CROSSING NUMBER OF ALTERNATING KNOTS IN $S \times I$

Colin Adams, Thomas Fleming, Michael Levin, and Ari M.
Turner

One of the Tait conjectures, which was stated 100 years ago and proved in the 1980’s, said that reduced alternating projections of alternating knots have the minimal number of crossings. We prove a generalization of this for knots in $S \times I$, where $S$ is a surface. We use a combination of geometric and polynomial techniques.

1. Introduction.

A hundred years ago, Tait conjectured that the number of crossings in a reduced alternating projection of an alternating knot is minimal. This statement was proven in 1986 by Kauffman, Murasugi and Thistlethwaite, [6], [10], [11], working independently. Their proofs relied on the new polynomials generated in the wake of the discovery of the Jones polynomial.

We usually think of this result as applying to knots in the 3-sphere $S^3$. However, it applies equally well to knots in $S^2 \times I$ (where $I$ is the unit interval $[0, 1]$). Indeed, if one removes two disjoint balls from $S^3$, the resulting space is homeomorphic to $S^2 \times I$. It is not hard to see that these two balls do not affect knot equivalence. We conclude that the theory of knot equivalence in $S^2 \times I$ is the same as in $S^3$.

With this equivalence in mind, it is natural to ask if the Tait conjecture generalizes to knots in spaces of the form $S \times I$ where $S$ is any compact surface.

More rigorously, consider the projection surface $S_0 = S \times \{1/2\}$. Let $\pi : S \times I \to S_0$ be the natural projection. We define crossing number, alternating projections and alternating knots in the obvious way. Given some choice of a definition of reduced, we want to know whether reduced alternating projections of alternating knots have minimal crossing number.

In other words, if $c(\pi(K))$ represents the crossing number of a projection, we want to know if it is always the case that if $K$ and $K'$ represent two spatial configurations of the same knot, so $\pi(K)$, $\pi(K')$ are two projections of the knot and $\pi(K)$ is “reduced” and alternating, then

\[ c(\pi(K)) \leq c(\pi'(K)). \]
In [4], Kamada showed that if two projections of a knot in $S \times I$ are both “properly reduced” alternating projections with the same supporting genus, then they have the same number of crossings. A projection is properly reduced if the four regions that meet at each crossing of the projection are distinct. This is a generalization of a reduced projection in the plane. The supporting genus of a projection is the genus of the surface that results if each region of the projection surface is replaced with a disk.

The result presented in this paper extends Kamada’s result in three ways. First, our notion of “reduced” is more general than Kamada’s, and it is a more natural generalization of the definition in $S^3$.

We define a knot projection to be reduced on $S_0$ if there are no trivial simple closed curves on $S_0$ that intersect the knot projection exactly once (a trivial curve is a curve that is homotopic to the constant curve). This is natural, because curves like this exist exactly when one can perform the “untwisting” operation to reduce the number of crossings. Note that such a curve intersects the projection at a crossing, with two strands of the knot coming out of the intersection to either side of the curve.

Second, we consider arbitrary projections $\pi(K)$, $\pi(K')$, not just projections with the same supporting genus. Third, we show that the crossing numbers of the reduced alternating projections are not just equal to one another, but that they are minimal.

Indeed, we prove the following:

**Theorem 1.1.** Let $S$ be a compact surface. Let $\pi(K)$ be a reduced alternating projection of an alternating knot in $S \times I$ and let $\pi(K')$ be an arbitrary projection (of the same knot). Then

$$c(\pi(K)) \leq c(\pi(K')).$$

Unlike the proof of the original Tait conjecture, polynomial techniques were not enough to establish our result. These techniques are only strong enough to give results analogous to those of Kamada. We use Menasco’s geometrical techniques to show something analogous to the supporting genus restriction always holds and to complete the proof. Independently, in [5], the author announced a version of Theorem 1.1 for knots and links, however, only an outline of the proof has appeared. The techniques utilized differ substantially from those presented here.

The specific breakdown of the paper is as follows. The second section of this paper presents the geometrical argument. The main result of this section is that the general result follows from the special case that $S$ is a punctured compact orientable surface, and $\pi(K')$ cuts this surface into disks and punctured disks. This special case is analogous to Kamada’s result.
The third section of this paper presents the polynomial argument. We define polynomial invariants for knots in $S \times I$ and use them to prove (2) in the special case.

Combining these two results gives the main theorem, the proof of which appears in the third section. The fourth and final section discusses extensions of the theorem, and other related questions and conjectures.

Acknowledgements. The first author was supported by NSF Grant DMS-9803362. The remaining authors were supported by NSF Grant DMS-9820570 as participants in the Williams College SMALL REU program in the summer of 1999.

2. The geometric argument.

In this section we show that in some sense, alternating projections of a knot in $S \times I$ wrap around the projection surface $S_0$ less than any other projections of the same knot. The rigorous statement of this idea is given by the next theorem. (Hayashi has proved a related result in the case that $S$ is a torus [2].)

**Theorem 2.1.** Let $S$ be an orientable surface. Let $K$ be a configuration of a knot in $S \times I$ such that $\pi(K)$ is regular and alternating. Let $K'$ be any other configuration of the knot, and let $H : (S \times I) \times I \to (S \times I)$ be the isotopy carrying $K'$ to $K$. Suppose that there exists a simple closed curve $\gamma \in \pi(K')^c$ which does not bound a disk in $\pi(K')^c$ (where $\pi(K')^c$ denotes the complement of the knot projection in $S_0$). Let $A_0$ be the annulus $\gamma \times I$. Let $A = H(A_0, 1)$ be the isotoped annulus. Then we may continue the isotopy, so that afterwards:

(i) The knot projection is $\pi(K)$ up to planar isotopy (i.e., there exists an isotopy of $S_0$ that takes the knot projection to $\pi(K)$).

(ii) The annulus is in its original position $A_0$.

Before giving the proof of this theorem, we first show how it can be applied to our main problem, the generalized Tait conjecture. Recall that we wish to show that if $S$ is a compact surface, $K$ a configuration of a knot in $S \times I$ with reduced alternating projection, $K'$ an arbitrary configuration of the knot, then

$$c(\pi(K)) \leq c(\pi(K')).$$

For the time being, we will assume that the surface is orientable and then we will finish off the argument for nonorientable surfaces by taking their double covers. Using Theorem 2.1, we can reduce this problem to a much simpler one.
Claim 2.2. To prove Theorem 1.1 in the orientable case, it suffices to show (3) in the case of a compact orientable surface $S$ possibly with boundary, where the regions in $\pi(K')^c$ are disks and disks with holes.

Proof. Consider a regular neighborhood $N$ of $\pi(K')$. The boundary of this neighborhood, $\partial N$ consists of a number of simple closed curves that lie in $\pi(K')^c$. Take the subset of these curves which do not bound disks in $\pi(K')^c$ and consider the vertical annuli (obtained by crossing each curve with $I$) corresponding to them. By Theorem 2.1 we can assume that after we perform the isotopy that takes $K$ to $K'$, these annuli will return to their original positions. We can interpret this as follows. First, cut the space $S \times I$ along these vertical annuli, and paste separate annuli onto each of the resulting boundary components. The result will be a number of separate spaces, each of the form $F \times I$ where $F$ is a compact surface with boundary. The knot $K'$ will lie in one of these spaces, $F_0 \times I$. The entire isotopy between $K$ and $K'$ will take place within a copy of $F_0 \times I$, although the actual position of $F_0 \times I$ within $S \times I$ will change over time.

The interpretation of the result from Theorem 2.1 is that the position of $F_0 \times I$ at the end of the isotopy is the same as at the beginning of the isotopy. By combining this result with an appropriate choice of identification between the continuously deforming $F_0 \times I$ and the original copy, one can see that the entire isotopy between $K$ and $K'$ can be performed within a fixed copy of $F_0 \times I$. In other words, we can think of $K, K'$ as being two configurations of a knot within a space $F_0 \times I$. By our construction, the projection $\pi(K')$ cuts the projection surface $F_0 \times \{1/2\}$ into disks and disks with holes. This means that, by thinking of the two configurations of the knot, $K, K'$, as lying in the space $F_0 \times I$, we only have to deal with the simpler case described above. \hfill \Box

Now that we have shown the importance of Theorem 2.1 in establishing the generalized Tait conjecture, we give the proof.

Proof of Theorem 2.1. We prove the theorem in the case that $\gamma$ is nontrivial on $S_0$. The case where $\gamma$ is trivial can be dealt with using the same arguments but with a few simplifications.

The first step is to isotope the knot $K$ into Menasco form (cf. [7], [8]). That is, we flatten $K$ onto the surface $S_0$, creating “bubbles” at the crossings. We arrange $A$ so that it meets these bubbles only in saddle shaped disks.

Denote the surface with the equatorial disks of the bubbles replaced by the upper(lower) hemispheres by $S^+_0 (S^-_0)$.

The proof can be subdivided into four parts:

1. We show that if $c$ is a curve of intersection of the annulus $A$ and $S^+_0$ that meets a saddle $s$ on both sides, then $c$ must be trivial on $S^+_0$. 

Moreover, the disk that $c$ bounds on $S_0^\pm$ cannot intersect the part of the bubble lying directly above (below) $s$.

2. We isotope $A$ so that it no longer meets any bubbles.

3. We isotope $A$ and $K$, removing all the intersection curves of $A \cap S_0$ except a single nontrivial curve.

4. We finish the isotopy so that the annulus returns to its original position $A_0$.

Before proceeding with the first step, we make the following observations:

(i) A curve in $A \cap S_0^\pm$ is nontrivial in $S_0^\pm$ if and only if it is nontrivial in $A$. This is equivalent to showing that the inclusion map $i_\ast : \pi_1(A) \to \pi_1(S \times I)$ is an injection. This follows immediately from the fact that $i_\ast$ sends the generator of the infinite cyclic group, $\pi_1(A)$, to the homotopy class of $\gamma$.

(ii) The simple closed nontrivial curves in $A \cap S_0^\pm$ are homotopic to either $\gamma$ or $\gamma^{-1}$. This follows from (i), and from the fact that simple closed nontrivial curves on an annulus have winding number $\pm 1$ around the center.

Now we proceed with (1).

1). We show that if $c$ is a curve of intersection of the annulus $A$ and $S_0^\pm$ that meets a saddle $s$ on both sides, then $c$ must be trivial. Moreover, the disk that $c$ bounds on $S_0^\pm$ cannot intersect the part of the bubble lying directly above (below) $s$.

Assume without loss of generality that $c$ lies on $S_0^+$. We will eliminate three different configurations for $c$. It will then follow that $c$ satisfies the required conditions.

Case 1. The curve $c$ runs between opposite corners of the saddle (see Figure 1).

Let $G$ be the graph consisting of a single point $p$ on $s$ together with four non-intersecting edges connecting $p$ to the four “corners” of $s$ (see Figure 2). Consider the graph $H = \partial s \cup C \cup G$. Then we see immediately that $H$ is
Figure 2. The graph $G$ on the saddle $s$.

Figure 3. $H$ is isomorphic to $K_5$.

Figure 4. Case 2.

isomorphic to $K_5$ (see Figure 3). But $H$ lies in the annulus, by construction. This is a contradiction since $K_5$ is a non-planar graph.

Case 2. The curve $c$ runs between adjacent vertices on the saddle (see Figure 4).
Let $c_1, c_2$ be the two arcs of $c$ that run between these vertices, and let $a_1, a_2$ be the arcs of $\partial s$ in $S_0^+$ that join these two pairs of vertices together.

**Case 2a.** One of the two curves, $c_1 \cup a_1, c_2 \cup a_2$ is trivial.

Assume without loss of generality that $c_1 \cup a_1$ is trivial. Observe that the linking number between $c_1 \cup a_1$ and the knot $K$ is $\pm 1$ (depending on which orientation we choose). To see this, note that $c_1$ lies on $S_0^+$ so that all the crossings between $K$ and $c_1$ are undercrossings (for $K$). Since they alternate in the orientation of $K$ and since there is an odd number of them, their total contribution to the linking number is $\pm 1$. The single overcrossing of $K$ with $A_1$ contributes $\pm 1$, giving a total of $(\pm 1 \pm 1)/2 = \pm 1$.

But $c_1 \cup a_1$ is trivial on $S_0^+$ and therefore trivial on the annulus $A$. We conclude that it bounds a disk $D \subset A$. $D$ doesn’t intersect $K$, so by the alternative definition of linking number, the linking number of $K$ and $c_1 \cup a_1$ is 0. This is a contradiction.

**Case 2b:** Both $c_1 \cup a_1$ and $c_2 \cup a_2$ are nontrivial and $c$ is nontrivial.

By Observation (ii), $c_1 \cup a_1, c_2 \cup a_2, c$ are all homotopic to $\gamma$ or $\gamma^{-1}$. But it is clear from the picture that

$$[c_1 \cup a_1] \cdot [c_2 \cup a_2] = [c]$$

(4)

(where $[\ ]$ denotes homotopy class). This is a contradiction, since it is impossible for

$$[\gamma^{\pm 1}] \cdot [\gamma^{\pm 1}] = [\gamma^{\pm 1}].$$

(5)

Combining Cases 1, 2a, 2b, we conclude that $c$ must be trivial, while $c_1 \cup a_1, c_2 \cup a_2$ must both be nontrivial. This means that the disk that $c$ bounds lies “outside” $c$. That is, it doesn’t intersect the portion of the bubble lying above (below) $s$. This completes Part (1).

**2). We isotope $A$ so that it no longer meets any bubbles.**

We first show that by isotopeing $A$, if necessary, we can always reduce the number of saddles that touch trivial intersection curves (of $A$ with $S_0^+$).

Suppose that some trivial curves do touch saddles. Assume without loss of generality that some of the curves lie in $S_0^+$. Choose an “innermost” trivial curve $j$. That is, choose $j$ such that $j = \partial D$ for a disk $D \subset S_0^+$, and such that $D$ doesn’t contain any trivial curves of $A \cap S_0^+$ that meet saddles.

It is easy to see that since $K$ is alternating it appears alternately on the left and right of $j$ as we traverse successive bubbles met by $j$. We may therefore choose a bubble and an arc $j_1$ of $j$ lying on the bubble, such that the knot is on the same side of $j_1$ as the disk $D$.

In general, the intersection between the bubble and $D$ will consist of some number of strips.
By drawing the arc $j_2$ that makes up the other half of the strip containing our original arc $j_1$, we get one of the following cases (see Figure 5).

a) The strip extends all the way across the bubble.
b) The arcs $j_1, j_2$ lie on opposite sides of $K$.
c) The arcs $j_1, j_2$ lie on the same side of $K$.

To see (a) is impossible, note that the saddle containing $j_1$ must meet another curve on the opposite side of the knot, and in Case (a) this curve is contained in $D$. This is a contradiction since we choose $j$ to be the innermost such curve.

To see that (b) is impossible we first note that $j_1, j_2$ must belong to the same saddle. This follows from the same reasoning as in (a). Next, we apply Part (1). By Part (1), the only possible configuration for $j$ is where $D$ does not meet the portion of the bubble lying directly above the saddle. This is a contradiction.

We are left with (c). We consider the disk $D$ and the appearance of the strip in $D$ (see Figures 6, 7).

Let $a$ be the arc joining $u$ to $v$, along the bubble. Let $l$ be the arc of $c$ joining $x$ and $y$. Then, by examining the strip on the disk $D$, we see that $uxyv = j_1 \cup l \cup j_2 \cup a$ bounds a disk $D'$. Note that $D'$ contains the strip and that it doesn’t meet $A \cap S_0^+$ except at its ($D'$'s) boundary and possibly at trivial curves contained in its interior.

We now show that by isotoping $A$, we can remove the saddles touching $j_1$ and $j_2$. The argument is taken from Adams [1].

The isotopy is accomplished in two steps. Consider the part of the annulus lying directly above the curve $j_1 \cup l \cup j_2$, and between $S \times \{ \frac{1}{2} + \epsilon \}$ and $S \times \{ \frac{1}{2} + 2\epsilon \}$. Take this portion of the annulus and push it horizontally towards the arc $a$, keeping the rest of the annulus fixed. Continue pushing until the annulus is just beyond the arc $a$. At this point, the annulus will still be vertical between $S_0$ and $S \times \{ \frac{1}{2} + \epsilon \}$, and vertical above $S \times \{ \frac{1}{2} + 2\epsilon \}$.
However, it will lie essentially horizontally, just above $S \times \{ \frac{1}{2} + \epsilon \}$ and just below $S \times \{ \frac{1}{2} + 2\epsilon \}$. In other words, the annulus will form a mouth that lies directly above the disk $D'$ with a “roof” at height $\frac{1}{2} + 2\epsilon$, and a “bottom” at height $\frac{1}{2} + \epsilon$. The “back” of the mouth will be vertical and will lie just beyond the arc $a$ (see Figure 8).

In the process of creating this mouth, one may encounter other pieces of the annulus that lie above $D'$. These pieces will necessarily be parts of “tubes” that lie above intersection curves in $D'$. As we push the annulus beyond $a$, we can push these tubes along with us. The end result will be that the tubes will be vertical below $S \times \{ \frac{1}{2} + \epsilon \}$ and above $S \times \{ \frac{1}{2} + 2\epsilon \}$, but they will make a long detour around the mouth in the intervening region.
Now that we have created the mouth, we can proceed with the second part of the isotopy. Take the portion of the bottom of the mouth that lies directly above the strip $uxyv$ and pull it under the knot, through the bubble, so that it lies on the opposite side of the bubble. If there are tubes, we also pull through the part of the tubes lying above $uxyv$.

The end result is that the saddles touching $j_1$ and $j_2$ no longer exist. Furthermore, no new saddles have been created. We have therefore reduced the number of saddles that touch trivial curves by at least two.

We have established that we can always reduce the number of saddles that touch trivial intersection curves. Hence, we may assume that no trivial intersection curves meet saddles on either $S^+_0$ or $S^-_0$. Note that this, together with Part (1), implies that no curves of any kind can touch the same saddle on opposite sides. It turns out that these two facts are enough to show that the intersection curves of $A \cap S^\pm_0$ do not meet any saddles.

Indeed, suppose that the set of intersection curves that touch saddles is nonempty. We know that these intersection curves are nontrivial, so by Observation (ii) they must have homotopy type $\gamma^\pm1$. Now consider the curves of $A \cap S^+_0$ that intersect saddles. If we drew them on $A$ they would appear as in Figure 9.

Take the “outermost” $S^+_0$ intersection curve $c$ that touches a saddle. (To define outermost rigorously, we embed the annulus $A$ in a disk $D$ such that
$\partial D = A \cap S \times \{1\}$. We say that $c$ is outermost if all the other curves are contained in the interior of the disk it bounds in $D$.)

The curve $c$ touches some saddle $s$ (see Figure 10). Let $c_1, c_2$ be the two intersection curves on $S_0$ obtained by “switching” $s$ (see Figure 11), or rather by viewing the intersection curves from below $S_0$ rather than above. Note that $c_1, c_2$ must be distinct curves since no curve touches the same saddle twice.

Now since $c$ is outermost, there are line segments on the disk $D$ from $c_1$ to $\partial D$ and from $c_2$ to $\partial D$ that do not cross $c_2$ or $c_1$ respectively. Hence the disk on $D$ bounded by $c_1$ does not contain $c_2$ and the disk on $D$ bounded by
Figure 11. The appearance of $c_1$ and $c_2$ on the annulus.

c_2$ does not contain $c_1$. This contradicts the fact these are disjoint nontrivial curves on the annulus.

Thus, no intersection curves touch saddles. This completes Part (2).

3). We remove all but a single nontrivial intersection curve.

First we remove all the trivial intersection curves. We accomplish this one curve at a time. Let $c$ be an innermost intersection curve on $S_0$. Let $D_1$ be the disk that $c$ bounds on $S_0$, and $D_2$ the disk $c$ bounds on $A$.

We isotope $D_2$ onto $D_1$. We then pull $D_2$ through the surface $S_0$, eliminating the trivial curve $c$. If necessary we pull the knot projection along, too, without changing its combinatorial structure (that is, without changing the knot projection up to planar isotopy). By this we mean that if the knot is in the way of the isotopy then the knot projection lives entirely in the disk $D_1$. We can assume that it lies in a disk $D_1'$ which is contained in $D_1$ and which is a distance $\epsilon$ from the boundary of $D_1$. We may now let $B$ be the $\frac{\epsilon}{2}$-neighborhood of $D_1$. This contains the knot $K$. As we isotope $D_2$ through $D_1$, we pull the ball $B$ along, all the while keeping the knot frozen within it. After we have removed this intersection curve of $A$ with $S_0$, we continue the isotopy to move this ball back down to $S_0$ until the disk $D_1'$ again sits on $S_0$. The knot has now been returned to $S_0$ with the same combinatorial projection it had before. We repeat this process until there are no more trivial curves.

We will now be left with a number of parallel nontrivial curves of intersection on $A$. Note that they are also parallel on $S_0$ since the annulus on $A$ that any two of them bound can be homotoped into $S_0$ by collapsing out the $I$ in $S \times I$. We eliminate these curves in pairs, using the same technique as above. Let $c_1$, $c_2$ be adjacent parallel nontrivial curves on $A$. Let $M$ be the annulus they bound on $A$ and $N$ the annulus they bound on $S_0$. Since both
annuli live in $S \times I$, and share boundary on $S_0$, $M$ can be isotoped onto $N$ and then pulled through $S_0$, eliminating a pair of nontrivial curves. Again, we may have to push the knot along during the isotopy but in that case, the knot projection was contained entirely in $N$. In fact, we may assume that the knot projection lies entirely in an annulus $N'$ which is contained in $N$ and which is a distance $\epsilon$ from the boundary of $N$. Then if $V$ is the solid torus $\frac{\epsilon}{2}$-neighborhood of $N'$, it contains the knot $K$. As we isotope $M$ through $N$, we pull the solid torus $V$ along, keeping the knot frozen within it. After removing the two intersection curves of $A$ with $S_0$, we continue the isotopy to move $V$ back down to $S_0$ until the annulus $N'$ again sits on $S_0$. Notice that to do so, we can slide $V$ along the annulus until the annulus again intersects $S_0$, and then set $N'$ down on $S_0$. Since all of the intersections of $A$ with $S_0$ are parallel on $S_0$, the resulting projection is isotopic on $S_0$ to the original projection of $K$. Repeating this process, we can remove all but a single nontrivial curve (initially there must be an odd number of nontrivial curves).

This completes (3).

4). We return the annulus to its initial position, $A_0$.

To prove (4), consider an isotopy $H : S_0 \times I \to S_0$ that takes $A \cap S_0$ back to $A_0 \cap S_0 = \gamma$. We know that such an isotopy exists, since $A \cap S_0$ consists of a simple closed curve homotopic to $\gamma$. Extend $H$ to an isotopy of the full space $S \times I$ by requiring that $H$ preserve the product structure of the space. This isotopy will take $A \cap S_0$ to $\gamma$ and it will preserve the combinatorial aspect of the knot projection. Next, flatten the knot onto the surface $S_0$ and straighten the portions of the annulus lying above and below the projection surface $S_0$ so that they are vertical. The result is that the annulus will be in its original vertical position, $\gamma \times I$. Moreover, at all times we have preserved the combinatorial aspect of the knot projection, so the projection will differ from $\pi(K)$ only by planar isotopy.

This completes (4) and proves Theorem 2.1. □

3. The polynomial argument.

We will now define a set of polynomials which we will use to prove the special case of the theorem. These polynomials generalize Kauffman’s bracket polynomial.

Because of the existence of the projection $\pi$, the equivalence of knots in $S \times I$ is the same as the equivalence of their diagrams by Reidemeister moves [4]. We may therefore define polynomials for such knots and links by
the skein relation

$$\langle \chi \rangle = A \langle \chi \rangle + A^{-1} \langle \chi \rangle$$

as in the plane. In the above equation, we call the first splitting of the link at the crossing an A-split and the second splitting a B-split.

The main difference between planar projections and projections to a surface is that on the surface, the curves to which the link is reduced (that is, the curves with no crossings) can have different isotopy types, and these are preserved by Reidemeister moves. This means that, in the expansion of the knot in terms of knots without crossings, the coefficient of each isotopy class is preserved separately, producing a family of polynomials \([3]\). Related polynomials also appear in Kamada’s proof [4].

Our precise definition of the polynomials is in terms of states. For a surface \(S\), define \(F(S)\) to be the set of families of non-intersecting nontrivial simple closed curves on \(S\) up to isotopy. Thus if \(S\) is a torus, \(F(S)\) is in one-to-one correspondence with the pairs \((p, q)\) of integers. If \(d = \gcd(p, q)\) then the family of curves corresponding to \((p, q)\) consists of \(d\) non-intersecting \((\frac{p}{d}, \frac{q}{d})\) torus knots. For each element of \(F(S)\) there will be a polynomial.

A state \(s\) of a knot (or link) \(K\) is a splitting of the knot at each crossing; such a state consists of non-intersecting curves. We make several definitions:

\[
\begin{align*}
N(s) &= \{\text{nontrivial curves of } s \} \in F \\
a(s) &= \text{number of A-splittings} \\
b(s) &= \text{number of B-splittings} \\
t(s) &= \text{number of trivial components of } s \\
|s| &= \text{number of components of } s \\
p(s) &= \text{number of components of } s \text{ which bound a disk or disk with holes on } S, \text{ whose other boundaries lie in } \partial S. \text{ (The distinction between different types of curves will be useful in the application of polynomials to knots on general surfaces.)}
\end{align*}
\]

For each \(F \in F\), let

\[
Q_F(K) = \sum_{s \in \{s | N(s) = F\}} A^{a(s) - b(s)}(-A^2 - A^{-2})^{t(s) - 1}.
\]

By redefining this recursively, we see (as discussed above) that all \(Q_F\)'s are invariant under Reidemeister moves of Types II and III, and that all are multiplied by the same power of \(A\) when a Type I move is applied.

The \(Q_F\)'s are the most general invariant polynomials of this type. However, for our purposes we specialize slightly. \(|s| - t(s)\) is the number of nontrivial curves in \(s\); that is, it is the number of curves in \(F = N(s)\).
Therefore, when we multiply $Q_F$ by $(-A^2 - A^{-2})^{|F|}$ we obtain the polynomial

$$P_F(K) = \sum_{s \in \{ s | N(s) = F \}} A^{a(s) - b(s)}(-A^2 - A^{-2})^{|s| - 1}. \quad (6)$$

This set of polynomials is invariant under Type II and III Reidemeister moves, and all are multiplied by the same factor when a Type I Reidemeister move is applied.

We now apply these polynomials to prove the following modification of our theorem, to which the original theorem reduces:

**Theorem 3.1.** Let $S$ be an orientable surface possibly with boundary. Let $K$ and $K'$ be equivalent knots in $S \times I$, with projections $\pi(K)$ and $\pi(K')$ such that:

1. $\pi(K)$ is alternating, reduced, and has $c$ crossings.
2. The complement of $\pi(K')$ consists of disks, possibly with holes. Only one boundary component of each disk with holes is on $\pi(K')$. The other components are boundary components of $S \times \{ \frac{1}{2} \}$.

Then $c(\pi(K)) \leq c(\pi(K'))$.

The proof closely parallels Kauffman’s proof of the original Tait conjecture [6]. Let us begin with a lemma on the result of splitting the projection $\pi(K')$ in the A and B directions simultaneously. This is analogous to Kauffman’s Lemma 2.11. Our proof is different, however. It does not use induction.

**Lemma 3.2.** Let $K'$ be a knot in a projection $\pi(K')$ (or a link with a connected diagram). Let $s'_A$ be the all-A split and $s'_B$ be the all-B split (see Figure 12). Then $p(s'_A) + p(s'_B) \leq R'$, where $R'$ is the number of regions in $\pi(K')^c$.

**Proof.** Consider two vector spaces of formal sums (modulo 2) of the edges and regions of the graph formed by $\pi(K')$ in $S \times \{ \frac{1}{2} \}$.

- $C_1 = \text{vector space over } \mathbb{Z}_2 \text{ generated by the edges of the projection } \pi(K')$.
- $C_2 = \text{vector space over } \mathbb{Z}_2 \text{ generated by the regions of } \pi(K')$.

We define a linear mapping $\delta : C_2 \rightarrow C_1$ as follows. Define $\delta(r)$ to be the formal sum of the edges of $r$, for an $r$ which consists of a single region of $\pi(K')^c$. Then, define $\delta(r)$ on the full space $C_2$ by extending linearly.

Note that the curves of $s'_A$ and of $s'_B$ can be thought of as elements of $C_1$. Indeed, each curve may be associated to the formal sum of edges along which the curve passes. A curve will form the boundary of a piece of surface precisely when the corresponding element of $C_1$ lies in $\delta(C_2)$. 
Figure 12. The knot $K$ together with the $A$ and $B$ curves, $s'_A$ and $s'_B$.

In particular, the curves of $s'_A$ and $s'_B$ which bound punctured disks on the surface span a vector subspace $V$ of $C_1$ which is entirely contained in $\delta(C_2)$. We conclude

$$\dim V \leq \dim \delta(C_2).$$

Note that:

$$\dim \delta(C_2) = R' - 1$$

since the kernel of $\delta$ is 1-dimensional (it includes only the sum of no regions and the sum of all regions).

We now find a lower bound to the dimension of $V$. Consider a relation between the curves spanning $V$, that is consider a family of curves from $s'_A$ and $s'_B$ which bound punctured disks on the surface and which, as elements of $C_1$, sum to zero. Since summation is modulo 2, each edge of the projection is passed over an even number of times by curves of the family. But then either all of the curves or none of the curves at each vertex must belong to the family, since otherwise one of the edges at the vertex would have only one curve from the family along it.

But if all the curves at some vertex belong to the family then, since these curves also pass through the neighboring vertices, all the curves at the neighboring vertices must belong to the family as well. Repeating this argument, since $\pi(K')$ is connected, either all of the curves or none of the curves from $s'_A$ and $s'_B$ must belong to the family. This shows that there is only one nontrivial relation between the curves of $s'_A$ and $s'_B$, and so there is certainly no more than one relationship between the curves generating $V$, which are restricted to those bounding disks with holes. Thus

$$\dim V \geq p(s'_A) + p(s'_B) - 1.$$
This, with Equations (7) and (8) shows that \( p(s'_A) + p(s'_B) \leq R' \). □

We now give the proof of Theorem 3.1.

Proof. We must first define a notion of span, as in Kauffman’s proof. This notion, however, is more technical, and depends on our projections. It is constructed particularly for this proof.

We use the projection of \( K \) to fix two polynomials. Let \( s_A \) be the all-A split of \( \pi(K) \) and \( s_B \) be the all-B split of \( \pi(K) \). Let \( F_A = \mathcal{N}(s_A) \) and \( F_B = \mathcal{N}(s_B) \). We now focus on the fixed polynomials \( P_{FA} \) and \( P_{FB} \).

Let \( \max(P) \) be the highest degree of any term of \( P \) and \( \min(P) \) the lowest. Notice that \( \max(P_{FA}(K)) - \min(P_{FB}(K)) = \max(P_{FA}(K')) - \min(P_{FB}(K')) \), since we are considering the same pair of polynomials in either case. Let \( c = c(\pi(K)) \) and \( c' = c(\pi(K')) \).

We prove the following two inequalities:

(i) \( \max(P_{FA}(K)) - \min(P_{FB}(K)) \geq 4c - 4g + 2N; \)
(ii) \( \max(P_{FA}(K')) - \min(P_{FB}(K')) \leq 4c' - 4g + 2N. \)

Here \( g \) is the genus of \( S \), and \( N \) is the number of curves in \( F_A \) and \( F_B \) which do not bound disks with holes. Note that once these inequalities have been proved, they together imply \( c \leq c' \), which will finish the proof.

Proof of (i). The proof of Statement (i) is in two parts. First we show that \( \max(P_{FA}(K)) \) is the degree of a term from the all-A split, and similarly for B, and then we calculate these degrees.

The highest degree contributed by a certain state is \( a(s) - b(s) + 2(|s| - 1) \) (provided \( \mathcal{N}(s) = F_A \)). Let us start with the state \( s_A \) and change to the state \( s \) by switching one A-crossing at a time to a B-crossing. We must show that all states which do contribute to \( P_{FA} \) contribute a strictly lower exponent than \( s_A \).

Every time an A-split is switched to a B-split, \( a(s) - b(s) \) decreases by two. \(|s|\) cannot increase by more than 1, and so the exponent \( a(s) - b(s) + 2(|s| - 1) \) cannot increase. Now, suppose that \( s \) contributes a term which cancels with the term from \( s_A \). Then the term from \( s \) must have the same degree and belong to the same polynomial as the term contributed by \( s_A \). Thus \(|s|\) must increase by one each time, and \( \mathcal{N}(s) \) must equal \( \mathcal{N}(s_A) \). The possibilities when the split of a given crossing is switched may be enumerated as follows, since some curve must split into two at each stage.

1) A trivial curve splits into two trivial curves.
2) A trivial curve splits into two nontrivial curves.
3) A nontrivial curve splits into two nontrivial curves.
4) A nontrivial curve splits into a nontrivial curve and a trivial curve.
Neither (1) nor (4) can occur at the first stage since $K$ is reduced (see Figure 13). If (2) or (3) occurs at the first stage the number of nontrivial curves increases and none of (1)–(4) occurring at a later stage can reduce this number to its original value, so it is impossible for $N(s) = N(s_A)$. So it is impossible for a cancellation to occur after all.

Thus $a(s) - b(s) + 2(|s| - 1)$ for $s = s_A$ is strictly the maximum exponent appearing in $P_{FA}$. The smallest exponent appearing in $P_{FB}$ is found similarly. Therefore,

$$\max P_{FA} - \min P_{FB} = c + 2(|s_A| - 1) - (-c - 2(|s_B| - 1))$$

$$= 2c + 2(|s_A| + |s_B|) - 4.$$

By the definition of $N$, this can be written as

$$\max P_{FA}(K) - \min P_{FB}(K) = 2c + 2N + 2(p(s_A) + p(s_B)) - 4.$$

(10)

If we let $N_1$ be the number of disk with holes components of $\pi(K)^c$ which have only one boundary component formed by the knot, then we may follow Kauffman, noting that since the knot is alternating, the boundaries of such regions become curves of $s_A$ and $s_B$. By the definition of $p$ we therefore obtain

$$N_1 \leq p(s_A) + p(s_B).$$

(11)

Now we use the Euler characteristic to relate $N_1$ to the crossing number of $\pi(K)$.

Let $W = \text{number of components of } \partial S$.

$W(r) = \text{number of boundary components of a region } r \text{ of } \pi(K)^c$, which are not formed by $\pi(K)$.
ALTERNATING KNOTS IN $S \times I$

$g(r)$ = the genus of region $r$ and let $\chi$ be the Euler characteristic. Euler’s formula generalized to the case where the regions are not necessarily disks gives

$$-c + \sum_r \chi(r) = \chi(S).$$

(12)

Also, $\chi(S) = 2 - 2g - W$ and $\sum_r W(r) = W$, so

$$-c + \sum_r (\chi(r) + W(r)) = 2 - 2g.$$

(13)

Now $\chi(r) + W(r) \leq 0$ unless $r$ has only one boundary formed by the knot and has genus zero. There are $N_1$ such regions and for each, $\chi(r) + W(r) = 1$, so by Equation (13)

$$-c + N_1 \geq 2 - 2g.$$

(14)

Combining this last equation with Equation (11) we see that

$$p(s_A) + p(s_B) \geq c + 2 - 2g.$$

(15)

From Equation (10) we now see that

$$\max(P_{F_A}(K)) - \min(P_{F_B}(K)) \geq 4c + 2N - 4g.$$

Proof of (ii). We now continue with the proof of Statement (ii), which concerns the non-alternating version of the knot, $K'$.

By lemma (3.2), $p(s'_A) + p(s'_B) \leq R'$. Now it follows by induction that for an arbitrary pair of states $s'_1, s'_2$,

$$p(s'_1) + p(s'_2) \leq R' + b_1 + a_2.$$

(17)

In fact, this follows by switching A-splits to B-splits as in (i) to turn $s'_A$ into $s'_1$ and $s'_B$ into $s'_2$. (For the argument, notice that if a curve does not bound a disk with holes, then it cannot split into curves which do bound disks with holes.)

Now apply this inequality to a pair of states $s'_1$ and $s'_2$ which are assumed to contribute to $P_{F_A}(K')$ and to $P_{F_B}(K')$, respectively. The difference between the exponents they contribute is:

$$a_1 - a_2 + b_2 - b_1 + 2(|s'_1| + |s'_2|) - 4$$

$$= a_1 - a_2 + b_2 - b_1 + 2(p(s'_1) + p(s'_2)) - 4 + 2(|s'_1| + |s'_2| - p(s_1) - p(s'_2))$$

$$= a_1 - a_2 + b_2 - b_1 + 2(p(s'_1) + p(s'_2)) - 4 + 2N$$

$$= 2c' - 2a_2 - 2b_1 + 2p(s'_1) + 2p(s'_2) - 4 + 2N$$

$$\leq 2c' + 2R' - 4 + 2N.$$

The second equality follows since the nontrivial curves in $s'_1$ and $s'_2$ are just the curves $F_a$ and $F_b$, and $N$ of these curves do not bound punctured disks. The inequality follows from Equation (17).
By Euler’s formula, and the assumption that all regions of $K'$ are genus zero, $R' = c' + 2 - 2g$, so the difference between an exponent of $P_{FA}(K')$ and one of $P_{FB}(K')$ is at most $4c' + 2N - 2g$, proving (ii).

The inequalities (i) and (ii) together imply Theorem 3.1, since the quantities on the left side of the inequalities are the same by the invariance properties of the polynomials.

We now restate Theorem 1.1:

**Theorem 1.1.** Let $S$ be a compact surface. Let $\pi(K)$ be a reduced alternating projection of an alternating knot in $S \times I$ and let $\pi(K')$ be an arbitrary projection (of the same knot). Then

\[
(18) \quad c(\pi(K)) \leq c(\pi(K')).
\]

**Proof.** On account of Claim 2.2, Theorem 3.1 which we have just proved implies this more general theorem in the orientable case. The nonorientable case follows immediately by taking double covers. □

### 4. Conclusion.

The statement of Theorem 1.1 can be strengthened. It is unnecessary to restrict the theorem to knots. Indeed, the proof works equally well for non-splittable links, and with a few modifications it extends to links in general.

The other possible extensions of Theorem 1.1 are more difficult. We have shown that if $\pi(K)$ is a reduced alternating projection of a knot in $S \times I$, and $\pi(K')$ is any other projection of that knot, then

\[
(19) \quad c(\pi(K)) \leq c(\pi(K')).
\]

It remains to be shown that

\[
(20) \quad c(\pi(K)) < c(\pi(K')),
\]

if $\pi(K)$ is non-alternating, and the knot is prime (Murasagi and Thistlethwaite established this strict inequality in $S^3$, [10], [11]).

Also, Tait conjectured that any two reduced alternating projections of the same knot can be converted to one another through a series of special moves called flypes. This statement was proved for knots in $S^3$ by Thistlethwaite and Menasco [9]. A natural extension of Theorem 1.1 would be to prove the flyping conjecture for knots in $S \times I$.

It is natural to ask questions about knots in more complicated spaces which contain subspaces of the form $S \times I$.

One possibility is to investigate the Tait conjecture for knots in (solid) handlebodies, where we project knots onto the boundary. If our definition of reduced is used, then the case in Figure 14 is possible. The knot projection
on the left is reduced and alternating, but the number of crossings can be lowered by pulling a crossing through the center of the solid torus, and then using a Type II Reidemeister move. The resulting knot with fewer crossings is shown on the right.

Thus, the conjecture doesn’t hold with our notion of reduced. However, it may hold if we use Kamada’s notion of reduced-properly reduced.

Alternatively, one could ask whether the Tait conjecture holds for knots lying on an incompressible surface $S$ in a 3-manifold $M$. We conjecture that if two projections of a knot are equivalent in the 3-manifold, then they are equivalent up to a homeomorphism of $S \times I$. If true, this would reduce the problem to the result proved here.

References


Received March 17, 2000 and revised April 30, 2001.

DEPARTMENT OF MATHEMATICS
WILLIAMS COLLEGE
WILLIAMSTOWN, MA 01267
E-mail address: colin.adams@williams.edu

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF CALIFORNIA
LA JOLLA, CA 92093

DEPARTMENT OF PHYSICS
MASSACHUSETTS INSTITUTE OF TECHNOLOGY
CAMBRIDGE, MA 02139

DEPARTMENT OF PHYSICS
HARVARD UNIVERSITY
CAMBRIDGE, MA 02138
CHOIX DES SIGNES POUR LA FORMALITÉ DE M. KONTSEVICH

D. ARNAL, D. MANCHON, AND M. MASMoudi

The existence of star products on any Poisson manifold $M$ is a consequence of Kontsevich’s formality theorem, the proof of which is based on an explicit formula giving a formality quasi-isomorphism in the flat case $M = \mathbb{R}^d$. We propose here a coherent choice of orientations and signs in order to carry on Kontsevich’s proof in the $\mathbb{R}^d$ case, i.e., prove that Kontsevich’s formality quasi-isomorphism verifies indeed the formality equation with all the signs precised.

Introduction.

La conjecture de formalité a été introduite par M. Kontsevich [K2]: elle affirme l’existence d’un $L_\infty$-quasi-isomorphisme de $g_1$ vers $g_2$, où $g_1$ et $g_2$ sont les deux algèbres de Lie différentielles graduées naturellement associées à une variété $M$: précisément $g_1$ est l’algèbre de Lie différentielle graduée des multi-champs de vecteurs munie de la différentielle nulle et du crochet de Schouten, et $g_2$ est l’algèbre de Lie différentielle graduée des opérateurs polydifférentiels munie de la différentielle de Hochschild et du crochet de Gerstenhaber.

Les éléments de degré $n$ dans $g_1$ sont les $(n+1)$-champs de vecteurs, et les éléments de degré $n$ dans $g_2$ sont les opérateurs $(n+1)$-différentiels. Dans les espaces gradués décalés $g_1[1]$ et $g_2[1]$ ce sont les $(n+2)$-champs de vecteurs (resp. les opérateurs $(n+2)$-différentiels) qui sont de degré $n$.

Toute algèbre de Lie différentielle graduée est une $L_\infty$-algèbre. Cela signifie en particulier que les structures d’algèbres de Lie différentielles graduées sur $g_1$ et $g_2$ induisent des codérivations $Q$ et $Q'$ de degré 1 sur des cogèbres $C(g_1) = S^+(g_1[1])$ et $C(g_2) = S^+(g_2[1])$ respectivement (cf. §II.3 et II.4), vérifiant toutes deux l’équation maitresse:

$$[Q,Q] = 0, \quad [Q',Q'] = 0.$$ 

Un $L_\infty$-quasi-isomorphisme de $g_1$ vers $g_2$ est par définition un morphisme de cogèbres:

$$U : C(g_1) \longrightarrow C(g_2)$$
de degré zéro et commutant aux codérivations, c'est-à-dire vérifiant l'équation:

\[ U \circ Q = Q' \circ U, \]

et dont la restriction à \( g_1 \) est un quasi-isomorphisme de complexes de \( g_1 \) dans \( g_2 \). M. Kontsevich démontre dans [K1] la conjecture de formalité, c'est-à-dire l'existence d'un \( L_\infty \)-quasi-isomorphisme de \( g_1 \) vers \( g_2 \), pour toute variété \( M \) de classe \( C^\infty \). La première étape de la preuve (et même l'essentiel du travail) consiste en la construction explicite du \( L_\infty \)-quasi-isomorphisme \( U \) pour \( M = \mathbb{R}^d \). Le \( L_\infty \)-quasi-isomorphisme \( U \) est uniquement déterminé par ses coefficients de Taylor:

\[ U : S^n(g_1[1]) \longrightarrow g_2[1] \]

(cf. §III.2). Si les \( \alpha_k \) sont des \( s_k \)-champs de vecteurs, ils sont de degré \( s_k - 2 \) dans l'espace décalé \( g_1[1] \), et donc \( U_n(\alpha_1 \cdots \alpha_n) \) est d'ordre \( s_1 + \cdots + s_n - 2n \) dans \( g_2[1] \). C'est donc un opérateur \( m \)-différentiel, avec:

\[ \sum_{k=1}^{n} s_k = 2n + m - 2. \]

Les coefficients de Taylor sont construits à l'aide de poids et de graphes: on désigne par \( G_{n,m} \) l'ensemble des graphes étiquetés et orientés ayant \( n \) sommets du premier type (sommets aériens) et \( m \) sommets du deuxième type (sommets terrestres) tels que:

1. Les arêtes partent toutes des sommets aériens.
2. Le but d'une arête est différent de sa source (il n'y a pas de boucles).
3. Il n'y a pas d'arêtes multiples.

A tout graphe \( \Gamma \in G_{n,m} \) muni d'un ordre sur l'ensemble de ses arêtes, et à tout \( n \)-uple de multi-champs de vecteurs \( \alpha_1, \ldots, \alpha_n \) on peut associer de manière naturelle un opérateur \( m \)-différentiel \( B_\Gamma(\alpha_1 \otimes \cdots \otimes \alpha_n) \) lorsque pour tout \( j \in \{1, \ldots, n\} \), \( \alpha_j \) est un \( s_j \)-champ de vecteurs, où \( s_j \) désigne le nombre d'arêtes qui partent du sommet aérien numéro \( j \) [K1, §6.3].

Le coefficient de Taylor \( U_n \) est alors donné par la formule:

\[ U_n(\alpha_1 \cdots \alpha_n) = \sum_{\Gamma \in G_{n,m}} W_\Gamma B_\Gamma(\alpha_1 \otimes \cdots \otimes \alpha_n), \]

où l'entier \( m \) est relié à \( n \) et aux \( \alpha_j \) par la formule (*) ci-dessus.

Le poids \( W_\Gamma \) est nul sauf si le nombre d'arêtes \( |E_\Gamma| \) du graphe \( \Gamma \) est précisément égal à \( 2n + m - 2 \). Il s'obtient en intégrant une forme fermée \( \omega_\Gamma \) de degré \( |E_\Gamma| \) sur une composante connexe de la compactification de Fulton-McPherson d'un espace de configuration \( C^+_A,B \), qui est précisément de dimension \( 2n + m - 2 \) [FM], [K1, §5]. Il dépend lui aussi d'un ordre sur l'ensemble des arêtes, mais le produit \( W_\Gamma B_\Gamma \) n'en dépend plus.
Pour prouver le théorème de formalité M. Kontsevich montre que le morphisme de cogèbres $U$ dont les coefficients de Taylor sont les $U_n$ définis ci-dessus est un $L_\infty$-quasi-isomorphisme. La méthode consiste à ramener l’équation de formalité $U \circ Q = Q' \circ U$, qui se développe à l’aide des coefficients de Taylor de $U$, $Q$ et $Q'$:

$$QU_n(\alpha_1, \ldots, \alpha_n) + \frac{1}{2} \sum_{I,J=\{1, \ldots, n\} \setminus \emptyset} \pm Q'_{\emptyset}(U_{|I|}(\alpha_I) \cup U_{|J|}(\alpha_J))$$

$$= \sum_{k=1}^{n} \pm U_n(Q_k(\alpha_k) \cdot \alpha_1 \cdots \hat{\alpha_k} \cdots \alpha_n)$$

$$+ \frac{1}{2} \sum_{k \neq l} \pm U_{n-1}(Q_k(\alpha_k, \alpha_l) \cdot \alpha_1 \cdots \hat{\alpha_k} \cdots \hat{\alpha_l} \cdots \alpha_n)$$

à l’application de la formule de Stokes pour les formes $\omega_1$ sur l’ensemble des faces de codimension 1 du bord des espaces de configuration.

Nous proposons dans la première partie de ce travail un choix d’orientation des espaces de configuration (ou plus exactement d’une composante connexe de ceux-ci) $C_{A,B}^+$, et un choix cohérent d’orientation pour chacune des faces de codimension 1 du bord de la compactification.

Au Chapitre II nous explicitons l’isomorphisme $\Phi : S^n(g[1]) \xrightarrow{\sim} \Lambda^n(g)[n]$ mentionné dans [K1, §4.2] pour tout espace vectoriel gradué $g$, afin de préciser le passage du langage des algèbres de Lie différentielles graduées et des $L_\infty$-algèbres au langage des $Q$-variétés formelles graduées pointées ([AKSZ], [K1, §4.1]).

Nous donnons au Chapitre III une formule explicite pour un champ de vecteurs sur une variété formelle graduée pointée ou un morphisme de variétés formelles graduées pointées en fonction de leurs coefficients de Taylor respectifs. La démonstration est de nature combinatoire et se fait en explicitant la restriction à la puissance symétrique $n$-ième par récurrence sur $n$.

Dans le Chapitre IV nous exprimons les deux algèbres de Lie différentielles graduées qui nous intéressent comme $Q$-variétés formelles graduées pointées. L’isomorphisme d’espaces gradués $\Phi$ explicité au Chapitre II est ici essentiel. Pour la suite nous sommes amenés à modifier l’algèbre de Lie différentielle graduée des multi-champs de vecteurs: nous utilisons un crochet de Lie gradué $[ , ]'$ lié au crochet de Schouten par la formule:

$$[x, y]' = -[y, x]_{\text{Schouten}}.$$

Les deux crochets coïncident modulo un changement de signe en présence de deux éléments impairs. Nous précisons au paragraphe IV.4 les signes (du type Quillen) qui apparaissent dans l’équation de formalité.
Enfin nous montrons au Chapitre VI que modulo tous les choix effectués précédemment $U$ est bien un $L_\infty$-morphisme.

Le Chapitre V est assez largement indépendant du reste de l'article bien que directement relié à [K1]: nous y donnons une démonstration détaillée du théorème de quasi-inversion des quasi-isomorphismes donné dans [K1, §4.4-4.5]. Enfin nous rappelons en appendice le lien entre formalité et quantification par déformation.


I. Orientation des espaces de configuration.

I.1. Trois choix de paramétrage.

Définition (Espaces de configuration). Soit $\mathcal{H}$ le demi-plan de Poincaré ($\mathcal{H} = \{z \in \mathbb{C}, \Im z > 0\}$). Appelons $\text{Conf}^+ (\{z_1, \ldots, z_n\}; \{t_1, \ldots, t_m\})$ l’ensemble des nuages de points:

$$\left\{ (z_1, \ldots, z_n; t_1, \ldots, t_m), \text{ t. q. } z_i \in \mathcal{H}, t_j \in \mathbb{R}, \right. $$

$$ \left. z_i \neq z_{i'} \text{ si } i \neq i', t_1 < \ldots < t_m \right\}$$

et $C^+_{\{p_1, \ldots, p_n\};\{q_1, \ldots, q_m\}}$ le quotient de cette variété sous l’action du groupe $G$ de toutes les transformations de la forme:

$$z_i \mapsto az_i + b, t_j \mapsto at_j + b \quad (a > 0, b \in \mathbb{R}).$$

La variété $C^+_{\{p_1, \ldots, p_n\};\{q_1, \ldots, q_m\}}$ est donc de dimension $2n + m - 2$. Cette variété est par convention orientée par le passage au quotient de la forme:

$$\Omega_{\{z_1, \ldots, z_n\};\{t_1, \ldots, t_m\}} = dx_1 \wedge dy_1 \wedge \ldots \wedge dx_n \wedge dy_n \wedge dt_1 \wedge \ldots \wedge dt_m$$

où $z_j = x_j + iy_j$. Le groupe des transformations considérées préserve l’orientation. On en déduit une orientation des espaces $C^+_{\{p_1, \ldots, p_n\};\{q_1, \ldots, q_m\}}$. Plus précisément, si $2n + m > 0$, on peut choisir des représentants pour paramétrer notre espace. Nous considérons trois méthodes:

Choix 1.

On choisit l’un des $z_i$ (disons $z_{j_0} = x_{j_0} + iy_{j_0}$) et on le place au point $i$ par une transformation de $G$. Les autres points sont alors fixés:

$$p_{j_0} = i, \quad p_j = \frac{z_j - x_{j_0}}{y_{j_0}}, \quad q_l = \frac{t_l - x_{j_0}}{y_{j_0}}.$$ 

Dans ce cas, on paramètre $C^+_{\{p_1, \ldots, p_n\};\{q_1, \ldots, q_m\}}$ par les coordonnées des $p_j = a_j + ib_j$ ($j \neq j_0$) et les $q_l$, l’orientation, dans ces coordonnées de
CHOIX DES SIGNES POUR LA FORMALITÉ DE M. KONTSEVICH

\( C^+ \{p_1, \ldots, p_n\}; \{q_1, \ldots, q_m\} \) est celle donnée par la forme:

\[
\Omega = \bigwedge_{j \neq j_0} (da_j \wedge db_j) \wedge dq_1 \wedge \ldots \wedge dq_m.
\]

(L’ordre sur les indices \( j \) n’importe pas car les 2-formes \( da_j \wedge db_j \) commutent entre elles.)

**Choix 2.**

On choisit l’un des \( t_l \) (disons \( t_{l_0} \)) et on le place en 0 par une translation, puis on fait une dilatation pour forcer le module de l’un des \( z_j \) (disons \( z_{j_0} \)) à valoir 1:

\[
p_{j_0} = \frac{z_{j_0} - t_{l_0}}{|z_{j_0} - t_{l_0}|} = e^{i\theta_{j_0}}, \quad p_j = \frac{z_j - t_{l_0}}{|z_j - t_{l_0}|}, \quad q_{l_0} = 0, \quad q_l = \frac{t_l - t_{l_0}}{|z_{j_0} - t_{l_0}|}.
\]

On paramètre alors \( C^+ \{p_1, \ldots, p_n\}; \{q_1, \ldots, q_m\} \) par l’argument \( \theta_{j_0} \) de \( p_{j_0} \) (compris entre 0 et \( \pi \)) et par les coordonnées des \( p_j \) (\( j \neq j_0 \)) et les \( q_l \) (\( l \neq l_0 \)). L’orientation, dans ces coordonnées de \( C^+ \{p_1, \ldots, p_n\}; \{q_1, \ldots, q_m\} \) est celle donnée par la forme:

\[
\Omega = (-1)^{l_0-1} d\theta_{j_0} \wedge \bigwedge_{j \neq j_0} (da_j \wedge db_j) \wedge dq_1 \wedge \ldots \wedge dq_{l_0} \wedge \ldots \wedge dq_m.
\]

En effet, on part de la forme \( \Omega \) du cas 1, avec \( j_0 = 1 \), puisque l’ordre des \( p \) n’intervient pas, on place \( q_{l_0} \) “en tête”:

\[
\Omega = (-1)^{l_0-1} dq_{l_0} \wedge da_2 \wedge db_2 \wedge \ldots \wedge da_n \wedge db_n \wedge dq_1 \wedge \ldots \wedge dq_{l_0} \wedge \ldots \wedge dq_1 \wedge \ldots \wedge dq_n,
\]

puis on effectue le changement de variables:

\[
p'_1 = \frac{i - q_{l_0}}{|i - q_{l_0}|} = e^{i\theta_1}, \quad p'_j = \frac{p_j - q_{l_0}}{|i - q_{l_0}|} = a'_j + ib'_j \quad (2 \leq j \leq n),
\]

\[
q'_k = \frac{q_k - q_{l_0}}{|i - q_{l_0}|} \quad (k \neq l_0).
\]

Dont le jacobien \( \frac{1}{(1 + q_{l_0}^2)^{n+m}} \) est strictement positif, pour obtenir la forme annoncée.

**Choix 3.**

On choisit deux points \( t_{l_0} < t_{l_1} \), on amène par une translation le premier en 0 et le second en 1 par une dilatation.

\[
p_j = \frac{z_j - t_{l_0}}{t_{l_1} - t_{l_0}}, \quad q_{l_0} = 0, \quad q_{l_1} = 1, \quad q_l = \frac{t_l - t_{l_0}}{t_{l_1} - t_{l_0}}.
\]
On paramètre $C^+_{\{p_1,\ldots, p_n\};\{q_1,\ldots, q_m\}}$ par les coordonnées des $p_j = a_j + ib_j$ et par les $q_l$ ($l \neq l_0$ et $l \neq l_1$). L’orientation est donnée par la forme:

$$\Omega = (-1)^{l_0+1} \frac{n}{\sum_{j=1}^{\infty} (da_j \wedge db_j) \wedge dq_1 \wedge \ldots \wedge dq_{l_0} \wedge \ldots \wedge dq_{l_1} \wedge \ldots \wedge dq_m}.$$ 

En effet, on part de la forme $\Omega$ du cas 1, on place $\Omega = (-1)^{l_0-1+l_1-2} dq_{l_0} \wedge dq_{l_1} \wedge \ldots \wedge dq_{l_n}$, puis on effectue le changement de variables:

$$p_j' = \frac{p_j - q_{l_0}}{q_{l_1} - q_{l_0}}, \quad q_k' = \frac{q_k - q_{l_0}}{q_{l_1} - q_{l_0}} \quad (k \neq l_0, k \neq l_1).$$

Dont le jacobien $\frac{q_{l_1} - q_{l_0}}{(q_{l_1} - q_{l_0})^{l_0+1+m}}$ est strictement positif, pour obtenir la forme annoncée.

I.2. Compactification des espaces de configuration. On plonge l’espace de configuration $C^+_{\{p_1,\ldots, p_n\};\{q_1,\ldots, q_m\}}$ dans une variété compacte de la façon suivante. Chaque fois que l’on prend deux points $A$ et $B$ du nuage de points $(z_j, z_j; t_l)$, on leur associe l’angle $\arg(B - A)$, à chaque triplet de points $(A, B, C)$ du nuage, on associe l’élément $[A - B, B - C, C - A]$ de l’espace projectif $\mathbb{P}^2(\mathbb{R})$ qu’ils définissent. On a ainsi une application:

$$\Phi : \text{Conf}^+(z_j, t_l) \longrightarrow \mathbb{T}^{(2n+m)(2n+m-1)} \times (\mathbb{P}^2(\mathbb{R}))^{(2n+m)(2n+m-1)(2n+m-2)}.$$ 

Cette application passe au quotient et il n’est pas difficile de montrer que l’on obtient ainsi un plongement

$$\Phi : C^+_{\{p_1,\ldots, p_n\};\{q_1,\ldots, q_m\}} \longrightarrow \mathbb{T}^{(2n+m)(2n+m-1)} \times (\mathbb{P}^2(\mathbb{R}))^{(2n+m)(2n+m-1)(2n+m-2)}.$$ 

On définit la compactification $\overline{C^+_{\{p_1,\ldots, p_n\};\{q_1,\ldots, q_m\}}}$ de $C^+_{\{p_1,\ldots, p_n\};\{q_1,\ldots, q_m\}}$ comme étant la fermeture dans

$$\mathbb{T}^{(2n+m)(2n+m-1)} \times (\mathbb{P}^2(\mathbb{R}))^{(2n+m)(2n+m-1)(2n+m-2)}$$

de $\Phi \left(C^+_{\{p_1,\ldots, p_n\};\{q_1,\ldots, q_m\}}\right)$. On obtient ainsi une variété à coins et on cherche son bord $\partial C^+_{\{p_1,\ldots, p_n\};\{q_1,\ldots, q_m\}}$.

Les points du bord s’obtiennent par une succession de collapses de points du nuage. On retrouve la description de M. Kontsevich à deux détails près : lorsque des points aériens (c’est à dire un ou des $p_j$) se rapprochent de $\mathbb{R}$, il faut distinguer entre quels $q_l$ ils arrivent, il y a trop de faces du bord, puisque les faces correspondant au rapprochement de points terrestres (des
$q_l$) non contigus est impossible sans que tous les points qui les séparent se rapprochent aussi. En codimension 1, on obtient deux types de faces:

**I.2.1. Faces de type 1.** Parmi les points aériens, $n_1$ points se rapprochent en un point $p$ qui reste aérien. Une telle face existe si $n \geq n_1 \geq 2$. À la limite, on obtient une variété produite:

\[(*) \quad F = \partial_{\{p_1, \ldots, p_{n_1}\}} C_{\{p_1, \ldots, p_n\}; \{q_1, \ldots, q_m\}}^+ = C_{\{p_1, \ldots, i_{n_1}\}} \times C_{\{p, p_1, \ldots, p_{n_1}, \ldots, p_n\}; \{q_1, \ldots, q_m\}}
\]

où l’espace $C_{\{p_1, \ldots, p_{n_1}\}}$ est le quotient de l’espace $Conf(z_1, \ldots, z_{n_1})$ par l’action du groupe $G'$ des transformations $z_j \mapsto az_j + b$ ($a > 0$ et $b \in \mathbb{C}$). C'est une variété de dimension $2n_1 - 3$ ($n_1 \geq 2$). On la plonge dans un produit de tores et d’espaces projectifs comme pour $C_{\{p_1, \ldots, p_n\}; \{q_1, \ldots, q_m\}}$. Enfin on l’oriente de la façon suivante; $z_1$ est placé en 0 par une translation complexe puis $|z_2|$ est normalisé à 1 par une dilatation,

$p_1 = 0, \quad p_2 = \frac{z_2 - z_1}{|z_2 - z_1|} = e^{i\theta_2}, \quad p_j = \frac{z_j - z_1}{|z_2 - z_1|} = a_j + ib_j$

et on prend l’orientation définie par la forme:

$$\Omega_1 = d\theta_2 \wedge \bigwedge_{j \geq 3} (da_j \wedge db_j).$$

Orientons maintenant la face $F$. On choisit la forme volume $\Omega_1 \wedge \Omega_2$ sur le produit $(*)$ où $\Omega_2$ est l’une des formes définies ci-dessus pour orienter $C_{\{p_j\}; \{q_l\}}^+$. L’orientation de la face à partir de celle de $\Omega$ est $\pm \Omega_1 \wedge \Omega_2$.

**Lemme I.2.1.** La face $F$ est orientée par $\Omega_F = -\Omega_1 \wedge \Omega_2$.

*Démonstration.* On a vu que l’on pouvait changer l’ordre des points $p_j$ de $C_{\{p_j\}; \{q_l\}}^+$ sans changer l’orientation. On renumérote les points $p_1, \ldots, p_{n_1}$ en $p_1, p_2, \ldots, p_{n_1}$, puis on fixe $p_1 = i$:

$$\Omega = \bigwedge_{j=2}^{n_1} (da_j \wedge db_j) \wedge \Omega_2,$$

ensuite on change de variables dans le premier facteur en posant:

$$p_j' = e^{i\theta_2}, \quad p_j' = \frac{p_j}{|p_2 - i|} = a_j' + ib_j' \quad (j = 3, \ldots, n_1).$$

Lorsque les $n_1$ premiers points collabent, on agrandit le petit nuage qu’ils forment en normalisant la distance qui sépare les 2 premiers à 1. Posons $\rho_2 = |p_2 - i|$. Le changement de variable donne pour $\Omega$ la forme:

$$\Omega' = dp_2 \wedge d\theta_2 \wedge \bigwedge_{j \geq 3} (da_j' \wedge db_j') \wedge \Omega_2.$$
La face est obtenue lorsque $\rho_2 \to 0$. Or $\rho_2 > 0$, on doit donc l’orienter avec
\[
\Omega_F = -d\theta_2 \wedge \bigwedge_{j \geq 3} (da'_j \wedge db'_j) \wedge \Omega_2 = -\Omega_1 \wedge \Omega_2.
\]

\[\square\]

**I.2.2. Face de type 2.** Parmi les points du nuage, $n_1$ points aériens et $m_1$ points terrestres se rapprochent en un point $q$ terrestre. Une telle face existe si $n + m > n_1 + m_1$ et $2n_1 + m_1 \geq 2$. À la limite, on obtient une variété produite:
\[
\begin{align*}
F &= \partial\{p_{i_1}, \ldots, p_{i_{n_1}}\};\{q_1, \ldots, q_{m_1}\} C_{\{p_1, \ldots, p_n\};\{q_1, \ldots, q_m\}}^+ \\
&= C_{\{p_1, \ldots, p_{n_1}\};\{q_1, \ldots, q_{n_1}\}} \times C_{\{p_{n_1+1}, \ldots, p_n\};\{q_1, \ldots, q_{m_1}, q_{m_1+1}, \ldots, q_m\}}.
\end{align*}
\]

On appelle $\Omega_1$ et $\Omega_2$ l’une des formes volumes de chacun des facteurs de ce produit. La forme $\Omega_1 \wedge \Omega_2$ est une forme volume sur $F$. On donne l’orientation de $F$ à partir de celle de l’espace de configuration de départ en terme de cette forme.

**Lemme I.2.2.** Avec nos notations, la face $F$ est orientée par:
\[
\Omega_F = (-1)^{m_1+l+m_1} \Omega_1 \wedge \Omega_2.
\]

**Démonstration.** Il faut considérer six types de nuages différents:

**Sous-cas 1.** $n > n_1 > 0$.
On suppose que $p_1, \ldots, p_n$ et $q_{l+1}, \ldots, q_{l+m}$ collapsent. On paramètre l’espace $C^+_{\{p_1, \ldots, p_n\};\{q_1, \ldots, q_m\}}$ par $p_{n+1} = i$. La forme d’orientation est:

$$
\Omega = (-1)^{lm_1} \frac{p_{j \cdot a_1}}{b_1} (2 \leq j \leq n_1), \quad q'_{k \cdot a_1} = \frac{q_{k \cdot a_1}}{b_1} (l + 1 \leq k \leq l + m_1).
$$

Alors on peut écrire de façon un peu abusive:

$$
\Omega = (-1)^{lm_1+l+m_1} \frac{p_{j \cdot a_1}}{b_1} (2 \leq j \leq n_1), \quad q'_{k \cdot a_1} = \frac{q_{k \cdot a_1}}{b_1} (l + 1 \leq k \leq l + m_1).
$$

et puisque $b_1 > 0$ et la face $F$ est obtenue pour $b_1 = 0$, son orientation est donnée par:

$$
\Omega_F = (-1)^{lm_1+l+m_1} \frac{p_{j \cdot a_1}}{b_1} (2 \leq j \leq n_1), \quad q'_{k \cdot a_1} = \frac{q_{k \cdot a_1}}{b_1} (l + 1 \leq k \leq l + m_1).
$$

Sous-cas 2. $n = n_1 > 0$ (et donc $m \geq m_1 + 1$) et $l > 0$

On suppose que $p_1, \ldots, p_n$ et $q_{l+1}, \ldots, q_{l+m}$ collapsent. On paramètre l’espace $C^+_{\{p_1, \ldots, p_n\};\{q_1, \ldots, q_m\}}$ par $q_l = 0, q_{l+1} = 1$. La forme d’orientation est:

$$
\Omega = (-1)^{l^2+2} \frac{p_{j \cdot a_1}}{b_1} (2 \leq j \leq n_1), \quad q'_{k \cdot a_1} = \frac{q_{k \cdot a_1}}{b_1} (l + 1 \leq k \leq l + m_1) \wedge

\Omega = (-1)^{l^2+2} \frac{p_{j \cdot a_1}}{b_1} (2 \leq j \leq n_1), \quad q'_{k \cdot a_1} = \frac{q_{k \cdot a_1}}{b_1} (l + 1 \leq k \leq l + m_1) \wedge
$$

et puisque $b_1 > 0$ et la face $F$ est obtenue pour $b_1 = 0$, son orientation est donnée par:

$$
\Omega_F = (-1)^{l^2+2} \frac{p_{j \cdot a_1}}{b_1} (2 \leq j \leq n_1), \quad q'_{k \cdot a_1} = \frac{q_{k \cdot a_1}}{b_1} (l + 1 \leq k \leq l + m_1).
$$
On change de variables en posant:

\[ a_1 + ib_1 - 1 = \rho_1 e^{i\theta_1} \quad p'_j = \frac{p_j - 1}{\rho_1} \quad (2 \leq j \leq n), \]

\[ q'_k = \frac{q_k - 1}{\rho_1} \quad (l + 2 \leq k \leq l + m_1). \]

Alors

\[
\Omega = (-1)^{lm_1+l+m_1+1} \rho_1^{-(n+m_1-5)} d\rho_1 \wedge d\theta_1 \wedge da_2 \wedge db_2 \wedge \ldots \\
\wedge da_n \wedge db_n \wedge dq_{l+2} \wedge \ldots \wedge dq_{l+m_1} \wedge \\
\wedge dq_1 \wedge \ldots \wedge dq_{l-1} \wedge dq_{l+m_1+1} \wedge \ldots \wedge dq_m.
\]

On peut donc écrire de façon un peu abusive:

\[
\Omega \simeq (-1)^{lm_1+l+m_1+1} d\rho_1 \wedge (\Omega_1 \wedge (-1)^{l-1+l+1-2} \Omega_2)
\]

et puisque \( \rho_1 > 0 \) et la face \( F \) est obtenue pour \( \rho_1 = 0 \), son orientation est donnée par:

\[
\Omega_F = (-1)^{lm_1+l+m_1} \Omega_1 \wedge \Omega_2.
\]

**Sous-cas 3.** \( n = n_1 > 0 \) (et donc \( m \geq m_1 + 1 \)) et \( l = 0 \).

C’est le même calcul que ci-dessus, on pose \( q_{m_1} = 0, q_{m_1+1} = 1 \), on obtient:

\[
\Omega = (-1)^{2m_1+2} da_1 \wedge db_1 \wedge \ldots \wedge da_n \wedge db_n \wedge dq_1 \wedge \ldots \wedge dq_{m_1} \wedge dq_{m_1+1} \wedge \ldots \wedge dq_m.
\]

On change de variables en posant:

\[ a_1 + ib_1 = \rho_1 e^{i\theta_1} \quad p'_j = \frac{p_j}{\rho_1} \quad (2 \leq j \leq n), \quad q'_k = \frac{q_k}{\rho_1} \quad (1 \leq k \leq m_1 - 1). \]

Alors

\[
\Omega = \rho_1 d\rho_1 \wedge d\theta_1 \wedge da_2 \wedge db_2 \wedge \ldots \wedge da_n \wedge db_n \wedge dq_1 \wedge \ldots \\
\wedge dq_{m_1-1} \wedge dq_{m_1+2} \wedge \ldots \wedge dq_m \\
\simeq d\rho_1 \wedge (-1)^{m_1-1} \Omega_1 \wedge (-1)^{1+2-2} \Omega_2.
\]
et puisque $\rho_1 > 0$ et la face $F$ est obtenue pour $\rho_1 = 0$, son orientation est donnée par:

$$\Omega_F = (-1)^{l m_1 + l + m_1} \Omega_1 \wedge \Omega_2.$$

**Sous-cas 4.** $n \geq n_1 = 0$ (et donc $m_1 > 1$) et $l > 0$.

On suppose que $q_{l+1}, \ldots, q_{l+m_1}$ collaprent. On paramètre l’espace $C^+_{\{p_1, \ldots, p_n\} \setminus \{q_1, \ldots, q_m\}}$ par $q_l = 0$, $q_{l+1} = 1$. La forme d’orientation est:

$$\Omega = (-1)^{2l+2} da_1 \wedge da_2 \wedge \ldots \wedge da_n \wedge db_1 \wedge \ldots \wedge dq_1 \wedge \ldots \wedge dq_{l+2} \wedge dq_{l+3} \wedge \ldots \wedge dq_{l+m_1} \wedge$$

$$\wedge da_1 \wedge db_1 \wedge \ldots \wedge da_n \wedge db_n \wedge dq_1 \wedge \ldots \wedge dq_{l-1} \wedge dq_{l+1} \wedge \ldots \wedge dq_m.$$

On change de variables en posant:

$$q_k' = \frac{q_k - q_{l+2}}{q_{l+2} - 1} \quad (l + 3 \leq k \leq l + m_1).$$

Alors:

$$\Omega \simeq (-1)^{l m_1 + l + m_1 + 1} dq_{l+2} \wedge dq_{l+3} \wedge \ldots \wedge dq_{l+m_1} \wedge$$

$$\wedge da_1 \wedge db_1 \wedge \ldots \wedge da_n \wedge db_n \wedge dq_1 \wedge \ldots \wedge dq_{l-1} \wedge dq_{l+1} \wedge \ldots \wedge dq_m.$$

On peut donc écrire de façon un peu abusive:

$$\Omega = (-1)^{l m_1 + l + m_1 + 1} dq_{l+2} \wedge dq_{l+3} \wedge \ldots \wedge dq_{l+m_1} \wedge$$

$$\wedge \Omega_1 \wedge (-1)^{l-1+l+1-2} \Omega_2$$

et puisque $q_{l+2} - 1 > 0$ et la face $F$ est obtenue pour $q_{l+2} - 1 = 0$, son orientation est donnée par:

$$\Omega_F = (-1)^{l m_1 + l + m_1} \Omega_1 \wedge \Omega_2.$$

**Sous-cas 5.** $n \geq n_1 = 0$ (et donc $m_1 > 1$), $l = 0$ et $m_1 < m$. 
On pose $q_{m_1} = 0$, $q_{m_1+1} = 1$, on obtient:

$$\Omega = (-1)^{2m_1+2} da_1 \wedge db_1 \wedge \ldots \wedge da_n \wedge db_n \wedge dq_1 \wedge \ldots \wedge dq_{m_1} \wedge dq_{m_1+1} \wedge \ldots \wedge dq_m.$$ 

On change de variables en posant:

$$q'_k = \frac{q_k - q_{m_1-1}}{-q_{m_1-1}} \quad (1 \leq k \leq m_1 - 2).$$

Alors:

$$\Omega \simeq dq'_1 \wedge \ldots \wedge dq'_{m_1-2} \wedge dq_{m_1-1} \wedge da_1 \wedge db_1 \wedge \ldots \wedge da_n \wedge db_n \wedge dq_{m_1+2} \wedge \ldots \wedge dq_m$$

$$= (-1)^{m_1} dq_{m_1-1} \wedge \Omega_1 \wedge (-1)^{1+2-1} \Omega_2.$$ 

Maintenant $q_{m_1-1} < 0$ et l’orientation de la face est encore:

$$\Omega_F = (-1)^{l} dq_{m_1-1} \wedge \Omega_1 \wedge \Omega_2.$$ 

**Sous-cas 6.** $n \geq n_1 = 0$ (et donc $m_1 > 1$), $l = 0$ et $m_1 = m$ et donc $n > 0$. 

![Diagram](image-url)
On pose \( q_1 = 0 \) et \( p_1 = e^{i\theta_1} \). La forme \( \Omega \) est

\[
\Omega = d\Theta \wedge d\alpha_2 \wedge d\beta_2 \wedge \ldots \wedge d\alpha_n \wedge d\beta_n \wedge dq_2 \wedge \ldots \wedge dq_m
\]

\[
= (-1)^{m-1} dq_2 \wedge \ldots \wedge dq_m \wedge d\Theta \wedge d\alpha_2 \wedge d\beta_2 \wedge \ldots \wedge d\alpha_n \wedge d\beta_n.
\]

On change de variables en posant:

\[
q'_k = \frac{q_k - q_2}{q_2} \quad (3 \leq k \leq m).
\]

Alors:

\[
\Omega \simeq (-1)^{m-1} dq_2 \wedge (-1)^{1+2} \Omega_1 \wedge \Omega_2.
\]

Puisque \( q_2 > 0 \) et \( F \) apparaît pour \( q_2 = 0 \), l’orientation de la face est encore:

\[
\Omega_F = (-1)^{l_1+l+m_1} \Omega_1 \wedge \Omega_2.
\]

Tout nuage de point correspondant à une face de type 2 relève d’un des six sous-cas. Ceci termine la démonstration du Lemme I.2.2. \( \square \)

II. Algèbres symétriques et extérieures sur les espaces gradués.

II.1. La catégorie des espaces gradués. Un espace vectoriel sur un corps \( k \) est gradué s’il est muni d’une \( \mathbb{Z} \)-gradation:

\[
V = \bigoplus_{n \in \mathbb{Z}} V_n.
\]

Le degré d’un élément homogène \( x \) sera noté \(|x|\). Un espace gradué sera toujours considéré comme un super-espace vectoriel, la \( \mathbb{Z}_2 \)-gradation étant déduite de la \( \mathbb{Z} \)-gradation:

\[
V_+ = \bigoplus_{n \in \mathbb{Z}} V_{2n} \quad V_- = \bigoplus_{n \in \mathbb{Z}} V_{2n+1}.
\]

Si \( V \) et \( W \) sont des espaces gradués, il existe une graduation naturelle sur \( V \oplus W, V \otimes W, \text{Hom}_k(V, W) \). Un morphisme d’espaces gradués entre \( V \) et \( W \) est par définition un élément de degré zéro dans \( \text{Hom}_k(V, W) \).

Une algèbre graduée est un espace gradué \( B \) muni d’une structure d’algèbre telle que la multiplication \( m : B \otimes B \to B \) est un morphisme d’espaces gradués, c’est-à-dire:

\[
B_iB_j \subset B_{i+j}.
\]

On définit de la même manière les notions de \( B \)-modules gradués à gauche ou à droite. Si \( A \) et \( B \) sont deux algèbres graduées, le produit:

\[
m_{A \otimes B} : A \otimes B \otimes A \otimes B \to A \otimes B
\]

\[
a \otimes b \otimes a' \otimes b' \mapsto (-1)^{|b||a'|} aa' \otimes bb'
\]

est associatif et munit \( A \otimes B \) d’une structure d’algèbre graduée. Si \( M \) (resp. \( N \)) est un \( A \)-module (resp. un \( B \)-module) gradué à gauche, la même règle des
signes (la règle de Koszul) permet de définir une structure de $A \otimes B$ – module gradué à gauche sur $M \otimes N$.

Si $A, A', B, B'$ sont des espaces gradués, l'identification de $\text{Hom}_k(A \otimes B, A' \otimes B')$ avec $\text{Hom}_k(A, A') \otimes \text{Hom}_k(B, B')$ se fait avec la même règle des signes:

$$(f \otimes g)(a \otimes b) = (-1)^{|g||a|} f(a) \otimes g(b).$$

Une algèbre graduée est dite commutative si on a:

$$xy - (-1)^{|x||y|}yx = 0.$$ 

Une cogèbre graduée $C$ se définit de manière similaire: la comultiplication doit vérifier:

$$\Delta C_j \subset \sum_{k+l=j} C_k \otimes C_l.$$

On définit une structure de cogèbre graduée sur le produit tensoriel de deux cogèbres graduées en appliquant la même règle sur les signes que dans le cas des algèbres.

Une dérivation de degré $i$ dans une algèbre graduée $B$ est un morphisme linéaire $d : B \rightarrow B$ de degré $i$ tel que:

$$d(xy) = dx.y + (-1)^{|x|}x.dy$$

ce qui s'écrit encore:

$$dm = m(d \otimes I + I \otimes d)$$

où $m$ désigne la multiplication de l'algèbre (attention à la règle des signes). Une codérivation de degré $i$ dans une cogèbre graduée $C$ est un morphisme linéaire $d : C \rightarrow C$ de degré $i$ tel que si $\Delta X = \sum_{(x)} x' \otimes x''$ on a:

$$\Delta dx = \sum_{(x)} dx' \otimes x'' + (-1)^{|x'|}x' \otimes dx''$$

ou encore:

$$\Delta d = (d \otimes I + I \otimes d)\Delta.$$

Enfin une algèbre de Lie graduée est un espace vectoriel gradué $g$ muni d’un crochet $[,]$ tel que:

1) $[g_i, g_j] \subset g_{i+j}$
2) $[x, y] = -(-1)^{|x||y|}[y, x]$
3) $(-1)^{|x||y||z|}[[x, y], z] + (-1)^{|y||x|}[[y, z], x] + (-1)^{|z||y|}[[z, x], y] = 0$

(Identité de Jacobi graduée).

L'identité de Jacobi graduée s'exprime aussi en disant que $ax = [x, .]$ est une dérivation (de degré $|x|$).

Une algèbre de Lie graduée est différentielle si elle est munie d’une différentielle $d$ de degré 1 ($d : g \rightarrow g[1]$), telle que:

$$d^2 = 0, \quad d ([x, y]) = [dx, y] + (-1)^{|x|}[x, dy].$$
II.2. La règle de Koszul. La raison profonde qui fait que “la règle des signes marche” est la suivante: la catégorie des espaces vectoriels $\mathbb{Z}_2$-gradués munie du produit tensoriel $\otimes$ usuel et des applications:

$$\tau_{A,B} : A \otimes B \longrightarrow B \otimes A$$

$$a \otimes b \longmapsto (-1)^{|a||b|} b \otimes a$$

est une catégorie tensorielle tressée, c'est-à-dire que les tressages $\tau_{A,B}$ sont fonctoriels:

$$
\begin{array}{ccc}
A \otimes B & \tau_{A,B} & B \otimes A \\
\downarrow f \otimes g & \downarrow & \downarrow 9 \otimes f \\
A' \otimes B' & \tau_{A',B'} & B' \otimes A'
\end{array}
$$

et vérifient:

$$\tau_{A \otimes B, C} = (\tau_{A, C} \otimes I_B)(I_A \otimes \tau_{B, C}) .$$

De ces deux propriétés on déduit facilement l’équation de l’hexagone, c’est-à-dire la commutativité du diagramme suivant:

$$
\begin{array}{ccc}
\mathcal{A} & \otimes & B \otimes C \\
\downarrow & & \downarrow \\
A \otimes C \otimes B & B \otimes A \otimes C & \\
\downarrow & & \downarrow \\
C \otimes A \otimes B & B \otimes C \otimes A & \\
\downarrow & & \downarrow \\
C \otimes B \otimes A
\end{array}
$$

La catégorie tensorielle tressée des espaces $\mathbb{Z}_2$-gradués peut aussi se voir comme la catégorie des modules sur l’algèbre de Hopf quasi-triangulaire $(H_2, R)$ où $H_2$ est l’algèbre du groupe $\mathbb{Z}_2$ munie de la multiplication et de la comultiplication usuelle, mais où la $R$-matrice est non triviale.

Dans cette catégorie le carré des tressages est toujours l’identité (c’est une catégorie tensorielle stricte). On peut faire de même avec des espaces $\mathbb{Z}_k$-gradués en remplaçant $-1$ par $e^{2i\pi/k}$. On obtient ainsi la catégorie des espaces vectoriels anyoniques, qui est tressée de manière effective pour $k \geq 3$ [M].

II.3. Décalages. Soit $V$ un espace gradué. On pose:

$$V[1] = V \otimes k[1]$$

où $k[1]$ est l’espace gradué tel que $k_n = \{0\}$ pour $n \neq -1$ et $k_{-1} = k$. Autrement dit $V[1]$ et $V$ ont même espace vectoriel sous-jacent, mais le degré d’un élément est baissé d’une unité dans $V[1]$. On posera en outre:

$$[n] = [1]^n$$

pour tout entier $n$. 
II.4. Algèbres symétriques et extérieures. L’algèbre symétrique \( S(V) \) (resp. l’algèbre extérieure \( \Lambda(V) \)) est définie par:
\[
S(V) = T(V)/\langle x \otimes y - (-1)^{|x||y|} y \otimes x \rangle
\]
resp. \( \Lambda(V) = T(V)/\langle x \otimes y + (-1)^{|x||y|} y \otimes x \rangle \).

Ce sont des espaces gradués de manière naturelle. La proposition suivante est implicite dans [K1]:

**Proposition II.4.1** (symétrisation). Pour tout espace vectoriel gradué \( V \) et pour tout \( n > 0 \) on a un isomorphisme naturel:
\[
\Phi_n : S^n(V[1]) \longrightarrow \Lambda^n(V)[n]
\]
donné par:
\[
\Phi_n (x_1, \ldots, x_n) = \alpha(x_1, \ldots, x_n) x_1 \wedge \cdots \wedge x_n
\]
 où, pour des \( x_i \) homogènes, \( \alpha(x_1, \ldots, x_n) \) désigne la signature de la permutation “unshuffle” qui range les \( x_i \) pairs dans \( V \) à gauche sans les permuter, et les \( x_i \) impairs dans \( V \) à droite sans les permuter.

**Démonstration.** Soit \( I \) (resp. \( J \)) l’ensemble des \( i \) tels que \( x_i \) soit de degré pair (resp. impair), et \( \alpha(I, J) = \alpha(x_1, \ldots, x_n) \) la signature de la permutation-rangement associée. L’isomorphisme \( \Phi_n \) est donné par la restriction à \( S^n(V[1]) \) de la composition des trois flèches du diagramme ci-dessous (la flèche supérieure est un isomorphisme d’algèbres):
\[
\begin{align*}
S(V[1]) & \longrightarrow S(V[1])_+ \otimes S(V[1])_- \\
x_1 \ldots x_n & \mapsto x_I \otimes x_J \\
\Lambda(V) & \longrightarrow \Lambda(V)_- \otimes \Lambda(V)_+ \\
\alpha(I, J)x_1 \wedge \cdots \wedge x_n & \mapsto x_{\wedge I} \otimes x_{\wedge J}.
\end{align*}
\]

Enfin si les \( x_j \) sont de degré \( d_j \) dans \( V[1] \), \( x_1 \ldots x_n \) est de degré \( d_1 + \cdots + d_n \) dans \( S^n(V[1]) \), donc de degré \( d_1 + \cdots + d_n + n \) dans \( \Lambda^n(V) \). \( \Phi_n(x_1 \ldots x_n) \) est donc de degré \( d_1 + \cdots + d_n + n \) dans \( \Lambda^n(V) \), donc de degré \( d_1 + \cdots + d_n \) dans \( \Lambda^n(V)[n] \).

**Remarque.** L’application \( \Phi = \oplus \Phi_n \) est un morphisme d’espace vectoriel gradué mais pas d’algèbre. Il est d’ailleurs vain de vouloir chercher un isomorphisme d’algèbres entre \( S(V[1]) \) et \( \bigoplus \Lambda^n(V)[n] \), car deux éléments de parité opposée commutent dans le premier cas, et anticommutent dans le second cas.

II.5. Un exemple: Tens \( (\mathbb{R}^d) \). L’algèbre des tenseurs contravariants totalement antisymétriques est une algèbre naturellement graduée par l’ordre des tenseurs. On aimerait la voir comme l’espace sous-jacent à une algèbre symétrique. Notons donc \( V \) l’espace vectoriel \( \mathcal{X}(\mathbb{R}^d) \) des champs de vecteurs sur \( \mathbb{R}^d \), gradué par \( V = V_0 \). On identifie \( \text{Tens}_n(\mathbb{R}^d) = \wedge^n V \) à \( S^n(V[1])[-n] \).
par $\Phi_n$, les puissances extérieures et symétriques étant prises au sens des $C^\infty(M)$-modules.

Dans la suite, on posera

$$T_{\text{poly}}\left(\mathbb{R}^d\right) = \text{Tens} \left(\mathbb{R}^d\right) [1].$$

### III. Variétés formelles graduées.

#### III.1. Variétés formelles.

On se place sur le corps des réels ou des complexes. On se donne un voisinage ouvert $U$ de $0$ dans $\mathbb{R}^d$. Une fonction analytique $\varphi$ sur $U$ à valeurs dans $\mathbb{C}$ est déterminée par son développement de Taylor en $0$:

$$\varphi(x) = \sum_{\alpha \in \mathbb{N}^d} \frac{x^\alpha}{\alpha!} (\partial^\alpha \varphi)(0).$$

On établit ainsi une dualité non dégénérée entre les fonctions analytiques sur $U$ et les distributions de support $\{0\}$. Plus abstraitement on peut remplacer les fonctions analytiques par les jets d’ordre infini au point $0$.

On appelle variété formelle, ou voisinage formel de $0$, l’espace $\mathbb{C}$ des distributions de support $\{0\}$. La structure d’algèbre commutative sur l’espace des fonctions analytiques sur $U$ détermine une structure de cogèbre cocommutative sur son dual restreint, qui est exactement $\mathbb{C}$. La comultiplication est donnée par:

$$\langle \Delta v, \varphi \otimes \psi \rangle = \langle v, \varphi \psi \rangle.$$

Considérant l’espace tangent $V$ à la variété $U$ en $0$, on a en fait un isomorphisme de cogèbres entre $\mathbb{C}$ et $S(V)$, où la comultiplication $\Delta$ de $S(V)$ est le morphisme d’algèbres tel que $\Delta(v) = v \otimes 1 + 1 \otimes v$ pour $v \in V$.

On considérera la version pointée:

$$\mathcal{C} = S^+(V) = \bigoplus_{n \geq 1} S^n(V).$$

C’est la cogèbre colibre cocommutative sans co-unité construite sur $V$. C’est aussi le dual restreint de l’algèbre des jets d’ordre infini qui s’annulent en $0$. On remarque que $\Delta v = 0$ si et seulement si $v$ appartient à $V$.

Un champ de vecteurs sur la variété formelle pointée est donné par une codérivation $Q : \mathcal{C} \rightarrow \mathcal{C}$ (c’est donc un champ de vecteurs qui s’annule en $0$). Un morphisme de variétés formelles pointées est donné par un morphisme de cogèbres. Tout morphisme $f$ de variétés pointées induit un morphisme de variétés formelles par transport des distributions de support $\{0\}$:

$$\langle f_* T, \varphi \rangle = \langle T, \varphi \circ f \rangle.$$

Or, par propriété universelle des cogèbres cocommutatives colibres, une codérivation $Q : S^+(V) \rightarrow S^+(V)$ (resp. un morphisme de cogèbres $\mathcal{F}$ :
$S^+(V_1) \rightarrow S^+(V_2)$ est entièrement déterminé(e) par sa composition avec la projection sur $V$ (resp. $V_2$), c’est à dire par une suite d’applications:

$$Q_n : S^nV \rightarrow V \quad \text{(resp. } F_n : S^nV_1 \rightarrow V_2\text{)}$$

qui sont par définition les coefficients de Taylor du champ de vecteurs $Q$ ou du morphisme $F$.

