CROSSING NUMBER OF ALTERNATING KNOTS IN $S \times I$

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One of the Tait conjectures, which was stated 100 years ago and proved in the 1980’s, said that reduced alternating projections of alternating knots have the minimal number of crossings. We prove a generalization of this for knots in $S \times I$, where $S$ is a surface. We use a combination of geometric and polynomial techniques.

1. Introduction.

A hundred years ago, Tait conjectured that the number of crossings in a reduced alternating projection of an alternating knot is minimal. This statement was proven in 1986 by Kauffman, Murasugi and Thistlethwaite, [6], [10], [11], working independently. Their proofs relied on the new polynomials generated in the wake of the discovery of the Jones polynomial.

We usually think of this result as applying to knots in the 3-sphere $S^3$. However, it applies equally well to knots in $S^2 \times I$ (where $I$ is the unit interval $[0,1]$). Indeed, if one removes two disjoint balls from $S^3$, the resulting space is homeomorphic to $S^2 \times I$. It is not hard to see that these two balls do not affect knot equivalence. We conclude that the theory of knot equivalence in $S^2 \times I$ is the same as in $S^3$.

With this equivalence in mind, it is natural to ask if the Tait conjecture generalizes to knots in spaces of the form $S \times I$ where $S$ is any compact surface.

More rigorously, consider the projection surface $S_0 = S \times \{\frac{1}{2}\}$. Let $\pi : S \times I \to S_0$ be the natural projection. We define crossing number, alternating projections and alternating knots in the obvious way. Given some choice of a definition of reduced, we want to know whether reduced alternating projections of alternating knots have minimal crossing number.

In other words, if $c(\pi(K))$ represents the crossing number of a projection, we want to know if it is always the case that if $K$ and $K'$ represent two spatial configurations of the same knot, so $\pi(K), \pi(K')$ are two projections of the knot and $\pi(K)$ is “reduced” and alternating, then

\[ c(\pi(K)) \leq c(\pi'(K)). \]
In [4], Kamada showed that if two projections of a knot in $S \times I$ are both “properly reduced” alternating projections with the same supporting genus, then they have the same number of crossings. A projection is properly reduced if the four regions that meet at each crossing of the projection are distinct. This is a generalization of a reduced projection in the plane. The supporting genus of a projection is the genus of the surface that results if each region of the projection surface is replaced with a disk.

The result presented in this paper extends Kamada’s result in three ways. First, our notion of “reduced” is more general than Kamada’s, and it is a more natural generalization of the definition in $S^3$.

We define a knot projection to be reduced on $S_0$ if there are no trivial simple closed curves on $S_0$ that intersect the knot projection exactly once (a trivial curve is a curve that is homotopic to the constant curve). This is natural, because curves like this exist exactly when one can perform the “untwisting” operation to reduce the number of crossings. Note that such a curve intersects the projection at a crossing, with two strands of the knot coming out of the intersection to either side of the curve.

Second, we consider arbitrary projections $\pi(K)$, $\pi(K')$, not just projections with the same supporting genus. Third, we show that the crossing numbers of the reduced alternating projections are not just equal to one another, but that they are minimal.

Indeed, we prove the following:

**Theorem 1.1.** Let $S$ be a compact surface. Let $\pi(K)$ be a reduced alternating projection of an alternating knot in $S \times I$ and let $\pi(K')$ be an arbitrary projection (of the same knot). Then

\begin{equation}
(2) \quad c(\pi(K)) \leq c(\pi(K')).
\end{equation}

Unlike the proof of the original Tait conjecture, polynomial techniques were not enough to establish our result. These techniques are only strong enough to give results analogous to those of Kamada. We use Menasco’s geometrical techniques to show something analogous to the supporting genus restriction always holds and to complete the proof. Independently, in [5], the author announced a version of Theorem 1.1 for knots and links, however, only an outline of the proof has appeared. The techniques utilized differ substantially from those presented here.

The specific breakdown of the paper is as follows. The second section of this paper presents the geometrical argument. The main result of this section is that the general result follows from the special case that $S$ is a punctured compact orientable surface, and $\pi(K')$ cuts this surface into disks and punctured disks. This special case is analogous to Kamada’s result.
The third section of this paper presents the polynomial argument. We define polynomial invariants for knots in $S \times I$ and use them to prove (2) in the special case.

Combining these two results gives the main theorem, the proof of which appears in the third section. The fourth and final section discusses extensions of the theorem, and other related questions and conjectures.

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2. The geometric argument.

In this section we show that in some sense, alternating projections of a knot in $S \times I$ wrap around the projection surface $S_0$ less than any other projections of the same knot. The rigorous statement of this idea is given by the next theorem. (Hayashi has proved a related result in the case that $S$ is a torus [2].)

**Theorem 2.1.** Let $S$ be an orientable surface. Let $K$ be a configuration of a knot in $S \times I$ such that $\pi(K)$ is regular and alternating. Let $K'$ be any other configuration of the knot, and let $H : (S \times I) \times I \to (S \times I)$ be the isotopy carrying $K'$ to $K$. Suppose that there exists a simple closed curve $\gamma \in \pi(K')^c$ which does not bound a disk in $\pi(K')^c$ (where $\pi(K')^c$ denotes the complement of the knot projection in $S_0$). Let $A_0$ be the annulus $\gamma \times I$. Let $A = H(A_0, 1)$ be the isotoped annulus. Then we may continue the isotopy, so that afterwards:

(i) The knot projection is $\pi(K)$ up to planar isotopy (i.e., there exists an isotopy of $S_0$ that takes the knot projection to $\pi(K)$).
(ii) The annulus is in its original position $A_0$.

