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RENORMALIZATION OF CERTAIN INTEGRALS DEFINING TRIPLE PRODUCT L-FUNCTIONS

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We obtain special values results for the triple product L-function attached to a Hilbert modular cuspidal eigenform over a totally real quadratic number field and an elliptic modular cuspidal eigenform, both of level one and even weight. Replacing the elliptic modular cusp form by a specified Eisenstein series, we renormalize the integral defining the triple product L-function in order to obtain an integral representation for a product of Asai L-functions. We hope in further work to extend these results to triple-product L-functions attached to automorphic representations and then study the critical values of this renormalized triple product.

1. Introduction.

This paper investigates Zagier's technique of *renormalization* ([**Z**]), applied to an integral defining a certain triple product L-function. The renormalized integral becomes the product of two Asai L-functions, one shifted by an integer. As a by-product of these results, under a certain weight restriction on the modular forms, special values results can be explicitly determined for the triple product L-function in question. Such special values issues have been studied in the representation-theoretic context by Piatetski-Shapiro and Rallis ([**PSR**]), Garrett and Harris ([**GH**]), and Harris and Kudla ([**HK**]). The foundation of this work is Garrett's groundbreaking study of the Rankin triple product L-function ([**G1**]).

The L-function in question, $L(f \otimes G, s)$, is a variation of the Rankin triple product L-function, defined for a holomorphic Hilbert modular cuspidal eigenform G and a holomorphic elliptic cuspidal eigenform f, both of level one and even weights. If we let E_3 denote the Siegel Eisenstein series of degree 3, then under a certain embedding $\iota_{2,1}$ of $\mathfrak{H}^3$ into the Siegel upper half-space of degree 3, following Garrett's techniques ([**G1**]) we show:

Theorem 1.1. *For* $\mathrm{Re}(s)$ *sufficiently large,*

$$\int_{\Gamma\backslash\mathfrak{H}^2}\int_{\mathrm{SL}(2,\mathbf{Z})\backslash\mathfrak{H}} E_3(\iota_{2,1}(Z,z_3);2k,s)\overline{G(\tilde z)}\,\overline{f(z_3)}(y_1y_2y_3)^{2k-2}dx_3\,dy_3\,d\tilde x\,d\tilde y$$
$$= L(f\otimes G, s+4k-2)\times(\textit{normalizing factors}).$$

For the result in the more general context of mixed weights, see Theorem 5.4. Such a theorem was proved in the representation-theoretic context by Garrett and Harris ([**GH**]) and leads to special values results.

Shimura developed a technique for determining special values for ratios of zeta functions associated with cusp forms by using Rankin product L-functions ([**S2**], [**S3**]). In the situations he considered, replacing one modular form in a Rankin product L-function by a specified Eisenstein series leads to a product of the zeta functions in question. One can then extend special values results for the product L-function to the product of the zeta functions. When attempting to use the same procedure for the triple product L-function, if we replace f by a specified Eisenstein series $\mathcal{E}$, then the integral no longer converges. However, by *renormalizing* the integral, we obtain:

Theorem 1.2. *For* $\mathrm{Re}(s)$ *sufficiently large,*

$$\begin{aligned} R.N. \int_{\Gamma\backslash\mathfrak{H}^2} \int_{\mathrm{SL}(2,\mathbf{Z})\backslash\mathfrak{H}} & E_3(\iota_{2,1}(Z, z_3); 2k, s)\overline{G(\tilde{z})}\,\overline{\mathcal{E}(z_3)} \\ & \cdot (y_1 y_2 y_3)^{2k-2} dx_3\, dy_3\, d\tilde{x}\, d\tilde{y} \\ = L_{\mathrm{Asai}}&(G, s+4k-2) L_{\mathrm{Asai}}(G, s+2k-1) \\ & \times (\textit{normalizing factors}). \end{aligned}$$

Refer to Theorem 6.4 for the result in the mixed-weight case.

The function $L_{\mathrm{Asai}}(G, s)$ is defined as in [**A**]. Namely, given the Hilbert modular form G of weight (k_1, k_2), $k_1 \geq k_2$, on a quadratic number field F with Fourier coefficients $b(\xi)$, the Asai L-function is constructed as a sort of "subseries" of the standard L-function attached to Hilbert modular forms, summing up only over the rational integers:

$$L_{\mathrm{Asai}}(G, s) = \zeta(2(s - k_1 + 1)) \sum_{n=1}^{\infty} b(n) n^{-s},$$

where $\zeta(s)$ is the Riemann zeta function.

Using the identities above, special values results for the triple product L-function can then extend to the product of Asai L-functions, and therefore to ratios of Asai L-functions, following the techniques of Shimura ([**S2**], [**S3**]). However, the special values results for the triple product L-function must be within a certain weight case for the modular forms, called the "indefinite case" by Harris and Kudla ([**HK**]).

Throughout this paper, we consider only modular forms of level one in the classical language. To follow Shimura's techniques for obtaining special values results, the generalization to higher levels is required, necessitating the representation-theoretic approach. The basic structure of the paper follows a similar method to that of Garrett ([**G1**]). After setting up notational preliminaries, we outline the various embeddings and coset decompositions required for the computations involving the Siegel Eisenstein series. We then compute the integral representation for the L-function, from which we derive the Euler product, functional equation, and special values result. Finally, replacing the elliptic cusp form by a specified Eisenstein series, we compute the renormalized integral, obtaining the product of Asai L-functions.

2. Preliminaries.

Let F be a real quadratic extension of $\mathbb{Q}$ with ring of integers $\mathfrak{o}$. We will assume that the quadratic number field F has narrow class number one. Let D_F denote the discriminant of F, δ^{-1} the inverse different, and let τ signify the nontrivial injection of F into $\mathbb{R}$. Write U^+ for the group of totally positive units of F.

For a commutative ring R, write $M(n, R)$ for the space of $n \times n$ matrices over R. $\mathrm{GL}(n, R)$ will signify the group of invertible $n \times n$ matrices, and $\mathrm{SL}(n, R)$ the group of matrices with determinant one. I_n is the $n \times n$ identity matrix, and let $J_n = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$.

The symbol $\mathfrak{H}$ will be used to denote the complex upper half-plane, and $\mathfrak{H}^n$ represents the product of n copies of the complex upper half-plane. The Siegel upper half-space of degree n is written

$$\mathfrak{H}_n = \{Z \in M(n, \mathbb{C}) | {}^tZ = Z, \mathrm{Im}(Z) > 0\}.$$

The Siegel Eisenstein series is then defined as follows. The symplectic group is given by

$$\mathrm{Sp}(n, R) = \{g \in \mathrm{GL}(2n, R) | {}^tg J_n g = J_n\}.$$

There is a natural action of $\mathrm{Sp}(n, \mathbb{R})$ on $\mathfrak{H}_n$ given by

$$Z \mapsto gZ = (AZ + B)(CZ + D)^{-1}, \qquad g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

Let

$$P_{n,0}(\mathbb{Z}) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(n, \mathbb{Z}) \middle| C = 0 \right\}.$$

For $s \in \mathbb{C}$, $k \in \mathbb{Z}$, and $Z = X + iY \in \mathfrak{H}_n$, define the Eisenstein series

$$E_n(Z; 2k, s) = \sum_{C,\, D} \det(CZ + D)^{-2k} \frac{(\det Y)^s}{|\det(CZ + D)|^{2s}}, \tag{2.1}$$

where the sum is over all representatives $\begin{pmatrix} * & * \\ C & D \end{pmatrix}$ for $P_{n,0}(\mathbb{Z})\backslash \mathrm{Sp}(n,\mathbb{Z})$. For $Z \in \mathfrak{H}_n$ and $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(n,R)$, we will often write

$$\mu(g,Z) = \det(CZ+D). \tag{2.2}$$

The Eisenstein series converges for $\mathrm{Re}(s)$ sufficiently large, and can be continued to a meromorphic function in the entire s-plane (see [**K**] or [**L**]). Its poles have been studied by Ikeda ([**I**]).

3. Embeddings of symplectic spaces and coset decompositions.

To proceed with the integral representation of the triple product L-function, we first need to determine an embedding from $\mathrm{Sp}(1,F)$ into $\mathrm{Sp}(2,\mathbb{Q})$, which takes a group $\Gamma \cong \mathrm{Sp}(1,\mathfrak{o})$ to $\mathrm{Sp}(2,\mathbb{Z})$. Write E_F for the determinant mapping $F^2 \times F^2 \to F$, and define

$$E_{\mathbb{Q}} = \frac{1}{2}\mathrm{Tr}_{F/\mathbb{Q}} \circ E_F : F^2 \times F^2 \to \mathbb{Q}.$$

Note that if we set $M = \mathfrak{o} \oplus \delta^{-1}$, then $E_{\mathbb{Q}} : M \times M \to \mathbb{Z}$ surjectively. Let

$$\Gamma = \Gamma(M, E_F) = \{g \in \mathrm{Sp}(1,F) \,|\, gM = M\}. \tag{3.1}$$

Then $\Gamma \cong \mathrm{Sp}(1,\mathfrak{o})$. One can check at once that over $\mathbb{Z}$, $\Gamma(M_{/\mathbb{Z}}, E_{\mathbb{Q}}) \cong \mathrm{Sp}(2,\mathbb{Z})$. Let $\beta \in F$ be an element such that $\{1,\beta\}$ is a $\mathbb{Z}$-basis of $\mathfrak{o}$, and put

$$B = \begin{pmatrix} 1 & \beta \\ 1 & \beta^\tau \end{pmatrix}.$$

We will define $\iota : \mathfrak{H}^2 \to \mathfrak{H}_2$ to be the embedding:

$$\iota(z_1,z_2) = B^{-1} \cdot \mathrm{diag}[z_1,z_2] \cdot {}^tB^{-1}.$$

It is compatible with the injection $\tilde{\iota}$ of $\mathrm{Sp}(1,F)$ into $\mathrm{Sp}(2,\mathbb{Q})$ defined by

$$\tilde{\iota}\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} B^{-1} & 0 \\ 0 & {}^tB \end{pmatrix}\begin{pmatrix} \Delta(a) & \Delta(b) \\ \Delta(c) & \Delta(d) \end{pmatrix}\begin{pmatrix} B & 0 \\ 0 & {}^tB^{-1} \end{pmatrix},$$

where $\Delta(a) = \mathrm{diag}[a, a^\tau]$, $(a \in F)$. Note that $\tilde{\iota} : \Gamma \to \mathrm{Sp}(2,\mathbb{Z})$. Define two more embeddings

$$\tilde{\iota}_{m,n} : \mathrm{Sp}(m,\mathbb{Q}) \times \mathrm{Sp}(n,\mathbb{Q}) \to \mathrm{Sp}(m+n,\mathbb{Q})$$
$$\text{and} \qquad \iota_{m,n} : \mathfrak{H}_m \times \mathfrak{H}_n \to \mathfrak{H}_{m+n}$$

by

$$\tilde{\iota}_{m,n}\left(\begin{pmatrix} A & B \\ C & D \end{pmatrix} \times \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix}\right) = \begin{pmatrix} A & 0 & B & 0 \\ 0 & A' & 0 & B' \\ C & 0 & D & 0 \\ 0 & C' & 0 & D' \end{pmatrix},$$

$$\text{and} \quad \iota_{m,n}(Z,Z') = \begin{pmatrix} Z & 0 \\ 0 & Z' \end{pmatrix}.$$