**III.2. Variétés formelles graduées pointées.** On fait la même construction algébrique dans la catégorie des espaces vectoriels gradués: une variété formelle graduée pointée est une cogèbre $C$ isomorphe à $S^+(V)$ où $V$ est cette fois-ci un espace gradué. Toutes les notions du §III.1 s’appliquent, à ceci près que l’on peut considérer des champs de vecteurs de différents degrés. Nous allons donner une formule explicite pour un champ de vecteurs ou un morphisme en fonction de ses coefficients de Taylor:

**Théorème III.2.1.** Soit $i$ un entier, soient $V, V_1, V_2$ des espaces gradués, et deux suites d’applications linéaires $Q_n : S^nV \rightarrow V$ de degré $i$, $F_n : S^nV_1 \rightarrow V_2$ de degré zéro. Alors il existe une unique codérivation $Q$ de degré $i$ de $S^+(V)$ et un unique morphisme $F : S^+(V_1) \rightarrow S^+(V_2)$ dont les $Q_n$ et les $F_n$ sont les coefficients de Taylor respectifs. $Q$ et $F$ sont donnés par les formules explicites:

$$Q(x_1 \ldots x_n) = \sum_{I \cup J = \{1, \ldots, n\}, I, J \neq \emptyset} \varepsilon_x(I, J) (Q_{|I|}(x_I))x_J$$

$$F(x_1 \ldots x_n) = \sum_{j\geq 1} \frac{1}{j!} \sum_{I_1 \cup \cdots \cup I_j = \{1, \ldots, n\}, I_1, \ldots, I_j \neq \emptyset} \varepsilon_x(I_1, \ldots, I_j) F_{|I_1|}(x_{I_1}) \cdots F_{|I_j|}(x_{I_j})$$

où $\varepsilon_x(I_1, \ldots, I_j)$ désigne la signature de l’effet sur les $x_i$ impairs de la permutation-battement associée à la partition $(I_1, \ldots, I_j)$ de $\{1, \ldots, n\}$.

**Démonstration.** Supposons que tous les coefficients de Taylor de la codérivation $Q$ sont nuls. En particulier $Q(x) = 0$ pour tout $x \in V$. Supposons que $Q(x_1 \ldots x_k) = 0$ pour tout $k \leq n$. Alors:

$$\Delta Q(x_1 \ldots x_{n+1}) = (Q \otimes I + I \otimes Q) \Delta(x_1 \ldots x_{n+1}) = 0,$$

compte tenu de l’hypothèse de récurrence et de l’expression explicite de $\Delta(x_1 \ldots x_{n+1})$:

$$\Delta(x_1 \ldots x_{n+1}) = \sum_{I \cup J = \{1, \ldots, n+1\}, I, J \neq \emptyset} \varepsilon_x(I, J)x_I \otimes x_J.$$
Nous vérifions directement les formules (les vérifications à l’ordre 2 ou 3 sont laissées au lecteur à titre d’exercice).

1. **Cas d’une codérivation:** On écrit la formule explicite pour $\Delta(x_1 \ldots x_n)$ en utilisant la cocommutativité graduée, ce qui permet de ne retenir que la moitié des partitions:

$$\Delta(x_1 \ldots x_n) = (1 + \tau) \sum_{K\cap L = \{1, \ldots, n\}, \ 1 \in K, \ L \neq \emptyset} \varepsilon_x(K, L)x_K \otimes x_L.$$ 

On a donc, en prenant pour $Q$ l’expression explicite du théorème:

$$\Delta Q(x_1 \ldots x_n)$$

$$= (1 + \tau) \sum_{I\cap J = \{1, \ldots, n\}, \ I, J \neq \emptyset} \varepsilon_x(I, J)\Delta(I|I|I) \Delta(J|J|J)$$

$$= (1 + \tau) \sum_{I\cap J = \{1, \ldots, n\}, \ I, J \neq \emptyset} \varepsilon_x(I, J)\varepsilon_x(J, K)\varepsilon_x(K, L)\Delta(I|I|I) \Delta(J|J|J) \Delta(K|K|K)$$

Par ailleurs on a:

$$(Q \otimes I + I \otimes Q)\Delta(x_1 \ldots x_n)$$

$$= (1 + \tau) \sum_{L\cap K = \{1, \ldots, n\}, \ L, K \neq \emptyset} \varepsilon_x(L, K)\Delta(I|I|I) \Delta(J|J|J) \Delta(K|K|K)$$

$$= (1 + \tau) \sum_{L\cap K = \{1, \ldots, n\}, \ L, K \neq \emptyset} \varepsilon_x(L, K)\varepsilon_x(J, K)\varepsilon_x(K, L)\Delta(I|I|I) \Delta(J|J|J) \Delta(K|K|K)$$

d’où le fait que $Q$ est bien une codérivation.

2. **Cas d’un morphisme:** le calcul est un peu plus compliqué: on commence par écrire $\Delta$ et $\mathcal{F}$ de manière redondante, en employant des permutations qui ne sont pas forcément des battements:

$$\Delta(x_1 \ldots x_n) = \sum_{\sigma \in S_n} \sum_{r=1}^n \frac{\varepsilon_x(\sigma)}{r!(n-r)!} x_{\sigma_1} \ldots x_{\sigma_r} \otimes x_{\sigma_{r+1}} \ldots x_{\sigma_n}$$

$$\mathcal{F}(x_1 \ldots x_n) = \sum_{j \geq 1} \frac{1}{j!} \sum_{k_1 + \ldots + k_j = n} \frac{1}{k_1! \ldots k_j!}$$

$$\cdot \sum_{\sigma \in S_n} \varepsilon_x(\sigma)\mathcal{F}_{k_1}(x_{\sigma_1} \ldots x_{\sigma_{k_1}}) \ldots \mathcal{F}_{k_j}(x_{\sigma_{k_{j-1}}} \ldots x_{\sigma_n}).$$
On vérifie directement l’égalité:
\[ \Delta F(x_1 \ldots x_n) = (F \otimes F) \Delta(x_1 \ldots x_n). \]

L’écriture par blocs: \( x_1 \ldots x_n = (x_1 \ldots x_{k_1}) \ldots (x_{k_1 + \ldots + k_{j-1}+1} \ldots x_n) \) induit par permutation des blocs un plongement du groupe de permutations \( S_j \) dans \( S_n \). On calcule:

\[
\Delta F(x_1 \ldots x_n) = \sum_{j \geq 2} \frac{1}{j!} \sum_{k_1 + \ldots + k_j = n} \frac{1}{k_1! \ldots k_j!} \cdot \sum_{\sigma \in S_n} \varepsilon_x(\sigma) \Delta(F_{k_1}(x_{\sigma_1} \ldots x_{\sigma_{k_1}}) \ldots F_{k_j}(x_{\ldots} x_{\sigma_n}))
\]

\[
= \sum_{j \geq 2} \frac{1}{j!} \sum_{k_1 + \ldots + k_j = n} \frac{1}{k_1! \ldots k_j!} \varepsilon_x(\sigma) \sum_{\tau \in S_j \subset S_n} \frac{1}{r!(j-r)!} F_{k_r}(\ldots) \otimes F_{k_{r+1}}(\ldots) \ldots F_{k_j}(\ldots).
\]

Dans le dernier membre de l’égalité ci-dessus, chaque terme se trouve répété autant de fois qu’il y a d’éléments dans \( S_j \). On a donc:

\[
\Delta F(x_1 \ldots x_n) = \sum_{j \geq 2} \frac{1}{j!} \sum_{k_1 + \ldots + k_j = n} \frac{1}{k_1! \ldots k_j!} \sum_{\sigma \in S_n} \varepsilon_x(\sigma) \cdot \sum_{r=1}^{j-1} \frac{1}{r!(j-r)!} F_{k_1}(\ldots) \ldots F_{k_r}(\ldots) \otimes F_{k_{r+1}}(\ldots) \ldots F_{k_j}(\ldots).
\]

Par ailleurs, on a:

\[
(F \otimes F) \Delta(x_1 \ldots x_n) = \sum_{\sigma \in S_n} \sum_{r=1}^{n-1} \frac{\varepsilon_x(\sigma)}{r!(n-r)!} F(x_{\sigma_1} \ldots x_{\sigma_r}) \otimes F(x_{\sigma_{r+1}} \ldots x_{\sigma_n})
\]

\[
= \sum_{\sigma \in S_n} \sum_{r=1}^{n-1} \sum_{\alpha \in S_r \times S_{n-r} \subset S_n} \frac{\varepsilon_x(\sigma)\varepsilon_{\sigma x}(\alpha)}{r!(n-r)!} \cdot \frac{1}{j!k!} \sum_{r_1+\ldots+r_j=r} \frac{1}{r_1! \ldots r_j!} \frac{1}{s_1! \ldots s_k!} \cdot F_{r_1}(x_{\alpha_{\sigma_1}} \ldots x_{\alpha_{\sigma_{r_1}}}) \ldots F_{r_j}(\ldots) \otimes F_{k_1}(\ldots) \ldots F_{k_k}(\ldots x_{\alpha_{\sigma_n}}).
\]
Dans le dernier membre de l’égalité ci-dessus, chaque terme se trouve répété autant de fois qu’il y a d’éléments dans $S_r \times S_{n-r}$. Donc:

\[
\begin{align*}
(F \otimes F) \Delta (x_1 \ldots x_n) &= \sum_{\sigma \in S_n} \sum_{r=1}^{n-1} \varepsilon_x(\sigma) \frac{1}{j!k!} \sum_{r_1 + \cdots + r_j = r, s_1 + \cdots + s_k = n-r} 1 \frac{1}{r_1! \ldots r_j! s_1! \ldots s_k!} \\
&\quad \cdot F_{r_1}(x_{\sigma_1} \ldots x_{\sigma_{r_1}}) \ldots F_{r_j}(\ldots) \otimes F_{s_1}(\ldots) \ldots F_{s_k}(\ldots x_{\sigma_n}).
\end{align*}
\]

Posant $l = j + k$ et procédant à la renumérotation $(s_1, \ldots, s_k) = (r_{j+1}, \ldots, r_l)$ on obtient:

\[
\begin{align*}
(F \otimes F) \Delta (x_1 \ldots x_n) &= \sum_{\sigma \in S_n} \varepsilon_x(\sigma) \sum_{l \geq 2, r_1 + \cdots + r_l = n} \sum_{k \geq 1} \frac{1}{k!(l-k)!} \frac{1}{r_1! \ldots r_l!} \\
&\quad \cdot F_{r_1}(x_{\sigma_1} \ldots x_{\sigma_{r_1}}) \ldots F_{r_k}(\ldots) \otimes F_{r_{k+1}}(\ldots) \ldots F_{r_l}(\ldots x_{\sigma_n}) \\
&= \Delta F(x_1 \ldots x_n)
\end{align*}
\]

compte tenu du calcul précédent, ce qui démontre le théorème. \qed

IV. L_∞-algèbres et L_∞-morphismes.

A tout espace vectoriel gradué $V$ on associe (attention au décalage!) la variété formelle $(V[1], 0)$ pointée, c’est à dire la cogèbre colibre sans co-unité:

\[ C(V) = S^+(V[1]) \xrightarrow{\Phi} \sum_{k \geq 1} (\Lambda^k V)[k] \]

où $\Phi$ est l’isomorphisme décrit au §II.4.

Un pré-L_∞-morphisme entre deux espaces gradués $V_1$ et $V_2$ est par définition un morphisme de variétés formelles, c’est-à-dire un morphisme de cogèbres:

\[ \mathcal{F} : C(V_1) \longrightarrow C(V_2) \]

qui est donc déterminé par ses coefficients de Taylor $\mathcal{F}_j$. Posant $\overline{\mathcal{F}}_j = \mathcal{F}_j \circ \Phi^{-1}$ on a:

\[
\begin{align*}
\overline{\mathcal{F}}_1 : V_1 &\longrightarrow V_2 \\
\overline{\mathcal{F}}_2 : \Lambda^2 V_1 &\longrightarrow V_2[-1] \\
\overline{\mathcal{F}}_3 : \Lambda^3 V_1 &\longrightarrow V_2[-2] \\
&\vdots
\end{align*}
\]
IV.1. Algèbres de Lie homotopiques. Par définition une $L_\infty$-algèbre, ou algèbre de Lie homotopique est une variété formelle graduée pointée du type $(\mathfrak{g}[1], 0)$, où $\mathfrak{g}$ est un espace vectoriel gradué, munie d’un champ de vecteurs $Q$ de degré 1 vérifiant l’équation maîtresse:

$$[Q, Q] = 2Q^2 = 0.$$ 

C’est-à-dire que $Q$ est une codérivation de carré nul de la cogèbre $C(\mathfrak{g})$. Les coefficients de Taylor $Q_k : S^k(\mathfrak{g}[1]) \to \mathfrak{g}[2]$ donnent naissance aux coefficients $\overline{Q}_k = Q_k \circ \Phi^{-1}$:

- $\overline{Q}_1 : \mathfrak{g} \to \mathfrak{g}[1]$
- $\overline{Q}_2 : \Lambda^2 \mathfrak{g} \to \mathfrak{g}$
- $\overline{Q}_3 : \Lambda^3 \mathfrak{g} \to \mathfrak{g}[-1]$

L’équation maîtresse se traduit par une infinité de relations quadratiques entre les $\overline{Q}_k$, qui s’obtiennent en écrivant explicitement pour tout $k$ l’équation:

$$\pi Q^2(x_1 \ldots x_k) = 0$$

où $\pi : C(\mathfrak{g}) \to \mathfrak{g}[1]$ est la projection canonique. On écrit explicitement les trois premières:

Première équation: $Q_1^2(x) = 0$ pour tout $x$ dans $\mathfrak{g}$. Donc $(\mathfrak{g}, Q_1)$ est un complexe de cochaînes.

Deuxième équation: $\pi Q^2(x, y) = 0$, soit:

$$Q_2(Q_1 x, y) + (-1)^{|x|-1} x . Q_1 y + Q_1 Q_2(x, y) = 0.$$  

(Remarque: $|x| - 1$ est bien le degré de $x$ dans la cogèbre $C(\mathfrak{g})$, à cause du décalage). Traduisant cette égalité en termes de $\overline{Q}_1$ et $\overline{Q}_2$ on obtient (cf. §II.4):

$$\alpha(\overline{Q}_1 x, y)\overline{Q}_2(\overline{Q}_1 x \wedge y) + (-1)^{|x|-1} \alpha(x, \overline{Q}_1 y)\overline{Q}_2(x \wedge \overline{Q}_1 y) + \alpha(x, y)\overline{Q}_1 \overline{Q}_2(x \wedge y) = 0.$$ 

Compte tenu de l’égalité:

$$\alpha(x, y) = (-1)^{|x|(|y|-1)}$$

on obtient:

$$(-1)^{|y|-1} \overline{Q}_2(\overline{Q}_1 x \wedge y) - \overline{Q}_2(x \wedge \overline{Q}_1 y) + \overline{Q}_1 \overline{Q}_2(x \wedge y) = 0.$$ 

Posant $dx = (-1)^{|x|}\overline{Q}_1 x$ et $[x, y] = \overline{Q}_2(x \wedge y)$ on obtient finalement:

$$d[x, y] = [dx, y] + (-1)^{|x|}[x, dy]$$

donc $\overline{Q}_2$ est un crochet antisymétrique pour lequel $d$ est une dérivation.
Remarque. On peut garder $\overline{Q}_1$ comme dérivation sans le modifier, à condition d’inverser le sens du crochet, c’est-à-dire de poser:

$$[x, y] = \overline{Q}_2(y \wedge x).$$

Nous choisirons la première solution.

Troisième équation: $\pi Q_3(x.y.z) = 0$ soit:

$$Q_3 \left( Q_1 x.y.z + (-1)^{|x|-1} x.Q_1 y.z + (-1)^{|x|+|y|-2} x.y.Q_1 z \right) + Q_1 Q_3(x.y.z) + Q_2(x.y).z + (-1)^{|y|-1}|z|^{-1} Q_2(x.z).y$$

soit:

$$Q_2 \left( Q_2(x.y).z + (-1)^{|y|-1} Q_2(z.x).y + (1)^{|z|-1} Q_2(y.z).x \right) + \text{termes en } Q_3 = 0.$$

Or on a:

$$Q_2(Q_2(x.y).z) = \alpha(Q_2(x.y), z)\alpha(x, y)Q_2(Q_2(x \wedge y) \wedge z)$$

$$= (-1)^{|x|+|y|+|z|} (-1)^{|y|-1} Q_2(Q_2(x \wedge y) \wedge z).$$

En reportant ceci dans l’équation précédente et en simplifiant par $(-1)^{|x|+|y|+|z|}$ on obtient finalement:

$$(1)^{|z|}[[x, y], z] + (1)^{|y|}[z, x] + (1)^{|z|} termes en Q_3 = 0.$$

Autrement dit le crochet fourni par $\overline{Q}_2$ vérifie l’identité de Jacobi graduée “à homotopie gouvernée par $\overline{Q}_2$ près”. En corollaire:

Théorème IV.1.1. Une algèbre de Lie différentielle graduée est la même chose qu’une $L_\infty$-algèbre pour laquelle tous les coefficients de Taylor sont nuls sauf les deux premiers.

IV.2. L’algèbre de Lie différentielle graduée des mutichamps de vecteurs. Sur $V = T_{poly}(\mathbb{R}^d)$, on dispose du crochet de Schouten défini par:

$$[\xi_1 \wedge \ldots \wedge \xi_k, \eta_1 \wedge \ldots \wedge \eta_l]_S$$

$$= \sum_{i=1}^k \sum_{j=1}^l (-1)^{i+j} [\xi_i, \eta_j] \wedge \xi_1 \wedge \ldots \wedge \hat{\xi}_i \wedge \ldots \wedge \xi_k \wedge \eta_1 \wedge \ldots \wedge \hat{\eta}_j \wedge \ldots \wedge \eta_l.$$

La symérisation de $Tens(\mathbb{R}^d)$ nous permet de définir une opération •. Si $\alpha_1$ est un $k_1$-tenseur antisymétrique:

$$\alpha_1 = \alpha_1^{i_1 \ldots i_{k_1}} \partial_{i_1} \wedge \partial_{i_2} \wedge \ldots \wedge \partial_{i_{k_1}} \in Tens^{k_1}(\mathbb{R}^d),$$

CHOIX DES SIGNES POUR LA FORMALITÉ DE M. KONTSEVICH 45
alors:

\[ \Phi_{k_1}^{-1}(\alpha_1) = \alpha_1^{i_1...i_{k_1}} \psi_{i_1} \ldots \psi_{i_{k_1}} \in S\left( \mathcal{X}\left( \mathbb{R}^d \right) \right) [-k_1] \]

où chaque \( \psi_i = \Phi_{k_1}^{-1}(\partial_i) \) est une variable de degré 1.

Si maintenant \( \alpha_2 \) est un \( k_2 \) tenseur antisymétrique, on posera:

\[ \alpha_1 \bullet \alpha_2 = \Phi_{k_1+k_2-1}\left( \sum_{i=1}^d \frac{\partial \Phi_{k_1}^{-1}(\alpha_1)}{\partial \psi_i} \cdot \frac{\partial \Phi_{k_2}^{-1}(\alpha_2)}{\partial x_i} \right) \]

en tenant compte du fait que \( \partial / \partial \psi_i \) est un opérateur de dérivation impair.

\textbf{Lemme IV.2.1 (Calcul de \( \alpha_1 \bullet \alpha_2 \)).} On a:

\[ \alpha_1 \bullet \alpha_2 = \sum_{l=1}^{k_1} (-1)^{l-1} \alpha_1^{i_1...i_{k_1}} \psi_{i_1} \ldots \psi_{i_{k_1}} \partial_{i_1} \psi_{j_1} \ldots \partial_{i_{k_1}} \psi_{j_{k_2}} \]

et

\[ [\alpha_1, \alpha_2]_S = (-1)^{k_1-1} \alpha_1 \bullet \alpha_2 - (-1)^{k_1(k_2-1)} \alpha_2 \bullet \alpha_1. \]

\textbf{Démonstration.} On a:

\[ \frac{\partial}{\partial \psi_i} \left( \alpha_1^{i_1...i_{k_1}} \psi_{i_1} \ldots \psi_{i_{k_1}} \right) = \sum_{l=1}^{k_1} (-1)^{l-1} \alpha_1^{i_1...i_{k_1}} \psi_{i_1} \ldots \psi_{i_{k_1}} \frac{\partial \psi_{i_1}}{\partial \psi_i} \ldots \frac{\partial \psi_{i_{k_1}}}{\partial \psi_i} \]

Donc:

\[ \sum_{i=1}^d \frac{\partial \Phi_{k_1}^{-1}(\alpha_1)}{\partial \psi_i} \cdot \frac{\partial \Phi_{k_2}^{-1}(\alpha_2)}{\partial x_i} \]

\[ = \sum_{l=1}^{k_1} (-1)^{l-1} \alpha_1^{i_1...i_{k_1}} \psi_{i_1} \ldots \psi_{i_{k_1}} \partial_{i_1} \alpha_2^{j_1...j_{k_2}} \psi_{j_1} \ldots \psi_{j_{k_2}} \]

\[ = \sum_{l=1}^{k_1} (-1)^{l-1} \alpha_1^{i_1...i_{k_1}} \partial_{i_1} \alpha_2^{j_1...j_{k_2}} \psi_{i_1} \ldots \psi_{i_{k_1}} \psi_{j_1} \ldots \psi_{j_{k_2}}. \]
D’autre part:

\[ [\alpha_1, \alpha_2]_S = [\alpha_1^{i_1 \ldots i_k} \partial_{i_1} \wedge \ldots \wedge \partial_{i_k}, \alpha_2^{j_1 \ldots j_k} \partial_{j_1} \wedge \ldots \wedge \partial_{j_k}] \]

\[ = [\alpha_1^{i_1 \ldots i_k} \partial_{i_1}, \alpha_2^{j_1 \ldots j_k} \partial_{j_1}] \wedge \partial_{i_2} \wedge \ldots \wedge \partial_{i_k} \wedge \partial_{j_2} \wedge \ldots \wedge \partial_{j_k} \]

\[ + \sum_{l=2}^{k_2} (-1)^{l+1} [\alpha_1^{i_1 \ldots i_k}, \partial_{i_1}, \partial_{j_1}] \wedge \partial_{j_2} \wedge \ldots \wedge \partial_{j_k} \wedge \partial_{i_k} \wedge \partial_{i_2} \wedge \ldots \wedge \partial_{i_l} \wedge \partial_{i_{l+1}} \wedge \ldots \wedge \partial_{i_k} \]

\[ + \sum_{l=2}^{k_1} [\partial_{i_1}, \alpha_2^{j_1 \ldots j_k} \partial_{j_1}] \wedge \alpha_1^{i_1 \ldots i_k} \partial_{i_1} \wedge \ldots \wedge \partial_{i_{l-1}} \wedge \partial_{i_l} \wedge \partial_{i_{l+1}} \wedge \ldots \wedge \partial_{i_k} \]

\[ = - \sum_{l=1}^{k_2} (-1)^{l+1} \alpha_2^{j_1 \ldots j_k} \partial_{j_1} \alpha_1^{i_1 \ldots i_k} \partial_{i_1} \wedge \ldots \wedge \partial_{i_{k-1}} \wedge \partial_{i_k} \wedge \partial_{j_2} \wedge \ldots \wedge \partial_{j_k} \]

\[ + \sum_{l=1}^{k_1} (-1)^{l+1} \alpha_1^{i_1 \ldots i_k} \partial_{i_1} \alpha_2^{j_1 \ldots j_k} \partial_{j_1} \wedge \partial_{i_2} \wedge \ldots \wedge \partial_{i_{k-1}} \wedge \partial_{i_k} \wedge \partial_{j_2} \wedge \ldots \wedge \partial_{j_k} \]

\[ = (-1)^{k_1-1} \alpha_1 \alpha_2 - (-1)^{(k_2-1)k_1} \alpha_2 \alpha_1. \]

**Corollaire IV.2.2.** L’espace gradué \( T_{\text{poly}}(\mathbb{R}^d) \), muni du crochet:

\[ [\alpha_1, \alpha_2]_S' = - [\alpha_2, \alpha_1]_S \]

est aussi une algèbre de Lie graduée et:

\[ [\alpha_1, \alpha_2]'_S = (-1)^{(k_1-1)k_2} \alpha_1 \alpha_2 + (-1)^{k_2} \alpha_2 \alpha_1. \]

Comme \([\ , \]_S \) définit sur \( T_{\text{poly}}(\mathbb{R}^d) \) une structure d’algèbre de Lie graduée, on aura, en prenant \( d = 0 \), une structure de \( L_\infty \) algèbre sur \( \mathcal{C}(T_{\text{poly}}(\mathbb{R}^d)) \).

Le champ de vecteurs \( Q \) est caractérisé par:

\[ Q_1 = 0, \quad Q_2(\alpha_1, \alpha_2) = (-1)^{(k_1-1)k_2} [\alpha_1, \alpha_2]'_S \]

\[ = \alpha_1 \alpha_2 + (-1)^{k_1k_2} \alpha_2 \alpha_1. \]

**IV.3.** L’algèbre de Lie différentielle graduée des opérateurs poly-différentiels. On considère l’espace vectoriel \( V' = D_{\text{poly}}(\mathbb{R}^d) \) des (combinaisons linéaires d’) opérateurs multidifférentiels gradué par \( |A| = m - 1 \) si \( A \) est \( m \)-différentiel.
Sur $D_{\text{poly}}(\mathbb{R}^d)$, l’opérateur de composition naturel $\circ$ s’écrit:
\[
(A_1 \circ A_2)(f_1, \ldots, f_{m_1+m_2-1}) = \sum_{j=1}^{m_1} (-1)^{(m_2-1)(j-1)} A_1(f_1, \ldots, f_{j-1}, A_2(f_j, \ldots, f_{j+m_2-1}, f_{j+m_2}, \ldots, f_{m_1+m_2-1}).
\]
On associe à cette composition d’une part le crochet de Gerstenhaber:
\[
[A_1, A_2]_G = A_1 \circ A_2 - (-1)^{|A_1||A_2|} A_2 \circ A_1,
\]
d’autre part l’opérateur de cobord:
\[
dA = -[\mu, A]
\]
où $\mu$ est la multiplication des fonctions: $\mu(f_1, f_2) = f_1 f_2$.

Remarque. Avec ce choix de $d$, $(D_{\text{poly}}(\mathbb{R}^d), [\ , \ ]_G, d)$ est une algèbre de Lie graduée différentielle, on vérifie en effet que $d \circ d = 0$ et
\[
d([A_1, A_2]) = [dA_1, A_2] + (-1)^{|A_1|}[A_1, dA_2].
\]
L’opérateur de cobord de Hochschild usuel $d_H$ donné par:
\[
(d_H A)(f_1, \ldots, f_m) = f_1 A(f_2, \ldots, f_m) - A(f_1 f_2, f_3, \ldots, f_m) + \cdots + (-1)^m A(f_1, \ldots, f_{m-1}) f_m
\]
\[
= (-1)^{|A|+1} dA(f_1, \ldots, f_m)
\]
n’est pas une dérivation de l’algèbre de Lie graduée $(D_{\text{poly}}(\mathbb{R}^d), [\ , \ ]_G)$.
Le champ de vecteurs $Q'$ sur la variété formelle $C(V')$ sera donc défini par:
\[
Q'_1(A) = (-1)^{|A|} dA = (-1)^{|A|+1} [\mu, A] = [A, \mu] = -d_H A
\]
et
\[
Q'_2(A_1, A_2) = (-1)^{|A_1||A_2|-1} [A_1, A_2]_G
\]
\[
= (-1)^{|A_1||A_2|-1} A_1 \circ A_2 - (-1)^{|A_1|} A_2 \circ A_1.
\]

IV.4. $L_\infty$-morphismes. Par définition un $L_\infty$-morphisme entre deux $L_\infty$-algèbres $(g_1, Q)$ et $(g_2, Q')$ est un morphisme de variétés formelles pointées:
\[
\mathcal{F} : C(g_1) \rightarrow C(g_2)
\]
vérifiant:
\[
\mathcal{F} Q = Q' \mathcal{F}.
\]
Cette équation induit une infinité de relations entre les coefficients de Taylor de $Q$, $Q'$ et $\mathcal{F}$, dont nous allons examiner les deux premières:

Première équation: $Q'_1 \mathcal{F}_1(x) = \mathcal{F}_1 Q_1(x)$, c’est-à-dire que $\mathcal{F}_1$ est un morphisme de complexes.
Deuxième équation: \( \pi Q'F(x,y) = \pi FQ(x,y) \) soit:

\[
\pi Q'(F_1 x.F_1 y + F_2(x,y)) = \pi F(Q_1 x.y + (-1)^{|x|-1} x.Q_1 y + Q_2(x,y))
\]

soit encore:

\[
Q_2'(F_1 x.F_1 y) + Q_1'F_2(x,y) = F_2(Q_1 x.y + (-1)^{|x|-1} x.Q_1 y) + F_1 Q_2(x,y).
\]

On traduit cette dernière égalité en termes de \( Q_1, Q_2, F_1 \), etc.:

\[
(-1)^{|x|(|y|-1)}[F_1 x, F_1 y] + (-1)^{|x|+|y|-1+|x|(|y|-1)} dF_2(x \land y)
\]

\[
= (-1)^(|x|-1)(|y|-1+|x|)[F_2(dx \land y) + (-1)^{|x||y|+|x|-1+|y|} F_2(x \land dy)
\]

soit, en multipliant par \((-1)^{|x|(|y|-1)}\):

\[
[F_1 ([x, y])] - [F_1 x, F_1 y]
\]

\[
= (-1)^{|x|+|y|-1} (dF_2(x \land y) - F_2(dx \land y) - (-1)^{|x|} F_2(x \land dy)).
\]

Dans le cas où \( g_1 \) et \( g_2 \) sont des algèbres de Lie différentielles graduées, \( F_1 \) n’est donc pas forcément un morphisme d’algèbres de lie différentielles graduées, mais le défaut est gouverné par le coefficient suivant, c’est-à-dire \( F_2 \).

**Proposition IV.4.1** (Equation de \( L_\infty \)-morphisme dans le cas des algèbres de Lie différentielles graduées). *Supposons que \( (V, [\ , \ ] , d) \) et \( (V', [ \ , \ ]', d') \) soient deux algèbres de Lie graduées. Notons \( (C(V), Q) \) et \( (C(V'), Q') \) les \( L_\infty \) algèbres correspondantes respectives. Soit \( F : C(V) \longrightarrow C(V') \) un morphisme de cogèbre. Alors \( F \) est un \( L_\infty \) morphisme si et seulement si:

\[
Q_1'F_n (\alpha_1, \ldots, \alpha_n) + \frac{1}{2} \sum_{I,J=\{1, \ldots, n\}} \varepsilon_{\alpha}(I,J)Q_2' (F_{[I]} (\alpha_I).F_{[J]} (\alpha_J))
\]

\[
= \sum_{k=1}^{n} \varepsilon_{\alpha}(k,1,\ldots,\hat{k},\ldots,n)F_n (Q_1(\alpha_k).\alpha_1,\ldots,\hat{\alpha}_k,\ldots,\alpha_n)
\]

\[
+ \frac{1}{2} \sum_{k \neq l} \varepsilon_{\alpha}(k,l,1,\ldots,\hat{k},l,\ldots,n)
\]

\[
F_{n-1} (Q_2(\alpha_k.\alpha_l).\alpha_1,\ldots,\hat{\alpha}_k,\ldots,\hat{\alpha}_l,\ldots,\alpha_n)
\]

où \(|I|\) et \( \varepsilon_{\alpha}(I,J) \) ont la même signification que dans le Théorème III.2.1, et où \( \varepsilon_{\alpha}(\ldots) \) désigne le signe de Quillen de la permutation indiquée entre parenthèses, c’est-à-dire la signature de la trace sur les \( \alpha_j \) impairs de cette permutation.*

Comme pour les codérivations \( Q \) et les morphismes de cogèbres \( F \), il est facile de voir que les applications \( Q'F \) et \( FQ \) sont uniquement déterminées par leur composition avec la projection sur \( V'[1] \). On déduit alors l’équation
de $L_\infty$-morphisme sous la forme $(Q'F)_n = (FQ)_n$ pour tout $n$. Puisqu’on est parti de deux algèbres de Lie différentielles graduées, tous les $Q_p$ et $Q'_p$ sont nuls pour $p \geq 3$.

V. Quasi-isomorphismes.

Par définition un quasi-isomorphisme entre deux $L_\infty$-algèbres $(\mathfrak{g}_1, Q_1)$ et $(\mathfrak{g}_2, Q_2)$ est un $L_\infty$-morphisme $F$ dont le premier coefficient de Taylor $F_1 : \mathfrak{g}_1[1] \rightarrow \mathfrak{g}_2[1]$ est un morphisme de complexes qui induit un isomorphisme en cohomologie (quasi-isomorphisme de complexes). Nous allons exposer la démonstration du théorème suivant ([K1] Theorem 4.4):

Théorème V.1. Pour tout quasi-isomorphisme $F$ d’une $L_\infty$-algèbre $(\mathfrak{g}_1, Q_1)$ vers une $L_\infty$-algèbre $(\mathfrak{g}_2, Q_2)$ il existe un $L_\infty$-morphisme $G$ de $(\mathfrak{g}_2, Q_2)$ vers $(\mathfrak{g}_1, Q_1)$ dont le premier coefficient de Taylor $G_1 : \mathfrak{g}_2[1] \rightarrow \mathfrak{g}_1[1]$ soit un quasi-inverse pour $F_1$.

V.1. Découpage des $L_\infty$-algèbres. Une $L_\infty$-algèbre $(\mathfrak{g}, Q)$ est minimale si $Q_1 = 0$. Une $L_\infty$-algèbre est linéaire contractile si $Q_j = 0$ pour $j \geq 2$ et si la cohomologie du complexe donné par $Q_1$ est triviale. On remarque que la première notion est invariante par $L_\infty$-isomorphismes, contrairement à la seconde notion.

Proposition V.2. Toute $L_\infty$-algèbre $(\mathfrak{g}, Q)$ est $L_\infty$-isomorphe à la somme directe d’une $L_\infty$-algèbre minimale et d’une $L_\infty$-algèbre linéaire contractile.

Démonstration. On décompose le complexe $(\mathfrak{g}, Q)$ en somme directe $(\mathfrak{g}', M_1) \oplus (\mathfrak{g}''', L_1)$ où $M_1$ est une différentielle nulle et où $(\mathfrak{g}'', L_1)$ est un complexe à cohomologie triviale (on néglige le décalage qui n’est pas essentiel ici). Pour ce faire on note comme d’habitude $Z_k$ et $B_k$ le noyau et l’image de la différentielle en degré $k$, on choisit un supplémentaire $g'_k$ de $B_k$ dans $Z_k$, et un supplémentaire $W_k$ de $Z_k$ dans $\mathfrak{g}_k$. Posant alors $g''_k = B_k \oplus W_k$ on a la décomposition cherchée.

Cette décomposition du complexe est le point de départ de la décomposition de la $L_\infty$-algèbre $(\mathfrak{g}, Q)$. La cogèbre associée à $\mathfrak{g} = \mathfrak{g}' \oplus \mathfrak{g}''$ s’écrit:

$$C(\mathfrak{g}) = C(\mathfrak{g}') \oplus C(\mathfrak{g}'') \oplus C(\mathfrak{g}') \otimes C(\mathfrak{g}'').$$

Il s’agit de construire un isomorphisme de cogèbres:

$$\mathcal{F} : C(\mathfrak{g}) \xrightarrow{\cong} C(\mathfrak{g})$$

tel que $\mathcal{F} \circ Q = Q \circ \mathcal{F}$, avec:

$$\overline{Q}_{|C(\mathfrak{g}')} = M$$

$$\overline{Q}_{|C(\mathfrak{g}'')} = L$$

$$\overline{Q}_{|C(\mathfrak{g}') \otimes C(\mathfrak{g}'')} = M \otimes I + I \otimes L$$
où $M_1 = 0$, $L_j = 0$ pour $j \geq 2$ et $L_1$ à cohomologie triviale. On pose donc pour commencer $\mathcal{F}_1 = \text{Id} : \mathfrak{g} \rightarrow \mathfrak{g}$, d’où forcément $\overline{Q}_1 = Q_1 = L_1$ au vu de la décomposition du complexe rappelée ci-dessus. Il est très facile de voir qu’un $L_\infty$-morphisme $\mathcal{F}$ vérifiant $\mathcal{F}_1 = \text{Id}$ s’écrit comme un produit infini:

$$\mathcal{F} = \ldots \mathcal{F}^k \mathcal{F}^{k-1} \ldots \mathcal{F}^2$$

où $\mathcal{F}^k$ est le $L_\infty$-morphisme ayant l’identité comme premier coefficient de Taylor, $\mathcal{F}_k$ comme $k$ième coefficient de Taylor, tous les autres coefficients étant nuls.

Chercher le coefficient $\mathcal{F}_k$ en supposant que les $\mathcal{F}_j$ sont connus pour $j < k$, c’est donc chercher un $L_\infty$-isomorphisme “lacunaire” comme le $\mathcal{F}^k$ ci-dessus, entre $(\mathfrak{g}' \oplus \mathfrak{g}'', Q)$ et $(\mathfrak{g}' \oplus \mathfrak{g}'', \overline{Q})$, où le champ de vecteurs impair $Q$ vérifie:

$$\begin{align*}
Q_1\vert_{\mathcal{C}(\mathfrak{g}')} &= 0 \\
Q_j(\mathcal{C}(\mathfrak{g}')) &\subset \mathfrak{g}' \text{ pour } j \leq k - 1 \\
Q_j(\mathcal{C}(\mathfrak{g}'')) &= 0 \text{ pour } 2 \leq j \leq k - 1 \\
Q_j(\mathcal{C}(\mathfrak{g}'') \otimes \mathcal{C}(\mathfrak{g}'')) &= 0 \text{ pour } j \leq k - 1
\end{align*}$$

et où le champ de vecteurs $\overline{Q}$ vérifie les mêmes conditions avec $k$ à la place de $k - 1$. On supposera également que les coefficients de Taylor de $Q$ et $\overline{Q}$ sont les mêmes jusqu’à l’ordre $k - 1$ et sont nuls à partir de l’ordre $k + 1$. Il s’agit donc simplement de trouver $\mathcal{F}_k$ et $\overline{Q}_k$.

La condition $\mathcal{F} \circ Q = \overline{Q} \circ \mathcal{F}$ s’écrit, en négligeant les signes provenant de la supersymétrie:

$$\begin{align*}
(\ast) \quad &\mathcal{F}_k(Q_1(x_1 \cdots x_k)) + Q_k(x_1 \cdots x_k) = Q_1\mathcal{F}_k(x_1 \cdots x_k) + \overline{Q}_k(x_1 \cdots x_k)
\end{align*}$$

où l’on a désigné par la même lettre $Q_1$ la dérivation de l’algèbre $S(\mathfrak{g}[1])$ valant $Q_1$ sur $\mathfrak{g}[1]$.

1). Si tous les $x_j, j = 1 \cdots k$ sont dans le noyau $Z$ de $Q_1$, l’équation $(\ast)$ se réduit à:

$$(\ast)_1 \quad Q_k(x_1 \cdots x_k) = Q_1\mathcal{F}_k(x_1 \cdots x_k) + \overline{Q}_k(x_1 \cdots x_k).$$

On choisit donc $\overline{Q}_k(x_1 \cdots x_k)$ comme étant la projection de $Q_k(x_1 \cdots x_k)$ sur le supplémentaire $\mathfrak{g}'$ de $B$ dans $Z$. Ceci permet de définir $\mathcal{F}_k(x_1 \cdots x_k)$ à un élément $z$ de $Z$ près.

De plus si $x_1 = Q_1 y_1 \in B$, l’équation maitresse $[Q, Q] = 0$ s’écrit (toujours en négligeant les problèmes de signes):

$$Q_k(Q_1 y_1 x_2 \cdots x_k) + \text{termes intermédiaires} + Q_1 Q_k(y_1 x_2 \cdots x_k) = 0.$$

Les termes intermédiaires sont une somme de termes du type:

$$Q_j(\cdots Q_l(\cdots \cdots), j, l < k.$$
L’élément $Q_1y_1$ se trouve dans une parenthèse interne ou dans la parenthèse externe. Dans les deux cas l’hypothèse de départ sur $Q$ entraîne l’annulation de ce terme. On a donc:

$$Q_k(Q_1y_1x_2\cdots x_k) = -Q_1Q_k(y_1x_2\cdots x_k),$$

ce qui montre que $\overline{Q}_k(Q_1y_1x_2\cdots x_k) = 0$.

2). Soit $x \in S^k(g[1])$, avec $k \geq 2$. On dit que $x$ est de type $j$, $0 \leq j \leq k$, si $x$ s’écrit $x_1\cdots x_k$ avec $x_1, \ldots, x_j \in W$ et $x_{j+1}, \ldots, x_k \in Z$. Nous allons déterminer $\mathcal{F}_k(x)$ par récurrence (finie) sur le type de $x$, le type 0 ayant été traité au 1). On remarque que $\overline{Q}_k(x) = 0$ si le type de $x$ est non nul.

L’équation $[Q,Q] = 0$ s’écrit:

$$Q_k(Q_1(x_1\cdots x_k)) + Q_1Q_k(x_1\cdots x_k) = 0,$$

les termes intermédiaires s’annulant pour la même raison que dans le 1). On a donc:

$$(M) \quad Q_1Q_k(x) + Q_kQ_1(x) = 0$$

pour tout $x \in S^k(g[1])$.

Soit $r \geq 1$. Supposons que $\mathcal{F}_k(x)$ soit déterminé pour tout $x$ de type $j \leq r - 2$, et déterminé à un $z \in Z$ près pour tout $x$ de type $r - 1$. Soit alors $x$ de type $r$. On veut déterminer $\mathcal{F}_k(x)$ à un élément $z' \in Z$ près et préciser $\mathcal{F}_k(y)$ pour tous les $y$ de type $r - 1$.

L’équation $(\ast)$ appliquée à $Q_1(x)$ s’écrit:

$$\mathcal{F}_kQ_1^2(x) + Q_1Q_k(x) = Q_1\mathcal{F}_kQ_1(x)$$

le terme $\overline{Q}_kQ_1(x)$ étant nul. En reportant $(M)$ dans cette équation on a donc:

$$Q_1\mathcal{F}_kQ_1(x) + Q_1Q_k(x) = 0,$$

d'où:

$$\mathcal{F}_kQ_1(x) + Q_k(x) \in Z.$$  

Comme $\mathcal{F}_kQ_1(x)$ est déterminé à un élément arbitraire de $Z$ près, on peut s’arranger pour que:

$$\mathcal{F}_kQ_1(x) + Q_k(x) = b(x)$$

où $b(x)$ appartient à $B$. L’équation $(\ast)$ appliquée à $x$ s’écrivant:

$$\mathcal{F}_kQ_1(x) + Q_k(x) = Q_1\mathcal{F}_k(x)$$

le choix d’un $b(x)$ nous permet de choisir $\mathcal{F}_k(x)$ à un élément $z' \in Z$ près. Le $b(x)$ doit obéir à la contrainte suivante: si $Q_1(x) = 0$, alors $b(x) = Q_k(x)$.

Supposons que $x = Q_1y$ où $y$ est de type $r + 1$. Alors, compte tenu de $(M)$ la contrainte sur $b$ s’écrit:

$$b(x) = -Q_1Q_k(y).$$
Ayant choisi un \( b(x) \) pour tout \( x \) de type \( r \) satisfaisant à la contrainte ci-dessus, on peut alors choisir \( F_k(x) \) à un élément \( z' \in Z \) près. Il reste donc simplement à démontrer le lemme ci-dessous:

**Lemme V.3.** Soit \( x \) de type \( r \geq 1 \). Alors si \( Q_1x = 0 \) il existe un \( y \) de type \( r + 1 \) tel que \( x = Q_1y \).

**Démonstration.** On considère l’application \( \delta : g \to g \) de degré \(-1\) définie par \( \delta(x) = 0 \) pour \( x \in g' \oplus W \), et \( \delta(Q_1x) = x \) pour tout \( x \) dans \( g \). On a alors:

\[
Q_1\delta + \delta Q_1 = \text{Id} - p,
\]

où \( p \) est la projection sur \( g' \) parallèlement à \( g'' \) (autrement dit \( \delta \) est une homotopie entre les deux endomorphismes de complexes \( \text{Id} \) et \( p \)).

Le Lemme V.3 est un corollaire du résultat suivant, dû à Quillen [Q, Appendix B]:

**Proposition V.4.**

1. La dérivation \( Q_1 \) de l’algèbre symétrique \( S(g) \) vérifie:

\[
Q_1^2 = 0.
\]

2. La cohomologie du complexe \( (S(g),Q_1) \) est isomorphe à \( S(g') \), et un supplémentaire de l’image de \( Q_1 \) dans le noyau de \( Q_1 \) est donné par \( S(g') \otimes 1 \) moyennant l’identification:

\[
S(g) = S(g') \otimes S(g'').
\]

**Démonstration.** 1). Comme \( Q_1 \) est impaire, \( Q_1^2 = \frac{1}{2}[Q_1,Q_1] \) est encore une dérivation de \( S(g) \). Comme \( Q_1^2|_g = 0 \) cette dérivation est nulle.

2). On a: \( Q_1(v'v'') = v'Q_1(v'') \) pour \( v' \in S(g') \) et \( v'' \in S(g'') \). On est ramené au cas où la cohomologie de \( g \) est triviale. On prolonge alors l’homotopie \( \delta \) à \( g' \) en une dérivation de \( S(g) \). On pose alors:

\[
E = [Q_1,\delta] = Q_1\delta + \delta Q_1.
\]

\( E \) est une dérivation telle que \( E|_g = \text{Id} \). On en déduit:

\[
E(x) = kx
\]

pour tout \( x \in S^k(g) \). Si maintenant \( x \) appartient à \( S^k(g) \) et \( Q_1x = 0 \), alors \( Ex = Q_1\delta x = kx \). Si \( k \geq 1 \) on a donc:

\[
x = Q_1\left(\frac{1}{k}\delta x\right).
\]

La cohomologie de \( S(g) \) est donc réduite au corps de base, qui est \( S(\{0\}) \). □

**Fin de la démonstration du Lemme V.3:** le complexe \( V = S(g) \) admet à son tour une décomposition:

\[
V = V' \oplus V''
\]

avec \( V' = S(g') \otimes 1 \). L’image de \( Q_1 \) dans \( S(g) \) est l’idéal engendré par \( B = Q_1(g) \). On peut donc choisir pour \( V'' \) l’idéal engendré par \( g'' \). Le
lemme provient alors du fait que tout élément de type \( r \geq 1 \) appartient à cet idéal, sur lequel la cohomologie est triviale.

\[ \square \]

V.2. Démonstration du Théorème V.1. On se donne deux \( L_\infty \)-algèbres \( (g_1, Q_1) \) et \( (g_2, Q_2) \) et un quasi-isomorphisme \( F \) de \( (g_1, Q_1) \) vers \( (g_2, Q_2) \). Appliquant la Proposition V.2 à ces deux \( L_\infty \)-algèbres on a le diagramme suivant, dans lequel toutes les flèches sont des quasi-isomorphismes:

\[ C(g_1') \xrightarrow{i} C(g_1' \oplus g''_1) \xrightarrow{=} C(g_1) \xrightarrow{F} C(g_2) \xrightarrow{p} C(g_1'). \]

On a ainsi construit un quasi-isomorphisme \( F' \) entre deux \( L_\infty \)-algèbres minimales. Son premier coefficient \( F'_{11} : g_1' \rightarrow g_2' \) étant inversible, \( F' \) lui-même est inversible. L’ajout du quasi-isomorphisme \( F'^{-1} \) dans le diagramme ci-dessus permet alors la construction d’un quasi-isomorphisme:

\[ G : C(g_2) \xrightarrow{\sim} C(g_1) \]

qui est un quasi-inverse pour \( F \).

\[ \square \]

VI. La formalité de Kontsevich.

Un \( L_\infty \) morphisme entre \( T_{poly} (\mathbb{R}^d) \) et \( D_{poly} (\mathbb{R}^d) \) qui soit aussi un quasi-isomorphisme c’est à dire un isomorphisme en cohomologie est une formalité.

M. Kontsevich a proposé dans [K1] une formalité \( \mathcal{U} \) explicite. Précisément, les applications \( \mathcal{U}_n \) sont donnés par:

\[ \mathcal{U}_n = \sum_{m \geq 0} \sum_{\hat{r} \in G_{n,m}} w_{\hat{r}} \mathcal{B}_{\hat{r}} \]

où \( G_{n,m} \) est l’ensemble des graphes orientés admissibles à \( n \) sommets aériens \( p_1, \ldots, p_n \) et \( m \) sommets terrestres \( q_1, \ldots, q_m \): de chaque sommet aérien est issu \( k_1, \ldots, k_n \) flèches aboutissant soit à un autre sommet aérien soit à un sommet terrestre. On ordonne les sommets aériens et terrestres du graphe et on oriente le graphe en ordonnant les flèches de façon compatible avec cet ordre, les flèches issues du sommet \( p_j \) ont les numéros \( k_1 + \cdots + k_{j-1} + 1, \ldots, k_1 + \cdots + k_j \). On les note:

\[ \text{Star } (p_j) = \{ \hat{p}_j a_1, \ldots, \hat{p}_j a_{k_j} \} \quad \text{ où } \quad \hat{v}^{k_1 + \cdots + k_{j-1} + i} = \hat{p}_j a_i. \]

Si \( \hat{r} \) est un graphe orienté, son poids \( w_{\hat{r}} \) est par définition l’intégrale sur l’espace de configuration \( C^+_{\{p_1, \ldots, p_n\}, \{q_1, \ldots, q_m\}} \) de la forme:

\[ \omega_{\hat{r}} = \frac{1}{(2\pi)^{k_1! \cdots k_n!}} d\Phi_{\hat{r}_1} \wedge \ldots \wedge d\Phi_{\hat{r}_{k_1 + \cdots + k_n}} \]

où

\[ \Phi_{\hat{r}_a} = \text{Arg} \left( \frac{a - \hat{p}_j}{a - \hat{p}_j} \right). \]
Enfin $B_{\Gamma}$ est un opérateur $m$-différentiel, nul sur $\alpha_1, \ldots, \alpha_n$ sauf si $\alpha_1$ est un $k_1$-tenseur, $\alpha_2$ un $k_2$-tenseur, \ldots, $\alpha_n$ un $k_n$-tenseur, auquel cas, on a:

$$B_{\Gamma}(\alpha_1, \ldots, \alpha_n)(f_1, f_2, \ldots, f_m) = \sum D_{p_1} \alpha_1^{i_1i_2\ldots i_{k_1}} \cdots D_{p_m} \alpha_n^{i_{k_1+\cdots+k_{n-1}+1+\cdots+k_n}} D_{q_1} f_1 \cdots D_{q_m} f_m$$

si $D_a$ est l’opérateur:

$$D_a = \prod_{i, \vec{v} = \vec{a}} \partial_i$$

et si la somme est étendue à tous les indices $i_j$ répétés. On notera aussi

$$U_n = \sum U(k_1, k_2, \ldots, k_n) = \sum U_{k(1, \ldots, n)}.$$

Maintenant, si on change l’ordre des flèches issues d’un sommet $p_j$, le produit $w_{\Gamma} B_{\Gamma}$ ne change pas. On prend la convention suivante: si $\Gamma$ est un graphe orienté de façon non compatible, on pose:

$$B_{\Gamma} = \varepsilon(\sigma) B_{\Gamma^\sigma}$$

où $\sigma$ est n’importe quelle permutation des flèches de $\Gamma$ qui le transforme en un graphe $\Gamma^\sigma$ orienté de façon compatible. Avec cette convention, on aura:

$$U_n = \sum_{m \geq 0} \sum_{\Gamma \in G_{n,m}} w'_{\Gamma} B_{\Gamma}$$

où $G'_{n,m}$ est l’ensemble de tous les graphes orientés de façon compatible ou non et $w'_{\Gamma}$ est l’intégrale de la forme:

$$\omega_{\Gamma} = \frac{1}{(2\pi)^{\sum k_i (\sum k_i)!}} d\Phi_{\vec{w}_1} \wedge \ldots \wedge d\Phi_{\vec{w}_{k_1+\cdots+k_n}}$$

où $\Phi_{\vec{w}_j} = Arg\left(\frac{a - p_j}{a - \overline{p}_j}\right)$.

Nous allons vérifier dans la suite que nos choix de signes sont cohérents.

**Théorème VI.1** (M.Kontsevich). L’application formelle $U$ est une formalité. En particulier c’est un $L_\infty$-morphisme.

**Démonstration.** Puisque $Q_1 = 0$, l’équation de formalité s’écrit:

$$0 = Q_1\left(U_{k(1, \ldots, n)}(\alpha_1, \ldots, \alpha_n)\right) + \frac{1}{2} \sum_{I, J = (1, \ldots, n)} \varepsilon_{\alpha}(I, J) Q_2\left(U_{k_1}(\alpha_I) \circ U_{k_2}(\alpha_J)\right)$$

$$- \frac{1}{2} \sum_{i \neq j} \varepsilon_{\alpha}(i, j, 1, \ldots, \widehat{i}, \widehat{j}, \ldots, n) U((k_i + k_j - 1, k_i, \ldots, k_i, k_j, \ldots, k_n))$$

$$\cdot \left( Q_2(\alpha_i, \alpha_j, \alpha_1, \ldots, \widehat{\alpha_i}, \ldots, \widehat{\alpha_j}, \ldots, \alpha_n) \right).$$
Remarquons maintenant que pour que $w_\Gamma$ ne soit pas nul, il faut que le degré de la forme $\omega_\Gamma$ soit égal à la dimension de l’espace de configuration $C^+\{p_1, \ldots, p_n\} \setminus \{q_1, \ldots, q_m\}$ sur lequel on intègre, c’est à dire:

$$\sum k_i = 2n + m - 2.$$ 

Dans ce cas,

$$(-1)^m = (-1)^{|U_k(1,\ldots,n)|} \sum k_i = (-1)^{|k_i(1,\ldots,n)|}.$$ 

Donc notre équation devient:

\begin{align*}
(1) & \quad 0 = U_{k_1(1,\ldots,n)}(\alpha_{1,\ldots,n}) \circ \mu - (-1)^{\sum k_i} \mu \circ U_{k_1(1,\ldots,n)}(\alpha_{1,\ldots,n}) + \\
(2) & \quad + \frac{1}{2} \sum_{I,J \neq \emptyset} \varepsilon_\alpha(I,J)(-1)^{|k_I|^1} |k_J| \varepsilon_\alpha(I,J) \circ U_{k_J}(\alpha_J) - \\
(3) & \quad + \frac{1}{2} \sum_{I,J \neq \emptyset} \varepsilon_\alpha(I,J)(-1)^{|k_J|} |k_I| \varepsilon_\alpha(I,J) \circ U_{k_I}(\alpha_I) - \\
(4) & \quad - \frac{1}{2} \sum_{i \neq j} \varepsilon_\alpha(i,j,1,\ldots,i,j,\ldots,n) U_{((k_i+k_j-1),k_i,\ldots,k_i,\ldots,k_j,\ldots,k_n)} \\
& \quad \cdot (\alpha_i \alpha_j) \alpha_1 \ldots \alpha_i \ldots \alpha_j \ldots \alpha_n - \\
(5) & \quad - \frac{1}{2} \sum_{i \neq j} \varepsilon_\alpha(i,j,1,\ldots,i,j,\ldots,n) U_{((k_i+k_j-1),\ldots,k_i,\ldots,k_j,\ldots,k_n)} \\
& \quad \cdot (-1)^{k_i k_j} (\alpha_i \alpha_j) \ldots \alpha_i \ldots \alpha_j \ldots \alpha_n.
\end{align*}

Montrons que $(2) = (3)$. En fait:

$$\varepsilon_\alpha(I,J) = \varepsilon_\alpha(J,I)(-1)^{|k_I| |k_J|}$$

car le nombre de $i$ de $I$ tel que $k_i - 2$ soit impair est congru modulo à 2 à $|k_I| = \sum k_i$. Donc:

\begin{align*}
(3) & \quad = \frac{1}{2} \sum_{I,J \neq \emptyset} \varepsilon_\alpha(I,J)(-1)^{|k_I| |k_J| + |k_I|} U_{k_J}(\alpha_J) \circ U_{k_I}(\alpha_I) \\
& \quad = \frac{1}{2} \sum_{I,J \neq \emptyset} \varepsilon_\alpha(I,J)(-1)^{|k_J| |k_I| - 1} U_{k_J}(\alpha_J) \circ U_{k_I}(\alpha_I) = (2).
\end{align*}
De même (5) = (4):

(4) = \(-\frac{1}{2} \sum_{i \neq j} \varepsilon_{\alpha}(i, j, 1, \ldots, i\hat{\to}, j\hat{\to}, \ldots, n) \mathcal{U}_{((k_i + k_j - 1), k_1, \ldots, k_i, \ldots, k_j, \ldots, k_n)} \times \cdot ((\alpha_i \bullet \alpha_j) \alpha_1, \ldots, \hat{\alpha}_i, \ldots, \hat{\alpha}_j, \ldots, \alpha_n)"

= \(-\frac{1}{2} \sum_{i \neq j} \varepsilon_{\alpha}(j, i, 1, \ldots, i\hat{\to}, j\hat{\to}, \ldots, n) (-1)^{k_i k_j} \mathcal{U}_{((k_i + k_j - 1), k_1, \ldots, k_i, \ldots, k_j, \ldots, k_n)} \times \cdot ((\alpha_i \bullet \alpha_j) \alpha_1, \ldots, \hat{\alpha}_i, \ldots, \hat{\alpha}_j, \ldots, \alpha_n)"

= \(-\frac{1}{2} \sum_{i \neq j} \varepsilon_{\alpha}(i, j, 1, \ldots, i\hat{\to}, j\hat{\to}, \ldots, n) \mathcal{U}_{((k_i + k_j - 1), k_1, \ldots, k_i, \ldots, k_j, \ldots, k_n)} \times \cdot (-1)^{k_i k_j} (\alpha_j \bullet \alpha_i) \alpha_1, \ldots, \hat{\alpha}_j, \ldots, \hat{\alpha}_i, \ldots, \alpha_n)"

= (5).

Posons enfin \(\mu = \mathcal{U}_0\). Alors (1) s’écrit:

(1) = \(-1)^{(|k_{\{1, \ldots, n\}| - 1)} \mathcal{U}_{k_{\{1, \ldots, n\}}} (\alpha_{\{1, \ldots, n\}}) \circ \mathcal{U}_0 + \(-1)^{(0 - 1)} \mathcal{U}_{\emptyset} \circ \mathcal{U}_{k_{\{1, \ldots, n\}}} (\alpha_{\{1, \ldots, n\}}).

Comme \(\varepsilon_{\alpha}(\{1, \ldots, n\}, \emptyset) = \varepsilon_{\alpha}(\emptyset, \{1, \ldots, n\}) = 1\), l’équation de formalité devient:

\[
\sum_{I \cup J = \{1, \ldots, n\}} \varepsilon_{\alpha}(I, J)(-1)^{|k_I| - 1)} |k_J|) \mathcal{U}_{k_I}(\alpha_I) \circ \mathcal{U}_{k_J}(\alpha_J) - \sum_{i \neq j} \varepsilon_{\alpha}(i, j, 1, \ldots, i\hat{\to}, j\hat{\to}, \ldots, n) \mathcal{U}_{((k_i + k_j - 1), k_1, \ldots, k_i, \ldots, k_j, \ldots, k_n)} \times \cdot ((\alpha_i \bullet \alpha_j) \alpha_1, \ldots, \hat{\alpha}_i, \ldots, \hat{\alpha}_j, \ldots, \alpha_n) = 0.
\]

Si on remplace les \(\mathcal{U}_{I}(\alpha_I)\) par les \(\sum_{p} w_{p} \mathcal{B}_{\Gamma}(\alpha_I)\) et qu’on développe tout, on obtient une somme d’opérateurs multi-différentiels de la forme:

\[
\sum_{\Gamma'} c_{\Gamma'} \mathcal{B}_{\Gamma'}(\alpha_1, \ldots, \alpha_n)
\]

où \(\Gamma'\) est un graphe à \(n\) sommets aériens, \(m\) sommets terrestres ayant \(2n + m - 3\) flèches. Si on se donne \(\Gamma'\) orienté et une face \(F\) de codimension 1 de \(\partial \mathcal{C}_{\{p_1, \ldots, p_m\}}(\{q_1, \ldots, q_m\})\), on associe à ce couple \((\Gamma', F)\) au plus un terme de l’équation de formalité. Plus précisément:
Cas 1: Si
\[ F = \partial_{\{p_1, \ldots, p_n\}; \{q_1, \ldots, q_m\}} C^+_{\{p_1, \ldots, p_n\}; \{q_1, \ldots, q_m\}} \]
\[ = C^+_{\{p_1, \ldots, p_n\}; \{q_1, \ldots, q_m\}} \times \]
\[ \times C^+_{\{p_1, \ldots, p_n\}; \{q_1, \ldots, q_m\}} \]
que l'on notera:
\[ \partial_{S, S'} C^+_{A, B} = C^+_{S, S'} \times C^+_{A \setminus S, B \setminus S' \cup \{q\}} \]
on associe au couple \( (\tilde{\Gamma}', F) \) l’unique terme:
\[ B'_{\tilde{\Gamma}', F} (\alpha_1, \ldots, \alpha_n) (f_1, \ldots, f_m) \]
\[ = B'_{\tilde{\Gamma}_2} (\alpha_{i_1}, \ldots, \alpha_{j_{n_2}}) \]
\[ \cdot \left( f_1, \ldots, f_i, \sum_{\alpha_{i_2} \in \partial \Gamma_1} (f_{i+1}, \ldots, f_{i+m_1}); f_{i+m_1+1}, \ldots, f_m \right) \]
où \( \tilde{\Gamma}_1 \) est la restriction à \( \{p_1, \ldots, p_{i_n}\} \cup \{q_1, \ldots, q_{i_{m_1}}\} \) (avec son ordre),
\( \tilde{\Gamma}_2 \) est le graphe obtenu en collapsant les points \( p_1, \ldots, p_{i_n} \) et \( q_1, \ldots, q_{i_{m_1}} \) en \( q \), on a posé \( \{1, \ldots, n\} \setminus \{i_1, \ldots, i_{n_1}\} = \{j_1 < j_2 < \ldots < j_{n_2}\} \). On note \( c_{\tilde{\Gamma}', F} \) le coefficient de cet opérateur.

Remarquons que l’application \( (\tilde{\Gamma}', F) \mapsto (\tilde{\Gamma}_1', \tilde{\Gamma}_2') \) est dans ce cas surjective mais pas injective. Si on se donne le couple \( (\tilde{\Gamma}_1, \tilde{\Gamma}_2) \), la face \( F \) est bien déterminée mais \( \tilde{\Gamma}' \) n’est pas unique: il y a d’abord la répartition des flèches allant d’un sommet de \( \tilde{\Gamma}_1 \) vers un sommet de \( \tilde{\Gamma}_2 \) (application de la règle de Leibniz) chaque répartition correspond à un graphe \( \Gamma_2 \) différent. Si cette répartition est donnée, il faut encore fixer l’ordre des flèches de \( \Gamma' \). Le nombre de choix est bien sûr le quotient du nombre d’orientations possibles pour \( \Gamma' \) par celui des orientations possibles de \( \Gamma_1 \) et \( \Gamma_2 \):

\[
\text{Nombre d’orientations de } \Gamma' = \frac{(\sum k_i)!}{(k_{i_1} + \ldots + k_{i_{n_1}})! (k_{j_1} + \ldots + k_{j_{n_2}})!} \]
\[ = \frac{|k_{[1, \ldots, n]}|!}{|k_I|! |k_J|!} \]

Cas 2: Si
\[ F = \partial_{\{p, p\}} C^+_{\{p_1, \ldots, p_n\}; \{q_1, \ldots, q_m\}} \]
\[ = C^+_{\{p_1, \ldots, p_n\}; \{q_1, \ldots, q_m\}} \times \]
\[ \times C^+_{\{p_1, \ldots, p_n\}; \{q_1, \ldots, q_m\}} \]
que l’on notera:
\[ \partial_S C^+_{A, B} = C_S \times C^+_{A \setminus S \cup \{p\}, B} \]
\( A (\tilde{\Gamma}', F) \), si la flèche \( \tilde{p}_{\tilde{p}} \) est une des flèches de \( \Gamma' \), on associe l’unique terme:
\[ B'_{\tilde{\Gamma}', F} (\alpha_1, \ldots, \alpha_n) (f_1, \ldots, f_m) = B'_{\tilde{\Gamma}_2} ((\alpha_1 \cdot \alpha_2), \alpha_3, \ldots, \alpha_{n_2}) \]

où $\bar{\Gamma}_2$ est le graphe obtenu en collapsant les sommets $p_i$ et $p_j$ du graphe $\Gamma'$ sur le point $p$ et en éliminant la flèche $\overrightarrow{p_i p_j}$. Si cette flèche n'existe pas dans $\Gamma'$, on associe l'opérateur nul à $(\bar{\Gamma}_1, F)$. On note $c_{\bar{\Gamma}_1, F}$ le coefficient de cet opérateur.

Dans ce cas, on considérera l'application $(\bar{\Gamma}_1, F) \mapsto (\bar{\Gamma}_1, \bar{\Gamma}_2)$ où $\bar{\Gamma}_1$ est le graphe tracé dans $C_{\{p_i, p_j\}}$ à une seule flèche: la flèche $\overrightarrow{p_i p_j}$. A part le cas 0, l'image réciproque d'un couple $(\bar{\Gamma}_1, \bar{\Gamma}_2)$ contient exactement:

Nombre d'orientations de $\Gamma'$

\[
\frac{(\sum k_i)!}{((k_i + k_j - 1) + k_1 + \ldots + \hat{k}_i + \ldots + \hat{k}_j + \ldots + k_n)!}
\]

Cas 3: Si

\[
F = \partial_S C_{A,B}^+ = C_S \times C_{A \setminus S \cup \{p\}, B}^+
\]

avec $|S| \geq 3$, dans ce cas aucun terme de l'équation de formalité n'est associé à $(\bar{\Gamma}_1, F)$. On pose donc $c_{\bar{\Gamma}_1, F} = 0$.

Pour chaque $\bar{\Gamma}_1$, on définit sur $C_{\{p_1, \ldots, p_n\}, \{q_1, \ldots, q_m\}}$ la forme:

\[
\omega_{\bar{\Gamma}_1} = \frac{1}{(2\pi)^{|k_{\{1, \ldots, n\}}|} |k_{\{1, \ldots, n\}}|!} d\Phi \cdot v_1 \wedge \ldots \wedge d\Phi \cdot v_{k_1 + \ldots + k_n}.
\]

Montrons qu'avec toutes ces notations, l'équation de formalité s'écrit:

\[
0 = \sum_{\bar{\Gamma}_1 \in G_{n,m}} \left[ \sum_{F \in \partial C_{A,B}^+} \int_{\bar{\Gamma}_1} \omega_{\bar{\Gamma}_1}^F \mathcal{B}_{\bar{\Gamma}_1}^F(\alpha_1, \ldots, \alpha_n) \right]
\]

\[
= \sum_{\bar{\Gamma}_1 \in G_{n,m}} \left[ \int_{C_{A,B}^+} d\omega_{\bar{\Gamma}_1}^F \mathcal{B}_{\bar{\Gamma}_1}^F(\alpha_1, \ldots, \alpha_n) \right] \cdot
\]

Le résultat est donc une simple conséquence du théorème de Stokes sur la variété à coins $C_{A,B}^+$ et pour les formes fermées $\omega_{\bar{\Gamma}_1}^F$.

Comparons donc terme par terme chaque coefficient $c_{\bar{\Gamma}_1, F}$ et l'intégrale sur la face orientée $\bar{F}$ de la forme $\omega_{\bar{\Gamma}_1}^F$. 


Cas 1: Avec nos notations, le coefficient $c_{\Gamma', F}$ est:

$$c_{\Gamma', F} = \varepsilon_\alpha(I, J)(-1)^{|k_I|-1|k_J|} \left| k_I \right|! \left| k_J \right|! \left| k_{\{1, \ldots, n\}} \right|! (\varepsilon_\alpha - 1)^{l(m_1 - 1)}$$

$$\int_{C^+} \omega'_{\Gamma_1} \int_{C^+_{A \setminus S, B \setminus S' \cup \{q\}}} \omega'_{\Gamma_2}.$$ 

(Le signe $(-1)^{l(m_1 - 1)}$ provient du développement de l’opération $\circ$. On rappelle que $S'$ est le segment $q_{l+1}, \ldots, q_{l+m_1}$ et que $q$ remplace $S'$ dans $B \setminus S' \cup \{q\}$. D’autre part, on a pour la forme

$$\omega'_{\Gamma'} = \varepsilon_\alpha(I, J) \left| k_I \right|! \left| k_J \right|! \left| k_{\{1, \ldots, n\}} \right|! \omega'_{\Gamma_1} \wedge \omega'_{\Gamma_2}$$

et la face $F'$ (en tenant compte de son orientation):

$$\int_{F'} \omega'_{\Gamma'} = \varepsilon_\alpha(I, J)(-1)^{l(m_1 + l + m_1)} \left| k_I \right|! \left| k_J \right|! \left| k_{\{1, \ldots, n\}} \right|! \int_{C^+} \omega'_{\Gamma_1} \int_{C^+_{A \setminus S, B \setminus S' \cup \{q\}}} \omega'_{\Gamma_2}.$$ 

Rappelons que $\left| k_I \right| = 2n_1 + m_1 - 2$, $\left| k_J \right| = 2(n - n_1) + (m - m_1 + 1) - 2$, le signe devant l’intégrale est donc:

$$\varepsilon_\alpha(I, J)(-1)^{l(m_1 + l + m_1)} = \varepsilon_\alpha(I, J)(-1)^{|k_I||k_J|}(-1)^{m_1 + m_1 + l}$$

$$= \varepsilon_\alpha(I, J)(-1)^{|k_I||k_J|}(-1)^{l(m_1 + l + 1)}(-1)^{|k_I|}.$$ 

Donc:

$$c_{\Gamma', F} = \int_{F'} \omega'_{\Gamma'}.$$ 

Cas 2: Avec nos notations, le coefficient $c_{\Gamma', F}$ est nul si $\Gamma'$ ne contient pas la flèche $p_i p_j$ et sinon:

$$c_{\Gamma', F} = -\varepsilon_\alpha(i, j, 1, \ldots, \widehat{i}, j, \ldots, n) \left( \left| k_{\{1, \ldots, n\}} \right| - 1 \right)! \left| k_{\{1, \ldots, n\}} \right|! \int_{C} \omega'_{\Gamma_1} \int_{C^+_{A \setminus S \cup \{p\}}} \omega'_{\Gamma_2}.$$ 

D’autre part, on a pour la forme

$$\omega'_{\Gamma'} = \varepsilon_\alpha(i, j, 1, \ldots, \widehat{i}, j, \ldots, n) \left( \left| k_{\{1, \ldots, n\}} \right| - 1 \right)! \left| k_{\{1, \ldots, n\}} \right|! \omega'_{\Gamma_1} \wedge \omega'_{\Gamma_2}.$$
et la face $\vec{F}$ (en tenant compte de son orientation):
\[ \int_{\vec{F}} \omega'_{\vec{\Gamma}} = -\varepsilon_{\alpha(i, j, 1, \ldots, i, j, \ldots, n)} \frac{|k\{1, \ldots, n\} - 1|!}{|k\{1, \ldots, n\}|!} \int_{C_S} \omega'_{\vec{\Gamma}_1} \int_{C^+_A(S, \omega(p), B)} \omega'_{\vec{\Gamma}_2}, \]
avec $S = \{p_i, p_j\}$. On a donc pour tout $\vec{\Gamma}'$ et $F$:
\[ c_{\vec{\Gamma}', F} = \int_{F} \omega'_{\vec{\Gamma}'}. \]

**Cas 3:** Il n’y a pas de termes dans notre équation de formalité dans ce cas, ou $c_{\vec{\Gamma}', F} = 0$. Mais dans ce cas, on a le lemme suivant de Kontsevich [K1, §6.6.1], [Kh]:
\[ \int_{C_S} \omega'_{\vec{\Gamma}_1} = 0 \]
si $|S| \geq 3$. On a donc de nouveau:
\[ c_{\vec{\Gamma}', F} = \int_{F} \omega'_{\vec{\Gamma}'}. \]

Et ceci finit la preuve de la validité de l’équation de formalité. □

**Appendice. Formalité et quantification par déformation.**

Nous expliquons dans ce paragraphe pourquoi la formalité de Kontsevich permet d’obtenir un étoile-produit à partir d’un 2-tenseur de Poisson. On considère une (limite projective d’) algèbre(s) nilpotente(s) de dimension finie $m$. Par exemple:
\[ m = h k[[h]] = \lim \left( h k[[h]] / h^k k[[h]] \right). \]

**A.1. Construction d’étoile-produits.** On se donne une $L_\infty$-algèbre $(g, Q)$ sur un corps $k$ de caractéristique zéro, que l’on voit comme une $Q$-variété formelle graduée pointée. Un $m$-point de la variété formelle $g$ est par définition un morphisme de cogèbres:
\[ p : m^* \longrightarrow C(g). \]

Le produit tensoriel (complété dans le cas d’une limite projective) $C(g) \hat{\otimes} m$, muni de la comultiplication de $C(g)$ étendue par $m$-linéarité, admet une structure de cogèbre (sans co-unité) sur $m$. On peut alors voir un $m$-point comme un élément non nul de type groupe de cette cogèbre, c’est-à-dire un élément $p \in C(g) \hat{\otimes} m$ vérifiant: $\Delta p = p \otimes p$.

**Proposition A.1.** Les $m$-points de la variété formelle $g$ sont donnés par:
\[ p_v = e^v - 1 = v + \frac{v^2}{2} + \cdots \]
on où $v$ est un élément pair de $g[1] \hat{\otimes} m$. 

Démonstration. La série a bien un sens dans $C(\mathfrak{g})\widehat{\otimes} \mathfrak{m}$. Si $p$ est un $\mathfrak{m}$-point, $p$ est forcément pair, et on voit que la série:

$$v = \Log(1 + p) = p - \frac{p^2}{2} + \cdots$$

a un sens dans $C(\mathfrak{g})\widehat{\otimes} \mathfrak{m}$ et définit un élément primitif (et pair), c'est-à-dire que l'on a: $\Delta v = 0$. Pour démontrer ce point on rajoute formellement la co-unité en considérant la cogèbre:

$$C_m = (k.1 \oplus \mathfrak{m}) \oplus C(\mathfrak{g})\widehat{\otimes} \mathfrak{m}.$$ 

Un élément de type groupe de cette cogèbre s'écrit toujours:

$$g = 1 + p$$ 

où $p$ est de type groupe dans la cogèbre sans co-unité $C(\mathfrak{g})\widehat{\otimes} \mathfrak{m}$. Il s'agit alors de montrer que le logarithme $v$ d’un tel élément est primitif, c'est-à-dire que l'on a dans $C_m$:

$$\Delta v = v \otimes 1 + 1 \otimes v.$$ 

Pour cela on remarque que la cogèbre $C_m$ est en fait une bigèbre. Le calcul formel suivant a alors un sens:

$$\Delta \Log g = \Log(\Delta g)$$

$$= \Log(g \otimes g)$$

$$= \Log((g \otimes 1)(1 \otimes g))$$

$$= \Log(g \otimes 1) + \Log(1 \otimes g)$$

$$= \Log g \otimes 1 + 1 \otimes \Log g.$$ 

Donc, forcément $v$ appartient à $\mathfrak{g}\widehat{\otimes} \mathfrak{m}$, et il est clair que $p = p_v$. □

Le champ de vecteurs $Q$ s'étend de manière naturelle à $C(\mathfrak{g})\widehat{\otimes} \mathfrak{m}$. Supposons que $Q$ s'annule au point $p_v$:

$$Q(e^v - 1) = 0.$$ 

On traduit ceci par le fait que $v$ vérifie l’équation de Maurer-Cartan généralisée :

(MCG) $Q_1(v) + \frac{1}{2}Q_2(v,v) + \cdots = 0.$

Si $\mathfrak{g}$ est une algèbre de Lie différentielle graduée, ça se réduit à l’équation de Maurer-Cartan:

$$dv - \frac{1}{2}[v,v] = 0.$$ 

(En effet $v$ est pair dans $\mathfrak{g}[1] \otimes \mathfrak{m}$, donc impair dans $\mathfrak{g} \otimes \mathfrak{m}$.) Si maintenant $\mathcal{F}$ est un $L_\infty$-morphisme entre $(\mathfrak{g}_1, Q)$ et $\mathfrak{g}_2, Q'$, et si $v \in \mathfrak{g} \otimes \mathfrak{m}$ est tel que $Q(p_v) = 0$, il est clair que:

$$Q'(\mathcal{F}(p_v)) = \mathcal{F}(Q(p_v)) = 0.$$
Or $\mathcal{F}(p_v) = e^w - 1$ avec $w \in g_2 \hat{\otimes} m$ d’après la Proposition A.1, puisque $\mathcal{F}(p_v)$ est de type groupe. Il est clair que $w$ est la projection canonique de $\mathcal{F}(p_v)$ sur $g_2 \hat{\otimes} m$, soit:

$$w = \sum_{n \geq 1} \frac{1}{n!} \mathcal{F}_n(v^n).$$

En résumé, si $v \in g_1 \hat{\otimes} m$ vérifie (MCG), alors l’élément $w \in g_2 \hat{\otimes} m$ donné par l’égalité ci-dessus vérifie (MCG). Dans le cas où les deux $L_\infty$-algèbres sont les algèbres de Lie différentielles graduées des multichamps de vecteurs et des opérateurs polydifférentiels, tout 2-tenseur de Poisson formel:

$$v = h \gamma_1 + h^2 \gamma_2 + \cdots$$

donne naissance grâce à ce processus à un opérateur bidifférentiel formel $w$ tel que $\mu + w$ soit un étoile-produit, $\mu$ désignant la multiplication usuelle de deux fonctions.

A.2. Equivalence des foncteurs de déformation. On suppose toujours le corps de base $k$ de caractéristique zéro. Soit $g$ une algèbre de Lie différentielle graduée. Rappelons [K1, §3.2] que le foncteur de déformation $\text{Def}_g$ associe à toute algèbre commutative nilpotente de dimension finie $m$ l’ensemble des classes de solutions de degré 1 de l’équation de Maurer-Cartan dans $g \otimes m$ modulo l’action du groupe de jauge, c’est-à-dire le groupe nilpotent $G_m = \exp(g^0 \otimes m)$, dont l’action (par des transformations affines de l’espace $g^1 \otimes m$) est donnée infinitésimalement par:

$$\alpha, \gamma = d\alpha + [\alpha, \gamma]$$

pour tout $\alpha \in g^0 \otimes m$ et pour tout $\gamma \in g^1 \otimes m$. Ce foncteur s’étend naturellement aux limites projectives d’algèbres commutatives nilpotentes de dimension finie: $\text{Def}_g(m)$ est dans ce cas défini comme l’ensemble des classes de solutions de degré 1 de l’équation de Maurer-Cartan dans le produit tensoriel complété $g \hat{\otimes} m$ modulo l’action du groupe pro-nilpotent $G_m = \exp(g^0 \hat{\otimes} m)$.

L’équivalence de jauge peut aussi se définir pour une $L_\infty$-algèbre quelconque $(g, Q)$: deux solutions de l’équation de Maurer-Cartan généralisée $\gamma_0$ et $\gamma_1$ dans $g^1 \otimes m$ sont équivalentes s’il existe une famille polynomiale $\xi(t)_{t \in k}$ de champs de vecteurs de degré $-1$ et une famille polynomiale $\gamma(t)_{t \in k}$ de solutions de l’équation de Maurer-Cartan généralisée dans $g^1 \otimes m$ telles que:

$$\frac{d\gamma(t)}{dt} = [Q, \xi(t)](\gamma(t))$$

$$\gamma(0) = \gamma_0, \quad \gamma(1) = \gamma_1.$$
On vérifie facilement que cette relation est une relation d’équivalence, ce qui permet de définir le foncteur de déformation $\text{Def}_g$ comme la correspondance qui à toute algèbre commutative nilpotente de dimension finie $m$ associe l’ensemble $\text{Def}_g(m)$ des classes de solutions de degré 1 de l’équation de Maurer-Cartan généralisée dans $g \otimes m$ modulo l’équivalence de jauge.

**Proposition A.2.1** (cf. [K1, §4.5.2]).

1. Dans le cas d’une algèbre de Lie différentielle graduée les deux notions d’équivalence de jauge (et donc de foncteur de déformation) coïncident.
2. Soient $g_1$ et $g_2$ deux algèbres de Lie différentielles graduées. Alors le foncteur $\text{Def}_{g_1 \otimes g_2}$ est naturellement équivalent au produit des foncteurs $\text{Def}_{g_1} \times \text{Def}_{g_2}$.
3. Le foncteur de déformation est trivial pour une $L_\infty$-algèbre linéaire contractile.

**Démonstration.** Les points 2) et 3) sont faciles à établir. Pour établir le premier point précisons d’abord la notion de champ de vecteurs: un champ de vecteurs sur une variété formelle graduée pointée $g[1] \otimes m$ est donné par une codérivation $A$ de la cogèbre $C(g) \otimes m$. Sa valeur en un point $\gamma$ de la variété formelle est donnée par la projection sur $g \otimes m$ de $A(e^{\gamma} - 1)$, que l’on notera $A(\gamma)$. Cette notation, bien que cohérente du point de vue géométrique, est un peu ambiguë, car $A(\gamma)$ ne coïncide en général pas avec la valeur de la codérivation $A$ prise en $\gamma$ vu comme un élément de la cogèbre $C(g) \otimes m$.

Revenons aux notations du début du §A.2. On remarque que l’action de $\alpha \in g^0 \otimes m$ sur $g^1 \otimes m$ est donnée par le champ de vecteurs: $D_\alpha = [Q, R_\alpha]$, où $R_\alpha$ est le champ de vecteurs constant égal à $\alpha$. Ce champ de vecteurs est bien de degré $-1$. L’équivalence de jauge au sens des algèbres de Lie différentielles graduées $\gamma_1 = (\exp \alpha) . \gamma_0$ entraîne donc l’équivalence de jauge au sens des $L_\infty$-algèbres, avec $\xi(t) = R_\alpha$ pour tout $t$ et $\gamma(t) = (\exp t\alpha) . \gamma_0$.

Supposons maintenant que $\gamma_0$ et $\gamma_1$ sont équivalents au sens des $L_\infty$-algèbres. Soient $\xi(t)$ et $\gamma(t)$ les familles polynomiales de champs de vecteurs de degré $-1$ et de solutions de l’équation de Maurer-Cartan généralisée respectivement, telles que l’équation ($\ast \ast$) soit vérifiée. Le champ de vecteurs de degré zéro $[Q, \xi(t)]$ s’écrit explicitement en tout $m$-point $\gamma \in g^1 \otimes m$:

$$[Q, \xi(t)](\gamma) = d\xi(t)(\gamma) + [\xi(t)(\gamma), \gamma].$$

On effectue ce calcul en appliquant la codérivation $[Q, \xi(t)]$ à l’élément de type groupe $e^\gamma - 1$. Compte tenu de l’équation ($\ast \ast$) on voit que le vecteur tangent $\frac{d\xi(t)}{dt}$ au point $\gamma(t)$ est donné par un champ de vecteurs provenant de l’action d’un élément de l’algèbre de Lie $g^0 \otimes m$. 
Choix des signes pour la formalité de M. Kontsevich

Montrons par récurrence sur le degré $d$ de $\gamma_t$ (en tant que polynôme) qu'il existe un entier $r$ (dépendant de $d$) tel que pour tout $t \in k$ il existe $g_t = \exp(tD_1 + \cdots + t^rD_r) \in G_m$ tel que $\gamma_t = g_t \cdot \gamma_0$. Si $d = 0$ on a $\gamma_t = \gamma_0$ et la constante $g_t = \text{Id}$ convient. Supposons donc que la propriété soit vraie au rang $d - 1$. Supposons que $\gamma_t$ soit un polynôme de degré $d$, que l'on peut écrire:

$$\gamma_t = \tilde{\gamma}_t + t^d \gamma_d.$$ 

Grâce à l'hypothèse de récurrence on peut écrire:

$$\gamma_t = e^{t \gamma_d} \tilde{\gamma}_t = e^{t \gamma_d} e^{t D_1 + \cdots + t^r D_r} \gamma_0.$$ 

Le terme $\gamma_d$ appartient à l’algèbre de Lie $\mathfrak{g}^0 \otimes \mathfrak{m}$. La série de Campbell-Hausdorff ne comprend qu’un nombre fini de termes dans le cas d’un groupe nilpotent, ce qui permet de conclure. Le passage aux limites projectives d’algèbres commutatives nilpotentes de dimension finie se fait sans difficulté.

Compte tenu de la Proposition V.2, la Proposition A.2.1 entraîne le résultat suivant:

Théorème A.2.2. Soient $\mathfrak{g}_1$ et $\mathfrak{g}_2$ deux $L_\infty$-algèbres quasi-isomorphes. Alors les foncteurs de déformation de $\mathfrak{g}_1$ et de $\mathfrak{g}_2$ sont isomorphes.

En particulier les classes d’équivalence de jauge de 2-tenseurs de Poisson formels sur une variété sont en bijection avec les classes d’équivalence de jauge d’étoile-produits.

References


[K1] M. Kontsevich, Deformation quantization of Poisson manifolds, q-alg. 9709040.


Received March 21, 2000.

Université de Metz
Département de Mathématiques
île du Saulcy
57045 Metz CEDEX 01
E-mail address: arnal@poncelet.univ-metz.fr

Institut Elie Cartan
CNRS
BP 239
54506 Vandœuvre CEDEX
E-mail address: manchon@iecn.u-nancy.fr

Université de Metz
Département de Mathématiques
île du Saulcy
57045 Metz CEDEX 01
E-mail address: masmoudi@poncelet.univ-metz.fr
THE DOMAIN ALGEBRA OF A CP-SEMIGROUP

WILLIAM ARVESON

A CP-semigroup (or quantum dynamical semigroup) is a semigroup \( \phi = \{ \phi_t : t \geq 0 \} \) of normal completely positive linear maps on \( B(H) \), \( H \) being a separable Hilbert space, which satisfies \( \phi_t(1) = 1 \) for all \( t \geq 0 \) and is continuous in the time parameter \( t \) the natural sense.

Let \( D \) be the natural domain of the generator \( L \) of \( \phi, \phi_t = \exp tL, t \geq 0 \). Since the maps \( \phi_t \) need not be multiplicative \( D \) is typically an operator space, but not an algebra. However, in this note we show that the set of operators

\[ \mathcal{A} = \{ A \in D : A^*A \in D, AA^* \in D \} \]

is a \( * \)-subalgebra of \( B(H) \), indeed \( \mathcal{A} \) is the largest self-adjoint algebra contained in \( D \). Examples are described for which the domain algebra \( \mathcal{A} \) is, and is not, strongly dense in \( B(H) \).

1. Basic properties of \( \mathcal{A} \).

Let \( \phi = \{ \phi_t : t \geq 0 \} \) be a CP-semigroup as defined in the abstract. We first recall four characterizations of the domain of the generator of \( \phi \).

**Lemma 1.** Let \( A \in B(H) \). The following are equivalent.

(i) The limit

\[ L(A) = \lim_{t \to 0^+} \frac{1}{t}(\phi_t(A) - A) \]

exists relative to the strong-* topology of \( B(H) \).

(ii) The limit

\[ L(A) = \lim_{t \to 0^+} \frac{1}{t}(\phi_t(A) - A) \]

exists relative to the weak operator topology of \( B(H) \).

(iii)

\[ \sup_{t > 0} \frac{1}{t} \| \phi_t(A) - A \| \leq M < \infty. \]

(iv) There is a sequence \( t_n \to 0^+ \) for which

\[ \sup_n \frac{1}{t_n} \| \phi_{t_n}(A) - A \| \leq M < \infty. \]
Proof. The implications (i) $\implies$ (ii) and (iii) $\implies$ (iv) are trivial, and (ii) $\implies$ (iii) is a straightforward consequence of the Banach-Steinhaus theorem.

Proof of (iv) $\implies$ (i). Since the unit ball of $B(H)$ is weakly sequentially compact, the hypothesis (iv) implies that there is a sequence $t_n \to 0^+$ such that

\[ \frac{1}{t_n}(\phi_{t_n}(A) - A) \to T \in B(H) \]

in the weak operator topology. We claim: for every $s > 0$,

\[(1.1) \quad \int_0^s \phi_\lambda(T) \, d\lambda = \phi_s(A) - A.\]

The integral on the left is interpreted as a weak integral; that is, for $\xi, \eta \in H$,

\[ \int_0^s \langle \phi_\lambda(T) \xi, \eta \rangle \, d\lambda = \langle \phi_s(A) \xi, \eta \rangle - \langle A \xi, \eta \rangle. \]

To see that, fix $\lambda > 0$. Since $\phi_\lambda$ is weakly continuous on bounded sets in $B(H)$ we have

\[ \frac{1}{t_n}(\phi_{\lambda+t_n}(A) - \phi_\lambda(A)) = \phi_\lambda \left( \frac{1}{t_n}(\phi_{t_n}(A) - A) \right) \to \phi_\lambda(T) \]

in the weak operator topology, as $n \to \infty$. By the bounded convergence theorem, we find that for fixed $\xi, \eta \in H$,

\[ \lim_{n \to \infty} \frac{1}{t_n} \left( \int_0^s \phi_{\lambda+t_n}(A) \xi, \eta \right) \, d\lambda - \int_0^s \langle \phi_\lambda(A) \xi, \eta \rangle \, d\lambda = \int_0^s \langle \phi_\lambda(T) \xi, \eta \rangle \, d\lambda. \]

Writing

\[ \int_0^s f(\lambda + t_n) \, d\lambda - \int_0^s f(\lambda) \, d\lambda = \int_0^{s+t_n} f(\lambda) \, d\lambda - \int_0^{t_n} f(\lambda) \, d\lambda, \]

the left side of the preceding formula becomes

\[ \lim_{n \to \infty} \left( \frac{1}{t_n} \int_0^{s+t_n} \langle \phi_\lambda(A) \xi, \eta \rangle \, d\lambda - \int_0^{t_n} \langle \phi_\lambda(A) \xi, \eta \rangle \, d\lambda \right) \]

which, because of continuity of $\phi$ in the time parameter, is $\langle \phi_s(A) \xi, \eta \rangle - \langle A \xi, \eta \rangle$, as asserted in (1.1).