Before giving the proof of this theorem, we first show how it can be applied to our main problem, the generalized Tait conjecture. Recall that we wish to show that if $S$ is a compact surface, $K$ a configuration of a knot in $S \times I$ with reduced alternating projection, $K'$ an arbitrary configuration of the knot, then

$$c(\pi(K)) \leq c(\pi(K')).$$

For the time being, we will assume that the surface is orientable and then we will finish off the argument for nonorientable surfaces by taking their double covers. Using Theorem 2.1, we can reduce this problem to a much simpler one.
Claim 2.2. To prove Theorem 1.1 in the orientable case, it suffices to show (3) in the case of a compact orientable surface \(S\) possibly with boundary, where the regions in \(\pi(K')^c\) are disks and disks with holes.

Proof. Consider a regular neighborhood \(N\) of \(\pi(K')\). The boundary of this neighborhood, \(\partial N\) consists of a number of simple closed curves that lie in \(\pi(K')^c\). Take the subset of these curves which do not bound disks in \(\pi(K')^c\) and consider the vertical annuli (obtained by crossing each curve with \(I\)) corresponding to them. By Theorem 2.1 we can assume that after we perform the isotopy that takes \(K\) to \(K'\), these annuli will return to their original positions. We can interpret this as follows. First, cut the space \(S \times I\) along these vertical annuli, and paste separate annuli onto each of the resulting boundary components. The result will be a number of separate spaces, each of the form \(F \times I\) where \(F\) is a compact surface with boundary. The knot \(K'\) will lie in one of these spaces, \(F_0 \times I\). The entire isotopy between \(K\) and \(K'\) will take place within a copy of \(F_0 \times I\), although the actual position of \(F_0 \times I\) within \(S \times I\) will change over time.

The interpretation of the result from Theorem 2.1 is that the position of \(F_0 \times I\) at the end of the isotopy is the same as at the beginning of the isotopy. By combining this result with an appropriate choice of identification between the continuously deforming \(F_0 \times I\) and the original copy, one can see that the entire isotopy between \(K\) and \(K'\) can be performed within a fixed copy of \(F_0 \times I\). In other words, we can think of \(K, K'\) as being two configurations of a knot within a space \(F_0 \times I\). By our construction, the projection \(\pi(K')\) cuts the projection surface \(F_0 \times \{\frac{1}{2}\}\) into disks and disks with holes. This means that, by thinking of the two configurations of the knot, \(K, K'\), as lying in the space \(F_0 \times I\), we only have to deal with the simpler case described above. \(\square\)

Now that we have shown the importance of Theorem 2.1 in establishing the generalized Tait conjecture, we give the proof.

Proof of Theorem 2.1. We prove the theorem in the case that \(\gamma\) is nontrivial on \(S_0\). The case where \(\gamma\) is trivial can be dealt with using the same arguments but with a few simplifications.

The first step is to isotope the knot \(K\) into Menasco form (cf. [7], [8]). That is, we flatten \(K\) onto the surface \(S_0\), creating “bubbles” at the crossings. We arrange \(A\) so that it meets these bubbles only in saddle shaped disks.

Denote the surface with the equatorial disks of the bubbles replaced by the upper(lower) hemispheres by \(S_0^+(S_0^-)\).

The proof can be subdivided into four parts:

1. We show that if \(c\) is a curve of intersection of the annulus \(A\) and \(S_0^\pm\) that meets a saddle \(s\) on both sides, then \(c\) must be trivial on \(S_0^\pm\).
Moreover, the disk that \( c \) bounds on \( S_0^\pm \) cannot intersect the part of the bubble lying directly above (below) \( s \).

2. We isotope \( A \) so that it no longer meets any bubbles.

3. We isotope \( A \) and \( K \), removing all the intersection curves of \( A \cap S_0 \) except a single nontrivial curve.

4. We finish the isotopy so that the annulus returns to its original position \( A_0 \).

Before proceeding with the first step, we make the following observations:

(i) A curve in \( A \cap S_0^\pm \) is nontrivial in \( S_0^\pm \) if and only if it is nontrivial in \( A \). This is equivalent to showing that the inclusion map \( i_* : \pi_1(A) \to \pi_1(S \times I) \) is an injection. This follows immediately from the fact that \( i_* \) sends the generator of the infinite cyclic group, \( \pi_1(A) \), to the homotopy class of \( \gamma \).

(ii) The simple closed nontrivial curves in \( A \cap S_0^\pm \) are homotopic to either \( \gamma \) or \( \gamma^{-1} \). This follows from (i), and from the fact that simple closed nontrivial curves on an annulus have winding number \( \pm 1 \) around the center.

Now we proceed with (1).

