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One of the Tait conjectures, which was stated 100 years ago and proved in the 1980's, said that reduced alternating projections of alternating knots have the minimal number of crossings. We prove a generalization of this for knots in $S \times I$, where S is a surface. We use a combination of geometric and polynomial techniques.

1. Introduction.

A hundred years ago, Tait conjectured that the number of crossings in a reduced alternating projection of an alternating knot is minimal. This statement was proven in 1986 by Kauffman, Murasugi and Thistlethwaite, [**6**], [**10**], [**11**], working independently. Their proofs relied on the new polynomials generated in the wake of the discovery of the Jones polynomial.

We usually think of this result as applying to knots in the 3-sphere S^3. However, it applies equally well to knots in $S^2 \times I$ (where I is the unit interval $[0,1]$). Indeed, if one removes two disjoint balls from S^3, the resulting space is homeomorphic to $S^2 \times I$. It is not hard to see that these two balls do not affect knot equivalence. We conclude that the theory of knot equivalence in $S^2 \times I$ is the same as in S^3.

With this equivalence in mind, it is natural to ask if the Tait conjecture generalizes to knots in spaces of the form $S \times I$ where S is any compact surface.

More rigorously, consider the projection surface $S_0 = S \times \{\frac{1}{2}\}$. Let $\pi : S \times I \to S_0$ be the natural projection. We define *crossing number*, *alternating projections* and *alternating knots* in the obvious way. Given some choice of a definition of *reduced*, we want to know whether reduced alternating projections of alternating knots have minimal crossing number.

In other words, if $c(\pi(K))$ represents the crossing number of a projection, we want to know if it is always the case that if K and K' represent two spatial configurations of the same knot, so $\pi(K)$, $\pi(K')$ are two projections of the knot and $\pi(K)$ is "reduced" and alternating, then

$$c(\pi(K)) \leq c(\pi'(K)). \tag{1}$$

In [**4**], Kamada showed that if two projections of a knot in $S \times I$ are both "properly reduced" alternating projections with the same supporting genus, then they have the same number of crossings. A projection is properly reduced if the four regions that meet at each crossing of the projection are distinct. This is a generalization of a reduced projection in the plane. The supporting genus of a projection is the genus of the surface that results if each region of the projection surface is replaced with a disk.

The result presented in this paper extends Kamada's result in three ways. First, our notion of "reduced" is more general than Kamada's, and it is a more natural generalization of the definition in S^3.

We define a knot projection to be reduced on S_0 if there are no *trivial* simple closed curves on S_0 that intersect the knot projection exactly once (a trivial curve is a curve that is homotopic to the constant curve). This is natural, because curves like this exist exactly when one can perform the "untwisting" operation to reduce the number of crossings. Note that such a curve intersects the projection at a crossing, with two strands of the knot coming out of the intersection to either side of the curve.

Second, we consider arbitrary projections $\pi(K)$, $\pi(K')$, not just projections with the same supporting genus. Third, we show that the crossing numbers of the reduced alternating projections are not just equal to one another, but that they are minimal.

Indeed, we prove the following:

Theorem 1.1. *Let S be a compact surface. Let $\pi(K)$ be a reduced alternating projection of an alternating knot in $S \times I$ and let $\pi(K')$ be an arbitrary projection* (*of the same knot*)*. Then*

$$c(\pi(K)) \leq c(\pi(K')). \tag{2}$$

Unlike the proof of the original Tait conjecture, polynomial techniques were not enough to establish our result. These techniques are only strong enough to give results analogous to those of Kamada. We use Menasco's geometrical techniques to show something analogous to the supporting genus restriction always holds and to complete the proof. Independently, in [**5**], the author announced a version of Theorem 1.1 for knots and links, however, only an outline of the proof has appeared. The techniques utilized differ substantially from those presented here.

The specific breakdown of the paper is as follows. The second section of this paper presents the geometrical argument. The main result of this section is that the general result follows from the special case that S is a punctured compact orientable surface, and $\pi(K')$ *cuts this surface into disks and punctured disks.* This special case is analogous to Kamada's result.

The third section of this paper presents the polynomial argument. We define polynomial invariants for knots in $S \times I$ and use them to prove (2) in the special case.

Combining these two results gives the main theorem, the proof of which appears in the third section. The fourth and final section discusses extensions of the theorem, and other related questions and conjectures.

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2. The geometric argument.

In this section we show that in some sense, alternating projections of a knot in $S \times I$ wrap around the projection surface S_0 less than any other projections of the same knot. The rigorous statement of this idea is given by the next theorem. (Hayashi has proved a related result in the case that S is a torus [**2**].)

Theorem 2.1. *Let S be an orientable surface. Let K be a configuration of a knot in $S \times I$ such that $\pi(K)$ is regular and alternating. Let K' be any other configuration of the knot, and let $H : (S \times I) \times I \to (S \times I)$ be the isotopy carrying K' to K. Suppose that there exists a simple closed curve $\gamma \in \pi(K')^c$ which does not bound a disk in $\pi(K')^c$ (where $\pi(K')^c$ denotes the complement of the knot projection in S_0). Let A_0 be the annulus $\gamma \times I$. Let $A = H(A_0, 1)$ be the isotoped annulus. Then we may continue the isotopy, so that afterwards:*

(i) *The knot projection is $\pi(K)$ up to planar isotopy (i.e., there exists an isotopy of S_0 that takes the knot projection to $\pi(K)$).*
(ii) *The annulus is in its original position A_0.*

Before giving the proof of this theorem, we first show how it can be applied to our main problem, the generalized Tait conjecture. Recall that we wish to show that if S is a compact surface, K a configuration of a knot in $S \times I$ with reduced alternating projection, K' an arbitrary configuration of the knot,then

$$c(\pi(K)) \leq c(\pi(K')). \tag{3}$$

For the time being, we will assume that the surface is orientable and then we will finish off the argument for nonorientable surfaces by taking their double covers. Using Theorem 2.1, we can reduce this problem to a much simpler one.

