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We construct a correspondence between the complex gauge equivalence classes of holomorphic $U(1)$-connections on a smooth semi-flat special Lagrangian torus fibration and the Hamiltonian deformation classes of Lagrangian sections Σ in the mirror manifold together with the gauge equivalence classes of flat $U(1)$-connections on Σ.

1. Introduction.

It was conjectured in [**SYZ**] that Calabi-Yau spaces can be often fibered by special Lagrangian tori and their mirrors can be constructed by dualizing these tori. It was further suggested by Vafa in [**V**] that the holomorphic vector bundles on a Calabi-Yau n-fold M correspond to the Lagrangian submanifolds in the mirror $\check{M}$ and the stable vector bundles correspond to the special Lagrangian submanifolds in $\check{M}$ together with flat $U(1)$-connections.

In this note, we will describe a correspondence between holomorphic $U(1)$-connections and Lagrangian cycles. We assume that M is a space admitting a *special Lagrangian torus fibration.* This is a topological fibration $\pi : M \to B$, where B is a compact n-dimensional manifold without boundary which is locally a Lagrangian section of π, whose fibers are special Lagrangian n-tori with respect to the Kähler form ω and a holomorphic n-form Ω on M (cf. Definition 2.2). We assume that the fibration does not possesses singular fibers and all fibers are flat with respect to the induced metric from M. Note that this is the case studied by Hitchin in [**H**], and the mirror manifold $\check{M}$ has been constructed and it can be identified with the cotangent bundle T^*B of quotient by a nondegenerate family of lattices. In particular, $\check{M}$ is a smooth special Lagrangian torus fibration over B as well. The symplectic form is the one induced by the canonical symplectic form on T^*B (cf. [**H**], [**G2**]). If degeneration of fibers possesses, the mathematically rigorous construction of the mirror manifolds remains one of the major challenges in the SYZ program (cf. [**G1**], [**G2**], [**G3**], [**R**]).

On the M side, we shall focus on the holomorphic connections on a $U(1)$-bundle E over M. On the mirror side, we consider the pair (Σ, α) where Σ is a Lagrangian section from B in $\check{M}$, and α is a flat $U(1)$-connection on a complex line bundle L over Σ. One can deform Σ in its Hamiltonian class which is denoted by $[\Sigma]$, i.e., through Lagrangian cycles which can be translated by the Hamiltonian diffeomorphisms from one to the other (cf. Definition 6.1), and deform the flat connection α on Σ in its gauge equivalence class $[\alpha]$.

Definition 1.1. Let Σ be a Lagrangian section from B in $\check{M}$ and α a flat $U(1)$-connection on Σ. The pair $([\Sigma], [\alpha])$ which consists of the Hamiltonian deformation class $[\Sigma]$ of Σ and the gauge equivalence class $[\alpha]$ of α is called a *Hamiltonian Lagrangian supersymmetric cycle* in $\check{M}$.

The main result of this note is:

Theorem 1.1. *Let M be a semi-flat special Lagrangian T^n-fibration over B with a Lagrangian section and let E be $U(1)$ vector bundle over M. Let $\check{M}$ be the mirror manifold of M. Suppose that $\mathcal{A} = \{A : A$ is a holomorphic connections on $E\}$ and $\pi_{\mathcal{G}_{\mathbf{C}}} : \mathcal{A} \to \mathcal{A}/\mathcal{G}_{\mathbf{C}}$ is the projection to the complex gauge equivalent classes. Let $\mathcal{S} = \{(\Sigma, \alpha) : \alpha$ is a flat $U(1)$-connection over a Lagrangian section Σ of $\check{M}\}$ and $\pi_{\mathcal{S}}$ be the projection of $\mathcal{S}$ to the set $\{([\Sigma], [\alpha])\}$ of Hamiltonian Lagrangian supersymmetric cycles in $\check{M}$. Then there is a map $\phi : \mathcal{A} \to \mathcal{S}$ which induces a map $\phi' : \mathcal{A}/\mathcal{G}_{\mathbf{C}} \to \mathcal{S}/\pi_{\mathcal{S}}$ such that $\pi_{\mathcal{S}} \circ \phi = \phi' \circ \pi_{\mathcal{G}_{\mathbf{C}}}$; and conversely there is an injective map $\psi : \mathcal{S} \to \mathcal{A}$ which induces a map $\psi' : \mathcal{S}/\pi_{\mathcal{S}} \to \mathcal{A}/\mathcal{G}_{\mathbf{C}}$ such that $\pi_{\mathcal{G}_{\mathbf{C}}} \circ \psi = \psi' \circ \pi_{\mathcal{S}}$.*

When the complex dimension of M is two, the special T^2-fibration M becomes an elliptic K3 surface by rotating the complex structure by $\frac{\pi}{2}$ and the Lagrangian fibers become holomorphic curves of genus one. In this context, Friedman-Morgan-Witten [**FMW**] studied extensively flat vector bundles through spectral curves. When M is an elliptic curve, Polishchuk-Zaslow in [**PZ**] described an isomorphism between the categories suggested by Kontsevich and a suitable version of Fukaya's category of Lagrangian submanifolds on $\check{M}$. There are also related works by Tyurin in [**Ty**] on the construction for Hermitian-Einstein bundles on Calabi-Yau n-folds with $n = 1, 2, 3$.

The results of this note grow out of extensive discussion with Gang Tian and they constitute partial progress of a general program of Tian and the author. These results and some of their extension to higher rank bundles have been reported by the author in several seminars and conferences. Finally, Tian informed the author that R. Thomas also obtained similar results.

The author is grateful to the referee for valuable comments.

2. Special Lagrangian torus fibration.

First, we describe the space we are interested in. Let $\pi : M \to B$ be a smooth proper map from a real $2n$-dimensional smooth compact manifold M to a real n-dimensional compact manifold B. Both M and B have no boundary.

Definition 2.1. $\pi : M \to B$ is a *topological torus fibration over* B if the fiber $\pi^{-1}(p)$ is a diffeomorphic to T^n for any point $p \in B$.

Recall that a real n dimensional submanifold N in a n dimensional Kähler manifold X is *Lagrangian* if the Kähler form of X restricts to zero everywhere on the submanifold N. A Lagrangian submanifold is *special* if it is Lagrangian and minimal. The later means that the mean curvature H of the submanifold vanishes identically. If X is a Calabi-Yau manifold, then there is a covariant constant holomorphic n-form Ω on X and a special Lagrangian submanifold N is characterized by

$$\omega|_N = 0 \tag{2.1}$$

$$\operatorname{Im}\Omega|_N = 0. \tag{2.2}$$

Definition 2.2. A compact Calabi-Yau n-fold (M, ω, J, Ω) is a *Lagrangian torus fibration over* B if for each $p \in B$, the fiber $\pi^{-1}(p)$ is a Lagrangian torus in M with respect to the symplectic structure ω, and M is *semi-flat* if each fiber is flat in the induced metric from M. Furthermore, M is a *special Lagrangian torus fibration over* B if in addition the Lagrangian fibers are special.

According to Hitchin's discussion (cf. [**H**], [**G2**]), the complex structure J on M acts on TM as follows

$$J\left(\frac{\partial}{\partial s_i}\right) = \frac{\partial}{\partial t_i}, \quad J\left(\frac{\partial}{\partial t_i}\right) = -\frac{\partial}{\partial s_i} \tag{2.3}$$

where t_i are the local coordinates in B and s_i are coordinates on the fiber tori for $i = 1, 2, \dots, n$. Note that the section B needs to be Lagrangian for the complex coordinates to exist, and this will be understood throughout the paper. Hitchin shows:

Proposition 2.1. *For the special Lagrangian torus fibration $\pi : M \to B$ without singular fibers, in the complex coordinates $s_j + \sqrt{-1}t_j$, the symplectic form of M can be written as*

$$\omega = \sum_{i,j} a_{ij} ds_i \wedge dt_j,$$

where a_{ij} only depend on $t \in B$.