Having determined the appropriate embeddings, following Garrett's lead ([**G1**]) we now investigate the coset decompositions that are used to rewrite the Siegel Eisenstein series. For a commutative ring R, consider the following subgroups of $\mathrm{Sp}(n,R)$, where I_m is the identity matrix of size m, 0_m is the zero matrix of size m, and $0 \le r < n$:

$$G_{n,r}(R) = \left\{ \begin{pmatrix} I_{n-r} & 0 & 0_{n-r} & 0 \\ 0 & * & 0 & * \\ 0_{n-r} & 0 & I_{n-r} & 0 \\ 0 & * & 0 & * \end{pmatrix} \in \mathrm{Sp}(n,R) \right\},$$

$$L_{n,r}(R) = \left\{ \begin{pmatrix} * & 0 & 0_{n-r} & 0 \\ 0 & I_r & 0 & 0 \\ 0_{n-r} & 0 & * & 0 \\ 0 & 0 & 0 & I_r \end{pmatrix} \in \mathrm{Sp}(n,R) \right\},$$

$$U_{n,r}(R) = \left\{ \begin{pmatrix} I_{n-r} & * & * & * \\ 0 & I_r & * & 0 \\ 0 & 0 & I_{n-r} & 0 \\ 0 & 0 & * & I_r \end{pmatrix} \in \mathrm{Sp}(n,R) \right\}.$$

Set $P_{n,r}(R) = G_{n,r}(R)L_{n,r}(R)U_{n,r}(R)$, and let

$$w_{n,r} = \begin{pmatrix} 0_{n-r} & 0 & -1_{n-r} & 0 \\ 0 & 1_r & 0 & 0 \\ 1_{n-r} & 0 & 0_{n-r} & 0 \\ 0 & 0 & 0 & 1_r \end{pmatrix},$$

$$w_n = w_{n,0}.$$

The following four results, which only involve the rational symplectic spaces, are proved in [**G1**].

Proposition 3.1. *The double coset space*

$$P_{n+1,0}(\mathbb{Q})\backslash \mathrm{Sp}(n+1,\mathbb{Q})/\tilde{\iota}_{n,1}(\mathrm{Sp}(n,\mathbb{Q}) \times \mathrm{Sp}(1,\mathbb{Q}))$$

has irredundant representatives I_{2n+2}, ξ, where

$$\xi = \xi_0 \tilde{\iota}_{n,1}(I_{2n}, w_1)$$

$$\xi_0 = \begin{pmatrix} I_n & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0_n & v & I_n & 0 \\ {}^t v & 0 & 0 & 1 \end{pmatrix}$$

$${}^t v = (0, \dots, 0, 1) \in \mathbb{Q}^n.$$

Proposition 3.2. *The coset space*

$$P_{n+1,0}(\mathbb{Q})\backslash \mathrm{Sp}(n+1,\mathbb{Q})$$

has irredundant representatives consisting of the disjoint union of representatives for

$$\tilde{\iota}_{n,1}(P_{n,0}(\mathbb{Q})\backslash \mathrm{Sp}(n,\mathbb{Q}) \times P_{1,0}(\mathbb{Q})\backslash \mathrm{Sp}(1,\mathbb{Q}))$$

and for

$$\xi\tilde{\iota}_{n,1}(P_{n,1}(\mathbb{Q})\backslash \mathrm{Sp}(n,\mathbb{Q}) \times \mathrm{Sp}(1,\mathbb{Q})),$$

where ξ is defined in Proposition 3.1*.*

Lemma 3.3. *Let $\gamma_1 \in \mathrm{Sp}(1,\mathbb{Z})$, $\varepsilon \in \mathbb{Z}$, $\varepsilon > 0$,*

$$A_\varepsilon = \begin{pmatrix} \varepsilon^{-1} & 0 \\ 0 & \varepsilon \end{pmatrix} \in \mathrm{Sp}(1,\mathbb{Q}),$$

and ξ be as in Proposition 3.1 *with $n = 2$. Then there is an element $p_\varepsilon \in P_{3,0}(\mathbb{Q})$ such that for all $\tilde{\gamma} \in \mathrm{Sp}(2,\mathbb{Z})$, $\gamma_2 \in \mathrm{Sp}(1,\mathbb{Z})$,*

$$p_\varepsilon \xi \tilde{\iota}_{2,1}(\tilde{\gamma}, \gamma_1 A_\epsilon \gamma_2) \in \mathrm{Sp}(3,\mathbb{Z}), \quad \textit{and} \quad \mu(p_\varepsilon, *) = \varepsilon,$$

where μ is defined by (2.2)*.*

Lemma 3.4. *The coset space $\mathrm{Sp}(1,\mathbb{Z})\backslash \mathrm{Sp}(1,\mathbb{Q})/\mathrm{Sp}(1,\mathbb{Z})$ has irredundant representatives*

$$\left\{ A_\varepsilon = \begin{pmatrix} \varepsilon^{-1} & 0 \\ 0 & \varepsilon \end{pmatrix} : 0 < \varepsilon \in \mathbb{Z} \right\}.$$

Now we incorporate the quadratic number field F into similar calculations.

Proposition 3.5. *The double coset space*

$$P_{2,1}(\mathbb{Q})\backslash \mathrm{Sp}(2,\mathbb{Q})/\tilde{\iota}(\mathrm{Sp}(1,F))$$

has just one orbit, and so one representative, I_4.

Proof. Since $P_{2,1}(\mathbb{Q})$ is the stabilizer of a line and $\mathrm{Sp}(2,\mathbb{Q})$ acts transitively on lines in $\mathbb{Q}^4$, the coset space $P_{2,1}(\mathbb{Q})\backslash \mathrm{Sp}(2,\mathbb{Q})$ is naturally $\mathbb{P}^3(\mathbb{Q})$. We may consider $\mathbb{Q}^4$ as F^2, so, as $\mathrm{Sp}(1,F)$ acts transitively on the nonzero vectors of F^2, there is only one orbit of $\mathrm{Sp}(1,F)$ on $\mathbb{P}^3(\mathbb{Q})$. □

Proposition 3.6. *The coset space $P_{2,1}(\mathbb{Q})\backslash \mathrm{Sp}(2,\mathbb{Q})$ has irredundant representatives consisting of the disjoint union of representatives for*

$$\tilde{\iota}(\widetilde{U}(F)\backslash \mathrm{Sp}(1,F)),$$

$$\textit{where } \widetilde{U}(F) = \left\{ \begin{pmatrix} q & * \\ 0 & q^{-1} \end{pmatrix} \in \mathrm{Sp}(1,F),\ q \in \mathbb{Q}^* \right\}.$$

Proof. We must find a subgroup H of $\mathrm{Sp}(1,F)$ so that $h \in H$ if and only if

$$P_{2,1}(\mathbb{Q})\tilde{\iota}(h) = P_{2,1}(\mathbb{Q}).$$

That is, $\tilde{\iota}(h) \in P_{2,1}(\mathbb{Q})$.

Looking at $P_{2,1}(\mathbb{Q})$ more closely, we see we can describe it as the following set of matrices:

$$P_{2,1}(\mathbb{Q}) = \left\{ \begin{pmatrix} \alpha & u\alpha & v\alpha & w\alpha \\ 0 & A & wA - uB & B \\ 0 & 0 & \gamma & 0 \\ 0 & C & wC - uD & D \end{pmatrix} \in \mathrm{Sp}(2,\mathbb{Q}) \middle| \right.$$

$$\left. \alpha, \gamma \neq 0, AD - BC \neq 0 \right\}.$$

If $h = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a_1 + a_2\beta & b_1 + b_2\beta \\ c_1 + c_2\beta & d_1 + d_2\beta \end{pmatrix} \in \mathrm{Sp}(1,F)$, then

$$\tilde{\iota}(h) = \begin{pmatrix} a_1 & \beta^2 a_2 & b_1/2 & b_2/2 \\ a_2 & a_1 & b_2/2 & b_1/2\beta^2 \\ 2c_1 & 2\beta^2 c_2 & d_1 & d_2 \\ 2\beta^2 c_2 & 2\beta^2 c_1 & \beta^2 d_2 & d_1 \end{pmatrix}.$$

Combining the explicit descriptions for $\tilde{\iota}(h)$ and the matrices in $P_{2,1}(\mathbb{Q})$, we obtain the required result. $\square$

Definition 3.7. Every totally positive element of $F^*/\mathbb{Q}^*$ has a unique representative α which can be written $\alpha = s + t\beta$, where $s, t \in \mathbb{Z}$, s is positive, and $(s,t) = 1$. We will call such a representative ***primitive***.

Lemma 3.8. *For $\alpha = s + t\beta$ a primitive element of F^*, set*

$$A = A_\alpha = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} \in \mathrm{Sp}(1,F).$$

Then there exists a matrix

$$p = p_\alpha = \begin{pmatrix} \mathrm{N}(\alpha)^{-1} & c_\alpha & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \mathrm{N}(\alpha) & 0 \\ 0 & 0 & d_\alpha & 1 \end{pmatrix} \in P_{2,0}(\mathbb{Q}) \cap P_{2,1}(\mathbb{Q})$$

such that for all $\gamma \in \Gamma$,

$$p_\alpha \tilde{\iota}(A_\alpha \gamma) \in \mathrm{Sp}(2,\mathbb{Z}).$$

Proof. Since

$$p_\alpha \tilde{\iota}(A_\alpha \gamma) = p_\alpha \tilde{\iota}(A_\alpha)\tilde{\iota}(\gamma),$$

and $\tilde{\imath}(\gamma) \in \mathrm{Sp}(2, \mathbb{Z})$, it suffices to consider the case where $\gamma = I_2$. To find c_α and d_α, using the fact that s and t are relatively prime, let $m, n \in \mathbb{Z}$ such that $sn + tm = 1$. Then set

$$c_\alpha = \frac{-t\beta^2 - m\mathrm{N}(\alpha)}{s\mathrm{N}(\alpha)}, \qquad d_\alpha = \frac{s - n\mathrm{N}(\alpha)}{t}$$

where $\mathrm{N}(\alpha) = \mathrm{Norm}_{F/\mathbb{Q}}(\alpha)$. Direct multiplying out gives the result. □

4. The integral representation.

We now obtain the integral representation of the triple product L-function $L(f \otimes G, s)$. Let $f(z)$ be a normalized holomorphic cuspidal eigenform of weight $2l$ on $\mathrm{SL}(2, \mathbb{Z})$, and let $G(z_1, z_2)$ be a normalized holomorphic cuspidal eigenform of weight $(2k_1, 2k_2)$ on Γ, where Γ is defined by Equation (3.1). For $\tilde{z} = (z_1, z_2) \in \mathfrak{H}^2$, write the Fourier expansions of f and G as

$$\begin{aligned} f(z) &= \sum_n a(n)\mathbf{e}(nz), \quad \mathbf{e}(z) = \exp(2\pi i z) \\ G(\tilde{z}) &= \sum_\xi b(\xi)\mathbf{e}_F(\xi\tilde{z}), \quad \mathbf{e}_F(\tilde{z}) = \exp\left(2\pi i(z_1 + z_2)\right), \end{aligned}$$

where ξ ranges over all totally positive elements of a lattice in F. If we write ξ_1 for the trivial injection of ξ into $\mathbb{R}$ and ξ_2 for the nontrivial injection of ξ into $\mathbb{R}$, then $\xi\tilde{z} = (\xi_1 z_1, \xi_2 z_2)$.