Claim 2.2. *To prove Theorem* 1.1 *in the orientable case, it suffices to show* (3) *in the case of a compact orientable surface S possibly with boundary, where the regions in $\pi(K')^c$ are disks and disks with holes.*

Proof. Consider a regular neighborhood N of $\pi(K')$. The boundary of this neighborhood, ∂N consists of a number of simple closed curves that lie in $\pi(K')^c$. Take the subset of these curves which do not bound disks in $\pi(K')^c$ and consider the vertical annuli (obtained by crossing each curve with I) corresponding to them. By Theorem 2.1 we can assume that after we perform the isotopy that takes K to K', these annuli will return to their original positions. We can interpret this as follows. First, cut the space $S \times I$ along these vertical annuli, and paste separate annuli onto each of the resulting boundary components. The result will be a number of separate spaces, each of the form $F \times I$ where F is a compact surface with boundary. The knot K' will lie in one of these spaces, $F_0 \times I$. The entire isotopy between K and K' will take place within a copy of $F_0 \times I$, although the actual position of $F_0 \times I$ within $S \times I$ will change over time.

The interpretation of the result from Theorem 2.1 is that the position of $F_0 \times I$ at the end of the isotopy is the same as at the beginning of the isotopy. By combining this result with an appropriate choice of identification between the continuously deforming $F_0 \times I$ and the original copy, one can see that the entire isotopy between K and K' can be performed within a fixed copy of $F_0 \times I$. In other words, we can think of K, K' as being two configurations of a knot within a space $F_0 \times I$. By our construction, the projection $\pi(K')$ cuts the projection surface $F_0 \times \{\frac{1}{2}\}$ into disks and disks with holes. This means that, by thinking of the two configurations of the knot, K, K', as lying in the space $F_0 \times I$, we only have to deal with the simpler case described above. $\square$

Now that we have shown the importance of Theorem 2.1 in establishing the generalized Tait conjecture, we give the proof.

Proof of Theorem 2.1. We prove the theorem in the case that γ is nontrivial on S_0. The case where γ is trivial can be dealt with using the same arguments but with a few simplifications.

The first step is to isotope the knot K into Menasco form (cf. [**7**], [**8**]). That is, we flatten K onto the surface S_0, creating "bubbles" at the crossings. We arrange A so that it meets these bubbles only in saddle shaped disks.

Denote the surface with the equatorial disks of the bubbles replaced by the upper(lower) hemispheres by $S_0^+ (S_0^-)$.

The proof can be subdivided into four parts:

1. We show that if c is a curve of intersection of the annulus A and $S_0^{\pm}$ that meets a saddle s on both sides, then c must be trivial on $S_0^{\pm}$.

Moreover, the disk that c bounds on $S_0^{\pm}$ cannot intersect the part of the bubble lying directly above (below) s.
2. We isotope A so that it no longer meets any bubbles.
3. We isotope A and K, removing all the intersection curves of $A \cap S_0$ except a single nontrivial curve.
4. We finish the isotopy so that the annulus returns to its original position A_0.

Before proceeding with the first step, we make the following observations:

(i) A curve in $A \cap S_0^{\pm}$ is nontrivial in $S_0^{\pm}$ if and only if it is nontrivial in A. This is equivalent to showing that the inclusion map $i_* : \pi_1(A) \to \pi_1(S \times I)$ is an injection. This follows immediately from the fact that i_* sends the generator of the infinite cyclic group, $\pi_1(A)$, to the homotopy class of γ.

(ii) The simple closed nontrivial curves in $A \cap S_0^{\pm}$ are homotopic to either γ or γ^{-1}. This follows from (i), and from the fact that simple closed nontrivial curves on an annulus have winding number ± 1 around the center.

Now we proceed with (1).

1). We show that if c is a curve of intersection of the annulus A and $S_0^{\pm}$ that meets a saddle s on both sides, then c must be trivial. Moreover, the disk that c bounds on $S_0^{\pm}$ cannot intersect the part of the bubble lying directly above (below) s.

Assume without loss of generality that c lies on S_0^{+}. We will eliminate three different configurations for c. It will then follow that c satisfies the required conditions.

Case 1. The curve c runs between opposite corners of the saddle (see Figure 1).

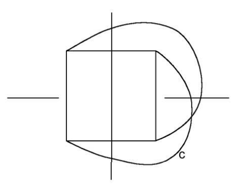

Figure 1. Case 1.

Let G be the graph consisting of a single point p on s together with four non-intersecting edges connecting p to the four “corners” of s (see Figure 2). Consider the graph $H = \partial s \cup C \cup G$. Then we see immediately that H is

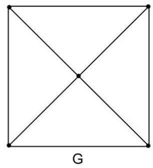

Figure 2. The graph G on the saddle s.

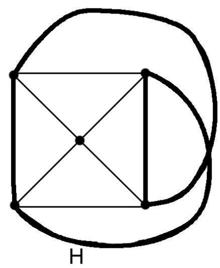

Figure 3. H is isomorphic to K_5.

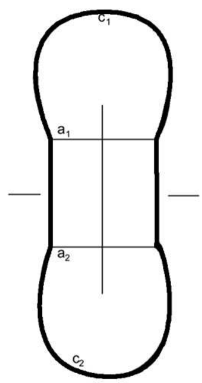

Figure 4. Case 2.

isomorphic to K_5 (see Figure 3). But H lies in the annulus, by construction. This is a contradiction since K_5 is a non-planar graph.