For each base point $t \in B$, set $L_t = \pi^{-1}(t)$. It is shown in [**H**] that the 1-form $\iota(\frac{\partial}{\partial t_j})\omega$ is harmonic on L_t when L_t is special Lagrangian, $j = 1, \dots, n$, hence $\iota(\frac{\partial}{\partial t_j})\omega$ and $*_t\iota(\frac{\partial}{\partial t_j})\omega$ are closed 1-form and $(n-1)$-form respectively, where $*_t$ is the Hodge star operator of the induced metric on L_t. Take a basis $A_1, \dots, A_n$ of the first homology group $H_1(L_t, \mathbf{Z})$. Evaluation of $\iota(\frac{\partial}{\partial t_j})\omega$ on

A_j yields a period matrix which depends on t:

$$\lambda_{ij} = \int_{A_i} \iota\left(\frac{\partial}{\partial t_j}\right)\omega. \tag{2.4}$$

The Poincaré dual of A_j provide a basis B_j of $H_{n-1}(L_t, \mathbf{Z})$ for $j = 1, \dots, n$. Then form a period matrix

$$\mu_{ij} = \int_{B_i} *_t \iota\left(\frac{\partial}{\partial t_j}\right)\omega. \tag{2.5}$$

Lemma 2.2. *For the special Lagrangian torus fibration $\pi : M \to B$ with the symplectic form $\omega = \sum_{i,j} a_{ij} ds_i \wedge dt_j$, then*

$$a_{ij} = V^{-1}\sum_k \lambda_{ik}\mu_{jk} \tag{2.6}$$

where V is the volume of the special Lagrangian fiber L_t and is independent of t.

Proof. By Proposition 2.1, a_{ij} only depend on t, then

$$\begin{aligned}\sum_k \lambda_{ik}\mu_{jk} &= \sum_{k,l_1,l_2} a_{il_1}a_{jl_2}\int_{A_k} ds_{l_1}\int_{B_k} *_t ds_{l_2} \\ &= \sum_{l_1,l_2} a_{il_1}a_{jl_2}\int_{L_t} ds_{l_1}\wedge *_t ds_{l_2}.\end{aligned}$$

On the other hand, s_j, t_j form a complex coordinates, so

$$g\left(\frac{\partial}{\partial s_i}, \frac{\partial}{\partial s_j}\right) = \omega\left(\frac{\partial}{\partial s_i}, \frac{\partial}{\partial t_j}\right) = a_{ij}.$$

Therefore,

$$\begin{aligned} ds_{l_1}\wedge *_t ds_{l_2} &= \langle ds_{l_1}, ds_{l_2}\rangle d\mu_{L_t} \\ &= a^{l_1 l_2} d\mu_{L_t}\end{aligned}$$

where a^{ij} denote the entries of the inverse matrix of (a_{ij}). It then follows that

$$\sum_k \lambda_{ik}\mu_{jk} = V(L_t)a_{ij}$$

where $V(L_t)$ is the volume of L_t and is independent of t since L_t is special Lagrangian. □

Since the special Lagrangian tori L_t are calibrated by $\operatorname{Im}\Omega$, from that $\operatorname{Re}\Omega$ is closed in M it follows easily:

Lemma 2.3. *The induced volume form $d\mu_{L_t}$ on the fiber L_t is independent of $t \in B$.*

3. Construction of the mirror manifold $(\check{M}, \check{\omega})$.

From now on, we shall fix a special Lagrangian torus fibration over B and denote it by (M, ω, J, Ω) with J defined in (2.3). The set $\mathcal{M}_{\mathrm{SL}}$ of all special Lagrangian submanifolds which can be deformed through special Lagrangian submanifolds to the fiber tori in (M, ω, J) is called *the moduli space of special Lagrangian submanifolds*. The deformations of special Lagrangian submanifolds were studied by McLean in [**M**]. As in [**SYZ**], we can construct the mirror manifold (the D-brane moduli space in the literatures of physics) over B by taking

$$\check{M} = \mathcal{M}_{\mathrm{SL}} \times_B \mathcal{M}_{\mathrm{FLAT}}$$

where $\mathcal{M}_{\mathrm{SL}}$ denotes local deformation space of the special Lagrangian fibers over B and $\mathcal{M}_{\mathrm{FLAT}}$ denotes the moduli space of the gauge equivalence classes of the flat $U(1)$-connections on the fibers over B. A point in $\check{M}$ is a pair $(L_t, [A])$ where L_t is a special Lagrangian fiber torus over $t \in B$ and $[A]$ is the gauge equivalence class of a flat $U(1)$-connection A on L_t. Note that $\mathcal{M}_{\mathrm{FLAT}}$ is diffeomorphic to $H^1(T^n, \mathbf{R})/H^1(T^n, \mathbf{Z})$ hence to T^n. Topologically, $\check{\pi} : \check{M} \to B$ is a torus fibration over B.

Recall that M is identified with T^*B quotient by the lattice Λ. We now describe the dual lattice $\check{\Lambda}$ (cf. [**H**], [**G2**]). Over a base point $t \in B$, consider a smooth fiber torus L_t. According to McLean's result, we know that

$$\dim \mathcal{M}_{\mathrm{SL}} = b_1(L_t) = \dim H^1(T^n, \mathbf{R}) = n.$$

Moreover, any tangent vector v of B can be identified with a harmonic 1-form on the fiber L_t as follows. Recall that we had a basis A_j of $H_1(L_t, \mathbf{Z})$ and a basis B_j of $H_{n-1}(L_t, \mathbf{Z})$ and A_j, B_j are dual to each other, for $j = 1, \ldots, n$. For each j, let α_j be the dual of A_j in $H^1(L_1, \mathbf{Z})$ hence they form a basis of $H^1(L_t, \mathbf{Z})$, and similarly let β_j be the dual of B_j in $H^{n-1}(L_t, \mathbf{R})$. Then the mapping

$$v \longrightarrow [\iota(v)\omega] = \sum_i \left(\int_{A_i} \iota(v)\omega \right) \alpha_i \tag{3.1}$$

identifies T_tB with $H^1(L_t, \mathbf{R})$. Define

$$\Lambda'_t = \left\{ v \in T_tB \;\middle|\; \int_\gamma \iota(v)\omega \in \mathbf{Z}, \text{ for any } \gamma \in H_1(L_t, \mathbf{Z}) \right\}.$$

Then we take

$$\check{\Lambda}_t = \left\{ [\iota(v)\omega] \,|\, v \in \Lambda'_t \right\} = H^1(L_t, \mathbf{Z}).$$

Let $\check{\Lambda} = \bigcup_{t \in B} \check{\Lambda}_t$ be the dual lattice over B. Then $\check{M} = T^*B/\check{\Lambda}$.