To prove the strong-* convergence asserted in (i), fix $\xi \in H$ and use (1.1) to write

\[ \left\| \frac{1}{s} (\phi_s(A) \xi - A \xi) - T \xi \right\| = \frac{1}{s} \left\| \int_0^s \phi_\lambda(T) \xi \, d\lambda - \int_0^s T \xi \, d\lambda \right\| \]

\[ \leq \frac{1}{s} \int_0^s \| \phi_\lambda(T) \xi - T \xi \| \, d\lambda \leq \left( \frac{1}{s} \int_0^s \| \phi_\lambda(T) \xi - T \xi \|^2 \, d\lambda \right)^{1/2}. \]
The integrand of the last term expands as follows
\[ \| \phi_\lambda(T)\xi - T\xi \|^2 = \langle \phi_\lambda(T)^*\phi_\lambda(T)\xi,\xi \rangle - 2\Re \langle \phi_\lambda(T)\xi,T\xi \rangle + \| T\xi \|^2, \]
the last inequality by the Schwarz inequality for unital CP maps. Since \( \phi_\lambda(T^*T) \) (resp. \( \phi_\lambda(T) \)) tends weakly to \( T^*T \) (resp. \( T \)) as \( \lambda \to 0^+ \), it follows that
\[ \limsup_{s \to 0^+} \frac{1}{s} \int_0^s \| \phi_\lambda(T)\xi - T\xi \|^2 \, d\lambda \leq \langle T^*T\xi,\xi \rangle - 2\Re \langle T\xi,T\xi \rangle + \| T\xi \|^2 = 0, \]
and we conclude that \( \frac{1}{s}(\phi_s(A) - A) \) tends strongly to \( T \) as \( s \to 0^+ \).

**Definition.** Let \( \mathcal{D} \) be the set of all operators \( A \in B(H) \) for which the four conditions of Lemma 1 are satisfied. \( L : \mathcal{D} \to B(H) \) denotes the generator of \( \phi \),
\[ L(A) = \lim_{t \to 0^+} \frac{1}{t}(\phi_t(A) - A), \quad A \in \mathcal{D}. \]

It is obvious that \( \mathcal{D} \) is a self-adjoint linear subspace of \( B(H) \), that \( L(A^*) = L(A)^* \) for \( A \in \mathcal{D} \), and a standard argument shows that \( \mathcal{D} \) is dense in \( B(H) \) in the \( \sigma \)-strong operator topology.

**Lemma 2.** For every operator \( A \in \mathcal{D} \) we have
\[ \| L(A) \| = \sup_{t > 0} \frac{1}{t} \| \phi_t(A) - A \|. \]

**Proof.** The inequality \( \leq \) is clear from the fact that \( L(A) \) is the weak limit of operators \( \frac{1}{t}(\phi_t(A) - A) \) near \( t = 0^+ \), i.e.,
\[ \| L(A) \| \leq \limsup_{t \to 0^+} \frac{1}{t} \| \phi_t(A) - A \| \leq \sup_{t > 0} \frac{1}{t} \| \phi_t(A) - A \|. \]

For \( \geq \), set \( T = L(A) \). Using (1.1), we can write for every \( t > 0 \)
\[ \frac{1}{t} \| \phi_t(A) - A \| = \frac{1}{t} \left\| \int_0^t \phi_\lambda(T) \, d\lambda \right\| \leq \frac{1}{t} \int_0^t \| \phi_\lambda(T) \| \, d\lambda \leq \| T \|, \]
since \( \| \phi_\lambda \| \leq 1 \) for every \( \lambda \geq 0 \). \( \square \)

**Theorem A.** \( \mathcal{A} = \{ A \in \mathcal{D} : A^*A \in \mathcal{D}, AA^* \in \mathcal{D} \} \) is a \( * \)-subalgebra of \( B(H) \).

**Proof.** \( \mathcal{A} \) is obviously a self-adjoint set of operators. We have to show that \( \mathcal{A} \) is a vector space satisfying \( \mathcal{A} \cdot \mathcal{A} \subseteq \mathcal{A} \).

Fix \( t > 0 \). By Stinespring’s theorem we can write
\[ \phi_t(X) = V_t^* \pi_t(X)V_t, \quad X \in B(H) \]
where $V_t$ is an isometry from $H$ into some other Hilbert space $H_t$ and where $\pi_t : \mathcal{B}(H) \to \mathcal{B}(H_t)$ is a normal $*\,$-homomorphism of von Neumann algebras. $P_t = V_t V^*_t$ is a self-adjoint projection in $\mathcal{B}(H_t)$.

For $t > 0$ we will consider the seminorms $p_t$, $q_t$ defined on $\mathcal{B}(H)$ as follows
\[
p_t(X) = t^{-1} \| \phi_t(X) - X \|, \\
q_t(X) = t^{-1/2} \| P_t \pi_t(X) - \pi_t(X) P_t \|, \quad X \in \mathcal{B}(H).
\]

**Lemma 3.** For every operator $X \in \mathcal{B}(H)$ we have the following characterizations.

(i) $X \in \mathcal{D}$ iff
\[
\sup_{t > 0} p_t(X) < \infty,
\]
and in that case \( \|L(X)\| = \sup_{t > 0} p_t(X) \).

(ii) $X \in \mathcal{A}$ iff both $\sup_{t > 0} p_t(X)$ and $\sup_{t > 0} q_t(X)$ are finite, and in that case
\[
\max(\|\sigma_L(dX^* dX)\|^{1/2}, \|\sigma_L(dX dX^*)\|^{1/2}) \leq \limsup_{t \to 0^+} q_t(X),
\]
where $\sigma_L(dX^* dX)$ and $\sigma_L(dX dX^*)$ are the operators in $\mathcal{B}(H)$ defined by
\[
\sigma_L(dX^* dX) = L(X^*X) - X^* L(X) - L(X^*) X, \\
\sigma_L(dX dX^*) = L(X X^*) - X L(X^*) - L(X) X^*.
\]

**Remark.** The second assertion of Lemma 3 requires clarification. By definition, an operator $X$ belongs to $\mathcal{A}$ iff all four operators $X, X^*, X X^*, XX^*$ belong to the domain of the generator $L$ of $\phi = \{ \phi_t : t \geq 0 \}$. In that case both operators $\sigma_L(dX^* dX)$ and $\sigma_L(dX dX^*)$ are well-defined by the above formulas. The “symbol” map $\sigma_L$ is discussed more fully in [2].

**Proof of Lemma 3.** The assertion (i) follows from Lemmas 1 and 2 above. In order to prove (ii) we require the following more concrete expression for the seminorm $q_t$.

\[
q_t(X) = \max \left( \left\| \frac{1}{t} (\phi_t(X^* X) - \phi_t(X)^* \phi_t(X)) \right\|^{1/2}, \left\| \frac{1}{t} (\phi_t(X X^*) - \phi_t(X)^* \phi_t(X^*)) \right\|^{1/2} \right).
\]

(1.3) To prove (1.3) we decompose the commutator $\pi_t(X) P_t - P_t \pi_t(X)$ into a sum
\[
\pi_t(X) P_t - P_t \pi_t(X) = (1 - P_t) \pi_t(X) P_t - P_t \pi_t(X) (1 - P_t).
\]
Since the first term \((1 - P_t)\pi_t(X)P_t\) has initial space in \(P_tH_t\) and final space in \((1 - P_t)H_t\), and the second term has the opposite property, it follows that

\[
\|\pi_t(X)P_t - P_t\pi_t(X)\| = \max(\|\pi_t(X)P_t\|, \|P_t\pi_t(X)\|).
\]

We have

\[
\|(1 - P_t)\pi_t(X)P_t\|^2 = \|V_t^*\pi_t(X^*)(1 - P_t)\pi_t(X)V_t\| = \|V_t^*\pi_t(X^*)V_t - V_t^*\pi_t(X)\|V_tV_t^*\pi_t(X)V_t\| = \|\phi_t(X^*) - \phi_t(X)^*\phi_t(X)\|.
\]

Similarly,

\[
\|P_t\pi_t(X)(1 - P_t)\|^2 = \|V_t^*\pi_t(X)(1 - P_t)\pi_t(X)^*V_t\| = \|\phi_t(X) - \phi_t(X)^*\phi_t(X)^*\|,
\]

and formula (1.3) follows from these two expressions.

Now if \(X \in \mathcal{A}\) then all four operators \(X, X^*, XX, XX^*\) belong to \(\mathcal{D}\), hence all four limits

\[
\lim_{t \to 0^+} \frac{1}{t}(\phi_t(X^*) - X^*X) = L(X^*X),
\]

\[
\lim_{t \to 0^+} \frac{1}{t}(\phi_t(XX^*) - X^*X) = L(XX^*),
\]

\[
\lim_{t \to 0^+} \frac{1}{t}(\phi_t(X) - X) = L(X),
\]

\[
\lim_{t \to 0^+} \frac{1}{t}(\phi_t(X^*) - X^*) = L(X^*)
\]

exist relative to the strong operator topology. Writing

\[
(1.4) \quad \phi_t(X^*) - \phi_t(X)^*\phi_t(X) = (\phi_t(X^*) - X^*) - X^*(\phi_t(X) - X) - (\phi_t(X^*) - X^*)\phi_t(X)
\]

and using strong continuity of multiplication on bounded sets, we find that the limit

\[
\lim_{t \to 0^+} \frac{1}{t}(\phi_t(X^*) - \phi_t(X)^*\phi_t(X)) = \sigma_L(dX^*\ dX) = \sigma_L(dX^*)
\]

exists relative to the strong operator topology.

In the same way we deduce the existence of the strong limit

\[
\lim_{t \to 0^+} \frac{1}{t}(\phi_t(XX^*) - \phi_t(X)\phi_t(X^*)) = \sigma_L(dX \ dX^*) = \sigma_L(dX \ dX^*)
\]
It follows that for every $X \in \mathcal{A}$ the seminorms $q_t(X)$ are bounded for $t > 0$, and for such $X$ we have

$$\max(\|\sigma_L(dX^* dX)\|^{1/2}, \|\sigma_L(dX dX^*)\|^{1/2}) \leq \limsup_{t \to 0^+} q_t(X).$$

Conversely, suppose we are given an operator $X \in D$ for which the seminorms $q_t(X)$ are bounded for $t > 0$. We have to show that $X^*X$ and $XX^*$ belong to $D$; since $D$ is self-adjoint and the seminorms $q_t$ are symmetric in that $q_t(X^*) = q_t(X)$, it is enough to show that $X^*X$ belong to $D$. (1.4) implies that for fixed $t > 0$,

$$\phi_t(X^*X) - X^*X = (\phi_t(X^*X) - \phi_t(X^*)\phi_t(X))$$
$$+ X^*(\phi_t(X) - X) + (\phi_t(X^*) - X^*)\phi_t(X).$$

Because of (1.3), the first term on the right of (1.5) is bounded in norm by $M_1 \cdot t$ where $M_1$ is a positive constant. Similarly, since $X$ and $X^*$ belong to $D$ the second and third terms are bounded in norm by terms of the form $M_2 \cdot t$ and $M_3 \cdot t$ respectively, hence

$$\|\phi_t(X^*X) - X^*X\| \leq (M_1 + M_2 + M_3) \cdot t.$$ 

By Lemma 1, $X^*X$ must belong to $D$. □

Turning now to the proof of Theorem A, (or more properly, to the proof that $\mathcal{A}$ is an algebra), Lemma 3 tells us that $\mathcal{A}$ consists of all operators $X \in \mathcal{B}(H)$ for which

$$\sup_{t > 0} p_t(X) < \infty, \quad \text{and} \quad \sup_{t > 0} q_t(X) < \infty.$$ 

Since $p_t$ and $q_t$ are both seminorms, it follows that $\mathcal{A}$ is a complex vector space which is obviously closed under the $*$-operation.

To see that $\mathcal{A}$ is closed under multiplication, pick $X,Y \in \mathcal{A}$. According to Lemma 3, it is enough to show

(1.6) $$\sup_{t > 0} q_t(XY) < \infty$$

and

(1.7) $$\sup_{t > 0} p_t(XY) < \infty.$$ 

To prove (1.6) we claim that

(1.8) $$q_t(XY) \leq q_t(X)\|Y\| + \|X\|q_t(Y).$$ 

Indeed, writing $[A,B]$ for the commutator $AB - BA$ we have

$$[P_t, \pi_t(XY)] = [P_t, \pi_t(X)]\pi_t(Y) + \pi_t(X)[P_t, \pi_t(Y)].$$
and hence
\[ q_t(XY) = t^{-1/2} \| [P_t, \pi_t(XY)] \| \]
\[ \leq t^{-1/2} \| [P_t, \pi_t(X)] \| \| \pi_t(Y) \| + t^{-1/2} \| [P_t, \pi_t(Y)] \|, \]
from which (1.8) is evident.

Finally, consider Condition (1.7). By definition of \( A \), \( A \in A \) implies \( A^*A \in D \). Since \( A \) is now known to be a linear space we can assert that if \( X, Y \in A \) then for every \( k = 0, 1, 2, 3 \) we have \( Y + i^k X \in A \), hence \((Y + i^k X)^*(Y + i^k X) \in D \) and by the polarization formula
\[ X^*Y = \frac{1}{4} \sum_{k=0}^{3} i^k (Y + i^k X)^*(Y + i^k X), \]
\( X^*Y \) must also belong to \( D \). Since \( A^* = A \), we can replace \( X^* \) with \( X \) to conclude that \( XY \in D \). (1.7) now follows from Lemma 3 (i).

**Corollary.** Let \( D \) be the domain of the generator of a CP-semigroup acting on \( B(H) \) and let \( A \) be a self-adjoint operator such that \( A^2 \in D \). Then \( p(A) \in D \) for every polynomial \( p(x) = a_0 + a_1 x + \ldots + a_n x^n \).

### 2. Examples, Remarks.

We describe two classes of examples which are in a sense at opposite extremes. In the first class of examples of CP-semigroups \( \phi = \{ \phi_t : t \geq 0 \} \), each \( \phi_t \) leaves the \( C^* \)-algebra \( \mathcal{K} \) of all compact operators invariant, \( \phi_t(\mathcal{K}) \subseteq \mathcal{K} \), its domain algebra \( A \) is strongly dense in \( B(H) \), and its generator restricts to a second order differential operator on \( A \) in the sense of [2]. In the second class of examples, the individual maps satisfy \( \phi_t(\mathcal{K}) \cap \mathcal{K} = \{0\} \) for \( t > 0 \), \( A \) is not strongly dense in \( B(H) \), and its generator is degenerate in the sense that it restricts to a derivation on \( A \).

We first recall the CP-semigroups of [1], including the heat flow of the CCR algebra. While for simplicity we confine the discussion to the case of one degree of freedom, the reader will note that everything carries over verbatim to the case of \( n \) degrees of freedom, \( n = 1, 2, \ldots \).

Let \( \{ W_z : z \in \mathbb{R}^2 \} \) be an irreducible Weyl system acting on a Hilbert space \( H \). Thus, \( z \in \mathbb{R}^2 \mapsto W_z \) is a strongly continuous mapping from \( \mathbb{R}^2 \) into the unitary operators on \( H \) which satisfies the canonical commutation relations in Weyl’s form
\[ W_{z_1} W_{z_2} = e^{i \omega(z_1, z_2)} W_{z_1 + z_2}, \quad z_1, z_2 \in \mathbb{R}^2, \]
\( \omega \) denoting the symplectic form on \( \mathbb{R}^2 \) given by
\[ \omega((x, y), (x', y')) = \frac{1}{2} (x'y - xy'). \]
Let \( \{ \mu_t : t \geq 0 \} \) be a one-parameter family of probability measures on \( \mathbb{R}^2 \) which is a semigroup under the natural convolution of measures
\[
\mu \ast \nu(S) = \int_{\mathbb{R}^2 \times \mathbb{R}^2} \chi_S(z + w) \, d\mu(z) \, d\nu(w),
\]
which satisfies \( \mu_0 = \delta_{(0,0)} \), and which is measurable in \( t \) in the natural sense.

It is convenient to define the Fourier transform of a measure \( \mu \) in terms of the symplectic form \( \omega \) as follows,
\[
\hat{\mu}(z) = \int_{\mathbb{R}^2} e^{i\omega(z,\zeta)} \, d\mu(\zeta), \quad z \in \mathbb{R}^2.
\]

Given such a semigroup of probability measures \( \{ \mu_t : t \geq 0 \} \) there is a unique \( \text{CP} \) semigroup \( \phi = \{ \phi_t : t \geq 0 \} \) acting on \( B(H) \) which satisfies
\[
\phi_t(W_z) = \hat{\mu}_t(z) W_z, \quad z \in \mathbb{R}^2, \quad t \geq 0
\]
see [1], Proposition 1.7. Two cases of particular interest are
(CCR heat flow) \( \phi_t(W_z) = e^{-t|z|^2} W_z, \quad t \geq 0 \)
(Cauchy flow) \( \phi_t(W_z) = e^{-t|z|^2} W_z, \quad t \geq 0 \).

For both of these examples a straightforward estimate shows that for fixed \( z \in \mathbb{R}^2 \) there is a constant \( M > 0 \) such that
\[
\|\phi_t(W_z) - W_z\| = |\hat{\mu}_t(z) - 1| \leq M \cdot t, \quad t > 0
\]
and hence \( W_z \in \mathcal{D} \). Since \( W_z \) is unitary, \( 1 = W_z^* W_z = W_z W_z^* \) belongs to \( \mathcal{D} \), and hence \( W_z \) belongs to the domain algebra \( \mathcal{A} \) of \( \phi \) for every \( z \in \mathbb{R}^2 \). We conclude that for these examples, the domain algebra is strongly dense in \( B(H) \).

Indeed, it can be seen that \( \mathcal{A} \) contains a \(*\)-algebra of compact operators that is norm-dense in the algebra \( \mathcal{K} \) of all compact operators. Unlike the examples to follow, these flows leave \( \mathcal{K} \) invariant in the sense that \( \phi_t(\mathcal{K}) \subseteq \mathcal{K} \) for all \( t \geq 0 \), and can therefore be considered as \( \text{CP}\)-semigroups which act on the separable \( \text{C}^*\)-algebra \( \mathcal{K} \), rather than than as \( \text{CP}\)-semigroups acting on \( B(H) \).

We now describe a class of examples of \( \text{CP} \) semigroups whose domain algebras are not strongly dense in \( B(H) \). The referee has kindly pointed out that there are previously known examples of singular Markov semigroups in the literature which exhibit a similar phenomenon ([15]). Consequently, we have omitted proofs of the results below. The examples we describe here are inspired by a class of \( \text{CP} \) semigroups that have emerged in recent work of Robert Powers, to whom we are indebted for useful discussions.

Let \( H = L^2(0,\infty) \) and let \( U = \{ U_t : t \geq 0 \} \) be the semigroup of isometries \( U_t \xi(x) = \xi(x - t) \) for \( x \geq t \), \( U_t \xi(x) = 0 \) for \( 0 \leq x < t \). Fix a real number
\[ \alpha > 0, \text{ and let } f \text{ be the unit vector in } L^2(0, \infty) \text{ obtained by normalizing the exponential function } u(x) = e^{-\alpha x}, x \geq 0. \] One has \( U_t^* f = e^{-\alpha t} f \) for every \( t \geq 0 \), hence the vector state \( \omega(A) = \langle Af, f \rangle \text{ satisfies } \omega(U_t AU_t^*) = e^{-2\alpha t} \omega(A), A \in \mathcal{B}(H). \)

We consider the family of unit-preserving normal completely positive maps \( \phi = \{ \phi_t : t \geq 0 \} \) defined on \( \mathcal{B}(H) \) by
\[
\phi_t(A) = \omega(A) E_t + U_t A U_t^*, \quad t \geq 0,
\]
where \( E_t = 1 - U_t U_t^* \) is the projection on the subspace \( L^2(0, t) \subseteq L^2(0, \infty) \).

Since \( \omega(E_t) = \omega(1) - \omega(U_t U_t^*) = 1 - e^{-2\alpha t} \), it follows that \( \omega(\phi_t(A)) = \omega(A) \) for every \( A \). A routine computation now shows that \( \phi \) satisfies the semigroup property \( \phi_s \circ \phi_t = \phi_{s+t} \), hence \( \phi \) is a \( CP \) semigroup.

Let \( \mathcal{D} \) be the domain of the generator of \( \phi \) and let \( \mathcal{A} \) be the domain algebra
\[
\mathcal{A} = \{ A \in \mathcal{D} : A^* A \in \mathcal{D}, AA^* \in \mathcal{D} \}.
\]
Theorem A implies that \( \mathcal{A} \) is a unital \( * \)-algebra, and its strong closure is described as follows.

**Proposition.** The strong closure of \( \mathcal{A} \) consists of all operators \( B \in \mathcal{B}(H) \) such that \( B \) commutes with the rank-one projection \( f \otimes f \).

Thus the strong closure \( \mathcal{A}^- \) of \( \mathcal{A} \) has the form \( \mathcal{B}(H_0) \oplus \mathbb{C} \) where \( H_0 \subseteq H \) is a subspace of codimension one in \( H \). The following consequence is easily deduced from the Proposition; it implies that these examples are “almost” \( E_0 \)-semigroups in the sense that there is an \( E_0 \)-semigroup \( \alpha = \{ \alpha_t : t \geq 0 \} \) acting on \( \mathcal{B}(H_0) \) such that \( \phi_t \) acts as follows on \( \mathcal{A}^- \),
\[
\phi_t(B \oplus \lambda) = \alpha_t(B) \oplus \lambda, \quad B \in \mathcal{B}(H_0), \quad \lambda \in \mathbb{C}.
\]

**Corollary.** Let \( \overline{\mathcal{A}} \) be the strong closure of \( \mathcal{A} \). Then \( \phi_t(\overline{\mathcal{A}}) \subseteq \overline{\mathcal{A}} \) for every \( t \geq 0 \) and \( \phi \) restricts to a semigroup of \( * \)-endomorphisms of this von Neumann algebra.

The Corollary implies that the semigroup \( \phi \) is degenerate in the sense that its generator is essentially a derivation, not a true “second order” noncommutative differential operator. Whether or not this degeneracy is related to the non-density of the domain algebra \( \mathcal{A} \) is an interesting question about which we as yet have very little information.

In particular, we do not know how small the domain algebra can be. For example, does there exist a \( CP \) semigroup whose domain algebra is just the scalars \( \mathbb{C} \cdot 1 \)?
References


Received May 30, 2000 and revised March 7, 2001. The author is on appointment as a Miller Research Professor in the Miller Institute for Basic Research in Science. Support is also acknowledged from NSF grant DMS-9802474.

**Department of Mathematics**  
**University of California, Berkeley**  
**Berkeley CA 94720**  
**E-mail address:** arveson@math.berkeley.edu
ON BOUNDARY AVOIDING SELECTIONS AND SOME EXTENSION THEOREMS

STOYU BAROV AND JAN J. DIJKSTRA

A theorem of Marc Frantz about controlled continuous extensions of functions inspired us to prove a general result concerning boundary avoiding continuous selections into Banach spaces, which has Frantz’ theorem as a corollary. In addition, with relatively simple means we improve upon some other results of Frantz involving extensions of products and of disjoint families of functions.

1. Introduction.

The following two extension theorems are presented in Frantz [3]. Let $I$ denote the interval $[0, 1]$.

**Theorem 1.** Let $X$ be a normal space, let $A$ be a closed subset of $X$, and let $Y_0, Y_1$ be disjoint closed $G_δ$-subsets of $X$. If $f : A → I$ is a continuous function such that for $i = 0, 1$, $f^{-1}(i) = Y_i ∩ A$ then there exists a continuous extension $\hat{f} : X → I$ of $f$ with $\hat{f}^{-1}(i) = Y_i$ for $i = 0, 1$.

**Theorem 2.** Let $X$ be a compact metric space and let $A$ be a closed subset of $X$. If $f : A → \mathbb{R}$, $g : A → [0, ∞)$, and $h : X → \mathbb{R}$ are continuous functions such that $f \cdot g = h|A$ and $g^{-1}(0) ⊂ f^{-1}(0)$ then there are continuous extensions $\hat{f} : X → \mathbb{R}$ and $\hat{g} : X → [0, ∞)$ of $f$ and $g$ with $\hat{f} \cdot \hat{g} = h$.

We present a general result (Theorem 4) about boundary avoiding continuous selections that has Theorem 1 as a corollary. We also give a very simple argument that shows that Theorem 2 is valid without any restrictions on the domain $X$ other than the necessary normality (see Corollary 8). In addition, with Corollary 12 and Example 3 we sharpen a result in [3] concerning the extension of pairwise disjoint collections of functions.

All spaces in this paper are assumed to be Tychonoff.

2. Boundary avoiding continuous selections.

If $Y$ is a set then $2^Y = \mathcal{P}(Y) \setminus \{\emptyset\}$. Let $X$ and $Y$ be topological spaces and let $\varphi : X → 2^Y$ be a set-valued function. If $A ⊂ Y$ then we put $\varphi^{-1}[A] = \{x ∈ X : \varphi(x) ∩ A ≠ \emptyset\}$. The function $\varphi$ is called lower semi-continuous (LSC for short) if for each open set $O$ in $Y$ the set $\varphi^{-1}[O]$ is
open in $X$. A function $f : X \to Y$ is called a selection of $\varphi$ if $f(x) \in \varphi(x)$ for every $x \in X$. If $Y$ is a metric space then we call $\varphi$ bounded if there is an $M > 0$ such that the diameter of every $\varphi(x)$ is less than $M$.

Let $(B, \| \cdot \|)$ be a Banach space and let $\varepsilon > 0$. Let $U_\varepsilon$ denote the open $\varepsilon$-ball $\{y \in B : \|y\| < \varepsilon\}$. If $C$ is a subset of $B$ then $\text{int} C$ denotes the interior of $C$ in $B$ and if $\varepsilon > 0$ then we put

$$\text{int}_\varepsilon C = \{y \in B : y + U_\varepsilon \subset C\}.$$ 

Note that $\text{int}_\varepsilon C$ is always closed and that if $C$ is convex then so is $\text{int}_\varepsilon C$.

A space $X$ is called countably paracompact if every countable open cover of the space has a locally finite open refinement that covers the space. For normal spaces this property is equivalent to the property that for every increasing sequence $U_1 \subset U_2 \subset \ldots$ of open sets with $\bigcup_{i=1}^\infty U_i = X$ there exist a sequence $F_1, F_2, \ldots$ of closed sets such that $F_i \subset U_i$ for $i \in \mathbb{N}$ and $\bigcup_{i=1}^\infty F_i = X$, see [2, Corollary 5.2.2]. Spaces that are normal but not countably paracompact are known as Dowker spaces, see Rudin [6].

**Lemma 3.** Let $X$ be a normal space, let $B$ be a Banach space, let $C$ be a convex subset of $B$, let $\varphi : X \to 2^C$ be LSC and bounded such that every $\varphi(x)$ is closed and convex in $B$, and let $F_1, F_2, \ldots$ be a sequence of closed subsets of $X$ such that $F_n \subset \varphi^{-1}([\text{int}_{1/n}C])$ for each $n \in \mathbb{N}$.

(a) If $B$ is separable and every $\varphi(x)$ is compact, or

(b) if $B$ is separable and $X$ is countably paracompact, or

(c) if $X$ is paracompact

then there is a continuous selection $f$ of $\varphi$ with $f(F_n) \subset \text{int} C$ for each $n$.

**Proof.** We may assume that $F_n \subset F_{n+1}$ for every $n$. Put $F_0 = \emptyset$ and $A = X \setminus \bigcup_{n=1}^\infty F_n$. Let $M > 1$ be an upper bound for the diameters of the $\varphi(x)$’s. For $n \in \mathbb{N}$ put $\delta(n) = 1/(Mn^2)$ and $C_n = \text{int}_{\delta(n)}C$. We define a function $\psi : X \to 2^C$ as follows:

$$\psi(x) = \begin{cases} \varphi(x), & \text{if } x \in A; \\ \varphi(x) \cap C_n, & \text{if } x \in F_n \setminus F_{n-1}. \end{cases}$$

Note that since $\text{int}_\varepsilon C$ is closed and convex, every $\psi(x)$ is closed (and in case (a) compact) and convex. If $\psi$ is LSC then according to Michael [5] it has a continuous selection $f$ which obviously has the property $f(F_n) \subset C_n \subset \text{int} C$ for each $n \in \mathbb{N}$.

It remains to prove that $\psi$ is LSC. Let $O$ be open in $B$ and let $x \in \psi^{-1}[O]$. Select a vector $a \in \psi(x) \cap O$. In order to prove that $x$ is an interior point of $\psi^{-1}[O]$ we distinguish two cases:

**Case I.** $x \notin A$. Let $n \in \mathbb{N}$ be such that $x \in F_n \setminus F_{n-1}$. So $\psi(x) = \varphi(x) \cap C_n$ and $a \in \varphi(x) \cap C_n \cap O$. Since by assumption $F_n \subset \varphi^{-1}([\text{int}_{1/n}C])$ we can find a vector $b \in \varphi(x) \cap \text{int}_{1/n}C$. Since $1/n > \delta(n)$ we have $b \in \text{int}_{1/n}C \subset \text{int} C_n$.
and hence $b + U_\varepsilon \subset C_n$ for some $\varepsilon > 0$. Let $t \in (0, 1]$ and note that by the convexity of $C_n$ we have $a + t(b - a) + U_\varepsilon \subset C_n$. Note that $a \in O$ so for some small enough $t \in (0, 1]$ the vector $c = a + t(b - a)$ is in $O \cap \text{int} C_n$. By convexity of $\varphi(x)$ we have $c \in \varphi(x)$. Define the open set $U = \varphi^{-1}[O \cap \text{int} C_n] \setminus F_{n-1}$. Note that $x \in U$. If $y \in U$ then there is a $d \in O \cap \varphi(y) \cap \text{int} C_n$. Since $y \notin F_{n-1}$ we have $\varphi(y) \cap C_n \subset \psi(y)$ and hence $d \in O \cap \psi(y)$. Conclusion: $y \in \psi^{-1}[O]$ and $U \subset \psi^{-1}[O]$.

Case II. $x \in A$. Let $n \in \mathbb{N}$ be such that $a + U_{2/n} \subset O$. Define the open set $U = \varphi^{-1}[a + U_{1/n}] \setminus F_n$. Since $x \in A$ we have $x \in U$. Let $y \in U$ and select $b \in \varphi(y)$ such that $\|b - a\| < 1/n$. If $y \in A$ then $\psi(y) = \varphi(y)$ and obviously $y \in \psi^{-1}[O]$. So we may assume that $y \in F_m \setminus F_{m-1}$ for some $m > n$. Since $F_m \subset \varphi^{-1}[\text{int}_{1/m} C]$ we can find a vector $c \in \varphi(y) \cap \text{int}_{1/m} C$. So $c + U_{1/m} \subset C$ and $b \in \varphi(y) \subset C$. Put $t = 1/(Mm)$ and note that by the convexity of $C$ we have $b + t(c - b) + U_{1/m} \subset C$. So $d = b + t(c - b)$ is in $\text{int}_{1/m} C = C_m$. Note that since $b$ and $c$ are in $\varphi(y)$ we have $\|c - b\| \leq M$ and hence $\|d - b\| \leq tM = 1/m < 1/n$. Also, by convexity of $\varphi(y)$ we have $d \in \varphi(y)$. So the distance between $d$ and $a$ is less than $2/n$ and hence $d \in O \cap \varphi(y) \cap C_m = O \cap \psi(y)$. Conclusion: $y \in \psi^{-1}[O]$ and $U \subset \psi^{-1}[O]$. □

**Theorem 4.** The following statements are equivalent:

1. $X$ is a normal and countably paracompact space.
2. For every separable Banach space $B$, every convex subset $C$ of $B$, every LSC function $\varphi: X \to 2^C$ such that each $\varphi(x)$ is compact and convex in $B$, and every $A \subset \varphi^{-1}[\text{int} C]$ that is an $F_\sigma$-subset of $X$ there exists a continuous selection $f$ of $\varphi$ with $A \subset f^{-1}(\text{int} C) \subset \varphi^{-1}[\text{int} C]$.
3. For every separable Banach space $B$, every convex subset $C$ of $B$, every LSC function $\varphi: X \to 2^C$ such that each $\varphi(x)$ is closed and convex in $B$, and every $A \subset \varphi^{-1}[\text{int} C]$ that is an $F_\sigma$-subset of $X$ there exists a continuous selection $f$ of $\varphi$ with $A \subset f^{-1}(\text{int} C) \subset \varphi^{-1}[\text{int} C]$.

**Theorem 5.** The following statements are equivalent:

1. $X$ is a paracompact space.
2. For every Banach space $B$, every convex subset $C$ of $B$, every LSC function $\varphi: X \to 2^C$ such that each $\varphi(x)$ is closed and convex in $B$, and every $A \subset \varphi^{-1}[\text{int} C]$ that is an $F_\sigma$-subset of $X$ there exists a continuous selection $f$ of $\varphi$ with $A \subset f^{-1}(\text{int} C) \subset \varphi^{-1}[\text{int} C]$.

**Proof.** We will prove both theorems at the same time. Note first that if we substitute $C = B$ then we have Michael’s selection theorems so if (2) is valid then $X$ is normal in Theorem 4 and paracompact in Theorem 5. Note that the implication (3) $\Rightarrow$ (2) in Theorem 4 is trivial.

In order to prove that condition (2) in Theorem 4 implies that $X$ is countably paracompact we consider an countable, monotone open cover
\[ U_1 \subset U_2 \subset \cdots \text{ of } X. \] Put \( U_0 = \emptyset \) and define the LSC function \( \varphi : X \to 2^I \) by
\[
\varphi(x) = [0,1/n] \quad \text{if } x \in U_n \setminus U_{n-1} \text{ for some } n \in \mathbb{N}.
\]
Let \( B = \mathbb{R}, C = I, \) and \( A = X = \varphi^{-1}[\text{int } C] \). According to condition (2) there is a continuous function \( f : X \to (0,1) \) such that \( f(X \setminus U_n) \subset [0,1/(n+1)] \) for each \( n \in \mathbb{N} \). Then \( F_n = f^{-1}([1/n, 1]) \), \( n \in \mathbb{N} \), is the closed cover of \( X \) that proves countable paracompactness.

Let us now turn to proving that (1) implies (3) in Theorem 4 and that (1) implies (2) in Theorem 5. So assume that \( X \) is normal and countably paracompact (respectively paracompact) and let \( B, C, \varphi, \) and \( A \) be as in the hypotheses of condition (3) in Theorem 4 (respectively (2) in Theorem 5). With Michael we choose a continuous selection \( g \) of \( \varphi \) and we define a function \( \psi : X \to 2^C \) by
\[
\psi(x) = \varphi(x) \cap \{a \in B : \|a - g(x)\| \leq 1\}.
\]

We intend to apply Lemma 3 to \( \psi \). It is obvious that \( \psi \) is bounded and LSC and that every \( \psi(x) \) is convex and compact (respectively closed). We verify that \( \psi^{-1}[\text{int } C] = \varphi^{-1}[\text{int } C] \) so that \( A \subset \psi^{-1}[\text{int } C] \). Let \( x \in \varphi^{-1}[\text{int } C] \). So there is a vector \( a \in \varphi(x) \cap \text{int } C \) and hence \( a + U_{\varepsilon} \subset C \) for some \( \varepsilon > 0 \). Note that \( g(x) \in \varphi(x) \subset C \) and pick a \( t \in (0,1] \) with \( t\|a - g(x)\| \leq 1 \). Let \( b = g(x) + t(a - g(x)) \in \varphi(x) \) and note that \( \|b - g(x)\| = t\|a - g(x)\| \leq 1 \). By convexity of \( C \) we have \( b + U_{\varepsilon} \subset C \) and hence \( b \in \text{int } C \). So \( b \in \psi(x) \cap \text{int } C \) and \( x \in \psi^{-1}[\text{int } C] \).

Since \( A \) is by assumption an \( F_\sigma \)-set we may choose a sequence \( H_1 \subset H_2 \subset \cdots \) of closed subsets of \( X \) such that \( \bigcup_{k=1}^\infty H_k = A \). For every \( n \in \mathbb{N} \) consider the open set \( U_n = \psi^{-1}[\text{int } \text{int }_n C] \) and note that the \( U_n \)‘s cover \( \psi^{-1}[\text{int } C] \) and hence \( A \). Since \( X \) is countably paracompact, which is a closed hereditary property, we can find for each \( n \in \mathbb{N} \) a closed covering \( K_{k_1} \subset K_{k_2} \subset \cdots \) of \( H_k \) such that \( K_{kn} \subset U_n \) for each \( n \in \mathbb{N} \). If we define \( F_n = \bigcup_{k=1}^n K_{kn} \) then the \( F_n \)‘s cover \( A \). Note that for each \( n \in \mathbb{N} \) we have \( F_n \subset U_n \subset \psi^{-1}[\text{int }_n C] \) so we may apply Lemma 3 to \( \psi \) to obtain a continuous \( f \) with the property \( f(A) = \bigcup_{n=1}^\infty f(F_n) \subset \text{int } C \). Since \( \psi(x) \subset \varphi(x) \) for each \( x \in X \), \( f \) is also a selection of \( \varphi \) and we trivially have \( f^{-1}(\text{int } C) \subset \varphi^{-1}[\text{int } C] \). \( \square \)

Theorem 1 now follows immediately from Theorem 4 with the slight flaw that Dowker spaces are not covered. To obtain the full strength of Theorem 1 we derive it from Lemma 3:

**Proof of Theorem 1.** Let \( X \) be a normal space, let \( A \) be a closed subset of \( X \), let \( Y_0, Y_1 \) be disjoint closed \( G_\delta \)-subsets of \( X \), and let \( f : A \to I \) be a continuous function such that for \( i = 0, 1, f^{-1}(i) = Y_i \cap A \). Choose a continuous extension \( g : X \to I \) of \( f \) such that \( g(Y_i) \subset \{i\} \) for \( i = 0, 1 \). Put
$G = g^{-1}(\{0, 1\})$ and let $H_2, H_3, \ldots$ be a sequence of closed subsets of $X$ such that $\bigcup_{n=2}^{\infty} H_n = X \setminus (Y_0 \cup Y_1)$. We define for $n \geq 2$ the closed sets

$$F_n = g^{-1}([1/n, 1 - 1/n]) \cup (H_n \cap G).$$

For the purpose of applying Lemma 3 the role of the Banach space $B$ is played by $\mathbb{R}$ and $C = I$ so $\text{int}_{1/n} C = [1/n, 1 - 1/n]$. Define the obviously bounded LSC function $\varphi : X \to 2^I$ by

$$\varphi(x) = \begin{cases} \{g(x)\}, & \text{if } x \in A \cup Y_0 \cup Y_1; \\ I, & \text{otherwise.} \end{cases}$$

If $x \in F_n$ then either $g(x) \in \varphi(x) \cap \text{int}_{1/n} C$ or $x \in H_n \cap G$ which means that $x \notin Y_0 \cup Y_1$ and $g(x) \in \{0, 1\}$. In the second case we have $x \notin A$ and $\varphi(x) = I$ which implies $1/2 \in \varphi(x) \cap \text{int}_{1/n} C$. So in either case we may conclude that $F_n \subset \varphi^{-1}[\text{int}_{1/n} C]$ for every $n \geq 2$. Observe that $\varphi$ satisfies all the hypotheses of Lemma 3 so there is a continuous selection $\hat{f}$ of $\varphi$ such that $\hat{f}(\bigcup_{n=2}^{\infty} F_n) \subset (0, 1)$. Note that $\hat{f}$ extends $g$ (and $f$) so $f(Y_i) \subset \{1\}$. Let $x \in X \setminus (Y_0 \cup Y_1)$. If $g(x) \in (0, 1)$ then $x$ is in some $g^{-1}([1/n, 1 - 1/n])$ and if $g(x) \in \{0, 1\}$ then $x$ is in some $H_m \cap G$. So $x$ is in some $F_k$ and $\hat{f}(x) \in (0, 1)$. We have shown that $\hat{f}^{-1}(i) = Y_i$ for $i = 0, 1$.

As to the question of whether it is necessary for $C$ to be convex in Theorems 4 and 5 note that if $C$ is any open set or any set with empty interior then (2) is always valid, the condition $A \subset f^{-1}(\text{int} C)$ being trivially satisfied. According to [1, p. TVS II.14], if $C$ is a convex set with nonempty interior then int $C$ is dense in $C$ and int $C = \text{int} C$, which means that the content of Theorems 4 and 5 does not change if we add the requirement that $C$ be closed. These observations suggest that the theorems are primarily of interest if $C$ is a closed set with dense interior so let us consider that case. It is obvious that (2) is valid if $C$ is for instance a union of two disjoint convex and closed sets so also in this case convexity is not strictly necessary. However, convexity plays an important role: The following proposition implies that if $C$ is a closed set with a dense and connected interior such that condition (2) is valid then $C$ must be convex.

**Proposition 6.** Let $B$ be a Banach space and let $C$ be a closed subset of $B$. If for every LSC function $\varphi : I \to 2^C$ such that each $\varphi(x)$ is compact and convex there is a continuous selection $f$ of $\varphi$ with $f^{-1}(\text{int} C) = \varphi^{-1}[\text{int} C]$ then each component of $\text{int} C$ is convex.

**Proof.** Let $O$ be a component of $\text{int} C$ and let $a$ and $b$ be two distinct vectors in $O$. Consider $\langle a, b \rangle$, the line segment $\{a + t(b - a) : t \in I\}$ that connects $a$ and $b$. Since we are in a Banach space $O$ is open and arcwise connected. We can find an embedding $\alpha : I \to O$ such that $\alpha(0) = a$ and $\alpha(1) = b$. Let $s = \sup\{t \in I : \langle a, \alpha(t) \rangle \subset O\}$. 


Since $a$ has convex neighbourhoods in $O$ we know that $s > 0$. Put $c = \alpha(s)$ and note that $\langle a, c \rangle$ is contained in the closure of $O$ and hence in the closed set $C$. Define the LSC function $\varphi : I \to 2^C$ by

$$
\varphi(t) = \begin{cases}
\{a\}, & \text{if } t = 0; \\
\langle a, c \rangle, & \text{if } 0 < t < 1; \\
\{c\}, & \text{if } t = 1.
\end{cases}
$$

Let $f : I \to C$ be a continuous selection of $\varphi$ such that $f^{-1}(\text{int} C) = \varphi^{-1}[\text{int} C] = I$. Since $f(I) \subseteq \langle a, c \rangle$, $f(0) = a$, and $f(1) = c$ the function $f$ must be surjective onto $\langle a, c \rangle$. So $\langle a, c \rangle$ is a subset of $\text{int} C$ and $O$. If $s = 1$ then $\langle a, b \rangle = \langle a, c \rangle \subseteq O$ and we are finished. Note that $\langle a, c \rangle$ must have a convex neighbourhood in $O$ so if $s < 1$ then there is an $\varepsilon > 0$ with $\langle a, \alpha(t) \rangle \subseteq O$ for all $t \in (s - \varepsilon, s + \varepsilon)$. This result contradicts the maximality of $s$.

\section*{3. Extending products.}

Put $\mathbb{R}^+ = [0, \infty)$ and $\mathbb{R}^- = (-\infty, 0]$.

\textbf{Theorem 7.} Let $X$ be a normal space and let $A$ be a closed subset of $X$. If $f : A \to \mathbb{R}^+$, $g : A \to \mathbb{R}^+$, and $h : X \to \mathbb{R}^+$ are continuous functions such that $f \cdot g = h|A$ then there are continuous extensions $\hat{f}, \hat{g} : X \to \mathbb{R}^+$ of $f$ and $g$ with $f \cdot \hat{g} = h$. If in addition $g^{-1}(0) \subseteq f^{-1}(0)$ then it can be arranged that $\hat{g}^{-1}(0) \subseteq \hat{f}^{-1}(0)$.

\textbf{Proof.} Let $\hat{f}, \hat{g} : X \to \mathbb{R}^+$ be Tietze extensions of $f$ and $g$. Define the obviously continuous functions $\hat{f}, \hat{g} : X \to \mathbb{R}^+$ by

$$
\hat{f}(x) = \frac{f(x) - \hat{g}(x) + \sqrt{(f(x) - \hat{g}(x))^2 + 4h(x)}}{2}
$$

and

$$
\hat{g}(x) = \frac{\hat{g}(x) - \hat{f}(x) + \sqrt{(\hat{f}(x) - \hat{g}(x))^2 + 4h(x)}}{2}.
$$

Some straightforward algebra shows that $\hat{f} \cdot \hat{g} = h$ and that whenever $\hat{f}(x) \cdot \hat{g}(x) = h(x)$ we have $\hat{f}(x) = \hat{f}(x)$ and $\hat{g}(x) = \hat{g}(x)$ which means that $\hat{f}$ and $\hat{g}$ are extensions of $f$ and $g$.

If we have $g^{-1}(0) \subseteq f^{-1}(0)$ or, equivalently, $f^{-1}(0) = h^{-1}(0) \cap A$ then we choose $\hat{g}$ as above but we let $\hat{f}$ be a Tietze extension of $f \cup (0|h^{-1}(0))$. We then define $\hat{f}$ and $\hat{g}$ as above. If $\hat{g}(x) = 0$ then $h(x) = \hat{f}(x) \cdot \hat{g}(x) = 0$ and hence $\hat{f}(x) = 0$. Substitution of this information into the definition of $\hat{f}$ gives $\hat{f}(x) = -\hat{g}(x) + \hat{g}(x) = 0$ and we may conclude that $\hat{g}^{-1}(0) \subseteq \hat{f}^{-1}(0)$.

The following result is Theorem 2 without the restrictions on the domain.
Corollary 8. Let $X$ be a normal space and let $A$ be a closed subset of $X$. If $f : A \to \mathbb{R}$, $g : A \to \mathbb{R^+}$, and $h : X \to \mathbb{R}$ are continuous functions such that $f \cdot g = h|A$ and $g^{-1}(0) \subset f^{-1}(0)$ then there are continuous extensions $\hat{f} : X \to \mathbb{R}$ and $\hat{g} : X \to \mathbb{R^+}$ of $f$ and $g$ with $\hat{f} \cdot \hat{g} = h$.

Proof. Let $\tilde{f}, \tilde{g} : X \to \mathbb{R^+}$ be continuous extensions of $|f|$ and $g$ such that $\tilde{f} \cdot \tilde{g} = |h|$ and $\tilde{f}^{-1}(0) = \tilde{h}^{-1}(0)$. If we put $\tilde{f} = (\tilde{f}|h^{-1}((\mathbb{R^+})) \cup (-\tilde{f}|h^{-1}((\mathbb{R^-})$) and $\tilde{g} = \tilde{g}$ then $\tilde{f}$ is continuous and $\tilde{f} \cdot \tilde{g} = h$. □

A natural question is how this corollary extends to the complex numbers. Let $\mathbb{C}^+$ stand for $\mathbb{C}$ with the negative real numbers removed.

Corollary 9. Let $X$ be a normal space and let $A$ be a closed subset of $X$. If $f : A \to \mathbb{C}$, $g : A \to \mathbb{C^+}$, and $h : X \to \mathbb{C}$ are continuous functions such that $f \cdot g = h|A$ and $g^{-1}(0) \subset f^{-1}(0)$ then there are continuous extensions $\hat{f} : X \to \mathbb{C}$ and $\hat{g} : X \to \mathbb{C^+}$ of $f$ and $g$ with $\hat{f} \cdot \hat{g} = h$.

Proof. Let $\tilde{f}, \tilde{g} : X \to \mathbb{R^+}$ be continuous extensions of $|f|$ and $|g|$ such that $\tilde{f} \cdot \tilde{g} = |h|$ and $\tilde{g}^{-1}(0) \subset \tilde{f}^{-1}(0)$. Put $O = \tilde{g}^{-1}((0, \infty))$ and $G_n = \tilde{g}^{-1}([1/n, \infty))$ for $n \in \mathbb{N}$. Since $g(A) \subset \mathbb{C^+}$ we can find a continuous function $\theta : A \cap O \to (-\pi, \pi)$ such that $g(x) = |g(x)|e^{i\theta(x)}$ for each $x \in A \cap O$. Let $\theta_1 : G_1 \to (-\pi, \pi)$ be a Tietze extension of $\theta|A \cap G_1$. Proceeding inductively, let $\theta_n : G_n \to (-\pi, \pi)$ be a Tietze extension of $\theta_{n-1} \cup (\theta|A \cap G_{n+1})$. Put $\tilde{\theta} = \bigcup_{n=1}^{\infty} \theta_n$ and note that since $O = \bigcup_{n=1}^{\infty} \operatorname{int} G_n$ we have that $\tilde{\theta} : O \to (-\pi, \pi)$ is a continuous extension of $\theta$.

Define for $x \in X$,

$$\hat{g}(x) = \begin{cases} \tilde{g}(x)e^{i\tilde{\theta}(x)}, & \text{if } x \in O; \\ 0, & \text{if } x \notin O, \end{cases}$$

and

$$\hat{f}(x) = \begin{cases} h(x)/\hat{g}(x), & \text{if } x \in O; \\ 0, & \text{if } x \notin O. \end{cases}$$

It is obvious that $\hat{f}$ and $\hat{g}$ extend $f$ and $g$, that $\hat{f} \cdot \hat{g} = h$, and that $\hat{f}$ and $\hat{g}$ are continuous at points in $O$. What remains is to verify the continuity at points in $X \setminus O$. Let $x \in X \setminus O$ and $y \in X$. Then $\hat{g}(x) = \hat{g}(x) = \hat{f}(x) = 0$ and since $\tilde{g}^{-1}(0) \subset \tilde{f}^{-1}(0)$ we have also $\tilde{f}(x) = 0$. Note that $\hat{g}(y) = \hat{f}(y) = 0$ or $|\tilde{g}(y) - \tilde{g}(x)| = |\tilde{g}(y)| = \tilde{g}(y) = |\tilde{g}(y) - \tilde{g}(x)|$ and $|\tilde{f}(y) - \tilde{f}(x)| = |\tilde{f}(y)| = |h(y)/\tilde{g}(y)| = f(y) = |\hat{f}(y) - \hat{f}(x)|$. Since $\tilde{g}$ and $\tilde{f}$ are continuous we have that $\hat{g}$ and $\hat{f}$ are continuous at $x$. □

The two restrictions, $g(A) \subset \mathbb{C^+}$ and $g^{-1}(0) \subset f^{-1}(0)$, are essential as the following examples show. Let $D$ be the unit disk $\{z \in \mathbb{C} : |z| \leq 1\}$. Choose $X = D$ and $A = \partial D = \{z \in \mathbb{C} : |z| = 1\}$. 
**Example 1.** For $z \in \partial D$, $f(z) = z$ and $g(z) = 1/z$ and let $h$ be the constant function 1 on $D$. If $\hat{f}$ extends $f$ over $D$ then according to Brouwer $\hat{f}(z) = 0$ for some $z \in D$ which contradicts $\hat{f} \cdot \hat{g} = 1$.

**Example 2.** For $z \in \partial D$, let $f(z) = z$ and $g(z) = 0$ and for $z \in D$ let $h(z) = 1 - |z|$. If $\hat{f}$ extends $f$ over $D$ then $\hat{f}(z) = 0$ for some $z \in D \setminus \partial D$ which contradicts $f(z) \cdot \hat{g}(z) = 1 - |z| > 0$.

4. Extending pairwise disjoint collections.

We call two functions $f, g : X \to \mathbb{R}$ disjoint if their product $f \cdot g$ is the zero function. Frantz [3] presents the following two propositions.

**Proposition 10.** Let $A$ be a closed subset of a normal space $X$ and let the functions $f_1, f_2, \ldots, f_n : A \to \mathbb{R}$ be continuous and pairwise disjoint. Then there exist pairwise disjoint continuous extensions $\hat{f}_1, \hat{f}_2, \ldots, \hat{f}_n$ of the respective $f_i$ over all of $X$.

**Proposition 11.** Let $A$ be a closed subset of a metric space $X$ and let $\{f_\gamma : \gamma \in \Gamma\}$ be a set of continuous and pairwise disjoint functions from $A$ to $\mathbb{R}$. Then there exist a set $\{\hat{f}_\gamma : \gamma \in \Gamma\}$ of pairwise disjoint continuous functions from $X$ to $\mathbb{R}$ such that $\hat{f}_\gamma|A = f_\gamma$ for each $\gamma \in \Gamma$.

Frantz states that Proposition 10 is also valid for countably infinite collections of functions but that the proof is rather technical and will be included in later work. We observe, however, that this result can easily be obtained as a corollary to Proposition 11.

**Corollary 12.** Let $A$ be a closed subset of a normal space $X$ and let the functions $f_1, f_2, \ldots : A \to \mathbb{R}$ be continuous and pairwise disjoint. Then there exist pairwise disjoint continuous extensions $\hat{f}_1, \hat{f}_2, \ldots$ of the respective $f_i$ over all of $X$.

**Proof.** Let $\tilde{f}_i : X \to \mathbb{R}$ be a Tietze extension of $f_i$ for each $i \in \mathbb{N}$. Consider the metric space $\mathbb{R}^\mathbb{N}$ and let $\pi_i : \mathbb{R}^\mathbb{N} \to \mathbb{R}$ be the projection on the $i$-th coordinate. Define the map $F : X \to \mathbb{R}^\mathbb{N}$ by $\pi_i \circ F = \tilde{f}_i$ for every $i \in \mathbb{N}$. Let $B$ stand for the closure of $F(A)$ in $\mathbb{R}^\mathbb{N}$. If $i \neq j$ then $\pi_i \cdot \pi_j|F(A)$ is the zero function and hence by continuity $\pi_i \cdot \pi_j|B$ is zero as well. So Proposition 11 implies that there are pairwise disjoint continuous extensions $g_i : \mathbb{R}^\mathbb{N} \to \mathbb{R}$ of $\pi_i|B$, $i \in \mathbb{N}$. Then the functions $\hat{f}_i = g_i \circ F$ are as required. □

**Example 3.** It can be shown that Proposition 11 fails for any space $X$ that contains an uncountable product of nontrivial spaces, which answers a question raised in [3]. The same examples also show that Corollary 12 does not extend to families of functions with cardinality $\mathfrak{N}_1$.

Let $X$ contain the space $Y = \prod_{\gamma \in \Gamma} Y_\gamma$, where $\Gamma$ is uncountable and every $Y_\gamma$ consists of at least two points. Let $\pi_\gamma : Y \to Y_\gamma$ be the projection. We
may assume that every $Y_{\gamma}$ contains only two points, $a_{\gamma}$ and $b_{\gamma}$. Define for each $\gamma \in \Gamma$ a point $x_{\gamma} \in Y$ by $\pi_{\gamma}(x_{\gamma}) = b_{\gamma}$ and $\pi_{\beta}(x_{\gamma}) = a_{\beta}$ for $\beta \neq \gamma$ and note that $D = \{x_{\gamma} : \gamma \in \Gamma\}$ is a discrete space. Define $a \in Y$ by $\pi_{\gamma}(a) = a_{\gamma}$ for all $\gamma \in \Gamma$ and note that $A = D \cup \{a\}$ is the one-point compactification of $D$ and hence $A$ is closed in $X$. Define for $\gamma \in \Gamma$, $f_{\gamma} : A \to \mathbb{R}$ as the characteristic function of the singleton $\{x_{\gamma}\}$. So $F = \{f_{\gamma} : \gamma \in \Gamma\}$ is an uncountable pairwise disjoint family of continuous functions. According to [4, Theorem 1.9] the Cantor cube $Y$ satisfies the countable chain condition which means that every pairwise disjoint collection of open sets in $Y$ is countable. So no continuous extension of the family $F$ over $Y$ (and hence over $X$) is pairwise disjoint.

References


Received May 5, 2000 and revised November 30, 2000.

Department of Mathematics
The University of Alabama
Tuscaloosa, AL 35487-0350
E-mail address: stoyu@hotmail.com

Ball State University
Muncie, IN 47306

Divisie der Wiskunde en Informatica
Vrije Universiteit
De Boelelaan 1081A
1081 HV Amsterdam
The Netherlands
E-mail address: dijkstra@cs.vu.nl
RENORMALIZATION OF CERTAIN INTEGRALS DEFINING TRIPLE PRODUCT $L$-FUNCTIONS

JENNIFER E. BEINEKE

We obtain special values results for the triple product $L$-function attached to a Hilbert modular cuspidal eigenform over a totally real quadratic number field and an elliptic modular cuspidal eigenform, both of level one and even weight. Replacing the elliptic modular cusp form by a specified Eisenstein series, we renormalize the integral defining the triple product $L$-function in order to obtain an integral representation for a product of Asai $L$-functions. We hope in further work to extend these results to triple-product $L$-functions attached to automorphic representations and then study the critical values of this renormalized triple product.

1. Introduction.

This paper investigates Zagier’s technique of renormalization ([Z]), applied to an integral defining a certain triple product $L$-function. The renormalized integral becomes the product of two Asai $L$-functions, one shifted by an integer. As a by-product of these results, under a certain weight restriction on the modular forms, special values results can be explicitly determined for the triple product $L$-function in question. Such special values issues have been studied in the representation-theoretic context by Piatetski-Shapiro and Rallis ([PSR]), Garrett and Harris ([GH]), and Harris and Kudla ([HK]). The foundation of this work is Garrett’s groundbreaking study of the Rankin triple product $L$-function ([G1]).

The $L$-function in question, $L(f \otimes G, s)$, is a variation of the Rankin triple product $L$-function, defined for a holomorphic Hilbert modular cuspidal eigenform $G$ and a holomorphic elliptic cuspidal eigenform $f$, both of level one and even weights. If we let $E_3$ denote the Siegel Eisenstein series of degree 3, then under a certain embedding $\tau_{2,1}$ of $\mathfrak{H}^3$ into the Siegel upper half-space of degree 3, following Garrett’s techniques ([G1]) we show:

**Theorem 1.1.** For $\text{Re}(s)$ sufficiently large,

$$\int_{\Gamma \backslash \mathfrak{H}^2} \int_{\text{SL}(2, \mathbb{Z}) \backslash \mathfrak{H}} E_3(\tau_{2,1}(Z, z_3); 2k, s) \overline{G(z)} f(z_3) y_1 y_2 y_3^{2k-2} dx_3 dy_3 d\tilde{x} d\tilde{y} = L(f \otimes G, s + 4k - 2) \times \text{(normalizing factors)}.$$
For the result in the more general context of mixed weights, see Theorem 5.4. Such a theorem was proved in the representation-theoretic context by Garrett and Harris ([GH]) and leads to special values results.

Shimura developed a technique for determining special values for ratios of zeta functions associated with cusp forms by using Rankin product $L$-functions ([S2], [S3]). In the situations he considered, replacing one modular form in a Rankin product $L$-function by a specified Eisenstein series leads to a product of the zeta functions in question. One can then extend special values results for the product $L$-function to the product of the zeta functions. When attempting to use the same procedure for the triple product $L$-function, if we replace $f$ by a specified Eisenstein series $E$, then the integral no longer converges. However, by renormalizing the integral, we obtain:

**Theorem 1.2.** For $\Re(s)$ sufficiently large,

\[
R.N. \int_{\Gamma \backslash \mathbb{H}^2} \int_{\text{SL}(2, \mathbb{Z}) \backslash \mathbb{H}} E_3(\tau_{2,1}(Z, z_3); 2k, s)G(\bar{z}) \overline{\mathcal{E}(z_3)} \\
\cdot (y_1 y_2 y_3)^{2k-2} dx_3 dy_3 d\bar{x} d\bar{y} \\
= L_{\text{Asai}}(G, s + 4k - 2)L_{\text{Asai}}(G, s + 2k - 1) \\
\times (\text{normalizing factors}).
\]

Refer to Theorem 6.4 for the result in the mixed-weight case.

The function $L_{\text{Asai}}(G, s)$ is defined as in [A]. Namely, given the Hilbert modular form $G$ of weight $(k_1, k_2)$, $k_1 \geq k_2$, on a quadratic number field $F$ with Fourier coefficients $b(\xi)$, the Asai $L$-function is constructed as a sort of “subseries” of the standard $L$-function attached to Hilbert modular forms, summing up only over the rational integers:

\[
L_{\text{Asai}}(G, s) = \zeta(2(s - k_1 + 1)) \sum_{n=1}^{\infty} b(n)n^{-s},
\]

where $\zeta(s)$ is the Riemann zeta function.

Using the identities above, special values results for the triple product $L$-function can then extend to the product of Asai $L$-functions, and therefore to ratios of Asai $L$-functions, following the techniques of Shimura ([S2], [S3]). However, the special values results for the triple product $L$-function must be within a certain weight case for the modular forms, called the “indefinite case” by Harris and Kudla ([HK]).

Throughout this paper, we consider only modular forms of level one in the classical language. To follow Shimura’s techniques for obtaining special values results, the generalization to higher levels is required, necessitating the representation-theoretic approach. The basic structure of the paper follows a similar method to that of Garrett ([G1]). After setting up notational
preliminaries, we outline the various embeddings and coset decompositions required for the computations involving the Siegel Eisenstein series. We then compute the integral representation for the $L$-function, from which we derive the Euler product, functional equation, and special values result. Finally, replacing the elliptic cusp form by a specified Eisenstein series, we compute the renormalized integral, obtaining the product of Asai $L$-functions.

2. Preliminaries.

Let $F$ be a real quadratic extension of $\mathbb{Q}$ with ring of integers $\mathfrak{o}$. We will assume that the quadratic number field $F$ has narrow class number one. Let $D_F$ denote the discriminant of $F$, $\delta^{-1}$ the inverse different, and let $\tau$ signify the nontrivial injection of $F$ into $\mathbb{R}$. Write $U^+$ for the group of totally positive units of $F$.

For a commutative ring $R$, write $M(n, R)$ for the space of $n \times n$ matrices over $R$. $GL(n, R)$ will signify the group of invertible $n \times n$ matrices, and $SL(n, R)$ the group of matrices with determinant one. $I_n$ is the $n \times n$ identity matrix, and let $J_n = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$.

The symbol $\mathcal{H}$ will be used to denote the complex upper half-plane, and $\mathcal{H}^n$ represents the product of $n$ copies of the complex upper half-plane. The Siegel upper half-space of degree $n$ is written

$$H_n = \{ Z \in M(n, \mathbb{C}) | \text{tr} = Z, \text{Im}(Z) > 0 \}.$$  

The Siegel Eisenstein series is then defined as follows. The symplectic group is given by

$$\text{Sp}(n, R) = \{ g \in GL(2n, R) | gJ_n g = J_n \}.$$  

There is a natural action of $\text{Sp}(n, \mathbb{R})$ on $\mathcal{H}_n$ given by

$$Z \mapsto gZ = (AZ + B)(CZ + D)^{-1}, \quad g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$  

Let

$$P_{n, 0}(Z) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{Sp}(n, \mathbb{Z}) \middle| C = 0 \right\}.$$  

For $s \in \mathbb{C}$, $k \in \mathbb{Z}$, and $Z = X + iY \in \mathcal{H}_n$, define the Eisenstein series

$$(2.1) \quad E_n(Z; 2k, s) = \sum_{C, D} \det(CZ + D)^{2k} \frac{(\det Y)^s}{|\det(CZ + D)|^{2s}},$$
where the sum is over all representatives \( \left( \begin{array}{cc} * & * \\ C & D \end{array} \right) \) for \( P_{n,0}(\mathbb{Z}) \setminus \text{Sp}(n, \mathbb{Z}) \).

For \( Z \in \mathfrak{h}_n \) and \( g = \left( \begin{array}{cc} A & B \\ C & D \end{array} \right) \in \text{Sp}(n, \mathbb{R}) \), we will often write

\[
\mu(g, Z) = \det(CZ + D). 
\] (2.2)

The Eisenstein series converges for \( \text{Re}(s) \) sufficiently large, and can be continued to a meromorphic function in the entire \( s \)-plane (see [K] or [L]). Its poles have been studied by Ikeda ([I]).

3. Embeddings of symplectic spaces and coset decompositions.

To proceed with the integral representation of the triple product \( L \)-function, we first need to determine an embedding from \( \text{Sp}(1, \mathfrak{o}) \) into \( \text{Sp}(2, \mathbb{Z}) \), which takes a group \( \Gamma \cong \text{Sp}(1, \mathfrak{o}) \) to \( \text{Sp}(2, \mathbb{Z}) \). Write \( \Delta(F) \) for the determinant mapping \( F^2 \times F^2 \to F \), and define \( \Delta(E) = \frac{1}{2} \text{Tr}_{F/\mathbb{Q}} \circ \Delta(F) : F^2 \times F^2 \to \mathbb{Q} \).

Note that if we set \( M = \mathfrak{o} \oplus \delta^{-1} \), then \( \Delta(E) : M \times M \to \mathbb{Z} \) surjectively. Let \( \Gamma = \Gamma(M, E) = \{ g \in \text{Sp}(1, \mathfrak{F}) \mid gM = M \} \).

Then \( \Gamma \cong \text{Sp}(1, \mathfrak{o}) \). One can check at once that over \( \mathbb{Z} \), \( \Gamma(M/\mathbb{Z}, E) \cong \text{Sp}(2, \mathbb{Z}) \). Let \( \beta \in \mathfrak{F} \) be an element such that \( \{ 1, \beta \} \) is a \( \mathbb{Z} \)-basis of \( \mathfrak{o} \), and put

\[
B = \left( \begin{array}{cc} 1 & \beta \\ 1 & \beta^\tau \end{array} \right).
\]

We will define \( \iota : \mathfrak{h}^2 \to \mathfrak{h}_2 \) to be the embedding:

\[
\iota(z_1, z_2) = B^{-1} \cdot \text{diag}[z_1, z_2] \cdot tB^{-1}.
\]

It is compatible with the injection \( \iota \) of \( \text{Sp}(1, \mathfrak{F}) \) into \( \text{Sp}(2, \mathbb{Q}) \) defined by

\[
\iota \left( \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \right) = \left( \begin{array}{cc} B^{-1} & 0 \\ 0 & tB \end{array} \right) \left( \begin{array}{cc} \Delta(a) & \Delta(b) \\ \Delta(c) & \Delta(d) \end{array} \right) \left( \begin{array}{cc} B & 0 \\ 0 & tB^{-1} \end{array} \right),
\]

where \( \Delta(a) = \text{diag}[a, a^\tau], (a \in \mathfrak{F}) \). Note that \( \iota : \Gamma \to \text{Sp}(2, \mathbb{Z}) \). Define two more embeddings

\[
\iota_{m,n} : \text{Sp}(m, \mathfrak{F}) \times \text{Sp}(n, \mathfrak{F}) \to \text{Sp}(m+n, \mathfrak{F})
\]

and

\[
\iota_{m,n} : \mathfrak{h}_m \times \mathfrak{h}_n \to \mathfrak{h}_{m+n}
\]

by

\[
\iota_{m,n} \left( \left( \begin{array}{cc} A & B \\ C & D \end{array} \right) \times \left( \begin{array}{cc} A' & B' \\ C' & D' \end{array} \right) \right) = \left( \begin{array}{ccc} A & 0 & B & 0 \\ 0 & A' & 0 & B' \\ C & 0 & D & 0 \\ 0 & C' & 0 & D' \end{array} \right).
\]
Having determined the appropriate embeddings, following Garrett’s lead ([G1]) we now investigate the coset decompositions that are used to rewrite the Siegel Eisenstein series. For a commutative ring \( R \), consider the following subgroups of \( \operatorname{Sp}(n, R) \), where \( I_m \) is the identity matrix of size \( m \), \( 0_m \) is the zero matrix of size \( m \), and \( 0 \leq r < n \):

\[
G_{n,r}(R) = \begin{cases} 
\left( \begin{array}{cccc}
I_{n-r} & 0 & 0_{n-r} & 0 \\
0 & * & 0 & * \\
0_{n-r} & 0 & I_{n-r} & 0 \\
0 & * & 0 & * 
\end{array} \right) & \in \operatorname{Sp}(n, R)
\end{cases},
\]

\[
L_{n,r}(R) = \begin{cases} 
\left( \begin{array}{cccc}
* & 0 & 0_{n-r} & 0 \\
0 & I_r & 0 & 0 \\
0_{n-r} & 0 & 0 & 0 \\
0 & 0 & 0 & I_r 
\end{array} \right) & \in \operatorname{Sp}(n, R)
\end{cases},
\]

\[
U_{n,r}(R) = \begin{cases} 
\left( \begin{array}{cccc}
I_{n-r} & * & * & * \\
0 & I_r & * & 0 \\
0 & 0 & I_{n-r} & 0 \\
0 & 0 & * & I_r 
\end{array} \right) & \in \operatorname{Sp}(n, R)
\end{cases}.
\]

Set \( P_{n,r}(R) = G_{n,r}(R)L_{n,r}(R)U_{n,r}(R) \), and let

\[
w_{n,r} = \left( \begin{array}{cccc}
0_{n-r} & 0 & -1_{n-r} & 0 \\
0 & 1 & 0 & 0 \\
1_{n-r} & 0 & 0_{n-r} & 0 \\
0 & 0 & 0 & 1_r 
\end{array} \right),
\]

\[
w_n = w_{n,0}.
\]

The following four results, which only involve the rational symplectic spaces, are proved in [G1].

**Proposition 3.1.** The double coset space

\[
P_{n+1,0}(\mathbb{Q}) \backslash \operatorname{Sp}(n+1, \mathbb{Q}) / \iota_{n,1}(\operatorname{Sp}(n, \mathbb{Q}) \times \operatorname{Sp}(1, \mathbb{Q}))
\]

has irredundant representatives \( I_{2n+2}, \xi \), where

\[
\xi = \xi_0 \iota_{n,1}(I_{2n}, w_1)
\]

\[
\xi_0 = \begin{pmatrix}
I_n & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0_n & v & I_n & 0 \\
\iota_v & 0 & 0 & 1
\end{pmatrix}
\]

\( \iota_v = (0, \ldots, 0, 1) \in \mathbb{Q}^n \).
Proposition 3.2. The coset space
\[ P_{n+1,0}(Q) \backslash \text{Sp}(n+1,Q) \]
has irredundant representatives consisting of the disjoint union of representatives for
\[ \tilde{\iota}_{n,1}(P_{n,0}(Q) \backslash \text{Sp}(n,Q) \times P_{1,0}(Q) \backslash \text{Sp}(1,Q)) \]
and for
\[ \xi \tilde{\iota}_{n,1}(P_{n,1}(Q) \backslash \text{Sp}(n,Q) \times \text{Sp}(1,Q)), \]
where \( \xi \) is defined in Proposition 3.1.

Lemma 3.3. Let \( \gamma_1 \in \text{Sp}(1,Z), \varepsilon \in Z, \varepsilon > 0 \),
\[ A_{\varepsilon} = \begin{pmatrix} \varepsilon^{-1} & 0 \\ 0 & \varepsilon \end{pmatrix} \in \text{Sp}(1,Q), \]
and \( \xi \) be as in Proposition 3.1 with \( n = 2 \). Then there is an element \( p_\varepsilon \in P_{3,0}(Q) \) such that for all \( \tilde{\gamma} \in \text{Sp}(2,Z), \gamma_2 \in \text{Sp}(1,Z) \),
\[ p_\varepsilon \xi \tilde{\iota}_{2,1}(\tilde{\gamma}, \gamma_1 A_\varepsilon \gamma_2) \in \text{Sp}(3,Z), \text{ and } \mu(p_\varepsilon, \ast) = \varepsilon, \]
where \( \mu \) is defined by (2.2).

Lemma 3.4. The coset space \( \text{Sp}(1,Z) \backslash \text{Sp}(1,Q)/\text{Sp}(1,Z) \) has irredundant representatives
\[ \left\{ A_{\varepsilon} = \begin{pmatrix} \varepsilon^{-1} & 0 \\ 0 & \varepsilon \end{pmatrix} : 0 < \varepsilon \in Z \right\}. \]

Now we incorporate the quadratic number field \( F \) into similar calculations.

Proposition 3.5. The double coset space
\[ P_{2,1}(Q) \backslash \text{Sp}(2,Q)/\tilde{\iota}(\text{Sp}(1,F)) \]
has just one orbit, and so one representative, \( I_4 \).

Proof. Since \( P_{2,1}(Q) \) is the stabilizer of a line and \( \text{Sp}(2,Q) \) acts transitively on lines in \( Q^4 \), the coset space \( P_{2,1}(Q) \backslash \text{Sp}(2,Q) \) is naturally \( P^3(Q) \). We may consider \( Q^4 \) as \( F^2 \), so, as \( \text{Sp}(1,F) \) acts transitively on the nonzero vectors of \( F^2 \), there is only one orbit of \( \text{Sp}(1,F) \) on \( P^3(Q) \). \( \square \)

Proposition 3.6. The coset space \( P_{2,1}(Q) \backslash \text{Sp}(2,Q) \) has irredundant representatives consisting of the disjoint union of representatives for
\[ \tilde{\iota}(\tilde{U}(F) \backslash \text{Sp}(1,F)), \]
where \( \tilde{U}(F) = \left\{ \begin{pmatrix} q & \ast \\ 0 & q^{-1} \end{pmatrix} \in \text{Sp}(1,F), q \in Q^* \right\}. \]
Proof. We must find a subgroup $H$ of $\text{Sp}(1, F)$ so that $h \in H$ if and only if

$$P_{2,1}(Q)i(h) = P_{2,1}(Q).$$

That is, $i(h) \in P_{2,1}(Q)$.

Looking at $P_{2,1}(Q)$ more closely, we see we can describe it as the following set of matrices:

$$P_{2,1}(Q) = \left\{ \begin{pmatrix} \alpha & u\alpha & v\alpha & w\alpha \\ 0 & A & wA - uB & B \\ 0 & 0 & \gamma & 0 \\ 0 & C & wC - uD & D \end{pmatrix} \in \text{Sp}(2, Q) \middle| \alpha, \gamma \neq 0, AD - BC \neq 0 \right\}.$$

If $h = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a_1 + a_2\beta & b_1 + b_2\beta \\ c_1 + c_2\beta & d_1 + d_2\beta \end{pmatrix} \in \text{Sp}(1, F)$, then

$$i(h) = \begin{pmatrix} a_1 & \beta^2 b_2 & b_1/2 & b_2/2 \\ a_2 & b_1/2 & b_2/2 & \beta^2 \beta^2 \\ 2c_1 & 2\beta^2 c_2 & d_1 & d_2 \\ 2\beta^2 c_1 & 2\beta^2 d_2 & \beta^2 d_2 & \beta d_2 \end{pmatrix}.$$

Combining the explicit descriptions for $i(h)$ and the matrices in $P_{2,1}(Q)$, we obtain the required result. □

Definition 3.7. Every totally positive element of $F^*/\mathbb{Q}^*$ has a unique representative $\alpha$ which can be written $\alpha = s + t\beta$, where $s, t \in \mathbb{Z}$, $s$ is positive, and $(s, t) = 1$. We will call such a representative primitive.

Lemma 3.8. For $\alpha = s + t\beta$ a primitive element of $F^*$, set

$$A = A_\alpha = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} \in \text{Sp}(1, F).$$

Then there exists a matrix

$$p = p_\alpha = \begin{pmatrix} N(\alpha)^{-1} & c_\alpha & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & N(\alpha) & 0 \\ 0 & 0 & d_\alpha & 1 \end{pmatrix} \in P_{2,0}(Q) \cap P_{2,1}(Q)$$

such that for all $\gamma \in \Gamma$,

$$p_\alpha i(A_\alpha \gamma) \in \text{Sp}(2, \mathbb{Z}).$$

Proof. Since

$$p_\alpha i(A_\alpha \gamma) = p_\alpha i(A_\alpha) i(\gamma),$$
and \(\tilde{\iota}(\gamma) \in \Sp(2, \mathbb{Z})\), it suffices to consider the case where \(\gamma = I_2\). To find \(c_\alpha\) and \(d_\alpha\), using the fact that \(s\) and \(t\) are relatively prime, let \(m, n \in \mathbb{Z}\) such that \(sn + tm = 1\). Then set
\[
c_\alpha = -t\beta^2 - mN(\alpha) \quad \text{and} \quad d_\alpha = s - nN(\alpha)
\]
where \(N(\alpha) = \Norm_F/\mathbb{Q}(\alpha)\). Direct multiplying out gives the result. \(\Box\)

4. The integral representation.

We now obtain the integral representation of the triple product \(L\)-function \(L(f \otimes G, s)\). Let \(f(z)\) be a normalized holomorphic cuspidal eigenform of weight \(2l\) on \(\SL(2, \mathbb{Z})\), and let \(G(z_1, z_2)\) be a normalized holomorphic cuspidal eigenform of weight \((2k_1, 2k_2)\) on \(\Gamma\), where \(\Gamma\) is defined by Equation (3.1).

For \(\tilde{z} = (z_1, z_2) \in \mathfrak{H}^2\), write the Fourier expansions of \(f\) and \(G\) as
\[
f(z) = \sum_n a(n)e(nz), \quad e(z) = \exp(2\pi i z)
\]
\[
G(\tilde{z}) = \sum_\xi b(\xi)e_F(\xi \tilde{z}), \quad e_F(\tilde{z}) = \exp(2\pi i (z_1 + z_2))
\]
where \(\xi\) ranges over all totally positive elements of a lattice in \(F\). If we write \(\xi^1\) for the trivial injection of \(\xi\) into \(\mathbb{R}\) and \(\xi^2\) for the nontrivial injection of \(\xi\) into \(\mathbb{R}\), then \(\xi \tilde{z} = (\xi^1 z_1, \xi^2 z_2)\).

In order to compute the integral, differential operators of Maass ([M]) and Shimura ([S1]) are needed to raise the weights of the forms so they are all equal. For \(z \in \mathfrak{H}, (z_1, z_2) \in \mathfrak{H}^2\), and integers \(\kappa, r, \lambda_\nu, s_\nu \geq 0\) with \(\nu = 1, 2\), define operators for the elliptic and Hilbert modular forms, respectively, by
\[
\delta^{(r)} = \left(\frac{1}{2\pi i}\right)^r \left(\kappa + 2r - 2\right)^\frac{1}{2iy} + \frac{\partial}{\partial z}\right) \cdots \left(\kappa + 2\right)^\frac{1}{2iy} \frac{\partial}{\partial z}\right)
\]
\[
\delta^{(s_1, s_2)}_{(\lambda_1, \lambda_2)} = \left(\frac{1}{2\pi i}\right)^{s_1 + s_2} \prod_{\nu=1}^2 \left(\frac{\lambda_\nu + 2s_\nu - 2}{2iy_\nu} + \frac{\partial}{\partial z_\nu}\right)
\]
\[
\cdots \left(\frac{\lambda_\nu + 2}{2iy_\nu} \frac{\partial}{\partial z_\nu}\right)
\]
where \(\partial/\partial z = (\partial/\partial x - i\partial/\partial y)/2\) as usual, and it is understood that \(\delta^{(0)}\) and \(\delta^{(0,0)}\) are the identity operators. It can be shown that \(\delta^{(r)}\) and \(\delta^{(s_1, s_2)}_{(\lambda_1, \lambda_2)}\) raise the corresponding weights of modular forms of weight \(\kappa\) and \((\lambda_1, \lambda_2)\) to \(\kappa + 2r\) and \((\lambda_1 + 2s_1, \lambda_2 + 2s_2)\), respectively ([S1], [S2]).
As Orloff demonstrated ([BO]), we may write
\[
\delta_k^{(r)} e(nz) = \sum_{j=0}^r \delta_j^{(r)} (4\pi y)^{-j} n^{-j} e(nz),
\]
with integers \(\delta_j^{(r)}\) defined by
\[
\delta_j^{(r)} = (-1)^j \binom{r}{j} \frac{\Gamma(\kappa + r)}{\Gamma(\kappa + r - j)}.
\]

Therefore, applying the operators given by (4.1) and (4.2) to the cusp forms \(f\) and \(G\) in the case where \(k_1 \geq l \geq k_2\), we may write
\[
\delta_{k_1 - l} f(z) = \sum_n a(n) \sum_{A=0}^{k_1 - l} A^{-A} n^{k_1 - l - A} e(nz),
\]
\[
\delta_{(0,k_1-k_2)} G(\tilde{z}) = \sum_{\xi} b(\xi) \sum_{B=0}^{k_1 - k_2} B^{-B} \xi^{k_1 - k_2 - B} e_F(\xi \tilde{z}).
\]

Define the Dirichlet series
\[
D_f^{(2)}(s) = \sum_n a(n^2) n^{-2s},
\]
\[
D_{f,G}(s) = \sum_{n, \alpha \in U^+} a(n) \overline{b(n \alpha)} (\alpha^2)^{k_1 - k_2} n^{1-l-k_2} (nN(\alpha)^2)^{1-s-2k_1},
\]
where \(U^+\) is the group of totally positive units of \(F\), \(n\) ranges over the positive integers, \(\alpha\) ranges over the primitive elements of \(F^\times\) modulo \(U^+\), and \(\overline{b(\xi)}\) is the complex conjugate of \(b(\xi)\).

For \(z, s \in \mathbb{C}, y \in \mathbb{R},\) and \(k \in \mathbb{Z},\) put
\[
q_{2k,s}(z) = |z + i|^{-2s} (z + i)^{-2k}
\]
\[
\hat{q}_{2k,s}(y) = \int_\mathbb{R} q_{2k,s}(x) e(-xy) dx
\]
\[
\chi_{2k,s}(z) = |z|^{-2s} z^{-2k}
\]
\[
\eta_{2k}(s) = D_{F^{1/2}}^{1/2} \sum_{A=0}^{k_1 - l - k_2} \sum_{B=0}^{k_1 - k_2} A^{A} B^{-B}
\]
\[
\cdot \int_0^\infty \int_0^\infty \int_0^\infty (y_1 y_2 y_3)^{s+2k-2} y_1^{-A} y_3^{-B} (y_1 + y_2 + y_3)^{1-2k-2s}
\]
\[
\cdot \hat{q}_{2k,s}(y_1 + y_2 + y_3) \exp(-2\pi(y_1 + y_2 + y_3)) dy_1 dy_2 dy_3.
\]
The integrals exist for Re(s) sufficiently large. Define \( \pi : \mathcal{H}_2 \to \mathcal{H} \) by

\[
\pi \left( \begin{pmatrix} * & * \\ * & z_{22} \end{pmatrix} \right) = z_{22}.
\]

The following lemma provides the integral computation with respect to \( f \).

**Lemma 4.1.** For \( w \in \mathcal{H} \), Re(s) sufficiently large, define

\[
f_s^*(w) = \sum_n a(n) \sum_{A=0}^{k_1-1} P_A(4\pi)^{-A} n^{k_1-l-A} e(n(Re(w))) \int_0^\infty y^{s+2k_1-2-A} \cdot (y + Im(w))^{1-2s-2k} \exp(-2\pi ny)q_{2k,a}(ny + nIm(w)) \, dy,
\]

where the coefficients are defined by (4.4) and (4.5). Then, for \( Z \in \mathcal{H}_2 \), \( z \in \mathcal{H} \),

\[
\int_{SL(2,\mathbb{Z}) \setminus \mathcal{H}} E_3(\nu_{2,1}(Z,z);2k_1,s)n^{k_1-l}f(z)y^{2k_1-2} \, dx \, dy
\]

\[
= \zeta(2s+2k_1)^{-1} D_f^{(2)}(s + k_1 + l - 1)
\]

\[
\cdot \sum_{\tilde{\gamma}} [\det Im(\tilde{\gamma}Z)]^s \mu(\tilde{\gamma},Z)^{-2k_1} f_s^*(\pi(\tilde{\gamma}Z)),
\]

where the sum is over \( \tilde{\gamma} \in P_{2,1}(\mathbb{Z}) \setminus Sp(2,\mathbb{Z}) \), and \( \zeta(z) \) is the Riemann zeta-function.

The proof is almost identical to that of Garrett ([G1]), requiring application of basic properties of the differential operators ([S1]).

Having integrated with respect to the elliptic cusp form, it remains to compute the integral with respect to the Hilbert modular cusp form.

**Proposition 4.2.** With notation as above, \( \tilde{z} = (z_2, z_3) \in \mathcal{H}_2 \), and \( \iota(\tilde{z}) = Z \),

\[
\int_{\Gamma \setminus \mathcal{H}_2} \int_{SL(2,\mathbb{Z}) \setminus \mathcal{H}} E_3(\nu_{2,1}(Z,z_3);2k_1,s)n^{k_1-l}f(z_1)
\]

\[
\cdot G(\tilde{z})(y_1y_2y_3)^{2k_1-2} \, dx_1 \, dy_1 \, dx_2 \, dy_2
\]

\[
= \sum_{\tilde{\gamma}} [\det Im(\tilde{\gamma}Z)]^s \mu(\tilde{\gamma},Z)^{-2k_1} f_s^*(\pi(\tilde{\gamma}Z)) \zeta(2s+2k_1)^{-1} D_f^{(2)}(s + k_1 + l - 1) D_{f,G}(s).
\]

**Proof.** By Lemma 4.1 the integral becomes

\[
\zeta(2s+2k_1)^{-1} D_f^{(2)}(s + k_1 + l - 1) \int_{\Gamma \setminus \mathcal{H}_2} [\det Im(\tilde{\gamma}Z)]^s
\]

\[
\cdot \mu(\tilde{\gamma},Z)^{-2k_1} f_s^*(\pi(\tilde{\gamma}Z)) \zeta(2s+2k_1)^{-1} D_f^{(2)}(s + k_1 + l - 1) D_{f,G}(s).
\]

with \( \tilde{\gamma} \in P_{2,1}(\mathbb{Z}) \setminus Sp(2,\mathbb{Z}) \).

Using the fact that

\[
P_{1,0}(F) \setminus Sp(1, F) \approx P_{1,0}(\mathfrak{o}) \setminus Sp(1, \mathfrak{o}) \approx P_{1,0}(\Gamma) \setminus \Gamma,
\]
and remarking that any element of $P_{1,0}(F)$ can be written as $A_\alpha$ times an element of $\tilde{U}(F)$ for some primitive $\alpha$, then by Propositions 3.5 and 3.6, we can write

$$\tilde{\gamma} \in \iota(A_\alpha \gamma),$$

where $\gamma \in P_{1,0}(\Gamma) \setminus \Gamma$.

With $p_\alpha$ as in Lemma 3.8, let $\varsigma \in P_{2,0}(\mathbb{Q}) \cap P_{2,1}(\mathbb{Q})$ be the matrix

$$\varsigma = \begin{pmatrix}
4\beta^2 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & (4\beta^2)^{-1}
\end{pmatrix}.$$

One computes that

$$\pi(p_\alpha \iota(A_\alpha \gamma) \iota(z_2, z_3)) = \pi\left(\left(\begin{array}{*{20}c}
\ast & \ast & t & s \\
\ast & t & s & \ast
\end{array}\right) \left(\begin{array}{*{20}c}
\frac{1}{4\beta^2} & (\beta^2\gamma z_2 + \gamma^\tau z_3) \\
\beta \gamma z_2 - \gamma^\tau z_3 & \gamma z_2 + \gamma^\tau z_3
\end{array}\right) \right) \cdot \left(\begin{array}{*{20}c}
4\beta^2 & \ast & \ast \\
\ast & t & s
\end{array}\right) = \alpha^2 \gamma z_2 + (\alpha^\tau)^2 \gamma^\tau z_3.$$

Therefore (4.13) is equal to

(4.14) \hspace{1cm} \zeta(2s + 2k_1)^{-1}D_f^{(2)}(s + k_1 + l - 1)

\hspace{1cm} \cdot \int_{\Gamma \setminus \mathfrak{H}} \sum_{\alpha} \sum_{\gamma} [\text{det}(p_\alpha \iota(A_\alpha \gamma) \iota(\tilde{z})))]^s \mu(p_\alpha \iota(A_\alpha \gamma), \iota(\tilde{z}))^{-2k_1}

\hspace{1cm} \cdot f_s^* \left(\alpha^2 \gamma z_2 + (\alpha^\tau)^2 \gamma^\tau z_3 \right) \delta^{(0,k_1-k_2)}(\tilde{z})(y_2 y_3)^{2k_1-2} d\tilde{x} d\tilde{y},

where $\alpha$ ranges over the primitive elements of $F^*$, and $\gamma$ is in $P_{1,0}(\Gamma) \setminus \Gamma$.