In order to compute the integral, differential operators of Maass ([**M**]) and Shimura ([**S1**]) are needed to raise the weights of the forms so they are all equal. For $z \in \mathfrak{H}$, $(z_1, z_2) \in \mathfrak{H}^2$, and integers κ, r, λ_ν, $s_\nu \geq 0$ with $\nu = 1, 2$, define operators for the elliptic and Hilbert modular forms, respectively, by

$$\delta_\kappa^{(r)} = \left(\frac{1}{2\pi i}\right)^r \left(\frac{\kappa + 2r - 2}{2iy} + \frac{\partial}{\partial z}\right) \cdots \left(\frac{\kappa + 2}{2iy} + \frac{\partial}{\partial z}\right) \left(\frac{\kappa}{2iy} + \frac{\partial}{\partial z}\right) \tag{4.1}$$

$$\begin{aligned} \delta_{(\lambda_1,\lambda_2)}^{(s_1,s_2)} = \left(\frac{1}{2\pi i}\right)^{s_1+s_2} \prod_{\nu=1}^{2} &\left(\frac{\lambda_\nu + 2s_\nu - 2}{2iy_\nu} + \frac{\partial}{\partial z_\nu}\right) \\ &\cdots \left(\frac{\lambda_\nu + 2}{2iy_\nu} + \frac{\partial}{\partial z_\nu}\right) \left(\frac{\lambda_\nu}{2iy_\nu} + \frac{\partial}{\partial z_\nu}\right), \end{aligned} \tag{4.2}$$

where $\partial/\partial z = (\partial/\partial x - i\partial/\partial y)/2$ as usual, and it is understood that $\delta_\kappa^{(0)}$ and $\delta_{(\lambda_1,\lambda_2)}^{(0,0)}$ are the identity operators. It can be shown that $\delta_\kappa^{(r)}$ and $\delta_{(\lambda_1,\lambda_2)}^{(s_1,s_2)}$ raise the corresponding weights of modular forms of weight κ and (λ_1, λ_2) to $\kappa + 2r$ and $(\lambda_1 + 2s_1, \lambda_2 + 2s_2)$, respectively ([**S1**], [**S2**]).

As Orloff demonstrated ([**BO**]), we may write

$$\delta_\kappa^{(r)}\mathbf{e}(nz) = \sum_{j=0}^{r} P_j^{(r)}(4\pi y)^{-j} n^{r-j}\mathbf{e}(nz), \tag{4.3}$$

with integers $P_j^{(r)}$ defined by

$$P_j^{(r)} = (-1)^j \binom{r}{j} \frac{\Gamma(\kappa+r)}{\Gamma(\kappa+r-j)}. \tag{4.4}$$

Therefore, applying the operators given by (4.1) and (4.2) to the cusp forms f and G in the case where $k_1 \geq l \geq k_2$, we may write

$$\delta_{2l}^{k_1-l} f(z) = \sum_n a(n) \sum_{A=0}^{k_1-l} P_A (4\pi y)^{-A} n^{k_1-l-A}\mathbf{e}(nz) \tag{4.5}$$

$$\delta_{(2k_1,2k_2)}^{(0,k_1-k_2)} G(\tilde{z}) = \sum_\xi b(\xi) \sum_{B=0}^{k_1-k_2} Q_B (4\pi y_2)^{-B} \xi_2^{k_1-k_2-B}\mathbf{e}_F(\xi\tilde{z}). \tag{4.6}$$

Define the Dirichlet series

$$D_f^{(2)}(s) = \sum_n a(n^2) n^{-2s}, \tag{4.7}$$

$$D_{f,G}(s) = \sum_{n,\,\alpha U^+} a(n)\overline{b(n\alpha^2)}(\alpha^2)^{k_1-k_2} n^{1-l-k_2}(n\mathrm{N}(\alpha)^2)^{1-s-2k_1}, \tag{4.8}$$

where U^+ is the group of totally positive units of F, n ranges over the positive integers, α ranges over the primitive elements of F^* modulo U^+, and $\overline{b(\xi)}$ is the complex conjugate of $b(\xi)$.

For $z, s \in \mathbb{C}$, $y \in \mathbb{R}$, and $k \in \mathbb{Z}$, put

$$q_{2k,s}(z) = |z+i|^{-2s}(z+i)^{-2k}$$

$$\hat{q}_{2k,s}(y) = \int_{\mathbb{R}} q_{2k,s}(x)\mathbf{e}(-xy)dx \tag{4.9}$$

$$\chi_{2k,s}(z) = |z|^{-2s} z^{-2k} \tag{4.10}$$

(4.11)

$$\begin{aligned}\eta_{2k}(s) = D_F^{1/2} &\sum_{A=0}^{k_1-l}\sum_{B=0}^{k_1-k_2} P_A Q_B (4\pi)^{-A-B} \\ &\cdot \int_0^\infty\int_0^\infty\int_0^\infty (y_1y_2y_3)^{s+2k-2} y_1^{-A} y_3^{-B} (y_1+y_2+y_3)^{1-2k-2s} \\ &\cdot \hat{q}_{2k,s}(y_1+y_2+y_3)\exp(-2\pi(y_1+y_2+y_3))dy_1\,dy_2\,dy_3.\end{aligned}$$

The integrals exist for $\mathrm{Re}(s)$ sufficiently large. Define $\pi : \mathfrak{H}_2 \to \mathfrak{H}$ by

$$\pi\left(\begin{pmatrix} * & * \\ * & z_{22} \end{pmatrix}\right) = z_{22}. \tag{4.12}$$

The following lemma provides the integral computation with respect to f.

Lemma 4.1. *For $w \in \mathfrak{H}$, $\mathrm{Re}(s)$ sufficiently large, define*

$$f_s^*(w) = \sum_n a(n) \sum_{A=0}^{k_1-l} P_A (4\pi)^{-A} n^{k_1-l-A} \mathbf{e}(n(\mathrm{Re}(w))) \int_0^\infty y^{s+2k_1-2-A} \cdot (y+\mathrm{Im}(w))^{1-2s-2k} \exp(-2\pi ny) \hat{q}_{2k,s}(ny + n\mathrm{Im}(w))\, dy,$$

where the coefficients are defined by (4.4) *and* (4.5). *Then, for $Z \in \mathfrak{H}_2$, $z \in \mathfrak{H}$,*

$$\begin{aligned}\int_{\mathrm{SL}(2,\mathbb{Z})\backslash\mathfrak{H}} & E_3(\iota_{2,1}(Z,z); 2k_1, s)\overline{\delta_{2l}^{k_1-l} f(z)} y^{2k_1-2} dx\, dy \\ &= \zeta(2s+2k_1)^{-1} D_f^{(2)}(s+k_1+l-1) \\ &\quad \cdot \sum_{\tilde\gamma} [\det \mathrm{Im}(\tilde\gamma Z)]^s \mu(\tilde\gamma, Z)^{-2k_1} f_s^*(\pi(\tilde\gamma Z)),\end{aligned}$$

where the sum is over $\tilde\gamma \in P_{2,1}(\mathbb{Z})\backslash \mathrm{Sp}(2,\mathbb{Z})$, and $\zeta(z)$ is the Riemann zeta-function.

The proof is almost identical to that of Garrett ([**G1**]), requiring application of basic properties of the differential operators ([**S1**]).

Having integrated with respect to the elliptic cusp form, it remains to compute the integral with respect to the Hilbert modular cusp form.

Proposition 4.2. *With notation as above, $\tilde z = (z_2, z_3) \in \mathfrak{H}^2$, and $\iota(\tilde z) = Z$,*

$$\begin{aligned}\int_{\Gamma\backslash\mathfrak{H}^2} \int_{\mathrm{SL}(2,\mathbb{Z})\backslash\mathfrak{H}} & E_3(\iota_{2,1}(Z,z_3); 2k_1, s)\overline{\delta_{2l}^{k_1-l} f(z_1)} \\ & \cdot \overline{\delta_{(2k_1,2k_2)}^{(0,k_1-k_2)} G(\tilde z)} (y_1y_2y_3)^{2k_1-2} dx_1\, dy_1\, d\tilde x\, d\tilde y \\ &= \eta_{k_1,k_2,l}(s) \zeta(2s+2k_1)^{-1} D_f^{(2)}(s+k_1+l-1) D_{f,G}(s).\end{aligned}$$

Proof. By Lemma 4.1 the integral becomes

$$\zeta(2s+2k_1)^{-1} D_f^{(2)}(s+k_1+l-1) \int_{\Gamma\backslash\mathfrak{H}^2} \sum_{\tilde\gamma} [\det \mathrm{Im}(\tilde\gamma Z)]^s \cdot \mu(\tilde\gamma, Z)^{-2k_1} f_s^*(\pi(\tilde\gamma Z)) \overline{\delta_{(2k_1,2k_2)}^{(0,k_1-k_2)} G(\tilde z)} (y_2y_3)^{2k_1-2} d\tilde x\, d\tilde y \tag{4.13}$$

with $\tilde\gamma \in P_{2,1}(\mathbb{Z})\backslash \mathrm{Sp}(2,\mathbb{Z})$.

Using the fact that

$$P_{1,0}(F)\backslash \mathrm{Sp}(1,F) \approx P_{1,0}(\mathfrak{o})\backslash \mathrm{Sp}(1,\mathfrak{o}) \approx P_{1,0}(\Gamma)\backslash\Gamma,$$

and remarking that any element of $P_{1,0}(F)$ can be written as A_α times an element of $\tilde{U}(F)$ for some primitive α, then by Propositions 3.5 and 3.6, we can write

$$\tilde{\gamma} \in \tilde{\iota}(A_\alpha \gamma),$$

where $\gamma \in P_{1,0}(\Gamma)\backslash\Gamma$.