We can use the basis $\alpha_1, \ldots, \alpha_n$ of $H^1(L_t, \mathbf{R})$ to give coordinates $x_1, \ldots, x_n$ on the universal covering of the torus $H^1(L_t, \mathbf{R}/\mathbf{Z})$. Let Ω be the covariant constant holomorphic n-form on M from the Calabi-Yau structure with a standard normalization

$$\frac{\omega^n}{n!} = (-1)^{\frac{n(n-1)}{2}} \left(\frac{i}{2}\right)^n \Omega \wedge \overline{\Omega}.$$

To fix a symplectic structure on $\check{M}$, Gross considered a holomorphic n-form Ω_n normalized by

$$\Omega_n = \frac{\Omega}{\int_{L_t} \Omega} = V^{-1}\Omega$$

where V is the volume of the special Lagrangian fiber torus L_t. Gross has shown (cf. Lemma 4.1 and Proposition 4.2 in [**G2**]):

Proposition 3.1. *Under the identification*

$$\check{\Lambda} \cong H^1(L_t, \mathbf{Z}) \cong H_{n-1}(L_t, \mathbf{Z}),$$

the image of $\check{\Lambda}$ *under the mapping* $F : H_{n-1}(L_t, \mathbf{Z}) \to T^*B$*, defined by*

$$F(\gamma)(v) = -\int_\gamma \iota(v) \operatorname{Im} \Omega_n$$

for any $v \in T_tB, \gamma \in H_{n-1}(L_t, \mathbf{Z})$*, is Lagrangian in* T^*B*. Moreover* $\check{M} = T^*B/\check{\Lambda}$ *inherits the symplectic form* $\check{\omega}$ *from* T^*B*, and*

$$\int_{A_j} \iota(v)\check{\omega} = \int_{B_j} \iota(v) \operatorname{Im} \Omega_n. \tag{3.2}$$

Next, we compute the symplectic form $\check{\omega}$ in coordinates $t_1, \dots, t_n, x_1, \dots, x_n$, determined by $\alpha_1, \dots, \alpha_n$.

Lemma 3.2. *Let* $\pi : M \to B$ *be a special Lagrangian torus fibration and* $\check{\pi} : \check{M} \to B$ *be its dual space. Then*

$$\check{\omega} = \sum_{i,j} \mu_{ij} dt_i \wedge dx_j. \tag{3.3}$$

Proof. By McLean's result [**M**],

$$\iota(v) \operatorname{Im} \Omega_n|_{L_t} = - *_t \iota(v)\omega|_{L_t}. \tag{3.4}$$

It follows that

$$\begin{aligned} \int_{B_j} \iota\left(\frac{\partial}{\partial t_i}\right) \operatorname{Im} \Omega_n &= -\int_{B_j} *_t \iota\left(\frac{\partial}{\partial t_i}\right)\omega \\ &= -\mu_{ij}. \end{aligned}$$

In the canonical coordinates $t_1, \dots, t_n, x'_1, \dots, x'_n$ on T^*B, the symplectic form has the form

$$\check{\omega} = \sum_j dt_j \wedge dx'_j.$$

The coordinates $x'_1, \dots, x'_n$ are determined by some basis $\alpha'_1, \dots, \alpha'_n$ of $H^1(L_t, \mathbf{Z})$, and in fact by $dx'_1, \dots, dx'_n$ which are harmonic 1-forms on L_t

by Corollary 5.15 in [**G2**] since $d\Omega = 0$. Lifted to the universal covering $H^1(L_t, \mathbf{R})$ of $H^1(L_t, \mathbf{R}/\mathbf{Z})$, $\alpha_1, \ldots, \alpha_n$ and $\alpha'_1, \ldots, \alpha'_n$ are related by

$$\alpha'_i = \sum_j b_{ij}\alpha_j$$

for some functions b_{ij} on B. Therefore,

$$\begin{aligned}\int_{A_j} \iota\left(\frac{\partial}{\partial t_i}\right)\check{\omega} &= -\int_{A_j} dx'_i \\ &= \sum_l b_{il}\int_{A_j}\alpha_l \\ &= -b_{ij}.\end{aligned}$$

We then have

$$b_{ij} = \mu_{ij}$$

as claimed in the lemma. □

From now on, we shall always assume that $\mathrm{Volume}(L_t) = 1$ for simplicity by normalizing the metric g on M.

We now explore the relationship between the coordinates s_i on L_t and x_i on the dual tori $\check{L}_t$, *i.e.*, the moduli space of the flat $U(1)$-connections on L_t.

Proposition 3.3. *Let* $(t_1, \ldots, t_n, s_1, \ldots, s_n)$ *and* $(t_1, \ldots, t_n, x_1, \ldots, x_n)$ *are the local coordinates on* M *and* $\check{M}$ *respectively as before. For any closed* 1*-form* $\sum_j c^j ds_j$ *on* L_t*, if its cohomology class* $[\sum_j c^j ds_j]$ *is expressed as* $\sum_j \tilde{c}^j \alpha_j$*, then*

$$\tilde{c}^i = \sum_k \mu^{ki}\int_{L_t} c^k d\mu_{L_t}. \tag{3.5}$$

Proof. Notice that by Proposition 2.1 and Lemma 2.2, we have

$$\omega = \sum_{i,j,k} \lambda_{ik}\mu_{jk} ds_i \wedge dt_j.$$

Then

$$\begin{aligned} ds_j &= \sum_{l,i} \lambda^{ji}\mu^{li}\iota\left(\frac{\partial}{\partial t_l}\right)\omega \\ &= \sum_l \mu^{jl}\alpha_l.\end{aligned}$$

Recall that 1-forms α_j and $(n-1)$-forms β_j are chosen such that

$$\alpha_i \wedge \beta_j = \delta_{ij} d\mu_{L_t}$$

and hence

$$\begin{aligned}\langle \alpha_i, \beta_j \rangle &= \int_{L_t} \alpha_i \wedge \beta_j \\ &= \delta_{ij} V.\end{aligned}$$

It follows that

$$\begin{aligned}\tilde{c}^k &= \left\langle \sum_j \tilde{c}^j \alpha_j, \beta_k \right\rangle \\ &= \left\langle \sum_j c^j ds_j, \beta_k \right\rangle \\ &= \int_{L_t} \sum_j c^j ds_j \wedge \beta_k \\ &= \sum_{j,l} \int_{L_t} c^j \mu^{jl} \alpha_l \wedge \beta_k \\ &= \sum_l \mu^{lk} \int_{L_t} c^l d\mu_{L_t}.\end{aligned}$$

This completes the proof. □

Let $Y = \sum_j Y^j ds_j$ be a differential 1-form on L_t. By the Hodge decomposition theorem,

$$Y = H(Y) + df + d^*\psi$$

where $H(Y)$ is the harmonic part of Y. In particular, $H(Y) + df$ defines a cohomology class and we denote it by $[Y]$ in $H^1(L_t, \mathbf{R})$.

Lemma 3.4. *If Y is a 1-form on the Lagrangian fiber L_t and $[Y]$ is its cohomology class in $H^1(L_t, \mathbf{R})$, then*

$$[Y] = \sum_j \left(\sum_l \mu^{lj} \int_{L_t} Y^l d\mu_{L_t} \right) \alpha_j \tag{3.6}$$

$$= \int_{L_t} Y d\mu_{L_t}. \tag{3.7}$$

Proof. By Proposition 3.3,

$$[Y] = \sum_{j,l} \left(\mu^{lj} \int_{L_t} \left(Y^l - (d^*\psi)^l \right) d\mu_{L_t} \right) \alpha_j.$$

Note that

$$\sum_{j,l} \mu^{lj} \int_{L_t} (d^*\psi)^l d\mu_{L_t} \alpha_j = \sum_l \left(\int_{L_t} (d^*\psi)^l d\mu_{L_t} \right) ds_l.$$

Also, we have

$$\begin{aligned} d^*\psi \wedge *\iota\left(\frac{\partial}{\partial t_l}\right)\omega &= \sum_{k,j}(d^*\psi)^k ds_k \wedge *a_{lj}ds_j \\ &= \sum_{k,j}(d^*\psi)^k a_{lj}a^{kj}d\mu_{L_t} \\ &= (d^*\psi)^l d\mu_{L_t}. \end{aligned}$$