Since $\mu(p_\alpha \iota(A_\alpha \gamma), \iota(\tilde{z})) = \mu(\gamma, z_2) \mu(\gamma^\tau, z_3)$, and $\text{det}(p_\alpha \iota(A_\alpha \gamma) \iota(\tilde{z})) = y_2 y_3$, by ‘unwinding’ as usual, we note that for fixed $\alpha$ the integral in (4.14) is now

$$\int_{P_{1,0}(\Gamma) \setminus \mathfrak{H}^2} f_s^* \left(\alpha^2 z_2 + (\alpha^\tau)^2 z_3 \right) \delta^{(0,k_1-k_2)}(\tilde{z})(y_2 y_3)^{s + 2k_1 - 2} d\tilde{x} d\tilde{y}.$$
From the Fourier expansions of \( f_j^* \) and \( G \), and summing over \( \alpha \), the Rankin method shows that the integral becomes

\[
\int_{\mathbb{R}^2/U^+} \left[ \int_{\mathbb{R}^2/\delta^{-1}} \sum_{\alpha} f_j^* (\alpha^2 z_2 + (\alpha^*)^2 z_3) \delta(0,k_1-k_2) G(\tilde{z}) d\tilde{x} \right] (y_2y_3)^{s+2k_1-2} d\tilde{y}
\]

\[
= \int_{\mathbb{R}^2/U^+} (\text{vol}(\mathbb{R}^2/\delta^{-1})) \int_0^\infty \sum_{\alpha, n} a(n) b(n\alpha^2) n^{k_1-l} \cdot \sum_{A=0}^{k_1-l} P_A(4\pi n)^{-A} \sum_{B=0}^{k_1-k_2} Q_B(4\pi \alpha^2 n)^{-B} (y_1 y_2 y_3)^{s+2k_1-2} y_1^A y_3^B 
\]

\[
\cdot (y_1 + \alpha^2 y_2 + (\alpha^*)^2 y_3)^{1-2k_1-2s} \exp(-2\pi n(y_1 + \alpha^2 y_2 + (\alpha^*)^2 y_3))
\]

\[
\cdot \hat{q}_{2k,s}(ny_1 + n\alpha^2 y_2 + n(\alpha^*)^2 y_3) dy_1 dy_2 dy_3.
\]

Replacing \( y_1 \) by \( y_1/n \), \( y_2 \) by \( y_2/n\alpha^2 \), and \( y_3 \) by \( y_3/n(\alpha^*)^2 \), the right side is simplified to become

\[
D_F^{1/2} \sum_{\alpha U^+, n} a(n) b(n\alpha^2) (\alpha^2)^{k_1-k_2} n^{1-l-k_2} (nN(\alpha)^2)^{1-s-2k_1}
\]

\[
\cdot \sum_{A=0}^{k_1-l} P_A(4\pi)^{-A} \sum_{B=0}^{k_1-k_2} Q_B(4\pi)^{-B} \int_0^\infty \int_0^\infty \int_0^\infty (y_1 y_2 y_3)^{s+2k_1-2}
\]

\[
\cdot y_1^A y_3^B (y_1 + y_2 + y_3)^{1-2k_1-2s} \exp(-2\pi n(y_1 + y_2 + y_3))
\]

\[
\cdot \hat{q}_{2k,s}(ny_1 + y_2 + y_3) dy_1 dy_2 dy_3.
\]

Substituting in the appropriate expressions provides the required result. \( \square \)

5. The Euler product.

Before determining the Euler factors of \( L(f \otimes G, s) \), we first extend the notion of primitive elements to ideals.

**Definition 5.1.** For \( \mathcal{I} \) an ideal of \( F \), write \( \mathcal{I} \) as a product of prime ideals:

\[ \mathcal{I} = \prod_i p_i^{n_i} \mathfrak{p}_i^{m_i} \]

Let \( \nu_\mathcal{I} = \prod_i (p_i, \mathfrak{p}_i)^{\min(n_i, m_i)} \), and let \( \mathcal{J} = \mathcal{I} \nu_\mathcal{I} \). We will call the ideal \( \mathcal{J} \) **primitive**.

Note that there is a one-to-one correspondence between primitive elements of \( F \) modulo \( U^+ \) and the primitive ideals. Defining the Fourier coefficients of \( G(\tilde{z}) \) on integral ideals \( (\xi) \) by \( b((\xi)) = b(\xi)e^{2\pi i \xi} \), we may now rewrite the Dirichlet series

\[
D_{f,G}(s) = \sum_{n, (\alpha)} a(n) b(n(\alpha^2))((\alpha^2)^{k_1-k_2} n^{1-l-k_2} (nN(\alpha)^2)^{1-s-2k_1},
\]

where \( (\alpha) \) ranges over the primitive ideals of \( F \).
Keeping the same notation as above, for each prime number $p$, define $\alpha_p$ and $\alpha'_p$ by

$$1 - a(p)X + p^{2l-1}X^2 = (1 - \alpha_pX)(1 - \alpha'_pX).$$

For each prime ideal $\mathfrak{p}$ of $F$, define $\beta_p$ and $\beta'_p$ by

$$1 - b(\mathfrak{p})X + N\mathfrak{p}^{k_1+k_2-1}X^2 = (1 - \beta_pX)(1 - \beta'_pX).$$

Recall that

$$a(p^m) = (\alpha_p^{m+1} - \alpha'_p^{m+1})/(\alpha_p - \alpha'_p),$$

and similarly for $b(p^m)$. Also, the Fourier coefficients are weakly multiplicative.

Now for $V = p^{-s}$ and $v = p^{k_1+k_2-1}$, put $L(f \otimes G, s) = \prod_p L_p(s)$, where

$$L_p(s)^{-1} = \begin{cases} 
(1 - \alpha_p\beta_p)(1 - \alpha_p\beta'_p)(1 - \alpha'_p\beta_p) & \text{if } p = \mathfrak{p} \\
\times(1 - \alpha'_p\beta_p)(1 - \alpha'_p\beta'_p) & \\
\times(1 - \alpha'_p\beta'_p) & \\
\times(1 - \alpha'_p\beta'_p) & \\
\times(1 - \alpha'_p\beta'_p) & \\
\times(1 - \alpha'_p\beta'_p) & \\
\times(1 - \alpha'_p\beta'_p) & \\
\times(1 - \alpha'_p\beta'_p) & \\
\times(1 - \alpha'_p\beta'_p) & \\
\times(1 - \alpha'_p\beta'_p) & \\
\times(1 - \alpha'_p\beta'_p) & \\
\times(1 - \alpha'_p\beta'_p) & \\
\times(1 - \alpha'_p\beta'_p) & \\
\times(1 - \alpha'_p\beta'_p) & \\
k \in \mathbb{Z}
\end{cases}
$$

(5.1)

Note that the above provides an explicit description of the Euler factors even at the ramified primes.

**Theorem 5.2.** The Dirichlet series

$$\zeta(2s + 2k_1)^{-1}D^{(2)}_{f}(s + k_1 + l - 1)D_{f,G}(s)$$

is equal to

$$\zeta(2s + 2k_1)^{-1}\zeta(4s + 4k_1 - 2)^{-1}L(f \otimes G, s + 2k_1 + k_2 + l - 2)$$

(first for $\text{Re}(s)$ sufficiently large, then by analytic continuation).

**Proof.** The proof uses the fact that the Dirichlet series $D^{(2)}_{f}(s + k_1 + l - 1)$ has an Euler product with $p$-factor

$$\frac{(1 + p^{1-2s-2k_1})}{(1 - \alpha_p^2p^{2-2s-2k_1-2l})(1 - \alpha'_p^2p^{2-2s-2k_1-2l})},$$
which can be computed using the previous remarks in this section regarding the Fourier coefficients of $f$. The theorem will then follow from this factorization and the fact that the Dirichlet series $D_{f,G}(s)$ has an Euler product with $p$-factor

$$(1 - p^{1-2s-2k_1})(1 - \alpha_p^2 p^{2-2s-4k_1-2l})(1 - \alpha_p' p^{2-2s-4k_1-2l})$$

$$\times \quad L_p(s + 2k_1 + k_2 + l - 2),$$

where $L_p(s)$ is defined by (5.1) above.

We will prove the equality by investigating each $p$-factor of the Euler product separately. By the weak multiplicativity of the Fourier coefficients, we can write

$$D_{f,G}(s) = \prod_p D_{f,G}(s)_p$$

where the precise description of $D_{f,G}(s)_p$ depends on whether $p$ is inert, split, or ramified. Each case will be handled separately.

Case 1. $p$ is inert. Since $p = p$, $\alpha = 1$, and $N(p) = p^2$, so

$$D_{f,G}(s)_p = \sum_n a(p^n)b(p^n)V^n$$

where $V = p^{2s-2k_1-k_2-l}$.

$$D_{f,G}(s)_p = \sum_n \left( V^n \cdot \frac{\alpha_p^{n+1} - \alpha_p'^{n+1}}{\alpha_p - \alpha_p'} \cdot \frac{\beta_p^{n+1} - \beta_p'^{n+1}}{\beta_p - \beta_p'} \right)$$

$$= (\alpha_p - \alpha_p')^{-1}(\beta_p - \beta_p')^{-1}$$

$$\cdot \left( \frac{\alpha_p\beta_p}{1 - \alpha_p\beta_pV} - \frac{\alpha_p\beta_p'}{1 - \alpha_p\beta_p'V} - \frac{\alpha_p'\beta_p}{1 - \alpha_p'\beta_pV} + \frac{\alpha_p'\beta_p'}{1 - \alpha_p'\beta_p'V} \right)$$

$$= (\alpha_p - \alpha_p')^{-1}$$

$$\cdot \left( \frac{\alpha_p(1 - \alpha_p\beta_pV)(1 - \alpha_p'\beta_p'V) - \alpha_p'(1 - \alpha_p\beta_pV)(1 - \alpha_p'\beta_p'V)}{(1 - \alpha_p\beta_pV)(1 - \alpha_p'\beta_p'V)(1 - \alpha_p'\beta_p'V)} \right)$$

$$= 1 - \alpha_p\alpha_p'\beta_p\beta_p'V^2$$

$$= \frac{(1 - \alpha_p\beta_pV)(1 - \alpha_p'\beta_p'V)(1 - \alpha_p'\beta_p'V)}{1 - p^{1-2s-2k_1}}$$

$$= (1 - \alpha p\beta p V)(1 - \alpha p' \beta p' V)(1 - \alpha p' \beta p' V).$$

Case 2. $p$ is split. In this case, $p = p_1p_2$, $N(p_1) = N(p_2) = p$, and since $(\alpha)$ is primitive, either $p_1$ or $p_2$ can divide $(\alpha)$, not both. Then

$$D_{f,G}(s)_p = \sum_{n,\epsilon,\nu} a(p^n)b(p_1^{n+2\epsilon})b(p_2^{n+2\nu})N^{-n}X^{n+2\epsilon+2\nu}$$
where \( X = p^{1-s-2k_1} \), \( N = p^{l+k_2-1} \), and \( \inf(\varepsilon, \nu) = 0 \). We can write

\[
D_{f,G}(s)_p = \sum_{n,\alpha,\nu} \left( N^{-n} X^{n+2s+2\nu} \times \frac{\alpha_p^{n+1} - \alpha_p'^{n+1}}{\alpha_p - \alpha_p'} \right) \cdot \frac{\beta_p^{n+2s+2\nu + 1} - \beta_p'^{n+2s+2\nu + 1}}{\beta_p - \beta_p'}
\]

and this is the case Garrett considers ([G1]), obtaining the required result.

**Case 3. \( p \) is ramified.** Here we consider \( p = p^2 \), \( N(p) = p \), and the two sums below correspond to the cases where \( p \nmid (\alpha) \) and \( p \mid (\alpha) \). Then

\[
D_{f,G}(s)_p = \sum_n a(p^n) b(p^{2n}) V^n + \sum_n a(p^n) b(p^{2n+2}) V^n X
\]

where \( V = p^{2-s-2k_1-k_2-1} \), and \( X = p^{2s-3k_1-k_2} \).

\[
D_{f,G}(s)_p = \sum_n \left( V^n \times \frac{\alpha_p^{n+1} - \alpha_p'^{n+1}}{\alpha_p - \alpha_p'} \cdot \frac{\beta_p^{n+2s+2\nu + 1} - \beta_p'^{n+2s+2\nu + 1}}{\beta_p - \beta_p'} \right)
+ X \sum_n \left( V^n \times \frac{\alpha_p^{n+1} - \alpha_p'^{n+1}}{\alpha_p - \alpha_p'} \cdot \frac{\beta_p^{n+3} - \beta_p'^{n+3}}{\beta_p - \beta_p'} \right)
\]

\[
D_{f,G}(s)_p = (\alpha_p - \alpha_p')^{-1}(\beta_p - \beta_p')^{-1}
\]

\[
\left( \frac{\alpha_p \beta_p}{1 - \alpha_p \beta_p^2 V} - \frac{\alpha_p' \beta_p}{1 - \alpha_p' \beta_p^2 V} \right)
+ X \left( \frac{\alpha_p \beta_p^3}{1 - \alpha_p \beta_p^2 V} - \frac{\alpha_p' \beta_p^3}{1 - \alpha_p' \beta_p^2 V} \right)
\]

\[
= (\beta_p - \beta_p')^{-1}
\]

\[
\left( \frac{(\beta_p + X \beta_p^3)(1 - \alpha_p' \beta_p^2 V)(1 - \alpha_p' \beta_p^2 V)}{(1 - \alpha_p \beta_p^2 V)(1 - \alpha_p \beta_p^2 V)(1 - \alpha_p' \beta_p^2 V)(1 - \alpha_p' \beta_p^2 V)} \right)
- \left( \frac{(\beta_p + X \beta_p^3)(1 - \alpha_p' \beta_p^2 V)(1 - \alpha_p' \beta_p^2 V)}{(1 - \alpha_p \beta_p^2 V)(1 - \alpha_p \beta_p^2 V)(1 - \alpha_p' \beta_p^2 V)(1 - \alpha_p' \beta_p^2 V)} \right)
\]

\[
= \frac{1 + \alpha_p p^{1-s-k_1-l}(1 - p^{1-2s-2k_1})}{(1 - \alpha_p \beta_p^2 V)(1 - \alpha_p \beta_p^2 V)(1 - \alpha_p' \beta_p^2 V)(1 - \alpha_p' \beta_p^2 V)}
\]

\[
+ \frac{\alpha_p' p^{1-s-k_1-l}(1 - p^{1-2s-2k_1}) - p^{2-4s-4k_1}}{(1 - \alpha_p \beta_p^2 V)(1 - \alpha_p \beta_p^2 V)(1 - \alpha_p' \beta_p^2 V)(1 - \alpha_p' \beta_p^2 V)}
\]
Theorem 5.5. \( \text{generated over Petersson inner products, for} \end{equation*}

Theorem 5.4. \( \text{Proposition 5.3.} \)

Comparing the above expressions for \( D_{f,G}(s)_p \) with the definition of \( L_p(s + 2k_1 + k_2 + l - 2) \) in (5.1), where \( v = p^{k_1+k_2-1} \) and \( V = p^{2-s-2k_1-k_2-1} \), we obtain the desired equality. \( \square \)

The following results, which determine the functional equation for the triple-product \( L \)-function, follow from the calculations of Sections 4 and 5, and the combinatorial techniques of Garrett and Orloff ([G1], [BO]).

**Proposition 5.3.** The function \( \eta_{k_1,k_2,l}(s) \) defined by Equation (4.11) is computed to be

\[ \eta_{k_1,k_2,l}(s) = (-1)^{k_1} 2^{6 - 4s - 10k_1} \pi^{3 - s - 4k_1} D_F^{1/2} \]

\[ \cdot \Gamma(s + 2k_1 - k_2 - l) \Gamma(s + 2k_1 - 1) \Gamma(s + k_1 + k_2 - l - 1) \]

\[ \cdot \left( \frac{\Gamma(s + 2k_1 + l - k_2 - 1) \Gamma(s + k_1 + k_2 + l - 2)}{\Gamma(2s + 4k_1 - 2) \Gamma(s + 2k_1) \Gamma(s)} \right). \]

**Theorem 5.4.** For \( \text{Re}(s) \) sufficiently large,

\[ \int_{\Gamma \setminus \mathfrak{H}} \int_{SL(2,\mathbb{Z}) \setminus \mathfrak{H}} E_3(t_{2,1}(Z, z_3); 2k_1, s) \delta_{2l}^{k_1-l} f(z_1) \]

\[ \cdot \delta_{(2k_1,2k_2)} G(z) (y_1 y_2 y_3)^{2k_1-2} dx_1 dy_1 d\bar{y} \]

\[ = L(f \otimes G, s + 2k_1 + k_2 + l - 2) (-1)^{k_1} 2^{6 - 4s - 10k_1} \pi^{3 - s - 4k_1} \]

\[ \cdot D_F^{1/2} \zeta(2s + 2k_1)^{-1} \zeta(2s + 4k_1 - 2)^{-1} \]

\[ \cdot \Gamma(s + 2k_1 - k_2 - l) \Gamma(s + 2k_1 - 1) \Gamma(s + 2k_1 + k_2 - l - 1) \]

\[ \cdot \Gamma(s + 2k_1 + l - k_2 - 1) \Gamma(s + 2k_1 + k_2 + l - 2) \]

\[ \cdot \Gamma(s + 2k_1) \Gamma(2s + 4k_1 - 2)^{-1} \]

where \( z_j = x_j + y_j \) as usual. Hence, \( L(f \otimes G, s) \) has a meromorphic continuation to all of \( \mathbb{C} \); the above identity holds away from the poles of the Eisenstein series. Under the transformation \( s \to 2k_1 + 2l + 2k_2 - 2 - s \),

\[ (2\pi)^{-4s} \Gamma(s - 2k_1 + 1) \Gamma(s - 2l + 1) \Gamma(s - 2k_2 + 1) L(f \otimes G, s) \]

is multiplied by \( (-1)^{l} \).

The special values result proceeds as follows. Let \( \mathbb{Q}(f, G) \) denote the field generated over \( \mathbb{Q} \) by the Fourier coefficients of \( f \) and \( G \).

**Theorem 5.5.** With \( f \) and \( G \) as above, \( k_2 + l > k_1 \), and with the usual Petersson inner products, for \( 2k_1 \leq n \leq 2k_2 + 2l - 2 \), let

\[ A(n; f, G) = \pi^{2k_1+2l+2k_2-3-n} D_F^{1/2} (f, f)^{-1} (G, G)^{-1} L(f \otimes G, n). \]
Then \( A(n; f, G) \in \mathbb{Q}(f, G) \). Moreover, if \( \sigma \in \text{Aut}(\mathbb{C}) \), then
\[
A(n; f, G)^\sigma = A(n; f^\sigma, G^\sigma),
\]
where the action of the Galois group on modular forms is the action on Fourier coefficients.

Note that we must restrict the weights in Theorem 5.5, for if \( k_2 + l \leq k_1 \), gamma factors in Theorem 5.4 vanish at the critical points. The proof is similar to those of Garrett ([G1]) and Orloff ([BO]) and will not be reproduced here.

6. Renormalization.

In this section, where we obtain a product of Asai \( L \)-functions from the triple product \( L \)-function \( L(f \otimes G, s) \), we will consider the case where the normalized holomorphic cuspidal eigenform \( f \) of weight 2\( l \) is replaced by the holomorphic Eisenstein series of weight 2\( l \geq 4 \) given by

\[
E(z) = (2l-1)! \sum_{m,n=-\infty}^{\infty} (mz+n)^{-2l} = -\frac{B_{2l}}{2l} + \sum_{n=1}^{\infty} \sigma_{2l-1}(n)e(nz),
\]

\[
= \sum_{n=0}^{\infty} a(n)e(nz)
\]

where \( B_m \) is the \( m \)-th Bernoulli number, and

\[
\sigma_{2l-1}(n) = \sum_{0<d|n} d^{2l-1}.
\]

Then we may write

\[
\delta_{2l}^{k_1-l}E(z) = \sum_{n=0}^{\infty} a(n) \sum_{A=0}^{k_1-l} P_A(4\pi y)^{-A} n^{k_1-l-A} e(nz)
\]

\[
= -\frac{B_{2l}}{2l} P_{k_1-l}(4\pi y)^{-k_1+l} + \sum_{A=0}^{k_1-l} P_A(4\pi y)^{-A} n^{k_1-l-A} \sum_{n=1}^{\infty} \sigma_{2l-1}(n)e^{2\pi inz}.
\]

6.1. The Petersson inner product. Let \( f \) and \( g \) be two level one holomorphic elliptic modular forms of weight \( k \). Recall that if at least one of them is a cusp form, we can define the Petersson inner product by

\[
\langle f, g \rangle = \int_{\Gamma \backslash \mathbb{H}} f(z)\overline{g(z)} y^{k-2} dx dy.
\]
Rankin showed that the inner product is in fact equal to

\[
\langle f, g \rangle = \frac{\pi}{3} (k-1)! (4\pi)^{-k} \text{Res}_{s=k} \left( \sum_{n=1}^{\infty} a_n b_n n^{-s} \right),
\]

where \(a_n\) and \(b_n\) are the Fourier coefficients of \(f\) and \(g\) respectively ([R]). If neither \(f\) nor \(g\) is a cusp form, then the integral diverges. However we can renormalize the integrand \(f(z)g(z) y^k\), following Zagier’s ideas ([Z]). Note that the integrand is of slow growth, so by subtracting an appropriate polynomial piece, Zagier defines the corresponding Rankin-Selberg transform and proves analogous results to the classical case where the integrand is of rapid decay. In particular, he relates this renormalized integral to a residue of the Rankin-Selberg transform. Performing this computation in the situation above, Zagier shows that after renormalization, (6.2) still holds. Hence we may define \(\langle f, g \rangle\) as its renormalized integral, thus extending the Petersson inner product to the space of all modular forms. The technique will be made explicit in the proof of the following lemma, where \(R.N.\) is used to denote a renormalized integral.

**Lemma 6.1.** Let \(E_1(z; 2k_1, s)\) and \(E(z)\) be the Eisenstein series defined by Equations (2.1) and (6.1), respectively. Then, for \(\text{Re}(s)\) sufficiently large,

\[
\langle E_1(z; 2k_1, s), \delta_{2l}^{k_1-l} E(z) \rangle = R.N. \int_{SL(2,\mathbb{Z}) \setminus \delta} E_1(z; 2k_1, s) \delta_{2l}^{k_1-l} E(z) y^{2k_1-2} \, dx \, dy = 0.
\]

**Proof.** The integrand is of slow growth, that is, we can write

\[
H(z) = E_1(z; 2k_1, s) \delta_{2l}^{k_1-l} E(z) y^{2k_1} = \varphi(y) + O(y^{-N}) \quad (\forall N) \quad \text{as} \quad y \to \infty
\]

where \(\varphi\) is a function of the form

\[
\varphi(y) = \sum_{i=1}^{m} c_i n_i! y^{\alpha_i} \log^{n_i} y \quad (c_i, \alpha_i \in \mathbb{C}, 0 \leq n_i \in \mathbb{Z}).
\]

Using the Fourier expansion of \(H\)

\[
H(z) = \sum_{n=0}^{\infty} c_n(y) e^{2\pi i n z},
\]

define the Rankin-Selberg transform of \(H\) by

\[
R(H; t) = \int_{0}^{\infty} (c_0(y) - \varphi(y)) y^{t-2} \, dy \quad (\text{Re}(t) \gg 0).
\]

Then as in [Z], we can compute

\[
\langle E_1(z; 2k_1, s), \delta_{2l}^{k_1-l} E(z) \rangle = 2 \text{Res}_{t=1} R^*(H; t),
\]
where $R^*(H; t) = \zeta^*(2t)R(H; t) := \pi^{-1}\Gamma(t)\zeta(2t)R(H; t)$.

The explicit calculation proceeds as follows. Write the Fourier expansion of $E_1(z; 2k_1, s)$ as

$$E_1(z; 2k_1, s) = y^s + \frac{\sqrt{\pi} \Gamma \left( \frac{2s + 2k_1 - 1}{2} \right) \Gamma \left( \frac{2s + 2k_1}{2} \right) \zeta(2s + 2k_1 - 1)}{y^{2k_1 + s - 1}(-1)^{k_1} \Gamma(s + 2k_1) \zeta(2s + 2k_1)}$$

$$+ \frac{2^{2k_1} \pi^{s+2k_1} \sum_{n=1}^{\infty} (\sigma_{2k_1+2s-1}(n)n^{-s}e^{2\pi inz} \omega(4\pi ny; 2k_1 + s, s))}{(-1)^{k_1} \Gamma(s + 2k_1) \zeta(2s + 2k_1)}$$

$$+ \frac{\pi^s \sum_{n=1}^{\infty} (\sigma_{2k_1+2s-1}(n)n^{-s-2k_1}e^{2\pi in\pi} \omega(4\pi ny; s, 2k_1 + s))}{y^{2k_1} 2^{2k_1} (-1)^{k_1} \Gamma(s + 2k_1) \zeta(2s + 2k_1)},$$

where

$$\omega(z; \alpha, \beta) = \Gamma(\beta)^{-1} z^\beta \int_0^\infty e^{-zu} (u + 1)^{\alpha-1} u^{\beta-1} du.$$

Multiplying the expansions for $E_1(z; 2k_1, s)$ and $\delta_{2l}^{k_1-l} \mathcal{E}(z) y^{2k_1}$, we find that

$$a_0(y) = -\frac{B_{2l}}{2l} P_{k_1-l}(4\pi y)^{-k_1+l} y^{s+2k_1} - \frac{B_{2l}}{2l} P_{k_1-l}(4\pi y)^{-k_1+l}$$

$$\cdot \left( \frac{\sqrt{\pi} \Gamma \left( \frac{2s + 2k_1 - 1}{2} \right) \Gamma \left( \frac{2s + 2k_1}{2} \right) \zeta(2s + 2k_1 - 1)}{(-1)^{k_1} \Gamma(s + 2k_1) \zeta(2s + 2k_1)} y^{1-s} \right)$$

$$+ \frac{2^{2k_1} \pi^{s+2k_1} y^{2k_1}}{(-1)^{k_1} \Gamma(s + 2k_1) \zeta(2s + 2k_1)} \sum_{A=0}^{k_1-l} P_A(4\pi y)^{-A}$$

$$\cdot \sum_{n=1}^{\infty} (\sigma_{2l-1}(n) \sigma_{2k_1+2s-1}(n)n^{k_1-l-A-s} e^{-4\pi ny} \omega(4\pi ny; 2k_1 + s, s))$$

and

$$\varphi(y) = \frac{B_{2l}}{2l} P_{k_1-l}(4\pi y)^{-k_1+l} y^{s+2k_1} - \frac{B_{2l}}{2l} P_{k_1-l}(4\pi y)^{-k_1+l}$$

$$\cdot \left( \frac{\sqrt{\pi} \Gamma \left( \frac{2s + 2k_1 - 1}{2} \right) \Gamma \left( \frac{2s + 2k_1}{2} \right) \zeta(2s + 2k_1 - 1)}{(-1)^{k_1} \Gamma(s + 2k_1) \zeta(2s + 2k_1)} y^{1-s} \right).$$
Therefore

\[ R(H; t) = \frac{2^{2k_1 - s + 2k_1}}{(-1)^{k_1} \Gamma(s + 2k_1) \zeta(2s + 2k_1)} \sum_{A=0}^{k_1-l} P_A(4\pi)^{-A} n^{k_1-l-A} \]

\[ \times \sum_{n=1}^{\infty} \left( \sigma_{2l-1}(n) \sigma_{2k_1+2s-1}(n) n^{-s} \cdot \int_0^{\infty} e^{-4\pi ny} \omega(4\pi ny; 2k_1 + s, s) y^{2k_1+t-A-2} dy \right) \]

\[ = \frac{2^{2k_1 - s + 2k_1}}{(-1)^{k_1} \Gamma(s + 2k_1) \zeta(2s + 2k_1)} \frac{\Gamma(2k_1 + t + s - 1) \Gamma(t-s)}{\Gamma(t)} \cdot \times_3 F_2(1 - k_1 - l, -k_1 + l, 1 - t; 2 - 2k_1 - s - t, 1 + s - t; 1) \]

\[ \times \sum_{n=1}^{\infty} \sigma_{2l-1}(n) \sigma_{2k_1+2s-1}(n) n^{-k_1-t-s+1} \]

and we obtain

\[ R(H; t) = \frac{2^{-2k_1 - 2t + 2s - s + t + 1} \Gamma(2k_1 + t + s - 1) \Gamma(t-s)}{(-1)^{k_1} \Gamma(s + 2k_1) \Gamma(t) \zeta(2s + 2k_1)} \times_3 F_2(1 - k_1 - l, -k_1 + l, 1 - t; 2 - 2k_1 - s - t, 1 + s - t; 1) \]

\[ \times \zeta(k_1 + l + t + s) \zeta(l - k_1 + t - s) \zeta(k_1 + l + t + s - 1) \]

\[ \times \zeta(-k_1 - l + t - s + 1) \zeta(2t)^{-1}, \]

where \( _3 F_2(a, b, c; d; e; 1) \) denotes the generalized hypergeometric function with unit argument. Then

\[ R^*(H; t) = \pi^{-l} \Gamma(t) \zeta(2t) R(H; t) \]

\[ = \frac{(-1)^{k_1} 2^{-2k_1 - 2t + 2s - s + t + 1} \Gamma(2k_1 + t + s - 1) \Gamma(t-s)}{\Gamma(s + 2k_1) \zeta(2s + 2k_1)} \]

\[ \times_3 F_2(1 - k_1 - l, -k_1 + l, 1 - t; 2 - 2k_1 - s - t, 1 + s - t; 1) \]

\[ \times \zeta(k_1 + l + t + s) \zeta(l - k_1 + t - s) \zeta(k_1 + l + t + s - 1) \]

\[ \times \zeta(-k_1 - l + t - s + 1) \]
\begin{equation*}
\frac{(-1)^{k_1}2^{-2k_1-2l+2}2^{k_1-l+s+1}\Gamma(2k_1 + t + s - 1)\Gamma(t-s)}{\Gamma(s + 2k_1)\zeta(2s + 2k_1)}
\cdot \pFq{3}{2}{{1 - k_1 - l, -k_1 + l, 1 - t; 2 - 2k_1 - s - t, 1 + s - t; 1}}{1}
\cdot \frac{\Gamma\left(\frac{k_1 - l + t + s}{2}\right)^{-1} \Gamma\left(\frac{l - k_1 + t - s}{2}\right)^{-1}}{\Gamma\left(\frac{k_1 - l + t + s + 1}{2}\right)^{-1} \Gamma\left(\frac{k_1 - l - t - s + 1}{2}\right)^{-1}}
\cdot \zeta^*(k_1 - l + t + s)\zeta^*(l - k_1 + t - s)
\cdot \zeta^*(k_1 + l + t + s - 1)\zeta^*(-k_1 - l + t - s + 1).
\end{equation*}

Using properties of the $\Gamma$-function, we can compute that for $k_1 \equiv l \pmod{2}$,
\begin{equation*}
R^*(H; t) = \frac{2^{1-2k_1+2l}2^{k_1-l+s}}{\Gamma(s + 2k_1)\zeta(2s + 2k_1)} \prod_{j = k_1 - l}^{k_1 + l} \left(\frac{t + s + j}{2} + j\right) \left(\frac{1 - t + s + j}{2} + j\right)
\cdot \frac{\Gamma(t - s)\Gamma(t + s + 2k_1 - 1)}{\Gamma(t - s + l - k_1)\Gamma(t + s + k_1 + l - 1)}
\cdot \pFq{3}{2}{{1 - k_1 - l, -k_1 + l, 1 - t; 2 - 2k_1 - s - t, 1 + s - t; 1}}{1}
\cdot \zeta^*(k_1 - l + t + s)\zeta^*(l - k_1 + t - s)
\cdot \zeta^*(k_1 + l + t + s - 1)\zeta^*(-k_1 - l + t - s + 1).
\end{equation*}

The computation for the case where $k_1 \not\equiv l \pmod{2}$ is similar. Using relationships between hypergeometric series ([B], p. 18), the above description of $R^*(H; t)$ makes clear the functional equation $R^*(H; t) = R^*(H; 1 - t)$. Computing the residue at $t = 1$, we see that for $\text{Re}(s)$ significantly large,
\begin{equation*}
\langle E_1(z; 2k, s), \delta_{2l}^{k_1-l} L(z) \rangle = 2 \text{Res}_{t=1} R^*(H; t) = 0,
\end{equation*}
as required.

One may also note that certain values of $s$ provide instances where the inner product is finite and nonzero, due to cancellation of factors introducing poles. For example, at $s = -k_1 + l$, we obtain
\begin{equation*}
\langle E_1(z; 2k_1, -k_1 + l), \delta_{2l}^{k_1-l} L(z) \rangle = 2^{5-2k_1-2l} \pi^{1-2l}(2l - 2)\zeta(2l - 1).
\end{equation*}

\hfill \square

6.2. The integral representation. We can now extend the previous results to determine the renormalized integral representation. For the following, recall the definitions of the Dirichlet series $D^{(2)}(s)$ and $D_{\mathcal{E}, G}(s)$ given by (4.7) and (4.8), respectively. Then Lemma 4.1 has the following analogue:
Lemma 6.2. Let \( q \) be given by (4.9), and let \( a(n) \) be the Fourier coefficients of \( E \) defined in (6.1). For \( w \in \mathcal{F} \), \( \text{Re}(s) \) sufficiently large, define

\[
E^*_s(w) = \sum_{n} a(n) \sum_{A=0}^{k_1-1} P_A(4\pi)^{-A} n^{k_1-l-A} e(n \text{Re}(w)) \int_{0}^{\infty} y^{s+2k_1-2-A} \cdot (y + \text{Im}(w))^{1-2s-2k} \exp(-2\pi ny) \hat{q}_{2k,s}(ny + n\text{Im}(w)) \, dy,
\]

where the coefficients are defined by (4.4) and (4.5). Then, for \( Z \in \mathcal{F}_2 \), \( z \in \mathcal{F} \),

\[
\begin{align*}
R.N. \left( \int_{\SL(2,\Z) \backslash \mathcal{F}} & E_3(\gamma, Z; 2k_1, s) \delta^{k_1-l}_2 E(z) y^{2k_1-2} \, dx \, dy \right) \\
& = E_2(Z; 2k_1, s) (E_1(z; 2k_1, s), \delta^{k_1-l}_2 E(z)) \\
& + \zeta(2s + 2k_1)^{-1} D^{(2)}_{E}(s + k_1 + l - 1) \\
& \cdot \sum_{\gamma} [\det \text{Im}(\gamma Z)]^s \mu(\gamma, Z)^{-2k_1} E^*_s(\pi(\gamma Z)),
\end{align*}
\]

where the sum is over \( \gamma \in P_{2,1}(Z) \backslash \text{Sp}(2, \Z) \).

To see why this is true, let \( \chi_{2k_1,s} \) and \( \mu \) be defined as in (4.10) and (2.2) respectively, and note that for \( Z \in \mathcal{F}_{2n} \) and \( g \in \text{Sp}(n, \Z) \),

\[
\det(\text{Im}(gZ))^s \mu(g, Z)^{-2k_1} = \det(\text{Im}(Z))^s \chi_{2k_1,s}(\mu(g, Z)).
\]

Then using the coset decomposition of Proposition 3.2, we have

\[
E_3(\gamma, Z; 2k_1, s) = (\det \text{Im}(Z))^s \text{Im}(z)^s \sum_{\gamma} \chi_{2k_1,s}(\mu(\gamma, Z) \mu(\gamma, z))
\]

\[
+ (\det \text{Im}(Z))^s \text{Im}(z)^s \sum_{\gamma} \chi_{2k_1,s}(\mu(\xi \tilde{e}_{2,1}(\gamma', \gamma'), \gamma, 1; 2k_1(Z, z))),
\]

where in the first sum \( \gamma \in P_{2,0}(Z) \backslash \text{Sp}(2, \Z) \), \( \gamma \in P_{1,0}(Z) \backslash \text{Sp}(1, \Z) \), and in the second sum \( \gamma' \in P_{2,1}(Z) \backslash \text{Sp}(2, \Z) \), \( \gamma' \in \text{Sp}(1, \Q) \), \( \xi \) is defined as in Proposition 3.1, and for each \( \gamma' \), choose \( p \in P_{3,0}(\Q) \) such that

\[
p \xi \tilde{e}_{2,1}(\gamma', \gamma') \in \text{Sp}(3, \Z).
\]

The first sum is clearly equal to \( E_2(Z; 2k_1, s) E_1(z; 2k_1, s) \), as desired, and the rest of the lemma follows as before.

Now consider the integral over \( \Gamma \backslash \mathcal{F}_2 \). If we combine the results of Lemmas 6.1 and 6.2, and note that

\[
\int_{\Gamma \backslash \mathcal{F}_2} E_2(\xi(\gamma); 2k_1, s) \delta^{(0,k_1-k_2)}_{2k_1,2k_2} G(z) y^{2k_1-2} \, dy = 0,
\]

the rest of the proof follows.
then we see the first term on the right-hand side of Equation (6.5) will contribute nothing. Likewise, if we apply the Rankin method as in the proof of Proposition 5.4, the term of $E_s^*$ involving $a(0)$ will disappear. Thus the argument of Proposition 5.4 applies in the case where the cusp form $f(z)$ is replaced by the Eisenstein series $E(z)$, and yields the corresponding result:

**Proposition 6.3.** With notation as above and the Dirichlet series corresponding to (4.7) and (4.8), then for $\tilde{z} = (z_1, z_2) \in H_2$ and $\iota(\tilde{z}) = Z$,

$$R.N. \left( \int_{\Gamma \setminus H} \int_{SL(2,\mathbb{Z}) \setminus \mathbb{H}} E_3(\epsilon_2,1(Z,z_1);2k_1,s) \delta_{2l} x_1^l \delta(y) \frac{dy}{dy} \right)$$

$$= \eta_{k_1,k_2,l}(s) \zeta(2s+2k_1-1)D_{E,G}(s)$$

for the real part of $s$ sufficiently large.

**6.3. The Euler product.** Regarding the Euler product computation, the proof of Theorem 5.2 is still valid. We can compute the roots of the Euler $p$-factor of the $L$-function attached to $E(z)$ explicitly. Namely,

$$L(E,s) = \prod_p [(1 - p^{-s})(1 - p^{-s+2l-1})]^{-1},$$

so $\alpha_p = 1$ and $\alpha'_p = p^{2l-1}$. Substituting these values into Equation (5.1), for $V = p^{-s}$, $V' = p^{-s+2l-1}$, and $v = p^{2k_1-1}$, we obtain

$$L_p(s)^{-1} = \begin{cases} 
(1 - \beta_p V)(1 - \beta'_p V)(1 - \beta_p V') \\
(1 - \beta'_p V')(1 - v^2 V'^2)(1 - v^2 V'^2) & \text{if } p = p \end{cases}$$

$$L_{p_1p_2}(s)^{-1} = \begin{cases} 
(1 - \beta_{p_1} V)(1 - \beta'_{p_1} V)(1 - \beta_{p_1} V') \\
(1 - \beta'_{p_1} V')(1 - \beta_{p_1} V'^2)(1 - \beta_{p_1} V'^2) & \text{if } p = p_1p_2 \\
(1 - \beta^2_p V)(1 - \beta^2_p V')(1 - \beta^2_p V') \\
(1 - \beta^2_p V')(1 - vV)(1 - vV') & \text{if } p = p^2.
\end{cases}$$
As in [A], the Euler product of $L_{\text{Asai}}(G, s)$ has the following form. For $V = p^{-s}$ and $v = p^{k_1-1}$, $L_{\text{Asai}}(G, s) = \prod_p L_p(s)$, where

$$ L_p(s)^{-1} = \begin{cases} (1 - \beta_p V)(1 - \beta'_p V)(1 - v^2 V^2) & \text{if } p = p \vspace{1mm} \\ (1 - \beta_{p_1} p_{p_2} V)(1 - \beta'_{p_1} p_{p_2} V) & \text{if } p = p_1 p_2 \vspace{1mm} \\ (1 - \beta^2_p V)(1 - \beta'_p V)(1 - v V) & \text{if } p = p^2 \end{cases} $$

$$ (6.7) \quad (6.7) $$

Thus, we can see that

$$ L(E \otimes G, s) = L_{\text{Asai}}(G, s) L_{\text{Asai}}(G, s - 2l + 1). $$

Hence we have the analogue of Theorem 5.4:

**Theorem 6.4.** For Re $(s)$ sufficiently large,

$$ \begin{aligned} R.N. \left( \int_{\Gamma \backslash \mathfrak{H}^2} \int_{\text{SL}(2, \mathbb{Z}) \backslash \mathfrak{H}} E_3(t_{v,1}(Z, z_3); 2k_1, s) \delta^{k_1-1} \mathcal{E}(z_1) \\
\cdot \delta^{(0,k_1-k_2)} G(\bar{z}) (y_1 y_2 y_3)^{2k_1-2} d\bar{x}_1 d\bar{y}_1 d\bar{y}_2 \right) \\
= L_{\text{Asai}}(G, s + 2k_1 + k_2 + l - 2) L_{\text{Asai}}(G, s + 2k_1 + k_2 - l - 1) \\
\cdot (-1)^{k_1} 2^{6s-10k_1} \pi^{3s-4k_1} \\
\cdot D_F^{1/2} \zeta(2s + 2k_1)^{-1} \zeta(2s + 4k_1 - 2)^{-1} \\
\cdot \Gamma(s + 2k_1 - k_2 - l) \Gamma(s + 2k_1 - 1) \Gamma(s + 2k_1 + k_2 - l - 1) \\
\cdot \Gamma(s + 2k_1 + l - k_2 - 1) \Gamma(s + 2k_1 + k_2 + l - 2) \\
\cdot \Gamma(s)^{-1} \Gamma(s + 2k_1)^{-1} \Gamma(2s + 4k_1 - 2)^{-1}. \end{aligned} $$

The above identity holds away from the poles of the Eisenstein series.

7. Concluding remarks.

The next logical step is to generalize the above results to the setting of automorphic representations, in order to be able to derive the desired special values results. We can investigate Deligne’s conjecture for the critical values of the product of Asai $L$-functions, obtained above. Specializing the result in the Appendix of [BO] to our case, we can determine the critical strip for the triple product $L$-function. Given any three positive integers $k \geq l \geq m$, corresponding to the weights of the forms, there are always two cases to consider, depending on whether

1. $l + m > k$, or
2. $l + m \leq k$. 


The special values results of Garrett, Harris, and Orloff that deal with the triple product $L$-function attached to cusp forms ([G2], [G1], [GH], [BO]), all fall under Case (1) and conform with Deligne’s conjecture ([D]). However, once we replace the cusp form $f$ with the Eisenstein series $E$, we are always in the situation of Case (2). More precisely, suppose our Hilbert modular form $G$ is of weight $(k_1, k_2)$, $k_1 > k_2$, and $k_1 \equiv k_2 \pmod{2}$. The Eisenstein series $E$ will be of weight $l < k_1$. Set

$$w = k_1 + k_2 + l - 3$$

and

$$c_0 = k_2 + l - 1.$$ 

Then the critical strip is given by

$$CS_0 = [c_0, \ldots, w - c_0 + 1] = [k_2 + l - 1, \ldots, k_1 - 1].$$

The critical strips corresponding to the Asai $L$-functions appearing in Theorem 6.4 will be

$$CS_1 = [k_2, \ldots, k_1 - 1] \quad \text{and} \quad CS_2 = [k_2 + l - 1, \ldots, k_1 + l - 2].$$

Therefore

$$CS_0 = CS_1 \cap CS_2,$$

as one would expect.

Let $\omega$ signify the central character for $G$, and let $\langle G, G \rangle_B$ denote the Petersson inner product normalized by an appropriate factor, as in the Appendix to [BO]. Then by Deligne’s conjecture, for two primitive Dirichlet characters $\xi$ and $\chi$ and the Gauss sum $g$, we would expect that for $n \in CS_0 = [k_2 + l - 1, \ldots, k_1 - 1],$

$$\frac{L_{Asai}(G, n, \xi)L_{Asai}(G, n - l + 1, \chi)}{(2 \pi i)^{2(n+1)-w}g(\omega \xi \chi)^2D_F^{1/2}\langle G, G \rangle_B^2} \in \mathbb{Q}(G, \mathcal{E}).$$

In the situation of Case (2), Harris and Kudla ([HK]) have provided the only general special value result, for the center of the critical strip. Extending their results to the other integers in the critical strip and applying Shimura’s methods ([S2], [S3]) should then lead to algebraicity results for ratios of the Asai $L$-function at different integers, twisted by Hecke characters.

**References**


Received May 25, 2000 and revised October 30, 2000.

DEPARTMENT OF MATHEMATICS
TRINITY COLLEGE
HARTFORD, CT 06106

WESTERN NEW ENGLAND COLLEGE
SPRINGFIELD, MA 01119
E-mail address: jbeineke@wnec.edu
LAGRANGE MAPPINGS OF THE FIRST OPEN WHITNEY UMBRELLA

I.A. BOGAEVSKI AND G. ISHIKAWA

In this paper we give a classification of simple stable singularities of Lagrange projections of the first open Whitney umbrella, the simplest singularity of Lagrange varieties. Our classification extends the ADE-classification, due to Arnold, of simple stable singularities of Lagrange projections of smooth Lagrange submanifolds. We also prove a criterion of equivalence of stable Lagrange projections of open Whitney umbrellas which is analogous to Mather’s fundamental theorem on stable map-germs.

0. Introduction.

The systematic investigation of singularities of Lagrange mappings started in 1972 with V.I. Arnold’s paper [1]. A Lagrange mapping is the projection of a Lagrange variety in a cotangent fibration onto its base. In the paper [1] it was discovered that singularities of Lagrange mappings of nonsingular Lagrange varieties are classified by degenerations of critical points of smooth functions and the discrete part of their classification is indexed by Coxeter’s groups $A_\mu$, $D_\mu$, $E_\mu$. This is the so-called ADE-classification of simple stable singularities of Lagrange mappings.

However, in applications singular Lagrange varieties appear. Among them open swallow tails and open Whitney umbrellas are very frequently encountered. Open swallow tails occur in the so-called obstacle problem about singularities of the distance on Riemannian manifold with boundary [2], [9], [20], [16]. Concerning open Whitney umbrellas see [8], [9], [13], [15]. They appear naturally in various situations; for instance, as singularities of the generalized Cauchy problems [9], [11], singularities of Riemannian invariants [18], and singularities of tangent developables [19], [20], [14] as the Legendre counterpart. In [9] the discrete part of local classification of Lagrange mappings of open swallow tails is carried out and some singularities of Lagrange mappings of open Whitney umbrellas are found. See also [22].

In this paper, we give the full discrete part of local classification of Lagrange mappings of the open Whitney umbrella of type one, or the first open Whitney umbrella. More accurately, we classify simple stable singularities of mappings of the first open Whitney umbrella. This problem was inspired
by the classification problem of the composition of an isotropic mapping and a cotangent fibration. (A smooth mapping is called isotropic if the pullback of the symplectic structure is equal to zero.) If the isotropic mapping is an immersion then the discrete part of local classification of the above compositions is the ADE-classification. Otherwise, the isotropic mapping can have singularities, first of all, open Whitney umbrellas [8]. Therefore, our classification gives the answer for simple stable compositions of the first open Whitney umbrella and a cotangent fibration.

We recall, in the ADE-classification and its generalizations, that stable mappings of Lagrange manifolds are classified by means of families of functions – generating families [3], [25]. Also in our problem, namely, the classification problem of stable projections of Lagrange varieties under Lagrange fibrations, the usage of generating families remains a powerful method, however in a different manner. Namely, we fix a Lagrange variety, while Lagrange fibrations are taken arbitrarily. A Lagrange fibration are regarded as a family of Lagrange submanifolds, and each Lagrange submanifold has a generating function. Thus we describe the Lagrange fibration by means of the family of the generating functions of the Lagrangian fibers, that is, the generating family of the Lagrange fibration. Then we describe stability and simplicity of Lagrange fibrations with respect to symplectic diffeomorphisms preserving the Lagrange variety, by means of the generating families. Similar methods are applied to other problems [23], [24].

First we recall the notions needed to state our main result.

**Definition.** A smooth fibration \( \pi : E \to Y \) is called Lagrange if its space \( E \) is a symplectic manifold and the fibers of \( \pi \) are Lagrange submanifolds of \( E \).

For example, the cotangent fibration of a smooth manifold is Lagrange.

We consider the classification problem of singularities on a Lagrange variety under various Lagrange projections. A subset of a symplectic manifold of dimension \( 2n \) is called a Lagrange variety if it has a stratification with maximal dimension \( n \) such that the symplectic form vanishes on each stratum.

**Definition.** Let \( \Lambda \subset E \) be a Lagrange variety. Two Lagrange fibrations \( \pi, \pi' : E \to Y \) are called \( \Lambda \)-equivalent and denoted by \( \pi \sim_{\Lambda} \pi' \) if \( \pi' \circ \tau = \sigma \circ \pi \) where \( \sigma \) is a diffeomorphism of \( Y \) and \( \tau \) is a symplectic diffeomorphism of \( E \) which preserves \( \Lambda \).

There are two basic notions for the classification problem – stability and simplicity.

**Definition.** A germ of Lagrange fibration \( \pi : (E, z_0) \to Y \) is called stable with respect to \( \Lambda \)-equivalence, or \( \Lambda \)-stable, if for any sufficiently small Lagrange perturbation of any its representative \( \bar{\pi} : U \to Y \), \( z_0 \in U \subset E \)
there exists a point \( z \in U \) such that the germ of the perturbation at \( z \) is \( \Lambda \)-equivalent to the original germ \( \pi \). (See [9], [15].)

**Definition.** A germ of Lagrange fibration \( \pi : (E, z_0) \to Y \) is called **simple** with respect to \( \Lambda \)-equivalence, or \( \Lambda \)-**simple**, if there exists its representative \( \tilde{\pi} : U \to Y \), \( z_0 \in U \subset E \) such that the number of \( \Lambda \)-equivalence classes of the germs \((U, z) \to Y\) for all \( z \in U \) of all sufficiently small Lagrange perturbations of \( \tilde{\pi} \) is finite.

Let \( \Lambda_1 \) be the first open Whitney umbrella given by the following parametric form:

\[
\begin{align*}
p_1 &= x_2 t, & p_2 &= t^3/3, & p_3 = \cdots = p_n = 0, & x_1 &= t^2/2, \\
\end{align*}
\]

where \( t \) is the parameter, \((p, x) = (p_1, \ldots, p_n, x_1, \ldots, x_n)\) are local coordinates in \( E \) such that \( \omega = dp \wedge dx \), and \( n \geq 2 \). See [8], [9], [13], [15].

Then \( \Lambda_1 \) is an \( n \)-dimensional algebraic Lagrange variety (Lemma 3), whose singular locus \( \Sigma(\Lambda_1) = \{p = 0, x_1 = 0, x_2 = 0\} \) is a nonsingular submanifold of \( E \) of dimension \( n - 2 \).

Remark that, at a regular point \( z_0 \in \Lambda_1 \), there exists a system of symplectic coordinates \((p, x)\) around \((E, z_0)\) such that \( \Lambda_1 \) is defined by \( \{p_1 = 0, \ldots, p_n = 0\} \).

Now a Lagrange fibration is given by a family of Lagrange submanifolds. It is well-known that a smooth Lagrange submanifold is locally given by \( x_I = w_{p_I}(p_I, x_J), p_J = -w_{x_J}(p_I, x_J) \), where \( I, J \) is a decomposition of \( \{1, 2, \ldots, n\} \) and \( w(p_I, x_J) \) is a smooth function.

**Definition.** Let \( W : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \) be a smooth function of \( p_I, x_J, \) and \( y = (y_1, \ldots, y_n) \), which satisfies the condition of nondegeneracy:

\[
\det \left| \begin{array}{c}
W_{p_I y} \\
W_{x_J y}
\end{array} \right| \neq 0
\]

where \( I \cap J = \emptyset, I \cup J = \{1, \ldots, n\} \). Then \( W \) is called a **generating family** of the Lagrange fibration \( \pi : (p, x) \mapsto y \) whose symplectic structure and fibers are given by the formulas

\[
\omega = dp \wedge dx, \quad \pi^{-1}(y) = \{x_I = W_{p_I}(p_I, x_J, y), p_J = -W_{x_J}(p_I, x_J, y)\}.
\]

(Locally \( \pi \) is a fibration in consequence of nondegeneracy of \( W \).)

For example, the natural Lagrange projections \((p, x) \mapsto p \) and \((p, x) \mapsto x \) are given by the generating families \(-xy \) and \( py \) respectively.

Then the main result of this paper is the following:

**Theorem 1.** Let us consider a germ of a Lagrange fibration at a point of the first open Whitney umbrella \( \Lambda_1 \). If our germ is simple and stable with respect to \( \Lambda_1 \)-equivalence then it is \( \Lambda_1 \)-equivalent to the germ at the origin of the fibration defined by one of the following generating families:
Classification of simple stable projections at regular points $\Lambda_1 = \{p_1 = \cdots = p_n = 0\}$.

$I.A.$ BOGAEVSKI AND G. ISHIKAWA

and the Lagrange fibration is given by

$$W(x, y, z) = \pm y_1^2 + y_2^2 + \cdots + y_{m-1}^2 \pm y_{m}^2 + y_{m+1}^2 + \cdots + y_{n}^2,$$

where $n \geq m - 1 \geq 1$ and $A_m^+ \sim_{\Lambda_1} A_m^-$ if $m$ is even;

$$D_m^\pm (x_1, x_2, x_3, \ldots, x_n, y) = x_1^2 x_2^2 + x_3^2 x_4^2 + \cdots + y_{m-1}^2 - y_1 x_1 - y_2 x_2 + y_3 x_3 + \cdots + y_n,$$

where $n \geq m - 1 \geq 3$;

$$E_6^\pm (x_1, x_2, x_3, \ldots, x_n, y) = x_1^3 + x_2^3 + y_3 x_1 x_2 + y_4 x_1 x_2 + y_5 x_2^2 - y_1 x_1 - y_2 x_2 + y_3 x_3 + \cdots + y_n,$$

where $n \geq 5$;

$$E_7 (x_1, x_2, x_3, \ldots, x_n, y) = x_1^3 + x_2^3 + y_3 x_1 x_2 + y_4 x_1 x_2 + y_5 x_2^2 - y_1 x_1 - y_2 x_2 + y_3 x_3 + \cdots + y_n,$$

where $n \geq 6$;

$$E_8 (x_1, x_2, x_3, \ldots, x_n, y) = x_1^3 + x_2^3 + y_3 x_1 x_2 + y_4 x_1 x_2 + y_5 x_2^2 + y_7 x_2^2 - y_1 x_1 - y_2 x_2 + y_3 x_3 + \cdots + y_n,$$

where $n \geq 7$.

Classification of simple stable projections at singular points.

$$S_3 (x, y) = y_1 x_1 + \cdots + y_n,$$

where $n \geq 2$;

$$S_m^\pm (x_1, x_2, x_3, \ldots, x_n, y) = \pm y_1 x_1 + y_2 x_2 + \cdots + y_n,$$

where $e_{2k} = x_1^k$, $e_{2k+1} = x_1^{k-1} p_2$, $n \geq m - 1 \geq 3$, and $S_m^+ \sim_{\Lambda_1} S_m^-$ if $m$ is even;

$$T_5 (x_1, x_2, p_3, \ldots, x_n, y) = x_1^2 + y_3 x_1 x_2 + y_4 x_1 x_2 + y_5 x_2^2 + y_7 x_2^2 - y_1 x_1 - y_2 x_2 + y_3 x_3 + \cdots + y_n,$$

where $n \geq 4$;

$$U_6 (x_1, x_2, x_3, \ldots, x_n, y) = x_1^3 + y_3 x_1^2 + y_4 x_1 x_2 + y_5 x_2^2 + y_7 x_2^2 + y_1 x_1 + y_2 x_2 + y_3 x_3 + y_5 x_3 + \cdots + y_n,$$

where $n \geq 5$;

$$V_6 (x_1, x_2, x_3, \ldots, x_n, y) = x_1^3 + y_3 x_1 x_2 + y_4 x_1 x_2 + y_5 x_2^2 + y_7 x_2^2 + y_1 x_1 + y_2 x_2 + y_3 x_3 + y_5 x_3 + \cdots + y_n,$$

where $n \geq 5$.

Generic Lagrange fibration is stable with respect to $\Lambda_1$-equivalence if $n \leq 4$. If $n > 4$, there exists a Lagrange fibration such that any its sufficient small perturbation has an unstable germ.

Remark. The change $(t, p_2, x_2) \mapsto -(t, p_2, x_2)$ shows us that $S_m^+ \sim_{\Lambda_1} S_m^-$ if $m$ is even.

We examine examples from the list of Theorem 1.

(1) The singularity $S_4$, $n = 3$. The generating family is given by

$$W = x_1 p_2 + y_3 x_1^2 + y_1 x_1 + y_2 p_2 + y_3 p_3,$$

and the Lagrange fibration is given by

$$\begin{align*}
  y_1 & = -p_1 - p_2 - 2x_1 x_3, \\
  y_2 & = -x_1 + x_2, \\
  y_3 & = x_3.
\end{align*}$$
The composition with the parametrization of the open Whitney umbrella is given by
\[
\begin{align*}
y_1 &= -x_2 t - \frac{t^3}{3} - x_3 t^2, \\
y_2 &= -\frac{t^2}{2} + x_2, \\
y_3 &= x_3.
\end{align*}
\]

The caustic of the singularity $S_4$ is a surface with a cuspidal edge (Figure 1). The edge consists of $A_3$-points and the $S_4$-point (the origin). The tangent line to the edge at the origin consists of $S_3$-points and the origin. In fact, the caustic is the tangent developable of the cuspidal edge, consisting of the tangent lines to the cuspidal edge.

![Figure 1. S4-caustic.](image)

(2) The singularity $T_5$, $n = 4$. The generating family is given by
\[
W(p_1, x_2, p_3, p_4, y) = x_2^2 + y_3 p_1 x_2 + y_4 x_2^2 + y_1 p_1 + y_2 x_2 + y_3 p_3 + y_4 p_4.
\]
Then the Lagrange fibration is given by
\[
\begin{align*}
y_1 &= x_1 - x_2 x_3, \\
y_2 &= -p_2 - 3x_2^2 - x_3 p_1 - 2x_2 x_4, \\
y_3 &= x_3, \\
y_4 &= x_4,
\end{align*}
\]
and the composition with the parametrization of $\Lambda_1$ is given by
\[
\begin{align*}
y_1 &= t^2 - \frac{t^3}{2} - x_2 x_3, \\
y_2 &= -\frac{t^3}{3} - 3x_2^2 - x_2 x_3 t - 2x_2 x_4, \\
y_3 &= x_3, \\
y_4 &= x_4.
\end{align*}
\]
This provides the example of stable projection of corank two at the singular point in the smallest dimension.
Recall that, in the general singularity theory of mappings, the classification of $C^\infty$-stable map-germs is reduced to their classification up to the contact equivalence [17] IV. Also recall that, in Lagrange singularity theory, the Lagrange classification of Lagrange stable immersion-germs is reduced to the classification of function-germs up to the right equivalence. Naturally, in our classification problem, we need an analogous result to establish the actual classification.

**Definition.** Let $\pi : (E, z_0) \to (Y, y_0)$ be a germ of Lagrange fibration. We call the germ of Lagrange submanifold $(\pi^{-1}(y_0), z_0) \subset E$ the central fiber of the germ $\pi$.

**Definition.** Let $\Lambda \subset E$ be a Lagrange variety. Two germs of Lagrange submanifolds $L, L' \subset E$ are called $\Lambda$-equivalent and denoted by $L \sim_{\Lambda} L'$ if they are the same with respect to a symplectic diffeomorphism preserving $\Lambda$, namely if there exists a symplectic diffeomorphism $\tau : E \to E$ such that $\tau(\Lambda) = \Lambda$ and that $\tau(L) = L'$.

**Definition.** The germs $L$ and $L'$ are called formally $\Lambda$-equivalent and denoted by $j_{\infty}(L) \sim_{\Lambda} j_{\infty}(L')$ if their $\infty$-jets are the same with respect to a symplectic diffeomorphism preserving $\Lambda$. More accurately, $j_{\infty}(L) \sim_{\Lambda} j_{\infty}(L')$ if there exist parametrizations $i, i' : (\mathbb{R}^n, 0) \to E$ of $L, L'$ respectively, and a symplectic diffeomorphism $\tau : E \to E$ preserving $\Lambda$ such that $j_{\infty}i'(0) = j_{\infty}(\tau \circ i)(0)$.

We say that $\Lambda \subset E$ is an open Whitney umbrella if $\Lambda$ is the image of an open Whitney umbrella $f_{n,k} : (\mathbb{R}^n, 0) \to (E, 0)$ in the sense of [13]. We have open Whitney umbrellas $\Lambda_k, 0 \leq k \leq [n/2]$; $\Lambda_0$ is a Lagrange submanifold and $\Lambda_1$ is the first open Whitney umbrella already introduced.

Then we prove and use in this paper the following:

**Theorem 2.** Let $\Lambda \subset E$ be an open Whitney umbrella. Then two $\Lambda$-stable germs of Lagrange fibrations $\pi, \pi' : E \to Y$ are $\Lambda$-equivalent if and only if their central fibers are $\Lambda$-equivalent. Moreover, $\pi$ and $\pi'$ are $\Lambda$-equivalent if and only if their central fibers are formally $\Lambda$-equivalent.

Consider the case when $\Lambda$ is a Lagrange submanifold ($k = 0$). Then we may take symplectic coordinates $(p, x)$ with $\Lambda = \{x = 0\}$. Recall the fundamental theorem of Lagrange singularity theory [3]: Two germs of Lagrange submanifolds in $T^*\mathbb{R}^n$ are Lagrange equivalent for the canonical projection $T^*\mathbb{R}^n \to \mathbb{R}^n$ if and only if their generating families are stably $R^+$-equivalent. Moreover, a Lagrange submanifold is Lagrange stable if and only if its generating family $F(q,x)$, $q$ being the inner variables, is an $R^+$-versal deformation of $F(q,0)$ [3]. So, two Lagrange stable Lagrange submanifold are Lagrange equivalent if their generating families are deformations of stably $R^+$-equivalent function germs. Besides, we recall the notion of “contact
equivalence” for Lagrange manifolds due to Golubitsky and Guillemin [10]. Then two germs $L, L'$ of Lagrange submanifolds are contact equivalent via a symplectic diffeomorphism in the sense [10] if and only if $L, L'$ are $\Lambda$-equivalent in our sense. Let $L, L'$ be the graphs of the differentials of function germs $h(x), h'(x)$ which have an order $\geq 3$. Then the germs $L$ and $L'$ are $\Lambda$-equivalent if and only if $h$ and $h'$ are right equivalent ([10], Prop. 4.2.). In general $L, L'$ are $\Lambda$-equivalent if and only if $F(q, 0)$ and $F'(q', 0)$ are stably $R^+$-equivalent, for the generating families $F(q, x), F(q', x)$ of $L, L'$ respectively. Therefore, Theorem 2 is a quite natural generalization of the fundamental theorem of Lagrange singularity theory. Also it is a Lagrange counterpart of the Mather’s theorem “two $\mathcal{A}$-stable mappings are $\mathcal{A}$-equivalent if and only if they are $\mathcal{K}$-equivalent” [17] in ordinary singularity theory of stable mappings. See also [15] §6.

In the next section we give the formal classification of central fibers of simple stable projections of the first open Whitney umbrella $\Lambda_1$ up to $\Lambda_1$-preserving symplectic diffeomorphisms (Theorem 3). Theorem 3 follows from Theorem 4 which is reduced to technically key Lemma 1. In §2, we prepare Lemmas needed in the following section. In particular, we give explicit equations defining the first open Whitney umbrella. Then Lemma 1 is proved in §3. Theorems 2 and 1 are proved in §4. In §5 we describe relations between our study and simple stable compositions.

For the proof of Theorem 1 we use explicit equations defining the first open Whitney umbrella. In order to carry out classifications of simple stable projections of general open Whitney umbrellas applying the method used in the present paper, we need to get explicit equations for them. That problem is left open.

The authors would like to thank the referee for the helpful comments.

1. Normal forms of fibers.

We start to prove Theorem 1 with finding formal normal forms for separate fibers which pass through singular points of the first open Whitney umbrella $\Lambda_1$ with respect to symplectic diffeomorphisms preserving $\Lambda_1$ itself.

Definition. A germ of Lagrange submanifold $L \subset E$ is called simple with respect to $\Lambda$-equivalence, or $\Lambda$-simple, if the number of $\Lambda$-equivalence classes of all germs of the kind $\tau(L)$ is finite, where $\tau : E \to E$ is any sufficiently small symplectic perturbation of the identity diffeomorphism.

In particular, a $\Lambda$-stable germ of Lagrange fibration is called simple with respect to $\Lambda$-equivalence, or $\Lambda$-simple, if there exists its representative such that the number of $\Lambda$-equivalence classes of all its germs is finite.

In coordinates $(p, x)$ such that $\omega = dp \wedge dx$, any Lagrange submanifold is locally given by at least one of the $2^n$ generating functions $w(p_I, x_J)$ by the
formulas: 
\[ x_I = w_{p_I}(p_I, x_J), \quad p_J = -w_{x_J}(p_I, x_J) \]
where \( I \cap J = \emptyset \) and \( I \cup J = \{1, \ldots, n\} \).

**Theorem 3.** Let us consider a germ of a Lagrange submanifold at a singular point of the first open Whitney umbrella \( \Lambda_1 \). If our germ is \( \Lambda_1 \)-simple then it is formally \( \Lambda_1 \)-equivalent to the germ at the origin of the Lagrange submanifold defined by one of the following generating functions:

- \( S_3 \) \( w(p_1, \ldots, p_n) = 0; \)
- \( S^m_{-} \) \( w(x_1, p_2, \ldots, p_n) = \pm e_{m+1} \) where \( e_{2k} = x_1^k, e_{2k+1} = x_1^{k-1} p_2, m \geq 4 \), \( S^m_{+} \sim_{\Lambda_1} S^m_{-} \) if \( m \) is even;
- \( T_3 \) \( w(p_1, x_2, p_3, \ldots, p_n) = x_3^2; \)
- \( U_6 \) \( w(x_1, x_2, p_3, \ldots, p_n) = x_2^3; \)
- \( V_6 \) \( w(p_1, p_2, x_3, p_4, \ldots, p_n) = x_3^3 \) where \( n \geq 3 \).

Non-simple germs occur in families of Lagrange submanifolds depending generically on at least 5 parameters. In generic 4-parametric families such germ do not occur.

**Remark.** The change \((t, p_2, x_2) \mapsto -(t, p_2, x_2)\) shows us that \( S^m_{+} \sim_{\Lambda_1} S^m_{-} \) if \( m \) is even.

**Proof.** Theorem 3 follows from the following Theorem 4. \( \square \)

**Definition.** Let \( \Lambda \subset E \) be a Lagrange variety. Two germs \( w, w' \) of generating functions are called *formally \( \Lambda \)-equivalent* and denoted by \( w \sim_{\Lambda} w' \) if the corresponding germs of Lagrange submanifolds are formally \( \Lambda \)-equivalent.

**Theorem 4.**

1. A germ of a Lagrange submanifold at a singular point of the first open Whitney umbrella \( \Lambda_1 \) is \( \Lambda_1 \)-equivalent to the germ at the origin of the Lagrange submanifold defined by a generating function \( w(p_I, x_J) \) such that \( w_{p_I}(0) = w_{x_J}(0) = w_{x_J x_I}(0) = 0 \) and one of the following conditions is satisfied:

   1) \( J = \emptyset \) \( (c = 2) \);
   2) \( J = \{1\} \) \( (c = 3) \);
   3) \( J = \{2\}, w_{p_1 x_2}(0) = 0, w_{x_2 x_2 x_2}(0) \neq 0 \) \( (c = 4) \);
   4) \( J = \{2\}, w_{p_1 x_2}(0) = w_{x_2 x_2 x_2}(0) = 0 \) \( (c = 5) \);
   5) \( J = \{3\}, w_{p_1 x_3}(0) = w_{p_3 x_3}(0) = 0, w_{x_3 x_3 x_3}(0) \neq 0 \) \( (c = 5) \);
   6) \( J = \{3\}, w_{p_1 x_3}(0) = w_{p_3 x_3}(0) = w_{x_3 x_3 x_3}(0) = 0 \) \( (c = 6) \);
   7) \( J = \{1, 2\}, w_{x_2 x_2 x_2}(0) \neq 0 \) \( (c = 5) \);
   8) \( J = \{1, 2\}, w_{x_2 x_2 x_2}(0) = 0 \) \( (c = 6) \);
   9) \( J = \{1, 3\}, w_{p_2 x_3}(0) = 0 \) \( (c = 6) \);
   10) \( J = \{2, 3\}, w_{p_1 x_2}(0) = w_{p_1 x_3}(0) = 0 \) \( (c = 7) \);
   11) \( J = \{3, 4\}, w_{p_1 x_3}(0) = w_{p_1 x_4}(0) = w_{p_2 x_3}(0) = w_{p_2 x_4}(0) = 0 \) \( (c = 9) \);
   12) \( \#J \geq 3 \) \( (c = 8) \).
Such germs occur in families of Lagrange submanifolds depending generically on at least \( c \) parameters. Case 8 is adjacent to Case 4; Cases 10 and 11 are adjacent to Case 9.

II. In the above cases:
1) \( w \sim_{\Lambda_1} 0; \)
2) \( w \) is not \( \Lambda_1 \)-simple or \( \exists \) \( m \geq 4 \) such that \( w \sim_{\Lambda_1} \pm e_{m+1} \) where \( e_{2k} = x_i^k, e_{2k+1} = x_i^{k+1} p_2; \)
3) \( w \sim_{\Lambda_1} x_3; \)
4) \( w \sim_{\Lambda_1} x_4; \)
5) \( w \sim_{\Lambda_1} x_5; \)
6) \( w \) is not \( \Lambda_1 \)-simple.

The non-\( \Lambda_1 \)-simple germs from Case 2 have infinite codimension. They are adjacent to \( S^\pm_m \).

Proof. I. The singularities of the first open Whitney umbrella \( \Lambda_1 \) are defined by the equations \( p_1 = \cdots = p_n = x_1 = x_2 = 0 \). After a shift we get \( w_{p_i}(0) = w_{x_j}(0) = 0 \). If some principal minor \( \det \left[ w_{x_j x_j'}(0) \right] \neq 0 \) where \( J' \subset J \), then we can change \( I \mapsto I \cup J' \) and \( J \mapsto J \setminus J' \). Therefore, we assume \( w_{x_j x_j}(0) = 0 \). After renumbering among \( p_3, \ldots, p_n \) we reach one (but not only one) of the following cases: \( J = \emptyset, J = \{1\}, J = \{2\}, J = \{3\}, J = \{1, 2\}, J = \{1, 3\}, J = \{2, 3\}, J = \{3, 4\}, \) or \( \#J \geq 3 \). In order to eliminate some of these cases let us note that we can replace \( i \mapsto j \) in \( I \) and \( j \mapsto i \) in \( J \) if \( w_{p_i, x_j}(0) \neq 0 \) where \( i \in I \) and \( j \in J \).

The singularities of the first open Whitney umbrella \( \Lambda_1 \) form a submanifold of codimension \( n + 2 \). So, germs passing through singularities of \( \Lambda_1 \) occur in families of Lagrange submanifolds depending generically on at least \( 2 \) parameters. This is the case \( J = 0 \). The other cases require the extra number of parameters which is equal to the quantity of conditions for the second and third derivatives.

II. It is sufficient to prove Cases 1, 2, 3, 7 for \( n = 2 \) and Case 5 for \( n = 3 \). This follows from the equivalence \( w \sim_{\Lambda_1} w_0 \) where \( w_0(p_I, x_J) = w|_{p_I'' = 0}, I' = I \cap \{1, 2\}, \) and \( I'' = I \cap \{3, \ldots, n\} \). The equivalence is performed by the symplectic diffeomorphism
\[
(p_I, p_J, x_I, x_J) \mapsto (p_I, p_J + \bar{w}_{x_J}, x_I - \bar{w}_{p_I}, x_J)
\]
where \( \bar{w} = w - w_0 \). This diffeomorphism preserves \( \Lambda_1 \) because it shifts the plane \( p_I'' = 0 \) along only \( x_I'' \) (preserving \( p_I, p_J, x_J' \), and \( x_J \)) that follows from the equality \( \bar{w}|_{p_I'' = 0} = 0 \).

The following infinite chains
\[
1_2 \Rightarrow 1_3 \Rightarrow \ldots,
2_5 \Rightarrow 2_6 \Rightarrow \ldots \quad \text{or} \quad 2_5 \Rightarrow 2_6 \Rightarrow \ldots \Rightarrow 2_{m+1} \Rightarrow 2_{m+2} \Rightarrow 2_{m+3} \Rightarrow \ldots,
3_6 \Rightarrow 3_6^I \Rightarrow 3_6^S \Rightarrow \ldots,
\]
5_6 \Rightarrow 5_6^5 \Rightarrow 5_6^5 \Rightarrow \ldots,
7_3 \Rightarrow 7_3^4 \Rightarrow 7_3^5 \Rightarrow \ldots

of propositions of the following Lemma 1 prove Cases 1, 2, 3, 5, 7 respectively.

The Propositions 4, 6, and 9 of Lemma 1 imply the Propositions 4, 6, 8–11 of Theorem 4 because Case 8 is adjacent to Case 4 and Cases 10, 11 are adjacent to Case 9.

It remains to prove that Case 12, namely, \#J \geq 3 is not \Lambda_1\text{-simple}. Indeed, the tangent plane to the first open Whitney umbrella \Lambda_1 at the point
\[ p_1 = x_2t, \quad p_2 = t^3/3, \quad p_3 = \cdots = p_n = 0, \quad x_1 = t^2/2 \]
is defined by the equations:
\[ tdp_1 - x_2dx_1 - t^2dx_2 = 0, \quad dp_2 - tdx_1 = 0, \quad dp_3 = \cdots = dp_n = 0. \]
Along the curve
\[ p_1 = t^3, \quad p_2 = t^3/3, \quad p_3 = \cdots = p_n = 0, \]
\[ x_1 = t^2/2, \quad x_2 = t^2, \quad x_3 = \cdots = x_n = 0 \]
our tangent plane is defined by the equations
\[ dp_1 - tdx_1 - tdx_2 = 0, \quad dp_2 - tdx_1 = 0, \quad dp_3 = \cdots = dp_n = 0 \]
and tends to the plane dp_1 = \cdots = dp_n = 0 as t \to 0. Therefore, the case \#J \geq 3 is adjacent to the class of ordered pairs of germs of smooth Lagrange submanifolds whose tangent planes have three-dimensional intersection. This class is not simple up to symplectic equivalence because, according to [10], it corresponds to the so-called \text{P}_8 class consisting of the germs of smooth functions at critical points of corank 3. Moreover, the symplectic equivalence of ordered pairs of Lagrange germs corresponds to the stable right equivalence of the germs of smooth functions (see [10]). But the \text{P}_8 class contains a continuous invariant up to stable right equivalence [3]. This invariant comes from linear equivalence of cubic forms of three variables.

\textbf{Lemma 1.} Let \alpha_I = \deg p_I and \beta_J = \deg x_J be positive integer quasidergrees and \mathcal{A}_0 \supset \mathcal{A}_1 \supset \ldots be the corresponding quasihomogeneous filtration in the algebra of germs at 0 of smooth functions of p_I and x_J. Then

1) \ n = 2, \ J = 0, \ \alpha_1 = \alpha_2 = 1, \ w_l \in \mathcal{A}_l, \ l \geq 2 \Rightarrow w_l \sim_{\Lambda_1} 0 (\text{mod } \mathcal{A}_{l+1});
2) \ n = 2, \ J = \{1\}, \ \beta_1 = 2, \ \alpha_2 = 3, \ w_l \in \mathcal{A}_l, \ l \geq 4 \Rightarrow w_l \sim_{\Lambda_1} \pm e_\ell (\text{mod } \mathcal{A}_{l+1}) \quad \text{if } w_l \notin (9p_2^2 - 8x_3) + \mathcal{A}_{l+1}, \ and \ w_l \sim_{\Lambda_1} 0 (\text{mod } \mathcal{A}_{l+1}) \quad \text{if } w_l \in (9p_2^2 - 8x_3) + \mathcal{A}_{l+1};
3) \ n = 2, \ J = \{2\}, \ \alpha_1 = 3, \ \beta_2 = 2, \ w_6 \in \mathcal{A}_6, \ w_6, x_2^2, x_2 \neq 0 \Rightarrow w_6 \sim_{\Lambda_1} x_2^3 (\text{mod } \mathcal{A}_7);
It is well-known that a vector field which preserves the symplectic structure \( w = dp \wedge dx \) is locally defined by its Hamiltonian \( H \):

\[
\dot{x} = H_p, \quad \dot{p} = -H_x.
\]

Let \( L(w) \) be the Lagrange submanifold given by a generating function \( w(p_I, x_J) \):

\[
L(w) = \{ x_I = w_{p_I}(p_I, x_J), \quad p_I = -w_{x_I}(p_I, x_J) \}
\]

and \( H(w) \) denote the derivative of the generating function when the Lagrange submanifold is perturbed by the vector field with a Hamiltonian \( H \).

**Lemma 2.** \( H(w) = H|_{L(w)} + \text{const}. \)

**Proof.** Indeed, \( (p_I, x_J) \) are coordinates on \( L(w) \) and for any two points \( A, B \in L(w) \)

\[
w \bigg|_A^B = w(B) - w(A) = \int_A^B \psi_I
\]

along any path on \( L(w) \) joining \( A \) and \( B \), where \( \psi_I = x_I dp_I - p_I dx_I \). After differentiating along our Hamiltonian vector field we get

\[
(H(w) + w_{p_I} \dot{p}_I + w_{x_J} \dot{x}_J) \bigg|_A^B = \int_A^B H(\psi_I)
\]
where $H(\psi_I)$ is the derivative of $\psi_I$ along our Hamiltonian vector field. The Cartan formula implies that
\[
\int_{A}^{B} H(\psi_I) = \int_{A}^{B} d(x_I \dot{p}_I - p_J \dot{x}_J) - \dot{p} dx + \dot{x} dp \\
= \int_{A}^{B} d(x_I \dot{p}_I - p_J \dot{x}_J + H) = (x_I \dot{p}_I - p_J \dot{x}_J + H) \bigg|_{A}^{B}.
\]
Comparing the two last equalities and taking into account (2) we get $H(w) \big|_{A}^{B} = H \big|_{A}^{B}$. □

Open Whitney umbrellas $\Lambda = \Lambda_k = f_{n,k}(\mathbb{R}^n)$ are real algebraic sets in $\mathbb{R}^{2n}$. In fact, the complexification $f_C : \mathbb{C}^n \rightarrow \mathbb{C}^{2n}$ of the parametrization $f : \mathbb{R}^n \rightarrow \mathbb{R}^{2n}$ is proper and one to one. Therefore $f_C(\mathbb{C}^n)$ is a complex algebraic set in $\mathbb{C}^{2n}$, and $f_C(\mathbb{C}^n) \cap \mathbb{R}^{2n} = f(\mathbb{R}^n)$ is a real algebraic set. However, for the explicit classification, we need, furthermore, the explicit equation of $\Lambda$.

Let $\mathcal{I}(\Lambda_1) \subset \mathbb{R}[p, x]$ be the ideal consisting of all polynomials which vanish on the first open Whitney umbrella $\Lambda_1$.

**Lemma 3.**

$\mathcal{I}(\Lambda_1) = \langle p_1^2 - 2x_1x_2, 3p_1p_2 - 4x_1^2x_2, 2p_1x_1 - 3p_2x_2, 9p_2^2 - 8x_1^3, p_3, \ldots, p_n \rangle$.

**Proof.** It is sufficient to prove this when $n = 2$. Let $\mathcal{I}_t(\Lambda_1) = \langle t^3/3 - p_2, t^2/2 - x_1, tx_2 - p_1 \rangle \subset \mathbb{R}[t, p_1, p_2, x_1, x_2]$ be the ideal defining the parametric form (1) of the first open Whitney umbrella $\Lambda_1$ if $n = 2$. Then the nine polynomials $t^2 - 2x_1$, $tp_1 - 2x_1x_2$, $3tp_2 - 4x_1^2$, $2tx_1 - 3p_2$, $tx_2 - p_1$, $p_2^2 - 2x_1x_2^2$, $3p_1p_2 - 4x_1^2x_2$, $2p_1x_1 - 3p_2x_2$, $9p_2^2 - 8x_1^3$ form a Gröbner basis of the ideal $\mathcal{I}_t(\Lambda_1)$ with respect to the lexicographic order. Hence, the four last polynomials which do not depend on $t$ generate $\mathcal{I}(\Lambda_1) \subset \mathbb{R}[p_1, p_2, x_1, x_2]$. □

**3. Proof of Lemma 1.**

In Cases 1, 2, 2', 3', 6, 3', 5', 6, 5', 7, 7' we use the standard homotopy method. Namely, let $\omega_\tau$ be a family of generating functions depending smoothly on a parameter $\tau$ and $H_\tau$ be a smooth family of Hamiltonians satisfying the homological equation
\[
H_\tau(\omega_\tau) + \partial_\tau \omega_\tau \equiv 0
\]
on a segment $[0, 1]$. Besides, the corresponding Hamiltonian vector fields $v_{H_\tau}$ are assumed to be tangent to the first open Whitney umbrella $\Lambda_1$ and to preserve the origin ($v_{H_\tau}(0) \equiv 0$). For the Hamiltonians $H_\tau$ these conditions
mean $H_\tau|_{\Lambda_1} \equiv 0$ and $\partial_\tau H_\tau(0) \equiv \partial_\tau H_\tau(0) \equiv 0$ respectively. So, taking into account Lemma 2 we can rewrite the homological equation as

$$H_\tau|_{L(\omega_\tau)} + \partial_\tau \omega_\tau \equiv 0, \quad H_\tau \in \mathcal{H}(\Lambda_1) = \mathcal{E}_\tau \otimes (m^2 \cap T'(\Lambda_1))$$

where $\mathcal{E}_\tau$ is the algebra of smooth functions on the segment $[0, 1]$, $m$ is the maximal ideal in the algebra of germs at 0 of smooth functions on $E$, and $T'(\Lambda_1)$ is the ideal consisting of all germs which vanish on the first open Whitney umbrella $\Lambda_1$.

Now solving the Cauchy problem

$$\dot{g}_\tau(p, x) \equiv v_{H_\tau}(g_\tau(p, x)), \quad g_0(p, x) = (p, x)$$

with respect to a family of diffeomorphisms $g_\tau$ on the segment $[0, 1]$ for small $(p, x)$ we get the equivalence $\omega_0 \sim_{\Lambda_1} \omega_1$ performed by the local symplectic diffeomorphism $g_1$ preserving $\Lambda_1$ and 0.

Let $\mathcal{H}(\Lambda_1)|_{L(\omega_\tau)} \subset \mathcal{E}_\tau \otimes A_0$ be the restriction of the ideal $\mathcal{H}(\Lambda_1)$ onto the family of Lagrange submanifolds given by the family $\omega_\tau(p_1, x) \in \mathcal{E}_\tau \otimes A_0$ of generating functions and $\text{Gr} \mathcal{H}(\Lambda_1)|_{L(\omega_\tau)} \subset \mathcal{E}_\tau \otimes A_0$ be the quasihomogeneous ideal generated by the principal quasihomogeneous parts of the germs from $\mathcal{H}(\Lambda_1)|_{L(\omega_\tau)}$.

Let us note that

$$\text{Gr} \mathcal{H}(\Lambda_1)|_{L(\omega_\tau)} \cap (\mathcal{E}_\tau \otimes A_d) = \mathcal{H}(\Lambda_1)|_{L(\omega_\tau)} \cap (\mathcal{E}_\tau \otimes A_d) \quad (\text{mod } \mathcal{E}_\tau \otimes A_{d+1}).$$

Therefore, in the considered cases, the homological equation

$$H_\tau|_{L(\omega_\tau)} + \partial_\tau \omega_\tau \equiv 0 \quad (\text{mod } \mathcal{E}_\tau \otimes A_{d+1}), \quad H_\tau \in \mathcal{H}(\Lambda_1)$$

is solvable because

$$\partial_\tau \omega_\tau \in \text{Gr} \mathcal{H}(\Lambda_1)|_{L(\omega_\tau)} \cap (\mathcal{E}_\tau \otimes A_d)$$

that is shown below in each case.

1) Let $\omega_\tau = \tau w_1$. According to Lemma 3,

$$p_1^2 - 2x_1x_2^2|_{L(\omega_\tau)}, \quad 3p_1p_2 - 4x_1^2x_2|_{L(\omega_\tau)}, \quad 9p_2^2 - 8x_1^3|_{L(\omega_\tau)} \in \mathcal{H}(\Lambda_1)|_{L(\omega_\tau)}.$$

Since $L(\omega_\tau) = \{x_1 = x_2 = 0 (\text{mod } \mathcal{E}_\tau \otimes A_1)\}$, the principal parts of these polynomials are $p_1^2, 3p_1p_2, 9p_2^2$. Therefore, $\text{Gr} \mathcal{H}(\Lambda_1)|_{L(\omega_\tau)} \supset (p_1^2, p_1p_2, p_2^2) = \mathcal{E}_\tau \otimes A_2$.

2) Let $w_1 = ae_l + \bar{w}_l (\text{mod } A_{l+1})$ where $\bar{w}_l$ is a quasihomogeneous element of the ideal $(9p_2^2 - 8x_1^3)$ such that $\deg \bar{w}_l = l, \omega_\tau = a(\tau)e_l + \tau \bar{w}_l, a(0) = \text{sign}(a), a(1) = a$. According to Lemma 3,

$$2p_1x_1 - 3p_2x_2|_{L(\omega_\tau)}, \quad 9p_2^2 - 8x_1^3|_{L(\omega_\tau)} \in \mathcal{H}(\Lambda_1)|_{L(\omega_\tau)}.$$

Since $L(\omega_\tau) = \{p_1 = -\partial_{x_1}\omega_\tau, x_2 = \partial_{p_2}\omega_\tau\}$, the principal parts of these polynomials are $\pm \omega_\tau$, $9p_2^2 - 8x_1^3$. Therefore, $\text{Gr} \mathcal{H}(\Lambda_1)|_{L(\omega_\tau)} \supset (a(\tau)e_l, 9p_2^2 - 8x_1^3) \supset \partial_\tau \omega_\tau$ if $a(\tau) \equiv 0$ or $a(\tau) \neq 0$ for $\forall \tau \in [0, 1]$. 

LAGRANGE MAPPINGS OF THE FIRST OPEN WHITNEY UMBRELLA 127
2) Let \( \omega_r = \pm e_l + \tau w_d \). According to Lemma 3,
\[
3p_1p_2 - 4x_1^2x_2 \bigg|_{L(\omega_r)}, \quad 2p_1x_1 - 3p_2x_2 \bigg|_{L(\omega_r)}, \quad 9p_2^2 - 8x_1^3 \bigg|_{L(\omega_r)} \in \mathcal{H}(A_1) \bigg|_{L(\omega_r)}.
\]
Since
\[
L(\omega_r) = \{ p_1 = \mp \partial_{x_1} e_l \pmod{\mathcal{A}_{l-1}}, \quad x_2 = \pm \partial_{x_2} e_l \pmod{\mathcal{A}_{l-2}} \},
\]
the principal parts of these polynomials are
\[
\mp(3p_2 \partial_{x_1} e_l + 4x_1^2 \partial_{x_2} e_l), \quad \mp(2x_1 \partial_{x_1} e_l + 3p_2 \partial_{x_2} e_l) = \mp le_l, \quad 9p_2^2 - 8x_1^3.
\]
But
\[
3p_2 \partial_{x_1} e_l + 4x_1^2 \partial_{x_2} e_l = \begin{cases}
3k \epsilon_{l+1} & \text{if } e_l = x_1^k \\
(8k/3 + 4) \epsilon_{l+1} \pmod{9p_2^2 - 8x_1^3} & \text{if } e_l = x_1^k p_2
\end{cases}
\]
and
\[
\text{Gr } \mathcal{H}(A_1) \big|_{L(\omega_r)} \ni (\epsilon_{l+1}, e_l, 9p_2^2 - 8x_1^3) \ni \mathcal{E}_r \otimes \mathcal{A}_l.
\]
3) Let \( w_6 = a \tau^2 + bx_3^3 \pmod{\mathcal{A}_7} \), \( \omega_r = a(\tau)p_2^2 + b(\tau)x_3^3 \), \( a(0) = b(0) = \text{sign}(b) \), \( a(1) = a \), \( b(1) = b \). According to Lemma 3,
\[
p_1^2 - 2x_1x_2^2 \bigg|_{L(\omega_r)}, \quad 2p_1x_1 - 3p_2x_2 \bigg|_{L(\omega_r)} \in \mathcal{H}(A_1) \bigg|_{L(\omega_r)}.
\]
Since
\[
L(\omega_r) = \{ x_1 = 0 \pmod{\mathcal{A}_4}, \quad p_2 = -3x_2^2 \pmod{\mathcal{A}_5} \},
\]
the principal parts of these polynomials are \( p_1^2, -9p_1x_2^2, 9x_3^3 \). Therefore, \( \text{Gr } \mathcal{H}(A_1) \big|_{L(\omega_r)} \ni (p_1^2, p_1x_2^2, x_3^3) \ni \mathcal{E}_r \otimes \mathcal{A}_6 \).
5) Let \( w_6 = a_{11}p_1^2 + a_{12}p_1p_2 + a_{22}p_2^2 + bx_3^3 \pmod{\mathcal{A}_7} \), \( \omega_r = a_{11}(\tau)p_1^2 + a_{12}(\tau)p_1p_2 + a_{22}(\tau)p_2^2 + b(\tau)x_3^3 \), \( a_{11}(0) = a_{12}(0) = a_{22}(0) = b(0) = \text{sign}(b) \), \( a_{11}(1) = a_{11} \), \( a_{12}(1) = a_{12} \), \( a_{22}(1) = a_{22} \), \( b(1) = b \). According to Lemma 3,
\[
p_1^2 - 2x_1x_2^2 \bigg|_{L(\omega_r)}, \quad 3p_1p_2 - 4x_1^2x_2 \bigg|_{L(\omega_r)}, \quad 9p_2^2 - 8x_1^3 \bigg|_{L(\omega_r)} \in \mathcal{H}(A_1) \bigg|_{L(\omega_r)}.
\]
Since
\[
L(\omega_r) = \{ x_1 = 2a_{11}(\tau)p_1 + a_{12}(\tau)p_2, \quad x_2 = a_{12}(\tau)p_1 + 2a_{22}(\tau)p_2, \quad p_3 = -3b(\tau)x_3^3 \},
\]
the principal parts of these polynomials are \( p_1^2, 3p_1p_2, 9p_2^2, -3b(\tau)x_3^3 \). Therefore, \( \text{Gr } \mathcal{H}(A_1) \big|_{L(\omega_r)} \ni (p_1^2, p_1p_2, p_2^2, b(\tau)x_3^3) \ni \partial_{x_3} \omega_r \pmod{\mathcal{A}_7} \) if \( b(\tau) \neq 0 \) for \( \forall \tau \in [0, 1] \).
The change \((p_3, x_3) \mapsto -(p_3, x_3)\) shows us that \(x_3^3 \sim \Lambda_1 - x_3^3\).

