vector $w(S_i)$ on $B$, which we normalize by dividing by $\text{Area}(S_i)$ to obtain $w_i = w(S_i)/\text{Area}(S_i)$. These vectors are contained
in a compact subset of the cone of weight vectors \( \mathcal{C}(B) \), hence we may pass to a subsequence and assume \( w_i \to w \). The weight vector \( w \) determines a measured lamination \( B(w) \). Calculating Euler characteristic, which is a linear function on \( \mathcal{C}(B) \), we get

\[
\chi(B(w)) = \lim_{i \to \infty} \frac{\chi(S_i)}{\text{Area}(S_i)} = 0.
\]

Thus \( w \) is contained in the subset of \( \{ v \in \mathcal{C}(B) : \chi(B(v)) = 0 \} \). The equation \( \chi(v) = 0 \) can be written as a linear equation with integer coefficients in the components \( v_j \) of \( v \), hence there are integer solutions if there are any solutions. We can thus replace \( w \) by an integer weight vector \( \bar{w} \) with \( \chi(B(\bar{w})) = 0 \), and this represents a union of annuli and tori. The annuli and tori are essential by Theorem 2.2, so we have proved the theorem by contradiction. \( \square \)

Next we give a proof of the Theorem 1.4, which implies the theorem of Jaco-Sedgwick. The theorem concerns an irreducible, \( \partial \)-irreducible 3-manifold \( M \) whose boundary is a single torus. As in the introduction, we let \( \mathcal{S} \) denote the surface obtained by capping every boundary component of \( S \) by a disc, and we let \( \mathcal{R}(S) \) denote \( |\chi(\mathcal{S})|/\text{Area}(S) \).

**Proof of Theorem 1.4.** Suppose there is no uniform lower bound for the Euler ratio \( \mathcal{R}(S) \) of essential surfaces \( S \). Let \( \{ S_i \} \) be a sequence of surfaces such that \( \mathcal{R}(S_i) \to 0 \). Without loss of generality in view of Theorem 2.1, we may pass to a subsequence and assume that all surfaces \( S_i \) are fully carried by one essential branched surface \( B \hookrightarrow M \). As before, \( S_i \) can be taken to be least area normal surfaces carried by the normal branched surface \( B \). From Allen Hatcher’s result on boundary slopes, \([6]\), we know that all surfaces fully carried by \( B \) and with non-empty boundary, have boundary curves of the same slope, \( r \) say. Let \( \mathcal{M} \) denote the closed manifold obtained from \( M \) using slope \( r \) Dehn filling. Then each \( S_i \) can be capped in \( \mathcal{M} \) by a disc. Furthermore, one can construct a branched surface \( \mathcal{B} \) by attaching a meridian disc of the surgery solid torus to \( \partial B \). Note that the attaching map of the disc to \( \partial B \) is in general far from an embedding, and there can be highly non-generic branching where the disc is attached. Note also that we do not claim that \( \mathcal{B} \) is essential, though it might be.

Now, as in the previous proof, we use normalized weight vectors \( w_i = w(S_i)/\text{Area}(S_i) \). Clearly each \( w_i \) determines a weight vector \( \bar{w}_i \) on \( B \) which is the normalized weight vector \( \bar{w}_i = w(S_i)/\text{Area}(S_i) \). As in the previous proof, the weights \( \bar{w}_i \) are contained in a compact subset of \( \mathcal{C}(\mathcal{B}) \), and so after passing to a subsequence, we may assume \( \bar{w}_i \to \bar{w} \). As before, we calculate \( \chi(B(\bar{w})) = 0 \), and again we can replace \( \bar{w} \) by a projectively near weight vector \( \bar{w} \) with integer entries. Then \( \mathcal{B}(\bar{w}) \) is a union of tori \( \mathcal{T} \) and
$T = T \cap M$ is a union of genus one essential surfaces. The surface $T$ is carried by $B = B \cap M$, and is therefore essential by Theorem 2.2. \qed

Proof of Theorem 1.5. Suppose the theorem is false, and that there is a sequence of surfaces $\{S_n\}$ such that $R(S_n) \to 0$. Without loss of generality, in view of Theorem 2.1, we may pass to a subsequence and assume that all surfaces $S_n$ are fully carried by one essential branched surface $B \hookrightarrow M$. As usual, $S_n$ can be taken to be least area normal surfaces carried by the normal branched surface $B$.

In this proof we cannot appeal to Hatcher’s theorem on boundary slopes, therefore the analysis is more delicate. Let

$$M_n = \text{Max}\{\text{Length}(b) : b \text{ is a boundary component of } S_n\}$$

$$m_n = \text{Min}\{\text{Length}(b) : b \text{ is a boundary component of } S_n\}.$$

We begin with easier cases before considering the most general case.

Case 1. There is a uniform bound on $M_n$, so that $M_n \leq U$ for all $n$. Let $B(S_n)$ be the set of normal curves in $\partial M$ which occur in $\partial S_n$. Then there are just finitely many possibilities for $B(S_n)$ as $n$ is allowed to vary. Thus we can pass to a subsequence of $\{S_n\}$ and assume that $B(S_n) = B$ is the same for all $n$. The set $B$ can be regarded as an embedded curve system in $\partial M$ carried by $\partial B$ with no two closed curves of $B$ isotopic in $\partial M$. We can now construct a branched surface $\overline{B}$ with one disc appropriately attached for each curve in $B$. In fact, $\overline{B}$ is embedded in a 3-manifold $\overline{M}$ obtained by attaching one 2-handle along $b$ for each curve $b \in B$. The situation is now very similar to the situation in the proof of Theorem 1.4. Taking $A_n = \text{Area}(S_n)$, and passing to a subsequence, we take a limit of the sequence $\{S_n/A_n\}$ in $\overline{B}$, and the assumption that $R(S_i) \to 0$ ensures that the limiting weight vector has Euler characteristic 0 and can be approximated by a surface $T$ which is a union of tori. Again, the intersection with $M$ must be a union of genus 1 surfaces $T$, and these are essential because they are carried by an essential branched surface. This contradicts the hypotheses of the theorem.

Case 2. The set $\{m_n : n \geq 1\}$ is unbounded. This is, in some sense, the opposite of the previous case. After passing to a subsequence of $\{S_n\}$, we may assume that $m_n \to \infty$. We will again take a limit of a subsequence of $\{S_n/A_n\}$, but this time in the branched surface $B$. The limit will again be a measured lamination carried by $B$ (or a limiting weight vector on $B$). Each surface $S_n/A_n$ induces a weight vector on $B$, and the weight vectors are contained in a compact subset of $\mathcal{C}(B)$. In this case, we compare $\chi(S_n/A_n)$ to $\chi(S_n/A_n)$, where $\overline{S_n}$ is again the surface with all boundary components capped by discs. (Here the $\overline{S_n}$’s cannot be embedded in any manifold $\overline{M}$.)

Since we have normalized, the length of the weighted boundaries $\partial S_n$ are
uniformly bounded, say $\text{Length}(\partial S_n/A_n) = (1/A_n)\text{Length}(\partial S_n) \leq R$ for some constant $R$. Now we conclude that

$$|\chi(\mathcal{S}_n/A_n) - \chi(S_n/A_n)| \leq R/m_n \to 0.$$ 

To see this we note that $\mathcal{S}_n/A_n$ is obtained from $S_n/A_n$ by attaching at most a weight $\text{Length}(S_n/A_n)/m_n$ of discs. This attachment can change the Euler characteristic by at most $R/m_n$ which approaches 0. Thus $\chi(S_n/A_n) \to 0$, since clearly $\chi(S_n/A_n) \to 0$. It follows that when we find a limiting lamination by passing to a subsequence and taking limits of weights induced by $S_n/A_n$ on $B$, we obtain a measured lamination of Euler characteristic 0 in $M$, which can be approximated by a union of tori and annuli. This contradicts the hypotheses of the theorem.

**Case 3. General case.** We have an arbitrary sequence $\{S_n\}$ with $\mathcal{K}(S_i) \to 0$. If some normal curve $b_1$ occurs as a component of $\partial S_n$ for infinitely many $S_n$, we pass to a subsequence such that every $S_n$ has a boundary component $b_1$. In the new sequence $\{S_n\}$, if some normal curve $b_2 \neq b_1$ is a component of $\partial S_n$ for infinitely many $n$, we again pass to a subsequence such that $b_1$ and $b_2$ occur in $\partial S_n$ for every $n$. Repeating this argument, we must end with $b_1, \ldots, b_k$ occurring in $\partial S_n$ for all $n$, and with no other normal curve occurring infinitely often.

Let $\hat{M}$ be the manifold obtained by attaching $k$ 2-handles along the curves $b_1, \ldots, b_k$, and let $\hat{S}_n$ denote the surface obtained from $S_n$ by capping all curves isotopic to $b_i$ by discs in $\hat{M}$. We can assume that all the surfaces $\hat{S}_n$ are carried by one branched surface $\hat{B}$, obtained from $B$ by attaching discs using the carrying maps of $b_i$ as attaching maps, as in Case 2. Now define

$$m_n = \min\{\text{Length}(b) : b \text{ is a boundary component of } S_n, b \neq b_1, \ldots, b_k\},$$

or

$$m_n = \min\{\text{Length}(b) : b \text{ is a boundary component of } \hat{S}_n\}.$$ 

Clearly $m_n \to \infty$.

As in Case 2, we construct a limiting measured lamination $(\hat{L}, \hat{\mu}) \hookrightarrow \hat{M}$ from $\hat{S}_n$ and we show as in Case 2 that $\chi((\hat{L}, \hat{\mu})) = 0$. The measured lamination can be approximated by a surface $\hat{S}$ which is a union of annuli and tori in $\hat{M}$. Then we obtain a surface $S = \hat{S} \cap M$ carried by $B$ whose components have genus 1 or genus 0. The surface $S$ is essential because $B$ is essential. Again, this contradicts the hypotheses of the theorem. □

**References**


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In this paper we present some new properties of the metric dimension defined by Bouligand in 1928 and prove the following new projection theorem: Let $\dim_b(A - A)$ denote the Bouligand dimension of the set $A - A$ of differences between elements of $A$. Given any compact set $A \subseteq \mathbb{R}^N$ such that $\dim_b(A - A) < m$, then almost every orthogonal projection $P$ of $A$ of rank $m$ is injective on $A$ and $P|_A$ has Lipschitz continuous inverse except for a logarithmic correction term.

1. Introduction.

What we shall call Bouligand dimension is the dimensional order defined by Bouligand in [4] and further generalized by Assouad much later in [1] and [2]. In this paper we prove several results about Bouligand dimension and its relation to the Mañé type projection theorems of [16], [3] and [10]. The use of Bouligand dimension in studying projections was initiated by Movahedi-Lankarani in [18] where he constructs a set $A$ with finite fractal dimension for which there are no finite rank projections $P$ with $P|_A$ having Lipschitz continuous inverse. Here $A$ is a subset of a Hilbert space $H$. To do this, he exhibits a set with finite fractal but infinite Bouligand dimension. He then raises the question: What can happen in the case that the Bouligand dimension of $A$ is finite? This paper will attempt to shed some light on this question.

When $A$ is an attractor for a partial differential equation, knowing whether $A$ may be embedded into a finite dimensional space is of theoretical and computational interest. Work along these lines include Eden, Foias, Nicolaenko and Temam [8] on exponential attractors, Sauer, Yorke and Casdagli [23] on delay-coordinate maps, and Robinson [21], [20] on approximate attractors. Ideally, given an infinite dimensional dynamical system, we would like to construct a finite dimensional dynamical system that has the same long-term behavior. This is important, in particular, because any numerical simulation of the dynamics on a computer is by necessity finite dimensional. Since the long-term behavior of a dynamical system is largely governed by its global attractor, this question may be rephrased as whether it is possible to
construct a finite dimensional dynamical system with the same global attractor as the original dynamical system. To construct such a system requires embedding the attractor of the original infinite dimensional dynamical system into a finite dimensional space. This embedding should possess certain continuity properties so as to preserve the original dynamics.

Inertial manifolds, as discussed by Constantin in [5] and Constantin, Foias, Nicolaenko and Temam in [6] and [7], provide a bi-Lipschitz embedding of the global attractor into a finite dimensional space. As the Bouligand dimension of an attractor is preserved under bi-Lipschitz mappings, the following holds:

**Fact 1.1.** The Bouligand dimension of the global attractor of any dynamical system possessing an inertial manifold must be finite.

Thus, there exist a number of global attractors with finite Bouligand dimension. For example, the Kuramoto-Sivashinsky equation, the Kolmogorov-Sivashinsky equation and the Ginzburg-Landau equation in one space dimension [7], reaction diffusion equations in higher space dimensions [15], and nonlinear viscoelasticity equations [19] have inertial manifolds and therefore global attractors with finite Bouligand dimension. It is not known if there exists an inertial manifold for the Navier-Stokes equations in two space dimensions. In light of the main result in this paper, computing the Bouligand dimension of the global attractor directly and more specifically the Bouligand dimension of the set \( A - A \) of differences between elements of \( A \) is of great interest. Our main result is:

**Theorem 1.2.** Given \( A \subseteq \mathbb{R}^N \) such that \( \dim_b(A - A) < m \), then almost every orthogonal projection \( P \) of \( A \) of rank \( m \) is injective on \( A \) and \( P|_A \) has Lipschitz continuous inverse except for a logarithmic correction term.

This shows that almost every orthogonal projection of \( A \) has the same Lipschitz properties that the embeddings of an inertial manifold have except for a logarithmic correction. Here, almost every should be understood in terms of the Haar measure invariant with respect to orthogonal transformations on the space of all orthogonal projections of rank \( m \) in \( \mathbb{R}^n \). This provides a partial converse to Fact 1.1. Note that \( n \) may be chosen arbitrarily large. Thus, Theorem 1.2 embeds a fractal subset \( A \) of a large finite dimensional space into a smaller finite dimensional space whose dimension is controlled explicitly by the Bouligand dimension of \( A - A \). Our theorem is related to a result stated by Mañé in [16] which states under the assumption of finite Hausdorff dimension that the injective projections form residual set.

**Definition 1.3.** The Hausdorff dimension of the set \( A \) is defined by

\[
\dim_h(A) = \inf\{d : \mathcal{H}^d(A) = 0\} = \sup\{d : \mathcal{H}^d(A) = \infty\}
\]
where $\mathcal{H}^d$ is the $d$-dimensional Hausdorff measure
\[
\mathcal{H}^d(A) = \liminf_{\delta \to 0} \left\{ \sum_{i=1}^{\infty} |U_i|^d : \{U_i\} \text{ is a } \delta \text{-cover of } A \right\}.
\]
Here $|U_i| = \sup \{|x - y| : x, y \in U_i\}$ is the diameter of the set $U_i$ and a $\delta$-cover is a cover $\{U_i\}$ such that $|U_i| \leq \delta$ for all $i$.

**Definition 1.4.** A set is called residual if its complement is a set of first category; a set of first category is a countable union of nowhere dense sets; and a nowhere dense set is a set whose closure has no interior points.

**Theorem 1.5** (Mañé). If $E$ is a Banach space and $A \subset E$ is a countable union of compact subsets with $\dim_h(A) < \infty$ then for every subspace $B$ of $E$ with $\dim_h(A - A) + 1 < \dim(B) < \infty$ the set of projections $P : A \to B$ such that $P|A$ is injective is a residual subset of the space of projections of $A$ onto $B$ endowed with the norm topology.

Hölder continuity for the inverse in Mañé’s projection theorem has been proved under the hypothesis of finite fractal dimension by Ben-Artzi, Eden, Foias, and Nicolaenko in [3], and extended by Foias and Olson in [10].

**Definition 1.6.** Let $\mathcal{N}_A(\rho)$ be the minimum number of $\rho$-radius balls required to cover all of $A$. Then, the fractal dimension
\[
\dim_f(A) = \limsup_{\rho \to 0} \frac{\log \mathcal{N}_A(\rho)}{\log \rho}.
\]

**Theorem 1.7** (Foias and Olson). Let $H$ be a real Hilbert space and $A \subseteq H$ be such that $2 \dim_f(A) < m$. Then for any orthogonal projection $P$ of rank $m$ and $\delta > 0$ there is an orthogonal projection $\tilde{P}$ such that $\|\tilde{P} - P\| < \delta$ and $\tilde{P}|_A$ has Hölder inverse.

Hunt and Kaloshin in [13] have recently shown that such projections $\tilde{P}$ are in fact prevalent according to the sense of prevalence given by Hunt, Sauer and Yorke in [14]. Prevalence extends the notion of Lebesgue almost every from Euclidean spaces to infinite-dimensional spaces. In particular:

**Definition 1.8.** A Borel subset $S \subseteq B$ is prevalent if there exists a compactly supported probability measure $\mu$ such that $\mu(S + x) = 1$ for all $x \in B$. A set, in general, is prevalent if it contains a prevalent Borel set.

Note that Theorem 1.7 is stated with a hypothesis only on the fractal dimension of $A$. This simplification may be made because the fractal dimension of $A - A$ is controlled by the inequality $\dim_f(A - A) < 2 \dim_f(A)$. Such inequalities are not true for the Bouligand and Hausdorff dimensions. Therefore, Theorems 1.2 and 1.5 are stated under hypotheses about the set $A - A$ directly. In this paper we give an example for Bouligand dimension
for which $\dim_b(A) = 0$ and $\dim_b(A - A) = \infty$ to demonstrate this dramatic difference.

We also show the Bouligand dimension of a set may increase under an orthogonal projection. Note that for the fractal and Hausdorff dimensions, the dimension of the projected image is always less than or equal the dimension of the original set. Thus, Bouligand dimension gives us further insight over what can happen to the geometry of a set under orthogonal projection. On the other hand, there may be bad projections that make a global attractor appear more complicated than it really is.

This paper is organized as follows: First, we define Bouligand dimension and give an alternate characterization of it. In Section 3 we state some properties and show that Bouligand dimension agrees with other notions of dimension for self-similar fractals. In Section 4 we prove Theorem 4.7 and in Section 5 we prove Theorem 1.2.

2. The Bouligand dimension.

In this section we define the fractal and Bouligand dimensions. Fractal dimension is sometimes also called the box-counting dimension, the capacity, or the Minkowski dimension. We have already defined this dimension in the introduction via Definition 1.6. An equivalent characterization of fractal dimension in terms of an infimum follows easily from the definitions of infimum and limit superior.

**Fact 2.1.** The fractal dimension $\dim_f(A)$ is the infimum over all $d$ for which there exists $K$ such that

$$N_A(\rho) \leq K(1/\rho)^d \quad \text{for } 0 < \rho < 1.$$  \hspace{1cm} (2.1)

We now define Bouligand dimension. As already mentioned this dimension is the generalization of the dimensional order of Bouligand [4] discussed by Assouad in [1] and [2] and by Movahedi-Lankarani in [18].

**Definition 2.2.** The Bouligand dimension $\dim_b(A)$ is defined as

$$\dim_b(A) = \lim_{\epsilon \to 0} \lim_{t \to \infty} \Delta_{\epsilon,t}(A)$$

where

$$\Delta_{\epsilon,t}(A) = \sup \left\{ \frac{\log N(r,\rho)}{\log(r/\rho)} : 0 < \rho < r < \epsilon \text{ and } r > t\rho \right\}$$

and $N(r,\rho)$ is the number of $\rho$-balls required to cover any $r$-ball in $A$.