With p_α as in Lemma 3.8, let $\varsigma \in P_{2,0}(\mathbb{Q}) \cap P_{2,1}(\mathbb{Q})$ be the matrix

$$\varsigma = \begin{pmatrix} 4\beta^2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & (4\beta^2)^{-1} \end{pmatrix}.$$

One computes that

$$\begin{aligned} &\pi(\varsigma p_\alpha \tilde{\iota}(A_\alpha\gamma)\iota(z_2, z_3)) \\ &= \pi\left(\begin{pmatrix} * & * \\ t & s \end{pmatrix}\left(\frac{1}{4\beta^2}\begin{pmatrix} \beta^2(\gamma z_2 + \gamma^\tau z_3) & \beta(\gamma z_2 - \gamma^\tau z_3) \\ \beta(\gamma z_2 - \gamma^\tau z_3) & \gamma z_2 + \gamma^\tau z_3 \end{pmatrix}\right)\right. \\ &\qquad \left.\cdot \left(4\beta^2 \begin{pmatrix} * & t \\ * & s \end{pmatrix}\right)\right) \\ &= \alpha^2 \gamma z_2 + (\alpha^\tau)^2 \gamma^\tau z_3. \end{aligned}$$

Therefore (4.13) is equal to

$$\begin{aligned} (4.14)\quad &\zeta(2s + 2k_1)^{-1} D_f^{(2)}(s + k_1 + l - 1) \\ &\quad \cdot \int_{\Gamma\backslash\mathfrak{H}^2} \sum_\alpha \sum_\gamma [\det(\operatorname{Im}(p_\alpha \tilde{\iota}(A_\alpha\gamma)\iota(\tilde{z})))]^s \mu(p_\alpha\tilde{\iota}(A_\alpha\gamma), \iota(\tilde{z}))^{-2k_1} \\ &\quad \cdot f_s^*\left(\alpha^2 \gamma z_2 + (\alpha^\tau)^2 \gamma^\tau z_3\right) \overline{\delta_{(2k_1,2k_2)}^{(0,k_1-k_2)} G(\tilde{z})} (y_2 y_3)^{2k_1 - 2} d\tilde{x}\, d\tilde{y}, \end{aligned}$$

where α ranges over the primitive elements of F^*, and γ is in $P_{1,0}(\Gamma)\backslash\Gamma$.

Since $\mu(p_\alpha\tilde{\iota}(A_\alpha\gamma), \iota(\tilde{z})) = \mu(\gamma, z_2)\mu(\gamma^\tau, z_3)$, and $\det(\operatorname{Im}(p_\alpha\tilde{\iota}(A_\alpha\gamma)\iota(\tilde{z}))) = y_2 y_3$, by 'unwinding' as usual, we note that for fixed α the integral in (4.14) is now

$$\int_{P_{1,0}(\Gamma)\backslash\mathfrak{H}^2} f_s^*\left(\alpha^2 z_2 + (\alpha^\tau)^2 z_3\right) \overline{\delta_{(2k_1,2k_2)}^{(0,k_1-k_2)} G(\tilde{z})} (y_2 y_3)^{s + 2k_1 - 2} d\tilde{x}\, d\tilde{y}.$$

From the Fourier expansions of f_s^* and G, and summing over α, the Rankin method shows that the integral becomes

$$\int_{\mathbb{R}_+^2/U^+}\left[\int_{\mathbb{R}^2/\delta^{-1}}\sum_{\alpha} f_s^*\left(\alpha^2 z_2+(\alpha^\tau)^2 z_3\right)\overline{\delta_{(2k_1,2k_2)}^{(0,k_1-k_2)}G(\tilde{z})}d\tilde{x}\right](y_2y_3)^{s+2k_1-2}d\tilde{y}$$

$$= \int_{\mathbb{R}_+^2/U^+}(\mathrm{vol}(\mathbb{R}^2/\delta^{-1}))\int_0^\infty \sum_{\alpha,\,n} a(n)\overline{b(n\alpha^2)}n^{k_1-l}(n\alpha^2)^{k_1-k_2}$$
$$\cdot \sum_{A=0}^{k_1-l} P_A(4\pi n)^{-A}\sum_{B=0}^{k_1-k_2} Q_B(4\pi\alpha^2 n)^{-B}(y_1y_2y_3)^{s+2k_1-2}y_1^{-A}y_3^{-B}$$
$$\cdot (y_1+\alpha^2 y_2+(\alpha^\tau)^2 y_3)^{1-2k_1-2s}\exp(-2\pi n(y_1+\alpha^2y_2+(\alpha^\tau)^2y_3))$$
$$\cdot \hat{q}_{2k,s}(ny_1+n\alpha^2y_2+n(\alpha^\tau)^2y_3)dy_3\,d\tilde{y}.$$

Replacing y_1 by y_1/n, y_2 by $y_2/n\alpha^2$, and y_3 by $y_3/n(\alpha^\tau)^2$, the right side is simplified to become

$$D_F{}^{1/2}\sum_{\alpha U^+,\,n} a(n)\overline{b(n\alpha^2)}(\alpha^2)^{k_1-k_2}n^{1-l-k_2}(nN(\alpha)^2)^{1-s-2k_1}$$
$$\cdot\sum_{A=0}^{k_1-l}P_A(4\pi)^{-A}\sum_{B=0}^{k_1-k_2}Q_B(4\pi)^{-B}\int_0^\infty\int_0^\infty\int_0^\infty(y_1y_2y_3)^{s+2k_1-2}$$
$$\cdot y_1^{-A}y_3^{-B}(y_1+y_2+y_3)^{1-2k_1-2s}\exp(-2\pi n(y_1+y_2+y_3))$$
$$\cdot\hat{q}_{2k,s}(ny_1+y_2+y_3)dy_1\,dy_2\,dy_3,$$

Substituting in the appropriate expressions provides the required result. □

5. The Euler product.

Before determining the Euler factors of $L(f\otimes G,s)$, we first extend the notion of primitive *elements* to *ideals*.

Definition 5.1. For $\mathcal{I}$ an ideal of F, write $\mathcal{I}$ as a product of prime ideals: $\mathcal{I}=\prod_i \mathfrak{p}_i{}^{n_i}\overline{\mathfrak{p}_i}^{m_i}$. Let $\nu_{\mathcal{I}}=\prod_i(\mathfrak{p}_i\overline{\mathfrak{p}_i})^{\min(n_i,m_i)}$, and let $\mathcal{J}=\mathcal{I}\nu_{\mathcal{I}}$. We will call the ideal $\mathcal{J}$ ***primitive***.

Note that there is a one-to-one correspondence between primitive elements of F modulo U^+ and the primitive ideals. Defining the Fourier coefficients of $G(\tilde{z})$ on integral ideals (ξ) by $b((\xi))=b(\xi)\xi_2^{k_1-k_2}$, we may now rewrite the Dirichlet series

$$D_{f,G}(s)=\sum_{n,\,(\alpha)} a(n)\overline{b(n(\alpha^2))}((\alpha^2))^{k_1-k_2}n^{1-l-k_2}(n\mathrm{N}((\alpha))^2)^{1-s-2k_1},$$

where (α) ranges over the primitive ideals of F.

Keeping the same notation as above, for each prime number p, define α_p and α'_p by

$$1 - a(p)X + p^{2l-1}X^2 = (1-\alpha_p X)(1-\alpha'_p X).$$

For each prime ideal $\mathfrak{p}$ of F, define $\beta_{\mathfrak{p}}$ and $\beta'_{\mathfrak{p}}$ by

$$1 - \overline{b(\mathfrak{p})}X + N\mathfrak{p}^{k_1+k_2-1}X^2 = (1-\beta_{\mathfrak{p}} X)(1-\beta'_{\mathfrak{p}} X).$$

Recall that

$$a(p^m) = ({\alpha_p}^{m+1} - {\alpha'_p}^{m+1})/(\alpha_p - \alpha'_p),$$

and similarly for $b(\mathfrak{p}^m)$. Also, the Fourier coefficients are weakly multiplicative.

Now for $V = p^{-s}$ and $v = p^{k_1+k_2-1}$, put $L(f\otimes G, s) = \prod_p L_p(s)$, where

$$L_p(s)^{-1} = \begin{cases} (1-\alpha_p\beta_{\mathfrak{p}}V)(1-\alpha_p\beta'_{\mathfrak{p}}V)(1-\alpha'_p\beta_{\mathfrak{p}}V) & \text{if } p = \mathfrak{p} \\ \quad\times(1-\alpha'_p\beta'_{\mathfrak{p}}V)(1-\alpha_p^2v^2V^2) & \\ \quad\times(1-{\alpha'_p}^2v^2V^2) & \\ & \\ (1-\alpha_p\beta_{\mathfrak{p}_1}\beta_{\mathfrak{p}_2}V)(1-\alpha_p\beta'_{\mathfrak{p}_1}\beta_{\mathfrak{p}_2}V) & \text{if } p = \mathfrak{p}_1\mathfrak{p}_2 \\ \quad\times(1-\alpha_p\beta_{\mathfrak{p}_1}\beta'_{\mathfrak{p}_2}V)(1-\alpha_p\beta'_{\mathfrak{p}_1}\beta'_{\mathfrak{p}_2}V) & \\ \quad\times(1-\alpha'_p\beta_{\mathfrak{p}_1}\beta_{\mathfrak{p}_2}V)(1-\alpha'_p\beta'_{\mathfrak{p}_1}\beta_{\mathfrak{p}_2}V) & \\ \quad\times(1-\alpha'_p\beta_{\mathfrak{p}_1}\beta'_{\mathfrak{p}_2}V)(1-\alpha'_p\beta'_{\mathfrak{p}_1}\beta'_{\mathfrak{p}_2}V) & \\ & \\ (1-\alpha_p\beta_{\mathfrak{p}}^2V)(1-\alpha_p{\beta'_{\mathfrak{p}}}^2V)(1-\alpha'_p\beta_{\mathfrak{p}}^2V) & \text{if } p = \mathfrak{p}^2. \\ \quad\times(1-\alpha'_p{\beta'_{\mathfrak{p}}}^2V)(1-\alpha_p vV) & \\ \quad\times(1-\alpha'_p vV) & \end{cases} \tag{5.1}$$

Note that the above provides an explicit description of the Euler factors even at the ramified primes.

Theorem 5.2. *The Dirichlet series*

$$\zeta(2s+2k_1)^{-1}D_f^{(2)}(s+k_1+l-1)D_{f,G}(s)$$

is equal to

$$\zeta(2s+2k_1)^{-1}\zeta(4s+4k_1-2)^{-1}L(f\otimes G, s+2k_1+k_2+l-2)$$

(*first for* $\operatorname{Re}(s)$ *sufficiently large, then by analytic continuation*).

Proof. The proof uses the fact that the Dirichlet series $D_f^{(2)}(s+k_1+l-1)$ has an Euler product with p-factor

$$\frac{(1+p^{1-2s-2k_1})}{(1-\alpha_p^2p^{2-2s-2k_1-2l})(1-{\alpha'_p}^2p^{2-2s-2k_1-2l})},$$

which can be computed using the previous remarks in this section regarding the Fourier coefficients of f. The theorem will then follow from this factorization and the fact that the Dirichlet series $D_{f,G}(s)$ has an Euler product with p-factor

$$(1-p^{1-2s-2k_1})(1-\alpha_p^2 p^{2-2s-4k_1-2l})(1-{\alpha'_p}^2 p^{2-2s-4k_1-2l}) \times \quad L_p(s+2k_1+k_2+l-2),$$

where $L_p(s)$ is defined by (5.1) above.

We will prove the equality by investigating each p-factor of the Euler product separately. By the weak multiplicativity of the Fourier coefficients, we can write

$$D_{f,G}(s)=\prod_p D_{f,G}(s)_p$$

where the precise description of $D_{f,G}(s)_p$ depends on whether p is inert, split, or ramified. Each case will be handled separately.