5^t) Let \(\omega_\tau = x_3^3 + \tau w_d\). According to Lemma 3,

\[
p_1^2 - 2x_1 x_3^2 \bigg|_{L(\omega_\tau)}', \quad 3p_1p_2 - 4x_1^2x_2 \bigg|_{L(\omega_\tau)}', \quad 9p_2^2 - 8x_1^3 \bigg|_{L(\omega_\tau)}', \quad p_1p_3 \bigg|_{L(\omega_\tau)}', \quad p_2p_3 \bigg|_{L(\omega_\tau)}', \quad p_3x_3 \bigg|_{L(\omega_\tau)} \in \mathcal{H}(\Lambda_1) \bigg|_{L(\omega_\tau)}.
\]

Since \(L(\omega_\tau) = \{ x_1 = 0 (\text{mod } A_4) \}, \quad x_2 = 0 (\text{mod } A_4), \quad p_3 = -3x_3^2 (\text{mod } A_5) \}, \) the principal parts of these polynomials are \(p_1^2, 3p_1p_2, 9p_2^2, -3p_1x_3^2, -3p_2x_3^2, -3x_3^3 \). Therefore, \(\text{Gr} \mathcal{H}(\Lambda_1) \big|_{L(\omega_\tau)} \supset (p_1^2, p_1p_2, p_2^2, p_1x_3^2, p_2x_3^2, x_3^3) = \mathcal{E}_7 \otimes A_6\).

7^t) Let \(w_l = ax_3^3 + x_1w_{l-1} (\text{mod } A_{l+1})\) where \(w_{l-1} = l - 1, \quad \omega_\tau = a(\tau)x_3^3 + \tau x_1w_{l-1}, \quad a(0) = \text{sign}(a), \quad a(1) = a\). According to Lemma 3,

\[
p_1^2 - 2x_1 x_3^2 \bigg|_{L(\omega_\tau)}', \quad 3p_1p_2 - 4x_1^2x_2 \bigg|_{L(\omega_\tau)}', \quad 2p_1x_1 - 3p_2x_2 \bigg|_{L(\omega_\tau)}', \quad 9p_2^2 - 8x_1^3 \bigg|_{L(\omega_\tau)} \in \mathcal{H}(\Lambda_1) \bigg|_{L(\omega_\tau)}.
\]

Since \(L(\omega_\tau) = \{ p_1 = -\partial_{x_3} \omega_\tau, \quad p_2 = -\partial_{x_2} \omega_\tau \} \), the principal parts of these polynomials are \(-2x_1x_3^2, -4x_1^2x_2, -2x_1\partial_{x_3} \omega_\tau + 3\tau x_1x_2 + 3\tau x_1x_2 \partial_{x_2} \omega_{l-1}, -8x_1^3 \). Therefore, \(\text{Gr} \mathcal{H}(\Lambda_1) \big|_{L(\omega_\tau)} \supset (x_3^3, x_1^2x_2, x_1x_2^2, a(\tau)x_3^3) \ni \partial_\tau \omega_\tau \text{ if } a(\tau) \equiv 0 \text{ or } a(\tau) \not\equiv 0 \text{ for } \forall \tau \in [0, 1]\).

7^t) Let \(\omega_\tau = \pm x_3^3 + \tau w_d\). According to Lemma 3,

\[
p_1^2 - 2x_1 x_3^2 \bigg|_{L(\omega_\tau)}', \quad 3p_1p_2 - 4x_1^2x_2 \bigg|_{L(\omega_\tau)}', \quad 2p_1x_1 - 3p_2x_2 \bigg|_{L(\omega_\tau)}', \quad 9p_2^2 - 8x_1^3 \bigg|_{L(\omega_\tau)} \in \mathcal{H}(\Lambda_1) \bigg|_{L(\omega_\tau)}.
\]

Since \(L(\omega_\tau) = \{ p_1 = 0 (\text{mod } A_4) \}, \quad p_2 = \pm lx_3^{l-1} (\text{mod } A_4) \}, \) the principal parts of these polynomials are \(-2x_1x_3^2, -4x_1^2x_2, \pm 3lx_3^l, -8x_1^3 \). Therefore,

\[
\text{Gr} \mathcal{H}(\Lambda_1) \big|_{L(\omega_\tau)} \supset (x_3^3, x_1^2x_2, x_1x_2^2, x_3^3) \supset \mathcal{E}_7 \otimes A_4.
\]

In Cases 4, 6, and 9, we consider the Lie algebra of germs of Hamiltonian vector fields which are tangent to the first open Whitney umbrella \(\Lambda_1\) and preserve 0. For the Hamiltonians \(H\) these conditions mean \(H|_{\Lambda_1} = 0\) and \(\partial_p H(0) = \partial_q H(0) = 0\) respectively. So, our Lie algebra is the ideal \(m^2 \cap T'(\Lambda_1)\) where \(m\) is the maximal ideal in the algebra of germs at 0 of smooth functions on \(E\) and \(T'(\Lambda_1)\) is the ideal consisting of all germs which vanish on the first open Whitney umbrella \(\Lambda_1\).
Let $B = \mathcal{A}_4/\mathcal{A}_5$ in Cases 4*, 6*, and $B = \mathcal{A}_5'/\mathcal{A}_4$ in Case 9*. In all these cases our Lie algebra $m^2 \cap \mathcal{I}'(\Lambda_1)$ acts on $B$ by the formula from Lemma 2: $H(\omega) = H|_{L(\omega)}$. But it turns out that, for any $\omega \in B$,

$$\dim m^2 \cap \mathcal{I}'(\Lambda_1)|_{L(\omega)} < \dim B.$$  

It remains to check these inequalities.

4*) Let $\omega = a_1p_1^2 + b_1x_2^2 + cx_3^2 \in B$ where $B = \mathcal{A}_4/\mathcal{A}_5$. Then

$$L(\omega) = \{x_1 = 2a_1p_1 + bx_2^2, \quad p_2 = -2b_1x_2 - 4cx_2^3\},$$

$$p_1^2 - 2x_1x_2^2|_{L(\omega)} \in B, \quad 2p_1x_1 - 3p_2x_2|_{L(\omega)} \in B,$$

$$3p_1p_2 - 4x_1^2x_2|_{L(\omega)} = 9p_2^2 - 8x_1^3|_{L(\omega)} = 0 \quad (mod \ \mathcal{A}_5).$$

Therefore, according to Lemma 3,

$$\dim m^2 \cap \mathcal{I}'(\Lambda_1)|_{L(\omega)} \leq 2$$

but $\dim B = 3$.

6*) Let $\omega = a_1p_1^2 + a_2p_1p_2 + a_3p_2^2 + b_1p_1x_3^2 + b_2p_2x_3^2 + cx_3^4 \in B$ where $B = \mathcal{A}_4/\mathcal{A}_5$. Then

$$L(\omega) = \{x_1 = 2a_1p_1 + a_2p_2 + b_1x_3^2, \quad x_2 = a_2p_1 + 2a_3p_2 + b_2x_3^2, \quad p_3 = -2b_1p_1x_3 - 2b_2p_2x_3 - 4cx_3^3\},$$

$$p_1^2 - 2x_1x_2^2|_{L(\omega)} = p_1^2 \quad (mod \ \mathcal{A}_5) \in B,$$

$$3p_1p_2 - 4x_1^2x_2|_{L(\omega)} = 3p_1p_2 \quad (mod \ \mathcal{A}_5) \in B,$$

$$2p_1x_1 - 3p_2x_2|_{L(\omega)} \in B, \quad 9p_2^2 - 8x_1^3|_{L(\omega)} = 9p_2^2 \quad (mod \ \mathcal{A}_5) \in B, \quad p_3x_3|_{L(\omega)} \in B,$$

$$p_1p_3|_{L(\omega)} = p_2p_3|_{L(\omega)} = p_3^2|_{L(\omega)} = x_1p_3|_{L(\omega)} = x_2p_3|_{L(\omega)} = 0 \quad (mod \ \mathcal{A}_5).$$

Therefore, according to Lemma 3,

$$\dim m^2 \cap \mathcal{I}'(\Lambda_1)|_{L(\omega)} \leq 5$$

but $\dim B = 6$.

9*) Let $\omega = ax_1p_2 + b_1x_2^3 + b_2x_2^3x_3 + b_3x_1x_3^2 + b_4x_3^3 \in B$ where $B = \mathcal{A}_5'/\mathcal{A}_4$. Then

$$L(\omega) = \{p_1 = -ap_2 - 3b_1x_1^2 - 2b_2x_1x_3 - b_3x_3^2, \quad x_2 = ax_1, \quad p_3 = -b_2x_1^2 - 2b_3x_1x_3 - 3b_4x_3^2\},$$
\[ p_1^2 - 2x_1x_2^2 \mid_{L(\omega)} = -2a^2x_1^3 \pmod{\mathcal{A}_4} \in \mathcal{B}, \]
\[ 3p_1p_2 - 4x_1^2x_2 \mid_{L(\omega)} = -4ax_1^3 \pmod{\mathcal{A}_4} \in \mathcal{B}, \]
\[ 2p_1x_1 - 3p_2x_2 \mid_{L(\omega)} = -5ax_1p_2 - 6b_1x_1^3 - 4b_2x_1^2x_3 - 2b_3x_1x_3^2 \in \mathcal{B}, \]
\[ 9p_2^2 - 8x_1^3 \mid_{L(\omega)} = -8x_1^3 \pmod{\mathcal{A}_4} \in \mathcal{B}, \]
\[ x_1p_3 \mid_{L(\omega)} = -b_2x_1^3 - 2b_3x_1^2x_3 - 3b_4x_1x_3^2 \in \mathcal{B}, \quad x_2p_3 \mid_{L(\omega)} = ax_1p_3 \mid_{L(\omega)} \in \mathcal{B}, \]
\[ p_3x_3 \mid_{L(\omega)} = -b_2x_1^3x_3 - 2b_3x_1x_3^2 - 3b_4x_3^3 \in \mathcal{B}, \]
\[ p_1p_3 \mid_{L(\omega)} = p_2p_3 \mid_{L(\omega)} = p_3^2 \mid_{L(\omega)} = 0 \pmod{\mathcal{A}_4}. \]

Therefore, according to Lemma 3,\n\[ \dim \mathfrak{m}^2 \cap \mathcal{I}'(\Lambda_1)|_{L(\omega)} \leq 4 \]
but \( \dim \mathcal{B} = 5 \).

### 4. Stable Lagrange mappings.

In this Section we prove Theorems 1 and 2. Theorem 1 follows from Theorem 2, the proved Theorem 3, and the following Lemma 4.

**Definition.** Let \( \Lambda \subset E \) be a Lagrange variety. The germ at 0 of the Lagrange fibration given by a generating family \( W(p_I, x_J, y) \) such that \( w(p_I, x_J) = \hat{W}(p_I, x_J, 0) \) is called \( \Lambda \)-versal if
\[ \mathcal{I}'(\Lambda)|_{L(\omega)} + \langle W_y|_{y=0}, 1 \rangle_{\mathbb{R}} = \mathcal{A}_0 \]
where \( \mathcal{I}'(\Lambda) \) is the ideal consisting of the germs of all functions on \( E \) which vanish on the Lagrange variety \( \Lambda \), \( L(w) \subset E \) is the Lagrange submanifold defined by the generating function \( w \), and \( \mathcal{A}_0 \) is the algebra of germs at 0 of smooth functions of \( p_I \) and \( x_J \).

**Remark.** This is nothing but the Givental’ versality [9] for the Lagrange mapping \( \Lambda \subset E \to Y \) when the Lagrange fibration is defined by the generating family \( W \). Also, in the case \( \Lambda \) is an open Whitney umbrella, the Givental’ versality condition is equivalent to that the parametrization of \( \Lambda \) is Lagrange stable with respect to the Lagrange fibration in the sense of [13], [15] (Theorem 2 from [13], page 216).

**Lemma 4.** The germs of Lagrange fibrations from Theorem 1 are stable with respect to \( \Lambda_1 \)-equivalence.
Proof. Let $\alpha_I = \deg p_I$ and $\beta_J = \deg x_J$ be positive integer quasidegrees, $A_0 \supset A_1 \supset \ldots$ the corresponding quasihomogeneous filtration in the algebra of germs at 0 of smooth functions of $p_I$ and $x_J$, and Gr $I'(A_1)|_{L(\omega_r)} \subset A_0$ the quasihomogeneous ideal generated by the principal quasihomogeneous parts of the germs from $I'(A_1)|_{L(\omega_r)}$. Let us note that

$$\text{Gr } I'(A_1)|_{L(\omega_r)} \cap A_d = I'(A_1)|_{L(\omega_r)} \cap A_d \mod A_{d+1}.$$

By analogy with the cases 1, 2, 3, 4, 5 from the proof of Lemma 1 we get

$S_3)$ $w(p_1, p_2) = 0, \alpha_1 = \alpha_2 = 1 \Rightarrow \text{Gr } I'(A_1)|_{L(w)} \supset (p_1^2, p_1p_2, p_2^2) = A_2$;

$S_m)$ $w(x_1, p_2) = \pm e_{m+1}, \beta_1 = 2, \alpha_2 = 3 \Rightarrow \text{Gr } I'(A_1)|_{L(w)} \supset (e_{m+1}, e_{m+2}, 9p_2^2 - 8x_1^2) \supset A_{m+1}$;

$T_5)$ $w(p_1, x_2) = x_3^2, \beta_1 = 3, \alpha_2 = 2 \Rightarrow \text{Gr } I'(A_1)|_{L(w)} \supset (p_1^2, p_1x_2^2, x_3^2) = A_6$;

$U_6)$ $w(x_1, x_2) = \pm x_3^{m-3}, \beta_1 = \beta_2 = 1 \Rightarrow \text{Gr } I'(A_1)|_{L(w)} \supset (x_1^2, x_1x_2, x_1x_2^2, x_2^m) \supset A_{m-3}$;

$V_6)$ $w(p_1, p_2, x_3) = x_3^2, \alpha_1 = \alpha_2 = 3, \beta_3 = 2 \Rightarrow \text{Gr } I'(A_1)|_{L(w)} \supset (p_1^2, p_1p_2, p_2^2, x_3^2) \supset A_6$.

Therefore, the Nakayama lemma implies that

$S_3)$ $I'(A_1)|_{L(w)} \supset (p_1^2, p_1p_2, p_2^2)$;

$S_m)$ $I'(A_1)|_{L(w)} \supset (e_{m+1}, e_{m+2}, 9p_2^2 - 8x_1^2)$;

$T_5)$ $I'(A_1)|_{L(w)} \supset (p_1^2, p_1x_2^2, x_3^2)$;

$U_6)$ $I'(A_1)|_{L(w)} \supset (x_1^2, x_1x_2, x_1x_2^2, x_2^m)$;

$V_6)$ $I'(A_1)|_{L(w)} \supset (p_1^2, p_1p_2, p_2^2, x_3^2)$.

Hence, in the case of an arbitrary $n$:

$S_3)$ $I'(A_1)|_{L(w)} \supset (p_1^2, p_1p_2, p_2^2, p_3, \ldots, p_n)$;

$S_m)$ $I'(A_1)|_{L(w)} \supset (e_{m+1}, e_{m+2}, 9p_2^2 - 8x_1^2, p_3, \ldots, p_n)$;

$T_5)$ $I'(A_1)|_{L(w)} \supset (p_1^2, p_1x_2^2, x_3^2, p_3, \ldots, p_n)$;

$U_6)$ $I'(A_1)|_{L(w)} \supset (x_1^2, x_1x_2, x_1x_2^2, x_2^m, p_3, \ldots, p_n)$;

$V_6)$ $I'(A_1)|_{L(w)} \supset (p_1^2, p_1p_2, p_2^2, x_3^2, p_4, \ldots, p_n)$.

So, our germs of Lagrange fibrations are $A_1$-versal. According to Theorem 3 from [9], they are stable with respect to $A_1$-equivalence. \hfill \Box

Proof of Theorem 1. The classification at regular points is just a rewrite of Arnold’s theorem. The classification at singular points follows from Theorem 3, Lemma 4, and Theorem 2. \hfill \Box

Now we are going to prove Theorem 2.

Lemma 5. Let $r$ be a nonnegative integer or $r = \infty$. Let $i : (\mathbb{R}^n, 0) \rightarrow (E, 0), i' : (\mathbb{R}^n, 0) \rightarrow (E, 0)$ be germs of Lagrange immersions with $j'i(0) = \ldots$
inverse matrix of $j^ri'(0)$. Then there exists a germ of symplectic diffeomorphism $\tau$ such that $i' = \tau \circ i$, and that $j^ri(0) = j^r iE(0)$.

**Proof.** It is sufficient to show when $\pi$: $\mathbb{R}^n \rightarrow \mathbb{R}^n$ and $i': \mathbb{R}^n \rightarrow \mathbb{R}^n$ is defined as the graph of $dh$ for a function $h: \mathbb{R}^n \rightarrow \mathbb{R}$ with ord$h > r + 1$. Then $\tau$ may be defined by $(p, x) \mapsto (p + dh(x), x)$.

For a germ of symplectic manifold $(E, 0)$ and a germ of Lagrange submanifold $(L, 0) \subset (E, 0)$ at a base point 0, we denote by $Sp(E, L)$ the group consisting of germs of symplectic diffeomorphisms $(E, 0) \rightarrow (E, 0)$ preserving $L$. Take a Lagrange fibration $\pi: (E, 0) \rightarrow (Y, 0)$ having $L$ as the central fiber: $\pi^{-1}(0) = L$. We denote by $\text{Lag}(E, \pi)$ the group consisting of $\pi$-fiber preserving symplectic diffeomorphism-germs $(E, 0) \rightarrow (E, 0)$. Notice that $\text{Lag}(E, \pi) \subset Sp(E, L)$.

**Lemma 6.** $\text{Lag}(E, \pi)$ is a deformation retract of $Sp(E, L)$. More exactly, there exists a mapping $D: Sp(E, L) \times [0, 1] \rightarrow Sp(E, L)$ with the properties:

1. $D(\tau, 0) = \tau$, $D(\tau, 1) \in \text{Lag}(E, \pi)$, $(\tau \in Sp(E, L))$.
2. $D(\tau, t) = \tau$, $(\tau \in \text{Lag}(E, \pi))$, $t \in [0, 1])$.
3. $D(\tau, \cdot): E \times [0, 1] \rightarrow E$ is smooth on $E \times [0, 1]$ for each $\tau \in Sp(E, L)$ and continuous on a compact neighborhood of $0 \times [0, 1]$ in $E \times [0, 1]$ with respect to $C^\infty$-topology, when $\tau$ is considered as a variable.
4. $j^1(D(\tau, t)|_L)(0) = j^1 iE(0)$, $(\tau \in Sp(E, L), 0 \leq t \leq 1)$.

In particular, each element of $Sp(E, L)$ is connected to an element of $\text{Lag}(E, \pi)$ by a smooth path, fixing the 1-jet of the restriction to $L$, within $Sp(E, L)$.

**Proof.** It suffices to show when $E = T^*\mathbb{R}^n$ with the canonical coordinates $(p, x)$, $L = \{x = 0\}$ and $\pi: T^*\mathbb{R}^n \rightarrow Y = \mathbb{R}^n$ is the standard projection $\pi(p, x) = x$. Let $\tau \in Sp(E, L)$. Set $\tau(p, x) = (P(p, x), X(p, x))$. Then $X(p, 0) = 0$. Remark that the Jacobi matrix $A = (\partial X/\partial x)(0, 0)$ is regular. Now consider the graph $\Gamma(\tau)$ of $\tau$ in $T^*\mathbb{R}^n \times T^*\mathbb{R}^n$, with coordinates $= (p, x; p', x')$. Then $\Gamma(\tau)$ is a Lagrange submanifold with respect to the symplectic form $\Omega = \sum dp_i \wedge dx_i - \sum dp_i \wedge dx_i$ of $T^*\mathbb{R}^n \times T^*\mathbb{R}^n$. Consider the Lagrange projection $\Pi: T^*\mathbb{R}^n \times T^*\mathbb{R}^n \rightarrow \mathbb{R}^{2n}$ defined by $\Pi(p, x, p', x') = (p, x')$. Then $\Pi|_{\Gamma(\tau)}: \Gamma(\tau) \rightarrow \mathbb{R}^{2n}$ is a germ of diffeomorphism at 0. Also consider the projection $\Pi': T^*\mathbb{R}^n \times T^*\mathbb{R}^n \rightarrow T^*\mathbb{R}^n$ defined by $\Pi'(p, x, p', x') = (p, x)$, then $\Phi(\tau) = \Pi' \circ (\Pi|_{\Gamma(\tau)})^{-1}: \mathbb{R}^{2n} \rightarrow T^*\mathbb{R}^n$ is a germ of diffeomorphism at 0. If we set $\Phi(p, x') = (p, x(p, x'))$, then the condition that $\tau$ preserves $L$ is interpreted to the equation $x(p, 0) = 0$. Moreover $\tau \in \text{Lag}(E, \pi)$ if and only if $x(p, x')$ does not depend on $p$. Remark that $\left(\frac{\partial x}{\partial x'}\right)(0, 0)$ is equal to the inverse matrix of $A$. 

Theorem 1. Therefore $f \leq i$. I. A. BOGAEVSKI AND G. ISHIKAWA

to the tangent space $t_f$. If the generating function $I(x) \pi$ is a group of ordinary Lagrange equivalence with respect to the canonical fibration $H$ then they are Lag- $f, f_0$ are Lagrange stable and $f, f_0$ are Lagrange stable with respect to $p$, namely if $I(p, x') = 0$. Then we set $H_t(p, x') = h_0(x') + \sum_{i=1}^n h_i(x') p_i + (1 - t)I(p, x'), 0 \leq t \leq 1$. The restriction of $\Pi$ to the graph of $dH_t$ in $T^*(\mathbb{R}^n) \times T^*\mathbb{R}^n$ is a diffeomorphism, and therefore $dH_t$ defines a family of symplectic diffeomorphisms $\tau_t : (T^*\mathbb{R}^n, 0) \to (T^*\mathbb{R}^n, 0)$ in $\text{Sp}(E, L)$. Then we set $D(\tau, t) = \tau_t$. The points (1), (2), (3) and (4) are clear.

Now we set

$\mathcal{F} := \{f : (\mathbb{R}^n, 0) \to (T^*\mathbb{R}^n, 0) \mid f \text{ is isotropic of corank } \leq 1\},$

$\text{Sp-}A := \{(\sigma, \tau) \mid \sigma \in \text{Diff}(\mathbb{R}^n, 0), \tau \in \text{Sp}(T^*\mathbb{R}^n, 0)\},$

$\text{Lag-A} := \{(\sigma, \tau) \mid \sigma \in \text{Diff}(\mathbb{R}^n, 0), \tau \in \text{Lag}(T^*\mathbb{R}^n, \pi)\}$

$= \text{Diff}(\mathbb{R}^n, 0) \times \text{Lag}(T^*\mathbb{R}^n, \pi),$

group of ordinary Lagrange equivalence with respect to the canonical fibration $\pi : T^*\mathbb{R}^n \to \mathbb{R}^n$, $\pi(p, x) = x$, and set

$\text{Lag-K} := \{(\sigma, \tau) \in \text{Sp-A} \mid \tau(L) = L\} = \text{Diff}(\mathbb{R}^n, 0) \times \text{Sp}(T^*\mathbb{R}^n, L),$

where $L = \pi^{-1}(0)$, the central fiber. Then the group $\text{Lag-A}$ (resp. $\text{Lag-K}$) acts on $\mathcal{F}$ naturally: $(\sigma, \tau)f := \tau \circ f \circ \sigma^{-1}$. Moreover we recall that $J_f^1(n, 2n) := \{j^*f(0) \mid f \in \mathcal{F}\}$ is a submanifold of the ordinary jet space $J^r(n, 2n)$ [12]. Then $\text{Lag-A}$ (resp. $\text{Lag-K}$) acts on $J_f^1(n, 2n)$ naturally as well.

**Lemma 7.** Let $f, f' \in \mathcal{F}$. If $f, f'$ are Lagrange stable and $\text{Lag-K}$-equivalent, then they are $\text{Lag-A}$-equivalent.

**Proof.** Since $f, f'$ are $\text{Lag-K}$-equivalent, there is a $(\sigma, \tau) \in \text{Lag-K}$ with $f' = \tau \circ f \circ \sigma^{-1}$. By Lemma 6, there is a smooth path $\tau_t \in \text{Sp}(E, L), 0 \leq t \leq 1$, with $\tau_0 = \tau, \tau_1 \in \text{Lag}(E, \pi)$ and $j^1(\tau_t|L)(0) = j^1(\text{id}|L)(0)$. We set $f_t := \tau_t \circ f$. Then we have $f_0 = f' \circ \sigma$.

Remark that $f_t, 0 \leq t \leq 1$ are Lagrange stable ([13], Theorem 1.2, [15], Theorem 1.1). Therefore $f_t, (0 \leq t \leq 1)$ are finitely $\text{Lag-A}$-determined ([15], Theorem 1.3), so we may discuss on an isotropic jet space $J_f^1(n, 2n)$ of sufficiently high order. (This argument is analogous to the ordinary one. See [21].)

The vector field $v = \frac{\partial f_t}{\partial t}|_{t=t_0} : (\mathbb{R}^n, 0) \to T(T^*\mathbb{R}^n)$ along $f_{t_0}$ belongs to the tangent space $tf_0(m_n V_n) + w f_0(m_n V I_{2n})$ at $f_{t_0}$ to the $\text{Lag-A}$-orbit,
for each $t_0 \in [0, 1]$. (See [13], [15] for the notations.) In fact there exists $\eta \in m_n V_{2n}$ such that $v - \eta \circ f_{t_0}$ has null generating function. So we have $\xi \in V_n$ with $tf(\xi) = v - \eta \circ f_{t_0}$ (cf. [13], Lemma 4.3). Then $\xi$ is tangent to the stratum through 0, with respect to the stratification of $\mathbb{R}^n$, by the types of open Whitney umbrellas. Since $tf(\xi)$ must vanish at 0, and $f$ is immersive along each stratum, we see $v$ vanishes at 0. Thus, by Mather’s Lemma ([17] IV, pp. 534-535), we see that $j^r f_i(0), (0 \leq t \leq 1)$ belong to the single Lag-$\mathcal{A}$-orbit in $J^r_i(n, 2n)$. In particular $j^r f_0(0)$ and $j^r f_1(0)$ belong to the same Lag-$\mathcal{A}$-orbit. By the determinacy, we have $f_0$ and $f_1$ are Lag-$\mathcal{A}$-equivalent. Since $f_0$ and $f'$ are Lag-$\mathcal{A}$-equivalent, and $f_1$ and $f$ are Lag-$\mathcal{A}$-equivalent, we see that $f$ and $f'$ are Lag-$\mathcal{A}$-equivalent.

Proof of Theorem 2. Let $\pi, \pi' : (E, 0) \rightarrow (Y, 0)$ be $\Lambda$-stable Lagrange projections. Set $L = \pi^{-1}(0)$ and $L' = \pi'^{-1}(0)$, and assume $L$ and $L'$ are formally $\Lambda$-equivalent. By taking symplectic coordinates, we may assume $E = T^*\mathbb{R}^n$ and $\pi : T^*\mathbb{R}^n \rightarrow \mathbb{R}^n$ is the standard fibration.

Then $j^\infty i(0) = j^\infty (\tau \circ i')(0)$, for some parametrizations $i$ of $L$ and $i'$ of $L'$ and for a symplectic diffeomorphism $\tau$ preserving $\Lambda$. By Lemma 5, there exists a symplectic diffeomorphism $\tau'$ with $j^\infty \tau'(0) = j^\infty \text{id}(0)$ and $i = \tau' \circ \tau \circ i'$. Remark that $\tau'$ needs not preserve $\Lambda$. Set $f' = \tau' \circ f$ for a parametrization (open Whitney umbrella) $f : (\mathbb{R}^n, 0) \rightarrow (E, 0)$ of $\Lambda$. Then $f'$ is symplectically equivalent to $f$ [13], and $j^\infty f'(0) = j^\infty f(0)$. Moreover $(f, \pi)$ is Lagrange stable (Theorem 1.2 of [13]). Then as shown in [15] Theorem 1.3, $f$ is finitely Lag-$\mathcal{A}$-determined, and therefore $(f', \pi)$ and $(f, \pi)$ are Lagrange equivalent by $(\sigma, \tau'')$, namely, $\tau'' \circ f' = f \circ \sigma$ and $\tau'' \in \text{Lag}(E, \pi)$. Set $\tau_1 = \tau'' \circ \tau' \circ \tau$. Then $\tau_1(\Lambda) = \Lambda$ and $\tau_1(L') = L$. Therefore $L$ and $L'$ are $\Lambda$-equivalent.

Now take a symplectic diffeomorphism $\tau_2 : (E, 0) \rightarrow (E, 0)$ such that $\pi = \pi' \circ \tau_2$ [3]. Set $\tilde{f} = \tau_2^{-1} \circ f$. Since $\pi$ is $\Lambda$-stable, we see that $f$ is Lagrange stable with respect to $\pi$. So is $\tilde{f}$, since $\pi'$ is $\Lambda$-stable. Since $\tau_1 \circ \tau_2$ maps $L$ to $L$ and $f(\mathbb{R}^n)$ to $\tilde{f}(\mathbb{R}^n)$, $f$ and $\tilde{f}$ are Lag-$\mathcal{K}$-equivalent, by Lemma 8 below. Thus, by Lemma 7, $f$ and $\tilde{f}$ are Lag-$\mathcal{A}$-equivalent: For a $(\sigma, \tau_3) \in \text{Lag}-\mathcal{A}$, $\tilde{f} = \tau_3 \circ f \circ \sigma^{-1}$. Then $\tau_2 \circ \tau_3$ preserves $\Lambda$ and $\pi' \circ \tau_2 \circ \tau_3 = \pi$. Therefore $\pi$ and $\pi'$ are $\Lambda$-equivalent.

5. Simple stable compositions.

Here we remark that our result is applied to the classification problem for compositions of an isotropic mapping and a Lagrange fibration [13], when the isotropic mapping is the first open Whitney umbrella. For this, first recall the notion of $C^\infty$ normalization [6], [7] in the special case we need:
A map-germ \( f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0) \) is called a \( C^\infty \) normalization if \( f \) is \( C^\infty \)-right-left-equivalent to an analytic map-germ \( f' : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0) \) which is a normalization of the image.

Recall that a map-germ \( f : (\mathbb{R}^n, 0) \to (E, 0) \) is called an open Whitney umbrella of type \( k \) if \( f \) is symplectically equivalent to a polynomial map-germ \( f_{n,k} : (\mathbb{R}^n, 0) \to (T^*\mathbb{R}^n, 0) \) explicitly given in [13], namely \( f \circ \sigma = \tau \circ f_{n,k} \) for a diffeomorphism \( \sigma : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0) \) and a symplectic diffeomorphism \( \tau : (E, 0) \to (E, 0) \).

Remark that the normal form \( f_{n,k} \) of open Whitney umbrellas is an analytic normalization of the image, and therefore any open Whitney umbrella is a \( C^\infty \)-normalization. The following lemma is a special case of Theorem 1.11 of [7]:

**Lemma 8.** Let \( f : (\mathbb{R}^n, 0) \to (E, 0) \) be a \( C^\infty \) normalization. Denote by \( f(\mathbb{R}^n) \) the well-defined germ of the image of \( f \). If a germ of diffeomorphism \( h : (E, 0) \to (E, 0) \) preserves \( f(\mathbb{R}^n) \), namely if \( h(f(\mathbb{R}^n)) = f(\mathbb{R}^n) \), then there exists a germ of diffeomorphism \( \sigma : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0) \) such that \( h \circ f = f \circ \sigma \).

Then we have:

**Lemma 9.** Let \( f, f' : (\mathbb{R}^n, 0) \to (E, 0) \) be open Whitney umbrellas of same type \( k \), and \( \pi, \pi' : (E, 0) \to (Y, 0) \) Lagrange fibrations. Then the following conditions are equivalent:

1. There exist a diffeomorphism \( \sigma : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0) \), a symplectic diffeomorphism \( \tau : (E, 0) \to (E, 0) \) and a diffeomorphism \( \varphi : (Y, 0) \to (Y, 0) \) with \( f' \circ \sigma = \tau \circ f \) and \( \pi' \circ \tau = \varphi \circ \pi \). (Lagrange equivalence of \( (f, \pi) \) and \( (f', \pi') \), in the sense of [15].)
2. There exists a symplectic diffeomorphism \( \tau : (E, 0) \to (E, 0) \) and a diffeomorphism \( \varphi : (Y, 0) \to (Y, 0) \) satisfying \( \tau(f(\mathbb{R}^n)) = f'(\mathbb{R}^n) \) and \( \varphi \circ \tau = \varphi \).

**Proof.** The implication (1) \( \Rightarrow \) (2) is straightforward. (2) \( \Rightarrow \) (1): First assume \( \pi' = \pi \). Since both \( f \) and \( f' \) are symplectically equivalent to the parametrization \( f_{n,k} \), there exist diffeomorphisms \( \sigma' : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0) \) and symplectic diffeomorphisms \( \tau', \tau'' : (E, 0) \to (E, 0) \) such that \( f \circ \sigma' = \tau' \circ f_{n,k} \) and \( f' \circ \sigma'' = \tau'' \circ f_{n,k} \). We set \( \Lambda_k = f_{n,k}(\mathbb{R}^n) \). Then we see that \( \tau''^{-1} \circ \tau \circ \tau'(\Lambda_k) = \Lambda_k \). By Lemma 8, there exists a diffeomorphism \( \sigma'' : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0) \) such that \( \tau''^{-1} \circ \tau \circ \tau' \circ f_{n,k} = f_{n,k} \circ \sigma'' \). Then we have \( \tau \circ f = f' \circ \sigma'' \circ \sigma' \circ \sigma'' \circ \sigma'^{-1}. \)

In general case, apply the above argument to \( f \) and \( T \circ f' \), for a symplectic diffeomorphism \( T : (E, 0) \to (E, 0) \) satisfying \( \pi' = \pi \circ T \) ([3]).
References


Received May 16, 2000 and revised February 1, 2001. The first author was partially supported by RFBR 99-01-01109 & NWO-RFBR 047-008-005. The second author was partially supported by Grants-in-Aid for Scientific Research, The Ministry of Education, Science, Sports and Culture, Japan, No. 10440013.

**INDEPENDENT UNIVERSITY OF MOSCOW**  
**BOLSHOI VLASEVSII PER. 11, MOSCOW 121002**  
**RUSSIA**  
**E-mail address**: bogaevsk@mccme.ru  

**DEPARTMENT OF MATHEMATICS**  
**HOKKAIDO UNIVERSITY**  
**SAPPORO 060-0810, JAPAN**  
**E-mail address**: ishikawa@math.sci.hokudai.ac.jp
LAGRANGIAN SECTIONS AND HOLOMORPHIC U(1)-CONNECTIONS

Jingyi Chen

We construct a correspondence between the complex gauge equivalence classes of holomorphic U(1)-connections on a smooth semi-flat special Lagrangian torus fibration and the Hamiltonian deformation classes of Lagrangian sections Σ in the mirror manifold together with the gauge equivalence classes of flat U(1)-connections on Σ.

1. Introduction.

It was conjectured in [SYZ] that Calabi-Yau spaces can be often fibered by special Lagrangian tori and their mirrors can be constructed by dualizing these tori. It was further suggested by Vafa in [V] that the holomorphic vector bundles on a Calabi-Yau n-fold M correspond to the Lagrangian submanifolds in the mirror ˇM and the stable vector bundles correspond to the special Lagrangian submanifolds in ˇM together with flat U(1)-connections.

In this note, we will describe a correspondence between holomorphic U(1)-connections and Lagrangian cycles. We assume that M is a space admitting a special Lagrangian torus fibration. This is a topological fibration π : M → B, where B is a compact n-dimensional manifold without boundary which is locally a Lagrangian section of π, whose fibers are special Lagrangian n-tori with respect to the Kähler form ω and a holomorphic n-form Ω on M (cf. Definition 2.2). We assume that the fibration does not possesses singular fibers and all fibers are flat with respect to the induced metric from M. Note that this is the case studied by Hitchin in [H], and the mirror manifold ˇM has been constructed and it can be identified with the cotangent bundle T*B of quotient by a nondegenerate family of lattices. In particular, ˇM is a smooth special Lagrangian torus fibration over B as well. The symplectic form is the one induced by the canonical symplectic form on T*B (cf. [H], [G2]). If degeneration of fibers possesses, the mathematically rigorous construction of the mirror manifolds remains one of the major challenges in the SYZ program (cf. [G1], [G2], [G3], [R]).

On the M side, we shall focus on the holomorphic connections on a U(1)-bundle E over M. On the mirror side, we consider the pair (Σ, α) where Σ is a Lagrangian section from B in ˇM, and α is a flat U(1)-connection on a complex line bundle L over Σ. One can deform Σ in its Hamiltonian
class which is denoted by $[\Sigma]$, i.e., through Lagrangian cycles which can be translated by the Hamiltonian diffeomorphisms from one to the other (cf. Definition 6.1), and deform the flat connection $\alpha$ on $\Sigma$ in its gauge equivalence class $[\alpha]$.

**Definition 1.1.** Let $\Sigma$ be a Lagrangian section from $B$ in $\check{M}$ and $\alpha$ a flat $U(1)$-connection on $\Sigma$. The pair $([\Sigma], [\alpha])$ which consists of the Hamiltonian deformation class $[\Sigma]$ of $\Sigma$ and the gauge equivalence class $[\alpha]$ of $\alpha$ is called a *Hamiltonian Lagrangian supersymmetric cycle* in $\check{M}$.

The main result of this note is:

**Theorem 1.1.** Let $M$ be a semi-flat special Lagrangian $T^n$-fibration over $B$ with a Lagrangian section and let $E$ be $U(1)$ vector bundle over $M$. Let $\check{M}$ be the mirror manifold of $M$. Suppose that $A = \{A: A$ is a holomorphic connections on $E\}$ and $\pi_{GC} : A \to A/\mathbb{C}G$ is the projection to the complex gauge equivalent classes. Let $S = \{([\Sigma], \alpha): \alpha$ is a flat $U(1)$-connection over a Lagrangian section $\Sigma$ of $\check{M}\}$ and $\pi_S$ be the projection of $S$ to the set $\{([\Sigma], [\alpha])\}$ of Hamiltonian Lagrangian supersymmetric cycles in $\check{M}$. Then there is a map $\phi : A \to S$ which induces a map $\phi' : A/\mathbb{C}G \to S/\pi_S$ such that $\pi_S \circ \phi = \phi' \circ \pi_{GC}$; and conversely there is an injective map $\psi : S \to A$ which induces a map $\psi' : S/\pi_S \to A/\mathbb{C}G$ such that $\pi_{GC} \circ \psi = \psi' \circ \pi_S$.

When the complex dimension of $M$ is two, the special $T^2$-fibration $M$ becomes an elliptic K3 surface by rotating the complex structure by $\pi_2$ and the Lagrangian fibers become holomorphic curves of genus one. In this context, Friedman-Morgan-Witten [FMW] studied extensively flat vector bundles through spectral curves. When $M$ is an elliptic curve, Polishchuk-Zaslow in [PZ] described an isomorphism between the categories suggested by Kontsevich and a suitable version of Fukaya’s category of Lagrangian submanifolds on $\check{M}$. There are also related works by Tyurin in [Ty] on the construction for Hermitian-Einstein bundles on Calabi-Yau $n$-folds with $n = 1, 2, 3$.

The results of this note grow out of extensive discussion with Gang Tian and they constitute partial progress of a general program of Tian and the author. These results and some of their extension to higher rank bundles have been reported by the author in several seminars and conferences. Finally, Tian informed the author that R. Thomas also obtained similar results.

The author is grateful to the referee for valuable comments.

### 2. Special Lagrangian torus fibration.

First, we describe the space we are interested in. Let $\pi : M \to B$ be a smooth proper map from a real $2n$-dimensional smooth compact manifold $M$ to a real $n$-dimensional compact manifold $B$. Both $M$ and $B$ have no boundary.
Definition 2.1. \( \pi : M \to B \) is a topological torus fibration over \( B \) if the fiber \( \pi^{-1}(p) \) is a diffeomorphic to \( T^n \) for any point \( p \in B \).

Recall that a real \( n \)-dimensional submanifold \( N \) in a \( n \)-dimensional Kähler manifold \( X \) is Lagrangian if the Kähler form of \( X \) restricts to zero everywhere on the submanifold \( N \). A Lagrangian submanifold is special if it is Lagrangian and minimal. The later means that the mean curvature \( H \) of the submanifold vanishes identically. If \( X \) is a Calabi-Yau manifold, then there is a covariant constant holomorphic \( n \)-form \( \Omega \) on \( X \) and a special Lagrangian submanifold \( N \) is characterized by

\[
\omega|_N = 0 \quad \text{(2.1)}
\]
\[
\text{Im } \Omega|_N = 0. \quad \text{(2.2)}
\]

Definition 2.2. A compact Calabi-Yau \( n \)-fold \((M, \omega, J, \Omega)\) is a Lagrangian torus fibration over \( B \) if for each \( p \in B \), the fiber \( \pi^{-1}(p) \) is a Lagrangian torus with respect to the symplectic structure \( \omega \), and \( M \) is semi-flat if each fiber is flat in the induced metric from \( M \). Furthermore, \( M \) is a special Lagrangian torus fibration over \( B \) if in addition the Lagrangian fibers are special.

According to Hitchin's discussion (cf. [H], [G2]), the complex structure \( J \) on \( M \) acts on \( TM \) as follows

\[
J \left( \frac{\partial}{\partial s_i} \right) = \frac{\partial}{\partial t_i}, \quad J \left( \frac{\partial}{\partial t_i} \right) = -\frac{\partial}{\partial s_i} \quad \text{(2.3)}
\]

where \( t_i \) are the local coordinates in \( B \) and \( s_i \) are coordinates on the fiber tori for \( i = 1, 2, \ldots, n \). Note that the section \( B \) needs to be Lagrangian for the complex coordinates to exist, and this will be understood throughout the paper. Hitchin shows:

Proposition 2.1. For the special Lagrangian torus fibration \( \pi : M \to B \) without singular fibers, in the complex coordinates \( s_j + \sqrt{-1} t_j \), the symplectic form of \( M \) can be written as

\[
\omega = \sum_{i,j} a_{ij} ds_i \wedge dt_j,
\]

where \( a_{ij} \) only depend on \( t \in B \).

For each base point \( t \in B \), set \( L_t = \pi^{-1}(t) \). It is shown in [H] that the 1-form \( \iota(\frac{\partial}{\partial t_j}) \omega \) is harmonic on \( L_t \) when \( L_t \) is special Lagrangian, \( j = 1, \ldots, n \), hence \( \iota(\frac{\partial}{\partial t_j}) \omega \) and \( *_t(\frac{\partial}{\partial t_j}) \omega \) are closed 1-form and \((n-1)\)-form respectively, where \( *_t \) is the Hodge star operator of the induced metric on \( L_t \). Take a basis \( A_1, \ldots, A_n \) of the first homology group \( H_1(L_t, \mathbb{Z}) \). Evaluation of \( \iota(\frac{\partial}{\partial t_j}) \omega \) on
A_j yields a period matrix which depends on t:

\[ \lambda_{ij} = \int_{A_i} t \left( \frac{\partial}{\partial t_j} \right) \omega. \]  

(2.4)

The Poincaré dual of A_j provide a basis B_j of H_{n-1}(L_t, \mathbb{Z}) for j = 1, \ldots, n. Then form a period matrix

\[ \mu_{ij} = \int_{B_i} * t \left( \frac{\partial}{\partial t_j} \right) \omega. \]  

(2.5)

**Lemma 2.2.** For the special Lagrangian torus fibration \( \pi : M \to B \) with the symplectic form \( \omega = \sum_{i,j} a_{ij} ds_i \wedge dt_j \), then

\[ a_{ij} = V^{-1} \sum_k \lambda_{ik} \mu_{jk} \]  

(2.6)

where \( V \) is the volume of the special Lagrangian fiber \( L_t \) and is independent of \( t \).

**Proof.** By Proposition 2.1, \( a_{ij} \) only depend on \( t \), then

\[ \sum_k \lambda_{ik} \mu_{jk} = \sum_{k,l_1,l_2} a_{il_1} a_{jl_2} \int_{A_k} ds_{l_1} \int_{B_k} * t ds_{l_2} \]

\[ = \sum_{l_1,l_2} a_{il_1} a_{jl_2} \int_{L_t} ds_{l_1} \wedge * t ds_{l_2}. \]

On the other hand, \( s_j, t_j \) form a complex coordinates, so

\[ g \left( \frac{\partial}{\partial s_i}, \frac{\partial}{\partial s_j} \right) = \omega \left( \frac{\partial}{\partial s_i}, \frac{\partial}{\partial t_j} \right) = a_{ij}. \]

Therefore,

\[ ds_{l_1} \wedge * t ds_{l_2} = \langle ds_{l_1}, ds_{l_2} \rangle d\mu_{L_t} \]

\[ = a^{l_1l_2} d\mu_{L_t} \]

where \( a^{ij} \) denote the entries of the inverse matrix of \( (a_{ij}) \). It then follows that

\[ \sum_k \lambda_{ik} \mu_{jk} = V(L_t) a_{ij} \]

where \( V(L_t) \) is the volume of \( L_t \) and is independent of \( t \) since \( L_t \) is special Lagrangian. \( \square \)

Since the special Lagrangian tori \( L_t \) are calibrated by \( \text{Im} \Omega \), from that \( \text{Re} \Omega \) is closed in \( M \) it follows easily:

**Lemma 2.3.** The induced volume form \( d\mu_{L_t} \) on the fiber \( L_t \) is independent of \( t \in B \).
3. Construction of the mirror manifold \((\tilde{M}, \tilde{\omega})\).

From now on, we shall fix a special Lagrangian torus fibration over \(B\) and denote it by \((M, \omega, J, \Omega)\) with \(J\) defined in (2.3). The set \(\mathcal{M}_{SL}\) of all special Lagrangian submanifolds which can be deformed through special Lagrangian submanifolds to the fiber tori in \((M, \omega, J)\) is called the moduli space of special Lagrangian submanifolds. The deformations of special Lagrangian submanifolds were studied by McLean in [M]. As in [SYZ], we can construct the mirror manifold (the \(D\)-brane moduli space in the literatures of physics) over \(B\) by taking

\[
\tilde{M} = \mathcal{M}_{SL} \times_B \mathcal{M}_{FLAT}
\]

where \(\mathcal{M}_{SL}\) denotes local deformation space of the special Lagrangian fibers over \(B\) and \(\mathcal{M}_{FLAT}\) denotes the moduli space of the gauge equivalence classes of the flat \(U(1)\)-connections on the fibers over \(B\). A point in \(\tilde{M}\) is a pair \((L_t, [A])\) where \(L_t\) is a special Lagrangian fiber torus over \(t \in B\) and \([A]\) is the gauge equivalence class of a flat \(U(1)\)-connection \(A\) on \(L_t\). Note that \(\mathcal{M}_{FLAT}\) is diffeomorphic to \(H^1(T^n, \mathbb{R})/H^1(T^n, \mathbb{Z})\) hence to \(T^n\). Topologically, \(\tilde{\pi} : \tilde{M} \to B\) is a torus fibration over \(B\).

Recall that \(M\) is identified with \(T^*B\) quotient by the lattice \(\Lambda\). We now describe the dual lattice \(\tilde{\Lambda}\) (cf. [H], [G2]). Over a base point \(t \in B\), consider a smooth fiber torus \(L_t\). According to McLean’s result, we know that

\[
\dim \mathcal{M}_{SL} = b_1(L_t) = \dim H^1(T^n, \mathbb{R}) = n.
\]

Moreover, any tangent vector \(v\) of \(B\) can be identified with a harmonic 1-form on the fiber \(L_t\) as follows. Recall that we had a basis \(A_j\) of \(H^1(L_t, \mathbb{Z})\) and a basis \(B_j\) of \(H_{n-1}(L_t, \mathbb{Z})\) and \(A_j, B_j\) are dual to each other, for \(j = 1, \ldots, n\). For each \(j\), let \(\alpha_j\) be the dual of \(A_j\) in \(H^1(L_t, \mathbb{Z})\) hence they form a basis of \(H^1(L_t, \mathbb{Z})\), and similarly let \(\beta_j\) be the dual of \(B_j\) in \(H^{n-1}(L_t, \mathbb{R})\). Then the mapping

\[
v \longrightarrow [\iota(v)\omega] = \sum_i \left(\int_{A_i} \iota(v)\omega\right) \alpha_i
\]

identifies \(T_t B\) with \(H^1(L_t, \mathbb{R})\). Define

\[
\Lambda'_t = \left\{ v \in T_t B \left| \int_{\gamma} \iota(v)\omega \in \mathbb{Z}, \text{ for any } \gamma \in H_1(L_t, \mathbb{Z}) \right\}. \right.
\]

Then we take

\[
\tilde{\Lambda}_t = \left\{ [\iota(v)\omega] \mid v \in \Lambda'_t \right\} = H^1(L_t, \mathbb{Z}).
\]

Let \(\tilde{\Lambda} = \bigcup_{t \in B} \tilde{\Lambda}_t\) be the dual lattice over \(B\). Then \(\tilde{M} = T^*B/\tilde{\Lambda}\).

We can use the basis \(\alpha_1, \ldots, \alpha_n\) of \(H^1(L_t, \mathbb{R})\) to give coordinates \(x_1, \ldots, x_n\) on the universal covering of the torus \(H^1(L_t, \mathbb{R}/\mathbb{Z})\). Let \(\Omega\) be the covariant constant holomorphic \(n\)-form on \(M\) from the Calabi-Yau structure with
a standard normalization
\[ \frac{\omega^n}{n!} = (-1)^{\frac{n(n-1)}{2}} \left( \frac{i}{2} \right)^n \Omega \wedge \overline{\Omega}. \]

To fix a symplectic structure on \( \tilde{M} \), Gross considered a holomorphic \( n \)-form \( \Omega \) normalized by
\[ \Omega_n = \frac{\Omega}{\int_{L_t} \Omega} = V^{-1} \Omega \]
where \( V \) is the volume of the special Lagrangian fiber torus \( L_t \). Gross has shown (cf. Lemma 4.1 and Proposition 4.2 in [G2]):

**Proposition 3.1.** Under the identification
\[ \tilde{\Lambda} \cong H^1(L_t, \mathbb{Z}) \cong H_{n-1}(L_t, \mathbb{Z}), \]
the image of \( \tilde{\Lambda} \) under the mapping \( F : H_{n-1}(L_t, \mathbb{Z}) \to T^*B \), defined by
\[ F(\gamma)(v) = - \int_{\gamma} \iota(v) \text{Im} \Omega_n \]
for any \( v \in T_t B, \gamma \in H_{n-1}(L_t, \mathbb{Z}) \), is Lagrangian in \( T^*B \). Moreover \( \tilde{M} = T^*B/\tilde{\Lambda} \) inherits the symplectic form \( \tilde{\omega} \) from \( T^*B \), and
\[ \int_{A_j} \iota(v) \tilde{\omega} = \int_{B_j} \iota(v) \text{Im} \Omega_n. \]

Next, we compute the symplectic form \( \tilde{\omega} \) in coordinates \( t_1, \ldots, t_n, x_1, \ldots, x_n \), determined by \( \alpha_1, \ldots, \alpha_n \).

**Lemma 3.2.** Let \( \pi : M \to B \) be a special Lagrangian torus fibration and \( \tilde{\pi} : \tilde{M} \to B \) be its dual space. Then
\[ \tilde{\omega} = \sum_{i,j} \mu_{ij} dt_i \wedge dx_j. \]

**Proof.** By McLean’s result [M],
\[ \iota(v) \text{Im} \Omega_n|_{L_t} = - \ast_t \iota(v) \omega|_{L_t}. \]
It follows that
\[ \int_{B_j} \iota \left( \frac{\partial}{\partial t_i} \right) \text{Im} \Omega_n = - \int_{B_j} \ast_t \iota \left( \frac{\partial}{\partial t_i} \right) \omega = -\mu_{ij}. \]
In the canonical coordinates \( t_1, \ldots, t_n, x_1', \ldots, x_n' \) on \( T^*B \), the symplectic form has the form
\[ \tilde{\omega} = \sum_j dt_j \wedge dx_j'. \]
The coordinates \( x_1', \ldots, x_n' \) are determined by some basis \( \alpha_1', \ldots, \alpha_n' \) of \( H^1(L_t, \mathbb{Z}) \), and in fact by \( dx_1', \ldots, dx_n' \) which are harmonic 1-forms on \( L_t \).
by Corollary 5.15 in [G2] since $d\Omega = 0$. Lifted to the universal covering $H^1(L_t, \mathbb{R})$ of $H^1(L_t, \mathbb{R}/\mathbb{Z})$, $\alpha_1, \ldots, \alpha_n$ and $\alpha'_1, \ldots, \alpha'_n$ are related by

$$\alpha'_i = \sum_j b_{ij} \alpha_j$$

for some functions $b_{ij}$ on $B$. Therefore,

$$\int_{A_j} t \left( \frac{\partial}{\partial t_i} \right) \check{\omega} = - \int_{A_j} dx'_i$$

$$= \sum_l b_{il} \int_{A_i} \alpha_l$$

$$= -b_{ij}.$$

We then have

$$b_{ij} = \mu_{ij}$$

as claimed in the lemma. \qed

From now on, we shall always assume that $\text{Volume}(L_t) = 1$ for simplicity by normalizing the metric $g$ on $M$.

We now explore the relationship between the coordinates $s_i$ on $L_t$ and $x_i$ on the dual tori $\check{L}_t$, i.e., the moduli space of the flat $U(1)$-connections on $L_t$.

**Proposition 3.3.** Let $(t_1, \ldots, t_n, s_1, \ldots, s_n)$ and $(t_1, \ldots, t_n, x_1, \ldots, x_n)$ are the local coordinates on $M$ and $\check{M}$ respectively as before. For any closed 1-form $\sum_j c^j ds_j$ on $L_t$, if its cohomology class $[\sum_j c^j ds_j]$ is expressed as $\sum_j \check{c}^j \alpha_j$, then

$$\check{c}^i = \sum_k \mu_k^{ki} \int_{L_t} c^k d\mu_{L_t}. \quad (3.5)$$

**Proof.** Notice that by Proposition 2.1 and Lemma 2.2, we have

$$\omega = \sum_{i,j,k} \lambda_{ik} \mu_{jk} ds_i \wedge dt_j.$$ Then

$$ds_j = \sum_{l,i} \lambda^{ji} \mu_{li} \left( \frac{\partial}{\partial t_l} \right) \omega$$

$$= \sum_l \mu^{jl} \alpha_l.$$ Recall that 1-forms $\alpha_j$ and $(n-1)$-forms $\beta_j$ are chosen such that

$$\alpha_i \wedge \beta_j = \delta_{ij} d\mu_{L_t}$$
and hence
\[ \langle \alpha_i, \beta_j \rangle = \int_{L_t} \alpha_i \wedge \beta_j = \delta_{ij} V. \]

It follows that
\[\tilde{c}^k = \langle \sum_j \tilde{c}^j \alpha_j, \beta_k \rangle = \langle \sum_j c^j ds_j, \beta_k \rangle = \int_{L_t} \sum_j c^j ds_j \wedge \beta_k = \sum_{j,l} \frac{\mu_{lj}}{l} \int_{L_t} c^j d\mu_{L_t} \].

This completes the proof. \[\Box\]

Let \( Y = \sum_j Y^j ds_j \) be a differential 1-form on \( L_t \). By the Hodge decomposition theorem,
\[ Y = H(Y) + df + d^* \psi \]
where \( H(Y) \) is the harmonic part of \( Y \). In particular, \( H(Y) + df \) defines a cohomology class and we denote it by \( [Y] \) in \( H^1(L_t, \mathbb{R}) \).

**Lemma 3.4.** If \( Y \) is a 1-form on the Lagrangian fiber \( L_t \) and \([Y]\) is its cohomology class in \( H^1(L_t, \mathbb{R}) \), then
\begin{align*}
\langle [Y] \rangle & = \sum_j \left( \sum_l \mu^{lj} \int_{L_t} Y^l d\mu_{L_t} \right) \alpha_j \quad (3.6) \\
\int_{L_t} Y d\mu_{L_t} & = \int_{L_t} Y \alpha_j \quad (3.7)
\end{align*}

**Proof.** By Proposition 3.3,
\[ [Y] = \sum_{j,l} \left( \mu^{lj} \int_{L_t} \left( Y^l - (d^* \psi)^l \right) d\mu_{L_t} \right) \alpha_j. \]
Note that
\[ \sum_{j,l} \mu^{lj} \int_{L_t} (d^* \psi)^l d\mu_{L_t} \alpha_j = \sum_l \left( \int_{L_t} (d^* \psi)^l d\mu_{L_t} \right) ds_l. \]
Also, we have
\[ d^* \psi \wedge \ast \left( \frac{\partial}{\partial t_l} \right) \omega = \sum_{k,j} (d^* \psi)^k ds_k \wedge \ast a_{lj} ds_j \]
\[ = \sum_{k,j} (d^* \psi)^k a_{lj} a^{kj} d\mu_{L_t} \]
\[ = (d^* \psi)^l d\mu_{L_t}. \]

It follows that
\[
\int_{L_t} (d^* \psi)^l d\mu_{L_t} = \int_{L_t} d^* \psi \wedge \ast \left( \frac{\partial}{\partial t_l} \right) \omega = \int_{L_t} \psi \wedge \ast dt \left( \frac{\partial}{\partial t_l} \right) \omega = 0 \]

since \( \nu \left( \frac{\partial}{\partial t_l} \right) \omega \) is closed. \( \Box \)

4. Holomorphic connections.

Let \( E \) be a complex vector bundle over \( M \) and \( A \) be a unitary connection. The curvature 2-form \( F \) of \( A \) can be decomposed according to the complex structure on \( E \) into \((2, 0), (1, 1), (0, 2)\) parts:
\[ F = F^{2,0} + F^{1,1} + F^{0,2}. \]

In terms of real local coordinates \( s_1, \ldots, s_n, t_1, \ldots, t_n \),
\[
F = \sum_{i,j} \left( F_{ij} dt_i \wedge dt_j + F_{i(j+n)} dt_i \wedge ds_j + F_{(i+n)(j+n)} ds_i \wedge ds_j \right) \tag{4.1}
\]
where the indices \( i, j + n \) stand for the \( t_i \)-component and \( s_j \)-component of the connection correspondingly. Recall that a unitary connection \( A \) on \( E \) over \( (M, J, \omega) \) gives rise to a holomorphic connection if and only if
\[ F^{0,2}_A = 0. \tag{4.2} \]

Therefore we obtain the following curvature equations for the holomorphic connections in real coordinates.

**Proposition 4.1.** Let \( E \) be a complex vector bundle over a smooth special Lagrangian fibration \((M, J, \omega)\) over \( B \). Then the curvature of a holomorphic connection on \( E \) satisfies
\[ F_{i(j+n)} - F_{j(i+n)} = 0 \tag{4.3} \]
\[ F_{ij} - F_{(i+n)(j+n)} = 0. \tag{4.4} \]
Proof. In terms of the complex coordinates $z_i = s_i + \sqrt{-1}t_i$, $i = 1, \ldots, n$, we can rewrite (4.1) as

$$F = - \sum_{i,j=1}^{n} F_{ij} (dz_i - d\overline{z}_i) \wedge (dz_j - d\overline{z}_j)$$

$$- \sqrt{-1} \sum_{i,j=1}^{n} F_{i(j+n)} (dz_i - d\overline{z}_i) \wedge (dz_j + d\overline{z}_j)$$

$$+ \sum_{i,j=1}^{n} F_{(i+n)(j+n)} (dz_i + d\overline{z}_i) \wedge (dz_j + d\overline{z}_j).$$

This local expression leads to

$$F^{0,2} = \sum_{i,j=1}^{n} (-F_{ij} + F_{(i+n)(j+n)} + \sqrt{-1}F_{i(j+n)}) \, d\overline{z}_i \wedge d\overline{z}_j$$

and

$$F^{1,1} = \sum_{i,j=1}^{n} (2F_{ij} + 2F_{(i+n)(j+n)} + \sqrt{-1}(F_{i(j+n)} + F_{j(i+n)})) \, dz_i \wedge d\overline{z}_j.$$

Then $F^{0,2} = 0$ implies the two desired equations. □

The holomorphic connections are preserved by the complex gauge transformations. Recall that the complex gauge group $G_c$ consists of all general linear automorphisms of the complex vector bundle $E$ which cover the identity map on the base manifold $M$. If $g \in G_c$, the action of $g$ is given by

$$\partial g(A) = \partial_A - (\partial_A g) g^{-1}$$

$$\bar{\partial} g(A) = \bar{\partial}_A + (\bar{\partial}_A g) g^{-1}.$$  

The unitary gauge group is contained as a subgroup in $G_c$ and it preserves the Hermitian metric on $E$. In particular, if $E$ is a complex line bundle, then a connection $A'$ is $C$-gauge equivalent to another connection $A$ if there exist real valued functions $u$ and $v$ such that

$$A' = A + \sqrt{-1}(\bar{\partial} - \partial)u + (\bar{\partial} + \partial)v.$$

When $u = 0$, we obtain the ordinary $U(1)$ gauge action.

5. Holomorphic line bundles vs. Lagrangian cycles with flat line bundles.

In this section, we shall start from a holomorphic connection on a complex line bundle $E$ over $M$ to construct a Lagrangian submanifold $\Sigma$ in $\mathcal{M}$ and a flat $U(1)$-connection $\alpha$ on $\Sigma$. Then we shall demonstrate how to reconstruct a holomorphic connection on $E$ from $(\Sigma, \alpha)$.
5.1. Construction of Lagrangian cycles with flat line bundles. Let $E$ be a holomorphic line bundle over a smooth special Lagrangian $T^n$ fibration $(M, J, \omega, g)$ and $A$ be a $U(1)$-connection whose curvature satisfies $F_A^{0,2} = 0$. It is a standard fact that each bundle trivialization, with a trivializing cover $\{U_j\}$ and $f_j : U_j \rightarrow \mathbb{C}$ satisfying the compatibility conditions

$$f_j = h_{jk} f_k \quad \text{on} \quad U_j \cap U_k \neq \emptyset,$$

where $h_{jk}$ are the transition functions, defines a global section $f \in \Gamma(M, E)$; and in the gauge determined by $f$ the connection $A$ can be viewed as an $E$-valued 1-form, which decomposes into its fiber component and its base component as follows

$$A = \sum_{i=1}^n \left( X^i dt_i + Y^i ds_i \right). \quad (5.1)$$

Here as in the previous sections we use $t_1, \ldots, t_n$ for the local coordinates on the base $B$ and $s_1, \ldots, s_n$ for the fiber torus $L_t = \pi^{-1}(t)$, for any $t \in B$. $X^i$ and $Y^i$ are $\mathbb{C}$-valued functions.

On the fiber tori $L_t$, the gauge equivalent class of $Y = A|_{L_t}$ is just the cohomology class $[Y]$ of the $E$-valued 1-form $Y$. The image of the single valued map

$$\Phi : t \rightarrow (t, [Y(\cdot, t)]) \quad (5.2)$$

defines an embedded submanifold of real dimension $n$ in $\tilde{M}$:

$$\Sigma = \{(t, [Y(\cdot, t)) : t \in B\}. \quad (5.3)$$

**Proposition 5.1.** Let $E$ be a $U(1)$-bundle over a special Lagrangian torus fibration $\pi : M \rightarrow B$ and let $\tilde{\pi} : \tilde{M} \rightarrow B$ be the dual space. If $A$ is a connection on $E$, then for any $v \in T_t B$ and $t \in B$,

$$\iota(v)(\omega|_{\Sigma}) = \int_{L_t} \iota(Jv) \text{Re} F_A^{0,2} \wedge d\mu_{L_t}. \quad (5.4)$$

In particular, if $A$ is holomorphic, then $\Sigma$ is an embedded Lagrangian submanifold in $\tilde{M}$.

**Proof.** In the local coordinates $(x_1, \ldots, x_n)$ on $\mathcal{M}_{\text{FLAT}}$, let $[Y]^1, \ldots, [Y]^n$ be the local expression of $[Y]$ in $H^1(L_t, \mathbb{R})$. We claim that the restriction of
\( \tilde{\omega} \) on \( \Sigma \) vanishes. In fact,

\[
\tilde{\omega}\big|_{\Sigma} = \Phi^* \tilde{\omega}
\]

\[
= \sum_{i,j} \tilde{\omega} \left( \frac{\partial}{\partial t_i} + \sum_\alpha \frac{\partial [Y^\alpha]}{\partial x_\alpha} \frac{\partial}{\partial t_j} + \sum_\beta \frac{\partial [Y^\beta]}{\partial x_\beta} \frac{\partial}{\partial t_j} \right) dt_i \land dt_j
\]

\[
= 2 \sum_{i,j,\alpha} \mu_{\alpha j} \frac{\partial [Y^\alpha]}{\partial t_i} dt_i \land dt_j
\]

by Proposition 3.3. Recall that the volume element \( d\mu_{Lt} \) is independent of \( t \) by Lemma 2.3. Since \( \tilde{\omega} = \sum_{i,j} \mu_{ij} dt_i \land dx_j \) is closed, we have

\[
\frac{\partial \mu_{ik}}{\partial t_j} = \frac{\partial \mu_{jk}}{\partial t_i}
\]

and it follows that

\[
\sum_\alpha \left( \mu_{\alpha j} \frac{\partial \mu^\alpha_k}{\partial t_i} - \mu_{\alpha i} \frac{\partial \mu^\alpha_k}{\partial t_j} \right) = \sum_\alpha \left( -\frac{\partial \mu_{\alpha j}}{\partial t_i} \mu^\alpha_k + \frac{\partial \mu_{\alpha i}}{\partial t_j} \mu^\alpha_k \right) = 0.
\]

Now we conclude from (5.5) that

\[
\tilde{\omega}\big|_{\Sigma} = \sum_{i,j} \left( \int_{Lt} \frac{\partial Y^j}{\partial t_i} d\mu_{Lt} \right) dt_i \land dt_j.
\]

On the other hand,

\[
\sum_{i,j} \left( \int_{Lt} \frac{\partial Y^j}{\partial t_i} d\mu_{Lt} \right) dt_i \land dt_j
\]

\[
= \sum_{i,j} \left( \int_{Lt} \left( F_{i(j+n)} - \frac{\partial X^i}{\partial s_j} \right) d\mu_{Lt} \right) dt_i \land dt_j
\]

\[
= \sum_{i,j} \left( \int_{Lt} F_{i(j+n)} d\mu_{Lt} \right) dt_i \land dt_j
\]

\[
- \sum_{i,j} \left( \int_{Lt} d_{Lt} X^i \land \ast t \left( \frac{\partial}{\partial t_j} \right) \omega \right) dt_i \land dt_j.
\]
Because $L_t$ is special Lagrangian, $\ast(\frac{\partial}{\partial t_j})\omega$ is closed on $L_t$, and in turn the last integral in (5.9) vanishes. This leads to

\begin{equation}
\tilde{\omega}|_{\Sigma} = \sum_{i,j} \left( \int_{L_t} F_{i(j+n)} d\mu_{L_t} \right) dt_i \wedge dt_j.
\end{equation}

If $A$ is a holomorphic $U(1)$-connection, it follows immediately from (4.3) in Proposition 4.1 that $\tilde{\omega}$ restricts to zero on $\Sigma$ and $\Sigma$ is Lagrangian.

It is straightforward to find

\begin{equation}
\Re F^{0,2} = \sum_{i,j} F_{i(j+n)} (dt_i \wedge ds_j + ds_i \wedge dt_j)
\end{equation}

\begin{equation*}
+ \sum_{i,j} (F_{(i+n)(j+n)} - F_{ij})(ds_i \wedge ds_j - dt_i \wedge dt_j).
\end{equation*}

Then

\begin{equation}
\iota \left( \frac{\partial}{\partial s_k} \right) \Re F^{0,2} \wedge d\mu_{L_t} = \sum_j (F_{j(k+n)} - F_{k(j+n)}) dt_j \wedge d\mu_{L_t}.
\end{equation}

Finally, we compute

\begin{equation}
\iota \left( \frac{\partial}{\partial t_k} \right) (\tilde{\omega}|_{\Sigma}) = \sum_j \left( \int_{L_t} (F_{j(k+n)} - F_{k(j+n)}) d\mu_{L_t} \right) dt_j
\end{equation}

and we are done. \hfill \Box

To investigate what the second curvature equation (4.4) leads to, we consider the 1-form defined by

\begin{equation}
\alpha = \int_{L_t} A \wedge \Re \Omega.
\end{equation}

**Proposition 5.2.** If $A$ is a holomorphic $U(1)$-connection over $M$, then $\alpha$ is a flat $U(1)$-connection over $\Sigma$. If $A$ is a $U(1)$-connection, then for any $v \in T_t B$ with $t \in B$,

\begin{equation}
\iota(v) d\iota \alpha = \int_{L_t} \iota(Jv) \Im F^{0,2}_A \wedge d\mu_{L_t}.
\end{equation}
Proof. By Lemma 2.3, the exterior differentiation on $B$ is

\begin{align*}
\frac{d}{dt}\alpha &= \int_{L_t} \left( d_t A \right) \wedge d\mu_{L_t} \\
&= \sum_i \int_{L_t} \left( d_t (X^i dt_i) \right) \wedge d\mu_{L_t} \\
&= \sum_{i,k} \left( \int_{L_t} \frac{\partial X^i}{\partial s_k} d\mu_{L_t} \right) dt_i \wedge dt_k \\
&= \sum_{i,k} \left( \int_{L_t} \frac{\partial Y^i}{\partial s_k} d\mu_{L_t} \right) dt_i \wedge dt_k \\
&= \sum_{i,k} \left( \int_{L_t} \frac{\partial Y^i}{\partial s_k} d\mu_{L_t} \right) dt_i \wedge dt_k \\
&= \sum_{i,k} \left( \int_{L_t} d_{L_t} Y^i \wedge \ast_{L_t} \left( \frac{\partial}{\partial t_k} \right) \omega \right) dt_i \wedge dt_k \\
&= 0
\end{align*}

where we have used $F_{ik} = F_{(i+n)(k+n)}$ in the fourth equality, that $\ast_{L_t} \left( \frac{\partial}{\partial t_k} \right) \omega$ is closed and Stokes’ theorem. The imaginary part of $F^{0,2}$ is given by

\begin{equation}
\text{Im} F^{0,2} = F_{(i+n)(j+n)} (ds_i \wedge ds_j - dt_i \wedge dt_j) + (F_{(i+n)(j+n)} - F_{ij}) (dt_i \wedge ds_j + ds_i \wedge dt_j).
\end{equation}

Then we can deduce

\begin{equation}
\text{Im} F^{0,2}_{A} \wedge d\mu_{L_t} = 2 \sum_i (F_{(i+n)(j+n)} - F_{ij}) dt_i \wedge d\mu_{L_t}.
\end{equation}

On the other hand, the previous computation shows

\begin{equation}
\frac{d}{dt} \omega = \sum_i \int_{L_t} \left( \frac{\partial X^i}{\partial t_j} - \frac{\partial X^j}{\partial t_i} \right) dt_i \wedge d\mu_{L_t} \\
= \sum_i \int_{L_t} (F_{(i+n)(j+n)} - F_{ij}) dt_i \wedge d\mu_{L_t}.
\end{equation}

Now the proof is complete by integrating (5.17) along $L_t$ and then substituting (5.18) into the result. \qed

It follows from Proposition 5.1 and Proposition 5.2 that:

**Proposition 5.3.** Let $E$ be a complex line bundle over a special Lagrangian torus fibration $M \to B$ and $A$ be a $U(1)$-connection on $E$. Then, for any
v ∈ T_1B and t ∈ B

\begin{equation}
\iota(v) \left( \tilde{\omega} + \sqrt{-1} \partial_t \alpha \right) = \int_{L_t} \iota(Jv) F_A^{0,2} \wedge \text{Re} \Omega.
\end{equation}

**Remark.** Assume that \(M\) is a Calabi-Yau 3-fold. The derivative of the holomorphic Chern-Simons functional is given by

\begin{equation}
\int_M \text{Tr}(\delta A \wedge F_0^0, A) \wedge \Omega.
\end{equation}

Its differential is given by the right side of (5.19). It was observed by Tian and myself that there is a useful version for the left side of (5.19). Consider the space \(L\) of all \((L, [N], B)\) where \(L\) is any 3-cycle homologous to a fixed 3-cycle \(L_0\), \(N\) is a 4-cycle with boundary \(\partial N = L - L_0\) and \(B\) is a \(U(1)\)-connection on \(L\) extendible to \(N\) with fixed boundary value along \(L_0\). Then one can integrate the left side of (5.19) to obtain a functional on \(L\)

\[ F(L, [N], B) = \int_N (\tilde{\omega} + \sqrt{-1} F_B)^2. \]

If \(L\) is a section, then it corresponds to the holomorphic Chern-Simons functional through (5.19). However, this functional \(F\) is well-defined on any Calabi-Yau 3-fold without knowing the mirrors. It is certainly interesting to explore more about \(F\).

### 5.2. Construction of holomorphic line bundles from \((\Sigma, \alpha)\)

We have just constructed a Lagrangian submanifold \(\Sigma\) in \(\tilde{M}\) and a flat \(U(1)\)-connection \(\alpha\). Strictly speaking, \(\alpha\) is a pull-back of a flat \(U(1)\)-connection on \(B\) via the projection \(\pi|_{\Sigma}: \Sigma \rightarrow B\). Conversely, given a pair \((\Sigma, \alpha)\) on the mirror side, we would like to construct a holomorphic connection \(A\) on a complex line bundle over \(M\).

The information encoded in \(\Sigma\) is \([Y]\). Let \(P: \Sigma \rightarrow B\) be the natural projection. The 1-form \(\alpha\) determines a flat complex line bundle over \(B\) which pulls back via \(P\) to the flat connection, still denoted by \(\alpha\), on \(\Sigma\). Since \(P: \Sigma \rightarrow B\) is diffeomorphic, the inverse map \(P^{-1}\) pulls back the flat bundle \((L, \alpha)\) to a flat bundle over \(B\), which will still be denoted by \((L, \alpha)\). Then we use \(\pi: M \rightarrow B\) to pull back \((L, \alpha)\) to \(X\) on \(M\) which satisfies (5.20) below. The desired connection \(A = X + Y\), where \(X = \sum_i X^i dt_i\) and \(Y = \sum_i Y^i ds_i\), should satisfy, in addition to being holomorphic, that

\begin{equation}
\int_{L_t} X^i d\mu_{L_t} = \alpha^i
\end{equation}

\begin{equation}
\int_{L_t} Y^i d\mu_{L_t} = \sum_k \mu_{ki}[Y]^k
\end{equation}

for \(i = 1, \ldots, n\), where \(\alpha = \alpha^i dt_i\) is over \(B\). To find a flat connection \(Y\) on \(L_t\), we recall that flat line bundles on \(L_t\) are classified by \(H^1(L_t, \mathbb{R}/\mathbb{Z})\). On
the universal covering \( H^1(L_t, \mathbb{R}) \) of \( H^1(L_t, \mathbb{R}/\mathbb{Z}) \), we consider the 1-form \( V(L_t)^{-1} \sum_{i,k} \mu_{ki}[Y]_k^i \alpha_i \), where \( \alpha_1, \ldots, \alpha_n \) is the basis of \( H^1(L_t, \mathbb{R}) \). This 1-form descents to \( Y \) which satisfies (5.21).

We shall concentrate on the special case that \( X^i \) and \( Y^i \) are smooth \( \mathbb{C} \)-valued functions on \( M \) which depend only on the base variable \( t \in B \).

**Proposition 5.4.** If \( X^i \) and \( Y^i \) depend only on \( t \) and satisfy (5.20) and (5.21) for all \( i \), then the 1-form \( A = \sum_i (X^i dt_i + Y^i ds_i) \) is holomorphic.

**Proof.** The volume of the fiber \( L_t \) is independent of \( t \), due to that \( L_t \) is special Lagrangian in \( M \), and without loss of any generality we may assume the fiber has unit volume by re-normalization. The \( \mathbb{C} \)-valued 1-form \( A \) satisfies

\[
F_{(i+n)(j+n)} = \frac{\partial Y^i}{\partial s_j} - \frac{\partial Y^j}{\partial s_i} = 0
\]

since \( Y^i \)'s are free of \( s \),

\[
F_{ij} = \frac{\partial X^i}{\partial t_j} - \frac{\partial X^j}{\partial t_i} = \frac{\partial \alpha^i}{\partial t_j} - \frac{\partial \alpha^j}{\partial t_i} = 0
\]

since \( \alpha \) is flat, and finally

\[
F_{i(i+n)} - F_{j(i+n)} = \frac{\partial X^i}{\partial s_j} - \frac{\partial Y^j}{\partial s_i} - \frac{\partial X^j}{\partial s_i} + \frac{\partial Y^i}{\partial t_j} = \frac{\partial Y^i}{\partial t_j} - \frac{\partial Y^j}{\partial t_i} = \sum_k \mu_{ki} \frac{\partial [Y]_k^i}{\partial t_j} - \sum_k \mu_{kj} \frac{\partial [Y]_k^i}{\partial t_i} = 0
\]

because \( \Sigma \) is Lagrangian. We conclude that \( A \) is holomorphic and satisfies (5.20) and (5.21). \( \square \)

6. Deformations of \( U(1) \)-connections and Lagrangian cycles.

In this section, we first investigate how deformation of holomorphic \( U(1) \)-connections \( A \) on \( E \) affects the Lagrangian cycles \( \Sigma \) and the flat \( U(1) \)-connections \( \alpha \) on the mirror side constructed in the last section. Then we examine how the \( U(1) \)-connection on \( E \) constructed from \((\Sigma, \alpha)\) varies if \( \Sigma \) and \( \alpha \) are deformed.
6.1. Deforming $A$ by the complex gauge groups. Let $A'$ be a $U(1)$-connection on $E$ which is $C$-gauge equivalent to $A$:

\[
A' = A + \sqrt{-1}(\bar{\partial} - \partial)u + (\bar{\partial} + \partial)v
\]

where $u$ and $v$ are $\mathbb{R}$-valued functions on $M$. If $A$ is holomorphic, so is $A'$. If $u = 0$, then $A'$ is gauge equivalent to $A$. It is straightforward to verify

\[
\sqrt{-1}(\bar{\partial} - \partial)u = -\sum_i \left( \frac{\partial u}{\partial t_i} ds_i - \frac{\partial u}{\partial s_i} dt_i \right)
\]

\[
(\bar{\partial} + \partial)v = \sum_i \left( \frac{\partial v}{\partial s_i} ds_i + \frac{\partial v}{\partial t_i} dt_i \right).
\]

The fiberwise component $Y'$ of $A'$ is given by

\[
Y' = Y + \sqrt{-1}\sum_i \left( \frac{\partial u}{\partial t_i} + \frac{\partial v}{\partial s_i} \right) ds_i.
\]

The Hodge decomposition yields

\[
\sum_i \left( \frac{\partial u}{\partial t_i} + \frac{\partial v}{\partial s_i} \right) ds_i = \psi + d^\ast \eta,
\]

where $\psi$ is the sum of a harmonic 1-form and an exact 1-form on $L_t$ and $\eta$ is a 2-form on $L_t$. Hence, the graph of the mapping $t \to [Y'(t, \cdot)]$ defines a Lagrangian submanifold in $\tilde{\mathcal{M}}$:

\[
\Sigma' = \{(t, [Y(t, \cdot)] + [\psi(t, \cdot)]) : t \in B\}.
\]

Let us recall:

**Definition 6.1.** A diffeomorphism $F : (\tilde{M}, \tilde{\omega}) \to (\tilde{M}, \tilde{\omega})$ is called Hamiltonian if there exists a smooth function $H : [0, 1] \times \tilde{M} \to \mathbb{R}$ and a family $f^t, t \in \mathbb{R}$, of symplectic diffeomorphisms of $\tilde{M}$ such that

\[
df^t = X_H(t, f^t)
\]

\[
f^0 = \text{id}
\]

\[
f^1 = F
\]

where the Hamiltonian vector field $X_H$ is determined by

\[
\iota(X_H)\tilde{\omega} = -dH.
\]

We then introduce:

**Definition 6.2.** Two Lagrangian submanifolds $\Sigma$ and $\Sigma'$ in $\tilde{M}$ are Hamiltonian equivalent if there exists a piecewise smooth and continuous family of Hamiltonian diffeomorphisms $F_\sigma : \tilde{M} \to \tilde{M}$ for $\sigma \in [0, 1]$ such that $F_0$ is the identity map and $F_1(\Sigma) = \Sigma'$. The equivalence class determined by $\Sigma$ is denoted by $[\Sigma]$. 

Proposition 6.1. Let $A$ be a holomorphic $U(1)$-connection on the complex line bundle $E$ over the special Lagrangian torus fibration $\tau : M \to B$. Suppose that another $U(1)$-connection $A'$ is $C$-gauge equivalent to $A$. Then $\Sigma = \{(t, [Y(t, \cdot)]) : t \in B\}$ and $\Sigma' = \{(t, [Y'(t, \cdot)]) : t \in B\}$ are Hamiltonian equivalent Lagrangian submanifolds in $\tilde{M}$. If $A'$ is gauge equivalent to $A$, then $\Sigma = \Sigma'$.

Proof. For each $t \in B$, we connect the two points $[Y(t)]$ and $[Y(t) + \psi(t)]$ by a path in the torus $L_t = H^1(L_t, R/\mathbb{Z})$. On the universal covering $H^1(L_t, R)$, the path may be taken to be $C_t(\tau) = [Y(t)] + \tau[\psi(t)]$ for $\tau \in [0, 1]$. Along the path $C_t$, the infinitesimal deformation vector field is equals to

$$\sum_k [\psi]_k^j \frac{\partial}{\partial x_k} = \sum_k \left( \mu^{kj} \int_{L_t} \psi_j d\mu_{L_t} \right) \frac{\partial}{\partial x_k}$$

$$= \sum_{k,j} \left( \mu^{kj} \int_{L_t} \left( \frac{\partial u}{\partial t_j} + \frac{\partial v}{\partial s_j} - (d^* \eta)^j \right) d\mu_{L_t} \right) \frac{\partial}{\partial x_k}$$

$$= \sum_{k,j} \left( \mu^{kj} \int_{L_t} \left( \frac{\partial u}{\partial t_j} d\mu_{L_t} + dv \wedge * t \left( \frac{\partial}{\partial t_j} \right) \omega - d^* \eta \wedge * t \left( \frac{\partial}{\partial t_j} \right) \omega \right) \right) \frac{\partial}{\partial x_k}$$

Integrating (6.5) yields the transformation from $\Sigma$ to $\Sigma'$. To construct the Hamiltonian deformation, it suffices to assume that $[\psi]$ is small enough so that $\Sigma'$ stays inside a tubular neighborhood $O$ of $\Sigma$. We may further assume that the closure of $O$ is contained in some larger tubular neighborhood $O'$ of $\Sigma$. The function $\int_{L_t} u d\mu_{L_t}$ is defined globally on $B$ hence may be viewed as a function on $\Sigma'$. On the mirror space $\tilde{\pi} : \tilde{M} \to B$, introduce a function $H : [0, 1] \times \tilde{M} \to \mathbb{R}$ as follows. If $t = \tilde{\pi}(y)$ for $y \in \tilde{M}$ set

$$H(\sigma, y) = \sigma h(y) \int_{L_t} u d\mu_{L_t}$$

where $h$ is a cut-off function on $\tilde{M}$ which equals to one on $O$ and zero outside $O'$. $H$ determines a Hamiltonian vector field $X_H$ for each fixed $\sigma$ by

$$\tilde{\omega}(X_H, \cdot) = -dH.$$

Using Lemma 3.2, we get

$$\tilde{\omega} \left( \sum_{j,k} \left( \mu^{kj} \int_{L_t} \frac{\partial u}{\partial t_j} d\mu_{L_t} \right) \frac{\partial}{\partial x_k} \right) = \sum_j \left( \int_{L_t} \frac{\partial u}{\partial t_j} d\mu_{L_t} \right) dt_j$$

$$= \int d\int_{L_t} u d\mu_{L_t}.$$
In particular, it follows that

\[ X_H|_O = \sigma \sum_{j,k} \left( \mu^{kj} \int_{L_t} \frac{\partial u}{\partial t_j} d\mu_{Lt} \right) \frac{\partial}{\partial x_k}. \]  

(6.7)

The Cauchy problem

\[ \frac{dF_\sigma}{d\sigma} = X_H(F_\sigma) \]  

(6.8)

\[ F_0 = id \]  

(6.9)

has a unique solution \( F_\sigma \). For each fixed parameter \( \sigma \), \( F_\sigma : \tilde{\Sigma} \rightarrow \tilde{\Sigma} \) is a Hamiltonian diffeomorphism and \( F_\sigma \) is equal to the identity map outside \( O' \) for any \( \sigma \). As \( \sigma \) moving from 0 to 1, any point \( p \) in \( \Sigma \) evolves along the curve \( F_\sigma(p) \) with velocity \( X_H \) evaluated at the point \( F_\sigma(p) \), and especially \( F_1(p) \in \Sigma' \).

Now we conclude that \( \Sigma \) and \( \Sigma' \) belong to the same Hamiltonian class, and moreover that if \( u = 0 \) then \( H \equiv 0 \) hence \( \Sigma = \Sigma' \).

\[ \square \]

Proposition 6.2. If \( A' \) is \( \mathbb{C} \)-gauge equivalent to \( A \), then \( \alpha' = \int_{L_t} A' \wedge \text{Re}\Omega \) differs from \( \alpha = \int_{L_t} A \wedge \text{Re}\Omega \) by an exact 1-form. In fact, for \( A' = A + \sqrt{-1} (\bar{\partial} - \partial)u + (\bar{\partial} + \partial)v, v = 0 \) implies \( \alpha' = \alpha \) and \( u = 0 \) implies that \( \alpha' - \alpha \) is an exact 1-form.

Proof. The 1-form defined by \( A' \) is

\[ \alpha' = \int_{L_t} A' \wedge \text{Re}\Omega \]

(6.10)

\[ = \alpha - \sqrt{-1} \sum_i \left( \int_{L_t} \frac{\partial u}{\partial s_i} d\mu_{Lt_i} \right) dt_i + \sqrt{-1} \sum_i \left( \int_{L_t} \frac{\partial v}{\partial t_i} d\mu_{Lt_i} \right) dt_i \]

\[ = \alpha - \sqrt{-1} \sum_i \left( \int_{L_t} d_{Lt_i} u \wedge \ast t_i \left( \frac{\partial}{\partial t_i} \omega \right) \right) dt_i + \sqrt{-1} \sum_i \left( \int_{L_t} \frac{\partial v}{\partial t_i} d\mu_{Lt_i} \right) dt_i \]

\[ = \alpha + \sqrt{-1} dt \left( \int_{L_t} v d\mu_{Lt} \right). \]

This completes the proof.

\[ \square \]

6.2. Deforming \( \Sigma \) in its Hamiltonian class and \( \alpha \) by the gauge group. To understand the effects of deformations of the pair \( (\Sigma, \alpha) \) on the reconstructed holomorphic connection \( A \), we first observe:

Proposition 6.3. The representatives of the cohomology class \([\alpha]\) yield gauge equivalent holomorphic connections \( A \).
Proof. If \( \alpha' = \alpha + dt \) for some function \( f \) on \( \Sigma \), the \( s \)-independent connection corresponds to \( \alpha' \) is
\[
A' = A + dt f = A + df.
\]
(6.11)
Thus \( A' \) is gauge equivalent to \( A \). \( \square \)

Next, we have:

**Proposition 6.4.** Assume that \( \Sigma' \) is in the Hamiltonian class determined by \( \Sigma \). Then \( A' \) induced from \( (\Sigma', \alpha) \) is \( C \)-gauge equivalent to \( A \) induced from \( (\Sigma, \alpha) \).

*Proof.* Let \( F_\sigma \) be the continuous family of Hamiltonian diffeomorphisms which deforms \( \Sigma \) to \( \Sigma' \). Take a finite collection of numbers \( 0 = \sigma_0 < \sigma_1 < \ldots < \sigma_m = 1 \) such that \( \Sigma_{\sigma_{i+1}} \) is contained in some open tubular neighborhood \( U_i \) of \( \Sigma_i \) in \( \tilde{M} \), and \( F_\sigma \) is smooth in \( \sigma \) for \( \sigma \in [\sigma_i, \sigma_{i+1}] \). Note that the Hamiltonian deformation classes of the Lagrangian section \( \Sigma_i \) are given by \( H^1(\Sigma_i, R) \cong H^1(B, R) \). Hence there exists a family of symplectic diffeomorphisms \( F_\sigma^{(i)} : U_i \rightarrow U_i \) which is Hamiltonian locally:
\[
\frac{dF_\sigma^{(i)}}{d\sigma} = X_{h^{(i)}}(F_\sigma^{(i)})
\]
(6.12)
for some function \( h^{(i)} \) on \( [\sigma_i - \epsilon, \sigma_{i+1} + \epsilon] \times U_i \) for some small \( \epsilon > 0 \) and \( h^{(i)} \) depends only on \( t \in B \), subject to
\[
\iota(X_{h^{(i)}}) \omega = -dh^{(i)}.
\]
It is straightforward to check
\[
X_{h^{(i)}} = \sum_{j,k} \mu^{jk}_i \frac{\partial h^{(i)}}{\partial t_j} \frac{\partial}{\partial x_k} - \sum_{j,l} \mu^{ij}_l \frac{\partial h^{(i)}}{\partial x_j} \frac{\partial}{\partial t_l}.
\]
(6.13)
That \( h^{(i)} \) depends only on \( t \) implies the second term above vanishes. It then follows from Proposition 3.3, (6.12) and (6.13) by taking the \( k \)th components of the corresponding vector fields that
\[
\int_{L_t} Y^k_{\sigma_{i+1}} \, d\mu_{L_t} - \int_{L_t} Y^k_{\sigma_i} \, d\mu_{L_t} = \sum_{j} \mu_{jk} ([Y_{\sigma_{i+1}}]^j - [Y_{\sigma_i}]^j)
\]
\[
= \sum_{j,l} \mu^{ij}_l \int_{\sigma_i}^{\sigma_{i+1}} \frac{\partial h^{(i)}}{\partial t_l} \, dt_l
\]
\[
= \frac{\partial}{\partial t_k} \int_{\sigma_i}^{\sigma_{i+1}} h^{(i)}.
\]
Extend the function \( h^{(i)} \) globally by introducing
\[
H^{(i)} = h^{(i)} \xi_i
\]
for some smooth cut-off function $\xi_i$ which equals 0 outside $U_i$ and equals 1 on a smaller open tubular neighborhood $U'_i \subset U_i$ of $\Sigma_i$ which still contains $\Sigma_{i+1}$.