The Bouligand dimension may be characterized as an infimum in a way similar to that of the fractal dimension. This is essentially the definition given by Assouad in [1]. This characterization highlights a scaling condition that will be important later. In particular, given $0 < \lambda \leq 1$ the Bouligand dimension requires the number of $\rho$ balls needed to cover an $r$ ball portion
of the set should be essentially the same as the number of \( \lambda \rho \) balls needed to cover a \( \lambda r \) ball portion.

**Theorem 2.3.** The Bouligand dimension \( \dim_b(A) \) is the infimum over all \( d \) for which there exists \( K \) such that

\[
N_A(r, \rho) \leq K(r/\rho)^d \quad \text{for} \quad 0 < \rho < r < 1.
\]

First note that the exact form of the upper bound for \( r \) in condition (2.2) is not critical. Equivalently we may require for some \( \epsilon > 0 \) that there exists \( K_\epsilon \) such that

\[
N_A(r, \rho) \leq K_\epsilon(r/\rho)^d \quad \text{for} \quad 0 < \rho < r < \epsilon.
\]

It is obvious that if \( \epsilon < 1 \), then condition (2.2) implies (2.3) with \( K_\epsilon = K \). Conversely suppose (2.3) holds for some \( 0 < \epsilon < 1 \). Since \( A \) is compact, then a finite number \( N \) of \( \epsilon/4 \) balls will cover it. Let \( r \) and \( \rho \) be such that \( 0 < \rho < r < 1 \). Consider the following cases:

Case \( \rho \geq \epsilon/2 \). Then

\[ N_A(r, \rho) \leq N_A(r, \epsilon/2) \leq N_A(\epsilon/4) \leq N. \]

Case \( \rho < \epsilon/2 \leq r \). Then

\[ N_A(r, \rho) \leq N_A(r, \epsilon/2) N_A(\epsilon/2, \rho) \leq N K_\epsilon \left( \frac{\epsilon/2}{\rho} \right)^d \leq N K_\epsilon \left( \frac{r}{\rho} \right)^d. \]

Thus taking \( K = N K_\epsilon \) we obtain that

\[ N_A(r, \rho) \leq K \left( \frac{r}{\rho} \right)^d \quad \text{for} \quad 0 < \rho < r < 1. \]

In the situation where \( \epsilon > 1 \) the argument is similar. Moreover, if there exists one \( \epsilon \) and \( K_\epsilon \) for which condition (2.3) holds, then for each \( \epsilon \) there is a \( K_\epsilon \) such that it holds.

**Proof of Theorem 2.3.** Suppose there is \( d \) and \( K \) such that (2.2) holds. Then

\[
\frac{\log N_A(r, \rho)}{\log(r/\rho)} \leq d + \frac{\log K}{\log(r/\rho)} \quad \text{for} \quad 0 < \rho < r < 1.
\]

It follows that

\[ \Delta_{1,t}(A) \leq d + \frac{\log K}{\log(t)} \]

and therefore

\[ \dim_b(A) = \lim_{\epsilon \to 0} \lim_{t \to \infty} \Delta_{1,t}(A) \leq d. \]
Conversely, suppose $d$ is chosen so that $\dim_b(A) < d$. Then by definition there are values of $\epsilon$ and $t$ such that $0 < \epsilon < 1 < t$ and $\Delta_{\epsilon,t}(A) < d$. It follows that
\[
\frac{\log N_A(r,\rho)}{\log(r/\rho)} < d \quad \text{for } 0 < t\rho < r < \epsilon
\]
and therefore
\[
N_A(r,\rho) < (r/\rho)^d \quad \text{for } 0 < t\rho < r < \epsilon.
\]
Now, suppose $0 < \rho < r < \epsilon$. Then $0 < t(\rho/t) < r < \epsilon$ and
\[
N_A(r,\rho) \leq N_A(r,\rho/t) \leq \left(\frac{r}{\rho/t}\right)^d = K \left(\frac{r}{\rho}\right)^d \quad \text{for } 0 < \rho < r < \epsilon
\]
where $K = t^d$. It follows that $\dim_b(A)$ is the infimum over all $d$ for which (2.2) holds. $\square$

It is often of interest to know whether a set has finite Bouligand dimension or not. The following lemma provides a simple test for determining this.

Lemma 2.4. If there exists $K$ such that $N_A(r,r/2) < K$ holds for all $r < 1$ then $A$ has finite Bouligand dimension. Moreover, $\dim_b(A) \leq \log_2 K$.

Proof. Given $r$ and $\rho$ such that $0 < \rho < r < 1$ choose $n$ so that $r/2^n \leq \rho < r/2^{n-1}$. Then
\[
N_A(r,\rho) \leq N_A(r,r/2)N_A(r/2,r/2^2)\cdots N_A(r/2^{n-1},r/2^n) \leq K^n
\]
and since $n - 1 \leq \log_2(r/\rho)$ it follows that
\[
N_A(r,\rho) \leq K(K^{n-1}) \leq K(r/\rho)^{\log_2 K}.
\]
Therefore $\dim_b(A) \leq \log_2 K < \infty$. $\square$


The Bouligand dimension satisfies many of the usual properties that a reasonable dimension should satisfy. In this section we state and prove a few of these properties. In particular, we prove the Bouligand dimension is well-behaved with respect to Cartesian products and that the Bouligand dimension agrees with the similarity dimension for self-similar fractals.

First, we will state for reference as Theorem 3.1 a few properties of the Bouligand dimension found in Movahedi-Lankarani [18] and Assouad [1].

Theorem 3.1. The Bouligand dimension has the following properties:

(i) If $A \subseteq B$ then $\dim_b(A) \leq \dim_b(B)$.
(ii) If $A$ is an open subset of $\mathbb{R}^N$ then $\dim_b(A) = N$.
(iii) If $A$ and $B$ are bi-Lipschitz isomorphic, then $\dim_b(A) = \dim_b(B)$.
We now discuss how the Bouligand dimension behaves with respect to Cartesian products. We would like
\[ \dim_b(A \times B) = \dim_b(A) + \dim_b(B). \]
However, as with any metric dimension, the Bouligand dimension depends on what metric is used. Therefore, care must be taken in choosing a metric. Let \( A \) and \( B \) be compact metric spaces with metrics \( d_A \) and \( d_B \). Define the metric spaces \( X_p = A \times B \) for \( 1 \leq p \leq \infty \) by the metrics \( d_p \) given by
\[
d_p((a_1, b_1), (a_2, b_2)) = \begin{cases} (d_A(a_1, a_2)^p + d_B(b_1, b_2)^p)^{1/p} & \text{for } 1 \leq p < \infty, \\ \max(d_A(a_1, a_2), d_B(b_1, b_2)) & \text{for } p = \infty. \end{cases}
\]

**Theorem 3.2.** \( \dim_b(X_p) = \dim_b(A) + \dim_b(B) \).

**Proof.** Since \( d_\infty \) is Lipschitz equivalent to \( d_p \) for \( 1 \leq p < \infty \), it follows from property (iii) of Theorem 3.1 that
\[ \dim_b(X_\infty) = \dim_b(X_p) \quad \text{for} \quad 1 \leq p < \infty. \]
Hence, we may prove the theorem for \( X = X_\infty \) without loss of generality.

For every \( \alpha \) and \( \beta \) such that \( \dim_b(A) < \alpha \) and \( \dim_b(B) < \beta \) there exists \( K_A \) and \( K_B \) such that
\[
N_A(r, \rho) < K_A \left( \frac{r}{\rho} \right)^\alpha \quad \text{and} \quad N_B(r, \rho) < K_B \left( \frac{r}{\rho} \right)^\beta
\]
for \( 0 < \rho < r < 1 \). Let \( B \) be a ball of radius \( r \) in \( X \). Then by definition, \( B = U \times V \) where \( U \) and \( V \) are balls of radius \( r \). Cover \( U \) by balls \( U_i \) of radius \( \rho \) in \( A \) and \( V \) by balls \( V_j \) of radius \( \rho \) in \( B \). Since \( U_i \times V_j \) form a cover of \( B \), it follows that
\[
N_X(r, \rho) \leq N_A(r, \rho)N_B(r, \rho) \leq K_AK_B \left( \frac{r}{\rho} \right)^{\alpha+\beta}.
\]
Hence \( \dim_b(X) \leq \alpha + \beta \) and so \( \dim_b(X) \leq \dim_b(A) + \dim_b(B) \).

Now, let \( n \) be the maximum number of disjoint \( \rho/2 \)-radius balls with centers \( u_i \) in \( U \). Let \( U_i \) be balls of radius \( \rho \) with the same centers. Thus the points \( u_i \) are at least a distance \( \rho/2 \) apart from each other and \( U_i \) covers \( U \).

Do the same for \( V \) to obtain \( m \) balls \( V_j \) of radius \( \rho \) whose centers \( v_j \) are at least a distance \( \rho/2 \) apart from each other which cover \( V \). Let \( z_k \in B \) be an enumeration of the points \( (u_i, v_j) \) and define \( Z = \{ z_k : k = 1 \ldots nm \} \). It follows that
\[
N_X(r, \rho/4) \geq N_Z(\rho/2) \geq nm \geq N_U(\rho)N_V(\rho).
\]
Choose \( U \) and \( V \) such that \( N_U(\rho) \) and \( N_V(\rho) \) attain their maximum values.

Given \( d \) such that \( \dim_b(X) < d \), there is \( K \) such that
\[
N_X(r, \rho) \leq K \left( \frac{r}{\rho} \right)^d \quad \text{for } 0 < \rho < r < 1.
\]
Thus
\[ N_A(r, \rho) N_B(r, \rho) \leq N_X(r, \rho/4) \leq 4^d K \left( \frac{r}{\rho} \right)^d, \]
and therefore by Theorem 2.3 \( \dim_b(A) + \dim_b(B) = \dim_b(X). \)

It is interesting to note that the Hausdorff dimension does not behave well under Cartesian products. For example, Falconer exhibits sets \( A \) and \( B \) in \([9]\) such that \( \dim_h(A) + \dim_h(B) < \dim_h(A \times B) \).

We end this section with a calculation of the Bouligand dimension for a self-similar fractal. In particular, we show the Bouligand dimension agrees with the similarity dimension for such sets, and therefore the fractal and Hausdorff dimensions as well. First note that the Hausdorff dimension is bounded by the fractal dimension which is in turn bounded by the Bouligand dimension. Thus given a compact metric space \( A \) we have that
\[
\dim_h(A) \leq \dim_f(A) \leq \dim_b(A). \tag{3.1}
\]

**Definition 3.3.** A transformation \( f: \mathbb{R}^n \to \mathbb{R}^n \) is said to be a similarity if there is a constant \( c \) such that \( |f(x) - f(y)| = c|x - y| \) for all \( x, y \) in \( \mathbb{R}^n \).

Let \( f_i: \mathbb{R}^N \to \mathbb{R}^N \), where \( i = 1, \ldots, L \) be similarities with ratios \( c_i \in (0, 1) \) and define \( F(M) = \bigcup f_i(M) \) for \( M \subseteq \mathbb{R}^N \). Let \( A \) be the compact invariant set such that \( F(A) = A \).

**Definition 3.4.** The similarity dimension of \( A \) is defined as \( \dim_s(A) = s \) where \( \Sigma c_i^s = 1 \).

**Theorem 3.5.** Let \( A \) be defined as above. Then \( \dim_b(A) = \dim_s(A) \).

**Proof.** Since \( A \) is compact there is \( \epsilon \) such that \( 0 < \epsilon < 1 \) and
\[
\text{dist}(f_i(A), f_j(A)) > \epsilon \quad \text{for} \quad i \neq j.
\]
Let \( c = \min \{c_i : i = 1, \ldots, L\} \). Let \( R > \epsilon/c \) be chosen so large that \( A \) fits inside an \( R \) ball. Let \( r \) and \( \rho \) be chosen so that \( 0 < \rho < r < \epsilon/2 \). Let \( I \) be the set of finite sequences \((i_1, \ldots, i_n)\) such that
\[
c_{i_1} \cdots c_{i_n} \epsilon \leq 2r < c_{i_1} \cdots c_{i_{n-1}} \epsilon.
\]
Since \( 2r < \epsilon \), then \( I \) is nonempty. Furthermore, \( A = \bigcup \{f_{i_1} \circ \cdots \circ f_{i_n}(A) : (i_1, \ldots, i_n) \in I\} \). Let \((i_1, \ldots, i_n)\) and \((j_1, \ldots, j_m)\) be distinct elements of \( I \). Suppose for definiteness that \( n \leq m \). Consider the sets \( f_{i_1} \circ \cdots \circ f_{i_n}(A) \) and \( f_{j_1} \circ \cdots \circ f_{j_m}(A) \). Let \( k \) be the largest index such that \( i_l = j_l \) for all \( l \leq k \). Obviously \( k < n \) for otherwise \( k = n \) would imply \( n < m \) leading to the contradiction
\[
c_{i_1} \cdots c_{i_n} \epsilon \leq 2r < c_{j_1} \cdots c_{j_{m-1}} \epsilon \leq c_{j_1} \cdots c_{j_n} \epsilon = c_{i_1} \cdots c_{i_n} \epsilon.
\]
Define $F = f_{i_1} \circ \cdots \circ f_{i_n}(A)$ and $G = f_{j_1} \circ \cdots \circ f_{j_m}(A)$. Then $i_{k+1} \neq j_{k+1}$ implies $\text{dist}(F, G) > \epsilon$. It follows that

$$\text{dist}(f_{i_1} \circ \cdots \circ f_{i_n}(A), f_{j_1} \circ \cdots \circ f_{j_m}(A)) > c_1 \cdots c_{n-1} \epsilon > 2r.$$ 

Now let $B$ be a ball of radius $r$. It follows that $B \cap f_{i_1} \circ \cdots \circ f_{i_n}(A) \neq \emptyset$ for at most one $(i_1, \ldots, i_n) \in I$. Let $(i_1, \ldots, i_n)$ be that one and fix it.

Hence $B \cap A = B \cap f_{i_1} \circ \cdots \circ f_{i_n}(A)$. Define $\gamma = c_1 \cdots c_n$ and let $J$ be the set of $(j_1, \ldots, j_m)$ such that

$$R_\gamma c_{j_1} \cdots c_{j_m} \leq \rho < R_\gamma c_{j_1} \cdots c_{j_{m-1}}.$$ 

Since $\rho < r \leq \gamma \epsilon / c \leq R_\gamma$, then $J$ is nonempty. Since

$$A = \bigcup \{f_{j_1} \circ \cdots \circ f_{j_m}(A) : (j_1, \ldots, j_m) \in J\}$$

it follows that

$$B \cap A \subseteq \bigcup \{f_{j_1} \circ \cdots \circ f_{j_m}(A) : (j_1, \ldots, j_m) \in J\}.$$ 

Furthermore, since $R_\gamma c_{j_1} \cdots c_{j_m} \leq \rho$, it follows that

$$f_{i_1} \circ \cdots \circ f_{i_n} \circ f_{j_1} \circ \cdots \circ f_{j_m}(A) \subseteq B_\rho(f_{i_1} \circ \cdots \circ f_{i_n} \circ f_{j_1} \circ \cdots \circ f_{j_m}(A)).$$

We now estimate the number of elements in $J$. Since $\sum c_i = 1$, it follows by induction that $\sum (c_j \cdots c_m)^s = 1$. Therefore $\sum (c \rho / (R_\gamma))^s \leq 1$ and so $J$ has no more than $(R_\gamma / (c \rho))^s$ elements. Hence, $N_{B \cap A}(\rho) \leq (R_\gamma / (c \rho))^s$. Since $\gamma = c_1 \cdots c_n \leq 2r / \epsilon$, it follows that $N_{A}(\rho, r / \epsilon) \leq K(r / \epsilon)^s$ where $K = (2R / (c \epsilon))^s$. As this holds for all $0 < \rho < r < \epsilon / 2$, it further follows that $\dim_b(A) \leq s$. By [9], Theorem 9.3, the Hausdorff dimension $\dim_h(A) = s$; therefore, in light of (3.1) we obtain that $\dim_b(A) = s$. \hfill \square

Thus, for self-similar sets, the Bouligand dimension agrees with the similarity, fractal, and Hausdorff dimensions. In the case that $c_i = c$ for all $i$, the similarity dimension has an easy to calculate form and we obtain the following corollary:

**Corollary 3.6.** Let $f_i : \mathbb{R}^N \to \mathbb{R}^N$ where $i = 1, \ldots, L$ be similarities with ratio $c \in (0, 1)$ and define $F(M) = \bigcup f_i(M)$ for $M \subseteq \mathbb{R}^N$. Take $A$ to be the compact invariant set such that $F(A) = A$. If the images under the $f_i$ are disjoint then $\dim_b(A) = -\log(L) / \log(c)$.

### 4. Dimension increasing projections.

The results in this section are motivated by examples involving orthogonal sequences in a Hilbert space. The fractal dimension of such sequences has been already studied by Ben-Artzi, Eden, Foias and Nicolaenko in [3] and by Ladislav Mišk Jr. and Tibor Žáčik in [17]. In some sense, an orthogonal sequence is the farthest thing possible from the regular self-similar fractals discussed in the previous section. Not surprisingly, it is in the treatment of...
these sets that the Bouligand dimension differs most dramatically from the fractal dimension. In particular, we exhibit a set $A$ such that $\dim_b(A) = 0$ and $\dim_b(A - A) = \infty$. This section closes with a proof of Theorem 4.7 concerning the existence orthogonal projections that increase the Bouligand dimension.

In Theorem 2.3 it was shown that the Bouligand dimension requires that the number of $\rho$ balls needed to cover an $r$ ball should be essentially the same as than the number of $\lambda \rho$ balls needed to cover a $\lambda r$ ball. The homogeneity of this scaling makes the Bouligand dimension sensitive to inhomogeneities in the set $A$. We will now consider a particularly inhomogeneous set: The closure of an orthogonal sequence converging to zero in a Hilbert space. Let $H$ be a Hilbert space and $e_i$ an orthonormal sequence.

**Lemma 4.1.** Let $A = \{0\} \cup \{e_n/n^\alpha : n \in \mathbb{N}\}$. Then $\dim_f(A) = 1/\alpha$ and $\dim_b(A) = \infty$.

**Proof.** The first fact appears in [3]. For the second, consider the ball of radius $r = 1/m^\alpha$ centered at the origin

$$B = \{a \in A : \|a\| < r\} = \{0\} \cup \{e_n/n^\alpha : n > m\}.$$ 

Cover $B$ by $r/2$ balls. Each point a distance more that $r/2$ from the origin will require a separate ball. Since

$$1/n^\alpha > r/2 \quad \text{implies} \quad n < (2/r)^{1/\alpha} = m2^{1/\alpha}$$

then

$$\mathcal{N}_A(r, r/2) \geq \mathcal{N}_B(r/2) > m2^{1/\alpha} - m - 1.$$ 

This is unbounded as $n \to \infty$; therefore, $\dim_b(A) = \infty$ by Lemma 2.4. \qed

The Bouligand dimension of a geometric sequence is finite because geometric sequences have the scaling property needed for $\mathcal{N}_A(r, r/2)$ in Lemma 2.4 to be bounded. Moreover, we have:

**Fact 4.2.** Let $A = \{0\} \cup \{a_n e_n : n \in \mathbb{N}\}$. If there exists $K$ and $\alpha$ such that $0 < \alpha < 1$ and

$$\frac{1}{K} \leq a_n \leq K \alpha^n,$$

then $\dim_b(A) = 0$.