It follows that

$$\begin{aligned} \int_{L_t}(d^*\psi)^l d\mu_{L_t} &= \int_{L_t} d^*\psi \wedge *\iota\left(\frac{\partial}{\partial t_l}\right)\omega \\ &= \int_{L_t} \psi \wedge *d\iota\left(\frac{\partial}{\partial t_l}\right)\omega \\ &= 0 \end{aligned}$$

since $\iota(\frac{\partial}{\partial t_l})\omega$ is closed. □

4. Holomorphic connections.

Let E be a complex vector bundle over M and A be a unitary connection. The curvature 2-form F of A can be decomposed according to the complex structure on E into $(2,0),(1,1),(0,2)$ parts:

$$F = F^{2,0} + F^{1,1} + F^{0,2}.$$

In terms of real local coordinates $s_1,\ldots,s_n,t_1,\ldots,t_n$,

$$F = \sum_{i,j}^{n}\left(F_{ij}dt_i\wedge dt_j + F_{i(j+n)}dt_i\wedge ds_j + F_{(i+n)(j+n)}ds_i\wedge ds_j\right) \tag{4.1}$$

where the indices $i, j+n$ stand for the t_i-component and s_j-component of the connection correspondingly. Recall that a unitary connection A on E over (M,J,ω) gives rise to a *holomorphic* connection if and only if

$$F_A^{0,2} = 0. \tag{4.2}$$

Therefore we obtain the following curvature equations for the holomorphic connections in real coordinates.

Proposition 4.1. *Let E be a complex vector bundle over a smooth special Lagrangian fibration (M,J,ω) over B. Then the curvature of a holomorphic connection on E satisfies*

$$F_{i(j+n)} - F_{j(i+n)} = 0 \tag{4.3}$$

$$F_{ij} - F_{(i+n)(j+n)} = 0. \tag{4.4}$$

Proof. In terms of the complex coordinates $z_i = s_i + \sqrt{-1}t_i,\ i = 1, \dots, n$, we can rewrite (4.1) as

$$\begin{aligned} F &= -\sum_{i,j=1}^{n} F_{ij}(dz_i - d\overline{z_i}) \wedge (dz_j - d\overline{z_j}) \\ &\quad -\sqrt{-1}\sum_{i,j=1}^{n} F_{i(j+n)}(dz_i - d\overline{z_i}) \wedge (dz_j + d\overline{z_j}) \\ &\quad +\sum_{i,j=1}^{n} F_{(i+n)(j+n)}(dz_i + d\overline{z_i}) \wedge (dz_j + d\overline{z_j}). \end{aligned}$$

This local expression leads to

$$F^{0,2} = \sum_{i,j=1}^{n} \left(-F_{ij} + F_{(i+n)(j+n)} + \sqrt{-1}F_{i(j+n)}\right) d\overline{z_i} \wedge d\overline{z_j}$$

and

$$F^{1,1} = \sum_{i,j=1}^{n} \left(2F_{ij} + 2F_{(i+n)(j+n)} + \sqrt{-1}(F_{i(j+n)} + F_{j(i+n)})\right) dz_i \wedge d\overline{z_j}.$$

Then $F^{0,2} = 0$ implies the two desired equations. □

The holomorphic connections are preserved by the complex gauge transformations. Recall that the *complex gauge group* $\mathcal{G}_c$ consists of all general linear automorphisms of the complex vector bundle E which cover the identity map on the base manifold M. If $g \in \mathcal{G}_c$, the action of g is given by

$$\overline{\partial}_{g(A)} = \overline{\partial}_A - (\overline{\partial}_A g)g^{-1} \tag{4.5}$$

$$\partial_{g(A)} = \partial_A + \overline{(\overline{\partial}_A g)g^{-1}}^{\,t}. \tag{4.6}$$

The unitary gauge group is contained as a subgroup in $\mathcal{G}_c$ and it preserves the Hermitian metric on E. In particular, if E is a complex line bundle, then a connection A' is $\mathbf{C}$-gauge equivalent to another connection A if there exist real valued functions u and v such that

$$A' = A + \sqrt{-1}(\overline{\partial} - \partial)u + (\overline{\partial} + \partial)v. \tag{4.7}$$

When $u = 0$, we obtain the ordinary $U(1)$ gauge action.

5. Holomorphic line bundles vs. Lagrangian cycles with flat line bundles.

In this section, we shall start from a holomorphic connection on a complex line bundle E over M to construct a Lagrangian submanifold Σ in $\check{M}$ and a flat $U(1)$-connection α on Σ. Then we shall demonstrate how to reconstruct a holomorphic connection on E from (Σ, α).

5.1. Construction of Lagrangian cycles with flat line bundles. Let E be a holomorphic line bundle over a smooth special Lagrangian T^n fibration (M, J, ω, g) and A be a $U(1)$-connection whose curvature satisfies $F_A^{0,2} = 0$. It is a standard fact that each bundle trivialization, with a trivializing cover $\{U_j\}$ and $f_j : U_j \to \mathbf{C}$ satisfying the compatibility conditions

$$f_j = h_{jk} f_k \quad \text{on } U_j \bigcap U_k \neq \emptyset,$$

where h_{jk} are the transition functions, defines a global section $f \in \Gamma(M, E)$; and in the gauge determined by f the connection A can be viewed as an E-valued 1-form, which decomposes into its fiber component and its base component as follows

$$A = \sum_{i=1}^{n} \left(X^i dt_i + Y^i ds_i \right). \tag{5.1}$$

Here as in the previous sections we use $t_1, \dots, t_n$ for the local coordinates on the base B and $s_1, \dots, s_n$ for the fiber torus $L_t = \pi^{-1}(t)$, for any $t \in B$. X^i and Y^i are $\mathbf{C}$-valued functions.

On the fiber tori L_t, the gauge equivalent class of $Y = A|_{L_t}$ is just the cohomology class $[Y]$ of the E-valued 1-form Y. The image of the single valued map

$$\Phi : t \to (t, [Y(\cdot, t)]) \tag{5.2}$$

defines an embedded submanifold of real dimension n in $\check{M}$:

$$\Sigma = \{(t, [Y(\cdot, t)]) : t \in B\}. \tag{5.3}$$

Proposition 5.1. *Let E be a $U(1)$-bundle over a special Lagrangian torus fibration $\pi : M \to B$ and let $\check{\pi} : \check{M} \to B$ be the dual space. If A is a connection on E, then for any $v \in T_t B$ and $t \in B$,*

$$\iota(v)(\check{\omega}|_\Sigma) = \int_{L_t} \iota(Jv) \mathrm{Re} F_A^{0,2} \wedge d\mu_{L_t}. \tag{5.4}$$

In particular, if A is holomorphic, then Σ is an embedded Lagrangian submanifold in $\check{M}$.