Case 2. The curve c runs between adjacent vertices on the saddle (see Figure 4).

Let c_1, c_2 be the two arcs of c that run between these vertices, and let a_1, a_2 be the arcs of ∂s in S_0^- that join these two pairs of vertices together.

Case 2a. One of the two curves, $c_1 \cup a_1, c_2 \cup a_2$ is trivial.

Assume without loss of generality that $c_1 \cup a_1$ is trivial. Observe that the linking number between $c_1 \cup a_1$ and the knot K is ± 1 (depending on which orientation we choose). To see this, note that c_1 lies on S_0^+ so that all the crossings between K and c_1 are undercrossings (for K). Since they alternate in the orientation of K and since there is an odd number of them, their total contribution to the linking number is ± 1. The single overcrossing of K with A_1 contributes ± 1, giving a total of $(\pm 1 \pm 1)/2 = \pm 1$.

But $c_1 \cup a_1$ is trivial on S_0^+ and therefore trivial on the annulus A. We conclude that it bounds a disk $D \subset A$. D doesn't intersect K, so by the alternative definition of linking number, the linking number of K and $c_1 \cup a_1$ is 0. This is a contradiction.

Case 2b: Both $c_1 \cup a_1$ and $c_2 \cup a_2$ are nontrivial and c is nontrivial.

By Observation (ii), $c_1 \cup a_1$, $c_2 \cup a_2$, c are all homotopic to γ or γ^{-1}. But it is clear from the picture that

$$[c_1 \cup a_1] \cdot [c_2 \cup a_2] = [c] \tag{4}$$

(where [ ] denotes homotopy class). This is a contradiction, since it is impossible for

$$[\gamma^{\pm 1}] \cdot [\gamma^{\pm 1}] = [\gamma^{\pm 1}]. \tag{5}$$

Combining Cases 1, 2a, 2b, we conclude that c must be trivial, while $c_1 \cup a_1$, $c_2 \cup a_2$ must both be nontrivial. This means that the disk that c bounds lies "outside" c. That is, it doesn't intersect the portion of the bubble lying above (below) s. This completes Part (1).

2). We isotope A so that it no longer meets any bubbles.

We first show that by isotoping A, if necessary, we can always reduce the number of saddles that touch trivial intersection curves (of A with $S_0^{\pm}$).

Suppose that some trivial curves do touch saddles. Assume without loss of generality that some of the curves lie in S_0^+. Choose an "innermost" trivial curve j. That is, choose j such that $j = \partial D$ for a disk $D \subset S_0^+$, and such that D doesn't contain any trivial curves of $A \cap S_0^+$ that meet saddles.

It is easy to see that since K is alternating it appears alternately on the left and right of j as we traverse successive bubbles met by j. We may therefore choose a bubble and an arc j_1 of j lying on the bubble, such that the knot is on the same side of j_1 as the disk D.

In general, the intersection between the bubble and D will consist of some number of strips.

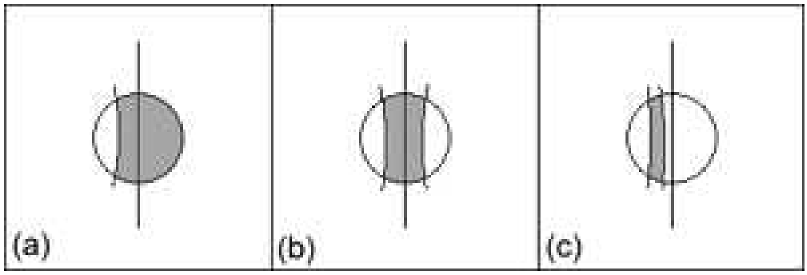

Figure 5. The three possible configurations for arc j_2.

By drawing the arc j_2 that makes up the other half of the strip containing our original arc j_1, we get one of the following cases (see Figure 5).

a) The strip extends all the way across the bubble.
b) The arcs j_1, j_2 lie on opposite sides of K.
c) The arcs j_1, j_2 lie on the same side of K.

To see (a) is impossible, note that the saddle containing j_1 must meet another curve on the opposite side of the knot, and in Case (a) this curve is contained in D. This is a contradiction since we choose j to be the innermost such curve.

To see that (b) is impossible we first note that j_1, j_2 must belong to the same saddle. This follows from the same reasoning as in (a). Next, we apply Part (1). By Part (1), the only possible configuration for j is where D does not meet the portion of the bubble lying directly above the saddle. This is a contradiction.

We are left with (c). We consider the disk D and the appearance of the strip in D (see Figures 6, 7).

Let a be the arc joining u to v, along the bubble. Let l be the arc of c joining x and y. Then, by examining the strip on the disk D, we see that $uxyv = j_1 \cup l \cup j_2 \cup a$ bounds a disk D'. Note that D' contains the strip and that it doesn't meet $A \cap S_0^+$ except at its (D''s) boundary and possibly at trivial curves contained in its interior.

We now show that by isotoping A, we can remove the saddles touching j_1 and j_2. The argument is taken from Adams [**1**].

The isotopy is accomplished in two steps. Consider the part of the annulus lying directly above the curve $j_1 \cup l \cup j_2$, and between $S \times \{\frac{1}{2} + \epsilon\}$ and $S \times \{\frac{1}{2} + 2\epsilon\}$. Take this portion of the annulus and push it horizontally towards the arc a, keeping the rest of the annulus fixed. Continue pushing until the annulus is just beyond the arc a. At this point, the annulus will still be vertical between S_0 and $S \times \{\frac{1}{2} + \epsilon\}$, and vertical above $S \times \{\frac{1}{2} + 2\epsilon\}$.