1). **We show that if \( c \) is a curve of intersection of the annulus \( A \) and \( S_0^\pm \) that meets a saddle \( s \) on both sides, then \( c \) must be trivial. Moreover, the disk that \( c \) bounds on \( S_0^\pm \) cannot intersect the part of the bubble lying directly above (below) \( s \).**

Assume without loss of generality that \( c \) lies on \( S_0^+ \). We will eliminate three different configurations for \( c \). It will then follow that \( c \) satisfies the required conditions.

*Case 1.* The curve \( c \) runs between opposite corners of the saddle (see Figure 1).

![Figure 1. Case 1.](image-url)
Figure 2. The graph $G$ on the saddle $s$.

Figure 3. $H$ is isomorphic to $K_5$.

Figure 4. Case 2.

isomorphic to $K_5$ (see Figure 3). But $H$ lies in the annulus, by construction. This is a contradiction since $K_5$ is a non-planar graph.

Case 2. The curve $c$ runs between adjacent vertices on the saddle (see Figure 4).
Let $c_1, c_2$ be the two arcs of $c$ that run between these vertices, and let $a_1, a_2$ be the arcs of $\partial s$ in $S_0^-$ that join these two pairs of vertices together.

**Case 2a.** One of the two curves, $c_1 \cup a_1, c_2 \cup a_2$ is trivial.

Assume without loss of generality that $c_1 \cup a_1$ is trivial. Observe that the linking number between $c_1 \cup a_1$ and the knot $K$ is $\pm 1$ (depending on which orientation we choose). To see this, note that $c_1$ lies on $S^+_{0}$ so that all the crossings between $K$ and $c_1$ are undercrossings (for $K$). Since they alternate in the orientation of $K$ and since there is an odd number of them, their total contribution to the linking number is $\pm 1$. The single overcrossing of $K$ with $A_1$ contributes $\pm 1$, giving a total of $(\pm 1 \pm 1)/2 = \pm 1$.

But $c_1 \cup a_1$ is trivial on $S^+_{0}$ and therefore trivial on the annulus $A$. We conclude that it bounds a disk $D \subset A$. $D$ doesn’t intersect $K$, so by the alternative definition of linking number, the linking number of $K$ and $c_1 \cup a_1$ is $0$. This is a contradiction.

**Case 2b:** Both $c_1 \cup a_1$ and $c_2 \cup a_2$ are nontrivial and $c$ is nontrivial.

By Observation (ii), $c_1 \cup a_1, c_2 \cup a_2, c$ are all homotopic to $\gamma$ or $\gamma^{-1}$. But it is clear from the picture that

$$(4) \quad [c_1 \cup a_1] \cdot [c_2 \cup a_2] = [c]$$

(where $[\ ]$ denotes homotopy class). This is a contradiction, since it is impossible for

$$(5) \quad [\gamma^{\pm 1}] \cdot [\gamma^{\pm 1}] = [\gamma^{\pm 1}].$$

Combining Cases 1, 2a, 2b, we conclude that $c$ must be trivial, while $c_1 \cup a_1, c_2 \cup a_2$ must both be nontrivial. This means that the disk that $c$ bounds lies “outside” $c$. That is, it doesn’t intersect the portion of the bubble lying above (below) $s$. This completes Part (1).

2). We isotope $A$ so that it no longer meets any bubbles.

We first show that by isotoping $A$, if necessary, we can always reduce the number of saddles that touch trivial intersection curves (of $A$ with $S_0^+$).

Suppose that some trivial curves do touch saddles. Assume without loss of generality that some of the curves lie in $S_0^+$. Choose an “innermost” trivial curve $j$. That is, choose $j$ such that $j = \partial D$ for a disk $D \subset S_0^+$, and such that $D$ doesn’t contain any trivial curves of $A \cap S_0^+$ that meet saddles.

It is easy to see that since $K$ is alternating it appears alternately on the left and right of $j$ as we traverse successive bubbles met by $j$. We may therefore choose a bubble and an arc $j_1$ of $j$ lying on the bubble, such that the knot is on the same side of $j_1$ as the disk $D$.

In general, the intersection between the bubble and $D$ will consist of some number of strips.
Figure 5. The three possible configurations for arc $j_2$.

By drawing the arc $j_2$ that makes up the other half of the strip containing our original arc $j_1$, we get one of the following cases (see Figure 5).

a) The strip extends all the way across the bubble.
b) The arcs $j_1, j_2$ lie on opposite sides of $K$.
c) The arcs $j_1, j_2$ lie on the same side of $K$.

To see (a) is impossible, note that the saddle containing $j_1$ must meet another curve on the opposite side of the knot, and in Case (a) this curve is contained in $D$. This is a contradiction since we choose $j$ to be the innermost such curve.

To see that (b) is impossible we first note that $j_1, j_2$ must belong to the same saddle. This follows from the same reasoning as in (a). Next, we apply Part (1). By Part (1), the only possible configuration for $j$ is where $D$ does not meet the portion of the bubble lying directly above the saddle. This is a contradiction.

We are left with (c). We consider the disk $D$ and the appearance of the strip in $D$ (see Figures 6, 7).

Let $a$ be the arc joining $u$ to $v$, along the bubble. Let $l$ be the arc of $c$ joining $x$ and $y$. Then, by examining the strip on the disk $D$, we see that $uxyv = j_1 \cup l \cup j_2 \cup a$ bounds a disk $D'$. Note that $D'$ contains the strip and that it doesn’t meet $A \cap S_0^+$ except at its ($D'$’s) boundary and possibly at trivial curves contained in its interior.