The proof of this fact follows from arguments similar to those used in the proof of Lemma 4.1. For the fractal dimension it is shown in [3] that $a_n \leq K/n^\alpha$ implies $\dim_f(A) \leq 1/\alpha$. To see that the lower bound in (4.1) is required consider

**Theorem 4.3.** There exist sequences converging arbitrarily fast to zero that have infinite Bouligand dimension.
Proof. Let \( \{b_j\} \) be a sequence converging to zero. Consider the sequence \( \{a_n e_n\} \) where
\[
a_n = b_j \quad \text{for } n = 2^{-j-1}, \ldots, 2^j - 1.
\]
Let \( B \) be the ball in \( A = \{0\} \cup \{a_n e_n : n \geq 1\} \) of radius \( r = b_j + \epsilon \) centered at the origin. Thus
\[
B = \{a_n e_n : n \geq 2^j - 1\}.
\]
Cover \( B \) by \( r/2 \) balls. Then
\[
\mathcal{N}_A(r, r/2) \geq \mathcal{N}_B(r/2) \geq 2^j - 2^{j-1} = 2^{j-1}
\]
is unbounded; therefore, \( \dim b(A) = \infty \). Since no conditions on \( \{b_j\} \) were imposed, then \( \{a_n\} \) may converge arbitrarily fast to zero. \( \square \)

**Corollary 4.4.** There exist a set \( A \) such that \( \dim b(A) = 0 \) and \( \dim b(A - A) = \infty \).

Proof. Let \( \{x_j\} \) be a sequence of the type given in Theorem 4.3 such that \( \|x_j\| \leq 4^{-j} \). We assume the complement of the closed linear span of \( \{x_j\} \) to be infinite dimensional and define \( \{y_j\} \) to be an orthogonal sequence in that complement such that \( \|y_j\| = 4^{-j} \). Let \( A \) be the closure of the set \( \{a_j\} \) where \( a_{2j} = y_j \) and \( a_{2j+1} = x_j + y_j \). Clearly the set of differences \( A - A \) contains the set \( \{x_j\} \) and therefore \( \dim b(A - A) = \infty \).

Claim \( \dim b(A) = 0 \). If \( k = 2j \) then
\[
\|a_k\| = \|y_j\| = 4^{-j} = 2^{-k}
\]
and so the condition (4.1) is satisfied with \( \alpha = 1/2 \). If \( k = 2j + 1 \), then
\[
\|a_k\| \leq \|x_j\| + \|y_j\| \leq 4^{-j} + 4^{-j} = 2(4^{-j} + 4^{-j}) = 2(4^{-j} + 4^{-j}) \leq 4(2^{-k})
\]
and
\[
\|a_k\| \geq \|y_j\| = 2(2^{-k}).
\]
Therefore, taking \( K = 4 \) and \( \alpha = 1/2 \) satisfies (4.1) for all terms of the sequence \( \{a_j\} \). It follows from Fact 4.2 that \( \dim b(A) = 0 \). \( \square \)

We will now use the same scaling properties exploited in the above examples to construct orthogonal projections that increase the Bouligand dimension. First, we need the following definitions and results.

**Definition 4.5.** Let \( \vee A \) denote the closed linear span of \( A \).

**Lemma 4.6.** There exists a projection \( Q \) such that \( QA \) contains an orthogonal sequence if and only if there exists \( V \subseteq A \) such that \( \dim(\vee V) = \infty \) and
\[
\vee(V \setminus \{v\}) \neq \vee V \quad \text{for all } v \in V.
\]
Proof. Suppose there is such a projection $Q$. Choose $v_n \in \mathcal{A}$ such that the $Qv_n$ form an orthogonal sequence. Define $V = \{v_n : n \in \mathbb{N}\}$. It follows that $Qv$ is orthogonal to $\vee(QV \setminus \{Qv\})$ for any $v \in V$. Therefore $\vee(QV \setminus \{Qv\}) \neq \vee(QV)$. Since orthogonal projections are continuous and linear, then $Q$ commutes with $\vee$ and we obtain $Q(\vee(V \setminus \{v\})) \neq Q(\vee V)$, which implies $\vee(V \setminus \{v\}) \neq \vee V$.

Conversely, by omitting some elements of $V$ if necessary, we write $V = \{v_n : n \in \mathbb{N}\}$ and suppose the orthogonal complement of $\vee V$ to be infinite dimensional. Let $f_i$ be an orthonormal sequence contained in that complement.

Define $V_1 = V$ and $W_1 = V_1 \setminus \{v_1\}$. For induction suppose $\vee W_n \neq \vee V_n$. Then there exists $e_n \in \vee V_n$ such that $\|e_n\| = 1$ and $e_n \perp \vee W_n$. Let $g_n = (e_n + f_n)/\sqrt{2}$. Define $V_{n+1} = V_n \cup \{g_n\}$ and $W_{n+1} = V_{n+1} \setminus \{v_{n+1}\}$.

Clearly $\vee W_{n+1} \neq \vee V_{n+1}$.

The above construction yields an orthonormal sequence $g_n$ such that $(g_i, v_i) \neq 0$ and $(g_i, v_j) = 0$ for $i \neq j$. Let $Q$ be the projection onto the space spanned by the $g_n$. It follows that $Qv_n = \alpha_n g_n$ where $\alpha_n \neq 0$ and so $QV \subseteq QA$ contains an orthogonal sequence.

Theorem 4.7. If $\mathcal{A}$ satisfies Lemma 4.6, then there exists an orthogonal projection $P$ such that the Bouligand dimension of $PA$ is infinite.

Note that Fact 4.2 provides an example of a set with finite Bouligand dimension that satisfies Lemma 4.6. However, a slight modification of the solution given by Halmos for Problem 11 in [12] yields an infinite dimensional compact set which does not satisfy these hypotheses. It is unknown whether the attractors of naturally occurring physical systems satisfy these hypotheses or not.

Proof of Theorem 4.7. By Lemma 4.6 there exists an orthogonal projection $Q$ such that $QA$ contains the orthogonal sequence $\alpha_n g_n$. Let

$$m_j = \min \{|\alpha_n| : n = 2^{j-1}, \ldots, 2^j - 1\}$$

and choose a monotone sequence $\beta_j$ strictly decreasing to zero such that $\beta_j \leq m_j$. We may assume that the orthogonal complement of $\vee \{\alpha_n g_n : n \in \mathbb{N}\}$ in $QH$ is infinite. Let $h_n$ be an orthonormal sequence in that complement and define

$$\tilde{g}_n = \sin(\theta_n) g_n + \cos(\theta_n) h_n$$

where $\theta_n$ has been chosen in such a way that

$$\alpha_n \sin(\theta_n) = \beta_j \quad \text{for} \quad n = 2^{j-1}, \ldots, 2^j - 1.$$ 

This can be done since $\beta_j < \alpha_n$. Let $P$ be the projection onto the space spanned by the $\tilde{g}_n$. It follows that $PA = PQA$ contains a sequence similar
to the one found in Theorem 4.3 and therefore
\[ \dim_b(PA) \geq \dim_b\{\alpha_n \sin(\theta_n)g_n : n \in \mathbb{N}\} = \infty. \]

\[ \square \]

5. The projection theorem.

In this section we prove Theorem 1.2 restated more explicitly in the form of Theorem 5.2 below. We want to find rank \( m \) orthogonal projections such that
\[ \|Px - Py\| \geq \epsilon f(\|x - y\|) \quad \text{for } x, y \in A \tag{5.1} \]
where \( f \) is linear with a logarithmic correction. Moreover, we show that the measure of all the projections which do not satisfy this condition for any constant \( \epsilon > 0 \) is zero. Writing \( Y = A - A \) simplifies (5.1) to the requirement that
\[ \|Py\| \geq \epsilon f(\|y\|) \quad \text{for } y \in Y. \]

Let \( G \) be the space of orthogonal projections in \( \mathbb{R}^N \) of rank \( m \) and \( \mu \) be the invariant measure on \( G \) with respect to orthogonal transformations. Define the shadow of a set \( B \) in \( \mathbb{R}^N \) to be
\[ S(B) = \{P \in G : 0 \in PB\}. \]

Showing that \( \mu(S(Y)) = 0 \) would prove almost every rank \( m \) orthogonal projection is injective. To obtain continuity of the inverse we construct a slightly larger set \( U \) containing \( Y \) and show that \( \mu(S(U)) = 0 \). We shall use the following estimate in our computations:

**Theorem 5.1.** The measure of the shadow of a \( \rho \)-ball \( B \) centered a distance \( r \) from the origin is bounded by \( \mu(S(B)) \leq C(\rho/r)^m \).

This result is given by Santalo in [22]. An explicit proof involving only elementary techniques has recently been given by by Friz and Robinson in [11].

**Theorem 5.2.** Given a bounded set \( A \subseteq \mathbb{R}^N \) such that \( A \subseteq B_R(0) \) and \( \dim_b(A - A) < d < m \). Let \( \eta > 1 \) and define
\[ f(x) = x \left( \frac{1}{\log_2(2R/x)} \right)^{\eta/(m-d)} \]
where \( R = 2\sup \{\|a\| : a \in A\} \). Then for almost every projection \( P \) of rank \( m \) there is a constant \( \epsilon \) such that (5.1) is satisfied.

**Proof.** Define \( r_n = R/2^n \) and \( \rho_n = f(r_{n-1}) \). Divide \( Y \) into shells
\[ Z_n = \{y \in Y : r_n \leq \|y\| \leq r_{n-1}\} \]
and cover \( Z_n \) by balls of radius \( \rho_n \).
Let $Y = A - A$ and choose $K$ such that $N_B(r, \rho) \leq K (r/\rho)^d$ for all $B \subseteq Y$. Further choose $c > 2$ so large that $c^{m-d} > K$. Cover each $\rho_n$-ball by $\rho_n/c$-balls and each of those by $\rho_n/c^2$-balls and so on. Label the centers of the $\rho_n/c^i$-balls that cover $Z_n$ by $a_{nij}$. Thus

$$Z_n \subseteq \bigcup_j B_{\rho_n/c^i}(a_{nij})$$

where the index $j$ ranges over at most $K^i + 1 \leq K^{i+1} c^id(r_{n-1}/\rho_n)^d$ balls.

Since $c > 2$, the covers formed by doubling the radius of the balls in each generation will be nested. Let $U_i = \bigcup_{n,j} B_{2\rho_n/c^i}(a_{nij})$ be the union of all the balls in the $i$-th generation with double the radius. For each $y \in Y$ there is a ball $B_\rho(a)$ of radius $\rho = \rho_n/c^i$ centered at $a$ such that $y \in B_\rho(a)$. If $P \not\in S(U_i)$ then $0 \not\in PB_{2\rho_n}(a)$ and hence

$$\|Py\| \geq \rho = \rho_n/c^i = f(r_{n-1})/c^i \geq f(\|y\|)/c^i$$

shows that (5.1) holds for the $i$-th generation with $\epsilon = 1/c^i$.

Let $U = \limsup_i U_i$. It remains to show the measure of $S(U)$ is zero. Estimate

$$\mu(S(U)) \leq \mu(S(U_i)) \leq \sum_{n,j} \mu(S(B_{2\rho_n/c^i}(a_{nij})))$$

$$\leq \sum_{n,j} C \left( \frac{2\rho_n}{c^3 r_n} \right)^m \leq \sum_n CK^{i+1} c^id \left( \frac{r_{n-1}}{\rho_n} \right)^d \left( \frac{2\rho_n}{c^3 r_n} \right)^m$$

$$= 4^m CK \left( \frac{K}{c^{m-d}} \right)^i \sum_n \left( \frac{f(r_{n-1})}{r_{n-1}} \right)^{m-d}$$

$$= 4^m CK \left( \frac{K}{c^{m-d}} \right)^i \sum_n \frac{1}{n^m} = L \left( \frac{K}{c^{m-d}} \right)^i$$

where $L$ is a constant. Letting $i$ become large, we obtain $\mu(S(U)) = 0$. Thus, almost every orthogonal projection of rank $m$ satisfies (5.1) for some $\epsilon > 0.

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References


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MULTIPLE-POINT FORMULAS — A NEW POINT OF VIEW

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On the basis of the Generalized Pontryagin-Thom construction (see Rimányi & Szucs, 1998) and its application in computing Thom polynomials (see Rimányi, 2001) here we introduce a new point of view in multiple-point theory. Using this approach we will first show how to reprove results of Kleiman and his followers (the corank 1 case) then we will prove some new multiple-point formulas which are not subject to the condition of corank \( \leq 1 \). We will concentrate on the case of complex analytic maps \( N^* \rightarrow P^{*+1} \), since this was the setting where the most formulas were known before. The scheme of the computation is similar to the one we used in computing Thom polynomials (see Rimányi, 2001), with an essential difference that here we need to compute nontrivial incidence classes.

1. Introduction.

Consider complex analytic maps \( f : M^* \rightarrow N^{*+k} (k > 0) \). The * is used to put an emphasis on the fact that what follows will not depend on the dimensions of the source and target manifolds, only on their differences. For a nice enough \( f \) the closure of the \( r \)-tuple points of \( f \) in the source and in the target determine cohomology classes \( m_r(f) \), \( n_r(f) \) — or simply \( m_r \), \( n_r \). The so called multiple-point formulas are cohomological identities in the cohomology ring of the source manifold involving these \( m_r \)'s, \( f^*(n_r) \)'s and the Chern classes of the map — valid for certain \( f \)'s. Of course, we like a formula more if it is valid for a bigger set of maps. Such a formula is Ronga’s result \( m_2(f) = f^*(n_1(f)) - c_k(f) \), which is valid for most maps. Another classical example is the Herbert-Ronga formula ([H], [Ro])

\[
m_r = f^*n_{r-1} - c_k m_{r-1}
\]

which is valid only for immersions. In the 80’s and 90’s it turned out that the Herbert-Ronga formulas can be “corrected” (by adding additional terms involving \( c_i \)'s for \( i > k \)) so that the new formulas hold for a bigger set of maps — maps of corank \( \leq 1 \).

The concrete determination of these formulas was a hot area in enumerative algebraic geometry — partly for their own beauty, partly because they
can be used to obtain other enumerative geometric results, such as e.g., in [C1], [C2], [Ka1], [Ka2], [J]. The modern history of the subject began with works of Laksov, Fulton and Le Barz. The best results are achieved by the two main approaches — i.e., iteration and Hilbert-scheme — of S. Kleiman ([K1], [K2], [K3], [K4], see also [Ka2]) whose papers also contain historical remarks and summaries of how these formulas yield old and new algebraic geometric formulas.

The multiple-point formulas can be considered as the Thom polynomials of multi-singularities (though they are not polynomials). In this paper we show that the approach which turned out to be very powerful in computing (ordinary) Thom polynomials is also capable to find multiple-point formulas. There, and here also, a part of the computation is the determination of certain “incidence (cohomology) classes” (see [R]). In fact, the situation here is more difficult, since in the case of Thom polynomials we only had to deal with incidence classes trivially 0 (for geometrical reasons), but in the case of multiple-point formulas we have to compute nontrivial incidence classes. Their computation for the corank 1 singularities leads to the results of [K4] and [Ka2]. We can, however, compute incidence classes involving higher corank singularities, and these lead us to some modification of Katz’ last formula (now valid not only for corank 1 maps) and some new results.

As we mentioned, the recent research in this area used the techniques of algebraic geometry. Here we use the techniques of “global singularity theory” (= singularity theory + differential topology), the great invention of Szücs, see the introduction and references in [RSz]. Therefore the reader is advised to get some familiarity with the generalized Pontryagin-Thom construction (gPTc) [RSz] and its application of computing Thom polynomials [R].

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2. Main results.

Following works of Kleiman, Katz [Ka2] proved the following result.

**Theorem 2.1.** There exist polynomials \( p_i \) such that for maps \( f : M^* \rightarrow N^{*+1} \) of corank \( \leq 1 \) the following formulas hold modulo torsion

\[
(1_r)
\]

\[
m_r = f^*n_{r-1} + \sum_{i=1}^{r-1} (-1)^ip_i(c(f))m_{r-i}.
\]

So the \( p_i \)'s are multivariable polynomials evaluated at the Chern classes

\[
c(f) = c(f^*TN - TM) = f^*c(N)/c(M)
\]

of the map.

Since in algebraic geometry some classes are counted with natural multiplicities, so Kleiman’s and Katz’s \( m_r \) is \( (r-1)! \) times ours, and their
\( f^* f_*(m_r) \) is \( r! \) times our \( f^*(n_r) \). Kleiman also proves so called refined formulas, which assert that the formulas \( (1_r) \) really hold, not only modulo torsion.

We will present a new method which reduces the problem of calculating \( p_i \)'s to solving linear equations. Using this method we will find the following polynomials:

\[
\begin{align*}
p_1 &= c_1 \\
p_2 &= c_2 \\
p_3 &= c_1c_2 + 2c_3 \\
p_4 &= c_1^2c_2 + c_2^2 + 5c_1c_3 + 6c_4 \\
p_5 &= c_1^2c_2 + 3c_1c_3^2 + 9c_1^2c_3 + 8c_2c_3 + 26c_1c_4 + 24c_5 \\
p_6 &= c_1^2c_2 + 6c_1^2c_3^2 + 2c_2^2 + 14c_1^3c_3 + 37c_1c_2c_3 + (56 - t)c_3^2 + tc_2c_4 + 71c_1^2c_4 + 154c_1c_5 + 120c_6 \\
p_7 &= c_1^2c_2 + 10c_1^3c_2^2 + 10c_1c_3^2 + 20c_1^2c_3 + 105c_2^2c_3 + 32c_2^3c_3 + (362 - u_1)c_1c_3^2 + 155c_1^3c_4 + u_1c_1c_2c_4 + (408 - u_2)c_3c_4 + 580c_1c_5 + u_2c_2c_5 + 1044c_1c_6 + 720c_7 \\
\end{align*}
\]

etc.

**Remark 2.2.** The polynomials \( p_1-p_6 \) appeared in \([K4]\) and \([Ka2]\) with a particular value \((42)\) of the parameter \( t \) (in \( p_6 \)). Here we state that the formulas hold with any value of \( t, u_1, u_2 \). Of course, this uncertainty of the polynomials is not a surprise, since the Thom polynomial of the singularity \( \text{III}_{2,2} \) (the simplest of Thom-Boardman type \( \Sigma^2 \)) is \( c_3^2 - c_2c_4 \). This explains the presence of the parameter \( t \). This, and the Thom polynomial of the singularity \( \text{I}_{2,2} \) (the second simplest of type \( \Sigma^2 \)) — i.e., \( c_1^3c_2 - c_1c_2c_4 + 2c_3c_4 = -2c_2c_5 \) (see \([R]\)) — together explains the presence of the parameters \( u_1, u_2 \). Indeed, \( u_1 \) corresponds to \( c_1 \cdot T.P. (\text{III}_{2,2}) \) and \( u_2 \) corresponds to \((T.P. (\text{I}_{2,2}) - c_1 \cdot T.P. (\text{III}_{2,2})))/2 \). These parameters \( t, u_1, u_2 \) will take concrete values if we allow the map \( f \) to have more difficult singularities (namely \( \text{III}_{2,2}, \text{I}_{2,2} \)), as follows.