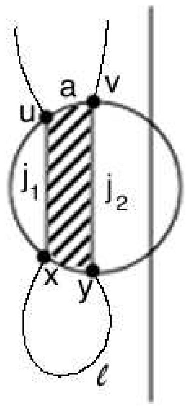

Figure 6. The appearance of the strip on the bubble in Case (c).

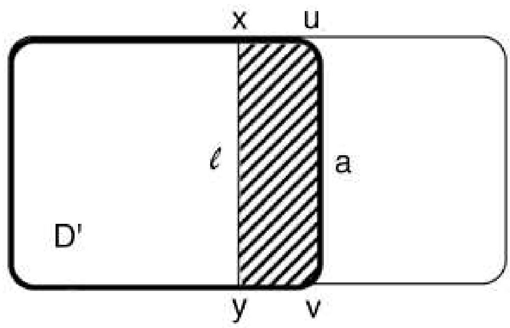

Figure 7. The appearance of the strip on the disk D in Case (c).

However, it will lie essentally horizontally, just above $S \times \{\frac{1}{2} + \epsilon\}$ and just below $S \times \{\frac{1}{2} + 2\epsilon\}$. In other words, the annulus will form a mouth that lies directly above the disk D' with a "roof" at height $\frac{1}{2} + 2\epsilon$, and a "bottom" at height $\frac{1}{2} + \epsilon$. The "back" of the mouth will be vertical and will lie just beyond the arc a (see Figure 8).

In the process of creating this mouth, one may encounter other pieces of the annulus that lie above D'. These pieces will necessarily be parts of "tubes" that lie above intersection curves in D'. As we push the annulus beyond a, we can push these tubes along with us. The end result will be that the tubes will be vertical below $S \times \{\frac{1}{2} + \epsilon\}$ and above $S \times \{\frac{1}{2} + 2\epsilon\}$, but they will make a long detour around the mouth in the intervening region.

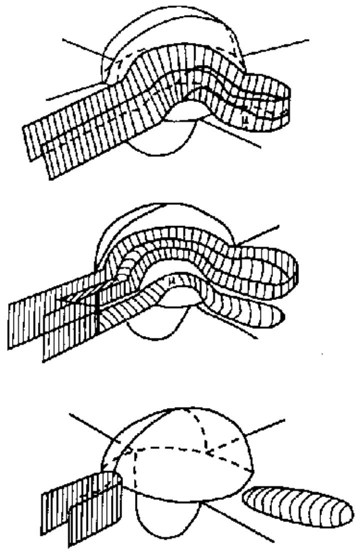

Figure 8. Creating the mouth (from Adams [**1**]).

Now that we have created the mouth, we can proceed with the second part of the isotopy. Take the portion of the bottom of the mouth that lies directly above the strip $uxyv$ and pull it under the knot, through the bubble, so that it lies on the opposite side of the bubble. If there are tubes, we also pull through the part of the tubes lying above $uxyv$.

The end result is that the saddles touching j_1 and j_2 no longer exist. Furthermore, no new saddles have been created. We have therefore reduced the number of saddles that touch trivial curves by at least two.

We have established that we can always reduce the number of saddles that touch trivial intersection curves. Hence, we may assume that no trivial intersection curves meet saddles on either S_0^+ or S_0^-. Note that this, together with Part (1), implies that no curves of any kind can touch the same saddle on opposite sides. It turns out that these two facts are enough to show that the intersection curves of $A \cap S_0^{\pm}$ do not meet any saddles.

Indeed, suppose that the set of intersection curves that touch saddles is nonempty. We know that these intersection curves are nontrivial, so by Observation (ii) they must have homotopy type $\gamma^{\pm 1}$. Now consider the curves of $A \cap S_0^+$ that intersect saddles. If we drew them on A they would appear as in Figure 9.

Take the "outermost" S_0^+ intersection curve c that touches a saddle. (To define outermost rigorously, we embed the annulus A in a disk D such that

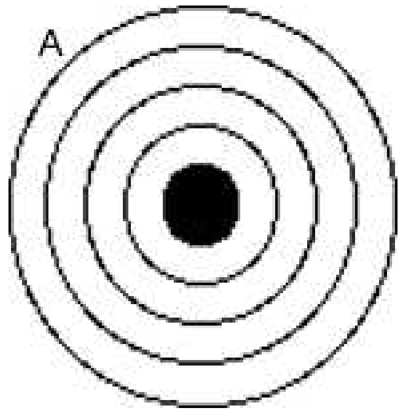

Figure 9. The appearance of the intersection curves on the annulus.

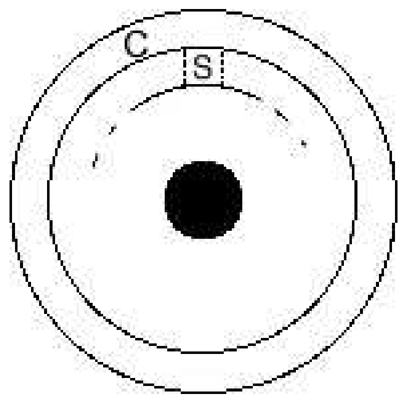

Figure 10. The appearance of c on the annulus.

$\partial D = A \cap S \times \{1\}$. We say that c is outermost if all the other curves are contained in the interior of the disk it bounds in D.)

The curve c touches some saddle s (see Figure 10). Let c_1, c_2 be the two intersection curves on S_0^- obtained by "switching" s (see Figure 11), or rather by viewing the intersection curves from below S_0 rather than above. Note that c_1, c_2 must be distinct curves since no curve touches the same saddle twice.