We now show that by isotoping $A$, we can remove the saddles touching $j_1$ and $j_2$. The argument is taken from Adams [1].

The isotopy is accomplished in two steps. Consider the part of the annulus lying directly above the curve $j_1 \cup l \cup j_2$, and between $S \times \{\frac{1}{2} + \epsilon\}$ and $S \times \{\frac{1}{2} + 2\epsilon\}$. Take this portion of the annulus and push it horizontally towards the arc $a$, keeping the rest of the annulus fixed. Continue pushing until the annulus is just beyond the arc $a$. At this point, the annulus will still be vertical between $S_0$ and $S \times \{\frac{1}{2} + \epsilon\}$, and vertical above $S \times \{\frac{1}{2} + 2\epsilon\}$. 
However, it will lie essentially horizontally, just above $S \times \{ \frac{1}{2} + \epsilon \}$ and just below $S \times \{ \frac{1}{2} + 2\epsilon \}$. In other words, the annulus will form a mouth that lies directly above the disk $D'$ with a “roof” at height $\frac{1}{2} + 2\epsilon$, and a “bottom” at height $\frac{1}{2} + \epsilon$. The “back” of the mouth will be vertical and will lie just beyond the arc $a$ (see Figure 8).

In the process of creating this mouth, one may encounter other pieces of the annulus that lie above $D'$. These pieces will necessarily be parts of “tubes” that lie above intersection curves in $D'$. As we push the annulus beyond $a$, we can push these tubes along with us. The end result will be that the tubes will be vertical below $S \times \{ \frac{1}{2} + \epsilon \}$ and above $S \times \{ \frac{1}{2} + 2\epsilon \}$, but they will make a long detour around the mouth in the intervening region.
Now that we have created the mouth, we can proceed with the second part of the isotopy. Take the portion of the bottom of the mouth that lies directly above the strip $uxyz$ and pull it under the knot, through the bubble, so that it lies on the opposite side of the bubble. If there are tubes, we also pull through the part of the tubes lying above $uxyz$.

The end result is that the saddles touching $j_1$ and $j_2$ no longer exist. Furthermore, no new saddles have been created. We have therefore reduced the number of saddles that touch trivial curves by at least two.

We have established that we can always reduce the number of saddles that touch trivial intersection curves. Hence, we may assume that no trivial intersection curves meet saddles on either $S_0^+$ or $S_0^-$. Note that this, together with Part (1), implies that no curves of any kind can touch the same saddle on opposite sides. It turns out that these two facts are enough to show that the intersection curves of $A \cap S_0^\pm$ do not meet any saddles.

Indeed, suppose that the set of intersection curves that touch saddles is nonempty. We know that these intersection curves are nontrivial, so by Observation (ii) they must have homotopy type $\gamma^\pm 1$. Now consider the curves of $A \cap S_0^+$ that intersect saddles. If we drew them on $A$ they would appear as in Figure 9.

Take the “outermost” $S_0^+$ intersection curve $c$ that touches a saddle. (To define outermost rigorously, we embed the annulus $A$ in a disk $D$ such that
\[ \partial D = A \cap S \times \{1\}. \] We say that \( c \) is outermost if all the other curves are contained in the interior of the disk it bounds in \( D \).

The curve \( c \) touches some saddle \( s \) (see Figure 10). Let \( c_1, c_2 \) be the two intersection curves on \( S_0 \) obtained by “switching” \( s \) (see Figure 11), or rather by viewing the intersection curves from below \( S_0 \) rather than above. Note that \( c_1, c_2 \) must be distinct curves since no curve touches the same saddle twice.

Now since \( c \) is outermost, there are line segments on the disk \( D \) from \( c_1 \) to \( \partial D \) and from \( c_2 \) to \( \partial D \) that do not cross \( c_2 \) or \( c_1 \) respectively. Hence the disk on \( D \) bounded by \( c_1 \) does not contain \( c_2 \) and the disk on \( D \) bounded by
Figure 11. The appearance of $c_1$ and $c_2$ on the annulus.

c_2$ does not contain $c_1$. This contradicts the fact these are disjoint nontrivial curves on the annulus.

Thus, no intersection curves touch saddles. This completes Part (2).

3). We remove all but a single nontrivial intersection curve.

First we remove all the trivial intersection curves. We accomplish this one curve at a time. Let $c$ be an innermost intersection curve on $S_0$. Let $D_1$ be the disk that $c$ bounds on $S_0$, and $D_2$ the disk $c$ bounds on $A$.

We isotope $D_2$ onto $D_1$. We then pull $D_2$ through the surface $S_0$, eliminating the trivial curve $c$. If necessary we pull the knot projection along, too, without changing its combinatorial structure (that is, without changing the knot projection up to planar isotopy). By this we mean that if the knot is in the way of the isotopy then the knot projection lives entirely in the disk $D_1$. We can assume that it lies in a disk $D'_1$ which is contained in $D_1$ and which is a distance $\epsilon$ from the boundary of $D_1$. We may now let $B$ be the $\frac{\epsilon}{2}$-neighborhood of $D_1$. This contains the knot $K$. As we isotope $D_2$ through $D_1$, we pull the ball $B$ along, all the while keeping the knot frozen within it. After we have removed this intersection curve of $A$ with $S_0$, we continue the isotopy to move this ball back down to $S_0$ until the disk $D'_1$ again sits on $S_0$. The knot has now been returned to $S_0$ with the same combinatorial projection it had before. We repeat this process until there are no more trivial curves.