Now we construct a 1-form $A^{(i+1)}$ by setting

\begin{equation}
A^{(i+1)} = X + Y_{\sigma_i} + \left( \frac{\partial}{\partial t_k} \int_{\sigma_i}^{\sigma_{i+1}} H^{(i)} \right) ds_k
\end{equation}

\[= A^{(i)} + \left( \frac{\partial}{\partial t_k} \int_{\sigma_i}^{\sigma_{i+1}} H^{(i)} \right) ds_k.\]

It is easy to see that $A^{(i+1)}$ is $C$-gauge equivalent to $A^{(i)}$. Repeat this procedure for each $i$. Finally, we obtain

\[A' = A^{(n)} = A + \left( \frac{\partial}{\partial t_k} \sum_{i=1}^{m} \int_{\sigma_i}^{\sigma_{i+1}} H^{(i)} \right) ds_k.\]

This implies that $A'$ is $C$-gauge equivalent to $A$. □

Summing up our discussion, we obtain Theorem 1.1.

References


Received May 16, 2000 and revised September 12, 2000. The author is supported partially by a NSERC grant and an Alfred P. Sloan Research Fellowship.

DEPARTMENT OF MATHEMATICS
THE UNIVERSITY OF BRITISH COLUMBIA
VANCOUVER, B.C.
CANADA V6T 1Z2
E-mail address: jychen@math.ubc.ca
POLYNOMIALS WITH GENERAL $C^2$–FIBERS ARE VARIABLES

SH. KALIMAN

Let $X'$ be a complex affine algebraic threefold with $H_3(X') = 0$ which is a UFD and whose invertible functions are constants. Let $Z$ be a Zariski open subset of $X'$ which has a morphism $p : Z \to U$ into a curve $U$ such that all fibers of $p$ are isomorphic to $C^2$. We prove that $X'$ is isomorphic to $C^3$ iff none of irreducible components of $X' \setminus Z$ has non-isolated singularities. Furthermore, if $X'$ is $C^3$ then $p$ extends to a polynomial on $C^3$ which is linear in a suitable coordinate system. This implies the fact formulated in the title of the paper.

1. Introduction.

A nonconstant polynomial on $C^n$ is a variable if it is linear in a suitable polynomial coordinate system on $C^n$. In 1961 Gutwirth [Gu] proved the following fact which was later reproved by Nagata [Na]: Every polynomial $p \in C^{[2]}$ whose general fibers are isomorphic to $C$ is a variable. In 1974–1975 Abhyankar, Moh, and Suzuki showed that a much stronger fact holds: Every irreducible polynomial $p \in C^{[2]}$ with $p^{-1}(0) \simeq C$ is variable [AbMo], [Su]. The Embedding conjecture formulated by Abhyankar and Sathaye [Sa1] suggests that the similar fact holds in higher dimensions:

*Every irreducible polynomial $p \in C^{[n]}$ with $p^{-1}(0) \simeq C^{n-1}$ is a variable.*

It seems that in the full generality the positive answer to the Embedding conjecture is not feasible in the near future but there is some progress for $n = 3$. In this dimension A. Sathaye, D. Wright, and P. Russell proved some special cases of this conjecture ([Sa1], [Wr], [RuSa], see also [KaSa1]). Then M. Koras and P. Russell proved the Linearization conjecture for $n = 3$ [KoRu2], [KaKoM-LRu] which implies the following theorem: If $p$ is an irreducible polynomial on $C^3$ such that it is quasi-invariant with respect to a regular $C^*$-action on $C^3$ and its zero fiber is isomorphic to $C^2$, then $p$ is a variable.\(^1\) This paper and paper [KaSa2] contain another step in the direction of the Embedding conjecture – we prove the analogue of the

\(^1\)In fact, P. Russell indicated to the author that the “hard-case” of the Linearization conjecture is equivalent to this theorem. This equivalence can be extracted from [KoRu1].
Gutwirth theorem in dimension 3, i.e., every polynomial with general $C^2$-fibers is a variable. It is worth mentioning that a special case of this theorem (when additionally all fibers are UFDs and the generic fiber is a plane) follows from more general results of Miyanishi [Miy1] and Sathaye [Sa2]. In fact, in our paper the analogue of the Gutwirth theorem in dimension 3 is also a consequence of the following more general result.

**Main Theorem.** Let $X'$ be an affine algebraic variety of dimension 3 such that

1. $X'$ is a UFD and all invertible functions on $X'$ are constants;
2. $X'$ is smooth and $H_3(X') = 0$;
3. there exists a Zariski open subset $Z$ of $X'$ and a morphism $p : Z \to U$ into a curve $U$ whose fibers are isomorphic to $C^2$;
4. each irreducible component of $X' \setminus Z$ has at most isolated singularities.

Then $U$ is isomorphic to a Zariski open subset of $C$ and $p$ can be extended to a regular function on $X'$. Furthermore, $X'$ is isomorphic to $C^3$ and $p$ is a variable.

The same conclusion remains true if we replace (1) and (3) by

1. the Euler characteristic of $X'$ is $e(X') = 1$;
3. each irreducible component of $X' \setminus Z$ is a UFD.

In the case when conditions (1) and (2) hold but (3') does not, $X'$ is an exotic algebraic structure on $C^3$ (that is, $X'$ is diffeomorphic to $R^6$ as a real manifold but not isomorphic to $C^3$) with a nontrivial Makar-Limanov invariant.

The Makar-Limanov invariant was introduced in [M-L1], [KaM-L1] (see also [KaM-L2], [Za], and [De]). For a reduced irreducible affine algebraic variety $X'$ this invariant is the subalgebra $ML(X')$ of the algebra of regular functions $C[X']$ on $X'$ that consists of all functions which are invariant under any regular $C^*_+$-action on $X'$. If $ML(X') \cong C$ then we call it trivial. This is so, for instance, when $X' \cong C^n$.

The proof of the Main Theorem consists of three Lemmas.

**Lemma I** (cf. [Miy1]). Let $X'$ be an affine algebraic variety of dimension 3 which satisfies assumption (0), (1), and (3) from the Main Theorem and

1. there exists a Zariski open subset $Z$ of $X'$ which is a $C^2$-cylinder over a curve $U$ (i.e., $Z$ is isomorphic to the $C^2 \times U$).

Then $U$ can be viewed as a subset of $C$, $X'$ is isomorphic to $C^3$, and the natural projection $Z \to U$ can be extended to a regular function on $X'$ which is a variable.

Miyanishi’s theorem (which can be also proved by the technique we develop below) claims the same fact with assumptions (1) and (3) replaced by (1') and (3'). The idea of the proof of Lemma I is as follows. Let $\sigma : X' \to X$
be an affine modification. The restriction \( \sigma \) to the complement of the exceptional divisor \( E \) of \( X' \) is an isomorphism between \( X' \setminus E \) and \( X \setminus D \) where \( D \) is a divisor of \( X \). We show that under the assumption of Lemma I \( X' \) is an affine modification of \( X = \mathbb{C}^3 \) and the divisor \( D \) is the union of a finite number of parallel affine planes in \( \mathbb{C}^3 \). Then the problem is reduced to the case when \( D \) consists of one plane only. We consider the set of so-called basic modification which preserve normality, contractibility, and for which \( C_0 = \sigma(E) \) is closed in \( X \) and \( E \) is naturally isomorphic to \( \mathbb{C}^k \times C_0 \). One of the central facts (Theorem 3.1) says that \( \sigma \) is the composition \( \sigma_1 \circ \cdots \circ \sigma_m \) where each \( \sigma_i : X_i \to X_{i-1} \) \((X' = X_m \) and \( X = X_0)\) is a basic modification. If \( m = 1 \) and \( C_0 \) is either a point or a straight line in \( D \), then it is easy to check that \( X \simeq \mathbb{C}^3 \) and the other statements of the Lemma I hold. When \( m > 1 \), using the control over topology, one can show that the center of \( \sigma_1 \) is either a point or a curve in \( D \) which is isomorphic to \( \mathbb{C} \). If the center is such a curve then it can be viewed as a straight line by \([\text{AbMo, Su}]\) whence \( X_1 \) is isomorphic to \( \mathbb{C}^3 \). Now the induction by \( m \) yields Lemma I.

**Lemma II.** Let \( X' \) be an affine algebraic variety of dimension 3 which satisfies assumptions (0), (1), and (2'), but does not satisfy assumption (3'). Then \( X' \) is an exotic algebraic structure on \( \mathbb{C}^3 \) with a nontrivial Makar-Limanov invariant.

Under the assumption of Lemma II \( X' \) is still an affine modification of \( X = \mathbb{C}^3 \); \( \sigma \) is still a composition of basic modifications, and one can reduce the problem to the case when \( D \) is a coordinate plane. It can be shown that \( C_0 \) is either a point or an irreducible contractible curve in \( D \). The remarkable Lin-Zaidenberg theorem \([\text{LiZa}]\) says that such a curve is given by \( x^n = y^m \) in a suitable coordinate system where \( n \) and \( m \) are relatively prime. This allows us to present explicitly a system of polynomial equations in some Euclidean space \( \mathbb{C}^N \) whose zero set is \( X' \). Here we use the fact that basic modifications of Cohen-Macaulay varieties are Davis modifications which were introduced in \([\text{KaZa1}]\) and which fit perfectly the aim of presenting explicitly the result of a modification as a closed affine subvariety of a Euclidean space. This explicit presentation of \( X' \) as a subvariety of \( \mathbb{C}^N \) enables us to compute \( \text{ML}(X') \), using the technique from \([\text{KaM-L1}], [\text{KaM-L2}]\). If condition (3') does not hold then \( \text{ML}(X') \neq \mathbb{C} \) whence \( X' \neq \mathbb{C}^3 \). We show also that \( X' \) is contractible whence it is diffeomorphic to \( \mathbb{R}^6 \) by the Dimca-Ramanujam theorem \([\text{ChDi}]\) which concludes Lemma II.

The Main Theorem follows from Lemmas I, II, Miyanishi’s theorem, and:

**Lemma III** ([\text{KaZa2}]). Assumptions (2) and (2') are equivalent.

**Acknowledgments.** It is our pleasure to thank M. Zaidenberg for his suggestion to check Lemma II and many fruitful discussions. Actually, the idea of this paper arose during the joint work of the author and M. Zaidenberg on
Later M. Zaidenberg decided not to participate in the project due to other obligations and the author had to finish it alone. It is also our pleasure to thank I. Dolgachev, P. Russell, and J. Lipman whose consultations were very useful for the author.

2. Affine modifications.

2.1. Notation and terminology. In this subsection we present central definitions and notation which will be used in the rest of the paper. The ground field in this paper will always be the field of complex numbers \( \mathbb{C} \).

Definition 2.1. Let \( X \) be a reduced irreducible affine algebraic variety, \( A = \mathbb{C}[X] \) be its algebra of regular functions, \( I \) be an ideal in \( A \), and \( f \in I \setminus \{0\} \). By the affine modification of \( A \) with locus \((I, f)\) we mean the algebra \( A' := A[I/f] \) together with the natural embedding \( A \hookrightarrow A' \). That is, if \( b_0, b_1, \ldots, b_s \) are generators of \( I \) then \( A' \) is the subalgebra of the field of fractions of \( A \) which is generated over \( A \) by the elements \( b_1/f, \ldots, b_s/f \).

It can be easily checked that \( A' \) is also an affine domain, i.e., its spectrum \( X' \) is an affine algebraic variety and the natural embedding \( A \hookrightarrow A' \) generates a morphism \( \sigma : X' \to X \). Sometimes we refer to \( \sigma \) as an affine modification or we say that \( X' \) is an affine modification of \( X \). The reduction \( D \) (resp. \( E \)) of the divisor \( f^*(0) \subset X \) (resp. \((f \circ \sigma)^*(0) \subset X'\)) will be called the divisor (resp. the exceptional divisor) of the modification. The (reduction of the) subvariety of \( X \) defined by \( I \) will be called the (reduced) center of the modification and \( \sigma(E) \) will be called the geometrical center of modification.

Definition 2.2. A morphism \( p : Y \to Z \) of algebraic varieties is called a \( \mathbb{C}^s \)-cylinder over \( Z \) if there exists an isomorphism \( \varphi : Y \to \mathbb{C}^s \times Z \) so that \( p \circ \varphi^{-1} \) is the projection to the second factor. Let \( \sigma(E) \) be an algebraic variety of pure dimension where \( \sigma \) and \( E \) are from Definition 2.1. We say that \( \sigma \) is a cylindrical modification of rank \( s \) if \( \sigma|_E : E \to \sigma(E) \) is a \( \mathbb{C}^s \)-cylinder where \( s + 1 = \text{codim}_X \sigma(E) \).

Definition 2.3. A sequence of generators \( b_0, \ldots, b_s \) of an ideal \( I \) of \( A = \mathbb{C}[X] \) is called semi-regular if the height of \( I \) is \( s + 1 \). If in addition \( b_0 = f \) then the affine modification \( A \hookrightarrow A' \) with locus \((I, f)\) is called semi-basic of rank \( s \). Furthermore, if \( C \) is the set of the common zeros of \( I \) in \( X \) this semi-regular sequence is called an almost complete intersection when every irreducible components \( G \) of \( C \) meets \text{reg} \( X \) and \( G \cap \text{reg} X \) contains a Zariski open subset which is a complete intersection given by \( b_0 = \cdots = b_s = 0 \). If, in addition, \( b_0 = f \) then the affine modification \( A \hookrightarrow A' \) is basic of rank \( s \).

Let \( S \) be a multiplicative system of \( A \) and \( S^{-1}A \) (resp. \( S^{-1}A' \)) the ring of fractions of \( A \) (resp. \( A' \)) with respect to \( S \). Every ideal \( I \) in \( A \) generates
an ideal $S^{-1}I$ in $S^{-1}A$. The following fact is an immediate consequence of
the definitions of affine modifications and rings of fractions.

**Proposition 2.1.** In the notation above we have $S^{-1}A' = (S^{-1}A)[S^{-1}I/f]$.

**Definition 2.4.** Suppose that $B$ is a localization of an affine domain, $J$ is an ideal in $B$ and $g \in B \setminus \{0\}$. By the local modification of $B$ with locus $(J, g)$ we mean the algebra $B' := B[J/g]$ together with the natural embedding $B \hookrightarrow B'$. By Proposition 2.1 $B \hookrightarrow B'$ can be obtained by the operation of taking fractions of an affine modification $A \hookrightarrow A'$. We call $B \hookrightarrow B'$ semi-basic (resp. basic) if the affine modification $A \hookrightarrow A'$ can be chosen semi-basic (resp. basic).

**Definition 2.5.** Let $A_M$ be the localization of an affine domain $A$ at a maximal ideal $M$, and $I_M$ be the extension of an ideal $I \subset A$ in $A_M$. An affine modification $A \hookrightarrow A'$ is locally semi-basic (resp. basic) if for every maximal ideal $M$ that vanishes at a point of the geometrical center the local modification $A_M \hookrightarrow A_M[I_M/f] = S^{-1}A'$ is semi-basic (resp. basic) where $S = A \setminus M$.

**Convention 2.1.** The algebra of regular functions of an affine algebraic variety $Y$ will be denoted by $C[Y]$. Further in this paper $X$ and $X'$ are always reduced irreducible affine algebraic varieties, $A = C[X]$, and $A' = C[X']$. We suppose that the notation $A \hookrightarrow A'$ is fixed throughout the paper. It will always mean an affine modification with locus $(I, f)$. The corresponding morphism is always denoted by $\sigma : X' \rightarrow X$. The divisor, the exceptional divisor, and the reduced center of the modification are always denoted by $D$, $E$, and $C$ respectively.

We shall also use the following notation in the rest of this section: If $Y$ is an affine algebraic variety and $B = C[Y]$ then for every closed algebraic subvariety $Z$ of $Y$ the defining ideal of $Z$ in $B$ will be denoted by $I_B(Z)$. For every ideal $J$ in $C[Y]$ we denote by $V_Y(J)$ the zero set of this ideal in $Y$.

**2.2. General facts about affine modifications.** The ideal $K = \{a \in A | a/f \in A'\}$ in $A$ is called the $f$-largest ideal of modification $A \hookrightarrow A'$. Clearly, $I \subset K$ and $A' = A[K/f]$. When $A$ and $A'$ are fixed we denote $K$ by $I_f$. Nullstellensatz implies that the geometrical center of an affine modification is always contained in the reduced center, in the case of an $f$-largest ideal one can see that it implies more.

**Proposition 2.2.** Let $A \hookrightarrow A'$ be an affine modification such that $I = I_f$. Then the reduced center of the modification is the closure of the geometrical one.
Proposition 2.3. Let \( g \in A \setminus \{0\} \) and \( f = g^n \) for a natural \( n \). Suppose that \( \mathcal{I}_A(E) \) coincides with the principal ideal in \( A' \) generated by \( g \). Then \((\mathcal{I}_A(C))^n \subset I_f \) (in particular, for \( n = 1 \) we have \( \mathcal{I}_A(C) = I_f \)). Furthermore, for every ideal \( J \) in \( A \) which is contained in \( \mathcal{I}_A(C) \) the algebra \( A_1 := A[J/g] \) is contained in \( A' \).

Proof. Note that for every \( a \in (\mathcal{I}_A(C))^n \) we have \( a \in (\mathcal{I}_A(E))^n \) whence \( a/f \in A' \). Thus \( a \in I_f \) which is the first statement. This implies that \( g^{n-1} J \subset I_f \). Hence \( A_1 = A[g^{n-1} J/f] \subset A[I_f/f] = A' \). □

Corollary 2.1 (cf. [WaWe, Prop. 1.2]). Suppose that \( J = \mathcal{I}_A(C) \) in Proposition 2.3. Then there exists an ideal \( K_1 \) in \( A_1 \) such that \( A' = A_1[K_1/g^{n-1}] \). That is, \( A_1 \hookrightarrow A' \) may be viewed as an affine modification with locus \((K_1,g^{n-1})\).

Proof. Let \( b_0 = g^n, b_1, \ldots, b_s \) be generators of \( I \). Note that \( b_i/g \in A_1 \) for every \( i \). The ideal \( K_1 \) in \( A_1 \) generated by \( g^{n-1}, b_1/g, \ldots, b_s/g \) is the desired ideal. □

Proposition 2.4. Let \( B \) be a UFD, \( J \) be an ideal in \( B, g \in J \setminus 0 \) be irreducible in \( B \), \( f = g^n \), \( B' = B[J/f] \), and \( B' \neq B[1/f] \). Then \( g \) is irreducible in \( B' \).

Proof. Let \( g^k = a'b' \) where \( a' = a/f^l, b' = b/f^m, a \in J^l, \text{ and } b \in J^m \). Hence \( g^{k+nl+nm} = ab \) in \( B \). Since \( B \) is a UFD we have \( a = ug^s \) and \( b = vg^r \) where \( s + r = k + nl + nm \) and \( u, v \) are units. If \( s < nl \) then \( a' = u/g^nl-s \) whence \( 1/f \in B' \) which contradicts \( B' \neq B[1/f] \). Thus \( s \geq nl \) and, similarly, \( r \geq nm \). Hence \( a' = ug^{s-nl} \) and \( b' = vg^{r-nm} \) are in \( B \) whence \( g \) is irreducible in \( B' \). □

Proposition 2.5. Let \( A \hookrightarrow A' \) be an affine modification and \( \forall k > 0 \) each divisor \( g \in A \) of \( f^k \) is not a unit in \( A' \) (i.e., \( (g \circ \sigma)^{-1}(0) \) is not empty). Then the units of \( A' \) and \( A \) are the same.

Proof. Since \( A' \) is a subalgebra of \( A[1/f] \) its units are also units of \( A[1/f] \). The units of the last algebra are the products of irreducible divisors of \( f^k \) and the units of \( A \). By the assumption these divisors are not invertible functions on \( X' \) whence the units of \( A' \) coincide with the units of \( A \). □

Proposition 2.6. Let \( I_j \) be an ideal in \( A \) for \( j = 1, \ldots, k \), and let \( f_j \in I_j \) \( \setminus \{0\} \). Suppose that \( f = f_1 \cdots f_k \) and \( I = (f/f_1)I_1 + \cdots + (f/f_k)I_k \). Let \( A_j = A[I_j/f_j] \) and let \( \delta_j : X_j \rightarrow X \) be the morphism of affine algebraic varieties associated with the affine modification \( A \hookrightarrow A_j \) with locus \((I_j,f_j)\). Suppose that \( E_j \) is the exceptional divisor of this modification. These morphisms define the affine variety \( Y = X_1 \times_X X_2 \times_X \cdots \times_X X_k \) and its subvariety \( Y^* = (X_1 \setminus E_1) \times_X \cdots \times_X (X_k \setminus E_k) \).
(1) $X'$ is isomorphic to the closure $\overline{Y^*}$ of $Y^*$ in $Y$ and under this isomorphism $\sigma$ coincides with the restriction of the natural projection $\tau : Y \to X$ to $\overline{Y^*}$.

(2) If $\forall j \neq l f_j$ and $f_l$ have no common zeros on $X$ then $X' \simeq Y$.

Proof. Let $D_j = f_j^{-1}(0)$. Then $D = \bigcup_{j=1}^{k} D_j$. As $\delta_j|_{X_j \setminus E_j}$ is an isomorphism between $X_j \setminus E_j$ and $X \setminus D_j$ we see that $Y^*$ is isomorphic to $X \setminus D$. Thus $B := \mathbb{C}[Y^*]$ is a subalgebra in the field of fractions of $A$. The natural projection $\overline{Y^*} \to X_j$ enables us to treat $A_j$ as a subalgebra of $B$. Note that $A_1, \ldots, A_k$ generate $B$ and $A'$ as $I = (f/f_1)I_1 + \ldots + (f/f_k)I_k$. Hence $A' = B$ which yields (1). For (2) it suffices to prove that $Y$ is irreducible. Assume that $Y$ has an irreducible component $Y_1 \neq Y^*$. Note that $\tau(Y_1) \subset D$ as $\tau^{-1}|_{X \setminus D}$ is an isomorphism between $X \setminus D$ and $Y^*$. We can suppose that $\tau(Y_1) \subset D_1$. Put $T = \bigcup_{j=2}^{k} D_j$ and consider $\theta : Y \setminus \tau^{-1}(T) \to X \setminus T$ where $\theta$ is the restriction of $\tau$. As for $j \geq 2$ the restriction of $\delta_j$ to $X_j \setminus \delta_j^{-1}(T)$ is an isomorphism between this variety and $X \setminus T$ we see that $Y \setminus \tau^{-1}(T)$ is isomorphic to $X_1 \setminus \delta_1^{-1}(T)$ and $\theta$ coincides with the restriction of $\delta_1$ to $X_1 \setminus \delta_1^{-1}(T)$ under this isomorphism. Thus $\delta_1^{-1}(X \setminus T) \simeq \tau^{-1}(X \setminus T)$. As $T$ does not meet $D_1$, $\tau^{-1}(X \setminus T)$ contains $Y_1$, i.e., it is not irreducible. But $\delta_1^{-1}(X \setminus T) \subset X_1$ is irreducible. Contradiction. □

Remark 2.1. We shall need the coordinate interpretation of Proposition 2.6 (2). Suppose for simplicity that $k = 2$. Let $X$ be a closed affine subvariety of $\mathbb{C}^n$ with a coordinate system $\bar{x}$ and let $X_j$ be a closed affine subvariety of $\mathbb{C}^{n_j}$ with a coordinate system $(\bar{x}, \bar{z}_j)$. Suppose that $X_j$ coincides with the zeros of a polynomial system of equations $P_j(\bar{x}, \bar{z}_j) = 0$ and $\sigma_j$ can be identified with the restriction of the natural projection $\mathbb{C}^{n_j} \to \mathbb{C}^n$. Consider the space $\mathbb{C}^{n_1 + n_2 - n}$ with coordinates $(\bar{x}, \bar{z}_1, \bar{z}_2)$. Then Proposition 2.6 (2) implies that the zero set of the system $P_1(\bar{x}, \bar{z}_1) = P_2(\bar{x}, \bar{z}_2) = 0$ in this space is isomorphic to $X'$.

2.3. Semi-basic modifications.

Lemma 2.1. Let $Z$ be a closed reduced subvariety of $X$ of codimension $s+1$ and let $J = \mathcal{I}_A(Z)$. Suppose that $f \in J \setminus \{0\}$.

(1) Then $J$ contains a semi-regular sequence $\mathcal{L} = \{f = b_0, \ldots, b_s\}$.

(2) Let this sequence generate an ideal $J_1$. If none of the irreducible components of $Z$ and $f^{-1}(0)$ is contained in $\text{sing} X$ then $\mathcal{L}$ can be chosen so that none of the irreducible components of $\mathcal{V}_X(J_1)$ is contained in $\text{sing} X$.

(3) If (2) holds and the zero multiplicity of $f$ at general points of each irreducible component of $Z$ is 1, then one can choose $\mathcal{L}$ to be an almost complete intersection.
(4) There exists a finite-dimensional subspace $S$ of $J$ such that (1)–(3) hold when $b_1, \ldots, b_s$ are general points of any finite-dimensional subspace of $J$ containing $S$.

The proof of the Lemma is straightforward. The first two statements is just an induction on $s$. For (3) and (4) consider generators $g_0 = f, g_1, \ldots, g_r$ of $J$ and a closed embedding $X \hookrightarrow \mathbb{C}^n$. Let $S$ consist of elements of the form $\sum_{j=0}^r l_j g_j$ where each $l_j$ is the restriction to $X$ of a linear function on $\mathbb{C}^n$. It is easy to see that the Lemma holds with this choice of $S$.

**Proposition 2.7.** Suppose that $A \hookrightarrow A'$ is a semi-basic modification of rank $s > 0$. Then it is a cylindrical modification of rank $s$. Furthermore, the reduced and geometrical centers of this modification coincide.

**Proof.** Let $J_0$ be the maximal ideal in $\mathbb{C}^{s+1} = \mathbb{C}[x_0, x_1, \ldots, x_s]$ that vanishes at the origin $o$ in $\mathbb{C}^{s+1}$. Put $B_0 = \mathbb{C}^{s+1}[J_0/x_0]$ and consider the modification $\mathbb{C}^{s+1} \hookrightarrow B_0$ with locus $(J_0, x_0)$. Then $B_0$ is isomorphic to $\mathbb{C}[x_0, y_1, \ldots, y_s]$ and $x_i = x_0 y_i$ for $i = 1, \ldots, s$. That is, $Z_0 := \text{spec} B_0$ may be viewed as the subvariety of $\mathbb{C}^{2s+1}$ (whose coordinates are $x_0, x_1, \ldots, x_s, y_1, \ldots, y_s$) given by the system of equations $x_i - x_0 y_i = 0$, $i \geq 1$. Let $\rho : \mathbb{C}^{2s+1} \rightarrow \mathbb{C}^{s+1}$ be the natural projection to the first $s + 1$ coordinates. Our modification is nothing but the restriction of $\rho$ to $Z_0$. Its reduced and geometrical centers are $o$ and the exceptional divisor $E_0 = \rho^{-1}(o) \simeq \mathbb{C}^s$. Let $Z = \mathbb{C}^{s+1} \times X$, $B = \mathbb{C}[Z]$, and $J = J_0 B$. Consider the modification $B \hookrightarrow B'$ with locus $(J, x_0)$. By [KaZa1, Cor. 2.1] we see that $Z' := \text{spec} B' = Z_0 \times X$ and the above modification is the restriction $\delta$ to $Z' \subset \mathbb{C}^{2s+1} \times X$ of the natural projection $(\rho, \text{id}) : \mathbb{C}^{2s+1} \times X \rightarrow \mathbb{C}^{s+1} \times X = Z$. In particular, its reduced and geometrical centers are $C^0 = o \times X$ and the exceptional divisor $E_0 = E_0 \times X$. Let $b_0 = f, b_1, \ldots, b_s$ be a semi-regular system of generators of $I$. Consider the embedding $i : X \hookrightarrow Z$ given by the system of equations $x_j - b_j = 0$, $j = 0, \ldots, s$. The restriction of $J$ to $X$ coincides with $I$. By [KaZa1, Cor. 2.1] we have the commutative diagram

$$
\begin{array}{ccc}
X' & \hookrightarrow & Z' \\
\downarrow \sigma & & \downarrow \delta \\
X & \hookrightarrow & Z
\end{array}
$$

where $i' : X' \hookrightarrow Z'$ is a closed embedding. The reduced center of $\sigma$ is $C = C^0 \cap i(X)$, and it is of codimension $s + 1$ in $X$ as $\sigma$ is semi-basic. As $\text{codim}_X E = 1$ each fiber $F$ of $\sigma|_E : E \rightarrow \sigma(E) \subset C$ must be at least of dimension $s$. But $F$ is contained in a fiber $F^0 \simeq C^s$ of $\delta|_{E^0} : E^0 \rightarrow C^0$. Hence $\dim F = s$ and $\sigma(E)$ is dense in $C$. Furthermore, as $i'$ is a closed
embedding $F = F^0$ and $\sigma(E) = C$ whence $E$ is a $C^n$-cylinder over $C$ and the reduced and geometrical centers of $\sigma$ coincide. $\square$

2.4. Davis modifications.

**Theorem 2.1** ([Da], see also [Ei, Ex. 17.14]). Let $f = b_0, b_1, \ldots, b_s$ be generators of an ideal $J$ in a Noetherian domain $B$. Consider the surjective homomorphism

$$\beta : B[s] = B[y_1, \ldots, y_s] \longrightarrow B[J/f] = B[b_1/f, \ldots, b_s/f] \simeq B'$$

where $y_1, \ldots, y_s$ are independent variables and $\beta(y_i) = b_i/f$, $i = 1, \ldots, s$. Denote by $J'$ the ideal of $B[s]$ generated by the elements $L_1, \ldots, L_s \in \ker \beta$ where $L_i = fy_i - b_i$. Then $\ker \beta$ coincides with $J'$ iff $J'$ is a prime ideal. The latter is true, for instance, if the system of generators $b_0 = f, b_1, \ldots, b_s$ of the ideal $J$ is regular.

**Definition 2.6.** Let $B$ be (a localization of) an affine domain. When $J'$ from Theorem 2.1 is prime the (local) affine modification $B \hookrightarrow B'$ with locus $(J, f)$ is called Davis, and $b_0 = f, b_1, \ldots, b_s$ is its representative system of generators.

**Remark 2.2.** It is easy to see that in the case of a nonempty reduced center every (local) affine Davis modification is automatically semi-basic.

**Proposition 2.8.** Let $A \hookrightarrow A'$ be an affine modification, $b_0 = f, b_1, \ldots, b_s$ be a system of generators of $I$, $M$ be a maximal ideal in $A$, and $A_M, I_M$, $S$ be as in Definition 2.5, i.e., $A_M \hookrightarrow S^{-1}A'$ is the local modification with locus $(I_M, f)$. Let for every maximal ideal $M$ this local modification be Davis and $b_0, \ldots, b_s$ be a representative system of generators. Then $A \hookrightarrow A'$ is a Davis modification.

**Proof.** Let $Y = C^n \times X$, i.e., $C[Y] = A[s] = A[y_1, \ldots, y_s]$. Let $I'$ be the ideal in $A[s]$ generated by $L_i = y_if - b_i$, $i = 1, \ldots, s$ and $Y_1 = V_Y(I')$. Show that $I'$ is prime, i.e., $Y_1$ is reduced irreducible. Choose a maximal ideal $M'$ in $A[s]$ which vanishes at $x' \in Y_1$. Let $x$ be the image of $x'$ in $X$ under the natural projection and let $M$ be the maximal ideal of $A$ that vanishes at $x$. Then $A \setminus M \subset A[s] \setminus M'$ and $A_M[s]$, is a further localization of $S^{-1}A[s]$. Since $A_M \hookrightarrow S^{-1}A'$ is a Davis modification and $b_0, \ldots, b_s$ is a representative system of generators of this modification, the ideal $S^{-1}I'$ is prime in $S^{-1}A[s]$ whence the localization $I'_M$ of this ideal is also prime, i.e., the germ of $Y_1$ at $x'$ is reduced irreducible. It remains to show that $Y_1$ is connected. As $E_1 = Y_1 \cap f^{-1}(0) \simeq C^n \times C$ and the localizations of our modification are Davis the codimension of irreducible component of $C$ in $X$ must be $s + 1$ by Remark 2.2. Hence $\dim E_1 = \dim X - 1$ unless $E_1$ is empty. By construction $Y_1 \setminus E_1$ is isomorphic to $X \setminus D$ and, therefore, irreducible. As $\text{codim}_Y Y_1 = s$
the numbers of irreducible components of $Y_1$ and $Y_1 \setminus E_1$ are the same. Thus $Y_1$ is irreducible.

**Proposition 2.9.**

1. Let $A$ be Cohen-Macaulay and $A \hookrightarrow A'$ be semi-basic. Then this modification is Davis.

2. Suppose that $A \hookrightarrow A'$ is a Davis modification. Let $A$ be Cohen-Macaulay. Then $A'$ is also Cohen-Macaulay.

**Proof.** (1) Let $M$ be a maximal ideal in $A$. Then $A_M$ is also Cohen-Macaulay [Ma, Th. 30]. In the local ring $A_M$ every semi-regular sequence is regular [Ma, Th. 31]. Thus the modification $A_M \hookrightarrow A_M[I_M/f] = S^{-1}A'$ (where $S = A \setminus M$) is Davis by Theorem 2.1. Hence $A \hookrightarrow A'$ is Davis by Proposition 2.8. For (2) consider $L_1, \ldots, L_s \in A[s]$ as in the proof of Proposition 2.8. Since $A$ is Cohen-Macaulay $A^{[s]}$ is Cohen-Macaulay as well [Ei, Prop. 18.9]. The ideal $I'$ generated by $L_1, \ldots, L_s$ has height $s$. Hence $A' \simeq A^{[s]}/I'$ is Cohen-Macaulay by [Ei, Prop. 18.13].

### 2.5. Basic modifications.

**Remark 2.3.** Let $A \hookrightarrow A'$ be a basic modification and $b_0 = f, \ldots, b_s$ be a system of generators of $I$ which is an almost complete intersection. Note $b_0, \ldots, b_s$ may be viewed as elements of a local holomorphic coordinate system at a general point $x$ of $C$. That is, in a neighborhood of $x$ the basic modification is nothing but a usual (affine) monoidal transformation. This implies that every point $y \in \sigma^{-1}(x)$ is a smooth point of $X'$ and the zero multiplicity of $f \circ \sigma$ at $y$ is 1.

Remark 2.3 and [KaZa1, Th. 3.1 and Prop. 3.1] imply the following fact.

**Proposition 2.10.** Let $A \hookrightarrow A'$ be a basic modification, $C$ and, therefore, $E$ be irreducible topological manifolds, and the natural embedding of $C$ into $D$ generate an isomorphism of the homology of $C$ and $D$. Then $\sigma$ generates isomorphisms of the fundamental groups and the homology groups of $X$ and $X'$.

**Proposition 2.11.** Let $A \hookrightarrow A'$ be a basic modification. Suppose that $A$ is normal and Cohen-Macaulay. Then $A'$ is normal and Cohen-Macaulay.

**Proof.** By Proposition 2.9 this modification is Davis and $A'$ is Cohen-Macaulay. Note that if the singularities of $X'$ is at least of codimension 2 then $X'$ is normal by [Ha, Ch. 2, Prop. 8.23]. Since $X$ is normal the codimension of $\sigma^{-1}(\text{sing } X \setminus D) = \sigma^{-1}(\text{sing } X \setminus C)$ in $X'$ is at least 2 whence we can ignore this subvariety. Let $C^0$ be the subset of the reduced center $C$, at the points of which the gradients of a system of generators of $I$ (which are an almost a complete intersection) are linearly independent. The codimension of $C \setminus C^0$...
in $C$ is at least 1. Since $\sigma$ is cylindrical the codimension of $\sigma^{-1}(C \setminus C^0)$ in $E$ is at least 1, and in $X'$ it is at least 2, and we can ignore these points again. The other points of $X'$ are smooth by Remark 2.3. \qed

2.6. Preliminary decomposition. We shall fix first notation for this subsection.

Convention 2.2.  
(1) In this subsection $A$ is normal Cohen-Macaulay. When we speak about the modification $A \hookrightarrow A'$ then $I = I_f$, $f = g^n$ where $g \in A$ generates $\mathcal{I}_A(D)$, $E$ is nonempty irreducible and $\mathcal{I}_{A'}(E)$ is also generated by $g$. 
(2) We consider affine domains $A_i = \mathbb{C}[X_i]$, $i \geq 0$ such that $A \hookrightarrow A_i \hookrightarrow A'$. These embeddings generate morphisms $\delta_i : X_i \rightarrow X$ and $\rho_i : X' \rightarrow X_i$ where $\sigma = \delta_i \circ \rho_i$. It is easy to see that there exist ideals $I_i$ in $A$ and $K_i$ in $A_i$ such that the locus of $A \hookrightarrow A_i$ is $(I_i, g^{n_i})$ and the locus of $A_i \hookrightarrow A'$ is $(K_i, f)$. Hence the exceptional divisor $E_i$ of $\delta_i$ coincides with the divisor $D_i$ of $\rho_i$. 
(3) We suppose that $K_i$ is the $f$-largest ideal of $\rho_i$ whence by Proposition 2.2 the closure of the geometrical center $C_i$ of $\rho_i$ coincides with its reduced center $\overline{C_i}$.

Lemma 2.2. Let $A_1 \hookrightarrow A'$ be an affine modification as in Convention 2.2. Suppose that $A_1$ is normal and the closure of $C_1 = \rho_1(E)$ in $X_1$ is an irreducible component $D_1$ of $D_1$. Let $E_0$ be the Zariski open subset of $E$ that consists of all points $x' \in E$ such that $x'$ is a connected component of $\rho_1^{-1}(\rho_1(x'))$. Put $D_0 = \rho_1(E_0)$ and let $D_2$ be the union of irreducible components of $D_1$ different from $D_1$. Then

(i) $D_0 = D_1 \setminus D_2$ and $E_0 = \rho_1^{-1}(D_0)$;
(ii) the restriction of $\rho_1$ to $(X' \setminus E) \cup E^0$ is an isomorphism between this variety and $(X_1 \setminus D_1) \cup D_0$;
(iii) in particular, if $E = E_0$ (this is so, for instance, when $D_2$ does not meet $D_1$) then $\rho_1$ is an embedding, and if $D_1 = D_2$ then $\rho_1$ is an isomorphism.

Proof. Put $x_1 = \rho_1(x')$ for $x' \in E_0$. As $X_1$ is normal $x_1$ cannot be a fundamental point of the birational map $\rho_1^{-1}$ by the Zariski Main Theorem [Ha, Ch. 5, Th. 5.2]. That is, $\rho_1^{-1}$ an embedding in a neighborhood of $x_1$ which proves (ii). Put $X_0^1 = (X_1 \setminus D_1) \cup (D_1 \setminus D_2)$. The complement to $(X_1 \setminus D_1) \cup D_0$ in $X_0^1$ is a constructive subset of codimension at least 2. Since $X'$ is affine and $X_0^0$ is normal we can extend morphism $\rho_1^{-1}$ to a morphism from $X_0^1$ to $X'$ [Dan, Sect. 7.1]. This implies that $\rho_1^{-1}|_{X_0^1} : X_0^1 \rightarrow X'$ is an embedding whence $D_0 \supset D_1 \setminus D_2$. Assume that $x' \in E_0$ and $x_1 = \rho_1(x') \in D_1 \cap D_2$. As $\rho_1^{-1}$ is an embedding in a neighborhood of $x_1$ we see that the
exceptional divisor of $\rho_1$ must contain a component different from $E$. This contradiction yields (i). The last statement follows immediately from (i) and (ii).

**Definition 2.7.** Let $A \hookrightarrow A'$ be an affine modification, $A \hookrightarrow A_1$ be a basic modification such that $A_1 \subset A', h \in A \setminus \{0\}$ and $S = \{h^n \mid n \in \mathbb{N}\}$. Suppose that $(h \circ \sigma)^{-1}(0)$ does not contain $E$ and $S^{-1}A_1 = S^{-1}A'$. Then we call $A \hookrightarrow A'$ a pseudo-basic modification (relative to $A \hookrightarrow A_1$).

Note that if the assumption of Lemma 2.2 holds and $A \hookrightarrow A_1$ is basic then it follows from this Lemma that $A \hookrightarrow A'$ is pseudo-basic.

**Proposition 2.12.** Let Convention 2.2 hold, $C$ be not contained in $\text{sing} \, X$, $\text{codim}_X C \geq 2$, and the zero multiplicity of $g$ at general points of $C$ is 1. Let

$$A = A_0 \hookrightarrow \cdots \hookrightarrow A_{k-1} \hookrightarrow A_k, \, k \geq 0$$

be a strictly increasing sequence of affine domains such that $A_k \subset A'$, and $\forall \, i \leq k$

(i) affine modification $A_{i-1} \hookrightarrow A_i$ is basic with locus $(J_i, g)$ and of rank $s_{i-1}$ where $s_{i-1} + 1 = \text{codim}_{X_{i-1}} C_{i-1}$.

(1) Then $k \leq n$ (recall that $f = g^n$) and this sequence can be extended to a strictly increasing sequence of affine domains

$$A_0 \hookrightarrow \cdots \hookrightarrow A_{m-1} \hookrightarrow A_m, \, k \leq m \leq n$$

so that (i) holds $\forall \, i \leq m$ and $A_{m-1} \hookrightarrow A'$ is pseudo-basic relative to $A_{m-1} \hookrightarrow A_m$.

(2) Suppose that $\sigma_i : X_i \to X_{i-1}$ is the morphism associated with the affine modification $A_{i-1} \hookrightarrow A_i$. Then $\sigma_i(C_i) = C_{i-1}$ for $i \leq m - 1$, and $\rho_{m-1}(E) \subset C_{m-1}$.

(3) Let the closure $E_m^1$ of $\rho_n(E)$ be a connected component of $E_m$ (resp. $E_m$ be irreducible). Then $\rho_{m-1}$ is a locally basic (resp. basic) modification.

**Proof.** Let us show (2) first. By Convention 2.2 the exceptional divisor of $\rho_i$ is $E$ whence $\rho_i(E) = C_i$. As $\sigma = \rho_0$ we have $\sigma(E) = C_0$. As $\sigma = \rho_i \circ \delta_i$ we see that $\delta_i(C_1) = C_0$. Hence $\sigma_i(C_1) = C_{i-1}$ since $\delta_i = \delta_{i-1} \circ \sigma_i$.

For (1) note first that as $A$ is normal Cohen-Macaulay so is each $A_i$ by Proposition 2.11. Hence if $s_k = 0$ then we put $m = k$ and (1) follows from Lemma 2.2. Let $s_k > 0$. By Remark 2.3 $\sigma_1^{-1}(x) \subset \text{reg} \, X_1$ for a general point $x \in C_0$. As $\sigma_1(C_1) = C_0$ we see that $\sigma_1^{-1}(x)$ contains general points of $C_1$ whence $C_1$ meets $\text{reg} \, X_1$, and the zero multiplicity of $g \circ \delta_1$ at general points of $C_1$ is 1 by Remark 2.3. By induction the similar facts are true for $C_k$ and $g \circ \delta_k$. By Lemma 2.1 and Proposition 2.3 we can choose a basic modification $A_k \hookrightarrow A_{k+1}$ with locus $(J_{k+1}, g)$ such that its rank is $s_k$ and $A_{k+1} \subset A'$. Thus we can extend our strictly increasing sequence of affine domains and we can always suppose that $k \geq 1$ in (1). There are two possibilities: Either this sequence is infinite or there exists $m$ such that
The locus of the restriction of $\tau$ under the exceptional divisor $M_i^2$.

**Lemma 3.1.** Let $C$ be a threefold with an embedding. Hence:

Claim (3) is now a consequence of Lemma 2.2.

Let $C_{m-1}^*$ be the complement in $C_{m-1}$ to the intersection of $C_{m-1}$ with the other components of the reduced center of $\sigma_m$. Then the exceptional divisor of $\sigma_m$ contains $E_m^* \simeq C^{s_{m-1}} \times C_{m-1}^*$. By Lemma 2.2 under the assumption of Proposition 2.12 the restriction of $\rho_m^{-1}$ to $(X_m \setminus E_m) \cup E_m^*$ is an embedding. Hence:

**Corollary 2.2** (cf. [Miy2, Lemma 2.3]). Under the assumption of Proposition 2.12 the exceptional divisor $E$ contains a Zariski open cylinder $E_m^* \simeq C^{s_{m-1}} \times C_{m-1}^*$ such that $\rho_m^{-1}|_{E_m^*}$ is the projection to the second factor.

3. The geometry of the exceptional divisor and the reduced center.

3.1. The exceptional divisor. In this section we shall strengthen Proposition 2.12 in the case when $X$ is a threefold. Our main aim is to make $A_m = A'$.

**Lemma 3.1.** Let $X'$ be an affine threefold with $H_3(X') = 0$ and $E$ be a closed irreducible surface in $X'$ which admits a surjective morphism $\tau : E \rightarrow C_{m-1}$ into an irreducible curve $C_{m-1}$ such that for a Zariski open subset $C_{m-1}^* \subset C_{m-1}$ and $E^* = \tau^{-1}(C_{m-1}^*)$ the morphism $\tau|_{E^*} : E^* \rightarrow C_{m-1}^*$ is a $C$-cylinder and $L := E \setminus E^*$ is a curve. Let $H_3(X' \setminus E) = H_3(X' \setminus E) = 0$. Let $z \in C_{m-1} \setminus C_{m-1}^*$ and $C^*$ be the punctured germ of $C_{m-1}$ at $z$. Put $L^z = \tau^{-1}(z)$. Then there exists an isomorphism $H_0(\text{reg } L) \simeq H_1(E^*)$ such

\[ s_m = 0 \text{ which implies (1). Show by induction that the first possibility does not hold and that } m \leq n. \text{ Assume first that (1) holds for } n - 1 > 0. \text{ Let } b_0 = g, b_1, \ldots, b_s \text{ be a system of generators of } J_1 \text{ which is an almost complete intersection. By Definition 2.3 there exists } h \in A \text{ such that } h^{-1}(0) \text{ does not contain } C, X \setminus h^{-1}(0) \text{ is smooth, } C \setminus h^{-1}(0) \text{ is a complete intersection in } X \setminus h^{-1}(0) \text{ given by } b_0 = \cdots = b_s = 0. \text{ If } S = \{ h^j \mid j \in \mathbb{N} \} \text{ then the affine modification } S^{-1}A \hookrightarrow S^{-1}A' \text{ satisfies the analogue of assumption of this proposition and, furthermore, } S^{-1}J_1 \text{ is the defining ideal of } C \setminus h^{-1}(0) \text{ in } S^{-1}A. \text{ By Proposition 2.1 the locus of } S^{-1}A \hookrightarrow S^{-1}A_1 \text{ is } (S^{-1}J_1, g), \text{ and by Corollary 2.1 the locus of } S^{-1}A_1 \hookrightarrow S^{-1}A' \text{ can be chosen in the form } (A_1, g^{n-1}). \text{ By the induction assumption the codimension of the reduced center of the modification } S^{-1}A_m \hookrightarrow S^{-1}A' \text{ is } 1 \text{ for some } m \leq n. \text{ Hence the same is true for the reduced center of } A_m \hookrightarrow A', \text{ i.e., } s_m = 0 \text{ which concludes this step of induction. The next step is for } n = 1. \text{ By Proposition 2.3 in this case } S^{-1}J_1 \text{ coincides with the } g\text{-largest ideal of the affine modification } S^{-1}A \hookrightarrow S^{-1}A'. \text{ Hence } S^{-1}A_1 = S^{-1}A'. \text{ As } h \text{ is chosen so that } h^{-1}(0) \text{ does not contain } C \text{ this implies (1) which concludes induction. Claim (} 3) \text{ is now a consequence of Lemma 2.2.} \]

\[ \square \]
that for every germ $C^z$ as above the restriction of this isomorphism generates an isomorphism $H_0(\text{reg } L^z) \simeq H_1(C^z)$ (i.e., the number of irreducible components of $L^z$ is the same as the number of connected components of $C^z$). Furthermore, the normalization of $C_{m-1}$ is $C$.

**Proof.** Consider the following exact homology sequences of pairs:

$$\rightarrow H_{j+1}(X') \rightarrow H_{j+1}(X', X' \setminus L) \rightarrow H_j(X' \setminus L)$$

$$\rightarrow H_j(X') \rightarrow H_j(X', X' \setminus L) \rightarrow$$

and

$$\ldots \rightarrow H_j(X' \setminus E) \rightarrow H_j(X' \setminus L) \rightarrow H_j(X' \setminus L, X' \setminus E) \rightarrow H_{j-1}(X' \setminus E) \rightarrow \ldots$$

Note that $H_4(X') = 0$ since $X'$ is an affine algebraic variety [Mil, Th. 7.1]. Replace $X'$ (resp. $E$, resp. $L$) with the complement to sing $L$ in $X'$ (resp. $E$, resp. $L$). Though $X'$ is no more affine, this replacement does not affect $H_3(X'), H_4(X')$, and $H_i(X \setminus E)$, and the advantage is that $L$ is smooth now. From the above sequences and Thom’s isomorphisms (e.g., see [Do, Ch. 8, 11.21]) we have

$$H_0(L) \simeq H_4(X', X' \setminus L) \simeq H_3(X' \setminus L) \simeq H_3(X' \setminus L, X' \setminus E) \simeq H_1(E^*).$$

As $H_1(C_{m-1}^*) \simeq H_1(E^*)$ we have an isomorphism $H_0(L) \simeq H_1(C_{m-1}^*)$. Let $L_i$ be an irreducible component of $L$ (which is now a connected smooth component of $L$), and $V$ be a tubular neighborhood of $L_i$ in $X'$. Consider the germ $S'_i$ of a smooth complex surface whose image under a natural retraction $V \rightarrow L_i$ is a point $z' \in L_i$, i.e., $S_i$ is transversal to $L_i$ at $z'$. We can suppose that $S'_i$ is diffeomorphic to a ball and its boundary $\partial S'_i$ in $X'$ is diffeomorphic to a three-sphere which meets $E^*$ transversally along a smooth real curve $\gamma_i$. Let $[S_i] \in H_4(V, V \setminus L_i)$ be generated by $S_i$. Thom’s isomorphism $H_4(V, V \setminus L_i) \rightarrow H_0(L_i)$ sends $[S_i]$ to the positive generator of $H_0(L_i)$ [FoFu, Ch. 4, Sect. 30, p. 262]. As Thom’s isomorphisms are functorial under open embeddings [Do, Ch. 8, 11.5], isomorphism $H_4(X', X' \setminus L) \rightarrow H_0(L)$ sends $[S_i]$ to the positive generator of $H_0(L_i)$. Isomorphism $H_4(X', X' \setminus L) \simeq H_3(X' \setminus L)$ sends $[S'_i]$ to the element $[\partial S'_i] \in H_3(X' \setminus L)$ generated by $\partial S'_i$. Let $T_i$ be a small tubular neighborhood of $\gamma_i$ in $\partial S'_i$. By the excision theorem $T_i$ generates an element $[T_i]$ of $H_3(X' \setminus L, X' \setminus E)$ which coincides with $[\partial S'_i]$. Thus the constructed isomorphism $H_0(L) \simeq H_3(X' \setminus L, X' \setminus E)$ sends the positive generator of $H_0(L_i)$ to $[T_i]$. The same argument as above implies that under isomorphism $H_3(X' \setminus L, X' \setminus E) \rightarrow H_1(E^*)$ the cycle $[T_i]$ goes to the element $[\gamma_i] \in H_1(E^*) \simeq H_1(C_{m-1}^*)$ generated by $\gamma_i$. Note that $H_1(C_{m-1}^*) = \oplus_{z \in C_{m-1} \setminus C^*} H_1(C^z) \oplus N$ where the group $N$ is not trivial provided that either $C_{m-1}$ is of positive genus or $C_{m-1}$ has more than one puncture. As $\tau(z') = z$ and $\gamma_i$ is contained in a small neighborhood of $z'$, the image of the generator of $H_0(L_i)$ under isomorphism $H_0(L) \simeq H_1(C_{m-1}^*)$ is contained in $H_1(C^z)$. Hence the image of $H_0(L)$ is
the image of a way to compute these coefficients

\[ H_1(C^2) \]. Thus \( N \) is trivial and \( H_0(L^2) \simeq H_1(C^2) \) which is the desired conclusion. \( \square \)

**Remark 3.1.** If \( E \) is a UFD then there is no need to assume in Lemma 3.1 that \( X' \) is smooth and \( H_3(X') = H_2(X' \setminus E) = H_2(X' \setminus E) = 0 \). One can show that the fibers of \( \tau \) are irreducible whence the Euler characteristics \( e(E) = e(C_{m-1}) \leq 1 \). Thus in order to make \( C_{m-1} \) contractible one need \( e(E) = 1 \).

The proof of Lemma 3.1 implies more. Fix \( z \in C_{m-1} \setminus C_{m-1}^* \). Let \( C_j, j = 1, \ldots, k \) be the irreducible components of \( C^2 \), i.e., \( C_j \) corresponds to a generator \( \alpha_j \) of \( H_1(C^2) \). Each irreducible component \( L_i \) of \( \tau^{-1}(z) \) corresponds to a generator \( \beta_i \) of \( H_0(L^2) \). By Lemma 3.1 the image of \( \beta_i \) under isomorphism \( H_0(L^2) \simeq H_1(C^2) \) is \( \sum_j m^j_i \alpha_j \). One can extract from Lemma 3.1 a way to compute these coefficients \( m^j_i \).

**Lemma 3.2.** Let the notation above hold and \( \tau = \rho_{m-1}|_E \) where \( \rho_{m-1} : X' \to X_{m-1} \) is the same as in Proposition 2.12. Let \( S'_i \) be the Euclidean germ of a smooth algebraic surface transversal to \( L_i \) at a smooth point \( z' \in L_i \), and \( S_i = \rho_{m-1}(S'_i) \). Then for every point \( x \in C^j \) the germ of \( S_i \) at \( x \) consists of \( m_i^j \) smooth branches which meet the divisor \( D_{m-1} \) of modification \( \rho_{m-1} \) transversally along the germ of \( C^j \) at \( x \).

**Proof.** Suppose that \( S'_i \) meets \( \tau^{-1}(C^j) \) along the germ \( \Gamma^j_i \) of a curve. As \( S'_i \) is transversal to \( L_i \), it is transversal to \( E \) at every \( x' \in \Gamma^j_i \). Let \( B' \) be the germ of \( S'_i \) at \( x' = \rho_{m-1}(x') \), and \( B = \rho_{m-1}(B') \). As \( \rho_{m-1} \) is pseudo-basic it is basic in a neighborhood of \( x' \), i.e., it can be viewed as a monoidal transformation at \( x \), by Remark 2.3. Hence \( B \) is smooth and transversal to \( D_{m-1} \) at \( x \). It remains to show that the number of such branches \( B \) is \( m^j_i \) which is equivalent to the fact that the mapping \( \tau|_{\Gamma^j_i} : \Gamma^j_i \to C^j \) is \( m^j_i \)-sheeted. One can suppose that \( S'_i \) is the same as in the proof of Lemma 3.1.

The boundary of \( \Gamma^j_i \) may be viewed as a smooth real curve \( \gamma^j_i \) and \( \bigcup_j \gamma^j_i = \gamma_i \) where \( \gamma_i = \partial S'_i \cap E \). Let \( E^2 = \tau^{-1}(C^2) \). It was shown in the proof of Lemma 3.1 that the image of \( \beta_i \) under the isomorphism \( H_0(L^2) \simeq H_1(E^2) \) is \( [\gamma_i] = \sum_j [\gamma^j_i] \) where \( [\gamma^j_i] \) is the cycle in \( H_1(E^2) \) generated by \( \gamma^j_i \). Then the image of \( [\gamma^j_i] \) under the isomorphism \( H_1(E^2) \simeq H_1(C^2) \) coincides with \( m^j_i \alpha_j \) where \( m^j_i \) is the winding number of \( \tau(\gamma^j_i) \) in \( C^j \) around \( z \). As \( \gamma^j_i \) is the boundary of the punctured disc \( \Gamma^j_i \) this implies that \( \tau|_{\Gamma^j_i} : \Gamma^j_i \to C^j \) is \( m^j_i \)-sheeted. \( \square \)

### 3.2. The reduced center

We shall describe some condition under which the reduced center of an affine modification is a complete intersection.
Lemma 3.3. Let $C$ be an affine reduced irreducible curve, $v$ be a coordinate on the first factor of $D_1 = C \times C$, and $\theta : D_1 \to C$ be the natural projection. Let $o$ be a singular point of $C$, $\mathcal{V}$ be the germ of $C$ at $o$, $D_1 = \theta^{-1}(\mathcal{V})$, and $F_{\mathcal{V}}$ (resp. $\mathcal{O}_\mathcal{V}$) be the ring of complex-valued (resp. holomorphic) functions on $\mathcal{V}$.

1) Suppose that a function $h \in F_{\mathcal{V}}[v]$ is holomorphic everywhere in $D_1$ except for a finite number of points. Then $h$ is holomorphic in $D_1$.

2) Let $h \in F_{\mathcal{V}}[v]$ be holomorphic in $D_1$, $h^{-1}(0)$ not contain $\theta^{-1}(o)$, and the zero multiplicity of $h$ at general points of $h^{-1}(0)$ be $n$. Then $h^{1/n}$ is holomorphic.

3) Let $C_1$ be a reduced irreducible curve in $D_1$ so that projection $\theta|_{C_1} : C_1 \to C$ is finite and for each singular point $o \in C$ there exist $\mathcal{V}$ and $D_1$ as in (1) for which the defining ideal of $C_1 \cap D_1$ in $\mathcal{O}_\mathcal{V}[v]$ is principal. Then the defining ideal of $C_1$ in $C[D_1]$ is generated by a function $b \in C[D_1]$ which is a monic polynomial in $v$.

Proof. The argument is of local analytic nature whence it is enough to consider the case when $C$ is contractible. Let $\nu_0 : C \simeq C' \to C$ be a normalization, $t$ be a coordinate on $C'$, and $\nu = (\nu_0, \text{id}) : C^2 \simeq C \times C' \to D_1$, i.e., $(v, t)$ is a coordinate system on $C^2$. Note that $\gamma = h \circ \nu$ is of form $r_k(t)u^k + r_{k-1}(t)u^{k-1} + \cdots + r_0(t)$. One can suppose that $o$ is the origin of $C^n \supset C$. As $h$ is holomorphic everywhere on $D_1$ except for a finite number of points implies that for every fixed $v = v_0$ except for a finite number of values, $\gamma(v_0, t)$ is contained in the ring of convergent power series of the coordinate functions $x_1(t), \ldots, x_n(t)$ of $\nu_0$. Hence each $r_i(t)$ belongs to this ring whence $h$ is holomorphic in $D_1$ which is (1). For (2) it suffices to note that the function $h^{1/n}$ is holomorphic everywhere in $D_1$ except for possibly points from the finite set $h^{-1}(0) \cap \theta^{-1}(a)$. Let $C''_1 = \nu^{-1}(C_1)$ be the zero fiber of an irreducible polynomial $\beta(v, t)$ on $C^2$. The projection of $C''_1$ to the $t$-axis is finite as $\theta|_{C_1}$ is finite whence $\beta(v, t)$ is monic in $v$. The function $b = \beta \circ \nu^{-1}$ is rational on $D_1$, and in order to show that it is regular, it suffices to show that $b$ is holomorphic at each point of $D_1$ (e.g., see [Kal1]). Let $o$ be a singular point of $C$. It is enough to check that $b$ is holomorphic at the points of $\theta^{-1}(a)$. Let $O$ be the ring of germs of analytic functions at $o \in C^n$ and $h$ be the generator of the defining ideal of $C_1 \cap D_1$ in $\mathcal{O}_\mathcal{V}[v]$. By Cartan’s theorems (e.g., see [GuRo, Ch. 8A, Th. 18]) we can extend each coefficient of $h$ to a holomorphic function in a neighborhood of $o \in C^n$ whence we can treat $h$ as an element of $O[v]$. Applying the Weierstrass Preparation Theorem [Rem, Ch. 1, Th. 1.4] one can show that $h = \omega e$ where $\omega \in O[v]$ is a monic polynomial and $e \in O[v]$ is invertible on $D_1$. Thus $\omega|_{D_1}$ generates the same ideal as $h$ whence we can suppose that $h$ is a monic polynomial in $v$. Thus $\gamma = h \circ \nu$ is monic as a polynomial in $v$ over the ring of germs of analytic functions at $\nu_0^{-1}(o) \subset C$. Note that $\gamma = \beta \alpha$.
where $\alpha$ does not vanish since the zero multiplicity of $\gamma$ and $\beta$ at general points $C_i'\cap C_j'$ is 1. Hence $\alpha$ is is constant on each line parallel to the $v$-axis. This constant is 1 since both $\gamma$ and $\beta$ are monic (look at the quotient $\gamma/\beta$ as $v$ approaches $\infty$ along any of these lines). Thus $\beta = \gamma$ whence $b$ coincides with $h$ in $D_1$ and, therefore, $b$ is holomorphic.

**Definition 3.1.** We say that $X$ is a locally analytic UFD of for every $x \in X$ the ring of germs of holomorphic functions on $X$ at $x$ is a UFD.

**Proposition 3.1.** Let the assumptions of Convention 2.2 and Proposition 2.12 hold. Suppose that $\dim X = 3$, $m \geq 2$ where $m$ is from Proposition 2.12, $X'$ is smooth, and $H_3(X') = H_2(X'\setminus E) = H_3(X'\setminus E) = 0$. Suppose that for $i = 1, \ldots, r$ the divisor $D_i$ is naturally isomorphic to $\mathbf{C} \times C_{i-1}^{-}$ (i.e., $C_{i-1}$ is a curve), the natural projection $\sigma_i: C_i \to C_{i-1}$ is finite, $D_0 = (D)$ is smooth. Let $X$ be a locally analytic UFD. Then the defining ideal of $C_r$ in $\mathbf{C}[D_r]$ is principal.

**Proof.** Let $L, L', L^2$ be as in Lemma 3.1 and let $L_i$ be an irreducible component of $L^2$. Suppose that for each $z \in C_{m-1} \setminus C_m^{-1}$ the objects $S_i, S_j, C^j, m_i^j$ are the same as in Lemma 3.2. As these objects depend on $z$, the notation, say, $m_i^j(z)$ has an obvious meaning. By Lemma 3.1 the matrix $(m_i^j(z))$ is invertible whence there exists a vector $\sigma(z)$ with integer entries $v^i(z)$ such that each entry of the vector $(m_i^j(z))\sigma(z)$ is equal to 1. Consider the Euclidean germ $T' = \sum_{z \in C_{m-1} \setminus C_m^{-1}} \sum_{L_i \subset L^2} v^i(z)S_i\sigma(z)$ of a divisor and its strict transforms $T_i = \rho_i(T')$. By Lemma 3.2 the germ of $T_{m-1}$ at any point of $C^j$ consists of smooth branches transversal to $D_{m-1}$ and the sum of multiplicities of these branches is 1. For $i \leq m-1$ put $\gamma_i = \rho_i \circ \rho_{m-1}^{-1} = \sigma_{i+1} \circ \cdots \circ \sigma_{m-1}$ and $z_i = \gamma_i(z)$. Let $C_i$ be the germ of $C_i$ at $z_i$ and morphism $\gamma_i: C_{m-1} \to C_i$ be $k_i$-sheeted. As $\gamma_i$ is a composition of basic modifications (which can be viewed locally as monoidal transformations by Remark 2.3) the germ of $T_i$ at a general point of $C_i$ consists of smooth branches which meet $D_i$ transversally and the sum of multiplicities of these branches is $k_i$. Let $V_0$ be the Euclidean germ of $X = X_0$ at $z_0$ and let $S^0 \subset V_0$ be an irreducible component of $T_0$ (i.e., $S_0 = \rho_0(S_i)$ for some $i$). One can check that $S^0 \setminus D$ is closed in $V_0 \setminus D$ (i.e., $S^0 \setminus D$ is an analytic hypersurface in $V_0 \setminus D$ as the restriction of $\rho_0$ to $X' \setminus E$ is an embedding) and the closure $S^0$ of $S^0 \setminus D$ in $V_0$ is $S_0$. Hence $S^0 \cap D \subset C$ (in particular, the intersection of $T_0$ and $D_0 \cap V_0$ is $C_0$) and by [BeNa, Th. 1.2] $S^0$ is an analytic hypersurface in $V_0$. As $X$ is locally analytic UFD, in $V_0$ the divisor $T_0$ coincides with the divisor of a meromorphic function $h_0$ on $V_0$. By the theorem about deleting singularities the restriction of $h_0$ to $V_0 \cap D_0$ is a holomorphic function whose divisor is $k_0C_0$. Put $V_1 = \sigma_{1}^{-1}(V_0)$ and $D_1 = \sigma_{1}^{-1}(C_0)$. As $\sigma_1$ is basic $e_1 = h_0 \circ \sigma_1$ is a meromorphic function on $V_1$ whose divisor is $T_1 + k_0D_1$. 


By Remark 2.3 the divisor of \( q_1 = g \circ \sigma_1 \) on \( V_1 \) is \( D_1 \) whence \( T_i \) is the divisor of the meromorphic function \( h_1 := e/g^k_1 \). As for general \( x_1 \in C_1 \) each branch of \( T_i \) at \( x_1 \) meets \( D_1 \) transversally and \( T_1 \cap D_1 = C_1 \). Lemma 3.3 (1) implies that \( h_1|D_1 \) is a holomorphic function with divisor \( k_1C_1 \). Put \( e_2 = h_1 \circ \sigma_2, V_2 = \sigma_2^{-1}(V_1) = \delta_2^{-1}(V_0) \), and \( D_2 = \sigma_2^{-1}(C_1) = \delta_2^{-1}(C_0) \) (where \( \delta_i \) is as in Convention 2.2 (2)). Repeating the procedure we get the germ of a meromorphic function \( h_2 \) whose divisor is \( T_2 \). Induction yields a meromorphic function \( h_i \) on \( V_i = \delta_i^{-1}(V_0) \) whose divisor is \( T_i \). Hence \( \forall i \leq r \) the restriction of \( h_i \) to \( D_i = \delta_i^{-1}(C_0) \) is a holomorphic function whose divisor is \( k_iC_i \). By Lemma 3.3 the defining ideal of \( C_i \) is principal, and, therefore, the defining ideal of \( C_i \) in \( C[D_1] \) is also principal. \( \Box \)

3.3. Decomposition.

**Lemma 3.4.** Let Convention 2.2 hold, \( A_1 \hookrightarrow A' \) be a basic modification, \( D_1 \simeq C \times C \) where \( C \) is a curve, \( z \) be an irreducible singular point of \( C \), and \( C_1 \) meet \( C \times z \subset D_1 \) at \( z_1 = 0 \times z \) but \( C_1 \neq C \times z \). Then \( z_1 \) is a singular point of \( C_1 \).

**Proof.** Assume the contrary, i.e., \( C_1 \) is smooth at \( z_1 \). As the situation is local we can suppose that \( C \) is a closed curve in \( C^n \). Consider a normalization \( \nu_0 : C^n \to C \) and \( \nu = (\text{id}, \nu_0) : C \times C^n \to C \times C \subset C^{n+1} \). Let \((y, \bar{x}) = (y, x_1, \ldots, x_n) \) be a coordinate system in \( C^{n+1} \), and \( g, b_1 \) be an almost complete intersection in \( A_1 \) which generates modification \( A_1 \hookrightarrow A' \), i.e., \( b_1 \) generates the defining ideal of \( C_1 \) in \( C[D_1] \). We treat \( b_1 \) as a polynomial \( b_1(y, \bar{x}) \) on \( C^{n+1} \). Let \( \beta = b_1|_{D_1} \circ \nu, C'_1 \) be the proper transform of \( C_1 \) (i.e., \( C'_1 = \beta^{-1}(0) \)), and let \( o = \nu^{-1}(z_1) \). As \( C_1 \) is smooth and \( \nu|_{C'_1} : C'_1 \to C \) is a homeomorphism, \( C'_1 \) is biholomorphic to \( C_1 \) by [Pe, Cor. 1.5] whence \( C'_1 \) is smooth at \( o \). As \( A_1 \hookrightarrow A' \) is basic the gradient of \( \beta \) does not vanish at general points of \( C'_1 \) and also at \( o \) as \( C'_1 \) is smooth at \( o \). Let \((v, t) \) be a local coordinate system at \( o \) where \( t \) is a coordinate on the second factor of \( C \times C'_1 \) and \( v \) is a coordinate on the first one. The Taylor series of \( \beta(v, t) = b_1(v, \bar{x}(t)) \) at \( o \) does not have a nonzero linear term \( ct \) since \( z \) is a singular point of \( C \). The linear part of this power series must be nonzero (otherwise the gradient of \( \beta \) at \( o \) is zero). Thus the Taylor series of \( b_1 \) at \( z_1 \) has a nonzero linear term \( cv \). The implicit function theorem implies that the germ of \( C_1 \) at \( z_1 \) is biholomorphic to the germ of \( C \) at \( z \), i.e., \( C_1 \) is singular at \( z_1 \). Contradiction. \( \Box \)

**Theorem 3.1.** Let Convention 2.2 (1) hold for an affine modification \( A \hookrightarrow A' \), \( X' \) be a threefold, and \( H_3(X') = H_2(X' \setminus E) = H_3(X' \setminus E) = 0 \). Let

(i) \( D \) be isomorphic to \( C^2 \);
(ii) \( X \) be a locally analytic UFD, and
(iii) \( C \) be not contained in the singularities of \( X \).
Let \( m, A, C, \) and \( C_i \) be the same as in Proposition 2.12 \(^3\) and Convention 2.2 (3).

(1) Then the algebras \( A_i \)’s can be chosen so that \( A_m = A' \), \( C_i = C_i \) for every \( i \), and if \( C_i \) is a curve its defining ideal in \( C[D_i] \) is principal.

(2) Each \( C_i \) is either a point or an irreducible contractible curve, and in the case when \( E \) has at most isolated singularities these curves are smooth contractible.

Proof. We use induction on \( m \). Suppose first that \( C_0 \) is a point. Assumption (i) allows us to choose \( b_1, b_2 \in A \) such that \( g, b_1, b_2 \) generate the defining ideal \( I_1 \) of \( C_0 \) in \( A \). Hence the exceptional divisor \( E_1 = D_1 \) of modification \( A \hookrightarrow A_1 = A[I_1/g] \) with locus \( (I_1, g) \) is isomorphic to \( C^2 \). By Proposition 2.3 and Convention 2.2 (1) \( A_1 \subset A' \). If \( m = 1 \) in Proposition 2.12 then Lemma 2.2 implies that \( \rho_1 \) is an isomorphism, i.e., \( A_1 = A' \). Let \( m \geq 2 \). Note that \( E_1 \simeq C^2 \) and it is the divisor of the modification \( A \hookrightarrow A' \) from Convention 2.2.

By (iii) \( C_0 \) is a smooth point of \( X \) whence \( \sigma^{-1}_1(C_0) \) is contained in the smooth part of \( X_1 \) by Remark 2.3, i.e., \( X_1 \) is a locally analytic UFD and normal Cohen-Macaulay. Thus the assumptions of this Theorem hold also for the modification \( A \hookrightarrow A' \). The decomposition of \( A \hookrightarrow A' \) contains \( m - 1 \) factors and induction implies the desired conclusion in this case. Let \( C_0 \) be a curve. As there is a surjective morphism \( C_{m-1} \twoheadrightarrow C_0 \) the normalization of \( C_0 \) is \( C \) by Lemma 3.1. This implies that \( C_0 \) is closed in \( X \), i.e., \( C = C_0 \). The defining ideal of \( C_0 \subset D \simeq C^2 \) is generated by \( \hat{b} \in C[D] \) whence the defining ideal \( I_1 \) of \( C \) in \( A \) is generated by \( g \) and \( b \) where \( \hat{b} \) is an extension of \( b \) to \( X \). Thus the exceptional divisor \( E_1 \simeq C \times C_0 \) of \( A \hookrightarrow A_1 = A[I_1/g] \) is irreducible and by Proposition 2.11 \( A_1 \) is normal Cohen-Macaulay. By Proposition 3.1 the defining ideal of \( C_1 \) in \( C[D_1] \) (recall \( D_1 = E_1 \) by Convention 2.2) is principal. Let \( \hat{b}_1 \) be its generator and \( b_1 \) be an extension of \( \hat{b}_1 \) to \( X_1 \). The defining ideal \( I_2 \) of \( C_1 \) in \( A_1 \) is generated by \( g \) and \( b_1 \). Thus the exceptional divisor \( E_2 \) of \( A \hookrightarrow A_2 = A[I_2/g] \) is again irreducible. Repeating this procedure we can construct basic modifications \( \sigma_i \) from Proposition 2.12 so that \( E_m \) is irreducible. As \( \rho_m \) is pseudo-basic relative to \( \sigma_m \), Lemma 2.2 implies that \( X' \simeq X_m \) and \( \rho_m \) coincides with \( \sigma_m \) under this isomorphism. If \( C_i \) has a double point then one can check that \( C_{i+1} \subset D_{i+1} \simeq C \times C_i \) has also a double point as the defining ideal of \( C_{i+1} \) in \( C[D_{i+1}] \) is principal. Thus \( C_{m-1} \) has a double point. For every \( z \in C_{m-1} \) the number of components in \( \sigma^{-1}_m(z) \) is one, since \( \sigma^{-1}_m(z) \simeq C \). By Lemma 3.1 the number of irreducible components of the germ of \( C_{m-1} \) at \( z \) is one, i.e., \( C_{m-1} \) and, therefore, each \( C_i \) have no double points. By the same Lemma the normalization of \( C_{m-1} \) is \( C \) whence \( C_{m-1} \) and similarly \( C_i \)’s are contractible. If \( C_i \) has an irreducible singularity then, by Lemma 3.4,

\(^3\)If we allow \( \sigma_i \) in Proposition 2.12 to be only locally basic then instead of (i) one can suppose that \( D \) is only smooth, or it is a cylinder over a curve.
4. Applications of the decomposition.

4.1. The proof of Lemma I. We shall reduce first Lemma I to a problem about affine modifications.

Lemma 4.1. Let $X'$ be a UFD of dimension 3 which contains a $\mathbb{C}^2$-cylinder $Z$ over a smooth affine curve $U$.

(i) Then $U$ is rational and the natural projection $p_0 : Z \to U$ can be extended to a function $p \in \mathbb{C}[X']$ whose general fibers are still isomorphic to $\mathbb{C}^2$.

(ii) Furthermore, let $x, y, z$ be coordinates on $X = \mathbb{C}^3$. Then there exists an affine modification $\sigma : X' \to X$ such that its coordinate form is $\sigma = (p, p_1, p_2)$ and the divisor of this modification coincides with the zeros of some polynomial $f(x)$ on $\mathbb{C}^3$.

Proof. Let $F_c$ be the closure of the fiber $F_c = \{p_0 = c\} \subset Z$ in $X'$ (where $c \in U$). Assume that $F_c \cap F_{c'} \neq \emptyset$ for some $c \neq c' \in U$. Since $X'$ is a UFD there exists $g \in \mathbb{C}[X']$ whose zero fiber is $F_{c'}$. Thus the zero locus of $g|_{F_c}$ is $F_c \cap F_{c'}$. But $g|_{F_c}$ is nowhere zero on $F_c \setminus (F_c \cap F_{c'}) = F_c \simeq \mathbb{C}^2$ whence this function must be a nonzero constant on $F_c$ and, therefore, $F_c$. Contradiction. Thus $F_c \cap F_{c'} = \emptyset$ for every $c' \neq c \in U$. Assume $F_c \simeq \mathbb{C}^2$ if different from $F_c$. Let one of the irreducible components of $F_c \setminus F_c$ be a point. Then a normalization $G$ of $F_c$ contains $\mathbb{C}^2$ and one of the irreducible components of $G \setminus \mathbb{C}^2$ is also a point $o$. By [Rem, Ch. 13] every holomorphic function on $\mathbb{C}^2$ can be extended to $o$ whence $\mathbb{C}^2$ is not Stein. Contradiction. Thus $F_c \setminus F_c$ is a curve. As $F_c \cap F_{c'} = \emptyset$ the closure of $\bigcup_{c \in U}(F_c \setminus F_c)$ is a divisor in $X'$. As $X'$ is a UFD there exists $h \in \mathbb{C}[X']$ whose zero fiber is this divisor. Thus the zero locus of $h|_{F_c}$ is $F_c \setminus F_c$ and we get a contradiction in the same way we did for function $g$. Hence $F_c = F_c$. This implies that $Z \to U$ can be extended to continuous map $p$ from $X'$ to the completion $\overline{U}$ of $U$, and $p^{-1}(U) = Z$. In particular, general fibers of $p$ are isomorphic to $\mathbb{C}^2$. As $X'$ is a UFD $p$ must be holomorphic [Rem, Ch. 13] and, therefore, regular (e.g., see [Ka1]). Since $X'$ is a UFD $Z$ is also a UFD whence $U$ is a UFD. This implies that $U$ is rational, i.e., $\overline{U} = \mathbb{P}^1$. Assume that $p : X' \to \mathbb{P}^1$ is surjective. Let $X_0 = p^{-1}(\mathbb{C})$ and $q = p|_{X_0}$. We can suppose that $Z \subset X_0$, i.e., $U \subset \mathbb{C}$. Extend the isomorphism $Z \simeq U \times \mathbb{C}^2 \subset \mathbb{C} \times \mathbb{C}^2$ to a rational map from $X_0$ to $\mathbb{C}^3$ (with coordinate $x, y, z$) and then multiply the two last coordinates by polynomials in $q$ to make this mapping regular. We obtain a birational morphism $\sigma : X_0 \to \mathbb{C}^3$ which is an affine modification by $C_{i+1}$ and, therefore, $C_{m-1}$ have singularities. But if $C_{m-1}$ is singular then $E \simeq C \times C_{m-1}$ has non-isolated singularities which yields the last statement of (2).
Clearly, \( q = x \circ \sigma \) and the divisor of this modification is the zero fiber of \( f \in \mathbb{C}[x] \). As \( q : X_0 \to \mathbb{C} \) is surjective, every invertible function on \( X_0 \) is constant by Proposition 2.5. But \( p^{-1}(\infty) \) is the zero divisor of \( g \in \mathbb{C}[X] \) as \( X' \) is a UFD. Hence \( g|_{X_0} \) is invertible and nonconstant. Contradiction. Thus one can suppose that \( p = q \) and \( X' = X_0 \).

**Lemma 4.2.** Let \( A \simeq \mathbb{C}[x, y, z] \), \( f = x^n \), and \( A \hookrightarrow A' \) be an affine modification. Let \( A' \) be a UFD, \( E \neq \emptyset \), \( X' \) be smooth, and \( H_3(X') = 0 \). If \( E \) have at most isolated singularities then \( A' \) is also a polynomial ring which contains \( x \) as variable.

**Proof.** By Proposition 2.4 \( x \circ \sigma \) is irreducible in \( A' \) whence \( E \) is irreducible as \( A' \) is a UFD. Thus Convention 2.2 (1) holds which makes Theorem 3.1 applicable to modification \( \sigma : X' \to X \). Let the notation from Theorem 3.1 and Proposition 2.12 hold. Then the reduced center \( C_i \) of each element \( \sigma_{i+1} \) of the decomposition is either a point or a smooth contractible irreducible curve. Let \( C_0 = C \) be a point (say, the origin \( o = \{x = y = z = 0\} \)) and \( M \) be the maximal ideal in \( A \) that vanishes at \( o \). By Theorem 3.1 \( A_1 = A[M/x] \) whence \( A_1 \simeq \mathbb{C}[x, y/x, z/x] \). Suppose that \( j \) is the first number for which \( C_j \) and, therefore, by Proposition 2.12 (2) every \( C_k \) with \( k > j \) are curves. By induction we can suppose that \( A_j \simeq \mathbb{C}[x, \xi, \zeta] \) (in particular, the divisor \( D_{j+1} \) of \( \sigma_{j+1} \) is the \( \xi \zeta \)-plane). By Theorem 3.1 \( C_j \) is isomorphic to \( C \) and by the Abhyankar-Moh-Suzuki theorem one can assume that it is given by \( x = \xi = 0 \). Let \( I_j \) be the ideal generated by \( x \) and \( \xi \). By Theorem 3.1 \( A_{j+1} = A_j[I_j/x] \) whence \( A_{j+1} \simeq \mathbb{C}[x, \xi/x, \zeta] \). Induction concludes the proof.

**Lemma 4.3.** Let \( A = \mathbb{C}[x, y, z] \), \( f \in \mathbb{C}[x] \), and \( A \hookrightarrow A' \) be an affine modification. Let \( A' \) be a UFD and \( f - c \) be a non-unit in \( A' \) for every root \( c \) of \( f \). If \( X' \) is smooth, \( H_3(X') = 0 \), and every irreducible component of \( E \) has at most isolated singularities then \( X' \simeq \mathbb{C}^3 \) and \( x \circ \sigma \) is a variable on this sample of \( \mathbb{C}^3 \).

**Proof.** Let \( f(x) = x^n q(x) \) where \( q(0) \neq 0 \), \( J = I[1/q] \) and \( B = A[1/q] \). By Proposition 2.1 \( B' = B[J/x^n] \) coincides with \( A'[1/q] \). Hence the exceptional divisor \( E^0 \) of modification \( B \hookrightarrow B' \) is not empty by the assumption on \( f \circ \sigma \). Let \( L \) be the ideal in \( A \) generated by \( I \) and \( x^n \), i.e., \( I[1/q] = L[1/q] = J \). Put \( A^1 = A[L/x^n] \). Note that \( B' = A^1[1/q] \) whence the exceptional divisor of \( A \hookrightarrow A^1 \) is not empty. By Lemma 4.2 \( A^1 \) is a polynomial ring in three variables. Let \( K \) be the ideal in \( A^1 \) generated by \( I/x^n \). By [KoZa1, Prop. 1.2] \( A' = A^1[K/q] \). Now the induction by the degree of \( f \) implies the desired conclusion.

Lemmas 4.1 and 4.3 yield Lemma 1.

**Remark 4.1.** Miyanishi’s theorem can be proven by this technique as follows. Assumption (3') implies that \( E^0 \) (from the proof of Lemma 4.3) is a
UFD, and it is enough show that \( e(E_0) = 1 \) (see Remark 3.1). By Proposition 2.12 we can present \( B \leftrightarrow B' \) as a composition of basic modifications. Hence either \( E^0 \) is isomorphic to \( C_{m-1} \times C^2 \) where \( C_{m-1} \) is a point or it is as in Remark 3.1, i.e., \( e(E_0) \leq 1 \). Let \( D_0 \) be the plane \( x = 0 \). Then by the additivity of Euler characteristics \([\text{Du}] \) \( e(X') \) differs from \( e(X) = e(C^3) = 1 \) by the sum of terms of form \( e(E_0) - e(D_0) \) (which should be considered for each root of \( f \)). As \( e(X') = 1 \) we have \( e(E_0) = e(D_0) = 1 \) which makes Lemma 4.2 applicable.

### 4.2. How to present \( X' \) as a closed algebraic subvariety of \( C^N \).

**Proposition 4.1.** Let \( A = C[x, y, z], \ f \in C[x] \) have roots \( c_0 = 0, c_1, \ldots, A \leftrightarrow A' \) an affine modification, and \( X' \) satisfy assumptions (0) and (1) of the Main Theorem. Then \( X' \) is contractible and either \( X' \simeq C^3 \) or there exists a root of \( f \) (say \( c_0 \)) so that \( X' \) can be viewed as the subvariety of \( C^N \) given by polynomial equations

\[
\begin{align*}
xy_1 - q_0(y, z) &= 0 \\
xy_2 - q_1(y, z, v_1) &= 0 \\
\ldots \\
xy_m - q_m(y, z, v_1, \ldots, v_{m-1}) &= 0 \\
(x - c_1)u_{1,1} - r_{1,0}(y, z) &= 0 \\
(x - c_1)u_{1,2} - u_{1,2}^{n_1} + r_{1,1}(y, z, u_{1,1}) &= 0 \\
\ldots \\
(x - c_1)u_{n_1} - u_{n_1}^{n_1} + r_{n_1,1}(y, z, u_{1,1}, \ldots, u_{n_1-1}) &= 0 \\
(x - c_2)u_{2,1} - r_{2,0}(y, z) &= 0 \\
\ldots
\end{align*}
\]

where \( q_0(y, z) = y^k - z^l \), \( (k, l) = 1, k > l \geq 2, m > 1 \), the standard degree of \( q_j \) with respect to \( v_i \) is less than \( n_i \) \( \forall i = 1, \ldots, j \), and the standard degree of \( r_{s,j} \) with respect to \( u_{s,i} \) is less than \( n_{s,i} \) \( \forall i = 1, \ldots, j \). Furthermore, the defining ideal \( I' \) of \( X' \) in \( C^N \) is prime and generated by the left-hand sides of the equations above.

**Proof.** We can suppose that \( f(x) = x^n \) since Remark 2.1 reduces the problem to the case when \( f \) has one root only. As \( X' \setminus E \simeq C^3 \setminus \{x = 0\} \) we have \( H_2(X' \setminus E) = H_3(X' \setminus E) = 0 \). By Theorem 3.1 \( \sigma : X' \rightarrow X \) is a composition of basic modifications \( X' = X_m \sigma_{m-1} \cdots \sigma_0 X \). Let \( A_j = C[X_j] \) and \( C_j \) be as in Convention 2.2 (3). By Theorem 3.1 each \( C_j \) is either a point or an irreducible contractible curve. If \( C_0 \) is either a point or a smooth curve then \( X_1 \simeq C^3 \) and one can use induction on \( m \) (see the proof of Lemma 4.2). When \( C_0 \) is not a smooth curve, by the Lin-Zaidenberg theorem \([\text{LiZa}] \) one can assume \( C_0 \) is given in \( C^3 \) by \( x = y^k - z^l = 0 \) where \( (k, l) = 1 \) and \( k > l \geq 2 \). Let \( I_1 \) be the ideal in \( A \).
generated by $x$ and $y^k - z^l$. By Theorem 3.1 $A_1 = A[I_1/x]$. By Theorem 2.1
that $A_1 = \mathbb{C}[x, y, z, (y^k - z^l)/x]$ and $X_1$ is the irreducible hypersurface in
$\mathbb{C}^4$ with coordinates $(x, y, z, v_1)$ given by $xv_1 = q_0(y, z) := y^k - z^l$. By
Theorem 3.1 and by Lemma 3.3 $C_1$ is the zero fiber of a regular function on
the exceptional divisor $E_1 = X_1 \cap \{ x = 0 \}$ which is of form $v_1^{n_1} + q_1(y, z, v_1)$
where the standard degree of $q_1$ with respect to $v_1$ is at most $n_1 - 1$. Let $I_2$ be
the ideal in $A_1$ generated by $x$ and $v_1^{n_1} + q_1(y, z, v_1)$. By Theorem 3.1 $A_2 =
A_1[I_2/x]$. Therefore, by Theorem 2.1 $X_2$ may be viewed as the irreducible
complete intersection in $\mathbb{C}^5$ (with coordinates $(x, y, z, v_1, v_2)$) given by the
equations $xv_1 - q_0(y, z) = xv_2 - v_1^{n_1} + q_1(y, z, v_1) = 0$. Repeating the above
argument we see that $X'$ can be viewed as the desired irreducible complete
intersection in $\mathbb{C}^{3+m}$. Contractibility of $X'$ follows from Proposition 2.10.
In order to check that $m > 1$ when $X'$ is smooth it is enough to note that
$X_1$ is singular at the origin. □

In combination with Lemma 4.1 and [ChDi] we get:

**Corollary 4.1.** Suppose that $X'$ satisfies Assumptions (0) and (1) of the
Main Theorem and Assumption (2') of Lemma I. Then either $X' \sim \mathbb{C}^3$
or $X'$ is diffeomorphic to $\mathbb{R}^6$ and given by the system of equations from
Proposition 4.1.

5. The Makar-Limanov invariant.

5.1. General facts about locally nilpotent derivations. Recall that a
derivation $\partial$ on $A$ is called locally nilpotent if for each $a \in A$ there exists
an $k = k(a)$ such that $\partial^k(a) = 0$. For $t \in \mathbb{C}$ the mapping
$\exp(t\partial) : A \rightarrow A$
is an automorphism whence $\partial$ generates a $\mathbb{C}_+$-action on $X$ [Ren]. Every
locally nilpotent derivation defines a degree function $\deg_\partial$ on the domain
$A$ with natural values (e.g., see [FLN]) given by the formula $\deg_\partial(a) =
\max\{k \mid \partial^k(a) \neq 0\}$ for every nonzero $a \in A$.