Case 1. *p is inert.* Since $p=\mathfrak{p}$, $\alpha=1$, and $\mathrm{N}(\mathfrak{p})=p^2$, so

$$D_{f,G}(s)_p=\sum_n a(p^n)\overline{b(\mathfrak{p}^n)}V^n$$

where $V=p^{2-s-2k_1-k_2-l}$.

$$\begin{aligned}
D_{f,G}(s)_p &= \sum_n \left(V^n\cdot\frac{\alpha_p^{n+1}-{\alpha'_p}^{n+1}}{\alpha_p-\alpha'_p}\cdot\frac{\beta_{\mathfrak{p}}^{n+1}-{\beta'_{\mathfrak{p}}}^{n+1}}{\beta_{\mathfrak{p}}-\beta'_{\mathfrak{p}}}\right)\\
&=(\alpha_p-\alpha'_p)^{-1}(\beta_{\mathfrak{p}}-\beta'_{\mathfrak{p}})^{-1}\\
&\quad\cdot\left(\frac{\alpha_p\beta_{\mathfrak{p}}}{1-\alpha_p\beta_{\mathfrak{p}}V}-\frac{\alpha_p\beta'_{\mathfrak{p}}}{1-\alpha_p\beta'_{\mathfrak{p}}V}-\frac{\alpha'_p\beta_{\mathfrak{p}}}{1-\alpha'_p\beta_{\mathfrak{p}}V}+\frac{\alpha'_p\beta'_{\mathfrak{p}}}{1-\alpha'_p\beta'_{\mathfrak{p}}V}\right)\\
&=(\alpha_p-\alpha'_p)^{-1}\\
&\quad\cdot\left(\frac{\alpha_p(1-\alpha'_p\beta_{\mathfrak{p}}V)(1-\alpha'_p\beta'_{\mathfrak{p}}V)-\alpha'_p(1-\alpha_p\beta_{\mathfrak{p}}V)(1-\alpha_p\beta'_{\mathfrak{p}}V)}{(1-\alpha_p\beta_{\mathfrak{p}}V)(1-\alpha_p\beta'_{\mathfrak{p}}V)(1-\alpha'_p\beta_{\mathfrak{p}}V)(1-\alpha'_p\beta'_{\mathfrak{p}}V)}\right)\\
&=\frac{1-\alpha_p\alpha'_p\beta_{\mathfrak{p}}\beta'_{\mathfrak{p}}V^2}{(1-\alpha_p\beta_{\mathfrak{p}}V)(1-\alpha_p\beta'_{\mathfrak{p}}V)(1-\alpha'_p\beta_{\mathfrak{p}}V)(1-\alpha'_p\beta'_{\mathfrak{p}}V)}\\
&=\frac{1-p^{1-2s-2k_1}}{(1-\alpha_p\beta_{\mathfrak{p}}V)(1-\alpha_p\beta'_{\mathfrak{p}}V)(1-\alpha'_p\beta_{\mathfrak{p}}V)(1-\alpha'_p\beta'_{\mathfrak{p}}V)}.
\end{aligned}$$

Case 2. *p is split.* In this case, $p=\mathfrak{p}_1\mathfrak{p}_2$, $\mathrm{N}(\mathfrak{p}_1)=\mathrm{N}(\mathfrak{p}_2)=p$, and since (α) is primitive, either $\mathfrak{p}_1$ or $\mathfrak{p}_2$ can divide (α), not both. Then

$$D_{f,G}(s)_p=\sum_{n,\varepsilon,\nu} a(p^n)\overline{b(\mathfrak{p}_1^{n+2\varepsilon})}\,\overline{b(\mathfrak{p}_2^{n+2\nu})}N^{-n}X^{n+2\varepsilon+2\nu}$$

where $X = p^{1-s-2k_1}$, $N = p^{l+k_2-1}$, and $\inf(\varepsilon, \nu) = 0$. We can write

$$D_{f,G}(s)_p = \sum_{n,\varepsilon,\nu} \left(N^{-n} X^{n+2\varepsilon+2\nu} \times \frac{\alpha_p^{n+1} - {\alpha'_p}^{n+1}}{\alpha_p - \alpha'_p} \right.$$
$$\left. \cdot \frac{\beta_{\mathfrak{p}_1}^{n+2\varepsilon+1} - {\beta'_{\mathfrak{p}_1}}^{n+2\varepsilon+1}}{\beta_{\mathfrak{p}_1} - \beta'_{\mathfrak{p}_1}} \times \frac{\beta_{\mathfrak{p}_2}^{n+2\nu+1} - {\beta'_{\mathfrak{p}_2}}^{n+2\nu+1}}{\beta_{\mathfrak{p}_2} - \beta'_{\mathfrak{p}_2}} \right),$$

and this is the case Garrett considers ([**G1**]), obtaining the required result.

Case 3. *p is ramified.* Here we consider $p = \mathfrak{p}^2$, $\mathrm{N}(\mathfrak{p}) = p$, and the two sums below correspond to the cases where $\mathfrak{p} \nmid (\alpha)$ and $\mathfrak{p} \mid (\alpha)$. Then

$$D_{f,G}(s)_p = \sum_n a(p^n) \overline{b(\mathfrak{p}^{2n})} V^n + \sum_n a(p^n) \overline{b(\mathfrak{p}^{2n+2})} V^n X$$

where $V = p^{2-s-2k_1-k_2-l}$, and $X = p^{2-2s-3k_1-k_2}$.

$$D_{f,G}(s)_p = \sum_n \left(V^n \cdot \frac{\alpha_p^{n+1} - {\alpha'_p}^{n+1}}{\alpha_p - \alpha'_p} \cdot \frac{\beta_{\mathfrak{p}}^{2n+1} - {\beta'_{\mathfrak{p}}}^{2n+1}}{\beta_{\mathfrak{p}} - \beta'_{\mathfrak{p}}} \right)$$
$$+ \; X \sum_n \left(V^n \cdot \frac{\alpha_p^{n+1} - {\alpha'_p}^{n+1}}{\alpha_p - \alpha'_p} \cdot \frac{\beta_{\mathfrak{p}}^{2n+3} - {\beta'_{\mathfrak{p}}}^{2n+3}}{\beta_{\mathfrak{p}} - \beta'_{\mathfrak{p}}} \right)$$

$$\begin{aligned}
D_{f,G}(s)_p &= (\alpha_p - \alpha'_p)^{-1} (\beta_{\mathfrak{p}} - \beta'_{\mathfrak{p}})^{-1} \\
&\quad \cdot \left[\left(\frac{\alpha_p \beta_{\mathfrak{p}}}{1 - \alpha_p \beta_{\mathfrak{p}}^2 V} - \frac{\alpha_p \beta'_{\mathfrak{p}}}{1 - \alpha_p {\beta'_{\mathfrak{p}}}^2 V} - \frac{\alpha'_p \beta_{\mathfrak{p}}}{1 - \alpha'_p \beta_{\mathfrak{p}}^2 V} + \frac{\alpha'_p \beta'_{\mathfrak{p}}}{1 - \alpha'_p {\beta'_{\mathfrak{p}}}^2 V} \right) \right. \\
&\quad \left. + X \left(\frac{\alpha_p \beta_{\mathfrak{p}}^3}{1 - \alpha_p \beta_{\mathfrak{p}}^2 V} - \frac{\alpha_p {\beta'_{\mathfrak{p}}}^3}{1 - \alpha_p {\beta'_{\mathfrak{p}}}^2 V} - \frac{\alpha'_p \beta_{\mathfrak{p}}^3}{1 - \alpha'_p \beta_{\mathfrak{p}}^2 V} + \frac{\alpha'_p (\beta'_{\mathfrak{p}})^3}{1 - \alpha'_p {\beta'_{\mathfrak{p}}}^2 V} \right) \right] \\
&= (\beta_{\mathfrak{p}} - \beta'_{\mathfrak{p}})^{-1} \\
&\quad \cdot \left[\left(\frac{(\beta_{\mathfrak{p}} + X \beta_{\mathfrak{p}}^3)(1 - \alpha'_p \beta_{\mathfrak{p}}^2 V)(1 - \alpha'_p {\beta'_{\mathfrak{p}}}^2 V)}{(1 - \alpha_p \beta_{\mathfrak{p}}^2 V)(1 - \alpha_p {\beta'_{\mathfrak{p}}}^2 V)(1 - \alpha'_p \beta_{\mathfrak{p}}^2 V)(1 - \alpha'_p {\beta'_{\mathfrak{p}}}^2 V)} \right) \right. \\
&\quad \left. - \left(\frac{(\beta'_{\mathfrak{p}} + X {\beta'_{\mathfrak{p}}}^3)(1 - \alpha_p \beta_{\mathfrak{p}}^2 V)(1 - \alpha_p {\beta'_{\mathfrak{p}}}^2 V)}{(1 - \alpha_p \beta_{\mathfrak{p}}^2 V)(1 - \alpha_p {\beta'_{\mathfrak{p}}}^2 V)(1 - \alpha'_p \beta_{\mathfrak{p}}^2 V)(1 - \alpha'_p {\beta'_{\mathfrak{p}}}^2 V)} \right) \right] \\
&= \frac{1 + \alpha_p p^{1-s-k_1-l}(1 - p^{1-2s-2k_1})}{(1 - \alpha_p \beta_{\mathfrak{p}}^2 V)(1 - \alpha_p {\beta'_{\mathfrak{p}}}^2 V)(1 - \alpha'_p \beta_{\mathfrak{p}}^2 V)(1 - \alpha'_p (\beta'_{\mathfrak{p}})^2 V)} \\
&\quad + \frac{\alpha'_p p^{1-s-k_1-l}(1 - p^{1-2s-2k_1}) - p^{2-4s-4k_1}}{(1 - \alpha_p \beta_{\mathfrak{p}}^2 V)(1 - \alpha_p {\beta'_{\mathfrak{p}}}^2 V)(1 - \alpha'_p \beta_{\mathfrak{p}}^2 V)(1 - \alpha'_p (\beta'_{\mathfrak{p}})^2 V)}
\end{aligned}$$

$$= \frac{(1-p^{1-2s-2k_1})(1+\alpha_p p^{1-s-k_1-l})(1+\alpha'_p p^{1-s-k_1-l})}{(1-\alpha_p\beta_{\mathfrak{p}}^2 V)(1-\alpha_p{\beta'_{\mathfrak{p}}}^2 V)(1-\alpha'_p\beta_{\mathfrak{p}}^2 V)(1-\alpha'_p(\beta'_{\mathfrak{p}})^2 V)}.$$

Comparing the above expressions for $D_{f,G}(s)_p$ with the definition of $L_p(s+2k_1+k_2+l-2)$ in (5.1), where $v = p^{k_1+k_2-1}$ and $V = p^{2-s-2k_1-k_2-l}$, we obtain the desired equality. □

The following results, which determine the functional equation for the triple-product L-function, follow from the calculations of Sections 4 and 5, and the combinatorial techniques of Garrett and Orloff ([**G1**], [**BO**]).