**Theorem 2.3.** The above formulas hold if \( f \) is allowed to have \( \text{III}_{2,2} \) and \( \text{I}_{2,2} \) singularities, with the following values of the parameters in \( p_6, p_7 \)

\[
t = 43, \quad u_1 = 281, \quad u_2 = 278.
\]

Of course, allowing a bigger set of maps would be a more spectacular result — and in fact we can allow the maps to have more complicated singularities. Our method, however, works like a test: One can ask, whether the formulas hold if we allow also singularity \( \eta \), then we make some algebraic calculation with \( \eta \) and answer either yes or no. In fact, for any (stable) singularity the author tried the answer is yes, the formulas hold, therefore we conjecture that the above formulas hold for any stable maps. The singularities \( \text{III}_{2,2} \) and \( \text{I}_{2,2} \) are special in the theorem because they are needed to compute the values of the parameters: Allowing \( \text{III}_{2,2} \) sets the value of
Let $t = 43$ and the linear equation $u_2 = 2u_1 - 284$. Allowing also $I_{2,2}$ sets the values of $u_1$ and $u_2$.

3. A short review on gPTc.

In this section we give a brief review on the generalized Pontryagin-Thom construction which was invented decades ago by A. Szücs.

Let $k > 0$ (in our application $k = 1$) be fixed and consider the set of all stable germs $(\mathbb{C}^*, S) \rightarrow (\mathbb{C}^{*+k}, 0)$ where $S$ is any finite set. Divide this set by the equivalence relation generated by right-left equivalence (permutation of the elements of $S$ are allowed) and trivial unfolding (i.e., adding irrelevant coordinates to the source and to the target). The equivalence classes will be called multi-singularities, or singularities (mono-singularity if $|S| = 1$). Each singularity has a prototype (defined up to right-left equivalence) such that any other germ of the same singularity is right-left equivalent to an appropriate trivial unfolding of it. The source dimension of the prototype is called the codimension of the singularity. For example, the mono-singularities for $k = 1$ up to codim 10 is given in the following table (with their Thom-Boardman class):

<table>
<thead>
<tr>
<th>codim</th>
<th>$\Sigma^0$</th>
<th>$\Sigma^1$</th>
<th>$\Sigma^{2,0}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$A_0$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td></td>
<td>$A_1$</td>
<td></td>
</tr>
<tr>
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<td></td>
<td>$A_2$</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
<td></td>
<td>$I_{2,2}$</td>
</tr>
<tr>
<td>4</td>
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<td>$A_3$</td>
<td>$\Pi_{2,2}$</td>
</tr>
<tr>
<td>5</td>
<td></td>
<td></td>
<td>$I_{2,3}$</td>
</tr>
<tr>
<td>6</td>
<td></td>
<td>$A_4$</td>
<td>$\Pi_{2,3}$</td>
</tr>
<tr>
<td>7</td>
<td></td>
<td></td>
<td>$I_{2,3}$</td>
</tr>
<tr>
<td>8</td>
<td></td>
<td>$A_5$</td>
<td>$\Pi_{3,3}$</td>
</tr>
<tr>
<td>9</td>
<td></td>
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<tr>
<td>10</td>
<td></td>
<td>$A_5$</td>
<td>$\Pi_{3,3}$</td>
</tr>
</tbody>
</table>

Here $A_i$ is the unique singularity of Thom-Boardman class $\Sigma^1_i$ (i.e., the one with local algebra $\mathbb{C}[[x]]/(x^{i+1})$). For the singularities with local algebras

$$\mathbb{C}[[x,y]]/(x^a, y^b, xy), \quad \mathbb{C}[[x,y]]/(x^a + y^b, xy)$$

we adapted Mather’s notation $\Pi_{a,b}$ and $I_{a,b}$, respectively.

We can define the usual hierarchy of singularities: $\eta$ is more complicated than $\zeta$ ($\eta > \zeta$) if near 0 in the target of a representative of $\eta$ there is necessarily a $\zeta$ point $y$, i.e., the representative at $f^{-1}(y)$ is from $\zeta$. (This is the obvious definition in the case where there is no moduli, so for the sake of simplicity we will stay in that region.)
Let \( \tau \) be an ascending (\( \zeta < \eta, \eta \in \tau \Rightarrow \zeta \in \tau \)) set of singularities. A map \( f : N^0_n \to N^{n+k}_1 \) is called a \( \tau \)-map if for every \( y \in f(N_0) \) the map \( f \) near \( f^{-1}(y) \) has singularity from \( \tau \). For example, \( \{A_0\}\)-maps are the embeddings, \( \{rA_0\}_{r=1,2,\ldots} \)-maps are the immersions, or if \( \tau \) contains finite linear combinations of \( A_i \)'s (\( i = 0, 1, 2, \ldots \)) then we call a \( \tau \)-map a Morin-map or a corank 1 map.

In (local) singularity theory one can write up “normal forms” of (prototypes of) singularities. Here we will present a “global normal form” of singularities as follows. For simplicity we give it for a mono-singularity \( \eta \) with prototype \( \kappa : (\mathbb{C}^n, 0) \to (\mathbb{C}^{n+k}, 0) \). One can consider the maximal compact subgroup of \( \text{Aut} \kappa = \{ (\psi, \phi) \in \text{Diff}(\mathbb{C}^n, 0) \times \text{Diff}(\mathbb{C}^{n+k}, 0) \mid \phi \circ \kappa \circ \psi^{-1} = \kappa \} \).

Denote it by \( G_\eta \). It comes with two representations \( \lambda_0, \lambda_1 \) on the source and the target spaces (which are linear if \( \kappa \) is well chosen from its right-left equivalence class). Now associate vector bundles \( \xi_0(\eta), \xi_1(\eta) \) with the universal principal \( G_\eta \)-bundle using these representations. It can be easily checked that one has a fibrewise map from the total space of \( \xi_0 \) to the total space of \( \xi_1 \) which is (right-left equivalent to) \( \kappa \) in each fibre. We call this map \( f_\eta \) the \textit{global normal form} of \( \eta \).

It is showed in \([RSz]\) that using the global normal forms of the singularities from \( \tau \) one can put together a “universal \( \tau \)-map” \( f_\tau : X_0\tau \to X_1\tau \) — from which any \( \tau \)-map can be pulled back in a certain, more-or-less unique way. A consequence of its “universality” is that whenever a cohomological identity holds for \( f_\tau \) it must hold for all \( \tau \)-maps.

Of course, the cohomological structure of e.g., \( X_0\tau \) is not clear. The way we can check a cohomological identity for \( f_\tau \) has two levels: (1) If we know the “structure” of the identity, and only concrete coefficients are missing, then we can restrict it to the zero sections \( K_0(\eta) \) of the global normal form of \( \eta \). Usually the computations can be carried out easily there. (2) If we also need to \textit{prove} the identity for \( f_\tau \) then we need a Mayer-Vietoris argument, which is usually easy in the complex case, for more details see \([R]\).

**4. Morin singularities.**

The prototype of the Morin singularity \( A_i \) in the case \( M^* \to N^{*+1} \) is the miniversal unfolding of the map \( (\mathbb{C}^1, 0) \to (\mathbb{C}^2, 0) \), \( x \mapsto (x^{i+1}, 0) \), namely

\[
(x, y_1, \ldots, y_{i-1}, z_1, \ldots, z_i) 
\mapsto \left( x^{i+1} + \sum_{j=1}^{i-1} y_j x^j, \sum_{j=1}^i z_j x^j, y_1, \ldots, y_{i-1}, z_1, \ldots, z_i \right).
\]
The maximal compact symmetry group is $U(1) \times U(1)$ with the representation $\lambda_0 = \rho_1 \oplus \rho_V$, $\lambda_1 = \rho_1^{i+1} \oplus \rho_2 \oplus \rho_V$, where

$$\rho_V = \bigoplus_{j=1}^{i-1} \rho_1^{i+1-j} \oplus \bigoplus_{j=1}^{i} (\rho_1 \otimes \rho_2).$$

Here $\rho_1, \rho_2$ are the standard 1-dimensional representations of the 1st and 2nd factor of $U(1) \times U(1)$, and the powers mean tensor powers.

Now we compute the Chern classes of the map $f\tau$ restricted to $K_0(A_i)$:

$$c(f\tau)|_{K_0(A_i)} = \frac{f^*c(T(BG_{A_i}) \oplus \xi_1(A_i))}{c(T(BG_{A_i}) \oplus \xi_0(A_i))} = \frac{c(f^*(\xi_1(A_i)))}{c(\xi_0(A_i))} = \frac{(1 + (i + 1)a)(1 + b)}{1 + a} = 1 + (ia + b) + (-ia^2 + iab) + (ia^3 - ia^2b) + \cdots.$$ 

It seems to be a more difficult subject to compute $m_r(f\tau)$ (= the Poincaré-dual of the fundamental homology class carried by the closure of $K_0(rA_0)$) restricted to $K_0(A_i)$. This is called the incidence class $I(rA_0, A_i)$ in [R]. Of course, this is trivially zero if $r > i + 1$, since in this case there are no $r$-tuple points near $A_i$. It is interesting, that the analogue of this easy observation was enough when computed Thom polynomials, but it is not enough here.

**Lemma 4.1.**

$$m_r|_{K_0(A_i)} = \binom{i}{r-1} (b-a)(b-2a) \cdots (b-(r-1)a)$$

$$f\tau^*(n_r)|_{K_0(A_i)} = \binom{i+1}{r} b(b-a)(b-2a) \cdots (b-(r-1)a).$$

Let us concentrate on the computation of $m_r|_{K_0(A_i)} = I(rA_0, A_i)$ first. Since this is the first nontrivial incidence class computation, before giving the complete proof let us discuss easier special cases.

Consider the special case of $i = 2$. A prototype of $A_2$ is

$$(x, y, z_1, z_2) \mapsto (x^3 + yx, z_1x + z_2x^2, y, z_1, z_2),$$

with the maximal compact symmetry group $U(1) \times U(1)$ acting as

$$\kappa : \alpha \oplus \alpha^2 \oplus \bar{\alpha} \beta \oplus \bar{\alpha}^2 \beta \quad \text{and} \quad \alpha^3 \oplus \beta \oplus \alpha^2 \oplus \bar{\alpha} \beta \oplus \bar{\alpha}^2 \beta$$

($\alpha, \beta$ are the standard representations of the first and second $U(1)$, products are meant to be tensor products). Easy analysis of $\kappa$ shows that the closure of the triple point set (in the source) is $z_1 = 0, z_2 = 0$. Considering this set
in each fibre of $\xi_0(A_2)$ we get a subbundle $\xi$, so

$$m_3|K_0(A_2) = [\text{cl } K_0(3A_0)]|K_0(A_2) = e(\xi_0(A_2)/\xi) = \text{top}((1 + b - a)(1 + b - 2a)) = (b - a)(b - 2a).$$

This easy method works if the (closure of the) $r$-tuple point set of the prototype is smooth at 0. This is hardly ever the case. E.g., the (closure of the) double point set of the same $\kappa$ has the equation $z_1^2 + yz_2^2 = 0$. Although it is not smooth at 0, its tangent cone $z_1^2 = 0$ is a linear space with multiplicity 2, so the Euler class of the subbundle corresponding to $z_1$ in $\xi_0(A_2)$ has to be counted twice: $m_2|K_0(A_2) = 2(b - a)$.

This last computation relied on our ability to write up the (scheme theoretically) “correct equation system” for the $r$-tuple point set (with the appropriate number of equations). This seems to be impossible in general, already in the $r = 3, i = 3$ case. However we can easily give a parameterization (a desingularization) of the $r$-tuple point set, which gives us a complete proof.

**Proof.** Consider the map $g : \mathbb{C}^{2i} \rightarrow \mathbb{C}^{2i+1}$

$$(x, y_1, \ldots, y_{i-1}, z_1, \ldots, z_i) \mapsto \left(x^{i+1} + \sum_{j=1}^{i-1} y_jx^j, \sum_{j=1}^{i} z_jx^j, y_1, \ldots, y_{i-1}, z_1, \ldots, z_i\right)$$

which is a (representative of a) prototype of $A_i$. Let the first two coordinate functions — as functions of $x$ — be denoted by $e_1(x)$ and $f_1(x)$. The natural parameterization of the closure of the double point set would come from $\{(u, v) \in \mathbb{C}^{2i} \times \mathbb{C}^{2i} | g(u) = g(v)\}$, but of course there is no need to “double” the unfolding parameters $y_1, \ldots, y_{i-1}, z_1, \ldots, z_i$. Also, we have to get rid of the diagonal component, so we consider the set

$$\left\{(x_1, x_2, y_1, \ldots, y_{i-1}, z_1, \ldots, z_i) \mid e_2(x_1) := \frac{e_1(x_1) - e_1(x_2)}{x_1 - x_2}, \right. \left. f_2(x_1) := \frac{f_1(x_1) - f_1(x_2)}{x_1 - x_2}\right\}.$$ 

On one hand this set is smooth — it is a graph of a map $\left(x_1, x_2, y_2, \ldots, y_{i-1}, z_2, \ldots, z_i\right) \mapsto (y_1, z_1)$, on the other hand forgetting $x_2$ it projects to the closure of the double point set of $g$. So we get the following desingularization of the closure of the double point set:

$$(x_1, x_2, y_2, \ldots, y_{i-1}, z_2, \ldots, z_i) \mapsto (x_1, Y_1, y_2, \ldots, y_{i-1}, Z_1, z_2, \ldots, z_i),$$

where the function $Y_1$ is from $-x_2^i + m(x_1, y_2, \ldots, y_{i-1}, z_2, \ldots, z_i)$ and the function $Z_1$ is from $m(x_1, y_2, \ldots, y_{i-1}, z_2, \ldots, z_i)$. Here $m$ denoted the maximal ideal in the function algebra in the variables given in brackets. Now
it can be easily read off that the tangent cone of the closure of the double point set is the subspace spanned by the coordinates

\[(x_1, y_1, \ldots, y_{i-1}, z_2, \ldots, z_i)\]

with multiplicity \(i\). So we get that

\[m_2 | K_0(A_i) = i \cdot e(\text{the subbundle of } \xi_0(A_i) \text{ corresponding to } z_1) = i(b - a).\]

Now we turn to the closure of the triple point set, so we consider

\[\{(x_1, x_2, x_3, y_1, \ldots, y_{i-1}, z_1, \ldots, z_i) | e_2(x_1), f_2(x_1), e_3(x_1) := \frac{e_2(x_1) - e_2(x_3)}{x_1 - x_3}, f_3(x_1) := \frac{f_2(x_1) - f_2(x_3)}{x_1 - x_3}\}\]

This is again smooth — a graph of a map to the coordinates \(y_1, z_1, y_2, z_2\), and parametrizes the closure of the triple point set by forgetting \(x_2\) and \(x_3\). By calculation we get that the closure of the triple point set is parametrized by

\[(x_1, x_2, x_3, y_1, \ldots, y_{i-1}, z_1, \ldots, z_i)\]

\[\mapsto (x_1, Y_1, Y_2, y_3, \ldots, y_{i-1}, Z_1, Z_2, z_3, \ldots, z_i),\]

where the functions \(Z_1, Z_2\) are zero — modulo \(m(x_1, y_3, \ldots, y_{i-1}, z_3, \ldots, z_i)\) and the functions \(Y_1, Y_2\) are homogeneous degree \(i\) and \(i - 1\) polynomials in \(x_2, x_3\) — modulo the mentioned maximal ideal. Again, we can read off that the tangent cone of the closure of the triple point set is the subspace spanned by the coordinates

\[(x_1, y_1, \ldots, y_{i-1}, z_3, \ldots, z_i)\]

with multiplicity \(i(i - 1)\). So we get that

\[m_3 | K_0(A_i) = i(i - 1) \cdot e(\text{the subbundle of } \xi_0(A_i) \text{ corresponding to } z_1, z_2)\]

\[= \binom{i}{2} (b - a)(b - 2a).\]

Going on like this we obtain a (degree \((r - 1)!)\) parameterization of the closure of the \(r\)-tuple set as

\[(x_1, \ldots, x_r, y_r, \ldots, y_{i-1}, z_r, \ldots, z_i)\]

\[\mapsto (x_1, Y_1, \ldots, Y_{r-1}, y_r, \ldots, y_{i-1}, Z_1, \ldots, Z_{r-1}, z_r, \ldots, z_i),\]

where the functions \(Z_1, \ldots, Z_{r-1}\) are in the maximal ideal generated by the coordinates \(x_1, y_r, \ldots, y_{i-1}, z_r, \ldots, z_i\), while \(Y_1, \ldots, Y_{r-1}\) are homogeneous
degree $i, i - 1, \ldots, i - r + 2$ polynomials in $x_2, x_3, \ldots, x_r$. So the sought incidence class $m_r|_{K_0(A_i)}$ is

$$i(i - 1) \ldots (i - r + 2) \cdot \epsilon(\text{the subbundle of } \xi_0(A_i) \text{ corresponding to } z_1, \ldots, z_{r-1}) = \binom{i}{r-1} (b-a)(b-2a) \ldots (b-(r-1)a),$$

which was to be proved.

To prove the second equality we use the Gysin push-forward $f_*$ in cohomology, and its well-known property $f_*f^*v = v \cup f_1$ for all $v$. Let us apply this for $f = f(A_i)$, and pull it back by $f$:

$$f_*f_*(f^*v) = (f^*v) \cup f^*(n_1).$$

Since $f^*$ is isomorphism, we can put any cohomology class for $f^*v$, e.g., $m_r$:

$$f_*f_*m_r = m_r \cup f^*(n_1).$$

The left hand side is clearly $r \cdot f^*n_r$ and $f^*(n_1) = (i + 1)b$ (with the same method — it is given by parameterization). This implies the second equality. 

As an example let us show how to compute $p_1$. This must be weighted homogeneous of degree 2, so it is a constant $A$ times $c_1$. Consider the first multiple-point formula $m_2 = f^*n_1 - Ac_1m_1$ for $f = f_\tau$ (with $\tau$ large enough to consist $A_0$) and restrict it to $K_0(A_0)$. According to Lemma 4.1 and the Chern-class computation above it, the restrictions of $m_2$, $f^*n_1$ and $c_1(f_\tau)$ to $K_0(A_1)$ are $\binom{0}{1}(b-a), \binom{1}{1} b, b$ respectively. So our formula reduces to

$0 = b - A \cdot b \in H^1(BG_{A_0}) = \mathbb{Z}[a, b]$ (the equality is meant modulo torsion, but there is no torsion at all here). So $A = 1$, i.e., $p_1 = c_1$.

To compute $p_2 = Ac_1^2 + Bc_2$ we proceed similarly. Knowing the result above, restrict the triple point formula $m_3 = f^*n_2 - c_1m_2 + (Ac_1^2 + Bc_2)m_1$ (applied to $f_\tau$, where $\tau$ big enough to consist $A_1$) to $K_0(A_1)$. Using Lemma 4.1 and the Chern-class computation above it again, we get

$$\binom{1}{2} (b-a)(b-2a) = \binom{2}{2} b(b-a) - (a+b)\binom{1}{1} (b-a) + (A(a+b)^2 + B(-a^2 + ab)) \in \mathbb{Z}[a, b],$$

which gives an (overdetermined) system of linear equations on $A, B$, with the unique solution $A = 0, B = 1$. Therefore $p_2 = c_2$.

To obtain the next polynomial $p_3 = Ac_1^3 + Bc_1c_2 + Cc_3$ we use the values of $p_1, p_2$ just computed and restrict formula (14) (applied to an appropriate
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\[ 0 = b(b-a)(b-2a) - (3a+b)(b-a)(b-2a) + (-3a^2 + 3ab)2(b-a) \]
\[ - (A(3a+b)^3 + B(3a+b)(-3a^2 + 3ab) + C(3a^3 - 3a^2b)) \]

in \( \mathbb{Z}[a,b] \) which leads to an (overdetermined) system of linear equations with the unique solution \( A = 0, B = 1, C = 3 \), so \( p_3 = c_1 c_2 + 3c_3 \).