**Proposition 5.1** (cf. [Za], proof of Lemma 9.3). Let $\partial$ be a nonzero locally
nilpotent derivation of $A = \mathbb{C}[X]$ and let $F = (f_1, \ldots, f_s) : X \rightarrow Y \subset \mathbb{C}^s$
and $G : Y \rightarrow Z \subset \mathbb{C}^j$ be dominant morphisms of reduced affine algebraic
varieties. Put $H = G \circ F = (h_1, \ldots, h_j) : X \rightarrow Z$. Suppose that for general
point $\xi \in Z$ there exists a (Zariski) dense subset $T_\xi$ of $G^{-1}(\xi)$ such that the
image of any nonconstant morphism from $\mathbb{C}$ to $G^{-1}(\xi)$ does not meet $T_\xi$. If
$h_1, \ldots, h_j \in A^0$ then $f_1, \ldots, f_s \in A^0$.

**Proof.** Consider the $\mathbb{C}_+$-action on $X$ generated by $\partial$. Choose a general point
$\xi \in Z$. Let $O_\xi$ be the orbit of $\xi \in H^{-1}(\xi)$. As $h_1, \ldots, h_j \in A^0$ the fiber
$H^{-1}(\xi)$ is invariant under the action and $O_\xi \subset H^{-1}(\xi)$. Note that $F(O_\xi)$
is a point $\forall \zeta \in F^{-1}(T_\xi)$. As $F^{-1}(T_\xi)$ is dense in $H^{-1}(\xi)$ this is also true
$\forall \zeta \in H^{-1}(\xi)$ whence each orbit is contained in a fiber of $F$ which yields the
desired conclusion. □
Definition 5.1. The Makar-Limanov invariant of \( A \) is \( \text{ML}(A) = \bigcap_{\partial \in \text{LND}(A)} \text{Ker} \partial \) where \( \text{LND}(A) \) is the set of all locally nilpotent derivations on \( A \). Equivalently, \( \text{ML}(A) \) is the subset of \( A \) which consists of those regular functions on \( X \) that are invariant under any regular \( \mathbb{C}_+ \)-action.

5.2. The associated algebra. Let \( A' = \mathbb{C}^{[N]}/I' \) where \( I' \) is a prime ideal in \( \mathbb{C}^{[N]} \). For every \( a \in A' \) put \( [a] = \{ p \in \mathbb{C}^{[N]} | p|_{X'} = a \} \) and for every \( p \in \mathbb{C}^{[N]} \setminus \{0\} \) we denote by \( M(p) \) the set of monomials that are summands of \( p \).

Definition 5.2. A weight degree function on \( \mathbb{C}^{[N]} \) is a degree function \( d \) such that \( d(p) = \max\{d(\mu) | \mu \in M(p)\} \), where \( p \in \mathbb{C}^{[N]} \setminus \{0\} \). Let \( \bar{p} = \sum_{\mu \in M(p), d(\mu) = d(p)} \mu \) be the leading \( d \)-homogeneous part of \( p \). Consider the ideal \( \hat{I}_d \) generated by such \( p \) when \( p \) runs over \( I' \setminus \{0\} \) and the variety \( \hat{X}_d = \mathbb{C}^{[N]} \) defined by \( \hat{I}_d \). Then we call \( \hat{A}_d = \mathbb{C}[\hat{X}_d] \) the associated graded algebra.

Proposition 5.2 ([KaM-L2]). Let \( X' \subset \mathbb{C}^{N} \) contain the origin of \( \mathbb{C}^{N} \). Then:

1. \( \forall a \in A' \setminus \{0\} \) there exists \( p \in [a] \) so that \( \bar{p} \notin \hat{I}_d \). Furthermore, the map \( \text{gr}_d : A' \setminus \{0\} \to \hat{A}_d \setminus \{0\} \) given by \( \text{gr}_d(a) = \bar{p}|_{\hat{X}_d} \) is well-defined.

2. Every nonzero \( \partial \in \text{LND}(A') \) generates a nonzero (associated) \( \hat{\partial} \in \text{LND}(\hat{A}_d) \) such that for every \( a \in A' \setminus \{0\} \) we have \( \deg \partial(a) \geq \deg_{\hat{\partial}}(\text{gr}_d(a)) \).

Convention 5.1. Let \( q_0(y,z) = y^k - z^l, m_i, n_{j,i} \), and coordinates \((x,y,z,v_1, \ldots, v_m, u_{1,1}, \ldots, u_{j,i}, \ldots) \) in \( \mathbb{C}^{N} \) be as in Proposition 4.1. Put \( d_z = d(x), d_y = d(y), d_z = d(z), d_i = d(v_i) \) and \( d_{j,i} = d(u_{j,i}) \) where \( d \) is a weight degree function. From now on we suppose that

1. \( kd_y = ld_z \) (in particular, \( q_0 = q_0 = y^k - z^l \));
2. \( d_1 + d_z = kd_y \) and \( d_1, d_x \) are \( \mathbb{Q} \)-independent;
3. \( d_x < 0 \) and \( d_1 >> d_y > 0 \);
4. \( d_x + d_{i+1} = m_id_i \) for \( i \geq 1 \);
5. \( d_x + d_{j,i+1} = n_{j,i}d_{j,i} > 0 \) for every \( j, i \geq 1 \).

This Convention implies the following non-difficult Proposition:

Proposition 5.3. Let \( X' \) be the zero set of the system of polynomial equations from Proposition 4.1 and \( A' = \mathbb{C}[X'] \). Then under Convention 5.1 the associated graded algebra \( \hat{A}_d = \mathbb{C}[\hat{X}_d] \) where \( \hat{X}_d \) is isomorphic to the zero
set of the following system
\[\begin{align*}
xv_1 - q_0(y, z) &= 0, xv_2 - v_1^{n_1} = 0, \ldots, xv_m - v_{m-1}^{n_{m-1}} = 0 \\
-c_1u_{1,1} &= 0, -c_1u_{1,2} - u_{1,1}^{n_1} = 0, \ldots, -c_1u_{m_1} - u_{1,m_1-1}^{n_{m_1-1}} = 0 \\
-c_2u_{2,1} &= 0, -c_2u_{2,2} - u_{2,1}^{n_{2,1}} = 0, \ldots, -c_2u_{2,m_2} - u_{2,m_2-1}^{n_{2,m_2-1}} = 0
\end{align*}\]

Furthermore, the defining ideal \(\hat{P}'_d\) of \(\hat{X}'_d\) is prime and generated by the left-hand sides of the equations above.

**Remark 5.1.** The variety \(\hat{X}'_d\) is independent on the choice of \(d\) satisfying Convention 5.1 and it is isomorphic to the zero set of the following polynomial equations in \(\mathbb{C}^{3+m}\) with coordinates \((x, y, z, v_1, \ldots, v_m)\)
\[\begin{align*}
P_1(x, y, z, v_1) &= xv_1 - q_0(y, z) = 0 \\
P_2(x, v_1, v_2) &= xv_2 - v_1^{n_1} = 0 \\
\vdots \\
P_m(x, v_{m-1}, v_m) &= xv_m - v_{m-1}^{n_{m-1}} = 0.
\end{align*}\]
Therefore, we shall write further \(\hat{P}', \hat{A}', \text{ and } \hat{X}'\) instead of \(\hat{P}'_d, \hat{A}'_d, \text{ and } \hat{X}'_d\) provided it does not cause misunderstanding.

### 5.3. Locally nilpotent derivation of Jacobian type
We say that \(a \in \hat{A}'\) is \(d\)-homogeneous if \(a\) is the restriction to \(\hat{X}' \subset \mathbb{C}^N\) of a \(d\)-homogeneous polynomial. In the rest of the paper we denote \(q|_{\hat{X}}\), by \(\hat{q}\) for every \(q \in \mathbb{C}^{m+3}\).

**Lemma 5.1.** Let \(a \in \hat{A}'\) be an irreducible \(d\)-homogeneous element. Then up to a constant factor \(a\) is of one of the following elements \(\tilde{v}_i, \tilde{x}, \tilde{y}, \tilde{z}\), or \(\tilde{y}^k + cz^l\) where \(c \in \mathbb{C}^*\) and \(k, l\) are the same as in Proposition 4.1.

**Proof.** Let \(q\) be \(d\)-homogeneous and \(a = \hat{q}\) (in particular, \(q\) is irreducible). By Remark 5.1 we can suppose that \(\forall \mu \in M(q)\) is non-divisible by \(xv_i \forall i = 1, \ldots, m\). Each \(\tilde{v}_i\) coincides with a rational function on \(\hat{X}'\) of form \(q_0^i/x^j\) where \(s, j > 0\). If we extend \(d\) naturally to the field of rational functions then \(d(v_i) = d(q_0^i/x^j)\) by Convention 5.1. Assume that \(\mu_1, \mu_2 \in M(q)\) are such that \(\mu_1\) is divisible by \(x\) but \(\mu_2\) is not. Then \(\mu_1\) and \(\mu_2\) coincides with the restriction to \(\hat{X}'\) of functions \(x^j y^{a_1} z^{b_1}\) and \(y^{a_2} z^{b_2} q_0^i/x^{j_2}\) where \(j_1 > 0, j_2 \geq 0\). As \(d(\mu_1) = d(\mu_2)\) we have \((j_1 + j_2)d_x = d(y^{a_2-a_1} z^{b_2-b_1} q_0^i)\).
As \(d_y = (l/k)d_z\) and \(d(q_0) = kd_y\) we get \(\mathbb{Q}\)-dependence of \(d_x\) and \(d_y\) which contradicts Convention 5.1. Thus if \(q \neq cx, c \in \mathbb{C}^*\), none of \(\mu \in M(q)\) is divisible by \(x\). Let \(\mu_1, \mu_2 \in M(q)\) and \(\mu_i = y^{a_i} z^{b_i} v_i\) where \(v_i\) is a monomial which depends on \(v_1, \ldots, v_m\) only. The restriction of \(\mu_i\) to \(\hat{X}'\) coincides with \(y^{a_i} z^{b_i} q_0^i/x^{j_i}\). The same argument as above shows that \(j_1 = j_2\) since otherwise \(d_x\) and \(d_y\) are \(\mathbb{Q}\)-dependent. Hence \(d(y^{a_1} z^{b_1} q_0^i) = d(y^{a_2} z^{b_2} q_0^i)\).
As \(d(q_0) = kd_y = ld_z\) and \((k, l) = 1\) we have \(\alpha_i = \alpha_0 + ti, k\) and \(\beta_i = \beta_0 + \tau, l\)
where \(0 \leq \alpha_0 \leq k-1\), \(0 \leq \beta_0 \leq l-1\), and \(t_1 - t_2 + \tau_1 - \tau_2 = s_2 - s_1\). Therefore, the restriction of \(q\) to \(\hat{X}'\) coincides with \(y^{\alpha_0}z^{\beta_0}\varphi(y^k, z^l)q_0/x^j\) where \(\varphi(y^k, z^l)\) is \(d\)-homogeneous and the restriction of \(q_0/x^j\) to \(\hat{X}'\) coincides with the restriction of a monomial \(\nu\) which depends on \(v_1, \ldots, v_m\) only. Now the Lemma follows from the fact that that \(\varphi(y^k, z^l)\) is the product of factors of type \(c_1y^k + c_2z^l\) where \(c_1, c_2 \in \mathbb{C}\). \(\square\)

**Corollary 5.1.** Let \(a = \bar{q}\) where \(q \notin \mathbb{C}[y, z]\) is a \(d\)-homogeneous polynomial which does not depend on \(x\). Then \(q\) is divisible by some \(v_i\).

Note that \((\bar{x}, \bar{y}, \bar{z})\) is a local (holomorphic) coordinate system at each point of \(\hat{X}_0 = \hat{X}' \setminus \{x = 0\}\). For \(a_1, a_2, a_3 \in \hat{A}'\) we denote by \(\text{Jac}(a_1, a_2, a_3)\) the Jacobian of these regular functions on \(\hat{X}_0\) with respect to \(\bar{x}, \bar{y}, \bar{z}\). This is a rational function on \(\hat{X}'\) but \(\bar{x}^m\text{Jac}(a_1, a_2, a_3)\) is already regular on \(\hat{X}'\) since \(x^m\) is the determinant of the matrix \(\{\partial P_i/\partial v_j \mid i, j = 1, \ldots, m\}\) where \(P_i\) are as in Remark 5.1. Fix \(a_1, a_2 \in \hat{A}'\) and let \(a \in \hat{A}'\). Then one can see that \(\partial(a) = \bar{x}^m\text{Jac}(a_1, a_2, a)\) is a derivation on \(\hat{A}'\).

**Proposition 5.4.** Let \(m \geq 2\), \(a_1\) and \(a_2\) be \(d\)-homogeneous, and \(\partial(a) = \bar{x}^m\text{Jac}(a_1, a_2, a)\) be nontrivial locally nilpotent. Then:

1. If \(a_1\) and \(a_2\) are irreducible then \((a_1, a_2)\) coincides (up to the order) with one of the pairs \((\bar{x}, \bar{y})\) or \((\bar{x}, \bar{z})\).
2. \(\bar{x} \in \text{Ker} \partial\) and \(\deg \partial(\bar{v}_i) \geq 2\) for every \(i = 1, \ldots, m\).

**Proof.** If \((a_1, a_2)\) is one of the pairs in (1) it is easy to check that \(\partial\) is nontrivial and locally nilpotent, and (2) holds also. Show that if we use other possible irreducible \(d\)-homogeneous elements from Lemma 5.1 as \(a_1, a_2\) then \(\partial\) cannot be a nontrivial locally nilpotent derivation. Note that \(a_1\) and \(a_2\) are algebraically independent in \(\hat{A}'\) as otherwise \(\partial\) is trivial.

**Case 1.** Let \((a_1, a_2) = (\bar{y}, \bar{z})\). The direct computation shows that \(\partial(\bar{x}) = \bar{x}^m\) whence \(\partial\) cannot be locally nilpotent. Indeed, one can see that \(\deg \partial(\partial(\bar{x})) = \deg \partial(\bar{x}) - 1\). But \(\deg \partial(\bar{x}^m) = m\deg \partial(\bar{x})\) which yields a contradiction.

**Case 2.** Either \(a_1\) or \(a_2\) is of form \(\bar{y}^k + cz^l\) where \(c \in \mathbb{C}^*\) and \(k\) and \(l\) are as in Proposition 4.1. By [M-L2] \(\bar{y}, \bar{z} \in \text{Ker} \partial\) as \(\bar{y}^k + cz^l \in \text{Ker} \partial\). By [KaM-L1, Lemma 5.3] the derivation \(\bar{x}^m\text{Jac}(\bar{y}, \bar{z}, a)\) must be also nonzero locally nilpotent whence this case does not hold.

**Case 3.** Let \((a_1, a_2) = (\bar{v}_{i_1}, \bar{v}_{i_2})\) where \(i_1 < i_2\). Consider the identical morphism \(F : \hat{X}' \to \hat{X}' \subset \mathbb{C}^{m+3}\) and morphism \(G : \hat{X}' \to \mathbb{C}^2\) given by \((\bar{x}, \bar{y}, \bar{z}, \bar{v}_{i_1}, \ldots, \bar{v}_m) \to (\bar{v}_{i_1}, \bar{v}_{i_2})\). Recall that \(\bar{v}_{i_k} = \bar{q}_0^{\delta_k}/\bar{x}^{\delta_k}\). It is easy to check that \(\bar{v}_{i_1}\) and \(\bar{v}_{i_2}\) are algebraically independent in \(\hat{A}'\) which means that the pairs \((s_1, j_1)\) and \((s_2, j_2)\) are not proportional. Consider a general point \(\xi \in \mathbb{C}^2\). Each component of the fiber \(G^{-1}(\xi)\) is a curve in \(\mathbb{C}^{m+3}\) given by equations \(v_i = c_i, x = c',\) and \(q_0(y, z) = y^k - z^l = c\) where \(c_i, c' \in \mathbb{C}\).
and \( c \in \mathbb{C}^* \). This curve is hyperbolic and thus it does not admit nonconstant morphisms from \( \mathbb{C} \). By Proposition 5.1 if \( \partial \) is locally nilpotent then \( \hat{x}, \hat{y}, \hat{z}, \hat{v}_i \in \text{Ker} \partial \) whence \( \partial \) is trivial. Thus this case does not hold.

**Case 4.** Let \((a_1, a_2) = (\hat{x}, \hat{v}_i)\). The same argument is in Case 3 works.

**Case 5.** Let \((a_1, a_2) = (\hat{y}, \hat{v}_i)\) (or, similarly \((\hat{z}, \hat{v}_i)\)). Consider the identical morphism \( F : \tilde{X}' \to \tilde{X}' \) and \( G : \tilde{X}' \to \mathbb{C}^2 \) given by \((\hat{x}, \hat{y}, \hat{z}, \hat{v}_1, \ldots, \hat{v}_m) \to (\hat{y}, \hat{v}_i)\). As \( \hat{v}_i = \tilde{q}_i^a / \tilde{z}^j \) (where \( j \geq 2 \) if \( i \geq 2 \)) the fiber \( G^{-1}(\xi) \) where \( \xi = (c_1, c_2) \in \mathbb{C}^2 \) is isomorphic to the curve \((c_1^k - z^l)^s - c_2x^j = 0\). When \( j \geq 2 \) and \( s \) is not divisible by \( j \) the last curve have no contractible components for general \( \xi \). Proposition 5.1 implies that \( \partial \) must be trivial. If \( j \geq 2 \) and \( s \) is divisible by \( j \) then each irreducible component of \( G^{-1}(\xi) \) is contractible and contains double points of \( G^{-1}(\xi) \). As \( G^{-1}(\xi) \subset \tilde{X}' \) is invariant under the associated \( \mathbb{C}_+ \)-action and has singular points this action is trivial on \( G^{-1}(\xi) \) and, thus, on \( \tilde{X}' \). Hence \( \partial \) is trivial. Let \( j = 1 \), i.e., \((a_1, a_2) = (\hat{y}, \hat{v}_i) = (\hat{y}, \tilde{q}_i) \). The direct computation shows that \( \partial(\hat{x}) = c\hat{x}^{m-1}\hat{z}^{l-1}, c \in \mathbb{C} \). As \( m \geq 2 \), \( \partial \) cannot be nontrivial locally nilpotent (indeed, compare \( \text{deg}_\partial(\hat{x}) \) and \( \text{deg}_\partial(\partial(\hat{x})) \)) and we have to disregard this case. In order to see statement \((2)\) in the case when \( a_1 \) and \( a_2 \) are not irreducible, we note that one can replace \( a_1 \) and \( a_2 \) with their irreducible factors in the definition of \( \partial \) and obtain a locally nilpotent derivation equivalent to \( \partial \) [KaM-L2].

### 5.4. The computation of \( \text{ML}(A') \).

A locally nilpotent derivation \( \partial \) on \( A' \) is called perfect if its associated derivation \( \hat{\partial}_d \) is of form \( \hat{\partial}_d(a) = \hat{z}^m \text{Jac } (a_1, a_2, a) \) where \( a_1, a_2 \in A'_d \) are \( d \)-homogeneous and algebraically independent. The set of all perfect derivations will be denoted by \( \text{Per}(A') \).

**Proposition 5.5.** Let \( A' \) be as in Proposition 4.1 and let \( d \) satisfy Convention 5.1. For every \( \partial \in \text{Per}(A') \) we have \( x \in \text{Ker} \partial \).

**Proof.** Let \( a \in A' \) with \( \text{deg}_\partial(a) \leq 1 \). Show that there exist a polynomial \( q \) with \( q|_{x'} = a \) such that none of \( \mu \in M(q) \) is divisible by \( v_i \) or \( u_{s,j} \) for all \( i, s, j \), i.e., \( q \in \mathbb{C}[x, y, z] \). By Proposition 4.1 we can suppose that none of \( \mu \in M(q) \) is divisible by \( xv_i \) or \( xu_{s,j} \). Thus \( M(q) = \{M_1(q) \cup M_2(q)\} \) where \( M_1(q) \subset \mathbb{C}[x, y, z] \) and \( M_2(q) \) consists of monomials which do not depend on \( x \) and do not belong to \( \mathbb{C}[y, z] \). Assume that \( \mu \in M_2(q) \). Under Convention 5.1 one can keep \( d_y, d_z \) fixed, decrease \( d_x \), and increase \( d_i, d_{s,j} \) so that \( d(\mu) > d(\nu) \forall \nu \in M_1(q) \). Hence if \( \tilde{q}_d \) is the leading \( d \)-homogeneous part of \( q \) then \( M(\tilde{q}_d) \subset M_2(q) \). By Proposition 5.2 \( \text{deg}_\partial(\text{gr}_d(a)) \leq 1 \). The element \( \text{gr}_d(a) = \tilde{q}_d|_{x'} \) is the product of irreducible \( d \)-homogeneous elements of \( \hat{A}' \). By Corollary 5.1 one of them is \( \hat{v}_i \) whence \( \text{deg}_\partial(\hat{v}_i) \leq \text{deg}_\partial(\text{gr}_d(a)) \leq 1 \) which contradicts Proposition 5.4. Thus \( M_2(q) \) is empty. Let \( b \in A' \)
with $\deg \partial(b) = 1$. By [M-L1] there exist $a', a_0, \ldots, a_s \in \ker \partial$ such that

$$a' v_1 = \sum_{j=0}^s a_j b_j$$

where $s = \deg a' v_1$. Hence $v_1 = (q(x,y,z)/r(x,y,z))_{|X'}$, where $a' = r(x,y,z)_{|X'}$. But $A' \subset \mathbb{C}[x,y,z,1/f(x)]$ where $f$ is as in Proposition 4.1. Since $v_1 \not\in \mathbb{C}[x,y,z]$, we have $r(x,y,z)$ divisible by some $x-c$ where $c$ is a root of $f$. Therefore, $x-c \in \ker \partial$ as a divisor of an element from $\ker \partial$.

Proof of Lemma II and the Main Theorem. By Proposition 5.3 $\tilde{A}'$ is a domain whence $ML(A') = \bigcap_{\partial \in \text{Per}(A')} \ker \partial$ [KaM-L2]. By Propositions 5.5 $ML[A'] \supset \mathbb{C}[x]_{|X'}$ which implies Lemma II and, therefore, the Main Theorem.

References


4It can be shown that $ML[A'] = \mathbb{C}[x]_{|X'}$. 
POLYNOMIALS WITH GENERAL $C^2$-FIBERS ARE VARIABLES


Received April 21, 2000 and revised May 3, 2001. This work was partially supported by NSA grant MDA 904-00-1-0016.

Department of Mathematics
University of Miami
Coral Gables, FL 33124
E-mail address: kaliman@math.miami.edu
A COMBINATORIAL APPROACH TO THE QUANTIFICATION OF LIE ALGEBRAS

V.K. Kharchenko

We propose a notion of a quantum universal enveloping algebra for any Lie algebra defined by generators and relations which is based on the quantum Lie operation concept. This enveloping algebra has a PBW basis that admits a monomial crystallization by means of the Kashiwara idea. We describe all skew primitive elements of the quantum universal enveloping algebras for the classical nilpotent algebras of the infinite series defined by the Serre relations and prove that the above set of PBW-generators for each of these enveloping algebras coincides with the Lalonde–Ram basis of the ground Lie algebra with a skew commutator in place of the Lie operation. The similar statement is valid for Hall–Shirshov basis of any Lie algebra defined by one relation, but it is not so in the general case.

1. Introduction.

Quantum universal enveloping algebras appeared in the famous papers by Drinfeld [15] and Jimbo [18]. Since then a great deal of articles and number of monographs were devoted to their investigation. All of these publications are mainly concerned with a particular quantification of Lie algebras of the classical series. This is accounted for first by the fact that these Lie algebras have applications and visual interpretations in physical speculations, and then by the fact that a general, and commonly accepted as standard, notion of a quantum universal enveloping algebra is not elaborated yet (see a detailed discussion in [2], [33]).

In the present paper we propose a combinatorial approach to a solution of this problem by means of the quantum (Lie) operation concept [22], [24], [25]. In line with the main idea of our approach, the skew primitive elements must play the same role in quantum enveloping algebras as the primitive elements do in the classical case. By the Friedrichs criteria [13], [16], [32], [34], [35], the primitive elements form the ground Lie algebra in the classical case. For this reason we consider the space spanned by the skew primitive elements and equipped with the quantum Lie operations as a quantum analogue of a Lie algebra.
In the second section we adduce the main notions and consider some examples. These examples, in particular, show that the Drinfeld–Jimbo enveloping algebra as well as its modifications are quantum enveloping algebras in our sense.

In the third section with the help of the Heyneman–Radford theorem we introduce a notion of a \textit{combinatorial rank} of a Hopf algebra generated by skew primitive semi-invariants. Then we define the quantum enveloping algebra of an \textit{arbitrary rank} that slightly generalizes the definitions given in the preceding section.

The basis construction problem for the quantum enveloping algebras is considered in the fourth section. We indicate two main methods for the construction of \textit{PBW-generators}. One of them modifies the Hall–Shirshov basis construction process by means of replacing the Lie operation with a skew commutator. The set of the PBW-generators defined in this way, the values of \textit{hard super-letters}, plays the same role as the basis of the ground Lie algebra does in the PBW theorem. At first glance it would seem reasonable to consider the $k[G]$-module generated by the values of hard super-letters as a quantum Lie algebra. However, this extremely important module falls far short of being uniquely defined. It essentially depends on the ordering of the main generators, their degrees, and it is almost never antipode stable. Also we have to note the following important fact. Our definition of the hard super-letter is not constructive and, of course, it cannot be constructive in general. The basis construction problem includes the word problem for Lie algebras defined by generators and relations, while the latter one has no general algorithmic solution (see \cite{5}, \cite{8}).

The second method is connected with the Kashiwara crystallization idea \cite{20}, \cite{21} (see also a development in \cite{12}, \cite{27}). M. Kashiwara has considered the main parameter $q$ of the Drinfeld–Jimbo enveloping algebra as a temperature of some physical medium. When the temperature tend to zero, the medium crystallizes. The PBW-generators must crystallize as well. In our case under this process no one limit quantum enveloping algebra appears since the existence conditions normally include equalities of the form $\prod p_{ij} = 1$ (see \cite{24}). Nevertheless if we equate all quantification parameters to zero, the hard super-letters would form a new set of PBW-generators for the given quantum universal enveloping algebra. To put this another way, the PBW-basis defined by the super-letters admits a crystallization by means of the Kashiwara idea.

In the fifth section we bring a way to construct a Groebner–Shirshov relations system for a quantum enveloping algebra. This system is related to the main skew primitive generators, and, according to the Diamond Lemma (see \cite{4}, \cite{6}, \cite{41}), it determines the basis appeared in the above crystallization process. The usefulness of the Groebner–Shirshov systems depends upon the fact that such a system not only defines a basis of an associative
A combinatorial approach provides a simple diminishing algorithm for expansion of elements on this basis (see, for example [3], [8]).

In the sixth section we adapt a well-known method of triangular splitting to the quantification with constants. The original method appeared in studies of simple finite dimensional Lie algebras. Then it has been extended into the field of quantum algebra in a lot of publications (see, for example [9], [31], [42]). By means of this method the investigation of the Drinfeld–Jimbo enveloping algebra amounts to a consideration of its positive and negative homogeneous components, quantum Borel sub-algebras. Constructions of this type also appear in classification theorems for pointed Hopf algebras (see [1]).

In the seventh section we consider more thoroughly the quantum universal enveloping algebras of nilpotent algebras of the series \( A_n, B_n, C_n, D_n \) defined by the Serre relations. We adduce first lists of all hard super-letters in the explicit form, then Grobner–Shirshov relations systems, and next spaces \( L(U_P(\mathfrak{g})) \) spanned by the skew primitive elements (i.e., the Lie algebra quantifications \( \mathfrak{g}_P \) proper). In all cases the lists of hard super-letters (but not the hard super-letters themselves) turn out to be independent of the quantification parameters. This means that the PBW-generators result from the Hall–Shirshov basis of the ground Lie algebra by replacing the Lie operation with the skew commutator. The same is valid for the Grobner–Shirshov relations systems. Note that the Hall–Shirshov bases, under the name standard Lyndon bases, for the classical Lie series were constructed by P. Lalonde and A. Ram [28], while the Grobner–Shirshov systems of Lie relations were found by L.A. Bokut’ and A.A. Klein [7].

Furthermore, in all cases \( \mathfrak{g}_P \) as a quantum Lie algebra (in our sense) proves to be very simple in structure. Either it is a colored Lie super-algebra (provided that the parameter \( p_{11} \) equals 1), or values of all non-umary quantum Lie operations equal zero on \( \mathfrak{g}_P \). In particular, if \( \text{char}(k) = 0 \) and \( p'_{11} \neq 1 \) then the quantum Lie operations may be defined on \( \mathfrak{g}_P \), but all of them have zero values. Thus, in this case we have a reason to consider \( U_P(\mathfrak{g}) \) as an algebra of ‘commutative’ quantum polynomials, since the universal enveloping algebra of a Lie algebra with zero bracket is the algebra of ordinary commutative polynomials. Immediately afterwards a number of interesting questions appears. What is the structure of other algebras of ‘commutative’ quantum polynomials? When do the PBW-generators result from a basis of the ground Lie algebra by means of replacing the Lie operation with the skew commutator? These and other questions we briefly discuss in the last section.

It is well to bear in mind that the combinatorial approach is not free from flaws: The quantum universal enveloping algebra essentially depends on a combinatorial representation of the ground Lie algebra, i.e., a close connection with the abstract category of Lie algebras is lost.
2. Quantum enveloping algebras.

Recall that a variable $x$ is called a quantum variable if an element $g_x$ of a fixed Abelian group $G$ and a character $\chi^x \in G^*$ are associated with it. The parameters $g_x$ and $\chi^x$ associated with a quantum variable say that an element $a$ in a Hopf algebra $H$ may be considered as a value of this quantum variable only if $a$ is a skew primitive semi-invariant with the same parameters, that is

\[ \Delta(a) = a \otimes 1 + g_x \otimes a, \quad g^{-1}ag = \chi^x(g)a, \quad g \in G, \]

where we suppose that the elements of $G$ have some interpretation in $H$ as grouplike elements.

A noncommutative polynomial in quantum variables is called a quantum Lie operation if all of its values in all Hopf algebras are skew primitive for all values of the quantum variables.

Let $x_1, \ldots, x_n$ be a set of quantum variables. For each word $u$ in $x_1, \ldots, x_n$ we denote by $g_u$ an element of $G$ that appears from $u$ by replacing of all $x_i$ with $g_{x_i}$. In the same way we denote by $\chi^u$ a character that appears from $u$ by replacing of all $x_i$ with $\chi^{x_i}$. Thus on the free algebra $k\langle x_1, \ldots, x_n \rangle$ a grading by the group $G \times G^*$ is defined. For each pair of homogeneous elements $u, v$ we fix the denotations $p_{uv} = \chi^u(g_v) = p(u,v)$.

We define an action of $G$ on $k\langle x_1, \ldots, x_n \rangle \ast G$ has a natural Hopf algebra structure with the coproduct

\[ \Delta(x_i) = x_i \otimes 1 + g_{x_i} \otimes x_i, \quad 1 \leq i \leq 1, \quad \Delta(g) = g \otimes g, \quad g \in G. \]

Hence $x_i = x_i \in G(X)$ are correct values of quantum variables. By this means the quantum Lie operations can be identified with skew primitive polynomials in $G(X)$. Recall that the Hopf algebra $G(X)$ is called the free enveloping algebra for the set $X$ of quantum variables (see [22, Sect. 3] under denotation $H(X)$).

The free algebra $k\langle x_1, \ldots, x_n \rangle$ has a structure of braided bigraded Hopf algebra. Namely, let $\mathcal{H}$ be an associative algebra graded by the group $G \times G^*$:

\[ \mathcal{H} = \sum_{g \in G, \chi \in G^*} \mathcal{H}^g. \]

Define multiplication on the tensor product $\mathcal{H} \otimes \mathcal{H}$ of linear spaces by setting

\[ (a \otimes b) \cdot (c \otimes d) = (\chi^c(g_b))^{-1}(ac \otimes bd). \]

The result is an associative algebra, denoted by $\mathcal{H} \otimes \mathcal{H}$. Now if, in the definition of a Hopf algebra, we change the sign $\otimes$ by $\overline{\otimes}$, and assume coproduct, $\Delta^b$, counity, $\varepsilon^b$, and antipode, $S^b$, are homogeneous, we arrive at a definition of the braided bigraded Hopf algebra. In other words a braided
bigraded Hopf algebra is a graded by $G \times G^*$ Hopf algebra in braided category where the braiding is connected with the grading by the formula $c(u \otimes v) = (χ_{\psi}(g_u))^{-1}(v \otimes u)$.

The quantum Lie operation can be defined equivalently as a $G \times 1$-homogeneous polynomial that has only primitive values in all braided bigraded Hopf algebras provided that the correct value of a quantum variable $x = x^\chi_g$ is primitive and homogeneous, that is $a \in H^\chi_g$, $\Delta^b(a) = a \otimes 1 + 1 \otimes a$. The detailed discussion of the notion of quantum Lie operation and examples can be found in [22, Sect. 1–4].

Recall that a constitution of a word $u$ is a sequence of nonnegative integers $(m_1, m_2, \ldots, m_n)$ such that $u$ is of degree $m_1$ in $x_1$, $\deg_1(u) = m_1$; of degree $m_2$ in $x_2$, $\deg_2(u) = m_2$; and so on (see [39, Definition 3]). Since the group $G$ is Abelian, all constitution homogeneous polynomials are homogeneous with respect to the grading. Let us define a bilinear skew commutator on the set of graded homogeneous noncommutative polynomials by the formula

$$ [u, v] = uv - p_{uv}vu. $$

These brackets satisfy the following Jacobi and skew differential identities:

$$ [[u, v], w] = [u, [v, w]] + p_{uv}^{-1}([u, w], v) + (p_{uv} - p_{wu})[u, w] \cdot v; $$

$$ [[u, v], w] = [u, [v, w]] + p_{uv}([u, w], v) + p_{uv}(p_{vw}p_{uv} - 1)v \cdot [u, w]; $$

$$ [u, v \cdot w] = [u, v] \cdot w + p_{uv}v \cdot [u, w]; \quad [u \cdot v, w] = p_{uv}[u, w] \cdot v + u \cdot [v, w], $$

where by the dot we denote the usual multiplication. It is easy to see that the following conditional restricted identities are valid as well

$$ [u, v^n] = \ldots [u, [v, v] \ldots , v]; \quad [v^n, u] = [v, \ldots [v, u] \ldots ], $$

provided that $p_{uv}$ is a primitive $l$-th root of unit, and $n = t$ or $n = tl^k$ in the case of characteristic $l > 0$.

Suppose that a Lie algebra $\mathfrak{g}$ is defined by the generators $x_1, \ldots, x_n$ and the relations $f_i = 0$. Let us convert the generators into quantum variables. For this associate to them elements of $G \times G^*$ in arbitrary way. Let $P = ||p_{ij}||$, $p_{ij} = \chi^{x_i}(g_{x_j})$ be the quantification matrix.

**Definition 2.1.** A **braided quantum enveloping algebra of $\mathfrak{g}$** is a braided bigraded Hopf algebra $U^b_P(\mathfrak{g})$ defined by the variables $x_1, \ldots, x_n$ and the relations $f_i = 0$, where the Lie operation is replaced with (2), provided that in this way $f_i$ are converted into the quantum Lie operations $f^*_i$. The coproduct and the braiding are defined by

$$ \Delta^b(x_i) = x_i \otimes 1 + 1 \otimes x_i, $$

$$ (x_i \otimes x_j) \cdot (x_k \otimes x_m) = (\chi^{x_k}(g_{x_j}))^{-1} x_i x_k \otimes x_j x_m. $$


Definition 2.2. A simple quantification of $U(\mathfrak{g})$ or a quantum universal enveloping algebra of $\mathfrak{g}$ is an algebra $U_P(\mathfrak{g})$ that is isomorphic to the skew group algebra

$$U_P(\mathfrak{g}) = U^b_P(\mathfrak{g}) \ast G,$$

where the group action and the coproduct are defined by

$$g^{-1}x_i g = \chi^x_i(g)x_i, \quad \Delta(x_i) = x_i \otimes 1 + g x_i \otimes x_i, \quad \Delta(g) = g \otimes g. \tag{9}$$

Definition 2.3. A quantification with constants is a simple quantification where additionally some generators $x_i$ associated to the trivial character are replaced with the constants $\alpha_i(1 - g x_i)$.

The formulae (10) and (7) correctly define the coproduct since by definition of the quantum Lie operation $\Delta(f^*_i) = f^*_i \otimes 1 + g_i \otimes f^*_i$ in the case of ordinary Hopf algebras and $\Delta^b(f^*_i) = f^*_i \otimes 1 + 1 \otimes f^*_i$ in the braided case.

We have to note that the defined quantifications essentially depend on the combinatorial representation of the Lie algebra. For example, an additional relation $[x_1, x_1] = 0$ does not change the Lie algebra. At the same time if $\chi^{x_1}(g_1) = -1$ then this relation admits the quantification and yields a nontrivial relation for the quantum enveloping algebra, $2x_1^2 = 0$.

Example 1. Suppose that the Lie algebra is defined by a system of constitution homogeneous relations. If the characters $\chi^i$ are such that $p_{ij}p_{ji} = 1$ for all $i, j$ then the skew commutator itself is a quantum operation. Therefore on replacing the Lie operation all relations become quantum operations as well. This means that the braided enveloping algebra is the universal enveloping algebra $U(\mathfrak{g}^{col})$ of the colored Lie super-algebra which is defined by the same relations as the given Lie algebra is. The simple quantification appears as the Radford biproduct $U(\mathfrak{g}^{col}) \ast k[G]$ or, equivalently, as the universal $G$-enveloping algebra of the colored Lie super-algebra $\mathfrak{g}^{col}$ (see [37] or [22, Example 1.9]).

Example 2. Suppose that the Lie algebra $\mathfrak{g}$ is defined by the generators $x_1, \ldots, x_n$ and the system of nil relations

$$x_j(ad x_i)^{n_{ij}} = 0, \quad 1 \leq i \neq j \leq n. \tag{11}$$

Usually instead of the matrix of degrees (without the main diagonal), $||n_{ij}||$, the matrix $A = ||a_{ij}||$, $a_{ij} = 1 - n_{ij}$ is considered. The Coxeter graph $\Gamma(A)$ is associated to every such a matrix. This graph has the vertices $1, \ldots, n$, where the vertex $i$ is connected by $a_{ij}a_{ji}$ edges with the vertex $j$.

If $a_{ij} = 0$ then the relation $x_jad x_i = 0$ is in the list (11), and the relation $x_i(ad x_j)^{n_{ji}} = 0$ is a consequence of it. The skew commutator $[x_j, x_i]$ is a quantum Lie operation if and only if $p_{ij}p_{ji} = 1$. Under this condition we have $[x_i, x_j] = -p_{ij}[x_j, x_i]$. Therefore both in the given Lie algebra and in its quantification one may replace the relation $x_i(ad x_j)^{n_{ji}} = 0$ with $x_iad x_j = 0$. 
In other words, without loss of generality, we may suppose that \(a_{ij} = 0 \leftrightarrow a_{ji} = 0\). By the Gabber-Kac theorem [17] we get that the algebra \(g\) is the positive homogeneous component \(g_1^+\) of a Kac-Moody algebra \(g_1\).

The following theorem describes the conditions for a homogeneous polynomial in two variables which is linear in one of them to be a quantum operation.

**Theorem 2.4.** For quantum variables \(x_1\) and \(x_2\), there exists a nonzero linear in \(x_1\) quantum Lie operation \(W\) of degree \(n\) in \(x_2\) if and only if either \(p_{12}p_{21} = p_{22}^{1-n}\), or \(p_{22}\) is a primitive \(m\)-th root of unity, \(m|n\), and \(p_{12}^m p_{21}^m = 1\). If one of these conditions is satisfied, then all the operations have the form \(W = \alpha[[x_1 x_2] x_2] \ldots x_2\), \(\alpha \in k\), where the brackets are defined by (2).

**Proof.** It follows from Theorem 6.1 [22], and the conditional identity (6). 

From this theorem we have the following corollary.

**Corollary 2.5.** If \(n_{ij}\) is a simple number or unit and in the former case \(p_{ii}\) is not a primitive \(n_{ij}\)-th root of unit, then the relation (11) admits a quantification if and only if \(p_{ij} = p_{ii}^{a_{ij}}\).

Theorem 2.4 provides no essential restrictions on the non-diagonal parameters \(p_{ij}\): If the matrix \(P\) correctly defines a quantification of (11) then for every set \(Z = \{z_{ij}|z_{ij}z_{ji} = z_{ii} = 1\}\) the following matrix does as well:

\[
P_Z = \{p_{ij}z_{ij}|p_{ij} \in P, z_{ij} \in Z\}.
\]

**Example 3.** Let \(G\) be freely generated by \(g_1, \ldots, g_n\) and \(A\) be a generalized Cartan matrix symmetrized by \(d_1, \ldots, d_n\), while the characters are defined by \(p_{ij} = q^{-d_{ij}}\). In this case the simple quantification of \(g\) defined by (11) is the positive component of the Drinfeld–Jimbo enveloping algebra together with the group-like elements, \(U_P(g) = U_q^+(g) \ast G\). By means of an arbitrary deformation (12) one may define a ‘coloring’ of \(U_q^+(g) \ast G\).

The braided enveloping algebra equals \(U_q^+(g)\) where the coproduct and braiding are defined by (7) and (8) with the coefficient \(q^{d_{i_1 k_{ij}}}\). The formula (12) correctly defines its ‘coloring’ as well.

**Example 4.** If in the above example we complete the set of quantum variables by the new ones \(x^-_1, \ldots, x^-_n; z_1, \ldots, z_n\) such that

\[
\chi^x = (\chi^x)^{-1}, \quad g_{xx} = g_x, \quad z_i = \text{id}, \quad g_{zi} = g_{i2}^2;
\]

then, by Theorem 2.4, the Gabber–Kac relations (2), (3) of [17, Theorem 2], and \([e_i, f_j] = \delta_{ij} h_i\) under the identification \(e_i = x_i, f_i = x_i^-, h_i = z_i\) admit the quantification with constants \(z_i = \varepsilon_i (1 - g_i^2)\). (Informally, we may consider the obtained quantification as one of the Kac–Moody algebra identifying \(g_i\) with \(q^{h_i}\), where the rest of the Kac–Moody algebra relations, \([h_i, e_j] = a_{ij} e_i, [h_i, f_j] = -a_{ij} f_j\), is quantified to the \(G\)-action:
\( g_j x_i^+ g_j = q^{\mp d_{ij} a_j} x_i^+ \). This quantification coincides with the Drinfeld–Jimbo one under a suitable choice of \( x_i, x_i^- \), and \( \varepsilon_i \) depending up the particular definition of \( U_q(g) \):

\begin{align*}
[30] & x_i = E_i, \; g_i = K_i, \; x_i^- = F_i K_i, \; p_{ij} = v^{-d_{ai} a_j}, \; \varepsilon_i = (v^{-d_i} - v^{d_i})^{-1}; \\
[31] & x_i = E_i, \; g_i = K_i, \; x_i^- = F_i K_i, \; p_{ij} = v^{-\langle \mu_i, \ell_j \rangle}, \; \varepsilon_i = (v^{1} - v^{-1})^{-1}; \\
[20] & x_i = e_i, \; g_i = t_i, \; x_i^- = t_i f_i, \; p_{ij} = q^{-\langle h_j, a_i \rangle}, \; \varepsilon_i = (q_i - q_i^{-1})^{-1}; \\
[36] & x_i = E_i K_i, \; g_i = K_i^2, \; x_i^- = F_i K_i, \; p_{ij} = q^{-2d_{ai} a_j}, \; \varepsilon_i = (1 - q^{4d_i})^{-1}.
\end{align*}

By (13) the brackets \( [x_i, x_j^-] \) are the quantum Lie operation only if \( p_{ij} = p_{ji} \).

So in this case the ‘colorings’ (12) may be only black-white, \( z_{ij} = \pm 1 \).

In the perfect analogy the Kang quantification (19) of the generalized Kac-Moody algebras (10) is a quantification in our sense as well.

### 3. Combinatorial rank.

Recall that a Hopf algebra \( H \) is called character if the group \( G \) of all group-like elements is commutative and \( H \) is generated by skew primitive invariants \( a_i \):

\( \Delta(a_i) = a_i \otimes 1 + g_{a_i} \otimes a_i, \quad g^{-1} a_i g = \chi^a_i(g)a_i, \quad g \in G. \)

By the definitions of the above section the quantum enveloping algebras (with or without constants) are character Hopf algebras. In this section by means of a combinatorial rank notion we identify the quantum enveloping algebras in the class of character Hopf algebras.

Let \( H \) be a character Hopf algebra generated by the skew primitive invariants \( a_1, \ldots, a_n \). Let us associate a quantum variable \( x_i \) with the parameters \( (\chi^{a_i}, g_{a_i}) \) to \( a_i \). Denote by \( G(X) \) the free enveloping algebra defined by the quantum variables \( x_1, \ldots, x_n \). The map \( x_i \rightarrow a_i \) has an extension to a homomorphism of Hopf algebras \( \varphi : G(X) \rightarrow H \). Denote by \( I \) the kernel of this homomorphism. If \( I \neq 0 \) then by the Heyneman–Radford theorem (see [36, Corollary 5.4.7]), the Hopf ideal \( I \) has a nonzero skew primitive element. Let \( I_1 \) be an ideal generated by all skew primitive elements of \( I \). Clearly \( I_1 \) is a Hopf ideal as well. Now consider the Hopf ideal \( I/1_1 \) of the quotient Hopf algebra \( G(X)/I_1 \). This ideal also has nonzero skew primitive elements (provided \( I_1 \neq 1 \)). Denote by \( I_2/I_1 \) the ideal generated by all skew primitive elements of \( I/I_1 \), where \( I_2 \) is its preimage with respect to the projection \( G(X) \rightarrow G(X)/I_1 \). Continuing the process we will find a strictly increasing, finite or infinite, chain of Hopf ideals of \( G(X) \):

\( 0 = I_0 \subset I_1 \subset I_2 \subset \cdots \subset I_n \subset \cdots, \; \bigcup \alpha I_\alpha = I. \)

**Definition 3.1.** The length of (15) is called a combinatorial rank of \( H \).
By definition, the combinatorial rank of any quantum enveloping algebra (with constants) equals one. In the case of zero characteristic the inverse statement is valid as well.

**Theorem 3.2.** Each character Hopf algebra of the combinatorial rank 1 over a field of zero characteristic is isomorphic to a quantum enveloping algebra with constants of a Lie algebra.

**Proof.** By definition, $I$ is generated by skew primitive elements. These elements as noncommutative polynomials are the quantum Lie operations. Consider one of them, say $f$. Let us decompose $f$ into a sum of homogeneous components $f = \sum f_i$. All positive components belongs to $k\langle X \rangle$ and they are the quantum Lie operations themselves, while the constant component has the form $\alpha(1 - g)$, $g \in G$ (see [22, Sec. 3 and Prop. 3.3]). If $\alpha \neq 0$ then we introduce a new quantum variable $z_f$ with the parameters $(id, g)$. Each $f_i$ has a representation through the skew commutator. Indeed, by [22, Theorem 7.5] the complete linearization $f_i^{\text{lin}}$ of $f_i$ has the required representation. By the identification of variables in a suitable way in $f_i^{\text{lin}}$ we get the required representation for $f_i$ multiplied by a natural number, $m_i f_i = f_i^{[1]}$.

Now consider a Lie algebra $g$ defined by the generators $x_i, z_f$ and the relations $\sum m_i f_i^{[1]} + z_f = 0$, with the Lie multiplication in place of the skew commutator. It is clear that $H$ is the quantification with constants of $g$. □

In the same way one may introduce the notion of the combinatorial rank for the braided bigraded Hopf algebras. In this case all braided quantum enveloping algebras are of rank 1, and all braided bigraded algebras of rank 1 are the braided quantification of some Lie algebras.

Now we are ready to define a quantification of arbitrary rank. For this in the definitions of the above section it is necessary to change the requirement that all $f_i^*$ are quantum Lie operations with the following condition.

**Theorem 3.3.** The set $F$ splits in a union $F = \bigcup_{j=1}^{n} F_j$ such that $F_1^*$ consists of quantum Lie operations; the set $F_2^*$ consists of skew primitive elements of $G\langle X || F_1^* \rangle$; the set $F_3^*$ consists of skew primitive elements of $G\langle X || F_1^*, F_2^* \rangle$, and so on.

The quantum enveloping algebras of an arbitrary rank are character Hopf algebras also. Conversely, if a character Hopf algebra $H$ is homogeneous and the ground field has a zero characteristic, then $H$ is a quantification of some rank of a suitable Lie algebra (see [26]). It is not clear if there exist character Hopf algebras, or braided bigraded Hopf algebras, of infinite combinatorial rank; while it is easy to see that $\bigcup_{n=1}^{\infty} I_n = I$. Also it is possible to show that $F_1$ always contains all relations of a minimal constitution in $F$. For example, each of (11) is of a minimal constitution in (11). Therefore the quantification of arbitrary rank with the identification $g_i = \exp(h_i)$ of any
(generalized) Kac–Moody algebra $\mathfrak{g}$, or its nilpotent component $\mathfrak{g}^+$, is always a quantification in the sense of the above section.

4. PBW-generators and monomial crystallization.

The next result yields a PBW basis for the quantum enveloping algebras.

**Theorem 4.1.** Every character Hopf algebra $H$ has a linearly ordered set of constitution homogeneous elements $U = \{u_i \mid i \in I\}$ such that the set of all products $gu_1^{n_1}u_2^{n_2} \cdots u_m^{n_m}$, where $g \in G$, $u_1 < u_2 < \ldots < u_m$, $0 \leq n_i < h(i)$ forms a basis of $H$. Here if $p_{ii}^{df} = p_{ui,ui}$ is not a root of unity then $h(i) = \infty$; if $p_{ii} = 1$ then either $h(i) = \infty$ or $h(i) = l$ is the characteristic of the ground field; if $p_{ii}$ is a primitive $t$-th root of unity, $t \neq 1$, then $h(i) = t$.

The set $U$ is referred to as a set of **PBW-generators** of $H$. This theorem easily follows from [23, Theorem 2]. Let us recall necessary notions.

Let $a_1, \ldots, a_n$ be a set of skew primitive generators of $H$, and let $x_i$ be the associated quantum variables. Consider the lexicographical ordering of all words in $x_1 > x_2 > \ldots > x_n$. A beginning of a word is considered to be greater than the word itself, for example $x_1 > x_1x_2 > x_1x_2x_1$. A nonempty word $u$ is called standard if $vw > wv$ for each decomposition $u = vw$ with nonempty $v, w$. The following properties are well-known (see, for example [11], [14], [29], [40], [41]).

1. A word $u$ is standard if and only if it is greater than each of its ends.
2. Every standard word starts with a maximal letter that it has.
3. Each word $c$ has a unique representation $c = u_1^{n_1}u_2^{n_2} \cdots u_k^{n_k}$, where $u_1 < u_2 < \cdots < u_k$ are standard words (the Lyndon theorem).
4. If $u, v$ are different standard words and $u^n$ contains $v^k$ as a sub-word, $u^n = cv^kd$, then $u$ itself contains $v^k$ as a sub-word, $u = bv^ke$.

Recall that a nonassociative word is a word where brackets [ ] somehow arranged to show how multiplication applies. If $[u]$ denotes a nonassociative word then by $u$ we denote an associative word obtained from $[u]$ by removing the brackets (of course $[u]$ is not uniquely defined by $u$ in general).

The set of **standard nonassociative** words is defined as the smallest set $SL$ that contains all variables $x_i$ and satisfies the following properties.

1) If $[u] = [v][w] \in SL$ then $[v], [w] \in SL$, and $v > w$ are standard.
2) If $[u] = [[v_1][v_2]][w] \in SL$ then $v_2 \leq w$.

The following statements are valid as well.

5. Every standard word has the only alignment of brackets such that the appeared nonassociative word is standard (the Shirshov theorem [40]).
6. The factors $v, w$ of the nonassociative decomposition $[u] = [[v][w]]$ are the standard words such that $u = vw$ and $v$ has the minimal length ([41]).
**Definition 4.2.** A super-letter is a polynomial that equals a nonassociative standard word where the brackets mean (2). A super-word is a word in super-letters.

By 5s every standard word $u$ defines the only super-letter, in what follows we will denote it by $[u]$. For example, the words $x_1x_2^2$, $x_3^3x_3$, $x_1x_2x_3x_2$, $x_2x_3x_2x_3x_4$, $x_1x_2x_3^2x_2$ are standard and they define the following super-letters

$$[x_1x_2^2] = [(x_1x_2)x_2], \quad [x_3^3x_3] = [x_2[x_2x_3]], \quad [x_1x_2x_3x_2] = [(x_1[x_2x_3])x_2],$$

$$[x_2x_3x_2x_3x_4] = [(x_2x_3)[x_2[x_3x_4]]], \quad [x_1x_2x_3^2x_2] = [(x_1[[x_2x_3]x_3])x_2].$$

In Theorem 2.4 we have $W = \alpha[x_1x_2^3]$. If the variables are ordered in the opposite way, $x_2 > x_1$, then $x_1x_2^3$ is not a standard word, while $x_2^4x_2$ is, and one may see that $\ldots[[x_1x_2]x_2]\ldots x_2 = (-p_{21})^2p_2^{2}x_2^3x_1$ provided that one of the existence conditions is valid (see Corollary 4.10 below). Therefore the quantified relations (11) can be written in a form of equality to zero of some super-letters:

$$(16) \quad [x_jx_i^{n_ji}] = 0, \quad [x_j^{n_ji}x_i] = 0, \quad j < i.$$

Let $D$ be a linearly ordered Abelian additive group. Suppose that some positive $D$-degrees $d_1, \ldots, d_n \in D$ are associated to $x_1, \ldots, x_n$. We define the degree of a word to be equal to $m_1d_1 + \ldots + m_nd_n$ where $(m_1, \ldots, m_n)$ is the constitution of the word. The order and the degree on the super-letters are defined in the following way: $[u] > [v] \iff u > v; D([u]) = D(u)$.

**Definition 4.3.** A super-letter $[u]$ is called hard in $H$ provided that its value in $H$ is not a linear combination of values of super-words of the same degree in less than $[u]$ super-letters and $G$-super-words of a lesser degree.

**Definition 4.4.** We say that a height of a super-letter $[u]$ of degree $d$ equals $h = h([u])$ if $h$ is the smallest number such that: First $p_{uu}$ is a primitive $t$-th root of unity and either $h = t$ or $h = tl^r$, where $l = \text{char}(k)$; and then the value in $H$ of $[u]^h$ is a linear combination of super-words of degree $hd$ in less than $[u]$ super-letters and $G$-super-words of a lesser degree. If there exists no such number then the height equals infinity.

Clearly, if the algebra $H$ is $D$-homogeneous then one may omit the underlined parts of the above definitions.

**Theorem 4.5** ([23, Theorem 2]). The set of all values in $H$ of all $G$-super-words $W$ in the hard super-letters $[u_i]$,

$$(17) \quad W = g[u_1]^{n_1} [u_2]^{n_2} \cdots [u_m]^{n_m},$$

where $g \in G$, $u_1 < u_2 < \ldots < u_m$, $n_i < h([u_i])$ is a basis of $H$. 

A COMBINATORIAL APPROACH 201
In order to find the set $U$ of PBW-generators it is necessary first to include in $U$ the values of all hard super-letters, then for each hard super-letter $[u]$ of a finite height, $h([u]) = tl^k$, to add the values of $[u]^t, [u]^{tl}, \ldots, [u]^{tl(k-1)}$, and next for each hard super-letter of infinite height such that $p_{uu}$ is a primitive $t$-th root of unity to add the value of $[u]^t$.

Obviously the set of PBW-generators plays the same role as the basis of the Lie algebra in the PBW theorem does. Nevertheless the $k[G]$-bimodule generated by the PBW-generators is not uniquely defined. It depends on the ordering of the main generators, the $D$-degree, and under the action of antipode it transforms to a different bimodule of PBW-generators $k[G]S(U)$.

Another way to construct PBW-generators is connected with the M. Kashiwara crystallization idea [20], [21]. M. Kashiwara considered the main parameter of the Drinfeld–Jimbo enveloping algebra as the temperature of some physical medium. When the temperature tends to zero the medium crystallizes. By this means a ‘crystal’ bases must appear. If we replace $p_{ij}$ with zero then $[u,v]$ turns into a monomial $uv$, while $[u]$ turns into a monomial $u$.

**Lemma 4.6.** Under the above monomial crystallization the set of PBW-generators constructed in Theorem 4.5 turns into another set of PBW-generators.

**Proof.** See [23, Corollary 1].

**Lemma 4.7.** A super-letter $[u]$ is hard in $H$ if and only if the value of $u$ is not a linear combination of values of lesser words of the same degree and $G$-words of a lesser degree.

**Proof.** See [23, Corollary 2].

**Lemma 4.8.** Let $B$ be a set of the super-letters containing $x_1, \ldots, x_n$. If each pair $[u], [v] \in B$, $u > v$ satisfies one of the following conditions:

1) $[[u][v]]$ is not a standard nonassociative word;
2) the super-letter $[[u][v]]$ is not hard in $H$;
3) $[[u][v]] \in B$,

then the set $B$ includes all hard in $H$ super-letters.

**Proof.** Let $[w]$ be a hard super-letter of minimal degree such that $[w] \notin B$. Then $[w] = [[u][v]], u > v$ where $[u], [v]$ are hard super-letters. Indeed, if $[u]$ is not hard then by Lemma 4.7 we have $u = \sum \alpha_i u_i + S$, where $u_i < u$ and $D(u_i) = D(u), D(S) < D(u)$. We have $uv = \sum \alpha_i u_i v + Sv$, where $u_i v < uv$. Therefore by Lemma 4.7, the super-letter $[w] = [uv]$ can not be hard in $H$. Contradiction. Similarly, if $[v]$ is not hard then $v = \sum \alpha_i v_i + S, v_i < v, D(v_i) = D(v), D(S) < D(v)$. Therefore $uv = \sum \alpha_i w_i + uS, uv_i < uv$, and again $[w]$ can not be hard.
Thus, according to the choice of \([w]\), we get \([u],[v] \in B\). Since this pair satisfies neither condition 1) nor 2), the condition 3), \([uv]\) \(\in B\), holds. □

**Lemma 4.9.** If \(T \in H\) is a skew primitive element then 
\[
T = \alpha[u]^h + \sum \alpha_i W_i + \sum \beta_j g_j W'_j, \quad \alpha \neq 0,
\]
where \([u]\) is a hard super-letter, \(W_i\) are basis super-words in super-letters less than \([u]\), \(D(W_i) = hD([u])\), \(D(W'_j) < hD([u])\). Here if \(p_{uu}\) is not a root of unity then \(h = 1\); if \(p_{uu}\) is a primitive \(t\)-th root of unity then \(h = 1\), or \(h = t\), or \(h = tl^k\), where \(l\) is the characteristic.

**Proof.** Consider an expansion of \(T\) in terms of the basis (17)
\[
T = \alpha gU + \sum_{i=1}^{k} \gamma_i g_i W_i + W', \quad \alpha \neq 0,
\]
where \(gU, g_i W_i\) are different basis elements of maximal degree, and \(U\) is one of the biggest words among \(U, W_i\) with respect to the lexicographic ordering of words in the super-letters. On basis expansion of tensors, the element \(\Delta(T) - T \otimes 1 - g_t \otimes T\) has only one tensor of the form \(gU \otimes \ldots\) and this tensor equals \(gU \otimes \alpha(g - 1)\). Therefore \(g = 1\) and one may apply [23, Lemma 13]. □

**Corollary 4.10.** If one of the existence conditions in Theorem 2.4 holds then
\[
[\ldots[x_1x_2]x_2 \ldots x_2] = (-p_{12})^{n}p_{22}^{n-1}[x_2[x_2 \ldots [x_2 x_1] \ldots]].
\]

**Proof.** Let us introduce the opposite order, \(x_2 > x_1\). Since \([\ldots[x_1x_2]x_2 \ldots x_2]\) is a quantum Lie operation, it has a representation (18) where all addends have the same constitution, \((1, n)\). This implies \(h = 1\), \(u = x_2^n x_1\). All standard words of the constitution less than or equal to \((1, n)\) are \(x_2, x_2^k x_1, k \leq n\). By definition of the lexicographical order \(x_2 > x_2^k x_1\). Therefore \(x_2\) does not occur in (18) as a super-letter. Since every addend has degree 1 in \(x_1\), the equality (18) reduces to \(T = \alpha[x_2^n x_1]\). In order to find \(\alpha\) one may to compare the coefficients at \(x_2^n x_1\). □

5. Groebner–Shirshov relations systems.

Let \(x_1, \ldots, x_n\) be variables that have positive degrees \(d_1, \ldots, d_n \in D\). Recall that a Hall ordering of words in \(x_1, \ldots, x_n\) is an order when the words are compared firstly by the degree and then words of the same degree are compared by means of the lexicographic ordering. Consider a set of relations 
\[
w_i = f_i, \quad i \in I,
\]
where \(w_i\) is a word and \(f_i\) is a linear combination of Hall lesser words. The system (20) is said to be closed under compositions or a Groebner–Shirshov
relations system if first none of \( w_i \) contains \( w_j, i \neq j \in I \) as a sub-word, and then for each pair of words \( w_k, w_j \) such that some nonempty terminal of \( w_k \) coincides with an onset of \( w_j \), that is \( w_k = w_k'v, w_j = vw_j' \), the difference (a composition) \( f_kw_j' - w_k'f_j \) can be reduced to zero in the free algebra by means of a sequence of one sided substitutions \( w_i \to f_i, i \in I \).

**Lemma 5.1** (Diamond Lemma \[4],[6],[41\]). If the system (20) is closed under compositions then the words that have none of \( w_i \) as sub-words form a basis of the algebra \( H \) defined by (20).

If none of the words \( w_i \) has sub-words \( w_j, j \neq i \), then the converse statement is valid as well. Indeed, any composition by means of substitutions \( w_i \to f_i \) can be reduced to a linear combination of words that have no sub-words \( w_i \). Since \( f_iw_j' - w_i'f_j = (f_i - w_i)w_j' - w_i'(f_j - w_j) \), this linear combination equals zero in \( H \). Therefore all the coefficients have to be zero.

Since Lemma 4.6 provides the basis that consists of words, the above note gives a way to construct the Groebner–Shirshov relations system for any quantum enveloping algebra.

Let \( H \) be a character Hopf algebra generated by skew primitive semi-invariants \( a_1, \ldots, a_n \) (or a braided bigraded Hopf algebra generated by grading homogeneous primitive elements \( a_1, \ldots, a_n \)) and let \( x_1, \ldots, x_n \) be the related quantum variables. A non-hard in \( H \) super-letter \([w]\) is referred to as a minimal one if first \( w \) has no proper standard sub-words that define non-hard super-letters, and then \( w \) has no sub-words \( u^h \), where \([u]\) is a hard super-letter of the height \( h \).

By Lemma 4.7, for every minimal non-hard in \( H \) super-letter \([w]\) we may write a relation in \( H \)

\[
(21) \quad w = \sum \alpha_iw_i + \sum \beta_jg_jw_j,
\]

where \( w_j, w_i < w \) in the Hall sense, \( D(w_i) = D(w), D(w_j) < D(w) \). In the same way if \([u]\) is a hard in \( H \) super-letter of a finite height \( h \) then

\[
(22) \quad u^h = \sum \alpha_iu_i + \sum \beta_jg_ju_j,
\]

where \( u_j, u_i < u^h \) in the Hall sense, \( D(u_i) = hD(u), D(u_j) < hD(u) \). The relations (14) and the group operation provide the relations

\[
(23) \quad x_ig = \chi^{x_i}(g)gx_i, \quad g_1g_2 = g_3.
\]

**Theorem 5.2.** The set of relations (21), (22), and (23) forms a Groebner–Shirshov system that defines \( H \). The basis determined by this system in Diamond Lemma coincides with the PBW basis obtained via monomial crystallization.

**Proof.** The property 4s implies that none of the left hand sides of (21), (22), (23) contains another one as a sub-word. Therefore by Lemma 4.6 it is
sufficient to show that the set of all words $c$ determined in the Diamond Lemma coincides with the basis appeared in Lemma 4.6. By 3s we have $c = u_1^{n_1}u_2^{n_2}\cdots u_k^{n_k}$, where $u_1 < \ldots < u_k$ is a sequence of standard words. Every word $u_i$ define a hard super-letter $[u_i]$ since in the opposite case $u_i$, and therefore $c$, contains a sub-word $w$ that defines a minimal non-hard super-letter $[w]$. In the same way $n_i$ does not exceed the height of $[u_i]$. \hfill $\square$

**Lemma 5.3.** In terms of Lemma 4.8 the set of all super-letters $[u[v]]$ that satisfy the condition 2) contains all minimal non-hard super-letters, but non-hard generators $x_i$.

**Proof.** If $[w]$ is a minimal non-hard super-letter then $[w] = [u[v]]$, where $[u], [v]$ are hard super-letters. By Lemma 4.8 we have $[u], [v] \in B$, while $[[u[v]]$ neither satisfies 1) nor 3). \hfill $\square$

6. Quantification with constants.

By means of the Diamond Lemma in some instances the investigation of a quantification with constants can be reduced to one of a simple quantification.

Let $H_1 = G\langle x_1, \ldots, x_k||F_1\rangle$ be a character Hopf algebra defined by the quantum variables $x_1, \ldots, x_k$ and the grading homogeneous relations $\{f = 0 : f \in F_1\}$, while $H_2 = G\langle x_{k+1}, \ldots, x_n||F_2\rangle$ is a character Hopf algebra defined by the quantum variables $x_{k+1}, \ldots, x_n$ and the grading homogeneous relations $\{h = 0 : h \in F_2\}$. Consider the algebra $H = G\langle x_1, \ldots, x_n||F_1, F_2, F_3\rangle$, where $F_3$ is the following system of relations with constants

$$[x_i, x_j] = \alpha_{ij}(1 - g_ig_j), \quad 1 \leq i < j \leq n. \quad (24)$$

If the conditions below are met then the character Hopf algebra structure on $H$ is uniquely determined:

$$p_{ij}p_{ji} = 1, \quad 1 \leq i < j \leq n; \quad \chi^{x_i} \chi^{x_j} \neq 1 \implies \alpha_{ij} = 0. \quad (25)$$

Indeed, in this case the difference $w_{ij}$ between the left and right hand sides of $(24)$ is a skew primitive semi-invariant of the free enveloping algebra $G\langle x_1, \ldots, x_n\rangle$. Consider the ideals of relations $I_1 = \text{id}(F_1)$ and $I_2 = \text{id}(F_2)$ of $H_1$ and $H_2$ respectively. They are, in the present context, Hopf ideals of $G\langle x_1, \ldots, x_k\rangle$ and $G\langle x_{k+1}, \ldots, x_n\rangle$, respectively. Therefore $V = I_1 + I_2 + \sum k[G]w_{ij}$ is an antipode stable coideal of $G(X)$. Consequently the ideal generated by $V$ is a Hopf ideal. It remains to note that this ideal is generated in $G(X)$ by $w_{ij}$ and $F_1, F_2$.

**Lemma 6.1.** Every hard in $H$ super-letter belongs to either $H_1$ or $H_2$, and it is hard in the related algebra.

**Proof.** If a standard word contains at least one of the letters $x_i$, $i \leq k$ then it has to start with one of them (see 2s in §4). If this word contains a letter
$x_j$, $j > k$ then it has a sub-word of the form $x_ix_j$, $i \leq k < j$. Therefore by Lemma 4.7 and relations (24) this word defines a non-hard super-letter. □

The converse statement is not universally true. In order to formulate the necessary and sufficient conditions let us define partial skew derivatives:

$\partial_i(x_j) = \partial_j(x_i) = \alpha_{ij}(1 - g_j g_i), \quad i \leq k < j$;

$\partial_i(v \cdot w) = \partial_i(v) \cdot w + p(x_i, v)v \cdot \partial_i(w), \quad i \leq k, \quad v, w \in k\langle x_{k+1}, \ldots, x_n \rangle$;

$\partial_j(u \cdot v) = p(v, x_j)\partial_j(u) \cdot v + u \cdot \partial_j(v), \quad j > k, \quad u, v \in k\langle x_1, \ldots, x_k \rangle$.

**Lemma 6.2.** All hard in $H_1$ or $H_2$ super-letters are hard in $H$ if and only if $\partial_i(h) = 0$ in $H_2$ for all $i \leq k$, $h \in F_2$, and $\partial_j(f) = 0$ in $H_1$ for all $j > k$, $f \in F_1$. If these conditions are met then

$H \cong H_2 \otimes_{k[G]} H_1$

as $k[G]$-bimodules, and the space generated by the skew primitive elements of $H$ equals the sum of these spaces for $H_1$ and $H_2$.

**Proof.** By (5) and (26) the following equalities are valid in $H$:

$0 = [x_i, h] = \partial_i(h); \quad 0 = [f, x_j] = \partial_j(f), \quad i \leq k < j$.

If all hard in $H_1$ or $H_2$ super-letters are hard in $H$ then $H_1$, $H_2$ are sub-algebras of $H$. So (28) proves the necessity of the lemma conditions.

Conversely, let us consider an algebra $R$ defined by the generators $g \in G$, $x_1, \ldots, x_n$ and the relations (23), (24). Evidently this system is closed under the compositions. Therefore by Diamond Lemma the set of words $gvw$ forms a basis of $R$ where $g \in G$; $v$ is a word in $x_j$, $j > k$; and $w$ is a word in $x_i$, $i \leq k$. In other words $R$ as a bimodule over $k[G]$ has a decomposition

$R = G\langle x_{k+1}, \ldots, x_n \rangle \otimes_{k[G]} G\langle x_1, \ldots, x_k \rangle$.

Let us show that the two sided ideal of $R$ generated by $F_2$ coincides with the right ideal $I_2R = I_2 \otimes_{k[G]} G\langle x_1, \ldots, x_k \rangle$. It will suffice to show that $I_2R$ admits left multiplication by $x_i$, $i \leq k$. If $v$ is a word in $x_{k+1}, \ldots, x_n$, $h \in F_2$, $r \in R$ then $x_i vh r = [x_i, vh]r + p(x_i, vh)vhx_i r$. The second term belongs to $I_2R$, while the first one can be rewritten by (5): $[x_i, v]h + p(x_i, v)v[x_i, h]$. Both of these addends belong to $I_2R$ since $[x_i, v] = \partial_i(v) \in G\langle x_{k+1}, \ldots, x_n \rangle$ and $[x_i, h] = \partial_i(h) \in I_2$.

Furthermore, consider a quotient algebra $R_1 = R/I_2R$:

$R_1 = (G\langle x_{k+1}, \ldots, x_n \rangle \otimes_{k[G]} G\langle x_1, \ldots, x_k \rangle)/(I_2 \otimes_{k[G]} G\langle x_1, \ldots, x_k \rangle)$

$= H_2 \otimes_{k[G]} G\langle x_1, \ldots, x_k \rangle$,

where the equality means the natural isomorphism of $k[G]$-bimodules.

Along similar lines, the left ideal $R_1I_1 = H_2 \otimes_{k[G]} I_1$ of this quotient algebra coincides with the two-sided ideal generated by $F_1$. Therefore

$H = R_1/R_1I_1 = H_2 \otimes_{k[G]} G\langle x_1, \ldots, x_k \rangle/H_2 \otimes_{k[G]} I_1 = H_2 \otimes_{k[G]} H_1$. 

Thus the monotonous restricted $G$-words in hard in $H_1$ or $H_2$ super-letters form a basis of $H$. This, in particular, proves the first statement.

Now let $T = \sum \alpha_t g_t V_t W_t$ be the basis decomposition of a skew primitive element, $g_t \in G$, $V_t \in H_2$, $W_t \in H_1$, $\alpha_t \neq 0$. We have to show that for each $t$ one of the super-words $V_t$ or $W_t$ is empty. Suppose that it is not so. Among the addends with nonempty $V_t$, $W_t$ we choose the largest one in the Hall sense, say $g_s V_s W_s$. Under the basis decomposition of $\Delta(T) - T \otimes 1 - g(T) \otimes T$ the term $\alpha_s g_s g(V_s) W_s \otimes g_s V_s$ appears and cannot be canceled with other. Indeed, since the coproduct is homogeneous (see [23, Lemma 9]) and since under the basis decomposition the super-words are decreased (see [23, Lemma 7]) the product $\alpha_s (g_s \otimes g_s) \Delta(V_s) \Delta(W_s)$ has the only term of the above type. By the same reasons $\alpha_t (g_t \otimes g_t) \Delta(V_t) \Delta(W_t)$ has a term of the above type only if $V_t \geq V_s$ and $W_t \geq W_s$ with respect to the Hall ordering of the set of all super-words. However, by the choice of $s$, we have $D(V_t W_s) \geq D(V_t W_t)$. Hence $D(V_t) = D(V_s)$ and $D(W_t) = D(W_s)$. In particular $V_t$ is not a proper onset of $V_s$. Therefore $V_t = V_s$ since otherwise the inequality $V_t > V_s$ yields a contradiction $V_t W_t > V_s W_s$. The inequality $W_t > W_s$ yields the same contradiction. Therefore $V_t = V_s$ and $W_t = W_s$, in which case $g_t g(V_t) W_t \otimes g_t V_t = g_s g(V_s) W_s \otimes g_s V_s$. Thus $g_t = g_s$ and $t = s$. □

7. Quantification of the classical series.

In this section we apply the above general results to the infinite series $A_n$, $B_n$, $C_n$, $D_n$ of nilpotent Lie algebras defined by the Serre relations (11) or, equivalently, (16). Let $\mathfrak{g}$ be any such Lie algebra.

**Lemma 7.1.** If a standard word $u$ has no sub-words of the type

\[(30)\quad x_i^s x_j x_i^m, \text{ where } s + m = 1 - a_{ij}\]

then $[u]$ is a hard in $U_P(\mathfrak{g})$ super-letter.

**Proof.** Let $R$ be defined by the generators $x_1, \ldots, x_n$ and the relations

\[(31)\quad x_i^s x_j x_i^m = 0, \text{ where } s + m = 1 - a_{ij}.\]

Clearly (31) implies (16). Therefore $R$ is a homomorphic image of $U_P^b(\mathfrak{g})$. The system (31) is closed under compositions since a composition of monomial relations always has the form $0 = 0$.

Let $u$ have no sub-words (30). Then the value of $u$ in $R$ belongs to the basis of $R$ defined in Diamond Lemma. If $[u]$ is not hard then, by the homogeneous version of Lemma 4.7, $u$ is a linear combination of lesser words in $U_P^b(\mathfrak{g})$. Therefore $u$ is a linear combination of lesser words in $R$ as well. This contradicts the fact that $u$ belongs to the basis of $R$ defined in Diamond Lemma. □
Theorem A\text{\textsubscript{n}}. Suppose that $g$ is of the type $A\text{\textsubscript{n}}$, and $p_{ii} \neq -1$. Denote by $B$ the set of the super-letters given below:
\begin{equation}
[u_{km}] \overset{df}{=} [x_kx_{k+1} \ldots x_m], \quad 1 \leq k \leq m \leq n.
\end{equation}
The following statements are valid.
1. The values of $[u_{km}]$ in $U_P(g)$ form a PBW-generators set.
2. Each of the super-letters $[u_{1i}]$ has infinite height in $U_P(g)$.
3. The values of all non-hard in $U_P(g)$ super-letters equal zero.
4. The following relations with (23) form the Groebner–Shirshov relations system for $U_P(g)$:
\begin{equation}
[u_0] \overset{df}{=} [x_kx_m] = 0, \quad 1 \leq k < m - 1 < n;
\end{equation}
\begin{equation}
[u_1] \overset{df}{=} [x_kx_{k+1} \ldots x_mx_{k+1}] = 0, \quad 1 \leq k < m \leq n;
\end{equation}
\begin{equation}
[u_2] \overset{df}{=} [x_kx_{k+1} \ldots x_mx_{k+1} \ldots x_{m+1}] = 0, \quad 1 \leq k \leq m < n.
\end{equation}
5. If $p_{11} \neq 1$ then the generators $x_i$, the constants $1 - g$, $g \in G$, and, in the case that $p_{11}$ is a primitive $t$-th root of 1, the elements $x^t_i, x^{2t}_i$ form a basis of the space $g_P = L(U_P(g))$ generated by skew primitive elements. Here $l$ is the characteristic of the ground field.
6. If $p_{11} = 1$ then the elements (32) and, in the case $l > 0$, their $l^k$-th powers, together with $1 - g$, $g \in G$ form a basis of $g_P$.

By Corollary 2.5 the relations (11) with a Cartan matrix $A$ of type $A\text{\textsubscript{n}}$ admit a quantification if and only if
\begin{equation}
p_{ii} = p_{11}, \quad p_{ii+1p_{ii+1}} = p_{11}^{-1}; \quad p_{ij}p_{ji} = 1, \quad i - j > 1.
\end{equation}
In this case the quantified relations (16) take up the form
\begin{equation}
x_ix^2_{i+1} = p_{ii+1}(1 + p_{ii+1})x_{i+1}x_i - p_{ii+1}^2 p_{ii+1}x_{i+1}x_i,
\end{equation}
\begin{equation}
x^2_ix_{i+1} = p_{ii+1}(1 + p_{ii})x_{i+1}x_i - p_{ii+1}^2 p_{ii+1}x_{i+1}x_i,
\end{equation}
\begin{equation}
x_ix_j = p_{ij}x_jx_i, \quad i - j > 1.
\end{equation}

Definition 7.2. We introduce the congruence $u \equiv_k v$ on $G(X)$. This congruence means that the value of $u - v$ in $U_P^k(g)$ belongs to the subspace generated by values of all words with the initial letters $x_i, i \geq k$.

Clearly, this congruence admits right multiplication by arbitrary polynomials as well as left multiplication by the independent of $x_{k-1}$ ones (see (37)). For example, by (35) and (36) we have
\begin{equation}
x_ix^2_{i+1} \equiv_{i+1} 0; \quad x_ix_{i+1}x_i \equiv_{i+1} \alpha x^2_ix_{i+1}, \quad \alpha \neq 0.
\end{equation}

Lemma 7.3. If $y = x_i$, $m + 1 \neq i > k$ or $y = x^2_i$, $m + 1 = i > k$ then
\begin{equation}
u_{km}y \equiv_{k+1} 0.
\end{equation}
Proof. Let \( y = x^2 \) for \( m+1 > k \). By (38) and (37) we have that \( u_{km}y = u_{km-1}x^m x_{m+1} \equiv m+1 \). If \( y = x_i \) and \( m+1 \neq i > k \) then we get \( u_{km}y = a u_{ki-1}x^m x_{i+1} u_i u_{i+2m} \equiv i+1 \beta u_{ki-1}x^m u_{i+1} \equiv k+1 \) by the above case. □

Lemma 7.4. The brackets in \([u_{km}]\) are left-ordered, \([u_{km}] = [x_k u_{k+1}m] \).

Proof. The statement immediately follows from the properties 6s and 2s. □

Lemma 7.5. If a nonassociative word \([u_{km}] [u_{rs}]\) is standard then \( k = m \leq r \); or \( r = k+1, m \geq s \); or \( r = k, m < s \).

Proof. By definition, \( u_{km} > u_{rs} \) if and only if either \( k < r \); or \( k = r, m < s \). If \( k = m \) then \( u_{km} = x_k \) and \( m \leq r \). If \( k \neq m \) then \([u_{km}] = [x_k [u_{k+1}m]] \). Therefore \( u_{k+1}m \leq u_{rs} \), i.e., either \( k+1 > r \); or \( k+1 = r \) and \( m \geq s \). The former case contradicts \( k < r \) while the latter one does \( k = r \). Thus only the possibilities set in the lemma remain. □

Lemma 7.6. If \([w] = [u_{km}] [u_{rs}]\), \( n \geq 1 \) is a standard nonassociative word then the constitution of \([w]h\) does not equal the constitution of any super-word in less than \([w]\) super-letters from \( B \).

Proof. The inequalities at the last column of the following tableaux are valid for all \([u] \in B\) that are less than the super-letters located on the same row, where as above \( \deg_i(u) \) means the degree of \( u \) in \( x_i \).

\[
\begin{align*}
&[x_k u_{k+1}] \quad \deg_k(u) \leq \deg_{s+1}(u); \\
&[x_k u_{rs}] \quad k \leq r \neq k+1 \quad \deg_k(u) \leq \deg_{k+1}(u); \\
&[u_{km} u_{k+1}] \quad m \geq s \quad \deg_k(u) \leq \deg_{m+1}(u); \\
&[u_{km} u_k] \quad m < s \quad \deg_k(u) \leq \deg_{m+1}(u).
\end{align*}
\]

If all super-letters of a super-word \( U \) satisfy one of these inequalities then \( U \) does as well. Clearly, no one of the super-letters in the first column satisfies the degree inequality on the same row. Finally, by Lemma 7.5 the first column contains all standard nonassociative words of the type \([u_{km}] [u_{rs}]\). □

Lemma 7.7. If \( p_{11} \neq 0 \) then the values of \([u_{km}]^h\), \( k < m, h \geq 1 \) are not skew primitive, in particular they are nonzero.

Proof. The sub-algebra generated by \( x_2, \ldots, x_n \) is defined by the Cartan matrix of the type \( A_{n-1} \). This allows us to use induction on \( n \). If \( n = 1 \) then the lemma is correct in the sense that \([u_{km}]^h = x_1^h \neq 0 \).

Let \( n > 1 \). If \( k > 1 \) then we may use the inductive supposition directly. Consider the decomposition \( \Delta([u_{1m}]) = \sum u^{(1)} \otimes u^{(2)} \). Since

\[
[u_{1m}] = x_1 [u_{2m}] - p(x_1, u_{2m}) [u_{2m}] x_1,
\]

we have
(42) \[ \Delta([u_{1m}]) = (x_1 \otimes 1 + g_1 \otimes x_1)\Delta([u_{2m}]) - p(x_1, u_{2m})\Delta([u_{2m}]) (x_1 \otimes 1 + g_1 \otimes x_1). \]

Therefore the sum of all tensors \( u^{(1)} \otimes u^{(2)} \) with \( \deg_1(u^{(2)}) = 1, \deg_k(u^{(2)}) = 0, k > 1 \) has the form \( \varepsilon g_1[u_{2m}] \otimes x_1 \), where \( \varepsilon = 1 - p(x_1, u_{2m})p(u_{2m}, x_1) \) since \( [u_{2m}]g_1 = p(u_{2m}, x_1)g_1[u_{2m}] \). By (34) we have \( p_{ij}p_{ji} = 1 \) for \( i - 1 > j \). Therefore \( \varepsilon = 1 - p_{12}p_{21} = 1 - p_{11}^{-1} \neq 0 \).

This implies that in the decomposition \( \Delta([u_{1m}]^h) = \sum v^{(1)} \otimes v^{(2)} \) the sum of all tensors \( v^{(1)} \otimes v^{(2)} \) with \( \deg_1(v^{(2)}) = h, \deg_k(v^{(2)}) = 0, k > 1 \) equals \( \varepsilon^h[u_{2m}]^h \otimes x_1^h \). Thus \( [u_{1m}]^h \) is not skew primitive in \( U_P(g) \).

Proof of Theorem \( A_n \). Let us show firstly that \( B \) satisfies the conditions of Lemma 4.8. By Lemma 4.7 \( [w] = [[u_{km}][u_{rs}]] \) is non-hard if the value of \( u_{km}u_{rs} \) is a linear combination of lesser words. For \( k = m, r = k + 1 \) we have \( [w] = [u_{km}] \in B \). If \( k = m, r > k + 1 \) then the word \( x_ku_{rs} \) can be diminished by (36) or (37). If \( k \neq m \) then by Lemma 7.5 the word \( u_{km}u_{rs} \) has a sub-word of the type \( u_1 \) or \( u_2 \). Thus we need show only that the values in \( U_P(g) \) of \( u_1 \) and \( u_2 \) are linear combinations of lesser words.

The word \( u_1 \) has such a representation by Lemma 7.3. Consider the word \( u_2 \). Let us show by downward induction on \( k \) that

\[
(43) \quad u_{km}u_{k,m+1} \equiv_{k+1} \gamma u_{k,m+1}u_{km}, \quad \gamma \neq 0.
\]

If \( k = m \) then one may use (36) with \( i = k \). Let \( k < m \). Let us transpose the second letter \( x_k \) of \( u_2 \) as far to the left as possible by (37). We get

\[
u_2 = \alpha x_kx_{k+1}x_{k+2}\cdots x_m x_{k+1}\cdots x_{m+1}, \quad \alpha \neq 0.
\]

By (36) we have

\[
u_2 \equiv_{k+1} \beta x_k^2 (x_{k+1}x_{k+2}\cdots x_m x_{k+1}\cdots x_{m+1}), \quad \beta \neq 0.
\]

Let us apply the inductive supposition to the word in the parentheses. Since \( x_i, i > k + 1 \) commutes with \( x_k^2 \) according to the formulae (37), we get

\[
u_2 \equiv_{k+1} \gamma x_k^2 x_{k+1}x_{k+2}\cdots x_m x_{k+1}\cdots x_{m+1} x_{m+1} x_{k+1}\cdots x_m.
\]

Now it remains to replace the underlined sub-word according to (36) and then to transpose the second letter \( x_k \) to its former position by (37).

(Note. For the diminishing of \( u_1, u_2 \) we did not use, and we could not use, the relation \( [x_{n-1}x_n^2] = 0 \) since \( \deg_n(u_1) \leq 1, \deg_n(u_2) \leq 1 \).)

Thus \( B \) satisfies the conditions of Lemma 4.8. Since none of \( [u_{km}] \) has sub-words (30), Lemmas 7.1 and 4.8 show that the first statement is correct.

If \( [u_{km}] \) has a finite height \( h \) then the value of the polynomial \( [u_{km}]^h \) in \( U_P(g) \) is a linear combination of words in hard super-letters that are less than \( [u_{km}] \). However by Lemma 7.6 this linear combination is trivial,
\([u_{km}]^h = 0\), since the defining relations are homogeneous. By Lemma 7.7 the second statement is correct for \(p_{11} \neq 1\).

Similarly consider the skew primitive elements. Since both the defining relations and the coproduct are homogeneous, all the homogeneous components of a skew primitive element are skew primitive itself. Therefore it remains to describe all skew primitive elements homogeneous in each \(x_i\). Let \(T\) be such an element. By Lemma 4.9 we have

\[
T = [u]^h + \sum \alpha_i W_i,
\]

where \([u]\) is a hard super-letter, \(u = u_{km}\), and \(W_i\) are super-words in less than \([u]\) super-letters from \(B\). By the homogeneity all \(W_i\) have the same constitution as \([u_{km}]^h\) does. However by Lemma 7.6 there exist no such super-words. This means that the only possible case is \(T = [u_{km}]^h\). Thus, by Lemma 7.7 the fifth statement is valid as well.

If \(p_{11} = 1\) then \(p_{ij} = p_{ji} = p_{ii} = 1\) for all \(i, j\). So we are under the conditions of Example 1, that is \(U_{p}^b(\mathfrak{g})\) is the universal enveloping algebra of the color Lie algebra \(\mathfrak{g}^{\text{col}}\). Further, \([u_{km}] \in \mathfrak{g}^{\text{col}}\) and \([u_{km}]\) are linearly independent in \(\mathfrak{g}^{\text{col}}\) since they are hard super-letters and no one of them can be a linear combination of the lesser ones. Let us complete \(B\) to a homogeneous basis \(B'\) of \(\mathfrak{g}^{\text{col}}\). Then by the PBW theorem for the color Lie algebras the products \(b_1^{n_1} \cdots b_k^{n_k}, \ b_1 < \cdots < b_k\) form a basis of \(U(\mathfrak{g}^{\text{col}}) = U_{p}^b(\mathfrak{g})\). However, the monotonous restricted words in \(B\) form a basis of \(U_{p}^b(\mathfrak{g})\) also. Thus \(B' = B\) and all hard super-letters have the infinite height.

In particular, we get that the second statement is valid in complete extent. Moreover, if \(p_{11} = 1\) then \(p(u_{km}, u_{km}) = 1\), thus for \(l = 0\) all homogeneous skew primitive elements became exhausted by \([u_{km}]\), while for \(l > 0\) the powers \([u_{km}]^l\) are added to them (of course, here \(l \neq 2\) since \(-1 \neq p_{ii} = 1\)).

So we have proved all statements, but the third and fourth ones. These statements will follow Theorem 5.2 and Lemma 5.3 if we prove that all non-hard super-letters \([[u_{km}][u_{rs}]]\) equal zero in \(U_{p}(\mathfrak{g})\). By the homogeneous definition, \([[u_{km}][u_{rs}]]\) is a linear combination of super-words in lesser hard super-letters. However, by Lemma 7.6, there exist no such super-words of the same constitution. Therefore, by the homogeneity, the above linear combination equals zero.

\[\square\]

**Theorem B\(_n\).** Let \(\mathfrak{g}\) be of the type \(B_n\), and \(p_{ii} \neq -1, 1 \leq i < n, p_{3n}^{[3]} \stackrel{df}{=} p_{2n}^2 + p_{nn} + 1 \neq 0\). Denote by \(B\) the set of the super-letters given below:

\[
\begin{align*}
[u_{km}] &\stackrel{df}{=} [x_kx_{k+1} \cdots x_m], \quad 1 \leq k \leq m \leq n; \\
[w_{km}] &\stackrel{df}{=} [x_kx_{k+1} \cdots x_n, x_nx_{n-1} \cdots x_m], \quad 1 \leq k < m \leq n.
\end{align*}
\]

The following statements are valid.

1. The values of (44) in \(U_{p}(\mathfrak{g})\) form the PBW-generators set.
2. Every super-letter \([u] \in B\) has infinite height in \(U_P(\mathfrak{g})\).