We will now be left with a number of parallel nontrivial curves of intersection on $A$. Note that they are also parallel on $S_0$ since the annulus on $A$ that any two of them bound can be homotoped into $S_0$ by collapsing out the $I$ in $S \times I$. We eliminate these curves in pairs, using the same technique as above. Let $c_1$, $c_2$ be adjacent parallel nontrivial curves on $A$. Let $M$ be the annulus they bound on $A$ and $N$ the annulus they bound on $S_0$. Since both
annuli live in $S \times I$, and share boundary on $S_0$, $M$ can be isotoped onto $N$ and then pulled through $S_0$, eliminating a pair of nontrivial curves. Again, we may have to push the knot along during the isotopy but in that case, the knot projection was contained entirely in $N$. In fact, we may assume that the knot projection lies entirely in an annulus $N'$ which is contained in $N$ and which is a distance $\epsilon$ from the boundary of $N$. Then if $V$ is the solid torus $\frac{\epsilon}{2}$-neighborhood of $N'$, it contains the knot $K$. As we isotope $M$ through $N$, we pull the solid torus $V$ along, keeping the knot frozen within it. After removing the two intersection curves of $A$ with $S_0$, we continue the isotopy to move $V$ back down to $S_0$ until the annulus $N'$ again sits on $S_0$. Notice that to do so, we can slide $V$ along the annulus until the annulus again intersects $S_0$, and then set $N'$ down on $S_0$. Since all of the intersections of $A$ with $S_0$ are parallel on $S_0$, the resulting projection is isotopic on $S_0$ to the original projection of $K$. Repeating this process, we can remove all but a single nontrivial curve (intially there must be an odd number of nontrivial curves).

This completes (3).

4). We return the annulus to its initial position, $A_0$.

To prove (4), consider an isotopy $H : S_0 \times I \to S_0$ that takes $A \cap S_0$ back to $A_0 \cap S_0 = \gamma$. We know that such an isotopy exists, since $A \cap S_0$ consists of a simple closed curve homotopic to $\gamma$. Extend $H$ to an isotopy of the full space $S \times I$ by requiring that $H$ preserve the product structure of the space. This isotopy will take $A \cap S_0$ to $\gamma$ and it will preserve the combinatorial aspect of the knot projection. Next, flatten the knot onto the surface $S_0$ and straighten the portions of the annulus lying above and below the projection surface $S_0$ so that they are vertical. The result is that the annulus will be in its original vertical position, $\gamma \times I$. Moreover, at all times we have preserved the combinatorial aspect of the knot projection, so the projection will differ from $\pi(K)$ only by planar isotopy.

This completes (4) and proves Theorem 2.1. \qed

3. The polynomial argument.

We will now define a set of polynomials which we will use to prove the special case of the theorem. These polynomials generalize Kauffman’s bracket polynomial.

Because of the existence of the projection $\pi$, the equivalence of knots in $S \times I$ is the same as the equivalence of their diagrams by Reidemeister moves [4]. We may therefore define polynomials for such knots and links by
the skein relation

\[ \langle x \rangle = A \langle x \rangle + A^{-1} \langle x^- \rangle \]

as in the plane. In the above equation, we call the first splitting of the link at the crossing an A-split and the second splitting a B-split.

The main difference between planar projections and projections to a surface is that on the surface, the curves to which the link is reduced (that is, the curves with no crossings) can have different isotopy types, and these are preserved by Reidemeister moves. This means that, in the expansion of the knot in terms of knots without crossings, the coefficient of each isotopy class is preserved separately, producing a family of polynomials [3]. Related polynomials also appear in Kamada’s proof [4].

Our precise definition of the polynomials is in terms of states. For a surface \( S \), define \( \mathcal{F}(S) \) to be the set of families of non-intersecting nontrivial simple closed curves on \( S \) up to isotopy. Thus if \( S \) is a torus, \( \mathcal{F}(S) \) is in one-to-one correspondence with the pairs \((p, q)\) of integers. If \( d = \gcd(p, q) \) then the family of curves corresponding to \((p, q)\) consists of \( d \) non-intersecting \((\frac{p}{d}, \frac{q}{d})\) torus knots. For each element of \( \mathcal{F}(S) \) there will be a polynomial.

A state \( s \) of a knot (or link) \( K \) is a splitting of the knot at each crossing; such a state consists of non-intersecting curves. We make several definitions:

- \( \mathcal{N}(s) = \{ \text{nontrivial curves of } s \} \in \mathcal{F} \)
- \( a(s) = \text{number of A-splittings} \)
- \( b(s) = \text{number of B-splittings} \)
- \( t(s) = \text{number of trivial components of } s \)
- \( |s| = \text{number of components of } s \)
- \( p(s) = \text{number of components of } s \text{ which bound a disk or disk with holes on } S, \text{ whose other boundaries lie in } \partial S. \) (The distinction between different types of curves will be useful in the application of polynomials to knots on general surfaces.)