We can go on like that: When trying to compute \( p_i \) we use the already computed \( p_j \)'s \((j < i)\) and restrict formula \((1_{i+1})\) (when applied to an appropriate \( f \tau \)) to \( K_0(A_{i-1}) \). We obtain a system of linear equations on the coefficients of \( p_i \). From \( i = 6 \) the solution is not unique, contains a certain number of parameters — at which we are not surprised, see Remark 2.2.

**Remark 4.2.** By now it is clear how to compute the polynomials \( p_i \), once we know that formula \((1_r)\) holds (for corank 1 maps) with some polynomials. Of course, our method is also capable to prove that the formulas, with the computed polynomials really hold for corank 1 maps. For this we need to check three things.

First that they hold if restricted to \( K_0(A_i) \) for any \( i \), not just the small ones we used when determined the coefficients. This can be done individually (e.g., for \((1_2)\) we need to check that \( i(b-a) = (i+1)b - (ia+b) \) holds for any \( i \), or using a general argument, based on the fact that if a polynomial (in \( i \)) vanishes in \( i = 0, 1, \ldots, \deg \), then it vanishes for greater \( i \)'s, too, since it must be the 0 polynomial.

Secondly we need to check that the formulas hold restricted to \( K_0(\eta) \)'s, where \( \eta \) is a linear combination of \( A_i \)'s. The method we used in [R] (when dealing with immersion formulas) shows that modulo torsion this task can be reduced to verifying the formulas restricted to \( K_0(\zeta) \)'s, where the \( \zeta \)'s are the mono-singularities occurring in \( \eta \).

Thirdly we need to use a Mayer-Vietoris argument to deduce the following: If e.g., \((1_r)\) hold restricted to all blocks in \( X_0 \tau \) then it holds in the cohomology of \( X_0 \tau \), too. In the complex case these arguments are easy, see [R].

### 5. Singularities of higher corank.

Now consider the simplest singularity —denoted as \( \text{III}_{2,2} \) by Mather — of Thom-Boardman class \( \Sigma^{2,0} \), i.e., the one with local algebra \( \mathbb{C}[[x,y]]/(x^2, y^2, xy) \). Its prototype is the miniversal unfolding of \((x, y) \mapsto (x^2, y^2, xy) \). From this we can see that the maximal compact symmetry group \( G_{\text{III}_{2,2}} \) is \( U(2) \). So, according to our procedures we should now look for a miniversal unfolding which admits \( U(2) \) as a (right-left linear) symmetry group. Of course, such a prototype could be given, but the calculation is much simpler if we
ch{'1}: (x, y, u_1, u_2, v_1, v_2) \mapsto (x^2 + u_1x + u_2y, y^2 + v_1x + v_2y, xy, u_1, u_2, v_1, v_2)

as our prototype. On this germ only U(1) × U(1) (the maximal torus of U(2)) acts linearly, with the representations

\[ \lambda_0(\Pi_{2,2}) = \alpha \oplus \beta \oplus \mu_V \quad \lambda_1(\Pi_{2,2}) = \alpha^2 \oplus \beta^2 \oplus \alpha \beta \oplus \mu_V \]

where \( \mu_V = \alpha \oplus \beta \oplus \alpha \beta \oplus \beta \) (\( \alpha, \beta \) are the standard 1-dimensional representations of the first and the second \( U(1) \)). The reduction of the maximal symmetry group to a smaller one usually causes loss of information, but not now, as we will see below.

To obtain the equation of the (closure of the) double point set

\[ \{(x, y, u_1, u_2, v_1, v_2) \mid \exists (x_1, y_1) \neq (x, y) \text{ s.t.} \kappa(x, y, u_1, u_2, v_1, v_2) = \kappa(x_1, y_1, u_1, u_2, v_1, v_2) \} \]

in the source of \( \kappa \) let \( \lambda \) be \( x_1/x = y/y_1 \), i.e., let \( x_1 = \lambda x, y_1 = \frac{1}{\lambda} y \). Then the equation of the double point set is the resultant of the two polynomials

\[ \frac{\kappa_1(x, y) - \kappa_1(\lambda x, \frac{1}{\lambda} y)}{(\lambda - 1)}, \quad \frac{\kappa_2(x, y) - \kappa_2(\lambda x, \frac{1}{\lambda} y)}{(\lambda - 1)}. \]

These two polynomials are (by calculation)

\[ (\ast) \quad \lambda^2(-x^2) + \lambda(-u_1x - x^2) + (u_2y), \quad \lambda^2(-v_1x) + \lambda(v_2y + y^2) + (y^2). \]

Their resultant (after getting rid of the “false” factor \( x^2y^2 \)) is

\[ -(v_1x^3 + u_2y^3) - (v_2x^2y + u_1xy^2) - (2u_1v_1x^2 + 2u_2v_2y^2) \]
\[ -(u_1v_2xy) - (3v_1u_2xy) - (u_1^2v_1x + u_2v_2^2y) \]
\[ -(u_2v_1v_2x + u_1u_2v_2y) + (u_2v_1^2) - (u_1u_2v_1v_2). \]

This set coincides with its own tangent cone, which is now not a linear space (with multiplicity), so we cannot use the methods above to compute the cohomology class \( m_2 \) represented by the double points of the map \( f_\kappa : \xi_0 \to \xi_1 \) (recall that \( \xi_0, \xi_1 \) are vector bundles over \( U(1) \times U(1) \) associated with the universal principal \( U(1) \times U(1) \)-bundle using the representations \( \lambda_1, \lambda_2 \), and the map is fibrewise equivalent to \( \kappa \)). Our method now is the following. The class sought is from \( H^2(BU(1) \times BU(1)) = \mathbb{Z} \oplus \mathbb{Z} \), so it is \( Aa + Bb \), where \( a \) and \( b \) correspond to the two \( U(1) \)'s. Let us consider a 2-cycle in \( BU(1) \times BU(1) \), on which \( a \) takes the value 1, by abuse of language let us call this 2-cycle also \( a \). If we calculate the value of \( m_2 \) on this 2-cycle, we get the coefficient \( a \). Let us take a general perturbation of \( a \) in \( \xi_0 \). Then the coefficient \( A \) above will be the intersection number of this perturbation and the (closure of the) double-point locus of \( f_\kappa \). However we can make this perturbation in a special way: We can lift \( a \) in a universal direction (not contained in the double-point locus) everywhere but in one
point. The lifting will have a boundary over this point – the image of $S^1 \subset \mathbb{C}$ under the map: $z \mapsto (z, 1, z, z^2, \overline{z}, 1)$ – see the representation $\lambda_0(\text{III}_{2,2})$. To obtain a complete lifting (perturbation) of $a$, we have to extend this map from $S^1$ to the disc. This way we achieved that the intersection number of the double point locus and (a perturbation of) $a$ is to be counted in one fibre. Indeed we only have to calculate the intersection number of the curve $D^2 \rightarrow \mathbb{C}^6$, $z \mapsto (z, 1, z, z^2, \overline{z}, 1)$ – see the representation $\lambda_0(\text{III}_{2,2})$. To obtain a complete lifting (perturbation) of $a$, we have to extend this map from $S^1$ to the disc. This way we achieved that the intersection number of the double point locus and (a perturbation of) $a$ is to be counted in one fibre. Indeed we only have to calculate the intersection number of the curve $D^2 \rightarrow \mathbb{C}^6$, $z \mapsto (z, 1, z, z^2, \overline{z}, 1)$ and the double point locus, which is given by an equation above. To get this we substitute $x = z, y = 1, u_1 = z, u_2 = z^2, v_1 = \overline{z}, v_2 = 1$ in the equation. We get $z^2(|z|^4 - 10|z|^2 - 7) = 0$. Since we can clearly get rid of the second factor by a small perturbation, we obtain that the intersection number is (so $A$ equals to) 2. Very similarly, we substitute $x = 1, y = z, u_1 = 1, u_2 = z, v_1 = z^2, v_2 = z$ into the same equation and we get that $B = 2$. So the double point cohomology class represented by the double point locus of $f_\kappa$ is $2a + 2b$.

**Remark 5.1.** Technically what happened is that we calculated the equation of the double point locus in one fibre and — in some sense — we “substituted the representation” $\lambda_0$ for the variables (the value of $A$ and $B$ could have been obtained this way together). Of course, this method works for the singularities $A_i$ above, too.

Now let us deal with the (closure of the) triple point set. This is given by the condition that the two equations in $(\ast)$ have two common roots. This is equivalent to saying that the matrix

\[
\begin{pmatrix}
-x^2 & -u_1x - x^2 & u_2y \\
-v_1x & y^2 + v_2y & y^2
\end{pmatrix}
\]

has rank $\leq 1$. If this variety (in fact its ideal) were given by 2 equations then we could follow the same procedure as above. This is not the case, but there are ways to work with these kind of determinantal varieties. An easy way which works here is that we can write it as a difference of two varieties both given by two equations.

\[
\left\{ \det \begin{pmatrix} -x^2 & -u_1x - x^2 \\ -v_1x & y^2 + v_2y \end{pmatrix}, \quad \det \begin{pmatrix} -u_1x - x^2 \\ y^2 + v_2y \end{pmatrix} \right\} \setminus \left\{ -u_1 - x, y + v_2 \right\}.
\]

Now we can “substitute the representation $\lambda_0(\text{III}_{2,2})$” into these equations and obtain that the cohomology class represented by the (closure of the) triple point locus of $f_\kappa$ is $(2b)(2a) - ab = 3ab$.

Since there are no quadruple points near the singularity $\text{III}_{2,2}$, the cohomology classes corresponding to multiple points of $f_\kappa$ are as follows

\[
m_1 = 1, \quad m_2 = 2a + 2b, \quad m_3 = 3ab, \quad m_{\geq 4} = 0.
\]

Again, we can use the Gysin homomorphism to obtain from this classes the cohomology classes corresponding to multiple points in the target (pulled
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back), just as we did in case of $A_i$ singularities. We get that $f_κ^*(n_r) = 1/r \cdot 4(a + b)m_r$, that is
\[
f_κ^*(n_1) = 4(a + b), \quad f_κ^*(n_2) = 4a^2 + 8ab + 4b^2,
\]
\[
f_κ^*(n_3) = 4a^2b + 4ab^2, \quad f_κ^*(n_≥ 4) = 0.
\]

We also need the total Chern class of the map $f_κ$, which can be read from the representations $λ_0(III_{2,2}), λ_1(III_{2,2})$ as follows
\[
(1 + 2a)(1 + 2b)(1 + a + b)
\]
\[
= 1 + (2a + 2b) + (3ab) + (−a^2b − ab^2) + (a^3b − a^2b^2 + ab^3)
\]
\[
+ (−a^4b + a^3b^2 + a^2b^3 − ab^4) + \cdots.
\]

Now we have everything to substitute into the formulas $(1_r)$. Concrete calculation shows that $(1_r)$ holds for $r = 1, 2, 3, 4, 5, 6 (t = 42), 7 (u_2 = 2u_1 − 284)$. We also need the total Chern class of the map $f_κ$, which can be read from the representations $λ_0(III_{2,2}), λ_1(III_{2,2})$ as follows
\[
(1 + 2a)(1 + 2b)(1 + a + b)
\]
\[
= 1 + (2a + 2b) + (3ab) + (−a^2b − ab^2) + (a^3b − a^2b^2 + ab^3)
\]
\[
+ (−a^4b + a^3b^2 + a^2b^3 − ab^4) + \cdots.
\]

Recall that the whole maximal symmetry group of a prototype of $III_{2,2}$ is $U(2)$ and we worked only with its maximal torus $U(1) \times U(1)$, and found that some formulas hold in $H^*(BU(1) \times BU(1))$. However these formulas are the natural images of the analogous formulas for in $H^*(BU(2))$, and the map $H^*(BU(2)) \rightarrow H^*(BU(1) \times BU(1))$ induced by the inclusion $U(1) \times U(1) \subset U(2)$ is injective. So we can also conclude that formulas $(1_r) r = 2, 3, \ldots, 7$ hold with $t = 43, u_2 = 2u_1 − 284$ in the cohomology of the base space of the global normal form of $III_{2,2}$.

This means that if a formula like $(1_r)$ holds for maps with $III_{2,2}$ singularity then they must have the mentioned coefficients. To argue that they really hold for every map not having worse singularity than $III_{2,2}$ we would need that these formulas hold restricted to the base spaces of the global normal forms of multi-singularities, where the multi-singularity is put together from mono-singularities not worse than $III_{2,2}$. Easy cohomological analysis shows that if a formula holds in the base spaces of the global normal form of some mono-singularities, then the analogous formula holds (at least modulo torsion) in the base space of the global normal form of a multi-singularity put together from these mono-singularities.

According to the universal property of the gPTc this means that formulas $(1_r)$ for $r = 2, 3, 4, 5, 6 (t = 42), 7 (u_2 = 2u_1 − 284)$ hold for any map not having more complicated singularity than $III_{2,2}$.

Now, if we want to set the value of the parameter $u_1$ (and so $u_2$) we need to go through the analogous procedure with the stable singularities of codimension 7. In fact there is only one such, the one denoted by Mather as $I_{2,2}$.
(it is defined by its local algebra $\mathbb{C}[[x, y]]/(xy, x^2 + y^2)$). Recall that the procedure involves the following: Determination of the maximal compact symmetry group (or in some cases at least its maximal torus), its representations on the source and target spaces — then we have the global normal form. From this we can compute the total Chern class (basically $c = c(\lambda_1 - \lambda_0)$). More difficult is the determination of the cohomology classes represented by the multiple points — i.e., the incidence classes $I(rA_0, I_{2,2})$. However, the methods we used in the case of $A_i, III_{2,2}$ are sufficient to go through the calculation with $I_{2,2}$ (which we omit here), and get that

the total Chern class = \( \frac{(1 + 2a)(1 + 2b)(1 + c)}{(1 + a)(1 + b)} \) \( \in \mathbb{Z}[a, b, c] \) and

\[ m_1 = 1, \quad m_2 = 3c - a - b, \]
\[ m_3 = 3c^2 - 3(a + b)c + 3ab, \quad m_4 = (c - a)(c - b)(c - a - b). \]

As above we can substitute these values, and prove Theorem 2.3.


Let us make some remarks. First it would be nice to prove the so called refined formulas of Kleiman [K4] with our method, i.e., proving the multiple-point formulas without the comment “modulo torsion”. Although this does not seem to be impossible, we should make a much finer cohomological analysis, already in the case of immersion formulas — so cohomologies of the symmetric groups will get in the picture.

Another interesting question is the existence of analogous multiple-point formulas over the reals. The prototype of these formulas has been found by Szűcs in [Sz2] and [Sz3]. The fact is that our formulas can be translated blindly to the real world ($c_i \mapsto w_i$) with the only difference that $f^*(n_r)$, which does not make sense there should be replaced by suitable “linking cohomology class”, more details in a subsequent paper.

In [R] we computed Thom polynomials of mono-singularities no matter how complicated they are. Here we computed “Thom polynomials” of multi-singularities, whose mono-terms are all the simplest ones: $A_0$. So there is gap in between. One should understand and compute the Thom polynomials of multi-singularities of type, e.g., $A_1 + A_1, 2A_0 + 3A_1 + III_{2,2}$, etc. The author believes that these problems also can be attacked by the method presented in this paper.

References


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K-GROUPS AND CLASSIFICATION OF SIMPLE QUOTIENTS OF GROUP C*-ALGEBRAS OF CERTAIN DISCRETE 5-DIMENSIONAL NILPOTENT GROUPS

S. Walters

The \(K\)-groups, the range of trace on \(K_0\), and isomorphism classifications are obtained for simple infinite dimensional quotient \(C^*\)-algebras of the group \(C^*\)-algebras of six lattice subgroups, corresponding to each of the six non-isomorphic 5-dimensional connected, simply connected, nilpotent Lie groups. Connes’ non-commutative geometry involving cyclic cocycles and the Chern character play a key role in the proofs.

1. Introduction.

It is known that there are only six non-isomorphic 5-dimensional connected, simply connected, nilpotent Lie groups. These groups are denoted by \(G_{5,j}(j = 1, \ldots, 6)\), and were studied in great detail in Nielsen’s paper [10]. In [9], Milnes and the author have studied a natural lattice subgroup \(H_{5,j}\) of \(G_{5,j}\). These subgroups are higher dimensional analogues of the well-known discrete Heisenberg group \(H_3\), (but with more complicated multiplication rules inherited from \(G_{5,j}\)). The main result of [9] is an identification of all the simple infinite dimensional quotient \(C^*\)-algebras of the group \(C^*\)-algebra \(C^*(H_{5,j})\) – more specifically, they consist, respectively, of the ‘primary’ algebras \(A_{\theta}^{5,1}, A_{\theta,\varphi}^{5,2}, A_{\theta}^{5,3}, A_{\theta,\varphi}^{5,4}, A_{\theta}^{5,5}, A_{\theta,\varphi}^{5,6}\) (where \(\theta, \varphi\) are irrational and are independent in the 5, 2 and 5, 4 cases), other simple \(C^*\)-algebras isomorphic to matrix algebras over irrational rotation algebras (of any size and any irrational parameter), and a few more which are expressed as crossed products by the integers.

The objective of this paper is to find the \(K\)-groups, the range of the trace on \(K_0\), and obtain a classification for the ‘primary’ simple quotient \(C^*\)-algebras amongst themselves. Since each of these algebras is isomorphic to a crossed product by the integers, one uses the Pimsner-Voiculescu six term exact sequence [13] to compute their \(K\)-groups and Pimsner’s Theorem on the tracial range [12]. For the algebras \(A_{\theta}^{5,1} \cong A_{\theta} \otimes A_{\theta}, A_{\theta,\varphi}^{5,2}, A_{\theta}^{5,3}, A_{\theta,\varphi}^{5,4}\), application of the Pimsner-Voiculescu exact sequence is not hard. This is done briefly in Section 2, and included for comparison and completion.
For the algebras $A_5^{θ,5}$ and $A_5^{θ,6}$ however, the application of the Pimsner-Voiculescu sequence is not so straightforward, as the action of the underlying automorphism (of the crossed product) on $K_*$ requires some careful work. More specifically, in order to calculate this action we make use of Connes’ non-commutative geometry involving cyclic cocycles and the Connes Chern character [3] in order to decipher $K$-group elements. This is dealt with in Sections 3 and 4, which are the main parts of the paper. In summary, we obtain the following result on the $K$-groups and range of the trace on $K_0$:

<table>
<thead>
<tr>
<th>Algebra</th>
<th>$A_3^{θ}$</th>
<th>$A_4^{θ}$</th>
<th>$A_5^{θ,1}$</th>
<th>$A_5^{θ,2}$</th>
<th>$A_5^{θ,3}$</th>
<th>$A_5^{θ,4}$</th>
<th>$A_5^{θ,5}$</th>
<th>$A_5^{θ,6}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$K_0 = K_1$ =</td>
<td>$Z^2$</td>
<td>$Z^3$</td>
<td>$Z^8$</td>
<td>$Z^4$</td>
<td>$Z^6$</td>
<td>$Z^3$</td>
<td>$Z^4$</td>
<td>$Z^4$</td>
</tr>
<tr>
<td>$τ_*K_0 = Z^+$</td>
<td>$Zθ$</td>
<td>$Zθ$</td>
<td>$Zθ + Zθ^2$</td>
<td>$Zθ + Zθ^2$</td>
<td>$Zθ + Zθ^2$</td>
<td>$Zθ + Zθ^2$</td>
<td>$Zθ$</td>
<td>$Zθ + Zθ^2$</td>
</tr>
</tbody>
</table>

Here, we have included the well-known result for the irrational rotation algebra $A_θ = A_3^θ$ ([13] and [14]), as well as for the Heisenberg C*-algebra $A_4^θ$ studied by Packer in [11] (where it is referred to as class 2). (According to the convention adopted in [8] and [9], the superscripts on $A_3^θ, A_4^θ$ indicate the dimensions of the discrete nilpotent groups for which these are simple infinite dimensional quotients of the associate group C*-algebra – namely, the discrete Heisenberg group $H_3$ and the discrete group $H_4$ introduced in [8], respectively.)