Now since c is outermost, there are line segments on the disk D from c_1 to ∂D and from c_2 to ∂D that do not cross c_2 or c_1 respectively. Hence the disk on D bounded by c_1 does not contain c_2 and the disk on D bounded by

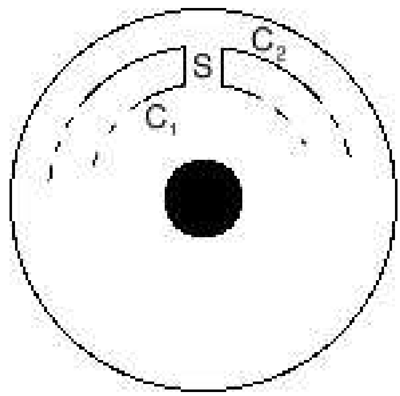

Figure 11. The appearance of c_1 and c_2 on the annulus.

c_2 does not contain c_1. This contradicts the fact these are disjoint nontrivial curves on the annulus.

Thus, no intersection curves touch saddles. This completes Part (2).

3). We remove all but a single nontrivial intersection curve.

First we remove all the trivial intersection curves. We accomplish this one curve at a time. Let c be an innermost intersection curve on S_0. Let D_1 be the disk that c bounds on S_0, and D_2 the disk c bounds on A.

We isotope D_2 onto D_1. We then pull D_2 through the surface S_0, eliminating the trivial curve c. If necessary we pull the knot projection along, too, without changing its combinatorial structure (that is, without changing the knot projection up to planar isotopy). By this we mean that if the knot is in the way of the isotopy then the knot projection lives entirely in the disk D_1. We can assume that it lies in a disk D_1' which is contained in D_1 and which is a distance ϵ from the boundary of D_1. We may now let B be the $\frac{\epsilon}{2}$-neighborhood of D_1. This contains the knot K. As we isotope D_2 through D_1, we pull the ball B along, all the while keeping the knot frozen within it. After we have removed this intersection curve of A with S_0, we continue the isotopy to move this ball back down to S_0 until the disk D_1' again sits on S_0. The knot has now been returned to S_0 with the same combinatorial projection it had before. We repeat this process until there are no more trivial curves.

We will now be left with a number of parallel nontrivial curves of interection on A. Note that they are also parallel on S_0 since the annulus on A that any two of them bound can be homotoped into S_0 by collapsing out the I in $S \times I$. We eliminate these curves in pairs, using the same technique as above. Let c_1, c_2 be adjacent parallel nontrivial curves on A. Let M be the annulus they bound on A and N the annulus they bound on S_0. Since both annuli live in $S \times I$, and share boundary on S_0, M can be isotoped onto N and then pulled through S_0, eliminating a pair of nontrivial curves. Again, we may have to push the knot along during the isotopy but in that case, the knot projection was contained entirely in N. In fact, we may assume that the knot projection lies entirely in an annulus N' which is contained in N and which is a distance ϵ from the boundary of N. Then if V is the solid torus $\frac{\epsilon}{2}$-neighborhood of N', it contains the knot K. As we isotope M through N, we pull the solid torus V along, keeping the knot frozen within it. After removing the two intersection curves of A with S_0, we continue the isotopy to move V back down to S_0 until the annulus N' again sits on S_0. Notice that to do so, we can slide V along the annulus until the annulus again intersects S_0, and then set N' down on S_0. Since all of the intersections of A with S_0 are parallel on S_0, the resulting projection is isotopic on S_0 to the original projection of K. Repeating this process, we can remove all but a single nontrivial curve (intially there must be an odd number of nontrivial curves).

This completes (3).

4). We return the annulus to its initial position, A_0.

To prove (4), consider an isotopy $H : S_0 \times I \rightarrow S_0$ that takes $A \cap S_0$ back to $A_0 \cap S_0 = \gamma$. We know that such an isotopy exists, since $A \cap S_0$ consists of a simple closed curve homotopic to γ. Extend H to an isotopy of the full space $S \times I$ by requiring that H preserve the product structure of the space. This isotopy will take $A \cap S_0$ to γ and it will preserve the combinatorial aspect of the knot projection. Next, flatten the knot onto the surface S_0 and straighten the portions of the annulus lying above and below the projection surface S_0 so that they are vertical. The result is that the annulus will be in its original vertical position, $\gamma \times I$. Moreover, at all times we have preserved the combinatorial aspect of the knot projection, so the projection will differ from $\pi(K)$ only by planar isotopy.

This completes (4) and proves Theorem 2.1. □

3. The polynomial argument.

We will now define a set of polynomials which we will use to prove the special case of the theorem. These polynomials generalize Kauffman's bracket polynomial.

Because of the existence of the projection π, the equivalence of knots in $S \times I$ is the same as the equivalence of their diagrams by Reidemeister moves [**4**]. We may therefore define polynomials for such knots and links by the skein relation

as in the plane. In the above equation, we call the first splitting of the link at the crossing an A-split and the second splitting a B-split.

The main difference between planar projections and projections to a surface is that on the surface, the curves to which the link is reduced (that is, the curves with no crossings) can have different isotopy types, and these are preserved by Reidemeister moves. This means that, in the expansion of the knot in terms of knots without crossings, the coefficient of each isotopy class is preserved separately, producing a family of polynomials [**3**]. Related polynomials also appear in Kamada's proof [**4**].

Our precise definition of the polynomials is in terms of states. For a surface S, define $\mathcal{F}(S)$ to be the set of families of non-intersecting nontrivial simple closed curves on S up to isotopy. Thus if S is a torus, $\mathcal{F}(S)$ is in one-to-one correspondence with the pairs (p, q) of integers. If $d = gcd(p, q)$ then the family of curves corresponding to (p, q) consists of d non-intersecting $(\frac{p}{d}, \frac{q}{d})$ torus knots. For each element of $\mathcal{F}(S)$ there will be a polynomial.