Proposition 5.3. *The function $\eta_{k_1,k_2,l}(s)$ defined by Equation (4.11) is computed to be*

$$\begin{aligned}\eta_{k_1,k_2,l}(s) = (-1)^{k_1} 2^{6-4s-10k_1}\pi^{3-s-4k_1} {D_F}^{1/2} \\ \cdot\Gamma(s+2k_1-k_2-l)\Gamma(s+2k_1-1)\Gamma(s+2k_1+k_2-l-1) \\ \cdot\left(\frac{\Gamma(s+2k_1+l-k_2-1)\Gamma(s+2k_1+k_2+l-2)}{\Gamma(2s+4k_1-2)\Gamma(s+2k_1)\Gamma(s)}\right).\end{aligned}$$

Theorem 5.4. *For* $\mathrm{Re}(s)$ *sufficiently large,*

$$\begin{aligned}&\int_{\Gamma\backslash\mathfrak{H}^2}\int_{SL(2,\mathbb{Z})\backslash\mathfrak{H}} E_3(\iota_{2,1}(Z,z_3);2k_1,s)\overline{\delta_{2l}^{k_1-l}f(z_1)} \\ &\qquad\cdot\overline{\delta_{(2k_1,2k_2)}^{(0,k_1-k_2)}G(\tilde z)}(y_1y_2y_3)^{2k_1-2}dx_1\,dy_1\,d\tilde x\,d\tilde y \\ &= L(f\otimes G, s+2k_1+k_2+l-2)(-1)^{k_1}2^{6-4s-10k_1}\pi^{3-s-4k_1} \\ &\qquad\cdot {D_F}^{1/2}\zeta(2s+2k_1)^{-1}\zeta(2s+4k_1-2)^{-1} \\ &\qquad\cdot\Gamma(s+2k_1-k_2-l)\Gamma(s+2k_1-1)\Gamma(s+2k_1+k_2-l-1) \\ &\qquad\cdot\Gamma(s+2k_1+l-k_2-1)\Gamma(s+2k_1+k_2+l-2) \\ &\qquad\cdot\Gamma(s)^{-1}\Gamma(s+2k_1)^{-1}\Gamma(2s+4k_1-2)^{-1}\end{aligned}$$

where $z_j = x_j + y_j$ as usual. Hence, $L(f\otimes G, s)$ has a meromorphic continuation to all of $\mathbb{C}$; the above identity holds away from the poles of the Eisenstein series. Under the transformation $s \to 2k_1+2l+2k_2-2-s$,

$$(2\pi)^{-4s}\Gamma(s)\Gamma(s-2k_1+1)\Gamma(s-2l+1)\Gamma(s-2k_2+1)L(f\otimes G,s)$$

is multiplied by (-1).

The special values result proceeds as follows. Let $\mathbb{Q}(f,G)$ denote the field generated over $\mathbb{Q}$ by the Fourier coefficients of f and G.

Theorem 5.5. *With f and G as above, $k_2+l > k_1$, and with the usual Petersson inner products, for $2k_1 \le n \le 2k_2+2l-2$, let*

$$A(n;f,G) = \pi^{2k_1+2l+2k_2-3-n}{D_F}^{1/2}\langle f,f\rangle^{-1}\langle G,G\rangle^{-1}L(f\otimes G,n).$$

Then $A(n; f, G) \in \mathbb{Q}(f, G)$. *Moreover, if* $\sigma \in \mathrm{Aut}(\mathbb{C})$, *then*

$$A(n; f, G)^\sigma = A(n; f^\sigma, G^\sigma),$$

where the action of the Galois group on modular forms is the action on Fourier coefficients.

Note that we must restrict the weights in Theorem 5.5, for if $k_2 + l \leq k_1$, gamma factors in Theorem 5.4 vanish at the critical points. The proof is similar to those of Garrett ([**G1**]) and Orloff ([**BO**]) and will not be reproduced here.

6. Renormalization.

In this section, where we obtain a product of Asai L-functions from the triple product L-function $L(f \otimes G, s)$, we will consider the case where the normalized holomorphic cuspidal eigenform f of weight $2l$ is replaced by the holomorphic Eisenstein series of weight $2l \geq 4$ given by

$$\begin{aligned}
\mathcal{E}(z) &= \frac{(2l-1)!}{2(2\pi i)^{2l}} \sum_{\substack{m,n=-\infty \\ (m,n)\neq(0,0)}}^{\infty} (mz+n)^{-2l} \qquad (6.1)\\
&= -\frac{B_{2l}}{2l} + \sum_{n=1}^{\infty} \sigma_{2l-1}(n)\mathbf{e}(nz), \\
&= \sum_{n=0}^{\infty} a(n)\mathbf{e}(nz)
\end{aligned}$$

where B_m is the m-th Bernoulli number, and

$$\sigma_{2l-1}(n) = \sum_{0<d|n} d^{2l-1}.$$

Then we may write

$$\begin{aligned}
\delta_{2l}^{k_1-l}\mathcal{E}(z) &= \sum_{n=0}^{\infty} a(n) \sum_{A=0}^{k_1-l} P_A(4\pi y)^{-A} n^{k_1-l-A}\mathbf{e}(nz) \\
&= -\frac{B_{2l}}{2l} P_{k_1-l}(4\pi y)^{-k_1+l} \\
&\quad + \sum_{A=0}^{k_1-l} P_A(4\pi y)^{-A} n^{k_1-l-A} \sum_{n=1}^{\infty} \sigma_{2l-1}(n) e^{2\pi i n z}.
\end{aligned}$$

6.1. The Petersson inner product. Let f and g be two level one holomorphic elliptic modular forms of weight k. Recall that if at least one of them is a cusp form, we can define the Petersson inner product by

$$\langle f, g\rangle = \int_{\Gamma\backslash\mathfrak{H}} f(z)\overline{g(z)} y^{k-2} dx\, dy.$$

Rankin showed that the inner product is in fact equal to

$$\langle f, g\rangle = \frac{\pi}{3}(k-1)!(4\pi)^{-k}\mathrm{Res}_{s=k}\left(\sum_{n=1}^{\infty} a_n\overline{b_n}n^{-s}\right), \tag{6.2}$$

where a_n and b_n are the Fourier coefficients of f and g respectively ([**R**]). If neither f nor g is a cusp form, then the integral diverges. However we can *renormalize* the integrand $f(z)\overline{g(z)}y^k$, following Zagier's ideas ([**Z**]). Note that the integrand is of slow growth, so by subtracting an appropriate polynomial piece, Zagier defines the corresponding Rankin-Selberg transform and proves analogous results to the classical case where the integrand is of rapid decay. In particular, he relates this renormalized integral to a residue of the Rankin-Selberg transform. Performing this computation in the situation above, Zagier shows that after renormalization, (6.2) still holds. Hence we may define $\langle f, g\rangle$ as its renormalized integral, thus extending the Petersson inner product to the space of all modular forms. The technique will be made explicit in the proof of the following lemma, where *R.N.* is used to denote a renormalized integral.

Lemma 6.1. *Let $E_1(z; 2k_1, s)$ and $\mathcal{E}(z)$ be the Eisenstein series defined by Equations* (2.1) *and* (6.1), *respectively. Then, for* $\mathrm{Re}\,(s)$ *sufficiently large,*

$$\begin{aligned}&\langle E_1(z; 2k_1, s), \delta_{2l}^{k_1-l}\mathcal{E}(z)\rangle\\ &= R.N. \int_{\mathrm{SL}(2,\mathbb{Z})\backslash\mathfrak{H}} E_1(z; 2k_1, s)\overline{\delta_{2l}^{k_1-l}\mathcal{E}(z)}y^{2k_1-2}dx\,dy\\ &= 0.\end{aligned}$$

Proof. The integrand is of slow growth, that is, we can write

$$H(z) = E_1(z; 2k_1, s)\overline{\delta_{2l}^{k_1-l}\mathcal{E}(z)}y^{2k_1} = \varphi(y) + O(y^{-N}) \qquad (\forall N) \text{ as } y \to \infty$$

where φ is a function of the form

$$\varphi(y) = \sum_{i=1}^{m} \frac{c_i}{n_i!}y^{\alpha_i}\log^{n_i} y \qquad (c_i, \alpha_i \in \mathbb{C}, 0 \le n_i \in \mathbb{Z}).$$

Using the Fourier expansion of H

$$H(z) = \sum_{n=0}^{\infty} c_n(y)e^{2\pi inx},$$

define the Rankin-Selberg transform of H by

$$R(H; t) = \int_0^{\infty} (c_0(y) - \varphi(y))y^{t-2}dy \qquad (\mathrm{Re}(t) \gg 0).$$

Then as in [**Z**], we can compute

$$\langle E_1(z; 2k_1, s), \delta_{2l}^{k_1-l}\mathcal{E}(z)\rangle = 2\mathrm{Res}_{t=1}R^*(H; t),$$

where $R^*(H;t) = \zeta^*(2t)R(H;t) := \pi^{-t}\Gamma(t)\zeta(2t)R(H;t)$.

The explicit calculation proceeds as follows. Write the Fourier expansion of $E_1(z;2k_1,s)$ as

$$\begin{aligned}
&E_1(z;2k_1,s)\\
&= y^s + \frac{\sqrt{\pi}\Gamma\left(\frac{2s+2k_1-1}{2}\right)\Gamma\left(\frac{2s+2k_1}{2}\right)\zeta(2s+2k_1-1)}{y^{2k_1+s-1}(-1)^{k_1}\Gamma(s)\Gamma(s+2k_1)\zeta(2s+2k_1)}\\
&\quad + \frac{2^{2k_1}\pi^{s+2k_1}\sum_{n=1}^{\infty}(\sigma_{2k_1+2s-1}(n)n^{-s}e^{2\pi inz}\omega(4\pi ny;2k_1+s,s))}{(-1)^{k_1}\Gamma(s+2k_1)\zeta(2s+2k_1)}\\
&\quad + \frac{\pi^s\sum_{n=1}^{\infty}(\sigma_{2k_1+2s-1}(n)n^{-s-2k_1}e^{2\pi in\overline{z}}\omega(4\pi ny;s,2k_1+s))}{y^{2k_1}2^{2k_1}(-1)^{k_1}\Gamma(s)\zeta(2s+2k_1)},
\end{aligned}$$

where

$$\omega(z;\alpha,\beta) = \Gamma(\beta)^{-1}z^{\beta}\int_0^{\infty}e^{-zu}(u+1)^{\alpha-1}u^{\beta-1}du. \tag{6.3}$$