The determination of the simple infinite dimensional quotients arising from 6-dimensional discrete nilpotent groups $H_{6,j}$ has been done by Milnes in [6] and [7] for $H_{6,4}$ and $H_{6,10}$ and by Junghenn and Milnes in [5] for $H_{6,7}$. It is known that there are exactly twenty-four non-isomorphic 6-dimensional connected, simply connected, nilpotent Lie groups $G_{6,j}$ ($j = 1, \ldots, 24$) (see Nielsen [10]), each of which contains a natural lattice subgroup $H_{6,j}$. In the 7-dimensional case, there are by contrast uncountably many non-isomorphic such Lie groups.

**Notation.** Throughout the paper we shall adopt Connes’ and Rieffel’s convention and write

$$e(t) := e^{2πi t}.$$

Briefly, recall Pimsner’s procedure [12] for finding the range of the trace in the case of crossed products by the integers. Let $A$ be a C*-algebra (for us unital) and $σ$ an automorphism of $A$. Let $τ$ be a trace state on $A ≀_σ Z$ and use it also to denote its restriction to $A$. Let $q : \mathbb{R} → R/τ_*K_0(A)$ denote the quotient map. For an element $[u]$ in $κ(σ_* − id_σ) ⊆ K_1(A)$, where $u$ is in $U_n(A)$ (the $n×n$ unitary matrices in $A$), consider the element of $\mathbb{R}/τ_*K_0(A)$.
called the “determinant” of \([u]\), given by
\[
\Delta[u] = q \left( \frac{1}{2\pi i} \int_a^b (\tau \otimes \text{Tr})(\xi(t)\xi(t)^{-1})dt \right)
\]
where \(\xi : [a, b] \to U_n(A)\) is a piecewise continuously differentiable path such that \(\xi(a) = 1\) and \(\xi(b) = \sigma(u)u^{-1}\). Pimsner’s result [12] (Theorem 3) is that the following is a short exact sequence:
\[
0 \longrightarrow r_*K_0(A) \longrightarrow \tau_*K_0(A \rtimes_{\sigma} \mathbb{Z}) \longrightarrow \Delta(\ker(\sigma_* - \text{id}_*)) \longrightarrow 0
\]
where \(\tau\) is the canonical inclusion (as subgroups of \(\mathbb{R}\)) and \(q\) is the restriction of the canonical map \(q\).

2. The \(C^*\)-algebra \(A_{\theta}^{5,k}\) for \(k = 1, 2, 3, 4\).

2.1. The \(C^*\)-algebra \(A_{\theta}^{5,1}\). Let us first look at the \(C^*\)-algebra \(A_{\theta}^{5,1}\) generated by unitaries \(U, V, W, X\) satisfying
\[
(2.1) \quad UV = \lambda VU, \quad WX = \lambda XW, \quad UW = WU,
\]
\[
UX = UX, \quad VW = WV, \quad VX = XV,
\]
where \(\lambda = e(\theta)\) and \(\theta\) is irrational (as in [9], Section 1). It is clear that it is isomorphic to the simple \(C^*\)-algebra \(A_{\theta} \otimes A_{\theta}\). We prefer to view it, however, as the crossed product \((A_{\theta} \otimes C(\mathbb{T})) \rtimes_{\sigma} \mathbb{Z}\) where \(A_{\theta}\) is generated by \(U, V, C(\mathbb{T})\) by \(W\), and \(\sigma = \text{Ad}_X\). So \(\sigma\) fixes \(U, V\) and \(\sigma(W) = \lambda W\).

Since \(\sigma\) is homotopic to the identity automorphism (in the sense of [1], 5.2.2), the Pimsner-Voiculescu sequence yields that \(K_j(A_{\theta}^{5,1})\) is isomorphic to \(K_j(A_{\theta} \otimes C(\mathbb{T}^2))\) \((j = 0, 1)\), which is isomorphic to \(\mathbb{Z}^8\). (One can also use the Künneth Theorem [16] to get \(K_j(A_{\theta} \otimes A_{\theta}) = \mathbb{Z}^8\).) From Pimsner’s range of trace formula, one needs to know the generators of the kernel of \(id_* - \sigma_*\) in \(K_0(A_{\theta} \otimes C(\mathbb{T}))\). But \(id_* - \sigma_* = 0\). It is easy to show that a basis for \(K_1(A_{\theta} \otimes C(\mathbb{T}))\) consists of the following set \([V, U, [W], [\xi]]\) where \(\xi = (1 - e) \otimes 1 + e \otimes W\) and \(e\) is a Powers-Rieffel projection in \(A_{\theta}\) of trace \(\theta\). This follows from the short exact sequence
\[
0 \longrightarrow A_{\theta} \otimes C_0(\mathbb{T}) \longrightarrow A_{\theta} \otimes C(\mathbb{T}) \longrightarrow A_{\theta} \longrightarrow 0
\]
where \(C_0(\mathbb{T})\) is the ideal of functions in \(C(\mathbb{T})\) vanishing at 1, \(e\) is evaluation at 1, and \(i\) is inclusion. Using the Bott periodicity isomorphism \(s^0 : K_0(A_{\theta}) \to K_1(A_{\theta} \otimes C_0(\mathbb{T}))\) (as given by Connes [4]) one has \(s^0(e) = [1 \otimes 1 + e \otimes (W - 1)] = [\xi]\), giving us the fourth basis element. For the range of trace, and since we already know that \(\tau_*K_0(A_{\theta}) = \mathbb{Z} + \mathbb{Z} \theta\), we need to compute the “determinant” of each basis element. From \(U, V, W\) we get determinants already in \(\mathbb{Z} + \mathbb{Z} \theta\), since \(\sigma(V)V^* = \sigma(U)U^* = 1\) and \(\sigma(W)W^* = \lambda = e(\theta)\). For \(\xi\) one has
\[
\sigma(\xi)\xi^* = ((1 - e) \otimes 1 + \lambda e \otimes W) \cdot ((1 - e) \otimes 1 + e \otimes W^*) = ((1 - e) + \lambda e) \otimes 1.
\]
A path of unitaries connecting this element to the identity is simply $\eta_t = ((1 - e) + e(t\theta)e) \otimes 1$ for $0 \leq t \leq 1$.

Thus

$$\frac{1}{2\pi i} \int_0^1 \tau(\eta_t\eta_t^*) \, dt = \frac{1}{2\pi i} \int_0^1 2\pi i\theta e \, dt = \theta^2,$$

since $\tau(e) = \theta$. From Pimsner’s trace formula one therefore obtains

$$\tau_*(K_0(A_{\theta,\phi}^{5,2})) = \mathbb{Z} + \mathbb{Z}\theta + \mathbb{Z}\phi.$$

One now has the isomorphism classification for the algebras $A_{\theta,\phi}^{5,1}$ for non-quartic irrationals $\theta$ (i.e., those that are not zeros of an integral polynomial of degree at most four). Therefore, for two such non-quartic irrationals $\theta, \theta'$, the algebras $A_{\theta,\phi}^{5,1}$ and $A_{\theta',\phi}^{5,1}$ are isomorphic if and only if $\theta' = n \pm \theta$ for some integer $n$.

**The C*-algebra $A_{\theta,\phi}^{5,2}$.**

The C*-algebra $A_{\theta,\phi}^{5,2}$ is generated by unitaries $U, V, W$ satisfying

$$(2.2) \quad UV = \lambda VU, \quad UW = \mu WU, \quad VW = WV,$$

where $\mu = e(\phi)$ and $\lambda = e(\theta)$ are assumed to be independent elements of the abelian group $\mathbb{T}$, so that in fact the algebra is simple. (See [9], Section 2.) This algebra can be realized as the crossed product $C(\mathbb{T}^2) \rtimes_\gamma \mathbb{Z}$ where $C(\mathbb{T}^2)$ is generated by $V, W$ and $\gamma(V) = \lambda V$, $\gamma(W) = \mu W$. Since this automorphism is homotopic to the identity, the Pimsner-Voiculescu sequence gives $K_j(A_{\theta,\phi}^{5,2}) = \mathbb{Z}^4$ since $K_j(C(\mathbb{T}^2)) = \mathbb{Z}^2$, for $j = 0, 1$. Since $K_1(C(\mathbb{T}^2))$ has basis $[V], [W]$ and since $\gamma(V)V^* = e(\theta)$ and $\gamma(W)W^* = e(\phi)$, one easily obtains the range of trace as

$$\tau_*(K_0(A_{\theta,\phi}^{5,2})) = \mathbb{Z} + \mathbb{Z}\theta + \mathbb{Z}\phi.$$

(This does in fact hold also for rational $\theta, \phi$, but one must use the canonical trace on the crossed product.) The classification for the algebras $A_{\theta,\phi}^{5,2}$ now follows:

**Proposition.** For independent irrationals $\theta, \phi$ the C*-algebras $A_{\theta,\phi}^{5,2}$ and $A_{\theta',\phi'}^{5,2}$ are isomorphic if, and only if there exists $X \in \text{GL}(2, \mathbb{Z})$ such that $[\theta' \, \phi'] = [\theta \, \phi]X$.

**Proof.** Given $X = [\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}]$ in $\text{GL}(2, \mathbb{Z})$ the substitutions $V' = V^a W^b$, $W' = V^c W^d$ satisfy the relations

$$UV' = \lambda^a \mu^b V' U, \quad UW' = \lambda^c \mu^d W' U, \quad V' W' = W' V',$$

so that $U, V', W'$, which already generate $A_{\theta,\phi}^{5,2}$, also generate $A_{\theta+b\phi, c\theta+d\phi}^{5,2}$; hence these algebras are isomorphic. Conversely, if $A_{\theta,\phi}^{5,2}$ and $A_{\theta',\phi'}^{5,2}$ are isomorphic then by the above one has $\mathbb{Z} + \mathbb{Z}\theta + \mathbb{Z}\phi = \mathbb{Z} + \mathbb{Z}\theta' + \mathbb{Z}\phi'$. Writing
each of $\theta', \phi'$ in terms of $\theta, \phi$ (modulo $\mathbb{Z}$) and vice versa, and using their rational independence, one easily obtains a matrix $X$ in $\text{GL}(2, \mathbb{Z})$ such that $[\theta' \phi'] = [\theta \phi]X$. □

The C*-algebra $A^{5,3}_\theta$. The C*-algebra $A^{5,3}_\theta$ is generated by unitaries $U, V, W, X$ satisfying

\begin{align}
UV &= XVU, & UX &= \lambda XU, & VX &= XV, \\
VW &= \lambda WV, & UW &= WU, & WX &= VW,
\end{align}

where $\lambda = e(\theta)$ and $\theta$ is irrational. (See [9], Section 3.) One can view this algebra as the crossed product $A^4_\theta \rtimes_\nu \mathbb{Z}$, where $A^4_\theta$ is the Heisenberg C*-algebra generated by the unitaries $U, V, X$ satisfying the three relations in the first line of (2.3), and $\nu(X) = X, \nu(U) = U, \nu(V) = \lambda V$. Since this automorphism is also homotopic to the identity, and since $K_j(A^5_\theta) = \mathbb{Z}^j$, the Pimsner-Voiculescu exact sequence immediately gives $K_j(A^{5,3}_\theta) = \mathbb{Z}^6$ for $j = 0, 1$. To find the range of the trace on $K_0$ using Pimsner’s Theorem we will need to do the following. The Pimsner-Voiculescu exact sequence applied to $A^4_\theta$, viewed as the crossed product $A_\theta \rtimes_\sigma \mathbb{Z}$ (where $A_\theta$ is generated by $U, X$ and $\sigma = \text{Ad}V$) is

\[
\begin{array}{cccc}
K_0(A^{4}_\theta) & \xrightarrow{id_*-\sigma_*=0} & K_0(A^{4}_\theta) & \xrightarrow{i_*} K_0(A^{5,3}_\theta) \\
\delta_1 & \uparrow & & \downarrow \delta_0 \\
K_1(A^{4}_\theta) & \xleftarrow{i_*} & K_1(A^{4}_\theta) & \xleftarrow{id_*-\sigma_*} K_1(A^{4}_\theta).
\end{array}
\]

Recall from Lemma 1.2 of [13] that the group $K_1(A^{4}_\theta \rtimes_\sigma \mathbb{Z})$ is generated by classes of unitaries of the form

\[
(1 \otimes I_n - F) + Fx(V^{-1} \otimes I_n)F,
\]

where $F$ is a projection in $M_n(A)$ and $x \in M_n(A)$. (Here, $V$ is the canonical unitary of the crossed product: $\sigma(a) = VaV^{-1}$.) In addition, from page 102 of [13], the connecting homomorphism $\delta_1 : K_1(A \rtimes_\sigma \mathbb{Z}) \rightarrow K_0(A)$ is given on classes of such unitaries by

\begin{align}
\delta_1[(1 \otimes I_n - F) + Fx(V^{-1} \otimes I_n)F] &= [F].
\end{align}

Lemma. A basis for $K_1(A^{4}_\theta)$ is $\{[U], [V], [\xi]\}$ where

\[
\xi := (1 - e) + ew^{-1}V^{-1}e
\]

and $e \in A_\theta = C^*(X, U)$ is a Powers-Rieffel projection of trace $\theta$ and $w$ is a unitary in $A_\theta$ such that $w e w^{-1} = V^{-1}e V$.

Proof. From the above exact sequence we see that $\delta_1$ is surjective and thus $K_1(A^{4}_\theta)$ contains elements that are mapped by $\delta_1$ to $[1]$ and $[e]$. Applying (2.4) with $F = 1$ one has $\delta_1[V^{-1}] = [1]$. To find an element that $\delta_1$ maps to
where \( (2.5) \) is Packer’s Heisenberg C*-algebra of class 3 \([\theta]\), this algebra is simple with a unique trace state. As in \( \nu \) satisfying \( W \) is in \( A_\theta = C^*(X, U) \), which is fixed by \( v \), one obtains for \( \xi \) (since \( Vw \) commutes with \( e \))

\[
\nu(\xi)\xi^{-1} = ((1 - e) + \lambda w^{-1}V^{-1}e) \cdot ((1 - e) + Vwe) = (1 - e) + \lambda e.
\]

This is exactly the same situation we had for the algebra \( A_{\theta, 5, 1} \) which yielded \( \theta^2 \) in the trace range. Therefore one concludes in the same manner that \( \tau_* K_0(A_{\theta, 5, 1}^5) = \mathbb{Z} + \mathbb{Z} \theta + \mathbb{Z} \theta^2 \) which also yields the same type of classification statement as for the case of the algebra \( A_{\theta, 5, 1}^5 \) above (namely, for non-quartic irrationals \( \theta \)).

**The C*-algebra** \( A_{\theta, \phi}^{5, 4} \). The C*-algebra \( A_{\theta, \phi}^{5, 4} \) is generated by unitaries \( U, V, W \) satisfying

\[
(2.5) \quad WV = UVW, \quad WU = \lambda UW, \quad VU = \mu UV,
\]

where \( \mu = e(\phi) \) and \( \lambda = e(\theta) \) are assumed to be independent. This algebra is Packer’s Heisenberg C*-algebra of class 3 \([11]\). As shown in \([9]\), Section 4, this algebra is simple with a unique trace state. As in \([9]\), we can view \( A_{\theta, \phi}^{5, 4} \) as the crossed product \( A_\phi \rtimes_\sigma \mathbb{Z} \), where \( A_\phi \) is generated by \( U, V, \) and \( \sigma \) is the “Anzai” automorphism \( \sigma(U) = \lambda U, \quad \sigma(V) = UV \). Since \( \sigma \) induces the identity map on \( K_0(A_{\theta, \phi}) \), and since on \( K_1(A_{\phi}) = \mathbb{Z}[U] + \mathbb{Z}[V] \) one has \((id_* - \sigma_*)[U] = 0, \quad (id_* - \sigma_*)[V] = -[U]\), the Pimsner-Voiculescu exact sequence gives \( K_0(A_{\theta, \phi}^{5, 4}) = K_1(A_{\theta, \phi}^{5, 4}) = \mathbb{Z}^3 \). Now Pimsner’s machine states that the range of trace is obtained from that of \( \tau_* K_0(A_{\phi}) = \mathbb{Z} + \mathbb{Z} \phi \) and from the class \([U]\). But \( \sigma(U)U \) so that one has

\[
\tau_* K_0(A_{\theta, \phi}^{5, 4}) = \mathbb{Z} + \mathbb{Z} \phi + \mathbb{Z} \theta.
\]

The classification for underlying algebras is the same as for the algebras \( A_{\theta, \phi}^{5, 2} \) above.

**Proposition.** For independent irrationals \( \theta, \phi \) the C*-algebras \( A_{\theta, \phi}^{5, 4} \) and \( A_{\theta', \phi'}^{5, 4} \) are isomorphic if, and only if there exists \( X \in GL(2, \mathbb{Z}) \) such that \([\theta', \phi'] = [\theta, \phi]X\).
Proof. Since the group $GL(2, \mathbb{Z})$ is generated by the matrices $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, it suffices to check that $A^{5.4}_{\theta, \phi}$ and $A^{5.4}_{\theta, \phi + \phi}$ are isomorphic to $A^{5.4}_{\theta, \phi}$. To get the algebra $A^{5.4}_{\theta, \phi}$, one lets $W' = V^*$, $V' = W^*$, $U' = \mu \lambda U^*$ so that the universal relations (2.5) are checked for $W', V', U'$ in place of $W, V, U$, respectively, and with $\mu$ and $\lambda$ switched. To get the algebra $A^{5.4}_{\theta, \phi + \phi}$, in the same manner one lets $W' = W$, $V' = WV$, $U' = \lambda U$ which satisfy (2.5) with $\lambda$ remaining the same and $\mu$ replaced by $\lambda \mu$. Conversely, if $A^{5.4}_{\theta, \phi}$ and $A^{5.4}_{\theta', \phi'}$ are isomorphic then exactly as in the proof of previous proposition one shows that there is a matrix $X$ in $GL(2, \mathbb{Z})$ such that $\begin{bmatrix} \theta' & \phi' \\ \phi & \theta \end{bmatrix} = \begin{bmatrix} \theta & \phi \\ \phi & \theta \end{bmatrix} X$. □

3. The C*-algebra $A^{5.5}_\theta$.

Let us view the commutative 3-torus $C(T^3)$ as generated by its three canonical unitaries $X, W, V$, where $X(r, s, t) = e(r)$, $W(r, s, t) = e(s)$, $V(r, s, t) = e(t)$. The C*-algebra $A^{5.5}_\theta$ can be viewed as the crossed product $C(T^3) \rtimes_\sigma \mathbb{Z}$ where

(3.1) $\sigma(X) = \lambda X, \quad \sigma(W) = XW, \quad \sigma(V) = WV.$ (where $\lambda = e(\theta)$) which was introduced in [9] (Section 5). When $\theta$ is irrational, $A^{5.5}_\theta$ is the unique C*-algebra generated by unitaries $X, W, V, U$ satisfying the relations

(3.2) $UV = WVU, \quad UW = XWU, \quad UX = \lambda XU,$

$VW = WV, \quad VX = XV, \quad WX = WX.$

We shall prove the following result.