For each \( F \in \mathcal{F} \), let

\[ Q_F(K) = \sum_{s \in \{s | \mathcal{N}(s) = F\}} A^{a(s) - b(s)}(-A^2 - A^{-2})^{t(s) - 1}. \]

By redefining this recursively, we see (as discussed above) that all \( Q_F \)’s are invariant under Reidemeister moves of Types II and III, and that all are multiplied by the same power of \( A \) when a Type I move is applied.

The \( Q_F \)’s are the most general invariant polynomials of this type. However, for our purposes we specialize slightly. \( |s| - t(s) \) is the number of nontrivial curves in \( s \); that is, it is the number of curves in \( F = \mathcal{N}(s) \).
Therefore, when we multiply $Q_F$ by $(-A^2 - A^{-2})^{|F|}$ we obtain the polynomial

$$P_F(K) = \sum_{s \in \{s | N(s) = F\}} A^{a(s) - b(s)}(-A^2 - A^{-2})^{|s| - 1}. \tag{6}$$

This set of polynomials is invariant under Type II and III Reidemeister moves, and all are multiplied by the same factor when a Type I Reidemeister move is applied.

We now apply these polynomials to prove the following modification of our theorem, to which the original theorem reduces:

**Theorem 3.1.** Let $S$ be an orientable surface possibly with boundary. Let $K$ and $K'$ be equivalent knots in $S \times I$, with projections $\pi(K)$ and $\pi(K')$ such that:

1. $\pi(K)$ is alternating, reduced, and has $c$ crossings.
2. The complement of $\pi(K')$ consists of disks, possibly with holes. Only one boundary component of each disk with holes is on $\pi(K')$. The other components are boundary components of $S \times \{\frac{1}{2}\}$.

Then $c(\pi(K)) \leq c(\pi(K'))$.

The proof closely parallels Kauffman’s proof of the original Tait conjecture [6]. Let us begin with a lemma on the result of splitting the projection $\pi(K')$ in the A and B directions simultaneously. This is analogous to Kauffman’s Lemma 2.11. Our proof is different, however. It does not use induction.

**Lemma 3.2.** Let $K'$ be a knot in a projection $\pi(K')$ (or a link with a connected diagram). Let $s'_A$ be the all-A split and $s'_B$ be the all-B split (see Figure 12). Then $p(s'_A) + p(s'_B) \leq R'$, where $R'$ is the number of regions in $\pi(K')^c$.

**Proof.** Consider two vector spaces of formal sums (modulo 2) of the edges and regions of the graph formed by $\pi(K')$ in $S \times \{\frac{1}{2}\}$.

- $C_1 = \text{vector space over } \mathbb{Z}_2 \text{ generated by the edges of the projection } \pi(K')$.
- $C_2 = \text{vector space over } \mathbb{Z}_2 \text{ generated by the regions of } \pi(K')$.

We define a linear mapping $\delta : C_2 \to C_1$ as follows. Define $\delta(r)$ to be the formal sum of the edges of $r$, for an $r$ which consists of a single region of $\pi(K')^c$. Then, define $\delta(r)$ on the full space $C_2$ by extending linearly.

Note that the curves of $s'_A$ and of $s'_B$ can be thought of as elements of $C_1$. Indeed, each curve may be associated to the formal sum of edges along which the curve passes. A curve will form the boundary of a piece of surface precisely when the corresponding element of $C_1$ lies in $\delta(C_2)$. 
Figure 12. The knot $K$ together with the $A$ and $B$ curves, $s'_A$ and $s'_B$.

In particular, the curves of $s'_A$ and $s'_B$ which bound punctured disks on the surface span a vector subspace $V$ of $C_1$ which is entirely contained in $\delta(C_2)$. We conclude

\begin{equation}
\dim V \leq \dim \delta(C_2).
\end{equation}

Note that:

\begin{equation}
\dim \delta(C_2) = R' - 1
\end{equation}

since the kernel of $\delta$ is 1-dimensional (it includes only the sum of no regions and the sum of all regions).

We now find a lower bound to the dimension of $V$. Consider a relation between the curves spanning $V$, that is consider a family of curves from $s'_A$ and $s'_B$ which bound punctured disks on the surface and which, as elements of $C_1$, sum to zero. Since summation is modulo 2, each edge of the projection is passed over an even number of times by curves of the family. But then either all of the curves or none of the curves at each vertex must belong to the family, since otherwise one of the edges at the vertex would have only one curve from the family along it.

But if all the curves at some vertex belong to the family then, since these curves also pass through the neighboring vertices, all the curves at the neighboring vertices must belong to the family as well. Repeating this argument, since $\pi(K')$ is connected, either all of the curves or none of the curves from $s'_A$ and $s'_B$ must belong to the family. This shows that there is only one nontrivial relation between the curves of $s'_A$ and $s'_B$, and so there is certainly no more than one relationship between the curves generating $V$, which are restricted to those bounding disks with holes. Thus

\begin{equation}
\dim V \geq p(s'_A) + p(s'_B) - 1.
\end{equation}
This, with Equations (7) and (8) shows that \( p(s'_A) + p(s'_B) \leq R' \). 

We now give the proof of Theorem 3.1.

**Proof.** We must first define a notion of span, as in Kauffman’s proof. This notion, however, is more technical, and depends on our projections. It is constructed particularly for this proof.