A state s of a knot (or link) K is a splitting of the knot at each crossing; such a state consists of non-intersecting curves. We make several definitions:

$\mathcal{N}(s) = \{\text{nontrivial curves of } s\} \in \mathcal{F}$
a(s) = number of A-splittings
b(s) = number of B-splittings
t(s) = number of trivial components of s
$|s|$ = number of components of s
$p(s)$= number of components of s which bound a disk or disk with holes on S, whose other boundaries lie in ∂S. (The distinction between different types of curves will be useful in the application of polynomials to knots on general surfaces.)

For each $F \in \mathcal{F}$, let

$$Q_F(K) = \sum_{s \in \{s | \mathcal{N}(s) = F\}} A^{a(s)-b(s)}(-A^2 - A^{-2})^{t(s)-1}.$$

By redefining this recursively, we see (as discussed above) that all Q_F's are invariant under Reidemeister moves of Types II and III, and that all are multiplied by the same power of A when a Type I move is applied.

The Q_F's are the most general invariant polynomials of this type. However, for our purposes we specialize slightly. $|s| - t(s)$ is the number of nontrivial curves in s; that is, it is the number of curves in $F = \mathcal{N}(s)$.

Therefore, when we multiply Q_F by $(-A^2 - A^{-2})^{|F|}$ we obtain the polynomial

$$P_F(K) = \sum_{s \in \{s|\mathcal{N}(s)=F\}} A^{a(s)-b(s)}(-A^2 - A^{-2})^{|s|-1}. \tag{6}$$

This set of polynomials is invariant under Type II and III Reidemesiter moves, and all are multiplied by the same factor when a Type I Reidemeister move is applied.

We now apply these polynomials to prove the following modification of our theorem, to which the original theorem reduces:

Theorem 3.1. *Let S be an orientable surface possibly with boundary. Let K and K' be equivalent knots in $S \times I$, with projections $\pi(K)$ and $\pi(K')$ such that:*

1. *$\pi(K)$ is alternating, reduced, and has c crossings.*
2. *The complement of $\pi(K')$ consists of disks, possibly with holes. Only one boundary component of each disk with holes is on $\pi(K')$. The other components are boundary components of $S \times \{\frac{1}{2}\}$.*

Then $c(\pi(K)) \leq c(\pi(K'))$.

The proof closely parallels Kauffman's proof of the original Tait conjecture [**6**]. Let us begin with a lemma on the result of splitting the projection $\pi(K')$ in the A and B directions simultaneously. This is analogous to Kauffman's Lemma 2.11. Our proof is different, however. It does not use induction.

Lemma 3.2. *Let K' be a knot in a projection $\pi(K')$ (or a link with a connected diagram). Let s'_A be the all-A split and s'_B be the all-B split (see Figure 12). Then $p(s'_A) + p(s'_B) \leq R'$, where R' is the number of regions in $\pi(K')^c$.*

Proof. Consider two vector spaces of formal sums (modulo 2) of the edges and regions of the graph formed by $\pi(K')$ in $S \times \{\frac{1}{2}\}$.

C_1 = vector space over Z_2 generated by the edges of the projection $\pi(K')$.
C_2 = vector space over Z_2 generated by the regions of $\pi(K')$.

We define a linear mapping $\delta : C_2 \to C_1$ as follows. Define $\delta(r)$ to be the formal sum of the edges of r, for an r which consists of a *single region* of $\pi(K')^c$. Then, define $\delta(r)$ on the full space C_2 by extending linearly.

Note that the curves of s'_A and of s'_B can be thought of as elements of C_1. Indeed, each curve may be associated to the formal sum of edges along which the curve passes. A curve will form the boundary of a piece of surface precisely when the corresponding element of C_1 lies in $\delta(C_2)$.

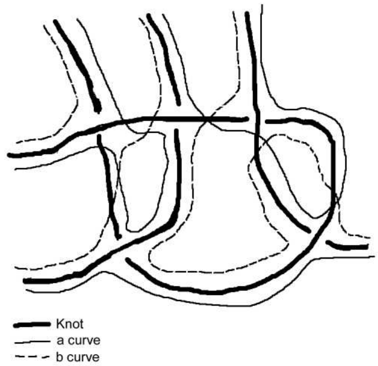

Figure 12. The knot K together with the A and B curves, s'_A and s'_B.

In particular, the curves of s'_A and s'_B which bound punctured disks on the surface span a vector subspace V of C_1 which is entirely contained in $\delta(C_2)$. We conclude

$$\dim\ V \leq \dim\ \delta(C_2). \tag{7}$$

Note that:

$$\dim\ \delta(C_2) = R' - 1 \tag{8}$$

since the kernel of δ is 1-dimensional (it includes only the sum of no regions and the sum of all regions).

We now find a lower bound to the dimension of V. Consider a relation between the curves spanning V, that is consider a family of curves from s'_A and s'_B which bound punctured disks on the surface and which, as elements of C_1, sum to zero. Since summation is modulo 2, each edge of the projection is passed over an even number of times by curves of the family. But then either all of the curves or none of the curves at each vertex must belong to the family, since otherwise one of the edges at the vertex would have only one curve from the family along it.