**Theorem 3.1.** For any $\theta$ (rational or irrational) one has

$K_0(A^{5.5}_\theta) = \mathbb{Z}^4, \quad K_1(A^{5.5}_\theta) = \mathbb{Z}^4.$

If $\theta$ is irrational, and if $\tau$ is the unique trace state on $A^{5.5}_\theta$, then $\tau_* K_0(A^{5.5}_\theta) = \mathbb{Z} + \mathbb{Z} \theta$. (This yields the usual isomorphism classification for irrational $\theta$ upon noting that $A^{5.5}_\theta \cong A^{5.5}_{1-\theta}$)

The Pimsner-Voiculescu exact sequence corresponding to the above crossed product is

(3.3) $K_0(C(T^3)) \xrightarrow{id_* - \sigma_*} K_0(C(T^3)) \xrightarrow{i_*} K_0(A^{5.5}_\theta) \xrightarrow{\delta_1} K_1(A^{5.5}_\theta) \xrightarrow{i} K_1(C(T^3)) \xrightarrow{id_* - \sigma_*} K_1(C(T^3))$

and our goal is to compute $id_* - \sigma_*$ at the $K_0$ and $K_1$ levels.
The Connes Chern character on $K_*(C(T^3))$. First we need to find concrete bases for the K-groups of $C(T^3)$. It is already known that $K_0(C(T^3)) = \mathbb{Z}^4$ and $K_1(C(T^3)) = \mathbb{Z}^4$. Let $B$ denote the Bott projection in $M_2(C(T^2))$ given by

$$B = \begin{bmatrix} 1 - f & g \\ g & f \end{bmatrix}$$

where $f, g \in C(T^2)$ are smooth functions satisfying

$$\phi#\text{Tr}(B, B, B) = \phi(f, g, g) = -\frac{6}{2\pi i} \int \int_{\mathbb{T}^2} f[g_x g_y - g_y g_x] \, dx \, dy = 1,$$

where $g_x := \partial g/\partial x$, which is just the Connes pairing of $[B]$ with $[\phi]$ (this number being often called the ‘twist’ of $B$ in the C*-literature), where $\phi$ is the fundamental cyclic cocycle on $T^2$:

$$\phi(f^0, f^1, f^2) = \frac{1}{2\pi i} \int \int_{\mathbb{T}^2} f^0[f^1_j f^2_j - f^1_j f^2_i] \, dx \, dy.$$ 

For $1 \leq i < j \leq 3$, let $P_{ij}$ denote the Bott projection in $M_2(C(T^3))$ in the variables $i, j$. More specifically,

$$P_{12}(r, s, t) = B(r, s), \quad P_{13}(r, s, t) = B(r, t), \quad P_{23}(r, s, t) = B(s, t).$$

Putting $b_{ij} = [P_{ij}] - [1]$ (the Bott elements), it is not hard to check that $\{[1], b_{12}, b_{13}, b_{23}\}$ is a basis for $K_0(C(T^3))$. Now the (numerical) Connes Chern character $c_0$ is the homomorphism

$$c_0 : K_0(C(T^3)) \to \mathbb{Z}^4$$

given by

$$c_0(x) = (x, \phi_{12}, \phi_{13}, \phi_{23})$$

where

$$\phi_{ij}(f^0, f^1, f^2) = \frac{1}{2\pi i} \int \int \int_{\mathbb{T}^3} f^0[f^1_i f^2_j - f^1_j f^2_i] \, dx_1 \, dx_2 \, dx_3$$

is a cyclic 2-cocycle on $C(T^3)$ and $f_k := \partial f/\partial x_k$. (Henceforth, all triple integrals are over the 3-torus.) From (3.4) one gets

$$\langle [P_{ij}], [\phi_{kl}] \rangle = \delta_{i,k} \delta_{j,l}$$

which gives

$$c_0[1] = (1, 0, 0, 0), \quad c_0[b_{12}] = (0, 1, 0, 0),$$
$$c_0[b_{13}] = (0, 0, 1, 0), \quad c_0[b_{23}] = (0, 0, 0, 1),$$

so that $c_0$ is injective on $K_0(C(T^3))$. 

Lemma 3.2. One has the following action of \( \sigma_* \) on \( K_0(C(\mathbb{T}^3)) \):

\[
\sigma_*[1] = [1], \quad \sigma_*(b_{12}) = b_{12}, \quad \sigma_*(b_{13}) = b_{12} + b_{13}, \quad \sigma_*(b_{23}) = b_{12} + b_{13} + b_{23}.
\]

Proof. For simplicity consider the change of variables \((u, v, w) = (r + \theta, r + s, s + t)\), and note that by the chain rule one has

\[
\frac{\partial}{\partial r} h(u, v, w) = h_1(u, v, w) + h_2(u, v, w),
\]

\[
\frac{\partial}{\partial s} h(u, v, w) = h_2(u, v, w) + h_3(u, v, w),
\]

\[
\frac{\partial}{\partial t} h(u, v, w) = h_3(u, v, w),
\]

which can be simplified by writing

\[
\frac{\partial}{\partial x_i} h(u, v, w) = h_i(u, v, w) + h_{i+1}(u, v, w)
\]

where \( h_4 = 0 \), and \( x_1 = r, x_2 = s, x_3 = t \). From this one gets

\[
\frac{\partial}{\partial x_i} g(u, v, w) \frac{\partial}{\partial x_j} g(u, v, w) = (g_i + g_{i+1})(\bar{g}_{j} + \bar{g}_{j+1})(u, v, w)
\]

and

\[
\frac{\partial}{\partial x_i} g(u, v, w) \frac{\partial}{\partial x_j} g(u, v, w) - \frac{\partial}{\partial x_j} g(u, v, w) \frac{\partial}{\partial x_i} g(u, v, w)
\]

\[
= [(g_i + g_{i+1})(\bar{g}_{j} + \bar{g}_{j+1}) - (g_j + g_{j+1})(\bar{g}_{i} + \bar{g}_{i+1})](u, v, w).
\]

Now if we write

\[
P_{ij} = \begin{bmatrix} 1 - f(u, v, w) & g(u, v, w) \\ \bar{g}(u, v, w) & f(u, v, w) \end{bmatrix}
\]

where \( f, g \) depend only on the \( i, j \) coordinates \((i < j)\), then

\[
\sigma(P_{ij}) = \begin{bmatrix} 1 - f(u, v, w) & g(u, v, w) \\ \bar{g}(u, v, w) & f(u, v, w) \end{bmatrix}
\]

for which one has

\[
\langle [\sigma(P_{ij})], [\phi_{\ell\ell}] \rangle = \langle \phi_{\ell\ell} \# \text{Tr}(\sigma(P_{ij}), \sigma(P_{ij}), \sigma(P_{ij}))
\]

\[
= -6\phi_{\ell\ell}(f(u, v, w), g(u, v, w), \bar{g}(u, v, w))
\]

\[
= -\frac{6}{2\pi i} \int \int \int f(u, v, w) \left[ \frac{\partial}{\partial x_k} g(u, v, w) \frac{\partial}{\partial x_l} g(u, v, w) \\
\frac{\partial}{\partial x_k} \bar{g}(u, v, w) \frac{\partial}{\partial x_l} \bar{g}(u, v, w) \right] drdsdt
\]

\[
= -\frac{6}{2\pi i} \int \int \int f(u, v, w) \left[ (g_k + g_{k+1})(\bar{g}_{\ell} + \bar{g}_{\ell+1}) \\
- (g_{\ell} + g_{\ell+1})(\bar{g}_k + \bar{g}_{k+1}) \right] (u, v, w) drdsdt.
\]
Now since the transformation \((u, v, w) = (r + \theta, r + s, s + t)\) has Jacobian determinant 1, the change of variables formula gives

\[
\langle [\sigma(P_{ij})], [\phi_{k\ell}] \rangle = -\frac{6}{2\pi i} \iiint f(r, s, t) \cdot \left[(g_k + g_{k+1})(\bar{g}_\ell + \bar{g}_{\ell+1})
- (g_\ell + g_{\ell+1})(\bar{g}_k + \bar{g}_{k+1})\right] (r, s, t) \, dr \, ds \, dt
\]

which yields

\[
\delta_{i,k}\delta_{j,\ell} + \delta_{i,k+1}\delta_{j,\ell+1} + \delta_{i,k+1}\delta_{j,\ell} + \delta_{i,k+1}\delta_{j,\ell+1}
\]

where \(\phi_{3,4} = \phi_{2,4} = 0\). One thus gets

\[
\langle [\sigma(P_{12})], [\phi_{12}] \rangle = 1, \quad \langle [\sigma(P_{12})], [\phi_{13}] \rangle = 0, \quad \langle [\sigma(P_{12})], [\phi_{23}] \rangle = 0,
\]

\[
\langle [\sigma(P_{13})], [\phi_{12}] \rangle = 1, \quad \langle [\sigma(P_{13})], [\phi_{13}] \rangle = 1, \quad \langle [\sigma(P_{13})], [\phi_{23}] \rangle = 0,
\]

\[
\langle [\sigma(P_{23})], [\phi_{12}] \rangle = 1, \quad \langle [\sigma(P_{23})], [\phi_{13}] \rangle = 1, \quad \langle [\sigma(P_{23})], [\phi_{23}] \rangle = 1,
\]

which yields

\[
\text{ch}_0[\sigma(P_{12})] = (1, 1, 0, 0), \quad \text{ch}_0[\sigma(P_{13})] = (1, 1, 1, 0),
\]

\[
\text{ch}_0[\sigma(P_{23})] = (1, 1, 1, 1)
\]

and the injectivity of \(\text{ch}_0\) thus yields the following equalities in \(K_0(C(\mathbb{T}^3))\)

\[
\sigma_*(b_{12}) = b_{12}, \quad \sigma_*(b_{13}) = b_{12} + b_{13}, \quad \sigma_*(b_{23}) = b_{12} + b_{13} + b_{23}.
\]

These give the desired result. \(\square\)

We now turn our attention to \(K_1\).

**Lemma 3.3.** A basis for \(K_1(C(\mathbb{T}^3))\) is \([[X], [W], [V], [\xi]]\), where \(\xi = I_2 + (V - 1) \otimes P_{12}\) is a unitary in \(M_2(C(\mathbb{T}^3))\) and \(P_{12}\) is the Bott projection in the variables \(X, W, V\).

**Proof.** This immediately follows from the Künneth Theorem applied to the tensor product expansion of \(K_1(C(\mathbb{T}^3)) = K_1(C(\mathbb{T}^2) \otimes C(\mathbb{T}))\) and using the individual generators of each factor. \(\square\)

**Lemma 3.4.** The action of \(\sigma\) on \(K_1(C(\mathbb{T}^3))\) is given by

\[
\sigma_*[X] = [X], \quad \sigma_*[W] = [X] + [W], \quad \sigma_*[V] = [W] + [V], \quad \sigma_*[\xi] = [\xi] + [W].
\]

**Proof.** The only nontrivial part is to show \(\sigma_*[\xi] = [\xi] + [W]\) (the rest follow trivially from the definition of \(\sigma\)). From Lemma 3.2 one has \(\sigma(P_{12}) = [P_{12}]\), and since \(C(\mathbb{T}^2)\) has the cancellation property, there is a unitary \(R\) in \(M_2(C(\mathbb{T}^3))\) (which depends only on the first two variables) such that
\[ \sigma(P_{12}) = RP_{12}R^* \]. Hence

\[
\sigma_* [\xi] = [I_2 + (WV - 1) \otimes \sigma(P_{12})] \\
= [I_2 + (WV - 1) \otimes RP_{12}R^*] \\
= [R(I_2 + (WV - 1) \otimes P_{12})R^*] \\
= [I_2 + (WV - 1) \otimes P_{12}] \\
= [I_2 + (W - 1) \otimes P_{12}] + [I_2 + (V - 1) \otimes P_{12}] \\
= [\xi] + [I_2 + (W - 1) \otimes P_{12}]
\]

and now we claim that \([I_2 + (W - 1) \otimes P_{12}] = [W] \). It is enough to show

that this equality holds in \(K_1(C(T^2)) \) (since all concerned variables here are
the first two – involving \(X, W\)). This is shown in the following remark. □

**Remark.** Let us view the 2-torus \(T^2\) as \(T \times [0, 1]\) with the endpoints of
the interval identified. Recall that the Bott projection in \(M_2(C(T^2))\) can be
given by \(P(x, s) = M(x, s) e_0 M(x, s)^*\) for \(0 \leq s \leq 1\), where

\[
M(x, s) = E^s \begin{bmatrix} \pi & 0 \\ 0 & 1 \end{bmatrix} E^{-s},
\]

\[
E^s = \begin{bmatrix} \cos(\pi s/2) & -\sin(\pi s/2) \\ \sin(\pi s/2) & \cos(\pi s/2) \end{bmatrix}, \quad e_0 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},
\]

where \(M : T \times [0, 1] \to U_2(\mathbb{C})\) is smooth (but clearly \(M\) is not in \(M_2(C(T^2))\)),
and \(E^s\) satisfies the usual exponential property. In our case above, the
unitary \(V\) corresponds to \(x\) and \(W(x, s) = e(s)\). The unitary \(\eta := I_2 + (W - 1) \otimes P\) can now be written as

\[
\eta = I_2 + (W - 1) \otimes M e_0 M^* = M(I_2 + (W - 1) \otimes e_0) M^* = M \begin{bmatrix} W & 0 \\ 0 & 1 \end{bmatrix} M^*.
\]

Now an explicit path of unitaries \(t \mapsto \eta_t\) in \(M_2(C(T^2))\) connecting \(\eta\) to
\(\begin{bmatrix} W & 0 \\ 0 & 1 \end{bmatrix}\) can be given by

\[
(3.5) \quad \eta_t(x, s) = M(x, ts) \begin{bmatrix} e(s) & 0 \\ 0 & 1 \end{bmatrix} M(x, ts)^*.
\]

(For each \(t\) one has \(\eta_t(x, 0) = \eta_t(x, 1) = I_2\) so that \(\eta_t \in M_2(C(T^2))\).) It
follows, in particular, that \(\eta\) and \(W\) give the same class in \(K_1(C(T^2))\). The
explicit form of the unitary path (3.5) is used in the trace computation
below.

In view of Lemmas 3.2 and 3.4 one obtains, from the Pimsner-Voiculescu
exact sequence (3.3), the \(K_0\) and \(K_1\) groups of \(A_9^{5,5}\) as stated in Theorem 3.1.
Tracial Range on $K_0(A^{5,5}_\theta)$. To complete the proof of Theorem 3.1 we now use Pimsner’s Theorem. In the present case, the quotient map is $q : \mathbb{R} \to \mathbb{R}/\mathfrak{r}_* (K_0(C(\mathbb{T}^3))) = \mathbb{R}/\mathbb{Z}$, since the range of the canonical trace state $\mathfrak{r}$ on $K_0(C(\mathbb{T}^3))$ is $\mathbb{Z}$. From Lemma 3.4 the kernel of $id_* - \sigma_*$ in $K_1(C(\mathbb{T}^3))$ is generated by the classes $[X]$ and $[\xi] - [V]$. For $[X]$, since $\sigma(X) = \lambda X$, one clearly has $\Delta[X] = q(\theta)$. For $[\xi] - [V]$, it suffices to show that $\Delta([\xi] - [V]) = 0$, and this will complete the proof that $\mathfrak{r}_* K_0(A^{5,5}_\theta) = \mathbb{Z} + \mathbb{Z}\theta$. As in the proof of Lemma 3.4, we noted that

$$\sigma(\xi) = R_0 (I + (W - 1) \otimes P_{12}) R^* = R \xi MW_1 M^* R^*$$

where $R$ is a unitary in $M_2(C(\mathbb{T}^2))$ such that $\sigma(P_{12}) = RP_{12} R^*$, and $I + (W - 1) \otimes P_{12} = MW_1 M^*$ (in the notation of the above remark). Writing $V_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and similarly for $W_1$, we have $[\xi] - [V] = [\xi V_1^*]$. Thus one has

$$\sigma \left( \begin{bmatrix} [\xi V_1^*] \\ I \end{bmatrix} \right) \begin{bmatrix} [\xi V_1^*] \\ I \end{bmatrix}^{-1} = \begin{bmatrix} \sigma(\xi) \\ I \end{bmatrix} \begin{bmatrix} V_1^* W_1^* \\ I \end{bmatrix} \begin{bmatrix} V_1 \\ I \end{bmatrix} \begin{bmatrix} \xi^* \\ I \end{bmatrix} = \begin{bmatrix} R \\ R^* \end{bmatrix} \begin{bmatrix} \xi \\ I \end{bmatrix} \begin{bmatrix} MW_1 M^* \\ I \end{bmatrix} \begin{bmatrix} R^* \\ R \end{bmatrix} \begin{bmatrix} W_1^* \\ I \end{bmatrix} \begin{bmatrix} \xi^* \\ I \end{bmatrix}.$$

Letting $t \mapsto \mathfrak{r}_t$ be the standard unitary path such that $\mathfrak{r}_0 = I$ and $\mathfrak{r}_1 = \begin{bmatrix} R \\ R^* \end{bmatrix}$ one considers the following path of unitaries in $M_4(C(\mathbb{T}^2))$

$$\gamma_t = \mathfrak{r}_t \begin{bmatrix} \xi \\ I \end{bmatrix} \begin{bmatrix} \eta_t \\ I \end{bmatrix} \mathfrak{r}_t^* \begin{bmatrix} W_1^* \\ I \end{bmatrix} \begin{bmatrix} \xi^* \\ I \end{bmatrix}$$

where $\eta_t$ is the path defined by (3.5) such that $\eta_0 = W_1$, $\eta_1 = MW_1 M^*$. The path $\gamma_t$ clearly connects the above element to the identity. Now it is straightforward to see that $(\mathfrak{r} \otimes \text{Tr}_4) (\gamma_t \gamma_t^*) = (\mathfrak{r} \otimes \text{Tr}_2) (\eta_t \eta_t^*)$, since the fact that both $\mathfrak{r}_t$ and $\mathfrak{r}_t^*$ appear in $\gamma_t$ leads to their cancellation under the trace. Since $\eta_t$ has the form (3.5), one similarly obtains $(\mathfrak{r} \otimes \text{Tr}_2) (\eta_t \eta_t^*) = 0$. Therefore, $\Delta([\xi V_1^*]) = 0$ which completes the proof of Theorem 3.1.

4. The $C^*$-algebra $A^{5,6}_\theta$.

The $C^*$-algebra $A^{5,6}_\theta$ can be characterized as the unique $C^*$-algebra (when $\theta$ is irrational) generated by unitaries $U, V, W, Z$ such that

$$ZV = \lambda VZ, \quad ZU = V^{-1}UZ, \quad ZW = WZ,$$

$$UV = WVU, \quad UW = \lambda UW, \quad VW = WV.$$  

(As in [9].) It will be convenient to present $A^{5,6}_\theta$ as the crossed product $(C(\mathbb{T}) \otimes A_\theta) \rtimes_\nu \mathbb{Z}$, where $C(\mathbb{T})$ is generated by $W$, $A_\theta$ is generated by $V, Z$, 

and $\nu = \text{Ad}_{U^*} := U(\cdot)U^*$ is the automorphism given by
$$\nu(W) = \lambda W, \quad \nu(Z) = VZ, \quad \nu(V) = W \otimes V = WV.$$}

The aim of this section is to prove the following.