We use the projection of \( K \) to fix two polynomials. Let \( s_A \) be the all-A split of \( \pi(K) \) and \( s_B \) be the all-B split of \( \pi(K) \). Let \( F_A = N(s_A) \) and \( F_B = N(s_B) \). We now focus on the fixed polynomials \( P_{FA} \) and \( P_{FB} \).

Let \( \max(P) \) be the highest degree of any term of \( P \) and \( \min(P) \) the lowest. Notice that \( \max(P_{FA}(K)) - \min(P_{FB}(K)) = \max(P_{FA}(K')) - \min(P_{FB}(K')) \), since we are considering the same pair of polynomials in either case. Let \( c = c(\pi(K)) \) and \( c' = c(\pi(K')) \).

We prove the following two inequalities:

(i) \[ \max(P_{FA}(K)) - \min(P_{FB}(K)) \geq 4c - 4g + 2N; \]
(ii) \[ \max(P_{FA}(K')) - \min(P_{FB}(K')) \leq 4c' - 4g + 2N. \]

Here \( g \) is the genus of \( S \), and \( N \) is the number of curves in \( F_A \) and \( F_B \) which do not bound disks with holes. Note that once these inequalities have been proved, they together imply \( c \leq c' \), which will finish the proof.

**Proof of** (i). The proof of Statement (i) is in two parts. First we show that \( \max(P_{FA}(K)) \) is the degree of a term from the all-A split, and similarly for \( B \), and then we calculate these degrees.

The highest degree contributed by a certain state is \( a(s) - b(s) + 2(|s| - 1) \) (provided \( N(s) = F_A \)). Let us start with the state \( s_A \) and change to the state \( s \) by switching one A-crossing at a time to a B-crossing. We must show that all states which do contribute to \( P_{FA} \) contribute a strictly lower exponent than \( s_A \).

Every time an A-split is switched to a B-split, \( a(s) - b(s) \) decreases by two. \( |s| \) cannot increase by more than 1, and so the exponent \( a(s) - b(s) + 2(|s| - 1) \) cannot increase. Now, suppose that \( s \) contributes a term which cancels with the term from \( s_A \). Then the term from \( s \) must have the same degree and belong to the same polynomial as the term contributed by \( s_A \). Thus \( |s| \) must increase by one each time, and \( N(s) \) must equal \( N(s_A) \). The possibilities when the split of a given crossing is switched may be enumerated as follows, since some curve must split into two at each stage.

1) A trivial curve splits into two trivial curves.
2) A trivial curve splits into two nontrivial curves.
3) A nontrivial curve splits into two nontrivial curves.
4) A nontrivial curve splits into a nontrivial curve and a trivial curve.
Neither (1) nor (4) can occur at the first stage since $K$ is reduced (see Figure 13). If (2) or (3) occurs at the first stage the number of nontrivial curves increases and none of (1)–(4) occurring at a later stage can reduce this number to its original value, so it is impossible for $N(s)$ to equal $N(s_A)$. So it is impossible for a cancellation to occur after all.

Thus $a(s) - b(s) + 2(|s| - 1)$ for $s = s_A$ is strictly the maximum exponent appearing in $P_{F_A}$. The smallest exponent appearing in $P_{F_B}$ is found similarly. Therefore,

$$\max P_{F_A}(K) - \min P_{F_B}(K) = c + 2(|s_A| - 1) - (-c - 2(|s_B| - 1))$$

$$= 2c + 2(|s_A| + |s_B|) - 4.$$ 

By the definition of $N$, this can be written as

$$\max P_{F_A}(K) - \min P_{F_B}(K) = 2c + 2N + 2(p(s_A) + p(s_B)) - 4.$$ 

If we let $N_1$ be the number of disk with holes components of $\pi(K)^c$ which have only one boundary component formed by the knot, then we may follow Kauffman, noting that since the knot is alternating, the boundaries of such regions become curves of $s_A$ and $s_B$. By the definition of $p$ we therefore obtain

$$N_1 \leq p(s_A) + p(s_B).$$

Now we use the Euler characteristic to relate $N_1$ to the crossing number of $\pi(K)$.

Let $W = \text{number of components of } \partial S$.

$W(r) = \text{number of boundary components of a region } r \text{ of } \pi(K)^c$, which are not formed by $\pi(K)$. 

**Figure 13.** A knot in which a trivial curve splits off when an A-split is changed to a B-split. The dark curve, bounding the gray region of the knot, is the original A-curve. The light curve parallel to the curve which splits off contradicts the definition of a reduced knot.
\[ g(r) = \text{the genus of region } r \text{ and let } \chi \text{ be the Euler characteristic. Euler's formula generalized to the case where the regions are not necessarily disks gives} \]
\begin{equation}
-c + \sum_r \chi(r) = \chi(S). \tag{12}
\end{equation}

Also, \( \chi(S) = 2 - 2g - W \) and \( \sum_r W(r) = W \), so
\begin{equation}
-c + \sum_r (\chi(r) + W(r)) = 2 - 2g. \tag{13}
\end{equation}

Now \( \chi(r) + W(r) \leq 0 \) unless \( r \) has only one boundary formed by the knot and has genus zero. There are \( N_1 \) such regions and for each, \( \chi(r) + W(r) = 1 \), so by Equation (13)
\begin{equation}
-c + N_1 \geq 2 - 2g. \tag{14}
\end{equation}

Combining this last equation with Equation (11) we see that
\begin{equation}
p(s_A) + p(s_B) \geq c + 2 - 2g. \tag{15}
\end{equation}

From Equation (10) we now see that,
\begin{equation}
\max(P_{F_A}(K)) - \min(P_{F_B}(K)) \geq 4c + 2N - 4g. \tag{16}
\end{equation}

**Proof of (ii).** We now continue with the proof of Statement (ii), which concerns the non-alternating version of the knot, \( K' \).