Proof. In the local coordinates $(x_1, \dots, x_n)$ on $\mathcal{M}_{\mathrm{FLAT}}$, let $[Y]^1, \dots, [Y]^n$ be the local expression of $[Y]$ in $H^1(\check{L}_t, \mathbf{R})$. We claim that the restriction of

$\breve{\omega}$ on Σ vanishes. In fact,

$$
\begin{aligned}
(5.5)\quad \breve{\omega}|_\Sigma &= \Phi^*\breve{\omega} \\
&= \sum_{i,j} \breve{\omega}\left(\frac{\partial}{\partial t_i} + \sum_\alpha \frac{\partial [Y]^\alpha}{\partial t_i}\frac{\partial}{\partial x_\alpha}, \frac{\partial}{\partial t_j} + \sum_\beta \frac{\partial [Y]^\beta}{\partial t_j}\frac{\partial}{\partial x_\beta}\right) dt_i \wedge dt_j \\
&= 2\sum_{i,j,\alpha} \mu_{\alpha j}\frac{\partial [Y]^\alpha}{\partial t_i} dt_i \wedge dt_j \\
&= 2\sum_{i,j,\alpha} \mu_{\alpha j}\frac{\partial}{\partial t_i}\left(\mu^{k\alpha}\int_{L_t} Y^k d\mu_{L_t}\right) dt_i \wedge dt_j
\end{aligned}
$$

by Proposition 3.3. Recall that the volume element $d\mu_{L_t}$ is independent of t by Lemma 2.3. Since $\breve{\omega} = \sum_{i,j} \mu_{ij} dt_i \wedge dx_j$ is closed, we have

$$
(5.6)\qquad \frac{\partial \mu_{ik}}{\partial t_j} = \frac{\partial \mu_{jk}}{\partial t_i}
$$

and it follows that

$$
\begin{aligned}
(5.7)\quad \sum_\alpha \left(\mu_{\alpha j}\frac{\partial \mu^{k\alpha}}{\partial t_i} - \mu_{\alpha i}\frac{\partial \mu^{k\alpha}}{\partial t_j}\right) &= \sum_\alpha \left(-\frac{\partial \mu_{\alpha j}}{\partial t_i}\mu^{k\alpha} + \frac{\partial \mu_{\alpha i}}{\partial t_j}\mu^{k\alpha}\right) \\
&= 0.
\end{aligned}
$$

Now we conclude from (5.5) that

$$
(5.8)\qquad \breve{\omega}|_\Sigma = \sum_{i,j}\left(\int_{L_t} \frac{\partial Y^j}{\partial t_i} d\mu_{L_t}\right) dt_i \wedge dt_j.
$$

On the other hand,

$$
\begin{aligned}
(5.9)\quad & \sum_{i,j}\left(\int_{L_t} \frac{\partial Y^j}{\partial t_i} d\mu_{L_t}\right) dt_i \wedge dt_j \\
&= \sum_{i,j}\left(\int_{L_t}\left(F_{i(j+n)} - \frac{\partial X^i}{\partial s_j}\right) d\mu_{L_t}\right) dt_i \wedge dt_j \\
&= \sum_{i,j}\left(\int_{L_t} F_{i(j+n)} d\mu_{L_t}\right) dt_i \wedge dt_j \\
&\quad - \sum_{i,j}\left(\int_{L_t} d_{L_t}X^i \wedge *\iota\left(\frac{\partial}{\partial t_j}\right)\omega\right) dt_i \wedge dt_j.
\end{aligned}
$$

Because L_t is special Lagrangian, $*\iota(\frac{\partial}{\partial t_j})\omega$ is closed on L_t, and in turn the last integral in (5.9) vanishes. This leads to

$$\breve{\omega}|_\Sigma = \sum_{i,j} \left(\int_{L_t} F_{i(j+n)} d\mu_{L_t} \right) dt_i \wedge dt_j. \tag{5.10}$$

If A is a holomorphic $U(1)$-connection, it follows immediately from (4.3) in Proposition 4.1 that $\breve{\omega}$ restricts to zero on Σ and Σ is Lagrangian.

It is straightforward to find

$$\begin{aligned} \mathrm{Re}F^{0,2} &= \sum_{i,j} F_{i(j+n)}(dt_i \wedge ds_j + ds_i \wedge dt_j) \\ &\quad + \sum_{i,j} (F_{(i+n)(j+n)} - F_{ij})(ds_i \wedge ds_j - dt_i \wedge dt_j). \end{aligned} \tag{5.11}$$

Then

$$\iota\left(\frac{\partial}{\partial s_k}\right) \mathrm{Re}F^{0,2} \wedge d\mu_{L_t} = \sum_j (F_{j(k+n)} - F_{k(j+n)}) dt_j \wedge d\mu_{L_t}. \tag{5.12}$$

Finally, we compute

$$\iota\left(\frac{\partial}{\partial t_k}\right) (\breve{\omega}|_\Sigma) = \sum_j \left(\int_{L_t} (F_{j(k+n)} - F_{k(j+n)}) d\mu_{L_t} \right) dt_j \tag{5.13}$$

and we are done. □

To investigate what the second curvature equation (4.4) leads to, we consider the 1-form defined by

$$\alpha = \int_{L_t} A \wedge \mathrm{Re}\,\Omega. \tag{5.14}$$

Proposition 5.2. *If A is a holomorphic $U(1)$-connection over M, then α is a flat $U(1)$-connection over Σ. If A is a $U(1)$-connection, then for any $v \in T_t B$ with $t \in B$,*

$$\iota(v) d_t \alpha = \int_{L_t} \iota(Jv) \mathrm{Im}F_A^{0,2} \wedge d\mu_{L_t}. \tag{5.15}$$

Proof. By Lemma 2.3, the exterior differentiation on B is

$$
\begin{aligned}
d_t\alpha &= \int_{L_t} (d_tA)\wedge d\mu_{L_t} \\
&= \sum_i \int_{L_t} (d_t(X^i dt_i))\wedge d\mu_{L_t} \\
&= \sum_{i,k}\left(\int_{L_t} \frac{\partial X^i}{\partial t_k} d\mu_{L_t}\right) dt_i\wedge dt_k \\
&= \sum_{i,k}\left(\int_{L_t} \frac{\partial Y^i}{\partial s_k} d\mu_{L_t}\right) dt_i\wedge dt_k \\
&= \sum_{i,k}\left(\int_{L_t} d_{L_t}Y^i \wedge *\iota\left(\frac{\partial}{\partial t_k}\right)\omega\right) dt_i\wedge dt_k \\
&= \sum_{i,k}\left(\int_{L_t} d_{L_t}\left(Y^i \wedge *\iota\left(\frac{\partial}{\partial t_k}\right)\omega\right)\right) dt_i\wedge dt_k \\
&= 0
\end{aligned}
$$

where we have used $F_{ik} = F_{(i+n)(k+n)}$ in the fourth equality, that $*\iota(\frac{\partial}{\partial t_k})\omega$ is closed and Stokes' theorem. The imaginary part of $F^{0,2}$ is given by

$$
\begin{aligned}
\operatorname{Im}F^{0,2} = {} & F_{i(j+n)}(ds_i\wedge ds_j - dt_i\wedge dt_j) \\
& + \left(F_{(i+n)(j+n)} - F_{ij}\right)(dt_i\wedge ds_j + ds_i\wedge dt_j).
\end{aligned} \tag{5.16}
$$

Then we can deduce

$$
\iota\left(\frac{\partial}{\partial s_j}\right)\operatorname{Im}F_A^{0,2}\wedge d\mu_{L_t} = 2\sum_i (F_{(i+n)(j+n)} - F_{ij})dt_i\wedge d\mu_{L_t}. \tag{5.17}
$$

On the other hand, the previous computation shows

$$
\begin{aligned}
\iota\left(\frac{\partial}{\partial t_j}\right)d_t\alpha &= \sum_i \int_{L_t}\left(\frac{\partial X^i}{\partial t_j} - \frac{\partial X^j}{\partial t_i}\right)dt_i\wedge d\mu_{L_t} \\
&= \sum_i \int_{L_t}\left(F_{(i+n)(j+n)} - F_{ij}\right)dt_i\wedge d\mu_{L_t}.
\end{aligned} \tag{5.18}
$$