Multiplying the expansions for $E_1(z;2k_1,s)$ and $\overline{\delta_{2l}^{k_1-l}\mathcal{E}(z)}y^{2k_1}$, we find that

$$\begin{aligned}
a_0(y) = &-\frac{B_{2l}}{2l}P_{k_1-l}(4\pi y)^{-k_1+l}y^{s+2k_1} - \frac{B_{2l}}{2l}P_{k_1-l}(4\pi y)^{-k_1+l}\\
&\cdot\left(\frac{\sqrt{\pi}\Gamma\left(\frac{2s+2k_1-1}{2}\right)\Gamma\left(\frac{2s+2k_1}{2}\right)\zeta(2s+2k_1-1)}{(-1)^{k_1}\Gamma(s)\Gamma(s+2k_1)\zeta(2s+2k_1)}y^{1-s}\right)\\
&+ \frac{2^{2k_1}\pi^{s+2k_1}y^{2k_1}}{(-1)^{k_1}\Gamma(s+2k_1)\zeta(2s+2k_1)}\sum_{A=0}^{k_1-l}P_A(4\pi y)^{-A}\\
&\cdot\sum_{n=1}^{\infty}(\sigma_{2l-1}(n)\sigma_{2k_1+2s-1}(n)n^{k_1-l-A-s}e^{-4\pi ny}\omega(4\pi ny;2k_1+s,s))
\end{aligned}$$

and

$$\begin{aligned}
\varphi(y) = &-\frac{B_{2l}}{2l}P_{k_1-l}(4\pi y)^{-k_1+l}y^{s+2k_1} - \frac{B_{2l}}{2l}P_{k_1-l}(4\pi y)^{-k_1+l}\\
&\cdot\left(\frac{\sqrt{\pi}\Gamma\left(\frac{2s+2k_1-1}{2}\right)\Gamma\left(\frac{2s+2k_1}{2}\right)\zeta(2s+2k_1-1)}{(-1)^{k_1}\Gamma(s)\Gamma(s+2k_1)\zeta(2s+2k_1)}y^{1-s}\right).
\end{aligned}$$

Therefore

$$R(H;t) = \frac{2^{2k_1}\pi^{s+2k_1}}{(-1)^{k_1}\Gamma(s+2k_1)\zeta(2s+2k_1)} \sum_{A=0}^{k_1-l} P_A(4\pi)^{-A} n^{k_1-l-A}$$

$$\cdot \sum_{n=1}^{\infty} \Bigg(\sigma_{2l-1}(n)\sigma_{2k_1+2s-1}(n)n^{-s}$$

$$\cdot \int_0^\infty e^{-4\pi ny}\omega(4\pi ny; 2k_1+s, s)y^{2k_1+t-A-2}dy \Bigg)$$

$$= \frac{2^{2k_1}\pi^{s+2k_1}}{(-1)^{k_1}\Gamma(s+2k_1)\zeta(2s+2k_1)} \left(\frac{\Gamma(2k_1+t+s-1)\Gamma(t-s)}{\Gamma(t)} \right)$$

$$\cdot \times {}_3F_2(1-k_1-l, -k_1+l, 1-t; 2-2k_1-s-t, 1+s-t; 1)$$

$$\cdot \sum_{n=1}^{\infty} \sigma_{2l-1}(n)\sigma_{2k_1+2s-1}(n)n^{-k_1-l-t-s+1}$$

and we obtain

$$\begin{aligned} R(H;t) = {} & \frac{2^{-2k_1-2t+2}\pi^{s-t+1}\Gamma(2k_1+t+s-1)\Gamma(t-s)}{(-1)^{k_1}\Gamma(s+2k_1)\Gamma(t)\zeta(2s+2k_1)} \\ & \times {}_3F_2(1-k_1-l, -k_1+l, 1-t; 2-2k_1-s-t, 1+s-t; 1) \\ & \times \zeta(k_1-l+t+s)\zeta(l-k_1+t-s)\zeta(k_1+l+t+s-1) \\ & \times \zeta(-k_1-l+t-s+1)\zeta(2t)^{-1}, \end{aligned}$$

where ${}_3F_2(a,b,c;d,e;1)$ denotes the generalized hypergeometric function with unit argument. Then

$$\begin{aligned} R^*(H;t) &= \pi^{-t}\Gamma(t)\zeta(2t)R(H;t) \\ &= \frac{(-1)^{k_1}2^{-2k_1-2t+2}\pi^{s-2t+1}\Gamma(2k_1+t+s-1)\Gamma(t-s)}{\Gamma(s+2k_1)\zeta(2s+2k_1)} \\ &\quad \cdot {}_3F_2(1-k_1-l, -k_1+l, 1-t; 2-2k_1-s-t, 1+s-t; 1) \\ &\quad \cdot \zeta(k_1-l+t+s)\zeta(l-k_1+t-s)\zeta(k_1+l+t+s-1) \\ &\quad \cdot \zeta(-k_1-l+t-s+1) \end{aligned}$$

$$= \frac{(-1)^{k_1} 2^{-2k_1-2t+2} \pi^{k_1-l+s+1} \Gamma(2k_1+t+s-1)\Gamma(t-s)}{\Gamma(s+2k_1)\zeta(2s+2k_1)}$$
$$\cdot\, {}_3F_2(1-k_1-l,-k_1+l,1-t;2-2k_1-s-t,1+s-t;1)$$
$$\cdot\, \Gamma\left(\frac{k_1-l+t+s}{2}\right)^{-1} \Gamma\left(\frac{l-k_1+t-s}{2}\right)^{-1}$$
$$\cdot\, \Gamma\left(\frac{k_1+l+t+s-1}{2}\right)^{-1} \Gamma\left(\frac{-k_1-l+t-s+1}{2}\right)^{-1}$$
$$\cdot\, \zeta^*(k_1-l+t+s)\zeta^*(l-k_1+t-s)$$
$$\cdot\, \zeta^*(k_1+l+t+s-1)\zeta^*(-k_1-l+t-s+1).$$

Using properties of the Γ-function, we can compute that for $k_1 \equiv l \pmod 2$,

$$R^*(H;t) = \frac{2^{1-2k_1+2l}\pi^{k_1-l+s}}{\Gamma(s+2k_1)\zeta(2s+2k_1)} \prod_{j=\frac{k_1-l}{2}}^{\frac{k_1+l}{2}-1} \left(\frac{t+s}{2}+j\right)\left(\frac{1-t+s}{2}+j\right)$$
$$\cdot\, \frac{\Gamma(t-s)\Gamma(t+s+2k_1-1)}{\Gamma(t-s+l-k_1)\Gamma(t+s+k_1+l-1)}$$
$$\cdot\, {}_3F_2(1-k_1-l,-k_1+l,1-t;2-2k_1-s-t,1+s-t;1)$$
$$\cdot\, \zeta^*(k_1-l+t+s)\zeta^*(l-k_1+t-s)$$
$$\cdot\, \zeta^*(k_1+l+t+s-1)\zeta^*(-k_1-l+t-s+1).$$

The computation for the case where $k_1 \not\equiv l \pmod 2$ is similar. Using relationships between hypergeometric series ([**B**], p. 18), the above description of $R^*(H;t)$ makes clear the functional equation $R^*(H;t) = R^*(H;1-t)$. Computing the residue at $t=1$, we see that for $\mathrm{Re}(s)$ significantly large,

$$\langle E_1(z;2k,s), \delta_{2l}^{k_1-l}\mathcal{E}(z)\rangle = 2\mathrm{Res}_{t=1} R^*(H;t) = 0,$$

as required.

One may also note that certain values of s provide instances where the inner product is finite and nonzero, due to cancellation of factors introducing poles. For example, at $s=-k_1+l$, we obtain

$$\langle E_1(z;2k_1,-k_1+l), \delta_{2l}^{k_1-l}\mathcal{E}(z)\rangle = 2^{5-2k_1-2l}\pi^{1-2l}(2l-2)!\zeta(2l-1).$$

$\square$

6.2. The integral representation. We can now extend the previous results to determine the renormalized integral representation. For the following, recall the definitions of the Dirichlet series $D_{\mathcal{E}}^{(2)}(s)$ and $D_{\mathcal{E},G}(s)$ given by (4.7) and (4.8), respectively. Then Lemma 4.1 has the following analogue:

Lemma 6.2. *Let $\hat{q}$ be given by* (4.9), *and let $a(n)$ be the Fourier coefficients of $\mathcal{E}$ defined in* (6.1). *For $w \in \mathfrak{H}$, $\mathrm{Re}(s)$ sufficiently large, define*

$$\mathcal{E}_s^*(w) = \sum_n a(n) \sum_{A=0}^{k_1-l} P_A (4\pi)^{-A} n^{k_1-l-A} \mathbf{e}(n(\mathrm{Re}(w))) \int_0^\infty y^{s+2k_1-2-A} \cdot (y+\mathrm{Im}(w))^{1-2s-2k} \exp(-2\pi n y) \hat{q}_{2k,s}(ny + n\mathrm{Im}\,(w))\, dy, \tag{6.4}$$

where the coefficients are defined by (4.4) *and* (4.5). *Then, for $Z \in \mathfrak{H}_2$, $z \in \mathfrak{H}$,*

$$\begin{aligned} R.N.&\left(\int_{\mathrm{SL}(2,\mathbb{Z})\backslash\mathfrak{H}} E_3(\iota_{2,1}(Z,z);2k_1,s)\overline{\delta_{2l}^{k_1-l}\mathcal{E}(z)}y^{2k_1-2}dx\,dy\right)\\ &= E_2(Z;2k_1,s)\langle E_1(z;2k_1,s), \delta_{2l}^{k_1-l}\mathcal{E}(z)\rangle\\ &\quad + \zeta(2s+2k_1)^{-1}D_{\mathcal{E}}^{(2)}(s+k_1+l-1)\\ &\quad\cdot \sum_{\tilde{\gamma}}[\det \mathrm{Im}(\tilde{\gamma}Z)]^s \mu(\tilde{\gamma},Z)^{-2k_1}\mathcal{E}_s^*(\pi(\tilde{\gamma}Z)), \end{aligned} \tag{6.5}$$

where the sum is over $\tilde{\gamma} \in P_{2,1}(\mathbb{Z})\backslash\mathrm{Sp}(2,\mathbb{Z})$.

To see why this is true, let $\chi_{2k_1,s}$ and μ be defined as in (4.10) and (2.2) respectively, and note that for $Z \in \mathfrak{H}_n$ and $g \in \mathrm{Sp}(n,\mathbb{Z})$,

$$\det(\mathrm{Im}(gZ))^s \mu(g,Z)^{-2k_1} = \det(\mathrm{Im}(Z))^s \chi_{2k_1,s}(\mu(g,Z)).$$

Then using the coset decomposition of Proposition 3.2, we have

$$\begin{aligned} &E_3(\iota_{2,1}(Z,z);2k_1,s)\\ &= (\det \mathrm{Im}(Z))^s \mathrm{Im}(z)^s \sum_{\tilde{\gamma},\gamma} \chi_{2k_1,s}(\mu(\tilde{\gamma},Z)\mu(\gamma,z))\\ &\quad + (\det \mathrm{Im}(Z))^s \mathrm{Im}(z)^s \sum_{\tilde{\gamma}',\gamma'} \chi_{2k_1,s}(\mu(p\xi\tilde{\iota}_{2,1}(\tilde{\gamma}',\gamma'), \iota_{2,1}(Z,z))), \end{aligned} \tag{6.6}$$

where in the first sum $\tilde{\gamma} \in P_{2,0}(\mathbb{Z})\backslash\mathrm{Sp}(2,\mathbb{Z})$, $\gamma \in P_{1,0}(\mathbb{Z})\backslash\mathrm{Sp}(1,\mathbb{Z})$, and in the second sum $\tilde{\gamma}' \in P_{2,1}\backslash\mathrm{Sp}(2,\mathbb{Z})$, $\gamma' \in \mathrm{Sp}(1,\mathbb{Q})$, ξ is defined as in Proposition 3.1, and for each γ', choose $p \in P_{3,0}(\mathbb{Q})$ such that

$$p\xi\tilde{\iota}_{2,1}(\tilde{\gamma}',\gamma') \in \mathrm{Sp}(3,\mathbb{Z}).$$

The first sum is clearly equal to $E_2(Z;2k_1,s)E_1(z;2k_1,s)$, as desired, and the rest of the lemma follows as before.