But if all the curves at some vertex belong to the family then, since these curves also pass through the neighboring vertices, all the curves at the neighboring vertices must belong to the family as well. Repeating this argument, since $\pi(K')$ is connected, either all of the curves or none of the curves from s'_A and s'_B must belong to the family. This shows that there is only one nontrivial relation between the curves of s'_A and s'_B, and so there is certainly no more than one relationship between the curves generating V, which are restricted to those bounding disks with holes. Thus

$$\dim\ V \geq p(s'_A) + p(s'_B) - 1. \tag{9}$$

This, with Equations (7) and (8) shows that $p(s'_A) + p(s'_B) \leq R'$. $\square$

We now give the proof of Theorem 3.1.

Proof. We must first define a notion of span, as in Kauffman's proof. This notion, however, is more technical, and depends on our projections. It is constructed particularly for this proof.

We use the projection of K to fix two polynomials. Let s_A be the all-A split of $\pi(K)$ and s_B be the all-B split of $\pi(K)$. Let $F_A = \mathcal{N}(s_A)$ and $F_B = \mathcal{N}(s_B)$. We now focus on the fixed polynomials P_{F_A} and P_{F_B}.

Let $\max(P)$ be the highest degree of any term of P and $\min(P)$ the lowest. Notice that $\max P_{F_A}(K) - \min P_{F_B}(K) = \max P_{F_A}(K') - \min P_{F_B}(K')$, since we are considering the same pair of polynomials in either case. Let $c = c(\pi(K))$ and $c' = c(\pi(K'))$.

We prove the following two inequalities:

$$\text{(i)} \qquad \max P_{F_A}(K) - \min P_{F_B}(K) \geq 4c - 4g + 2N;$$

$$\text{(ii)} \qquad \max P_{F_A}(K') - \min P_{F_B}(K') \leq 4c' - 4g + 2N.$$

Here g is the genus of S, and N is the number of curves in F_A and F_B which do not bound disks with holes. Note that once these inequaltities have been proved, they together imply $c \leq c'$, which will finish the proof.

Proof of (i). The proof of Statement (i) is in two parts. First we show that $\max P_{F_A}(K)$ is the degree of a term from the all-A split, and similarly for B, and then we calculate these degrees.

The highest degree contributed by a certain state is $a(s) - b(s) + 2(|s| - 1)$ (provided $\mathcal{N}(s) = F_A$). Let us start with the state s_A and change to the state s by switching one A-crossing at a time to a B-crossing. We must show that all states which do contribute to P_{F_A} contribute a strictly lower exponent than s_A.

Every time an A-split is switched to a B-split, $a(s) - b(s)$ decreases by two. $|s|$ cannot increase by more than 1, and so the exponent $a(s) - b(s) + 2(|s| - 1)$ cannot increase. Now, suppose that s contributes a term which cancels with the term from s_A. Then the term from s must have the same degree and belong to the same polynomial as the term contributed by s_A. Thus $|s|$ must increase by one each time, and $\mathcal{N}(s)$ must equal $\mathcal{N}(s_A)$. The possibilities when the split of a given crossing is switched may be enumerated as follows, since some curve must split into two at each stage.

1) A trivial curve splits into two trivial curves.
2) A trivial curve splits into two nontrivial curves.
3) A nontrivial curve splits into two nontrivial curves.
4) A nontrivial curve splits into a nontrivial curve and a trivial curve.

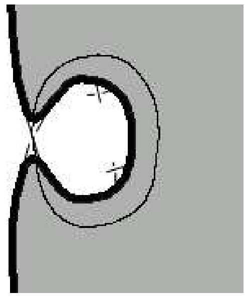

Figure 13. A knot in which a trivial curve splits off when an A-split is changed to a B-split. The dark curve, bounding the gray region of the knot, is the original A-curve. The light curve parallel to the curve which splits off contradicts the definition of a reduced knot.

Neither (1) nor (4) can occur at the first stage since K is reduced (see Figure 13). If (2) or (3) occurs at the first stage the number of nontrivial curves increases and *none* of (1)–(4) occuring at a later stage can reduce this number to its original value, so it is impossible for $\mathcal{N}(s)$ to equal $\mathcal{N}(s_A)$. So it is impossible for a cancellation to occur after all.

Thus $a(s) - b(s) + 2(|s| - 1)$ for $s = s_A$ is strictly the maximum exponent appearing in P_{F_A}. The smallest exponent appearing in P_{F_B} is found similarly. Therefore,

$$\begin{aligned} \max\, P_{F_A}(K) - \min\, P_{F_B}(K) &= c + 2(|s_A| - 1) - (-c - 2(|s_B| - 1)) \\ &= 2c + 2(|s_A| + |s_B|) - 4. \end{aligned}$$

By the definition of N, this can be written as

$$\max\, P_{F_A}(K) - \min\, P_{F_B}(K) = 2c + 2N + 2(p(s_A) + p(s_B)) - 4. \tag{10}$$

If we let N_1 be the number of disk with holes components of $\pi(K)^c$ which have only one boundary component formed by the knot, then we may follow Kauffman, noting that since the knot is alternating, the boundaries of such regions become curves of s_A and s_B. By the definition of p we therefore obtain

$$N_1 \le p(s_A) + p(s_B). \tag{11}$$

Now we use the Euler characteristic to relate N_1 to the crossing number of $\pi(K)$.

Let W = number of components of ∂S.

$W(r)$ = number of boundary components of a region r of $\pi(K)^c$, which are not formed by $\pi(K)$.