Now the proof is complete by integrating (5.17) along L_t and then substituting (5.18) into the result. □

It follows from Proposition 5.1 and Proposition 5.2 that:

Proposition 5.3. *Let E be a complex line bundle over a special Lagrangian torus fibration $M\to B$ and A be a $U(1)$-connection on E. Then, for any*

$v \in T_t B$ and $t \in B$

$$\iota(v)\left(\check{\omega}|_\Sigma + \sqrt{-1}d_t\alpha\right) = \int_{L_t} \iota(Jv)F_A^{0,2} \wedge \operatorname{Re}\Omega. \tag{5.19}$$

Remark. Assume that M is a Calabi-Yau 3-fold. The derivative of the holomorphic Chern-Simons functional is given by

$$\int_M \operatorname{Tr}(\delta A \wedge F_A^{0,2}) \wedge \Omega.$$

Its differential is given by the right side of (5.19). It was observed by Tian and myself that there is a useful version for the left side of (5.19). Consider the space $\mathcal{L}$ of all $(L, [N], B)$ where L is any 3-cycle homologous to a fixed 3-cycle L_0, N is a 4-cycle with boundary $\partial N = L - L_0$ and B is a $U(1)$-connection on L extendible to N with fixed boundary value along L_0. Then one can integrate the left side of (5.19) to obtain a functional on $\mathcal{L}$

$$F(L, [N], B) = \int_N \left(\check{\omega} + \sqrt{-1}F_B\right)^2.$$

If L is a section, then it corresponds to the holomorphic Chern-Simons functional through (5.19). However, this functional F is well-defined on any Calabi-Yau 3-fold without knowing the mirrors. It is certainly interesting to explore more about F.

5.2. Construction of holomorphic line bundles from (Σ, α)**.** We have just constructed a Lagrangian submanifold Σ in $\check{M}$ and a flat $U(1)$-connection α. Strictly speaking, α is a pull-back of a flat $U(1)$-connection on B via the projection $\check{\pi}|_\Sigma : \Sigma \to B$. Conversely, given a pair (Σ, α) on the mirror side, we would like to construct a holomorphic connection A on a complex line bundle over M.

The information encoded in Σ is $[Y]$. Let $P : \Sigma \to B$ be the natural projection. The 1-form α determines a flat complex line bundle over B which pulls back via P to the flat connection, still denoted by α, on Σ. Since $P : \Sigma \to B$ is diffeomorphic, the inverse map P^{-1} pulls back the flat bundle (L, α) to a flat bundle over B, which will still be denoted by (L, α). Then we use $\pi : M \to B$ to pull back (L, α) to X on M which satisfies (5.20) below. The desired connection $A = X + Y$, where $X = \sum_i X^i dt_i$ and $Y = \sum_i Y^i ds_i$, should satisfy, in addition to being holomorphic, that

$$\int_{L_t} X^i d\mu_{L_t} = \alpha^i \tag{5.20}$$

$$\int_{L_t} Y^i d\mu_{L_t} = \sum_k \mu_{ki}[Y]^k \tag{5.21}$$

for $i = 1, \dots, n$, where $\alpha = \alpha^i dt_i$ is over B. To find a flat connection Y on L_t, we recall that flat line bundles on L_t are classified by $H^1(L_t, \mathbf{R}/\mathbf{Z})$. On the universal covering $H^1(L_t, \mathbf{R})$ of $H^1(L_t, \mathbf{R}/\mathbf{Z})$, we consider the 1-form $V(L_t)^{-1}\sum_{i,k}\mu_{ki}[Y]^k\alpha_i$, where $\alpha_1, \ldots, \alpha_n$ is the basis of $H^1(L_t, \mathbf{R})$. This 1-form descents to Y which satisfies (5.21).

We shall concentrate on the special case that X^i and Y^i are smooth $\mathbf{C}$-valued functions on M which depend only on the base variable $t \in B$.

Proposition 5.4. *If X^i and Y^i depend only on t and satisfy* (5.20) *and* (5.21) *for all i, then the* 1*-form $A = \sum_i(X^i dt_i + Y^i ds_i)$ is holomorphic.*

Proof. The volume of the fiber L_t is independent of t, due to that L_t is special Lagrangian in M, and without loss of any generality we may assume the fiber has unit volume by re-normalization. The $\mathbf{C}$-valued 1-form A satisfies

$$F_{(i+n)(j+n)} = \frac{\partial Y^i}{\partial s_j} - \frac{\partial Y^j}{\partial s_i} = 0$$

since Y^i's are free of s,

$$\begin{aligned} F_{ij} &= \frac{\partial X^i}{\partial t_j} - \frac{\partial X^j}{\partial t_i} \\ &= \frac{\partial \alpha^i}{\partial t_j} - \frac{\partial \alpha^j}{\partial t_i} \\ &= 0 \end{aligned}$$

since α is flat, and finally

$$\begin{aligned} F_{i(j+n)} - F_{j(i+n)} &= \frac{\partial X^i}{\partial s_j} - \frac{\partial Y^j}{\partial t_i} - \frac{\partial X^j}{\partial s_i} + \frac{\partial Y^i}{\partial t_j} \\ &= \frac{\partial Y^i}{\partial t_j} - \frac{\partial Y^j}{\partial t_i} \\ &= \sum_k \mu_{ki}\frac{\partial [Y]^k}{\partial t_j} - \sum_k \mu_{kj}\frac{\partial [Y]^k}{\partial t_i} \\ &= 0 \end{aligned}$$

because Σ is Lagrangian. We conclude that A is holomorphic and satisfies (5.20) and (5.21). □

6. Deformations of $U(1)$-connections and Lagrangian cycles.

In this section, we first investigate how deformation of holomorphic $U(1)$-connections A on E affects the Lagrangian cycles Σ and the flat $U(1)$-connections α on the mirror side constructed in the last section. Then we examine how the $U(1)$-connection on E constructed from (Σ, α) varies if Σ and α are deformed.

6.1. Deforming A by the complex gauge groups. Let A' be a $U(1)$-connection on E which is $\mathbf{C}$-gauge equivalent to A:

$$A' = A + \sqrt{-1}(\overline{\partial} - \partial)u + (\overline{\partial} + \partial)v \tag{6.1}$$

where u and v are $\mathbf{R}$-valued functions on M. If A is holomorphic, so is A'. If $u = 0$, then A' is gauge equivalent to A. It is straightforward to verify

$$\begin{aligned}\sqrt{-1}(\overline{\partial} - \partial)u &= -\sum_i \left(\frac{\partial u}{\partial t_i} ds_i - \frac{\partial u}{\partial s_i} dt_i\right) \\ (\overline{\partial} + \partial)v &= \sum_i \left(\frac{\partial v}{\partial s_i} ds_i + \frac{\partial v}{\partial t_i} dt_i\right).\end{aligned}$$

The fiberwise component Y' of A' is given by

$$Y' = Y + \sqrt{-1}\sum_i \left(\frac{\partial u}{\partial t_i} + \frac{\partial v}{\partial s_i}\right) ds_i. \tag{6.2}$$

The Hodge decomposition yields

$$\sum_i \left(\frac{\partial u}{\partial t_i} + \frac{\partial v}{\partial s_i}\right) ds_i = \psi + d^*\eta, \tag{6.3}$$

where ψ is the sum of a harmonic 1-form and an exact 1-form on L_t and η is a 2-form on L_t. Hence, the graph of the mapping $t \to [Y'(t, \cdot)]$ defines a Lagrangian submanifold in $\check{M}$:

$$\Sigma' = \{(t, [Y(t, \cdot)] + [\psi(t, \cdot)]) : t \in B\}. \tag{6.4}$$

Let us recall:

Definition 6.1. A diffeomorphism $F : (\check{M}, \check{\omega}) \to (\check{M}, \check{\omega})$ is called *Hamiltonian* if there exists a smooth function $H : [0, 1] \times \check{M} \to \mathbf{R}$ and a family $f^t, t \in \mathbf{R}$, of symplectic diffeomorphisms of $\check{M}$ such that

$$\begin{aligned}\frac{df^t}{dt} &= X_H(t, f^t) \\ f^0 &= \mathrm{id} \\ f^1 &= F\end{aligned}$$

where the *Hamiltonian vector field* X_H is determined by

$$\iota(X_H)\check{\omega} = -dH.$$

We then introduce:

Definition 6.2. Two Lagrangian submanifolds Σ and Σ' in $\check{M}$ are *Hamiltonian equivalent* if there exists a piecewise smooth and continuous family of Hamiltonian diffeomorphisms $F_\sigma : \check{M} \to \check{M}$ for $\sigma \in [0, 1]$ such that F_0 is the identity map and $F_1(\Sigma) = \Sigma'$. The equivalence class determined by Σ is denoted by $[\Sigma]$.

Proposition 6.1. *Let A be a holomorphic $U(1)$-connection on the complex line bundle E over the special Lagrangian torus fibration $\pi : M \to B$. Suppose that another $U(1)$-connection A' is $\mathbf{C}$-gauge equivalent to A. Then $\Sigma = \{(t, [Y(t,\cdot)]) : t \in B\}$ and $\Sigma' = \{(t, [Y'(t,\cdot)]) : t \in B\}$ are Hamiltonian equivalent Lagrangian submanifolds in $\check{M}$. If A' is gauge equivalent to A, then $\Sigma = \Sigma'$.*

Proof. For each $t \in B$, we connect the two points $[Y(t)]$ and $[Y(t)]+[\psi(t)]$ by a path in the torus $L_t = H^1(L_t, \mathbf{R}/\mathbf{Z})$. On the universal covering $H^1(L_t, \mathbf{R})$, the path may be taken to be $C_t(\tau) = [Y(t)] + \tau[\psi(t)]$ for $\tau \in [0,1]$. Along the path C_t, the infinitesimal deformation vector field is equals to

(6.5)

$$\begin{aligned}\sum_k [\psi]^k \frac{\partial}{\partial x_k} &= \sum_k \left(\mu^{kj} \int_{L_t} \psi^j d\mu_{L_t}\right) \frac{\partial}{\partial x_k}\\ &= \sum_{k,j} \left(\mu^{kj} \int_{L_t} \left(\frac{\partial u}{\partial t_j} + \frac{\partial v}{\partial s_j} - (d^*\eta)^j\right) d\mu_{L_t}\right) \frac{\partial}{\partial x_k}\\ &= \sum_{k,j} \left(\mu^{kj} \int_{L_t} \left(\frac{\partial u}{\partial t_j} d\mu_{L_t} + dv \wedge *\iota\left(\frac{\partial}{\partial t_j}\right)\omega - d^*\eta \wedge *\iota\left(\left(\frac{\partial}{\partial t_j}\right)\omega\right)\right)\right) \frac{\partial}{\partial x_k}\\ &= \sum_{k,j} \left(\mu^{kj} \int_{L_t} \frac{\partial u}{\partial t_j} d\mu_{L_t}\right) \frac{\partial}{\partial x_k}.\end{aligned}$$

Integrating (6.5) yields the transformation from Σ to Σ'. To construct the Hamiltonian deformation, it suffices to assume that $[\psi]$ is small enough so that Σ' stays inside a tubular neighborhood O of Σ. We may further assume that the closure of O is contained in some larger tubular neighborhood O' of Σ. The function $\int_{L_t} u d\mu_{L_t}$ is defined globally on B hence may be viewed as a function on Σ'. On the mirror space $\check{\pi} : \check{M} \to B$, introduce a function $H : [0,1] \times \check{M} \to \mathbf{R}$ as follows. If $t = \check{\pi}(y)$ for $y \in \check{M}$ set

$$H(\sigma, y) \;=\; \sigma h(y) \int_{L_t} u d\mu_{L_t}$$

where h is a cut-off function on $\check{M}$ which equals to one on O and zero outside O'. H determines a Hamiltonian vector field X_H for each fixed σ by

$$\check{\omega}(X_H, \cdot) = -dH.$$

Using Lemma 3.2, we get

$$\begin{aligned}\check{\omega}\left(\sum_{j,k} \left(\mu^{kj} \int_{L_t} \frac{\partial u}{\partial t_j} d\mu_{L_t}\right) \frac{\partial}{\partial x_k}, \cdot\right) &= \sum_j \left(\int_{L_t} \frac{\partial u}{\partial t_j} d\mu_{L_t}\right) dt_j \\ &= d\int_{L_t} u d\mu_{L_t}.\end{aligned} \tag{6.6}$$

In particular, it follows that

$$X_H|_O = \sigma \sum_{j,k} \left(\mu^{kj} \int_{L_t} \frac{\partial u}{\partial t_j} d\mu_{L_t} \right) \frac{\partial}{\partial x_k}. \tag{6.7}$$

The Cauchy problem

$$\frac{dF_\sigma}{d\sigma} = X_H(F_\sigma) \tag{6.8}$$

$$F_0 = id \tag{6.9}$$

has a unique solution F_σ. For each fixed parameter σ, $F_\sigma : \check{M} \to \check{M}$ is a Hamiltonian diffeomorphism and F_σ is equal to the identity map outside O' for any σ. As σ moving from 0 to 1, any point p in Σ evolves along the curve $F_\sigma(p)$ with velocity X_H evaluated at the point $F_\sigma(p)$, and especially $F_1(p) \in \Sigma'$.

Now we conclude that Σ and Σ' belong to the same Hamiltonian class, and moreover that if $u = 0$ then $H \equiv 0$ hence $\Sigma = \Sigma'$. □

Proposition 6.2. *If A' is $\mathbf{C}$-gauge equivalent to A, then $\alpha' = \int_{L_t} A' \wedge \mathrm{Re}\Omega$ differs from $\alpha = \int_{L_t} A \wedge \mathrm{Re}\Omega$ by an exact 1-form. In fact, for $A' = A + \sqrt{-1}(\bar\partial - \partial)u + (\bar\partial + \partial)v$, $v = 0$ implies $\alpha' = \alpha$ and $u = 0$ implies that $\alpha' - \alpha$ is an exact 1-form.*

Proof. The 1-form defined by A' is

(6.10)

$$\begin{aligned}
\alpha' &= \int_{L_t} A' \wedge \mathrm{Re}\,\Omega \\
&= \alpha - \sqrt{-1} \sum_i \left(\int_{L_t} \frac{\partial u}{\partial s_i} d\mu_{L_t} \right) dt_i + \sqrt{-1} \sum_i \left(\int_{L_t} \frac{\partial v}{\partial t_i} d\mu_{L_t} \right) dt_i \\
&= \alpha - \sqrt{-1} \sum_i \left(\int_{L_t} d_{L_t} u \wedge *\iota \left(\frac{\partial}{\partial t_i} \right) \omega \right) dt_i + \sqrt{-1} \sum_i \left(\int_{L_t} \frac{\partial v}{\partial t_i} d\mu_{L_t} \right) dt_i \\
&= \alpha + \sqrt{-1} d_t \left(\int_{L_t} v d\mu_{L_t} \right).
\end{aligned}$$

This completes the proof. □

6.2. Deforming Σ in its Hamiltonian class and α by the gauge group. To understand the effects of deformations of the pair (Σ, α) on the reconstructed holomorphic connection A, we first observe:

Proposition 6.3. *The representatives of the cohomology class $[\alpha]$ yield gauge equivalent holomorphic connections A.*

Proof. If $\alpha' = \alpha + d_t f$ for some function f on Σ, the s-independent connection corresponds to α' is

$$A' = A + d_t f = A + df. \tag{6.11}$$

Thus A' is gauge equivalent to A. □

Next, we have:

Proposition 6.4. *Assume that Σ' is in the Hamiltonian class determined by Σ. Then A' induced from (Σ', α) is $\mathbf{C}$-gauge equivalent to A induced from (Σ, α).*

Proof. Let F_σ be the continuous family of Hamiltonian diffeomorphisms which deforms Σ to Σ'. Take a finite collection of numbers $0 = \sigma_0 < \sigma_1 < \ldots < \sigma_m = 1$ such that Σ_{i+1} is contained in some open tubular neighborhood U_i of Σ_i in $\check{M}$, and F_σ is smooth in σ for $\sigma \in [\sigma_i, \sigma_{i+1}]$. Note that the Hamiltonian deformation classes of the Lagrangian section Σ_i are given by $H^1(\Sigma_i, \mathbf{R}) \cong H^1(B, \mathbf{R})$. Hence there exists a family of symplectic diffeomorphisms $F_\sigma^{(i)} : U_i \to U_i$ which is Hamiltonian locally:

$$\frac{dF_\sigma^{(i)}}{d\sigma} = X_{h^{(i)}}(F_\sigma^{(i)}) \tag{6.12}$$

for some function $h^{(i)}$ on $[\sigma_i - \epsilon, \sigma_{i+1} + \epsilon] \times U_i$ for some small $\epsilon > 0$ and $h^{(i)}$ depends only on $t \in B$, subject to

$$\iota(X_{h^{(i)}})\check{\omega} = -dh^{(i)}.$$

It is straightforward to check

$$X_{h^{(i)}} = \sum_{j,k} \mu^{jk} \frac{\partial h^{(i)}}{\partial t_j} \frac{\partial}{\partial x_k} - \sum_{j,l} \mu^{lj} \frac{\partial h^{(i)}}{\partial x_j} \frac{\partial}{\partial t_l}. \tag{6.13}$$

That $h^{(i)}$ depends only on t implies the second term above vanishes. It then follows from Proposition 3.3, (6.12) and (6.13) by taking the kth components of the corresponding vector fields that

$$\begin{aligned} \int_{L_t} Y^k_{\sigma_{i+1}} d\mu_{L_t} - \int_{L_t} Y^k_{\sigma_i} d\mu_{L_t} &= \sum_j \mu_{jk}([Y_{\sigma_{i+1}}]^j - [Y_{\sigma_i}]^j) \\ &= \sum_{j,l} \mu_{jk} \int_{\sigma_i}^{\sigma_{i+1}} \mu^{lj} \frac{\partial h^{(i)}}{\partial t_l} \\ &= \frac{\partial}{\partial t_k} \int_{\sigma_i}^{\sigma_{i+1}} h^{(i)}. \end{aligned}$$

Extend the function $h^{(i)}$ globally by introducing

$$H^{(i)} = h^{(i)} \xi_i$$

for some smooth cut-off function ξ_i which equals 0 outside U_i and equals 1 on a smaller open tubular neighborhood $U_i' \subset U_i$ of Σ_i which still contains Σ_{i+1}.

Now we construct a 1-form $A^{(i+1)}$ by setting

$$A^{(i+1)} = X + Y_{\sigma_i} + \left(\frac{\partial}{\partial t_k} \int_{\sigma_i}^{\sigma_{i+1}} H^{(i)} \right) ds_k = A^{(i)} + \left(\frac{\partial}{\partial t_k} \int_{\sigma_i}^{\sigma_{i+1}} H^{(i)} \right) ds_k. \tag{6.14}$$

It is easy to see that $A^{(i+1)}$ is $\mathbf{C}$-gauge equivalent to $A^{(i)}$. Repeat this procedure for each i. Finally, we obtain

$$A' = A^{(n)} = A + \left(\frac{\partial}{\partial t_k} \sum_{i=1}^{m} \int_{\sigma_i}^{\sigma_{i+1}} H^{(i)} \right) ds_k.$$

This implies that A' is $\mathbf{C}$-gauge equivalent to A. □

Summing up our discussion, we obtain Theorem 1.1.

References

[AB] M.F. Atiyah and R. Bott, *The Yang-Mills equations over Riemann surfaces*, Phil. Tran. Roy. Soc. London, Series A, **308** (1982), 523-615, MR 85k:14006.

[BJPS] M. Bershadsky, A. Johansen, T. Pantev and V. Sadov, *On four-dimensional compactifications of F-theory*, Nucl. Phys. B, **505** (1997), 165-201, MR 99d:81107, Zbl 0925.14019.

[DK] S.K. Donaldson and P.B. Kronheimer, *The Geometry of Four-Manifolds*, Clarendon Press, Oxford, 1990, MR 92a:57036, Zbl 0820.57002.

[DT] S.K. Donaldson and R.P. Thomas, *Gauge theory in higher dimensions*, preprint.

[FMW] R. Friedman, J. Morgan and E. Witten, *Vector bundles and F theory*, Comm. Math. Phys., **187** (1997), 679-743, MR 99g:14052, Zbl 0919.14010.

[G1] M. Gross, *Special Lagrangian fibrations* I: *Topology*, in 'Integrable Systems and Algebraic Geometry' (M.-H. Saito, Y. Shimizu and K. Ueno, eds.), World Scientific, (1998), 156-193, MR 2000e:14066, Zbl 0964.14033.

[G2] ______, *Special Lagrangian fibrations* II: *Geometry*, Surveys in Differential Geometry, **5**, Int. Press, Boston, (1999), 341-403, MR 2001j:53065.

[G3] ______, *Topological mirror symmetry*, math.AG/9909015, 1999.

[HL] R. Harvey and H.B. Lawson, *Calibrated geometry*, Acta Math., **148** (1982), 47-157, MR 85i:53058.

[H] N. Hitchin, *The moduli space of special Lagrangian submanifolds*, Ann. Scuola Norm. Sup. Pisa CI. Sci. (4), **25**(**3-4**) (1997), 503-515, MR 2000c:32075.

[M] R.C. McLean, *Deformations of calibrated submanifolds*, Duke University, preprint, 1996.

[PZ] A. Polishchuk and E. Zaslow, *Categorical mirror symmetry: The elliptic curve*, math.AG/9801119.

[R] W.D. Ruan, *Lagrangian tori fibration of toric Calabi-Yau manifold* III: *Symplectic topological SYZ mirror construction for general quintics*, math.DG/9909126.

[SYZ] A. Strominger, S.T. Yau and E. Zaslow, *Mirror symmetry is T-duality*, Nuclear Phys., **B 479**(**1**-**2**) (1996), 243-259, MR 97j:32022, Zbl 0896.14024.

[Ti] G. Tian, *Gauge theory and calibrated geometry* I, Ann. of Math. (2), **151**(**1**) (2000), 193-268, MR 2000m:53074, Zbl 0957.58013.

[Ty] A. Tyurin, *Geometric quantization and mirror symmetry*, math.AG/9902027.

[V] C. Vafa, *Extending mirror conjecture to Calabi-Yau with bundles*, hep-th/9804131.

[W] R.O. Wells, *Differential Analysis on Complex Manifolds*, GTM 65, Springer-Verlag, 1980, MR 83f:58001, Zbl 0435.32004.

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