**Theorem 4.1.** For any $\theta$ (rational or irrational) one has
$$K_0(A^{5,6}_\theta) = \mathbb{Z}^4, \quad K_1(A^{5,6}_\theta) = \mathbb{Z}^4.$$ If $\theta$ is irrational, and if $\tau$ is the unique trace state on $A^{5,6}_\theta$, then $\tau_*K_0(A^{5,6}_\theta) = \mathbb{Z} + \mathbb{Z}\theta + \mathbb{Z}\theta^2$. (This yields the isomorphism classification for non-quartic irrationals $\theta$ upon noting that $A^{5,6}_\theta \cong A^{5,6}_{1-\theta}$.)

Since $ZV = \lambda VZ$, let $p = V^*g = f + gV$ be a Powers-Rieffel projection of trace $\theta$, where $f = f(Z)$, $g = g(Z)$ are $C^\infty$ real functions of $Z$ satisfying the usual properties that would make $p$ a projection. Among these, to be used below, are (after interpreting $f,g$ as functions of period 1 on $\mathbb{R}$)

\begin{equation}
\begin{aligned}
&f(t + \theta) = 1 - f(t) \text{ for } 0 \leq t \leq 1 - \theta, \\
g(t) - f(t)g(t) = g(t)f(t + \theta) \text{ for all } t,
\end{aligned}
\end{equation}

where we may assume, with no loss of generality, that $\frac{1}{2} < \theta < 1$, and where the dot indicates usual differentiation of a real function. Recall that $g$ is supported on $[0,1]$ on which it is given by $(f - f^2)^{1/2}$ and $f = 1$ on $[1-\theta,\theta]$.

**Lemma 4.2.** One has the following equalities in $K_0(C(T) \otimes A_\theta)$ for irrational $\theta$:
\begin{equation}
\begin{aligned}
\nu_*[p] &= [p] + ([P_{W,V}] - [1]) + ([P_{W,Z}] - [1]), \\
\nu_*[P_{W,Z}] &= [P_{W,Z}] + [P_{W,V}] - [1].
\end{aligned}
\end{equation}

**Proof.** We first compute the Connes Chern character for the algebra $C(T) \otimes A_\theta$. Consider the canonical cyclic 1-cocycles $\phi_0$ and $\phi_j$, $j = 1, 2$, of $C(T)$ and $A_\theta$, respectively, given by

\begin{equation}
\begin{aligned}
\phi_0(f^0, f^1) &= \frac{1}{2\pi i} \int_0^1 f^0 \frac{d}{dt}(f^1) dt, \\
\phi_1(x^0, x^1) &= \frac{1}{2\pi i} \tau(x^0 \delta_V(x^1)), \\
\phi_2(x^0, x^1) &= \frac{1}{2\pi i} \tau(x^0 \delta_Z(x^1))
\end{aligned}
\end{equation}

where $\delta_V, \delta_Z$ are the canonical derivations of $A_\theta$, and $\tau$ is the canonical trace. Let $\rho$ denote Connes’ canonical cyclic 2-cocycle of $A_\theta$

$$\rho(x^0, x^1, x^2) = \frac{1}{2\pi i} \tau \left( x^0 \left[ \delta_V(x^1) \delta_Z(x^2) - \delta_Z(x^1) \delta_V(x^2) \right] \right).$$

The Connes Chern character now takes the form of the group homomorphism

$$\text{ch}_0 : K_0(C(T) \otimes A_\theta) \to (\mathbb{Z} + \mathbb{Z}\theta) \oplus \mathbb{Z}^3.$$

given by taking the Connes pairings with the various cup products as
\[ ch_0(x) := (\tau(x), \langle x, \varphi_0 \# \varphi_1 \rangle, \langle x, \varphi_0 \# \varphi_2 \rangle, \langle x, \tau_0 \# \rho \rangle) \]
where \( \tau_0 \) is the canonical trace of \( C(\mathbb{T}) \). It is straightforward to check that
it assumes the following values on the basis for \( K_0(C(\mathbb{T}) \otimes A_\theta) \) given by the
classes \{[1], [1 \otimes p], [P_{W,V}], [P_{W,Z}]\}:
\[
\begin{align*}
ch_0[1] &= (1, 0, 0, 0) \\
ch_0[1 \otimes p] &= (\theta, 0, 0, 1) \\
ch_0[P_{W,V}] &= (1, 1, 0, 0) \\
ch_0[P_{W,Z}] &= (1, 0, 1, 0).
\end{align*}
\]
(This follows immediately from the multiplicative property of Connes’ canonical
pairing with respect to tensor products of algebras, see [3, III.3].) It is
immediate that \( ch_0 \) is injective on \( K_0 \) (for any \( \theta \)).

It is clear that \( \tau(\nu(p)) = \theta \). So to compute \( ch_0(\nu(p)) \), we will have to calculate
the above three 2-cocycles on \( \nu(p) \). First, we show that \( \langle [\nu(p)], \tau_0 \# \rho \rangle = 1 \).
Since \( p = V^*g + f + gV \), where \( f = f(Z), g = g(Z) \), one has \( \nu(p) = V^*W^*G + F + GWV \), where \( F = \nu(f) = f(VZ), G = \nu(g) = g(VZ) \), and hence
\[
\begin{align*}
\tau_0 \# \rho(\nu(p), \nu(p)) &= (\tau_0 \# \rho)(W^*V^*G + F + GWV, W^*V^*G + F + GWV) \\
&= \rho(F, F, F) + 3\rho(F, GV, V^*G) + 3\rho(F, V^*G, GV).
\end{align*}
\]
(In the expansion, the only possibly nonzero terms are ones of the form
\( (\tau_0 \# \rho)(W^a, \ldots, W^b, \ldots, W^c \ldots) \) where \( a + b + c = 0 \).) First, it is easy to verify that
\( \nu^{-1} \delta_Z \nu = \delta_Z, \nu^{-1} \delta_V \nu = \delta_V + \delta_Z \).

We thus see that \( \rho(F, F, F) = \rho(f, f, f) = 0 \) since \( \delta_V(f) = 0 \). Next, we have
\[
2\pi i \rho(F, GV, V^*G) = \tau(F[\delta_V(G)^0(G^0(G)) - \delta_Z(G)^0(G^0(G))])
\]
\[
= \tau(F[\delta_V(G) + 2\pi i G^0(G)^0(G) - \delta_Z(G)^0(G)^0(G)]g(2\pi i G^0(G)^0(G))])
\]
\[
= \tau(f[\delta_V(g) + \delta_Z(g) + 2\pi i g^0(g)^0(g) - \delta_Z(g)^0(g)^0(g)]g(2\pi i g^0(g)^0(g))])
\]
\[
= 2\pi i \tau(f[g^0(g) + \delta_Z(g)]g) = 2\pi i \tau(f \delta_Z(g^2)).
\]
Similarly, one checks that
\[
\rho(F, V^*G, GV) = -\tau(f V^* \delta_Z(g^2)).
\]
Therefore, using the properties (4.2) one gets
\[
(\tau_0 \# \rho)(\nu(p), \nu(p), \nu(p)) = 3\tau((f - VfV^*)\delta_Z(g^2)) = 1.
\]
Fix \(j = 1, 2\) and for simplicity let \(\psi = \varphi_0 \# \varphi_j\). From the definition of the cup product it can easily be shown that
\[
\psi(a^0 \otimes b^0, a^1 \otimes b^1, a^2 \otimes b^2)
= \varphi_0(a^2, a^0)\varphi_j(b^0, b^1) - \varphi_0(a^0, a^1)\varphi_j(b^2, b^0, b^1)
\]
for \(a^k \in C(T)\) and \(b^k \in A_0\). We want to calculate \(\psi(\nu(p), \nu(p), \nu(p))\). From \(p = V^*g + f + gV\) and \(\nu(p) = W^* \otimes V^*G + 1 \otimes F + W \otimes GV\) and upon expanding the expression \(\psi(\nu(p), \nu(p), \nu(p))\) we note that the only possibly nonzero terms are of the form \(\psi(W^a, W^b, W^c)\) for \(a + b + c = 0\). Hence using (4.4) and the cyclicity of \(\psi\) we get
\[
\psi(W^* \otimes V^*G + 1 \otimes F + W \otimes GV, W^* \otimes V^*G + 1 \otimes F + W \otimes GV,
W^* \otimes V^*G + 1 \otimes F + W \otimes GV)
= \psi(W^* \otimes V^*G, 1 \otimes F, W \otimes GV) + \psi(W^* \otimes V^*G, W \otimes GV, 1 \otimes F)
+ \psi(1 \otimes F, W^* \otimes V^*G, W \otimes GV) + \psi(1 \otimes F, 1 \otimes F, 1 \otimes F)
+ \psi(W \otimes GV, W^* \otimes V^*G, 1 \otimes F) + \psi(W \otimes GV, 1 \otimes F, W^* \otimes V^*G)
= 3\psi(W^* \otimes V^*G, 1 \otimes F, W \otimes GV)
+ 3\psi(W^* \otimes V^*G, W \otimes GV, 1 \otimes F) + \psi(1 \otimes F, 1 \otimes F, 1 \otimes F).
\]
Note that \(\psi(1 \otimes F, 1 \otimes F, 1 \otimes F) = 0\) since \(\varphi_0(x, 1) = 0\). Also, since \(\varphi_0(W^*, W) = 1\) one has
\[
\psi(W^* \otimes V^*G, 1 \otimes F, W \otimes GV) = -\varphi_0(W^*, W)\varphi_j(G^2, F)
= -\varphi_j(G^2, F),
\]
\[
\psi(W^* \otimes V^*G, W \otimes GV, 1 \otimes F) = \varphi_0(W^*, W)\varphi_j(V^*G^2V, F)
= \varphi_j(V^*G^2V, F)
\]
and hence
\[
\langle \nu(p), \varphi_0 \# \varphi_j \rangle = -3\varphi_j(G^2, F) + 3\varphi_j(V^*G^2V, F).
\]
First, for \(j = 1\), one has
\[
2\pi i \varphi_1(G^2, F) = \tau(\nu(g^2)\delta_V(\nu(f))) = \tau(g^2\delta_Z(f)) = -\tau(f\delta_Z(g^2))
\]
and
\[
2\pi i \varphi_1(V^*G^2V, F) = \tau(V^*\nu(g^2)\delta_V(\nu(f))) = \tau(V^*g^2V\delta_Z(f))
= \tau(g^2\delta_Z(VfV^*))
= -\tau(VfV^*\delta_Z(g^2))
\]
hence by (4.2)
\[ \langle \nu(p), \varphi_0 \# \varphi_1 \rangle = \psi(\nu(p), \nu(p), \nu(p)) = \frac{3}{2\pi i} \tau((f - V f V^*) \delta_Z(g^2)) = 1. \]

When \( j = 2 \), one similarly gets
\[ \varphi_2(C^2, F) = -\frac{1}{2\pi i} \tau(f \delta_Z(g^2)), \quad \varphi_2(V^* G^2 V, F) = -\frac{1}{2\pi i} \tau(V f V^* \delta_Z(g^2)) \]
and thus \( \langle \nu(p), \varphi_0 \# \varphi_2 \rangle = 1 \). Therefore, \( \text{ch}_0(\nu(p)) = (\theta, 1, 1, 1) \) from which one concludes the equality in the lemma. The proof of the second equality in the lemma follows in a similar way (in fact more like the proof of the third equality in Lemma 3.2 except with \( A_{\theta} \) in place of \( C(\mathbb{T}^2) \)).

Since
\[ K_1(C(\mathbb{T}) \otimes A_{\theta}) = [K_1(C(\mathbb{T})) \otimes K_0(A_{\theta})] \oplus [K_0(C(\mathbb{T})) \otimes K_1(A_{\theta})] = \mathbb{Z}^2 \oplus \mathbb{Z}^2 \]
it is easily seen that it has as basis the four elements \([W], [Z], [V], [\zeta] \)
where \( \zeta := W \otimes p + (1 - p) \). From Lemma 4.2 one has
\[ [\nu(p)] + [p_0] + [p_0] = [p] + [P] + [Q] \]
where \( P = P_{W,V}, Q = P_{W,Z} \), and \( p_0 = [1_0 0] \). Therefore, there exist integers \( m, n \) and an invertible matrix \( w \) over \( C(\mathbb{T}) \otimes A_{\theta} \) such that
\[ \nu(p) \oplus p_0 \oplus p_0 \oplus e_0 = w(p \oplus P \oplus Q \oplus e_0)w^{-1} \]
where \( e_0 = I_n \oplus O_m \). By suitably enlarging \( m \) one could assume that \( w \) is connectable to the identity by a smooth path of invertibles (upon replacing \( w \) by \( w \oplus w^{-1} \)). So let \( t \mapsto w_t \) be such a path with \( w_0 = I, w_1 = w \). Let
\[ p' = p \oplus p_0 \oplus a_0, \quad \text{and} \quad \zeta' = W \otimes p' + (I - p') \]
so that \( \nu(p') = w(p \oplus P \oplus Q \oplus e_0)w^{-1} \) and it is easily seen that \([\nu(\zeta')] = [\zeta'] \in K_1 \text{ of } C(\mathbb{T}) \otimes A_{\theta} \). Now since \([\zeta'] = [\zeta] + (n + 2)[W], \)
one gets \( \nu_*[\zeta] = [\zeta] \). It now follows that on \( K_1(C(\mathbb{T}) \otimes A_{\theta}) \) one has
\[ \ker(\nu_* - id_* \mathrel{\mapsto} Z[W] + Z[\zeta]), \quad \text{Im}(\nu_* - id_*) = Z[W] + Z[V]. \]
In view of the basis in (4.3) and the second equality in Lemma 4.2, on \( K_0(C(\mathbb{T}) \otimes A_{\theta}) \) one has
\[ \ker(\nu_* - id_* \mathrel{\mapsto} Z[1] + Z([P_{W,V}] - [1])), \quad \text{Im}(\nu_* - id_*) = Z([P_{W,V}] - [1]) + Z([P_{W,Z}] - [1]). \]
The Pimsner-Voiculescu exact sequence for \( A_{\theta}^{5,6} = C \rtimes_{\nu} \mathbb{Z} \), where \( C := C(\mathbb{T}) \otimes A_{\theta} \):
\[
\begin{array}{ccccccc}
K_0(C) & \xrightarrow{id_* - \nu_*} & K_0(C) & \xrightarrow{i_*} & K_0(A_{\theta}^{5,6}) \\
\uparrow & & & & \downarrow \\
K_1(A_{\theta}^{5,6}) & \xleftarrow{i_*} & K_1(C) & \xrightarrow{id_* - \nu_*} & K_1(C)
\end{array}
\]
now immediately yields $K_0(A_{5,6}^g) = K_1(A_{5,6}^g) = \mathbb{Z}^4$, as stated in Theorem 4.1.

It remains to obtain the range of the trace on $K_0$. For convenience, let us use the notation

$$\begin{bmatrix} X \\ Y \\ \vdots \end{bmatrix} := X \oplus Y \oplus \cdots$$

for block diagonal matrices. One then has (since $W$ is central in $C(\mathbb{T}) \otimes A_\theta$

(4.5)

$$\nu(\zeta')\zeta'^{-1} = (\lambda W \otimes w(p \oplus P \oplus Q \oplus e_0)w^{-1} + (I - w(p \oplus P \oplus Q \oplus e_0)w^{-1}))$$

$$\cdot (W^{-1} \otimes (p \oplus p_0 \oplus p_0 \oplus e_0) + (I - p \oplus p_0 \oplus p_0 \oplus e_0))$$

$$= w \begin{bmatrix} \lambda W \otimes p + (1 - p) \\ \lambda W \otimes P + (I - P) \\ \lambda W \otimes Q + (I - Q) \\ \lambda W \otimes e_0 + (I - e_0) \end{bmatrix} w^{-1} \begin{bmatrix} W^{-1} \otimes p + (1 - p) \\ W^{-1} \otimes P + (I - P) \\ W^{-1} \otimes Q + (I - Q) \\ W^{-1} \otimes e_0 + (I - e_0) \end{bmatrix}.$$

For $0 \leq t \leq 1$, let $t \mapsto a_t$ be a smooth path of invertibles in $C^*(W; V) \cong C(\mathbb{T}^2)$ such that $a_0 = \begin{bmatrix} W & 0 \\ 0 & 1 \end{bmatrix}$ and $a_1 = \lambda W \otimes P + (I - P)$, let $t \mapsto b_t$ be a smooth path of invertibles in $C^*(W; Z) \cong C(\mathbb{T}^2)$ such that $b_0 = \begin{bmatrix} W & 0 \\ 0 & 1 \end{bmatrix}$ and $b_1 = \lambda W \otimes Q + (I - Q)$, and let $t \mapsto c_t$ be a smooth path of invertibles in $C^*(W) \cong C(\mathbb{T})$ such that $c_0 = \begin{bmatrix} W & 0 \\ 0 & 1 \end{bmatrix}$ and $c_1 = \lambda W \otimes e_0 + (I - e_0)$. Let

$$\eta_t := \begin{bmatrix} e(\theta t)W \otimes p + (1 - p) \\ a_t \\ b_t \\ c_t \end{bmatrix}$$

and consider the smooth path

$$\gamma_t := (w_t \eta_t w_t^{-1} \eta_t^{-1}) \cdot \eta_t \xi$$

where $\xi$ is the right-most matrix in (4.5). Clearly, $\gamma_0 = I$ and $\gamma_1 = \nu(\zeta')\zeta'^{-1}$.

Now $v_t := w_t \eta_t w_t^{-1} \eta_t^{-1}$ being a commutator, one obtains (under the trace)

$$(\tau \otimes \text{Tr})(\hat{v}_t v_t^{-1}) = 0.$$ Hence

$$(\tau \otimes \text{Tr})(\hat{\gamma}_t \gamma_t^{-1}) = (\tau \otimes \text{Tr})(\hat{\eta}_t \eta_t^{-1})$$

$$= 2\pi i \theta (p) + (\tau \otimes \text{Tr})(\hat{a}_t a_t^{-1}) + (\tau \otimes \text{Tr})(\hat{b}_t b_t^{-1})$$

$$+ (\tau \otimes \text{Tr})(\hat{c}_t c_t^{-1})$$
and since $\tau(p) = \theta$, ones gets
\[
\Delta[\zeta'] = q\left(\frac{1}{2\pi i} \int_0^1 (\tau \otimes \text{Tr})(\dot{\gamma}_t \gamma_t^{-1}) dt\right)
\]
\[
= q\left(\frac{1}{2\pi i} \int_0^1 \left[2\pi i \theta^2 + (\tau \otimes \text{Tr})(\dot{a}_t a_t^{-1}) + (\tau \otimes \text{Tr})(\dot{b}_t b_t^{-1}) + (\tau \otimes \text{Tr})(\dot{c}_t c_t^{-1})\right] dt\right)
\]
\[
= q(\theta^2)
\]
since the last three integrals are integers (as $a_t, b_t, c_t$ are paths of invertibles in matrix algebras over $C(\mathbb{T}^2)$). Now as $q : \mathbb{R} \to \mathbb{R}/(\mathbb{Z} + \mathbb{Z}\theta)$, one deduces that $\tau_* K_0(A_{5,0}^\theta) = \mathbb{Z} + \mathbb{Z}\theta + \mathbb{Z}\theta^2$.

References


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