Now consider the integral over $\Gamma\backslash\mathfrak{H}^2$. If we combine the results of Lemmas 6.1 and 6.2, and note that

$$\int_{\Gamma\backslash\mathfrak{H}^2} E_2(\iota(\tilde{z});2k_1,s)\overline{\delta_{(2k_1,2k_2)}^{(0,k_1-k_2)}G(\tilde{z})}\tilde{y}^{2k_1-2}d\tilde{y} = 0,$$

then we see the first term on the right-hand side of Equation (6.5) will contribute nothing. Likewise, if we apply the Rankin method as in the proof of Proposition 5.4, the term of $\mathcal{E}_s^*$ involving $a(0)$ will disappear. Thus the argument of Proposition 5.4 applies in the case where the cusp form $f(z)$ is replaced by the Eisenstein series $\mathcal{E}(z)$, and yields the corresponding result:

Proposition 6.3. *With notation as above and the Dirichlet series corresponding to* (4.7) *and* (4.8), *then for* $\tilde{z} = (z_1, z_2) \in \mathfrak{H}^2$ *and* $\iota(\tilde{z}) = Z$*,*

$$\begin{aligned} R.N. \Bigg(\int_{\Gamma\backslash\mathfrak{H}^2} \int_{\mathrm{SL}(2,\mathbb{Z})\backslash\mathfrak{H}} & E_3(\iota_{2,1}(Z, z_1); 2k_1, s) \overline{\delta_{2l}^{k_1-l} \mathcal{E}(z_1)} \\ & \cdot \overline{\delta_{(2k_1,2k_2)}^{(0,k_1-k_2)} G(\tilde{z})} (y_1 y_2 y_3)^{2k_1-2} dx_1\, dy_1\, d\tilde{x}\, d\tilde{y} \Bigg) \\ = \eta_{k_1,k_2,l}(s) \zeta(2s + 2k_1)^{-1} & D_{\mathcal{E}}^{(2)}(s + k_1 + l - 1) D_{\mathcal{E},G}(s) \end{aligned}$$

for the real part of s sufficiently large.

6.3. The Euler product. Regarding the Euler product computation, the proof of Theorem 5.2 is still valid. We can compute the roots of the Euler p-factor of the L-function attached to $\mathcal{E}(z)$ explicitly. Namely,

$$L(\mathcal{E}, s) = \prod_p [(1 - p^{-s})(1 - p^{-s+2l-1})]^{-1},$$

so $\alpha_p = 1$ and $\alpha'_p = p^{2l-1}$. Substituting these values into Equation (5.1), for $V = p^{-s}$, $V' = p^{-s+2l-1}$, and $v = p^{2k_1-1}$, we obtain

$$L_p(s)^{-1} = \begin{cases} (1 - \beta_{\mathfrak{p}} V)(1 - \beta'_{\mathfrak{p}} V)(1 - \beta_{\mathfrak{p}} V') & \\ \qquad (1 - \beta'_{\mathfrak{p}} V')(1 - v^2 V^2)(1 - v^2 V'^2) & \text{if } p = \mathfrak{p} \\ & \\ (1 - \beta_{\mathfrak{p}_1}\beta_{\mathfrak{p}_2} V)(1 - \beta'_{\mathfrak{p}_1}\beta_{\mathfrak{p}_2} V)(1 - \beta_{\mathfrak{p}_1}\beta'_{\mathfrak{p}_2} V) & \\ \quad (1 - \beta'_{\mathfrak{p}_1}\beta'_{\mathfrak{p}_2} V)(1 - \beta_{\mathfrak{p}_1}\beta_{\mathfrak{p}_2} V')(1 - \beta'_{\mathfrak{p}_1}\beta_{\mathfrak{p}_2} V') & \text{if } p = \mathfrak{p}_1\mathfrak{p}_2 \\ \qquad (1 - \beta_{\mathfrak{p}_1}\beta'_{\mathfrak{p}_2} V')(1 - \beta'_{\mathfrak{p}_1}\beta'_{\mathfrak{p}_2} V') & \\ & \\ (1 - \beta_{\mathfrak{p}}^2 V)(1 - {\beta'_{\mathfrak{p}}}^2 V)(1 - \beta_{\mathfrak{p}}^2 V') & \\ \qquad (1 - {\beta'_{\mathfrak{p}}}^2 V')(1 - vV)(1 - vV') & \text{if } p = \mathfrak{p}^2. \end{cases}$$

As in [**A**], the Euler product of $L_{\mathrm{Asai}}(G,s)$ has the following form. For $V = p^{-s}$ and $v = p^{k_1-1}$, $L_{\mathrm{Asai}}(G,s) = \prod_p L_p(s)$, where

$$L_p(s)^{-1} = \begin{cases} (1-\beta_{\mathfrak{p}}V)(1-\beta'_{\mathfrak{p}}V)(1-v^2V^2) & \text{if } p = \mathfrak{p} \\ \\ (1-\beta_{\mathfrak{p}_1}\beta_{\mathfrak{p}_2}V)(1-\beta'_{\mathfrak{p}_1}\beta_{\mathfrak{p}_2}V) & \text{if } p = \mathfrak{p}_1\mathfrak{p}_2 \\ \qquad \times(1-\beta_{\mathfrak{p}_1}\beta'_{\mathfrak{p}_2}V)(1-\beta'_{\mathfrak{p}_1}\beta'_{\mathfrak{p}_2}V) & \\ \\ (1-\beta_{\mathfrak{p}}^2V)(1-{\beta'_{\mathfrak{p}}}^2V)(1-vV) & \text{if } p = \mathfrak{p}^2. \end{cases} \tag{6.7}$$

Thus, we can see that

$$L(\mathcal{E}\otimes G,s) = L_{\mathrm{Asai}}(G,s)L_{\mathrm{Asai}}(G,s-2l+1).$$

Hence we have the analogue of Theorem 5.4:

Theorem 6.4. *For* $\mathrm{Re}\,(s)$ *sufficiently large,*

$$\begin{aligned} R.N.&\Bigg(\int_{\Gamma\backslash\mathfrak{H}^2}\int_{\mathrm{SL}(2,\mathbf{Z})\backslash\mathfrak{H}} E_3(\iota_{2,1}(Z,z_3);2k_1,s)\overline{\delta_{2l}^{k_1-l}\mathcal{E}(z_1)} \\ &\qquad\cdot\overline{\delta_{(2k_1,2k_2)}^{(0,k_1-k_2)}G(\tilde{z})}\,(y_1y_2y_3)^{2k_1-2}dx_1\,dy_1\,d\tilde{x}\,d\tilde{y}\Bigg) \\ &= L_{\mathrm{Asai}}(G,s+2k_1+k_2+l-2)L_{\mathrm{Asai}}(G,s+2k_1+k_2-l-1) \\ &\quad\cdot(-1)^{k_1}2^{6-4s-10k_1}\pi^{3-s-4k_1} \\ &\quad\cdot {D_F}^{1/2}\zeta(2s+2k_1)^{-1}\zeta(2s+4k_1-2)^{-1} \\ &\quad\cdot\Gamma(s+2k_1-k_2-l)\Gamma(s+2k_1-1)\Gamma(s+2k_1+k_2-l-1) \\ &\quad\cdot\Gamma(s+2k_1+l-k_2-1)\Gamma(s+2k_1+k_2+l-2) \\ &\quad\cdot\Gamma(s)^{-1}\Gamma(s+2k_1)^{-1}\Gamma(2s+4k_1-2)^{-1}. \end{aligned}$$

The above identity holds away from the poles of the Eisenstein series.

7. Concluding remarks.

The next logical step is to generalize the above results to the setting of automorphic representations, in order to be able to derive the desired special values results. We can investigate Deligne's conjecture for the critical values of the product of Asai L-functions, obtained above. Specializing the result in the Appendix of [**BO**] to our case, we can determine the critical strip for the triple product L-function. Given any three positive integers $k \geq l \geq m$, corresponding to the weights of the forms, there are always two cases to consider, depending on whether

(1) $l+m > k$, or
(2) $l+m \leq k$.

The special values results of Garrett, Harris, and Orloff that deal with the triple product L-function attached to cusp forms ([**G2**], [**G1**], [**GH**], [**BO**]), all fall under Case (1) and conform with Deligne's conjecture ([**D**]). However, once we replace the cusp form f with the Eisenstein series $\mathcal{E}$, we are always in the situation of Case (2). More precisely, suppose our Hilbert modular form G is of weight (k_1, k_2), $k_1 > k_2$, and $k_1 \equiv k_2$ (modulo 2). The Eisenstein series $\mathcal{E}$ will be of weight $l < k_1$. Set

$$w = k_1 + k_2 + l - 3$$

and

$$c_0 = k_2 + l - 1.$$

Then the critical strip is given by

$$CS_0 = [c_0, \dots, w - c_0 + 1] = [k_2 + l - 1, \dots, k_1 - 1].$$

The critical strips corresponding to the Asai L-functions appearing in Theorem 6.4 will be

$$CS_1 = [k_2, \dots, k_1 - 1] \qquad \text{and} \qquad CS_2 = [k_2 + l - 1, \dots, k_1 + l - 2].$$

Therefore

$$CS_0 = CS_1 \cap CS_2,$$

as one would expect.

Let ω signify the central character for G, and let $\langle G, G\rangle_B$ denote the Petersson inner product normalized by an appropriate factor, as in the Appendix to [**BO**]. Then by Deligne's conjecture, for two primitive Dirichlet characters ξ and χ and the Gauss sum $\mathfrak{g}$, we would expect that for $n \in CS_0 = [k_2 + l - 1, \dots, k_1 - 1]$,

$$\frac{L_{\text{Asai}}(G, n, \xi) L_{\text{Asai}}(G, n - l + 1, \chi)}{(2\pi i)^{2((n+1)-w)} \mathfrak{g}((\omega\xi\chi)^2) D_F^{1/2} \langle G, G\rangle_B^2} \in \mathbb{Q}(G, \mathcal{E}).$$

In the situation of Case (2), Harris and Kudla ([**HK**]) have provided the only general special value result, for the center of the critical strip. Extending their results to the other integers in the critical strip and applying Shimura's methods ([**S2**], [**S3**]) should then lead to algebraicity results for ratios of the Asai L-function at different integers, twisted by Hecke characters.

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