By lemma (3.2), \( p(s'_A) + p(s'_B) \leq R' \). Now it follows by induction that for an arbitrary pair of states \( s'_1, s'_2 \),
\begin{equation}
p(s'_1) + p(s'_2) \leq R' + b_1 + a_2. \tag{17}
\end{equation}

In fact, this follows by switching A-splits to B-splits as in (i) to turn \( s'_A \) into \( s'_1 \) and \( s'_B \) into \( s'_2 \). (For the argument, notice that if a curve does not bound a disk with holes, then it cannot split into curves which do bound disks with holes.)

Now apply this inequality to a pair of states \( s'_1 \) and \( s'_2 \) which are assumed to contribute to \( P_{F_A}(K') \) and to \( P_{F_B}(K') \), respectively. The difference between the exponents they contribute is:
\[
\begin{align*}
a_1 - a_2 + b_2 - b_1 + 2(|s'_1| + |s'_2|) &- 4 \\
&= a_1 - a_2 + b_2 - b_1 + 2(p(s'_1) + p(s'_2)) - 4 + 2(|s'_1| + |s'_2|) - p(s_1) - p(s'_2) \\
&= a_1 - a_2 + b_2 - b_1 + 2(p(s'_1) + p(s'_2)) - 4 + 2N \\
&= 2c' - 2a_2 - 2b_1 + 2p(s'_1) + 2p(s'_2) - 4 + 2N \\
&\leq 2c' + 2R' - 4 + 2N.
\end{align*}
\]

The second equality follows since the nontrivial curves in \( s'_1 \) and \( s'_2 \) are just the curves \( F_a \) and \( F_b \), and \( N \) of these curves do not bound punctured disks.

The inequality follows from Equation (17).
By Euler’s formula, and the assumption that all regions of $K'$ are genus zero, $R' = c' + 2 - 2g$, so the difference between an exponent of $P_{F_4}(K')$ and one of $P_{F_3}(K')$ is at most $4c' + 2N - 2g$, proving (ii).

The inequalities (i) and (ii) together imply Theorem 3.1, since the quantities on the left side of the inequalities are the same by the invariance properties of the polynomials.

We now restate Theorem 1.1:

**Theorem 1.1.** Let $S$ be a compact surface. Let $\pi(K)$ be a reduced alternating projection of an alternating knot in $S \times I$ and let $\pi(K')$ be an arbitrary projection (of the same knot). Then

$$c(\pi(K)) \leq c(\pi(K')).$$

(18)

*Proof.* On account of Claim 2.2, Theorem 3.1 which we have just proved implies this more general theorem in the orientable case. The nonorientable case follows immediately by taking double covers.

□

4. Conclusion.

The statement of Theorem 1.1 can be strengthened. It is unnecessary to restrict the theorem to knots. Indeed, the proof works equally well for nonsplittable links, and with a few modifications it extends to links in general.

The other possible extensions of Theorem 1.1 are more difficult. We have shown that if $\pi(K)$ is a reduced alternating projection of a knot in $S \times I$, and $\pi(K')$ is any other projection of that knot, then

$$c(\pi(K)) \leq c(\pi(K')).$$

(19)

It remains to be shown that

$$c(\pi(K)) < c(\pi(K')),$$

if $\pi(K)$ is non-alternating, and the knot is prime (Murasagi and Thistlethwaite established this strict inequality in $S^3$, [10], [11]).

Also, Tait conjectured that any two reduced alternating projections of the same knot can be converted to one another through a series of special moves called flypes. This statement was proved for knots in $S^3$ by Thistlethwaite and Menasco [9]. A natural extension of Theorem 1.1 would be to prove the flyping conjecture for knots in $S \times I$.

It is natural to ask questions about knots in more complicated spaces which contain subspaces of the form $S \times I$.

One possibility is to investigate the Tait conjecture for knots in (solid) handlebodies, where we project knots onto the boundary. If our definition of reduced is used, then the case in Figure 14 is possible. The knot projection
Figure 14. A counterexample in the solid torus.

on the left is reduced and alternating, but the number of crossings can be lowered by pulling a crossing through the center of the solid torus, and then using a Type II Reidemeister move. The resulting knot with fewer crossings is shown on the right.

Thus, the conjecture doesn’t hold with our notion of reduced. However, it may hold if we use Kamada’s notion of reduced-properly reduced.

Alternatively, one could ask whether the Tait conjecture holds for knots lying on an incompressible surface $S$ in a 3-manifold $M$. We conjecture that if two projections of a knot are equivalent in the 3-manifold, then they are equivalent up to a homeomorphism of $S \times I$. If true, this would reduce the problem to the result proved here.

References


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