$g(r) =$ the genus of region r and let χ be the Euler characteristic. Euler's formula generalized to the case where the regions are not necessarily disks gives

$$-c + \sum_r \chi(r) = \chi(S). \tag{12}$$

Also, $\chi(S) = 2 - 2g - W$ and $\sum_r W(r) = W$, so

$$-c + \sum_r (\chi(r) + W(r)) = 2 - 2g. \tag{13}$$

Now $\chi(r) + W(r) \leq 0$ unless r has only one boundary formed by the knot and has genus zero. There are N_1 such regions and for each, $\chi(r)+W(r) = 1$, so by Equation (13)

$$-c + N_1 \geq 2 - 2g. \tag{14}$$

Combining this last equation with Equation (11) we see that

$$p(s_A) + p(s_B) \geq c + 2 - 2g. \tag{15}$$

From Equation (10) we now see that,

$$\max(P_{F_A}(K)) - \min(P_{F_B}(K)) \geq 4c + 2N - 4g. \tag{16}$$

Proof of (ii). We now continue with the proof of Statement (ii), which concerns the non-alternating version of the knot, K'.

By lemma (3.2), $p(s'_A) + p(s'_B) \leq R'$. Now it follows by induction that for an arbitrary pair of states s'_1, s'_2,

$$p(s'_1) + p(s'_2) \leq R' + b_1 + a_2. \tag{17}$$

In fact, this follows by switching A-splits to B-splits as in (i) to turn s'_A into s'_1 and s'_b into s'_2. (For the argument, notice that if a curve does not bound a disk with holes, then it cannot split into curves which *do* bound disks with holes.)

Now apply this inequality to a pair of states s'_1 and s'_2 which are assumed to contribute to $P_{F_A}(K')$ and to $P_{F_B}(K')$, respectively. The difference between the exponents they contribute is:

$$\begin{aligned}
&a_1 - a_2 + b_2 - b_1 + 2(|s'_1| + |s'_2|) - 4 \\
&= a_1 - a_2 + b_2 - b_1 + 2(p(s'_1) + p(s'_2)) - 4 + 2(|s'_1| + |s'_2| - p(s_1) - p(s'_2)) \\
&= a_1 - a_2 + b_2 - b_1 + 2(p(s'_1) + p(s'_2)) - 4 + 2N \\
&= 2c' - 2a_2 - 2b_1 + 2p(s'_1) + 2p(s'_2) - 4 + 2N \\
&\leq 2c' + 2R' - 4 + 2N.
\end{aligned}$$

The second equality follows since the nontrivial curves in s'_1 and s'_2 are just the curves F_a and F_b, and N of these curves do not bound punctured disks. The inequality follows from Equation (17).

By Euler's formula, and the assumption that all regions of K' are genus zero, $R' = c' + 2 - 2g$, so the difference between an exponent of $P_{F_A}(K')$ and one of $P_{F_B}(K')$ is at most $4c' + 2N - 2g$, proving (ii).

The inequalities (i) and (ii) together imply Theorem 3.1, since the quantities on the left side of the inequalities are the same by the invariance properties of the polynomials. □

We now restate Theorem 1.1:

Theorem 1.1. *Let S be a compact surface. Let $\pi(K)$ be a reduced alternating projection of an alternating knot in $S \times I$ and let $\pi(K')$ be an arbitrary projection (of the same knot). Then*

$$c(\pi(K)) \leq c(\pi(K')). \tag{18}$$

Proof. On account of Claim 2.2, Theorem 3.1 which we have just proved implies this more general theorem in the orientable case. The nonorientable case follows immediately by taking double covers. □

4. Conclusion.

The statement of Theorem 1.1 can be strengthened. It is unnecessary to restrict the theorem to knots. Indeed, the proof works equally well for non-splittable links, and with a few modifications it extends to links in general.

The other possible extensions of Theorem 1.1 are more difficult. We have shown that if $\pi(K)$ is a reduced alternating projection of a knot in $S \times I$, and $\pi(K')$ is any other projection of that knot, then

$$c(\pi(K)) \leq c(\pi(K')). \tag{19}$$

It remains to be shown that

$$c(\pi(K)) < c(\pi(K')), \tag{20}$$

if $\pi(K)$ is non-alternating, and the knot is prime (Murasagi and Thistlethwaite established this strict inequality in S^3, [**10**], [**11**]).

Also, Tait conjectured that any two reduced alternating projections of the same knot can be converted to one another through a series of special moves called flypes. This statement was proved for knots in S^3 by Thistlethwaite and Menasco [**9**]. A natural extension of Theorem 1.1 would be to prove the flyping conjecture for knots in $S \times I$.

It is natural to ask questions about knots in more complicated spaces which contain subspaces of the form $S \times I$.

One possibility is to investigate the Tait conjecture for knots in (solid) handlebodies, where we project knots onto the boundary. If our definition of reduced is used, then the case in Figure 14 is possible. The knot projection

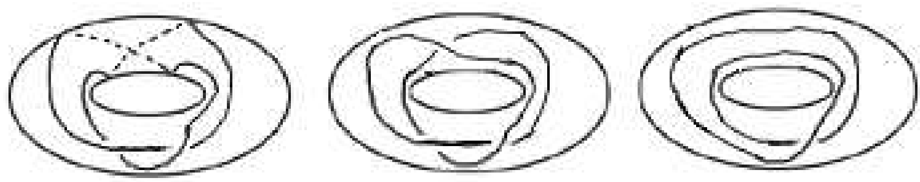

Figure 14. A counterexample in the solid torus.

on the left is reduced and alternating, but the number of crossings can be lowered by pulling a crossing through the center of the solid torus, and then using a Type II Reidemeister move. The resulting knot with fewer crossings is shown on the right.

Thus, the conjecture doesn't hold with our notion of reduced. However, it may hold if we use Kamada's notion of reduced-*properly reduced*.

Alternatively, one could ask whether the Tait conjecture holds for knots lying on an *incompressible* surface S in a 3-manifold M. We conjecture that if two projections of a knot are equivalent in the 3-manifold, then they are equivalent up to a homeomorphism of $S \times I$. If true, this would reduce the problem to the result proved here.

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