3. The relations (23) with the following ones form a Groebner–Shirshov system for \(U_P(\mathfrak{g})\).

\[
\begin{align*}
[u_0] & \triangleq [x_k x_m] = 0, & 1 \leq k < m - 1 < n; \\
[u_1] & \triangleq [u_{km} x_{k+1}] = 0, & 1 \leq k < m \leq n, \ k \neq n - 1; \\
[u_2] & \triangleq [u_{km} u_{k m+1}] = 0, & 1 \leq k \leq m < n; \\
[u_3] & \triangleq [w_{km} x_{k+1}] = 0, & 1 \leq k < m \leq n, \ k \neq m - 2; \\
[u_4] & \triangleq [w_{kk+1} x_{k+2}] = 0, & 1 \leq k < n - 1; \\
[u_5] & \triangleq [w_{km} w_{k m-1}] = 0, & 1 \leq k < m - 1 \leq n - 1; \\
[u_6] & \triangleq [w_{kn}^2 x_n] = 0, & 1 \leq k < n.
\end{align*}
\]  

(45)

4. If \(p_{11} \neq 1\) then the generators \(x_i\) and their powers \(x_i^t, x_i^{th}\), such that \(p_{ii}\) is a primitive \(t\)-th root of 1, together with the constants \(1 - g, g \in G\) form a basis of \(\mathfrak{g}_P = L(U_P(\mathfrak{g}))\). Here \(l\) is the characteristic of the ground field.

5. If \(p_{nn} = p_{11} = 1\) then the elements (44) and, for \(l > 0\), their \(l^k\)-th powers, together with \(1 - g, g \in G\) form a basis of \(\mathfrak{g}_P\). If \(p_{nn} = -p_{11} = -1\) then \([u_{kn}]^2, [u_{kn}]^{2l}\) are added to them.

Recall that in the case \(B_n\) the algebra \(U_P^b(\mathfrak{g})\) is defined by (35), (36), (37) where in (35) the last relation, \(i = n - 1\), is replaced with

\[
x_{n-1} x_n^3 = p_{n-1n} p_{nn}^3 x_n x_{n-1} x_n^2 - p_{n-1n} p_{nn} p_{nn}^3 x_n x_{n-1} x_n + p_{n-1n} p_{nn}^3 x_n^3 x_{n-1}.
\]

(46)

By Corollary 2.5 we get the existence conditions

\[
p_{ii} = p_{11}, \ p_{ii+1} p_{i+1i} = p_{11}^{-1} = p_{nn}^{-2}, \ 1 \leq i \leq n - 1; \ p_{ij} p_{ji} = 1, \ i < j > 1.
\]

The relations (35) and (46) show that

\[
x_i x_{i+1}^2 \equiv_{i+1} 0, \ i < n - 1; \quad x_{n-1} x_n^3 \equiv_{n} 0,
\]

while the relations (36) imply

\[
x_i x_{i+1} x_i \equiv_{i+1} \alpha x_i^2 x_{i+1}, \quad \alpha \neq 0.
\]

(49)

By means of these relations and (37), (46) we have

\[
x_{n-2} x_{n-1} x_n^2 x_{n-1} x_n \equiv_{n-1} 0.
\]

(50)

**Lemma 7.8.** The brackets in \([w_{km}]\) are set by the recurrence formulae:

\[
[w_{km}] = [x_k [w_{k+1m}]], \quad \text{if} \ 1 \leq k < m - 1 < n;
\]

\[
[w_{kk+1}] = [[w_{k k+2}] x_{k+1}], \quad \text{if} \ 1 \leq k < n.
\]

(51)

Here by the definition \(w_{k n+1} = u_{kn}\).
Proof. It is enough to use the property 6s and then 1s and 2s. □

Lemma 7.9. The nonassociative word \([w_{km}][w_{rs}]\) is standard only in the following two cases: 1) \(s \geq m > k + 1 = r; \) 2) \(s < m, r = k.\)

Proof. If \([w_{km}][w_{rs}]\) is standard then \(w_{km} > w_{rs}\) and by (51) either \(w_{k+1} \leq w_{rs}\), or \(m = k + 1\) and \(x_{k+1} \leq w_{rs}\). The inequality \(w_{km} > w_{rs}\) is correct only in two cases: \(k < r\) or \(k = r, m > s\). We get four possibilities:

1) \(k < r, k < m - 1, w_{k+1m} \leq w_{rs};\)
2) \(k < r, m = k + 1, x_{k+1} \leq w_{rs};\)
3) \(k = r, m > s, k < m - 1, w_{k+1m} \leq w_{rs};\)
4) \(k = r, m > s, m = k + 1, x_{k+1} \leq w_{rs}.\)

Only the first and third ones are consistent since in the second case \(x_{k+1} \leq w_{rs}\) implies \(k + 1 > r\), while in the fourth case \(r < s\) and \(k = r < s < m = k + 1\). If now we decode \(w_{k+1m} \leq w_{rs}\) in the first and third cases, we get the two possibilities mentioned in the lemma. □

Lemma 7.10. The nonassociative word \([u_{km}][w_{rs}]\) is standard only in the following two cases: 1) \(k = r; \) 2) \(k = m < r.\)

Proof. The inequality \(u_{km} > w_{rs}\) means \(k \leq r.\) Since \([u_{km}] = [x_{k}[u_{k+1m}]],\) for \(k \neq m\) we get \(u_{k+1m} \leq w_{rs}\), so \(k + 1 > r\) and \(k = r.\) If \(k = m \neq r\) then \(x_{m} > w_{rs}\) and \(m < r.\) □

Lemma 7.11. The nonassociative word \([w_{km}][u_{rs}]\) is standard only in the following two cases: 1) \(r = k + 1 < m; \) 2) \(r = k + 1 = m = s.\)

Proof. The inequality \(w_{km} > u_{rs}\) implies \(r > k.\) If \(k < m - 1\) then by the first formula (51) we have \(w_{k+1m} \leq u_{rs}\) that is equivalent to \(k + 1 \geq r.\) Therefore \(r = k + 1 < m.\) If \(k = m - 1\) then by the second formula (51) we get \(x_{k+1} \leq u_{rs},\) i.e., either \(k + 1 > r\) or \(k + 1 = r = s.\) The former case contradicts \(r > k\) while the latter one is mentioned in the lemma. □

Lemma 7.12. If \([u], [v] \in B\) then one of the statements below is correct.

1) \([u][v]\) is not a standard nonassociative word;
2) \(uv\) contains a sub-word of one of the types \(u_0, u_1, u_2, u_3, u_4, u_5, u_6;\)
3) \([u][v] \in B.\)

Proof. The proof results from Lemmas 7.5, 7.9, 7.10, 7.11. □

Lemma 7.13. If a super-word \(W\) equals one of the super-letters \([u_1] - [u_6]\) or \([u_{km}]^h, [w_{km}]^h, h \geq 1\) then its constitution does not equal the constitution of any super-word in less than \(W\) super-letters from \(B.\)

Proof. The proof is akin to Lemma 7.6 with the following tableaux:

\[
\begin{align*}
[u_{km}], [u_{km}x_{k+1}], [u_{km}u_{km+1}] & \quad \deg_k(u) \leq \deg_{m+1}(u); \\
[w_{km}], [w_{km}x_{k+1}], [w_{km}w_{km-1}] & \quad 2\deg_k(u) \leq \deg_{m-1}(u); \\
[w_{k+1km+1}], [w_{km}] & \quad \deg_k(u) = 0; \\
[u_{km}]^2 & \quad \deg_k(u) \leq \deg_m(u).
\end{align*}
\]
Lemma 7.14. If \( y = x_i, \ m - 1 \neq i > k \) or \( y = x_i^2, \ m - 1 = i > k \) then
\[
(53) \quad w_{km}y \equiv_{k+1} 0.
\]

Proof. If \( i < m - 1 \) then by means of (37) it is possible to permute \( y \) to the left beyond \( x_n^2 \) and use Lemma 7.3 with \( m' = n - 1 \). If \( y = x_i^2, \ m - 1 = i > k \) then by the above case, \( i < m - 1 \), we get
\[
(54) \quad w_{km}y = w_{km+1}x_mx_{m-1}^2 = w_{km+1}x_{m-1}(\alpha_{x_m}x_{m-1} + \beta_{x_{m-1}}x_m) \equiv_{k+1} 0,
\]
where for \( m = n \) by definition \( w_{kn+1} = u_{kn} \), and \( u_{kn}x_{n-1} \equiv_{n-1} 0 \).

If \( y = x_i, i = m > k \) then for \( m = n \) one may use the second equality (48). For \( m < n \) we have \( w_{km}y = w_{km+1}y_1 \) where \( y_1 = x_m^2 \). Therefore for \( k < n - 1 \) we may use (54) with \( m + 1 \) in place of \( m \). For \( k = n - 1 \) we have
\[
w_{km}x_n = x_{n-1}x_n^3 \equiv 0.
\]

Finally, if \( y = x_i, i > m > k \) then by (37) we have \( w_{km}y = \alpha w_{ki+1}x_i^2x_{i-1}x_i \cdot v \). For \( i = n \) one may use (50), while for \( i < n \), changing the underlined word according to (35), we may use the above considered cases: \( m' - 1 = i' \), where \( m' = i + 1, i' = i \); and \( i' < m' - 1 \), where \( m' = i + 1, i' = i - 1 \). □

Another interesting relation appears if we multiply (46) by \( x_{n-1} \) from the left and subtract (36) with \( i = n - 1 \) multiplied from the right by \( x_n^2 \):
\[
(55) \quad x_{n-1}x_nx_{n-1}x_n^2 \equiv_n \alpha x_{n-1}x_n^2x_{n-1}x_n,
\]
in which case \( \alpha = p_{n-1}p_n^{[3]} \neq 0 \).

Lemma 7.15. For \( k < s < m \leq n \) the following relation is valid.
\[
(56) \quad w_{km}w_{ks} \equiv_{k+1} \varepsilon w_{ks}w_{km}, \quad \varepsilon \neq 0.
\]

Proof. Let us use downward induction on \( k \). For this we first transpose the second letter \( x_k \) of \( w_{km}w_{ks} \) as far to the left as possible by means of (37), and then change the onset \( x_kx_{k+1}x_k \) according to (49). We get
\[
(57) \quad w_{km}w_{ks} \equiv_{k+1} \alpha x_k^2(w_{k+1}w_{k+1}s), \quad \alpha \neq 0.
\]

For \( k + 1 < s \) we apply the inductive supposition to the word in the parentheses and then by (49) and (37) transpose \( x_k \) to its former position.

The case \( k + 1 = s \), the basis of the induction on \( k \), we prove by downward induction on \( s \).

Let \( k + 1 = s = n - 1 \). Then \( m = n \). Let us first show that
\[
(58) \quad x_{n-1}x_n^2x_{n-1}x_nx_{n-1}x_n \equiv_n \alpha x_{n-1}x_n^2x_{n-1}^2x_n^2 + \beta x_{n-1}x_nx_{n-1}x_n^3, \quad \alpha \neq 0.
\]
For this in the left hand side we transpose the first letter $x_n$ by means of (55) to the penultimate position, and then replace the ending $x_n^3 x_{n-1}$ by (46). We get a linear combination of three words. One of them equals the second word of (58), while two other have the following forms.

$$x_{n-1} x_n x_{n-1} x_n x_{n-1} x_n^2, \ x_{n-1} x_n x_{n-1} x_n^2 x_{n-1} x_n.$$

The former word by (36) transforms into the form (58). The latter one, after the application of (55) and the replacing of $x_{n-1} x_n x_{n-1}$ by (36), will have an additional term $x_{n-1} x_n^3 x_{n-1} x_n$ to which it is possible to apply (48). The direct calculation of the coefficients shows that $\alpha = p_{n-1} p_{n-1} \neq 0$.

Now let us multiply (58) by $x_n^2$ from the left and use (36) with $i = n - 2$. We get that $w_{n-2} w_{n-2} w_{n-1}$ with respect to $\equiv_{n-1}$ equals

$$(n - 2) x_{n-2} x_{n-2} x_{n-1} x_n^2 x_{n-1} x_n^2 + \delta x_{n-2} x_{n-1} x_n x_{n-2} x_{n-1} x_n^2 x_{n-1} x_n^3, \ \gamma \neq 0.$$  

Let us apply (48) and then (49) and (48) to the second word. We get that this word with respect to $\equiv_{n-1}$ equals zero. The first word after application of (36) takes up the form

$$\varepsilon w_{n-2} w_{n-2} w_{n-1} + \varepsilon' w_{n-2} x_{n-1}^2 x_n^2 x_{n-1}^2, \ \varepsilon \neq 0.$$  

Thus, by Lemma 7.14, the basis of the induction on $s$ is proved.

Let us carry out the inductive step. Let $k + 1 = s < n - 1$. If $m > s + 1 = k + 2$ then by the inductive supposition on $s$ we may write

$$w_{km} w_{ks} = (w_{km} w_{kk+2}) x_{k+1} \equiv_{k+1} \alpha w_{kk+2} w_{km} x_{k+1} = \beta w_{kk+2} x_k x_{k+1} x_{k+2} x_{k+1} w_{k+1} w_{k+3} m.$$  

Taking into account (53) we may neglect the words starting with $x_{k+1}^2$, $x_{k+2}$ while transforming the underlined part:

$$x_k x_{k+1} x_{k+2} x_{k+1} \equiv x_k x_{k+1} x_{k+2} x_{k+2} \equiv \delta x_{k+1} x_k x_{k+1} x_{k+2}.$$  

In this way (60) is transformed into (56).

If $m = s + 1 = k + 2 < n$ then the relation (57) takes up the form

$$w_{km} w_{ks} \equiv_{k+1} \alpha x_k^2 (w_{k+1} x_{k+1} w_{k+1} x_{k+1}) x_{k+2} x_{k+1}.$$  

Let us apply the inductive supposition with $k' = k + 1$, $s' = k + 2$, $m' = k + 3$ to the word in the parentheses. We get

$$w_{km} w_{ks} \equiv_{k+1} \alpha \varepsilon x_k^2 w_{k+1} x_{k+1} w_{k+1} x_{k+1} x_{k+2} x_{k+1},$$  

or after an evident replacement

$$w_{km} w_{ks} \equiv_{k+1} \gamma x_k^2 w_{k+1} x_{k+1} x_{k+1} w_{k+1} x_{k+1} x_{k+1} x_{k+2}.$$  

In both terms we may transpose one letter $x_k$ to its former position by means of (49) and (37). We get

$$w_{km} w_{ks} \equiv_{k+1} \gamma' w_{kk+3} w_{kk+1} x_{k+2}^2 + \delta' w_{kk+3} x_{k+1} x_{k+2}.$$  

It is possible to apply (56) with \( m' = k + 3, s' = k + 1 \) to the first term since the case \( m > s + 1 \) is completely considered. Therefore it is enough to show that the second term equals zero with respect to \( \equiv k+1 \). When we transpose the third letter \( x_{k+1} \) as far to the left as possible we get the word
\[
(63) \quad w_{kk+3}x_kx_{k+1}x_{k+2}x_{k+1}w_{k+3}x_{k+2}.
\]
Taking into account (53) we may neglect the words starting with \( x_{k+1} \) while transforming the underlined part:
\[
(64) \quad x_kx_{k+1}x_{k+2}x_{k+1} \equiv x_kx_{k+2}x_{k+1}x_{k+1} \equiv x_kx_{k+1}w_{k+3}x_{k+2}.
\]
Therefore the word (63) equals \( w_{kk+1}w_{kk+3}x_{k+2} \) with respect to \( \equiv k+1 \) and it remains only to apply Lemma 7.14 twice. \( \square \)

**Lemma 7.16.** The set \( B \) satisfies the conditions of Lemma 4.8.

**Proof.** By Lemmas 7.12 and 4.7 it is sufficient to show that in \( U^h_p(\mathfrak{g}) \) all words of the form \( u_0, \ldots, u_6 \) are linear combinations of lesser ones. The words \( u_0 \) are diminished by (37). The words \( u_1, u_2 \) have been presented in this way, without using \([x_{n-1}x_n^2] = 0\), in the proof of the above theorem. The relation (53) shows that \( u_3 \equiv_k 0, u_4 \equiv_k 0 \). Lemma 7.15 with \( s = m - 1 \) yields the necessary representation for \( u_5 \).

Let us prove by downward induction on \( k \) that

\[
u_6 \equiv u_{kn}x_n \equiv_k \varepsilon u_{kn}x_n u_{kn}, \quad \varepsilon \neq 0.
\]

For \( k = n - 1 \) this equality takes up the form (55). Let \( k < n - 1 \). Let us transpose the second letter \( x_k \) of \( u_{kn}^2 x_n \) as far to the left as possible by means of (37) and then apply (35). We get

\[
u_{kn}^2 x_n \equiv_k \alpha x_k^2 (u_{kn+1}^2 x_n), \quad \alpha \neq 0.
\]

We may apply the inductive supposition to the term in the parentheses and then by (35), (37) transpose one of \( x_k \)’s to its former position. \( \square \)

**Lemma 7.17.** If \( p_{11} \neq 1 \) then the values of polynomials \( [v]^h \), where \([v] \in B, v \neq x_i h \geq 1\) are not skew primitive, in particular, they are nonzero.

**Proof.** Note that for \( n > 2 \) the sub-algebra generated by \( x_2, \ldots x_n \) is defined by the Cartan matrix of the type \( B_{n-1} \). This allows us to carry out the induction on \( n \) with additional supposition that the statements 1 and 2 of Theorem \( B_n \) are valid for lesser values of \( n \). It is convenient formally consider the one generated sub-algebras \( \langle x_i \rangle \) as algebras of the type \( B_1 \). In this case for \( n = 1 \) the lemma and the statements 1 and 2 are correct in the evident way. If \( v \) starts with \( x_k \neq x_1 \) then we may directly use the inductive supposition. If \( v = u_{1m} \), one may literally repeat the arguments of Lemma 7.7 starting at the formula (41). Let \( v = w_{1m} \). If \( m > 2 \) then by
Lemma 7.8 we have \( w_{1m} = [x_1[w_{2m}]] \). This provides a possibility to repeat
the same arguments of Lemma 7.7 with \( w \) in place of \( u \).

Consider the last case \( v = u_{12} \). By Lemma 7.8 we have

\[
\begin{align*}
[w_{12}] &= [w_{13}]x_2 - p(w_{13}, x_2)x_2[w_{13}], \\
[w_{13}] &= x_1[w_{23}] - p(x_1, w_{23})[w_{23}x_1].
\end{align*}
\]

Applying the coproduct first to (66) then to (65) we may find the sum \( \Sigma \)
of all tensors \( w^{(1)} \otimes w^{(2)} \) of \( \Delta([w_{12}]) \) with \( \deg_1(w^{(2)}) = 1 \), \( \deg_k(w^{(2)}) = 0 \),
\( k > 1 \) (in much the same way as (42)):

\[
\begin{align*}
\Sigma &= (\varepsilon g_1[w_{23}] \otimes x_1)(x_2 \otimes 1) - p(w_{13}, x_2)(x_2 \otimes 1)(\varepsilon g_1[w_{23}] \otimes x_1) \\
&= \varepsilon g_1([w_{23}]x_2 - p(w_{13}, x_2)p(x_2, x_1)x_2[w_{23}] \otimes x_1).
\end{align*}
\]

For \( n > 2 \), taking into account first the bicharacter property of \( p \), then
the equality \([x_2[w_{23}]] = x_2[w_{23}] - p(x_2, w_{23})[w_{23}x_2] \),
and next the following relations \( p_{ij} p_{ji} = 1 \), \( i - j > 1 \);
\( p_{11}^{-1} = p_{12} p_{21} = p_{21}^{-1} = p_{23} p_{32} \), we may write

\[
\begin{align*}
\Sigma &= \varepsilon g_1(-p(w_{13}, x_2)p_{21}[x_2w_{23}] + (1 - p_{11}^{-1})[w_{23}] \cdot x_2) \otimes x_1.
\end{align*}
\]

Consider the left hand side of this tensor on applying the inductive sup-
position. Note that \( x_2w_{23} \) is a standard word and \( [x_2w_{23}] = [x_2[w_{23}]] \).
This super-letter is non-hard in \( U_P(g) \) since \( x_2w_{23} \) contains the sub-word
\( x_2^2 x_3 \). Thus \([x_2w_{23}]\) is a linear combination of monotonous non-decreasing
super-words in lesser super-letters. Among these super-words there is no
\([w_{23}] \cdot x_2 \) since \( x_2 > x_2w_{23} \). On the other hand, \([w_{23}] \cdot x_2 \) is a monotonous
non-decreasing super-word and hence its value in \( U_P(g) \) is a basis element.
Therefore for \( n > 2 \) the left hand side \( W \) of \( \Sigma \) is nonzero.

For \( n = 2 \), by the definition \( w_{23} = x_2, w_{13} = x_1x_2 \), and the equality (67)
takes up the form \( \Sigma = \varepsilon g_1(1 - p_{12} p_{22} p_{21}) x_2^2 \otimes x_1 \). Since \( 1 \neq p_{11}^{-1} = p_{12} p_{21} = p_{22}^{-2} \), we get
\( 1 - p_{12} p_{22} p_{21} = 1 - p_{22}^{-1} \neq 0 \). Therefore in this case \( \Sigma \neq 0 \) as
well.

By [23, Corollary 10] and the inductive supposition the sub-algebra
generated by \( x_2, \ldots, x_n \) has no zero divisors. In particular \( W^h \neq 0 \) and \( \Sigma^h \neq 0 \)
in any case.

It remains to note that for \( n > 1 \) the sum of all tensors \( w^{(1)} \otimes w^{(2)} \) of
\( \Delta([w_{12}]) \) such that \( \deg_1(w^{(2)}) = h, \deg_k(w^{(2)}) = 0, k > 1 \) equals \( \Sigma^h \), hence
\([w_{12}]^h \) can not be skew-primitive.

\[ \square \]

\textbf{Proof of Theorem B}_n. Since none of \( u_{km}, w_{km} \) contains sub-words (30),
Lemmas 7.16, 7.1, 4.8 imply the first statement.

If \( [v] \in B \) is of finite height then by Lemma 7.13 and the homogeneous
version of Definition 4.4 we have \( [v]^h = 0 \). For \( p_{11} \neq 1 \) this contradicts
Lemma 7.17.
Along similar lines, by Lemma 4.9, every skew primitive homogeneous element has the form \([v]^h\). This, together with Lemma 7.17, proves the fourth statement and, for \(p_{11} \neq 1\), the second one too.

If \(p_{11} = 1\) then by (47) we have \(p_{22}^2 = 1\), \(p_{ii} = 1\), \(i < n\). Besides, \(p_{ij}p_{ji} = 1\) for all \(i, j\). This means that the skew commutator is a quantum Lie operation. Hence all elements of \(B\) are skew primitive. In the case \(p_{nn} = 1\) these elements span a color Lie algebra, while in the case \(p_{nn} = -1\) they span a color Lie super-algebra. Now as in Theorem \(A_n\), we may use the PBW-theorem for the color Lie super-algebras.

The third statement will follow Theorem 5.2 and Lemmas 5.3, 7.12 if we prove that all super-letters (45) are zero in \(U_P(g)\). We have already proved that these super-letters are non-hard. Therefore it remains to use the homogeneous version of Definition 4.3 and Lemma 7.13.

\(\square\)

**Theorem C\(_n\).** Suppose that \(g\) is of the type \(C_n\), and \(p_{ii} \neq -1\), \(1 \leq i \leq n\), \(p_{n-1,n-1} \neq 0\). Denote by \(B\) the set of the following super-letters:

\[
\begin{align*}
[u_{km}] & \overset{df}{=} [x_kx_{k+1} \ldots x_m], & 1 \leq k \leq m \leq n; \\
[v_{km}] & \overset{df}{=} [x_kx_{k+1} \ldots x_n \cdot x_{n-1} \ldots x_m], & 1 \leq k < m < n; \\
[w_k] & \overset{df}{=} [u_{kn-1}u_{kn}], & 1 \leq k < n.
\end{align*}
\]

The statements given below are valid.

1. The values of the super-letters (69) in \(U_P(g)\) form the PBW-generators set.
2. Each of these super-letters has the infinite height in \(U_P(g)\).
3. The following relations with (23) form a Groebner–Shirshov system for \(U_P(g)\).

\[
\begin{align*}
[u_0] & \overset{df}{=} [x_kx_m] = 0, & 1 \leq k < m - 1 < n; \\
[u_1] & \overset{df}{=} [u_{km}x_{k+1}] = 0, & 1 \leq k < m \leq n, (k, m) \neq (n - 2, n); \\
[u_2] & \overset{df}{=} [u_{km}u_{km+1}] = 0, & 1 \leq k \leq m < n - 1; \\
[v_3] & \overset{df}{=} [v_{kn}x_{k+1}] = 0, & 1 \leq k < m < n, k \neq m - 2; \\
[w_4] & \overset{df}{=} [v_{kk+1}x_{k+2}] = 0, & 1 \leq k < n - 1; \\
[w_5] & \overset{df}{=} [v_{km}v_{km+1}] = 0, & 1 \leq k < m - 1 \leq n - 1; \\
[w_6] & \overset{df}{=} [u_{kn-1}^3x_n] = 0, & 1 \leq k < n.
\end{align*}
\]

4. If \(p_{11} \neq 1\) then the generators \(x_i\) and their powers \(x_i^t, x_i^{tk}\), such that \(p_{ii}\) is a primitive \(t\)-th root of 1 together with the constants \(1 - g, g \in G\) form a basis of \(g_P = L(U_P(g))\). Here \(l\) is the characteristic of the ground field.
5. If \( p_{11} = 1 \) then the elements (69) and in the case of prime characteristic 1 theirs \( l^\text{th} \) powers, together with the constants \( 1-g, g \in G \) form a basis of \( g_P \).

In the case \( C_n \) the algebra \( U_P^b(g) \) is defined by the same relations (35), (36), (37), where in (36) the last relation, \( i = n-1 \), is replaced with

\[
x^{3}_{n-1}x_n = p_{n-1}^{-1}p_{n-1n-1}^{[3]}x_{n-1}x_nx_{n-1} +
- p_{n-1n}^{[2]}p_{n-1n-1}^{[3]}x_{n-1}x_nx_{n-1}^{2} + p_{n-1n}^{[3]}p_{n-1n-1}^{3}x_{n}x_{n-1}^{3}.
\]

By Corollary 2.5 we get the existence conditions

\[
(72) \quad p_{ii} = p_{11}, \quad p_{i-1}p_{ii-1} = p_{11}^{-1}, \quad 1 < i < n,
\]

\[
(73) \quad p_{n-1n}p_{nn-1} = p_{nn}^{-1} = p_{n-1n-1}^{-1}; \quad p_{ij}p_{ji} = 1, \quad i - j > 1.
\]

Therefore the following relations are correct

\[
(74) \quad x_{i}x_{i+1}x_{i} = 0, \quad 1 \leq i < n;
\]

\[
(75) \quad x_{n-1}x_{n}x_{n-1} = 0, \quad 1 \leq i < n, \quad \alpha = 1, \quad \beta \neq 0;
\]

\[
(76) \quad x_{n-2}x_{n-1}x_{n}x_{n-1} = 0.
\]

**Lemma 7.18.** The brackets in \( [v_{km}], [v_k] \) are set according to the following recurrence formulae, where by the definition \( v_{kn} = u_{kn} \).

\[
\begin{align*}
[v_{km}] &= [x_k[v_{k+1}m]], & & \text{if } 1 \leq k < m-1 < n-1; \\
[v_{kk+1}] &= [v_{kk+2}[x_{k+1}], & & \text{if } 1 \leq k < n-1; \\
[v_k] &= [u_{k-1}[u_{kn}]], & & \text{if } 1 \leq k < n.
\end{align*}
\]

**Proof.** It is enough to use the properties 6s, 1s and 2s.

**Lemma 7.19.** If \( [u],[v] \in B \) then one of the following statements is valid.

1) \( [u][v] \) is not a standard nonassociative word;
2) \( uv \) contains a sub-word of one of the types \( u_0, u_1, u_2, w_3, w_4, w_5, w_6; \)
3) \( [u][v] \in B \).

**Proof.** The first two formulae (77) coincide with (51) up to replacement of \( v \) with provided \( k+1 \neq n > m \). Obviously for \( m < n \) the inequality \( v_{km} > v_{rs} \) is equivalent to \( w_{km} > w_{rs} \), while \( v_{km} > u_{rs} \) is equivalent to \( w_{km} > w_{rs} \). Hence Lemmas 7.9, 7.10, 7.11 are still valid under the replacement of \( w \) with \( v \):

\[
\begin{align*}
[v_{km}][v_{rs}] & \quad \text{is standard } \iff \quad s \geq m > k+1 = r \lor (s < m \& r = k); \\
[u_{km}][v_{rs}] & \quad \text{is standard } \iff \quad k = r \lor k = m < r; \\
[u_{km}][u_{rs}] & \quad \text{is standard } \iff \quad r = k+1 < m \lor r = k+1 = m = s.
\end{align*}
\]
Further, $v_k > v_r$ if and only if $k < r$, and under this condition $[[v_k][v_r]]$ is not standard since $u_{kn} > u_{r,n-1}u_{rn}$.

In a similar manner $v_k > u_{rm}$ is equivalent to $k < r$, while $v_k > v_{rm}$ is equivalent to $k \leq r$. Therefore none of the words $[[v_k][u_{rm}]]$, $[[v_k][v_{rm}]]$ is standard since $u_{kn} > u_{rm}$ and $u_{kn} > v_{rm}$, respectively.

For the remaining two cases we have only two possibilities

$\begin{align*}
[[u_{km}][v_r]] &\text{ is standard } \iff r = k \leq m < n; \\
[[v_{km}][v_r]] &\text{ is standard } \iff r = k + 1 & k < m - 1.
\end{align*}$

The treatment in turn of the eight possibilities (78), (79) proves the lemma.

\[ \square \]

**Lemma 7.20.** If a super-word $W$ equals one of the super-letters (70) or $[v]^h$, $[v] \in B$, $h \geq 1$, then its constitution does not equal the constitution of any word in less then $W$ super-letters from $B$.

**Proof.** The proof is akin to Lemma 7.6 with the following tableaux:

$$
\begin{align*}
[u_{km}]^h, [u_{km}x_{k+1}], [u_{km}u_{km+1}] &\text{ deg}_k(u) \leq \text{ deg}_{m+1}(u); \\
[v_{km}]^h, [v_{km}x_{k+1}], [v_{km}v_{km-1}] &\text{ deg}_k(u) \leq \text{ deg}_{m-1}(u); \\
[v_{k+1}x_{k+2}] &\text{ deg}_k(u) = 0; \\
[v_k]^h, [v_{km}v_{km-1}] &\text{ deg}_k(u) \leq \text{ deg}_n(u); \\
[u_{k-1}^n x_{n}] &\text{ deg}_k(u) \leq 2\text{ deg}_n(u).
\end{align*}
$$

\[ \square \]

**Lemma 7.21.** If $y = x_i$, $m - 1 \neq i > k$ or $y = x_i^2$, $m - 1 = i > k$ then

$$
v_{km}y \equiv_{k+1} 0.
$$

**Proof.** For $i < m - 1$, we may transpose $y$ by means of (37) to the left across $x_n^2$ and then use Lemma 7.3 with $m' = n - 1$.

If $y = x_i^2$, $m - 1 = i > k$ then by the above case, $i < m - 1$, we get

$$
v_{km}y = v_{km+1}x_{m}x_{m-1}^2 = v_{km+1}x_{m-1}(\alpha x_{m}x_{m-1} + \beta x_{m-1}x_m) \equiv_{k+1} 0,
$$

where by definition $v_{kn} = u_{kn}$ and $u_{kn}x_{n-2} \equiv_{n-2} 0$, while $n - 2 = i > k$.

If $y = x_i$, $i = m > k$ then for $m = n - 1$ we may use the inequality (76), while for $m < n - 1$ we have $v_{km}y = v_{km+1}y_1$ where $y_1 = x_m^2$. Hence we may use (82) replacing $m$ by $m + 1$.

If $y = x_i$, $i > m > k$ then by (37) we get $v_{km}y = \alpha v_{ki+1}x_{i}x_{i-1}x_i \cdot w$. Changing the underlined by (35), we may apply the previously considered cases: $m' - 1 = i'$, where $m' = i + 1, i' = i$; and $i' < m' - 1$, where $m' = i + 1, i' = i - 1$.

\[ \square \]
If we multiply (71) by $x_n$ from the right and subtract (35) with $i = n - 1$ multiplied from the left by $x_{n-1}^2$, then by means of $p_{n-1}^{-2} = p_{nn-1}p_{n-1} = p_{nn}$ we get

\begin{equation}
(83)
x_{n-1}^2 x_n x_{n-1} x_n \equiv n p_{n-1} (p_{[3]}^{n-1} x_n - p_{n-1}^{x_n} x_{n-1}^2 x_n - p_{n-1}^{x_n} x_{n-1}^2 x_n x_{n-1}).
\end{equation}

Let us first multiply this relation by $x_{n-2}^2$ from the left and then apply (35) to the underlined sub-word. Taking into account the relation $x_{n-2}^2 x_{n-1}^3 \equiv n - 1 0$, we get that the left hand side of the multiplied (83) equals $p_{n-1} p_{nn} (1 + p_{nn})^{-1} x_{n-2} x_{n-1}^2 x_n x_{n-1}$ up to $\equiv n - 1$, i.e., it is proportional to the second term of the right hand side. As a result the relation below with $\alpha = p_{n-1}^{-1} (1 + p_{nn}) \neq 0$ is correct.

\begin{equation}
(84)
x_{n-2}^2 x_{n-1}^2 x_n x_{n-1} \equiv n - 1 \alpha x_{n-2}^2 x_{n-1}^x_n x_{n-1}^2 x_n.
\end{equation}

**Lemma 7.22.** If $k < s < m \leq n$ and as above $v_{kn} = u_{kn}$ then

\begin{equation}
(85)
v_{km} v_{ks} \equiv k + 1 \varepsilon v_{ks} v_{km}, \quad \varepsilon \neq 0.
\end{equation}

**Proof.** Let us use downward induction on $k$. For this we first transpose the second letter $x_k$ of $v_{km} v_{ks}$ as far to the left as possible by means of (37), and then change the onset $x_k x_{k+1} x_k$ according to (74). We get

\begin{equation}
(86)
v_{km} v_{ks} \equiv k + 1 \alpha x_k^2 (v_{k+1} v_{k+1}), \quad \alpha \neq 0.
\end{equation}

For $k + 1 < s$ we may apply the inductive supposition to the word in the parentheses, and then transpose $x_k$ to its former position by (74), (37).

For $k + 1 = s$ we will use downward induction on $s$.

Let $k + 1 = s = n - 1$. In this case $m = n$ and (86) becomes:

$$v_{n-2n+2} v_{n-2n-1} \equiv n - 1 \beta x_{n-2}^2 (x_{n-2}^2 x_{n-1}^2 x_n x_{n-1}).$$

Let us replace the underlined part according to (35). Since $x_{n-2}^2 x_{n-1}^2 x_n^2 \equiv n 0$, we may continue by (84):

$$\equiv n - 1 \beta x_{n-2}^2 x_{n-1}^2 x_n x_{n-1} \equiv n - 1 \beta x_{n-2}^2 x_{n-1}^2 x_n x_{n-1} \equiv n - 1 \beta x_{n-2}^2 x_{n-1}^2 x_n x_{n-1} \equiv n - 1 \beta x_{n-2}^2 x_{n-1}^2 x_n x_{n-1} \equiv n - 1 \beta x_{n-2}^2 x_{n-1}^2 x_n x_{n-1} \equiv n - 1 \beta x_{n-2}^2 x_{n-1}^2 x_n x_{n-1}.$$

With the help of (35) we get

$$= \varepsilon v_{n-2n+2} v_{n-2n} + \beta x_{n-2}^2 x_{n-1}^2 x_n x_{n-1} x_{n-2} x_n, \quad \varepsilon \neq 0.$$

By (75) and (73) we see that the second term equals zero up to $\equiv n - 1$.

The inductive step on $s$ coincides the inductive step on $s$ in Lemma 7.15 up to replacing both the citations of Lemma 7.14 with the citations of Lemma 7.21 and $w$ with $v$. \hfill \Box

**Lemma 7.23.** The set $B$ satisfies the Lemma 4.8 conditions.
Proof. According to Lemma 4.7 and Lemma 7.12 it is sufficient to show that words of the form $u_0, u_1, u_2, w_3, w_4, w_5, w_6$ are linear combinations of lesser words in $U_P(g)$. The words $u_0$ are diminished by (37). The words $u_1, u_2$ have been diminished in Theorem $A_n$ since in the case $C_n$ the words $u_2$ are independent of $x_n$, while $u_1$ depends on $x_n$ only if $u_1 = x_{n-1}x_n^2$. The relation (81) shows that $w_3 \equiv_{k+1} 0$, $w_4 \equiv_{k+1} 0$. Lemma 7.22 with $s = m - 1$ gives the required representation for $u_5$.

Consider the words $w_6$. For $k = n - 1$ the relation (71) defines the required decomposition. Let $k < n - 1$. Since $x_1, \ldots, x_{n-1}$ generate a sub-algebra of the type $A_{n-1}$, the decomposition of $u_6^2 n - 1 x_{n-1}$ in the basis defined by Lemma 4.7 has the form

$$u_6^2 n - 2 x_{n-1} = \sum \alpha u_{m_1 s_1} u_{m_2 s_2} \cdots u_{m_t s_t},$$

where $u_{m_1 s_1} \leq u_{m_2 s_2} \leq \cdots \leq u_{m_t s_t}$, that is $m_1 \geq m_2 \geq \cdots \geq m_t$, and $s_i \geq s_{i+1}$ if $m_i = m_{i+1}$. In particular, if $m_1 = k$ then $m_2 = \ldots = m_t = k$ and, due to the homogeneity, $t = 3, s_1 = n - 1, s_2 = s_3 = n - 2$. Therefore

$$u_6^2 n - 2 x_{n-1} \equiv_{k+1} \tilde{u}_k n - 1 u_k^2 n - 2.$$

Along similar lines, the following relations are valid as well

$$u_6^3 n - 2 x_{n-1} \equiv_{k+1} \mu u_k n - 1 u_k n - 2, \quad u_6^2 n - 2 x_{n-1} \equiv_{k+1} 0.$$

Now let us multiply (35) with $i = n - 2$ by $x_n$ from the right, and then add to the result the same relation multiplied by $p_n - 2 n - 1 1 + p_n - 1 n - 1) x_n$ from the left. We get the following relation with $\alpha = p_n^2 n - 2 n - 1 p_{n - 1 n - 1}^3 \neq 0$,

$$x_{n-2} x_{n-1}^3 = \alpha x_{n-1}^2 x_{n-2} x_{n-1} + \beta x_{n-1}^3 x_{n-2}.$$

Further, we may write

$$u_6^3 n - 1 = \beta_1 u_{k n - 2} u_{k n - 3} x_{n-1} x_{n-2} x_{n-1} u_{k n - 1}, \quad \beta_1 \neq 0,$$

where for $k = n - 2$ the term $u_{k n - 3}$ is absent. Let us apply (35) with $i = n - 2$ to the underlined word. Since $u_{k n - 2} u_{k n - 3} x_{n-1}^2 x_{n-1} \equiv_{n - 1} 0$, we have got

$$u_6^3 n - 1 \equiv_{n - 1} \beta_2 u_{k n - 2} u_{k n - 3} x_{n-1}^2 x_{n-1} x_{n-2} x_{n-1}.$$

Let us apply (90). Taking into account the second of (89) we get

$$u_6^3 n - 1 \equiv_{k+1} \beta_3 u_{k n - 2} x_{n-1}^3.$$

Let us multiply this relation from the right by $x_n$. By (71) we have

$$u_{k n - 1} x_n \equiv_{k+1} \alpha u_{k n - 2} x_{n-1} x_n^2 x_{n-1} + \beta u_{k n - 2} x_{n-1} x_n x_{n-1}.$$

By means of (88) and (89) we have got

$$u_{k n - 1} x_n \equiv_{k+1} \alpha u_{k n - 1} x_n u_{k n - 2} x_{n-1} + \beta_1 u_{k n - 1} x_n u_{k n - 2} x_{n-1},$$

and both of these words are less than $u_{k n - 1} x_n$. □
Lemma 7.24. If \( p_1 \neq 1 \) then the values of \([v]^h\), where \([v] \in B\), \( v \neq x_i\), \( h \geq 1 \) are not skew primitive. In particular they are nonzero.

Proof. Note that for \( n > 3 \) the algebra generated by \( x_2, \ldots x_n \) is a sub-algebra of the type \( C_{n-1} \). Therefore we may use induction on \( n \) with additional supposition that the theorem statements 1 and 2 are valid for the lesser values of \( n \). We will formally consider the sub-algebra generated by \( x_{n-1}, x_n \) as an algebra of the type \( C_2 \), and the sub-algebra generated by \( x_n \) as an algebra of type \( C_1 \). In this case for \( n = 1 \) the present lemma and the statements 1 and 2 are valid in obvious way.

If the first letter \( x_k \) of \( v \) is less than \( x_1 \) then we may use the inductive supposition directly. If \( v = u_{1m} \) then one may literally repeat arguments of Lemma 7.7 starting at (41).

If \( v = v_{1m} \) and \( n > 3 \) then we may repeat arguments of Lemma 7.17 starting at (65) up to replacing \( w \) with \( v \). For \( n = 3 \) in these arguments the formula (68) assumes the form

\[
\begin{align*}
\Sigma &= \varepsilon g_1 (-p(v_{13}, x_2)p_{21}[x_2^2 x_3] + (1 - p_{11}^{-1})[x_2 x_3] \cdot x_2) \otimes x_1. \\
\end{align*}
\]

Therefore the left component of the tensor \( \Sigma \) is a nonzero linear combination of the basis elements. For \( n = 2 \) the set \( B \) has no elements \( v_{1m} \) at all.

Consider the last case, \( v = v_1 = [u_{1n-1} x_n] \). Let \( S_k \) be the sum of all tensors of \( \Delta([u_{kn}]) = \sum u^{(1)} \otimes u^{(2)} \) with \( \text{deg}_n(w^{(1)}) = 1, \text{deg}_k(w^{(2)}) = 0 \), \( k < n \). Evidently \( S_n = x_n \otimes 1 \). Let us show by downward induction on \( k \) that \( S_k = (1 - p_{11}^{-1})g(u_{kn-1})x_n \otimes [u_{kn-1}] \) at \( k < n \). We have

\[
\begin{align*}
\Delta([u_{kn}]) &= \Delta(x_k)\Delta([u_{k+1n}]) - p(x_k, u_{k+1n})\Delta([u_{k+1n}])\Delta(x_k). \\
\end{align*}
\]

Consequently,

\[
\begin{align*}
S_k &= (g_k \otimes x_k)S_{k+1} - p(x_k, u_{k+1n})S_{k+1}(g_k \otimes x_k). \\
\end{align*}
\]

This implies the required formula since by (72) at \( k < n - 1 \) we have

\[
p(x_k, u_{k+1n})p(x_n, x_k) = p(x_k, u_{k+1n-1}),
\]

while at \( k = n - 1 \) we have \( p(x_{n-1}, x_n)p(x_n, x_{n-1}) = p_{11}^{-1} \).

In a similar manner, consider the sum \( S \) of all tensors of \( \Delta([u_{kn} x_n]) = \sum w^{(1)} \otimes w^{(2)} \) with \( \text{deg}_n(w^{(1)}) = 1, \text{deg}_i(w^{(2)}) = 0 \), at \( i < n \),

\[
\begin{align*}
\Delta([u_{1n-1}[u_{1n}]) &= \Delta([u_{1n-1}])\Delta([u_{1n}]) - p(u_{1n-1}, u_{1n})\Delta([u_{1n}])\Delta([u_{1n-1}]). \\
\end{align*}
\]

Since we know \( S_1 \), we may calculate \( S \):

\[
\begin{align*}
S &= (g(u_{1n-1}) \otimes [u_{1n}])S_1 - p(u_{1n-1}, u_{1n})S_1(g(u_{1n-1}) \otimes [u_{1n}]) \\
&= (1 - p_{11}^{-1})g(u_{1n}^2)x_n \otimes (1 - p(u_{1n-1}, u_{1n})p(x_n, u_{1n-1}))[u_{1n-1}]^2. \\
\end{align*}
\]
By (72), using the bicharacter property of $p$, we have

$$1 - p(u_{1n-1}, u_{1n})p(x_n, u_{1n-1})$$

$$= 1 - p(u_{1n-1}, u_{1n-1})p_{n-1}p_{nn-1}$$

$$= 1 - p_{n-1}p_{n-1} = 1 - p_{11}^{-1} \neq 0.$$ 

Because of this, $S \neq 0$ and the sum of all tensors $w^{(1)} \otimes w^{(2)}$ with $\deg_n(w^{(1)}) = h$, $\deg_k(w^{(1)}) = 0$, $k < n$ of the basis decomposition of $\Delta([v_1]^h)$ equals $S^h \neq 0$. Therefore $[v_1]^h$ is not skew primitive. 

Proof of Theorem $C_n$. For the first statement it will suffice to prove that all super-letters (69) are hard in $U_P(\mathfrak{g})$. Since none of $u_{km}$, $v_{km}$ contains a sub-word (30), Lemma 7.1 implies that $[u_{km}]$, $[v_{km}]$ are hard.

If $[v_k]$ is not hard then, by the homogeneous version of Definition 4.3, its value is a polynomial in lesser hard super-letters. In line with Lemmas 7.23 and 4.8, all hard super-letters belong to $B$. Therefore, by Lemma 7.20, $[v_k] = 0$. Since $\deg_n(v_k) = 1$ and $\deg_{n-1}(v_k) = 2$, the equality $[v_k] = 0$ is valid in the algebra $C'$ which is defined by all relations of $U_P(\mathfrak{g})$, except ones of degree greater than 1 in $x_n$ and ones of degree greater than 2 in $x_{n-1}$, that is in the algebra defined by (35), (36) with $i < n - 1$, and (37). These relations do not reverse the order of $x_{n-1}$ and $x_n$ in monomials since none of them has both $x_{n-1}$ and $x_n$. This implies that the sum of all monomials of $[v_k] = [u_{kn-1}] \cdot [u_{kn}] - p(u_{kn-1}, u_{kn})[u_{kn}] \cdot [u_{kn-1}]$ in which $x_n$ is prefixed to $x_{n-1}$ equals zero in the above defined algebra $C'$, that is $[u_{kn}] \cdot [u_{kn-1}] = 0$. Especially, this equality is valid in $U_P(\mathfrak{g})$. Since, by Theorem 4.5, the super-word $[u_{kn}] \cdot [u_{kn-1}]$ is a basis element, the first statement is proved.

If $[v] \in B$ is of finite height then, by Lemma 7.20 and the homogeneous version of Definition 4.4, we have $[v]^h = 0$. For $p_{11} \neq 1$ this contradicts Lemma 7.24. In a similar manner, according to Lemma 4.9, every skew primitive homogeneous element has the form $[v]^h$. This, together with Lemma 7.24, proves the fourth statement and, for $p_{11} \neq 1$, the second one too. If $p_{11} = 1$ then according to (72) we have $p_{ij} = p_{ij}p_{ji} = 1$ at all $i, j$. In particular, the skew commutator is a quantum Lie operation. Hence all elements of $B$ are skew primitive. These elements span a color Lie algebra. Now, as in Theorem $A_n$, we may use the colored PBW theorem.

The third statement will follow from Theorem 5.2 and Lemmas 5.3, 7.19 provided we note that all super-letters (70) are zero in $U_P(\mathfrak{g})$. We have proved already that these super-letters are non-hard. So it remains to use first the homogeneous version of Definition 4.3 and then Lemma 7.27. \(\square\)
Theorem $D_n$. Let $\mathfrak{g}$ be of the type $D_n$, and $p_{ii} \neq -1$, $1 \leq i \leq n$. Denote by $B$ the set of the following super-letters:

\begin{align}
[u_{km}] & \overset{df}{=} \ [x_kx_{k+1} \ldots x_m], \quad 1 \leq k \leq m < n; \\
[e_{km}] & \overset{df}{=} \ [x_kx_{k+1} \ldots x_{n-2} \cdot x_nx_{n-1} \ldots x_m], \quad 1 \leq k < m \leq n, \\
[e_{n-1n}] & \overset{df}{=} \ x_n.
\end{align}

The statements given below are valid.

1. The values of (100) in $U_P(\mathfrak{g})$ form the PBW-generators set.
2. Each of the super-letters (100) has infinite height in $U_P(\mathfrak{g})$.
3. The relations (23) together with the following ones form a Groebner–Shirshov system for $U_P(\mathfrak{g})$.

\begin{align}
[u_0] & \overset{df}{=} \ [x_kx_m] = 0, \quad 1 \leq k < m - 1 < n, \ (k, m) \neq (n - 2, n); \\
[u_1] & \overset{df}{=} \ [u_{km}x_{k+1}] = 0, \quad 1 \leq k < m < n; \\
[u'_1] & \overset{df}{=} \ [x_{n-2}x_m^2] = 0, \\
[u_2] & \overset{df}{=} \ [u_{km}u_{km+1}] = 0, \quad 1 \leq k \leq m < n - 1; \\
[u_3] & \overset{df}{=} \ [e_{km}x_{k+1}] = 0, \quad 1 \leq k < m \leq n, \ n - 1 \neq k \neq m - 2; \\
[u_4] & \overset{df}{=} \ [e_{kk+1}x_{k+2}] = 0, \quad 1 \leq k < n - 2; \\
[u'_4] & \overset{df}{=} \ [e_{n-3}x_{n-2}x_m] = 0, \\
[u_5] & \overset{df}{=} \ [e_{km}e_{km-1}] = 0, \quad 1 \leq k < m - 1 \leq n - 1; \\
[u_6] & \overset{df}{=} \ [u_{km}e_{kn}] = 0, \quad 1 \leq k \leq m < n, \ n - 2 \leq m.
\end{align}

4. If $p_{11} \neq 1$, then the generators $x_i$, their powers $x_i^t$, $x_i^{tk}$, such that $p_{ii}$ is a primitive $t$-th root of 1, together with the constants $1 - g$, $g \in G$ form a basis of $\mathfrak{g}_P = L(U_P(\mathfrak{g}))$. Here $l = \text{char}(k)$.

5. If $p_{11} = 1$, then the elements of $B$ and, for $l > 0$, their $t^k$-th powers together with the constants $1 - g$, $g \in G$ form a basis of $\mathfrak{g}_P$.

In the case $D_n$ the algebra $U_P^b(\mathfrak{g})$ can be defined by the condition that the sub-algebras $U_{n-1}$ and $U_n$ generated, respectively, by $x_1, \ldots, x_{n-1}$ and $x_1, \ldots, x_{n-2}, x'_{n-1} = x_n$ are quantum universal enveloping algebras of the type $A_{n-1}$, and by the only additional relation

\begin{align}
[x_{n-1}x_n] &= 0.
\end{align}

The existence conditions take up the form

\begin{align}
p_{ii} &= p_{nn} = p_{11}, \ p_{i+1;i+1}p_{ii+1} = p_{n-2}p_{nn-2} = p_{11}^{-1}, \ \text{if} \ 1 \leq i < n, \\
p_{n-1}p_{nn-1} = p_{ij}p_{ji} = 1, \ \text{if} \ i - j > 1 \& (i, j) \neq (n, n - 2).
\end{align}
Lemma 7.25. The brackets in (100) are set up by the recurrence formulae

\[
\begin{align*}
\epsilon_{km} &= [x_k[\epsilon_{k+1}]], & \text{if } 1 \leq k < m - 1 < n, \ k \neq n - 1; \\
\epsilon_{kk+1} &= [(\epsilon_{kk+2}x_{k+1}], & \text{if } 1 \leq k < n - 1.
\end{align*}
\]

Proof. It is enough to use the properties 6.1, 1.3, and 2.3.

Lemma 7.26. If \([u],[v] \in B\), then one of the statements below is correct.

1) \([uv]\) is not a standard nonassociative word;
2) \([uv]\) contains a sub-word of one of the types \(u_0, u_1, u_2, v_0, v_1, v_2, v_3, v_4, v_5, v_6; \)
3) \([uv]\) is standard.

Proof. The formulae (104) coincides with (51) at \(k \neq n - 1\) up to replacing \(e\) by \(w\). The inequality \(e_{km} > e_{rs}\) is set up by the same conditions, \(k < r \land (k = r \land \ m < s)\), as the inequality \(w_{km} > w_{rs}\) does. Likewise \(u_{km} > e_{rs}\) is set up by the same condition, \(k \leq r\), as \(u_{km} > w_{rs}\) does. Therefore Lemmas 7.9, 7.10, 7.11 remain valid with \(e\) in place of \(w\):

\[
\begin{align*}
\left[[e_{km}]\right]_{e_{rs}} \text{ is standard } & \iff s \geq m \geq k + 1 = r \lor (s < m \land r = k); \\
\left[[u_{km}]\right]_{e_{rs}} \text{ is standard } & \iff k = r \lor k = m < r; \\
\left[[e_{km}]\right]_{u_{rs}} \text{ is standard } & \iff r = k + 1 \land m \lor r = k + 1 = m = s.
\end{align*}
\]

By looking over all of these possibilities we get the lemma statement.

Lemma 7.27. If a super-word \(W\) equals one of the super-letters (101) or \([v]^h, [v] \in B, h \geq 1\) then its constitution does not equal the constitution of any super-word in less than \(W\) super-letters from \(B\).

Proof. The proof is similar to the one of Lemma 7.6 with the tableaux

\[
\begin{align*}
\left[u_{km}\right]^h, \left[u_{km}x_{k+1}\right], \left[u_{km}u_{km+1}\right], \left[u_{km}u_{km+1}\right] & \quad \text{deg}_k(u) \leq \text{deg}_{m+1}(u); \\
\left[e_{km}\right]^h, \left[e_{km}x_{k+1}\right], \left[e_{km}e_{km-1}\right], \left[e_{km}e_{km-1}\right] & \quad m < n \quad 2\text{deg}_k(u) \leq \text{deg}_{m-1}(u); \\
\left[e_{kn}\right]^h, \left[e_{kn}x_{k+1}\right], \left[e_{kn}e_{kn-1}\right], \left[e_{kn}e_{kn-1}\right] & \quad \text{deg}_k(u) \leq \text{deg}_{m-1}(u); \\
\left[e_{kk+1}x_{k+2}\right] & \quad \text{deg}_k(u) = 0; \\
\left[e_{n-3n-2x_{n}}\right] & \quad \text{deg}_{n-3}(u) = 0; \\
\left[u_{kn-2e_{kn}}\right] & \quad \text{deg}_k(u) \leq \text{deg}_{n-1}(u) + \text{deg}(u); \\
\left[u_{kn-1e_{kn}}\right] & \quad \text{deg}_k(u) \leq \text{deg}_{n}(u).
\end{align*}
\]

Lemma 7.28. If \(y = x_i, \ m - 1 \neq i > k\) or \(y = x_i^2, \ m - 1 = i > k\) then

\[
\epsilon_{km}y \equiv_{k+1} 0.
\]

Proof. If \(i < m - 1, m \neq n, \) or \(m = n, i < n - 2, \) then with the help of (37) and (102) it is possible to permute \(y\) to the left beyond \(x_n\) and then to use Lemma 7.3 for \(U_n-1\).

If \(m = n, i = n - 2\) then we may use Lemma 7.3 for \(U_n\).
If \( y = x_i^2 \), \( m - 1 = i > k \) then for \( m < n \) by the above case we get (108)
\[
e_{km}y = e_{km+1}x_mx_{m-1}^2 = e_{km+1}x_m(\alpha x_m x_{m-1} + \beta x_{m-1}x_m) \equiv_{k+1} 0.
\]
For \( m = n \) we have \( e_{kn}x_{n-1}^2 = \alpha u_{kn-2}x_{n-1}^2 \equiv_{n-1} 0 \) since the underlined part belongs to \( U_{n-1} \).

If \( y = x_i \), \( i = m > k \) then for \( m = n \) we may use Lemma 7.3 applied to \( U_n \); for \( m = n - 1 \) we may use the same lemma applied to \( U_{n-1} \) provided that beforehand we permute \( x_n \) with \( y \) by (102); for \( m < n - 1 \) we may first rewrite \( e_{km}y = e_{km+1}y_1 \), where \( y_1 = x_m^2 \), and then use (108) with \( m + 1 \) in place of \( m \).

If \( y = x_i \), \( i > m > k \) then for \( i < n \) we have \( e_{km}y = \alpha e_{ki+1}x_i x_{i-1}x_i \cdot v \).
Replacing the underlined word by (35) in \( U_{n-1} \), we may use the previously considered cases: \( m' - 1 = i' \), where \( m' = i + 1, i' = i \); and \( i' < m' - 1 \), where \( m' = i + 1, i' = i - 1 \). For \( i = n \), and \( m = n - 1 \) we have \( e_{kn-1}x_n = \alpha u_{kn-2}x_{n-1} \) and one may apply Lemma 7.3 to \( U_n \). Finally, for \( i = n \) and \( m < n - 1 \) we get
\[
e_{km}x_n = \beta_1 u_{kn-2}x_n x_{n-1}x_{n-2}x_n \cdot v = \beta_2 u_{kn-2}x_n x_{n-1}x_{n-2}x_n \cdot v = \\
\beta_3 u_{kn-2}x_n x_{n-1}x_{n-2}x_n^2 \cdot v + \beta_4 u_{kn-2}x_n x_{n-1}x_{n-2}x_n^2 \cdot v.
\]
One may apply first Lemma 7.3 for \( U_{n-1} \) to the underlined sub-word of the first term, and then, after (102), Lemma 7.3 for \( U_n \) to the second term. \( \square \)

**Lemma 7.29.** If \( k < s < m \leq n \) then \( e_{km}e_{ks} \equiv_{k+1} e_{ks}e_{km}, \ v \neq 0 \).

**Proof.** Let us carry out downward induction on \( k \). The largest value of \( k \) equals \( n - 2 \). In this case \( s = n - 1, m = n \) and we have
(109) \[
\frac{x_{n-2}x_n \cdot x_{n-2}x_{n-1} \equiv_n x_{n-2}x_{n-1} \equiv_n - x_{n-2}x_{n-1} \equiv_n - x_{n-2}x_{n-1}}{\beta x_{n-2}x_{n-1}x_{n-2}x_n \equiv_n x_{n-1}x_n \cdot x_{n-1}x_{n-2}x_n}.
\]
Let us first transpose the second letter \( x_k \) of \( e_{km}e_{ks} \) as far to the left as possible by (37), and then replace the onset \( x_kx_{k+1}x_k \) by (38). We get
(110) \[
e_{km}e_{ks} \equiv_{k+1} \alpha x_k^2(e_{k+1m}e_{k+1s}), \hspace{0.5cm} \alpha \neq 0.
\]
For \( k + 1 < s \) it suffices to apply the inductive supposition to the word in the parentheses and then by (38) and (37) to put \( x_k \) to the proper place. For \( k + 1 = s \) one may use downward induction on \( s \). The basis of this induction, \( s = n - 1 \), has been proved, see (109). For \( k < n - 3 \) the inductive step on \( s \) coincides with the one of Lemma 7.15 with \( e \) in place of \( w \) since in this case the active variables \( x_k, x_{k+1} q \)-commute with \( x_n \). If \( k = n - 3 \) then in consideration of Lemma 7.15 the variable \( x_{k+1} = x_{n-2} \) is transposed across \( x_n \) twice: In (60) and in the second word of (62).
In (60) with \( k = n - 3 \) we have \( s = n - 2, m = n \); and (60) becomes
\[
\epsilon_{n-3n} \epsilon_{n-3n-2} \equiv_{n-2} \beta \epsilon_{n-3n-1} x_{n-3} x_{n-2} x_n x_{n-2}.
\]
In view of Lemma 7.28, we may transform the underlined part in \( U_n \) neglecting the words starting with \( x_{n-2}^2 \) and \( x_n \) in much the same way as in (61), with \( x_n \) in place of \( x_{k+1} \). So (111) reduces to the required form.

The second word of (62) with \( k = n - 3 \) assumes the form \( \epsilon_{n-3n} x_{n-2} x_{n-1}^2 = \epsilon_{n-3n} x_{n-3} x_{n-2} x_n x_{n-2} x_n^2 \). By Lemma 7.3 applied to \( U_n \), the underlined word is a linear combination of words starting with \( x_{n-2} \) and \( x_n \). However, by Lemma 7.28 both \( \epsilon_{n-3n} x_{n-2} \) and \( \epsilon_{n-3n} x_n \) equal zero up to \( \equiv_{n-2} \).

\[\Box\]

**Lemma 7.30.** The set \( B \) satisfies the conditions of Lemma 4.8.

**Proof.** By Lemmas 7.26 and 4.7 one need show only that in \( U^b_p(g) \) the words (101) are linear combinations of lesser ones. The words \( v_6 \) with \( m = n - 2 \), and \( u_0, u_1, u'_1, u_2 \) have the required decomposition since they belong either to \( U_{n-1} \) or to \( U_n \). Lemma 7.28 shows that \( v_3 \equiv_{k+1} 0, v_4 \equiv_{k+1} 0, v'_4 \equiv_{k+1} 0 \). Lemma 7.29 with \( s = m - 1 \) yields the required representation for \( v_5 \). Consider \( v_6 \) with \( m = n - 1 \). Let us prove by downward induction on \( k \) that
\[
u_{k,n-1} e_{kn} \equiv_{k+1} \epsilon e_{kn} u_{k,n-1}, \quad \epsilon \neq 0.
\]
For \( k = n - 1 \) this equality assumes the form (102). Let \( k < n - 1 \). Let us transpose the second letter \( x_k \) of \( u_{k,n-1} e_{kn} \) as far to the left as possible in \( U_{n-1} \). After an application of (35) we get
\[
u_{k,n-1} e_{kn} \equiv_{k+1} \alpha x_k^2 (u_{k+1,n-1} e_{k+1,n}), \quad \alpha \neq 0.
\]
It suffices to apply the inductive supposition to the term in the parentheses, and then by (35) and (37) for \( U_n \) to move \( x_k \) to the proper place. \[\Box\]

**Lemma 7.31.** If \( p_{11} \neq 1 \) then the values of \([v]^h\), where \([v] \in B, v \neq x_i, h \geq 1\) are not skew primitive, in particular they are nonzero.

**Proof.** One need consider only super-letters that belong neither to \( U_{n-1} \) nor to \( U_n \). That is \([e_{km}]\) with \( m < n \). We use induction on \( n \).

For \( n = 3 \) the algebra of the type \( D_3 \) reduces to the algebra of the type \( A_3 \) with a new ordering of variables \( x_2 > x_1 > x_3 \). Therefore we may use Theorem \( A_n \), after the decomposition below of \( e_{12} \) in the PBW-basis:
\[
[x_1 x_3] x_2 = p_{12} p_{32} [x_2 x_1 x_3] + \beta [x_1 x_3] x_2.
\]

Let \( n > 3 \). If \( k > 1 \) then the inductive supposition works. For \( k = 1, m > 2 \) we have \( e_{1m} = [x_1 e_{2m}] \), and one may repeat the arguments of Lemma 7.7 with \( e \) in place of \( u \) starting at (41). If \( m = 2 \) then we may repeat the arguments of Lemma 7.17 with \( e \) on place of \( w \) starting at (65). \[\Box\]

**Proof of Theorem \( D_n \).** For the first statement it will suffice to prove that all super-letters (100) are hard in \( U^b_p(g) \).

Since none of \( u_{km} \) contains sub-words (30), \([u_{km}]\) are hard.
Suppose \([e_{km}]\) is non-hard. By Lemmas 7.30 and 4.8 all hard super-letters belong to \(B\). Thus, by Lemma 7.27, we get \([e_{km}] = 0\). Since \(\deg_n(e_{km}) = \deg_{n-1}(e_{km}) = 1\), the equality \([e_{km}] = 0\) is also valid in the algebra \(D'\) defined by the same relations as \(U'_p(g)\) is, except \([x_{n-2}x_n^2] = 0\) and \([x_{n-2}x_n^{2-1}] = 0\). Let us equate to zero all monomials in all the defining relations of \(D'\), except \([x_{n-1}x_n]\). Consider the algebra \(R'\) defined by (102) and by the resulting system of monomial relations. It is easy to verify that the mentioned relations system \(\Sigma\) of \(R'\) is closed under the compositions. Since \(e_{km}\) contains none of leading words of \(\Sigma\), the super-letter \([e_{km}]\) is nonzero in \(R'\), and so in \(D'\) too. This contradiction proves the first statement.

If \([v] \in B\) is of finite height then by Lemma 7.27 and the homogeneous version of Definition 4.4 we have \([v]^h = 0\). For \(p_{11} \neq 1\) this contradicts Lemma 7.31. In a similar manner, by Lemma 4.9, every skew primitive homogeneous element has the form \([v]^h\). This, together with Lemma 7.31, proves both the fourth statement and the second one with \(p_{11} \neq 1\).

If \(p_{11} = 1\) then by (103) we have \(p_{ii} = p_{ij}p_{ji} = 1\) for all \(i, j\). This means that the skew commutator itself is a quantum Lie operation. Hence all elements of \(B\) are skew-primitive. These elements span a color Lie super-algebra. Now, as in Theorem \(A_n\), one may use the PBW theorem for color Lie super-algebras.

For the third statement it will suffice to show that all super-letters (101) are zero in \(U_p(g)\). We have proved already that they are non-hard. Therefore it remains to use the homogeneous version of Definition 4.3 and Lemma 7.27.

\[
8. \text{Conclusion.}
\]

We see that in all Theorems \(A_n-D_n\) the lists of hard super-letters are independent of the parameters \(p_{ij}\). Therefore if we put \(p_{ij} = 1\), we get a basis of the ground Lie algebra \(g\). It is easy to see that this basis coincides with the basis defined by Lalonde and Ram in [28, Figure 1]. This fact signifies that the Lalonde-Ram basis of the ground Lie algebra with the skew commutator in place of the Lie operation coincides with the set of all hard super-letters of an arbitrary quantification. It is very interesting to clarify how general this statement is. On the one hand, this does not hold without exception for all quantum enveloping algebras since in Theorems \(A_n-D_n\) a restriction does exist. If \(p_{ii} = -1, 1 \leq i < n, n > 2\) then it is easy to see by means of Diamond Lemma that the sets of hard super-letters are infinite, while the ground Lie algebra is of finite dimension. On the other hand, this is not a specific property of Lie algebras defined by the Serre relations. By the Shirshov theorem [40] any Lie polynomial can be reduced to a linear combination of standard nonassociative words.
Corollary 8.1. If $g$ is defined by the only relation $f = 0$, where $f$ is a linear combination of standard nonassociative words, then the set of all hard in $U_P(g)$ super-letters coincides with the Hall–Shirshov basis of $g$ with the skew commutator in place of the Lie operation.

Proof. The only relation $f^* = 0$ forms a Groebner–Shirshov system since, according to 1st, none of onsets of its leading word, say $w$, coincides with a proper terminal of $w$. Consequently, a super-letter $[u]$ is hard if and only if $u$ does not contain $w$ as a sub-word. We see that this criteria is independent of $p_{ij}$ as well. □

Furthermore, the third statement of Theorem $A_n$ shows that $U_P^h(g)$ can be defined by the following relations in the PBW-generators $X_u = [u]$.

\begin{equation}
\begin{aligned}
[X_u,X_v] &= 0, \quad u > v, \quad [[u][v]] \notin B \\
[X_u,X_v] &= X_{uv}, \quad [[u][v]] \in B.
\end{aligned}
\end{equation}

This is an argument in favor of considering the super-letters PBW-generators $k[G]$-module as a quantum analogue of a Lie algebra. However in the cases $B_n$, $C_n$, $D_n$ the defining relations in the PBW-generators became more complicated. For example,

\begin{equation}
\begin{aligned}
B_n : \quad [[u_{k,n-1}][w_{k,n}]] &= \alpha[u_{k,n}]^2, \quad \alpha \neq 0 \text{ if } p_{nn} \neq 1; \\
C_n : \quad [[u_{k,n-2}][v_{k,n-1}]] &= \alpha[v_k] + \beta[u_{k,n}] \cdot [u_{k,n-1}], \quad \beta \neq 0 \text{ if } p_{1n} \neq 1; \\
D_n : \quad [[u_{k,n-2}][e_{k,n-1}]] &= \alpha[e_k] \cdot [u_{k,n-1}], \quad \alpha \neq 0 \text{ if } p_{1n} \neq \pm 1.
\end{aligned}
\end{equation}

Also it is interesting that for $p_{11} \neq 1$ the algebra $g_P$ turns out to be very simple in structure. Only unary quantum Lie operations can be nonzero. Other ones may be defined, but due to the homogeneity their values equal zero. In particular, if $p_{11}^1 \neq 1$ then without exception all quantum Lie operations have zero values. This provides reason enough to consider $U_P(g) = U(g_P)$ as an algebra of ‘commutative’ quantum polynomials or quantum ‘symmetric’ algebra. This statement is still retained for a large class of the quantum universal enveloping algebras of homogeneous components of other Kac–Moody algebras defined by the Gabber–Kac relations (11) (see M. Rosso [38, Theorem 15, and Remark 1])\footnote{We note, however, that Proposition 17 and Corollary 18 of [38] are wrong: The quantum shuffles may have finite heights.}. One may note that if a semigroup generated by $p_{ij}p_{ji}$ does not contain 1, then $G\langle x_1,\ldots,x_n \rangle$ itself is a ‘commutative’ quantum polynomial algebra merely since in this case there exists no nonzero quantum Lie operation at all. In another extreme case when $p_{ij}p_{ji} = 1$ for all $i,j$, the ‘commutative’ quantum variables commute by $x_i x_j = p_{ij} x_j x_i$ (see [38, Example 1, p. 409]).
Acknowledgments. The author is grateful to J.A. Montaraz, the director of the FES-C UNAM, S. Rodríguez-Romo, and A.V. Lara Sagahon for providing facilities for the research and also to L.A. Bokut’, R. Bautista, and N. Andruskiewitsch for helpful comments on the subject matter.

References


Received February 4, 2000 and revised February 20, 2001. The author was supported by SNI, México, exp. 18740, CONACyT México, Grant 32130-E, PAPIIT UNAM, Grant IN 102599, and in part by the NSF USA Grant DMS-9701755.

**Universidad Nacional Autonoma de México**  
Cuautitlán Izcalli  
Estado de México, 54768  
México

**Institute of Mathematics**  
Novosibirsk 630090  
Russia

E-mail address: vlad@servidor.unam.mx
We study the topology of codimension one taut foliations of closed orientable 3-manifolds which are smooth along the leaves. In particular, we focus on the lifts of these foliations to the universal cover, specifically when any set of leaves corresponding to nonseparable points in the leaf space can be totally ordered. We use the structure of branching in the lifted foliation to find conditions that ensure two nonseparable leaves are left invariant under the same covering translation. We also determine when the set of leaves nonseparable from a given leaf is finite up to the action of covering translations. The hypotheses for the results are satisfied by all Anosov foliations.

Introduction.

This paper concerns codimension one foliations of closed orientable 3-manifolds where the leaves are $C^1$ submanifolds. We focus on the topological structure of the lifts of these foliations to the universal cover, particularly when the leaf space of the lifted foliation (i.e., the quotient space obtained by identifying points on the same leaf) is non-Hausdorff.

When the lift of a foliation has a Hausdorff leaf space homeomorphic to $R$, the foliation is said to be “$R$-covered.” Foliations with this property are particularly nice in the sense that they are completely determined by the action of the fundamental group of the manifold on the leaf space of the universal cover [Pa]. These foliations are well-understood and have served as a powerful tool, particularly in the study of Anosov flows. For example, if an Anosov foliation is $R$-covered, then the other Anosov foliation associated with the flow also has this property and the Anosov flow can be shown to be transitive [So, Ba1, Ba2]. Certain restrictions on the manifold in combination with the $R$-covered property have been used to show the Anosov flow is conjugate to a standard model; that is, a geodesic flow or a suspension of an Anosov diffeomorphism [Pl1, Pl2, Gh]. In addition, Fenley has used the $R$-covered hypothesis to uncover the rich structure of metric and homotopy properties in flow lines of many Anosov flows [Fe1, Fe2].
In contrast, foliations without the $R$-covered property are not generally well-understood. The nonHausdorff leaf space of the lifted foliation can, in some cases, be very complex [Im]. Consequently, little is known about the structure of these foliations, except in the Anosov case, and it is only recently that the structure of non $R$-covered Anosov foliations has been fully understood. (Examples include Anosov foliations tangent to a nontransitive Anosov flow, the first of which was constructed by Franks and Williams [Fr-Wi], and the Anosov foliations tangent to the Bonatti and Langevin flow [Bo-La].)

In [Fe4], Fenley studies the lift of Anosov foliations to the universal cover and completely determines the structure of the set of “branching leaves” (i.e., leaves whose quotients in the leaf space are nonHausdorff points and which therefore correspond to positive or negative branching in the leaf space). He finds that this structure is very rigid and is strongly related to the topology of the ambient manifold below, the dynamics of the flow, and the metric behavior of the stable and unstable foliations. For example, Fenley shows that if two leaves $\hat{A}$ and $\hat{B}$ are nonseparable in the leaf space, then they are periodic; in fact, they cover leaves containing freely homotopic closed orbits of the Anosov flow, and $\hat{A}$ and $\hat{B}$ are left invariant under the same covering translation. This result is then used to show that, up to the action of covering translations, there are at most finitely many branching leaves. It is also a critical step in the proof that there is finite branching between the branching pair $(\hat{A}, \hat{B})$; that is, there are at most finitely many leaves “between” $\hat{A}$ and $\hat{B}$ which are nonseparable from $\hat{A}$ and $\hat{B}$. (A leaf $\hat{C}$ in the stable Anosov foliation is said to be “between” leaves $\hat{A}$ and $\hat{B}$ if there is a leaf $\hat{P}$ in the unstable Anosov foliation, through $\hat{C}$, such that $\hat{A}$ and $\hat{B}$ lie on opposite sides of $\hat{P}$.)

Here we shall consider the class of non $R$-covered taut foliations and show that by replacing the Anosov restriction with more general conditions we can ensure properties similar to those observed in Anosov foliations. To make sense of these properties in the more general setting, it will be necessary to develop a suitable definition of betweeness for nonseparable leaves in the universal cover. For this we use a flow which is transverse to the foliation, and betweeness will depend on the flow used. In the Anosov case, the definition of betweeness given here will coincide with the usual definition when the appropriate choice of transverse flow is made.

We ensure that for any 3 nonseparable leaves, precisely one of the leaves lies between the other two by requiring that every branching leaf in the universal cover contains a curve, called a “dividing curve,” whose saturation by orbits of the transverse flow satisfies certain conditions. (These conditions are given in Definition 1.2 and are shown to be sufficient in Proposition 1.5.)
Throughout, \( \hat{F} \) shall represent the lift of \( F \) to the universal cover. Given a branching leaf \( \hat{A} \in \hat{F} \), we will let \( E_-(\hat{A}) \) denote the set consisting of \( \hat{A} \) and all leaves nonseparable from \( \hat{A} \) on the negative side.

The main results are as follows:

**Theorem A.** Let \( F \) be a taut foliation with transverse flow \( \phi \) such that every branching leaf of \( \hat{F} \) contains a dividing curve for \( \hat{\phi} \). Given a branching leaf \( \hat{A} \in \hat{F} \), suppose \( E_-(\hat{A}) \) is order isomorphic to a subset of the integers. If there exists a covering translation \( d \) fixing \( \hat{A} \) such that \( d(\hat{A}) < d(\hat{B}) \) whenever \( \hat{B} \in E_-(\hat{A}) \) and \( \hat{A} < \hat{B} \), then each leaf in \( E_-(\hat{A}) \) is left invariant by \( d \). In particular, the quotient in \( F \) of each leaf in \( E_-(\hat{A}) \) contains a loop freely homotopic to a loop in \( A \).

Using \([Fe4]\), Theorems B and C] we shall see that the hypotheses of Theorem A are satisfied by all Anosov foliations. As a corollary, we obtain \([Fe4]\, Theorem D) under more general conditions.

By Theorem A, the set \( E_-(\hat{A}) \) is order isomorphic to a subset of \( \mathbb{Z} \) only in the case that every \( \hat{B} \in E_-(\hat{A}) \) is left invariant by certain covering translations fixing \( \hat{A} \). We shall give necessary and sufficient conditions on a branched surface constructed from \( \hat{F} \) to ensure that this is the case (Proposition 2.5).

In \([Go-Sh]\) it was shown that a branched surface \( W \) carrying a foliation exhibits a specific type of local behavior when there is a pair of planar nonseparable leaves in the universal cover; in particular, \( W \) lifts to a branched surface \( \hat{W} \) in the universal cover and contains an arc, called a “branching arc”, whose lift to \( \hat{W} \) links the images of the branching pair in \( \hat{W} \). A branching arc with restricted branchings along its interiors is said to be “simple”. These definitions will be may more precise in Section 2 where we prove:

**Theorem B.** Let \( F \) be a taut foliation with transverse flow \( \phi \) such that every branching leaf of \( \hat{F} \) contains a dividing curve for \( \hat{\phi} \). Given a branching leaf \( \hat{A} \in \hat{F} \), suppose \( E_-(\hat{A}) \) is order isomorphic to the integers. There exists a branched surface \( W \) carrying \( F \) such that some simple branching arc in \( W \) has at least 3 lifts linking distinct branching pairs in \( E_-(\hat{A}) \) if and only if there are at most finitely many leaves of \( F \) that lift to a leaf in \( E_-(\hat{A}) \); in other words, the set \( E_-(\hat{A}) \) is finite up to the action of covering translations.

Recall that if \( F \) is Anosov, then for every leaf \( \hat{A} \) in the universal cover the set \( E_-(\hat{A}) \) is order isomorphic to a subset of the integers. So by \([Fe4]\, Theorem F], the conditions in Theorem B are satisfied by all Anosov foliations.

As noted in \([Fe4]\), the number of leaves between a branching pair is a measure of the complexity of the branching. For example, finiteness of branching is, in some sense, a characteristic of foliations with a simple branching structure. Here, we show the following:
Theorem C. Let \( F \) be a taut foliation with transverse flow \( \phi \) such that each branching leaf of \( \hat{F} \) contains a dividing curve for \( \hat{\phi} \). For every leaf \( \hat{A} \) in \( \hat{F} \), the set \( E_-(\hat{A}) \) is order isomorphic to a subset of the rational numbers \( \mathbb{Q} \).

We also use branch surfaces to obtain the following information about the structure of the branching between two nonseparable leaves:

Theorem D. Let \( F \) be a taut foliation with transverse flow \( \phi \) such that each branching leaf of \( \hat{F} \) contains a dividing curve for \( \hat{\phi} \). For any positively branching pair \( \hat{A} \) and \( \hat{B} \) in \( \hat{F} \), there is an order preserving map \( f \) assigning each element of \( E_-(\hat{A}) \) between \( \hat{A} \) and \( \hat{B} \) to a connected component of \( [0,1] - \Sigma \), \( \Sigma \) a Cantor set, and such that \( f(\hat{A}) = \{0\} \) and \( f(\hat{B}) = \{1\} \). Moreover, for any finite subset \( \{\hat{C}_1, \ldots, \hat{C}_n\} \) of leaves between \( \hat{A} \) and \( \hat{B} \), there is an imbedded copy \( \hat{\gamma} \) of \( [0,1] \) in \( \hat{W} \) intersecting each \( \hat{\pi}(\hat{C}_i) \) in \( f(\hat{C}_i) \) with the property that for every \( \hat{C}' \in E_-(\hat{A}) \), either \( \hat{\pi}(\hat{C}') \cap \hat{\gamma} = \emptyset \) or \( \hat{\pi}(\hat{C}') \cap \hat{\gamma} = f(\hat{C}') \).

The author wishes to thank Annalisa Calini and Brenton LeMesurier for helpful conversations. She is also grateful to the referee for his careful reading of the manuscript and his comments which were very useful in improving this paper.

1. Betweeness and finite branching.

Throughout this paper, \( F \) will be a \( C^0 \) codimension one taut foliation of a closed orientable Riemannian 3-manifold \( M \neq S^2 \times S^1 \) where the leaves are \( C^1 \) submanifolds. Passing to a double cover of \( M \) if necessary, may assume that \( F \) is transversely orientable. We let \( \hat{F} \) denote the lift of \( F \) to the universal cover \( \hat{M} \). Likewise, for any leaf \( L \) (curve \( \gamma \)) in \( M \), \( \hat{L} \) (\( \hat{\gamma} \) respectively) will refer to one of its lifts to \( \hat{M} \).

Definition 1.1. Given two leaves \( \hat{A} \) and \( \hat{B} \) of \( \hat{F} \), we say \( (\hat{A}, \hat{B}) \) is a branching pair if the points in the leaf space of \( \hat{F} \) representing \( \hat{A} \) and \( \hat{B} \) respectively are nonseparable; that is, \( \hat{A} \) and \( \hat{B} \) do not have disjoint saturated neighborhoods. In this case \( \hat{A}(\hat{B}) \) corresponds to a nonHausdorff point in the leaf space and is called a branching leaf. We say the branching pair \( (\hat{A}, \hat{B}) \) is a positively (negatively) branching pair if \( \hat{A} \) and \( \hat{B} \) are nonseparable on their negative (respectively positive) sides; i.e., \( \hat{A} \) and \( \hat{B} \) correspond to positive (respectively negative) branching in the leaf space of \( \hat{F} \). When \( \hat{F} \) contains only finitely many branching leaves we say it has finite branching.

Recall that a nonsingular flow on a 3-manifold \( M \) is “Anosov” if for some Riemannian metric on \( M \) the tangent bundle to the manifold has a continuous flow-invariant splitting into a Whitney sum, \( TM = E^0 \oplus E^u \oplus E^s \), of the
line bundle $E^0$ tangent to the flow, and two other bundles, $E^u$ and $E^s$, so that the flow is exponentially expanding in the direction of $E^u$ and exponentially contracting in the direction of $E^s$. Associated with each Anosov flow is a foliation of $M$, called the “stable (unstable)” Anosov foliation, which is tangent to the subbundle spanned by $E^0$ and $E^s$ ($E^0$ and $E^u$ respectively) \([An]\). A leaf $\hat{C}$ in the stable (unstable) Anosov foliation of $\hat{M}$ that is non-separable from leaves $\hat{A}$ and $\hat{B}$ is said to be “between” $\hat{A}$ and $\hat{B}$ if $\hat{A}$ and $\hat{B}$ lie on opposite sides of a leaf, through $\hat{C}$, in the unstable (respectively stable) Anosov foliation. By \([Fe4]\), there are at most finitely many leaves between any branching pair ($\hat{A}, \hat{B}$) which are nonseparable from $\hat{A}$ and $\hat{B}$.

Here we adopt a notion of betweenness for leaves in the universal cover $\hat{M}$ which does not require a transverse foliation to $F$ in $M$. As in the Anosov case, we use a plane separating $\hat{M}$ which consists of a saturation by orbits in a transverse flow $\hat{\phi}$. Unlike the Anosov case, there is no natural choice for $\hat{\phi}$ and betweeness here will depend on this choice of flow. However, when the foliation is stable (unstable) Anosov, choosing $\hat{\phi}$ to be the flow in the unstable (stable respectively) direction will yield the usual notion of betweeness.

The results which follow concern the class of foliations with the property that for any 3 nonseparable leaves in the universal cover, precisely one of these leaves lies between the other two. We show that when there is finite branching between any branching pair ($\hat{A}, \hat{B}$) in $\hat{F}$, certain nontrivial covering translations fixing $\hat{A}$ will also leave $\hat{B}$ invariant (Theorem 1.7). We then show that for any foliation in this class, each set of nonseparable leaves in $\hat{F}$ is order isomorphic to a subset of the rational numbers $Q$ (Theorem 2.2) and use branched surfaces to describe the structure of this branching (Theorem 2.3).

The foliation $F$ is taut, so it is Reebless and therefore contains no vanishing cycles. Consequently, all leaves of $\hat{F}$ are simply connected and, since we are assuming that $M \neq S^2 \times S^1$, $\hat{F}$ is a foliation by planes of the universal cover $\hat{R}^3$. So there exists a foliation $F_0$ of $\hat{R}^2$ such that $\hat{F}$ is topologically conjugate to the product $F_0 \times \hat{R}$ ([Pa], [Ga-Ka]). For every $r \in \hat{R}$, we let $F_0(r)$ denote the image in $\hat{F}$ of $F_0 \times \{r\}$ under the conjugacy map and $P_0(r)$ denote the imbedded copy of $\hat{R}^2$ foliated by $F_0(r)$.

At any point $\hat{x} \in P_0(r)$ we may use the transverse orientation to $\hat{F}(r)$ as the first part of an orientation frame for $T\hat{M}$, and a vector both tangent to $\hat{F}$ and transverse to $P_0(r)$ in the direction of increasing $r$ as the second part of the frame. The third vector of the frame can be chosen tangent to the leaf of $F_0(r)$ through $\hat{x}$ and used to define an orientation for this leaf. Throughout this paper, we assume that for any $r \in \hat{R}$ the leaves of $F_0(r)$ are oriented in this manner.
Let \( \tilde{\chi} \) be a curve in a leaf of \( \tilde{F} \). Given a nonsingular flow \( \phi \) transverse to \( F \), let \( \hat{\phi} \) denote its lift to the universal cover, and let \( \hat{\phi}(\tilde{\chi}) \) represent the saturation of orbits through \( \tilde{\chi} \). Since \( F \) is Reebless, no orbit of \( \hat{\phi} \) can meet \( \tilde{\chi} \) more than once. So if \( \tilde{\chi} \) is a properly imbedded copy of \( R \), then \( \hat{\phi}(\tilde{\chi}) \) is a 2-manifold. It is also worth noting that \( \hat{\phi}(\tilde{\chi}) \) is nowhere tangent to \( \tilde{F} \).

Now suppose that for some \( r \), the intersection of a leaf \( \tilde{K} \) in \( F_0(r) \) with \( \hat{\phi}(\tilde{\chi}) \) is contained in some finite arc \( \gamma \) in \( \tilde{K} \) whose endpoints do not lie in \( \hat{\phi}(\tilde{\chi}) \). We say \( \tilde{K} \) and \( \hat{\phi}(\tilde{\chi}) \) have “intersection number equal to 1” if the ends of \( \gamma \) lie on opposite sides of \( \hat{\phi}(\tilde{\chi}) \). In other words, the intersection number of \( \tilde{K} \) with \( \hat{\phi}(\tilde{\chi}) \) is equal to 1 if all points in \( \hat{\phi}(\tilde{\chi}) \cap \tilde{K} \) lie between two points \( x \) and \( y \) in \( \tilde{K} \), where \( x \) and \( y \) are on opposite sides of \( \hat{\phi}(\tilde{\chi}) \).

**Definition 1.2.** A curve \( \tilde{\chi} \) in a branching leaf \( \tilde{C} \) of \( \tilde{F} \) is a *dividing curve* for \( \hat{\phi} \) if the following conditions are satisfied:

1. \( \hat{\phi}(\tilde{\chi}) \) is a properly imbedded plane and
2. there exist an \( r_\chi \in R \) such that for almost every \( r \geq r_\chi \), any leaf \( \tilde{K} \) of \( F_0(r) \) sufficiently close to \( \tilde{C} \cap P_0(r) \) has intersection number with \( \hat{\phi}(\tilde{\chi}) \) equal to 1.

When \( \tilde{F} \) is a stable Anosov foliation, the orbit space, obtained by collapsing each orbit of the Anosov flow \( \hat{\sigma} \) to a point, is homeomorphic to \( R^2 \) [Fe1]. In this case, we may choose \( P_0(0) \) to be an imbedded copy of this orbit space which is transverse to \( \hat{\sigma} \) and let \( P_0(r) = \{ \hat{\sigma}(x,r) : x \in P_0(0) \} \) for every \( r \in R \). If we then choose \( \hat{\phi} \) to be the flow in the unstable direction \( E^u \), we ensure that any branching leaf \( \tilde{C} \) contains a dividing curve for \( \hat{\phi} \). To see this we first note that by [Fe4, Theorem B] the quotient leaf \( C \) in \( M \) contains a closed orbit \( \chi \) of the Anosov flow. For some lift \( \tilde{\chi} \) of \( \chi \) to \( \tilde{C} \), the surface \( \hat{\phi}(\tilde{\chi}) \) is a leaf in the unstable Anosov foliation, so \( \tilde{\chi} \) satisfies Condition (1) above and is nowhere tangent to \( P_0(r) \) for any \( r \in R \). In fact, for every \( r \in R \), \( \hat{\phi}(\tilde{\chi}) \cap P_0(r) \) is a leaf in the restriction of the unstable Anosov foliation to \( P_0(r) \). So \( \tilde{\chi} \) also satisfies Condition (2) above.

We note that Condition (2) is not satisfied by finite depth foliations. It is also worth noting that Condition (1) in Definition 1.2 is fundamental to ensuring that \( \hat{\phi}(\tilde{\chi}) \) separates \( \tilde{M} \) into two disjoint open sets.

We now find alternative conditions which also guarantee Condition (1) is satisfied. In particular, we show the following:

**Proposition 1.3.** Let \( \tilde{\chi} \) be a properly imbedded curve in a branching leaf of \( \tilde{F} \) such that \( \hat{\phi}(\tilde{\chi}) \) is an imbedded plane without boundary. If the induced foliation of \( \hat{\phi}(\tilde{\chi}) \) has finite branching (i.e., if there are finitely many leaves of \( \tilde{F} \cap \hat{\phi}(\tilde{\chi}) \) which correspond to nonseparable points in the leaf space of the induced foliation) and if there is a flow on \( \tilde{M} \) transverse to \( \hat{\phi}(\tilde{\chi}) \) which is
nowhere tangent to \( \hat{\phi} \), then \( \hat{\chi} \) satisfies Condition (1) for a dividing curve; that is, \( \hat{\phi}(\hat{\chi}) \) is a plane separating \( \hat{M} \) into two open components.

Proof. It is sufficient to show that if points in \( \hat{\phi}(\hat{\chi}) \) accumulate on some point \( \hat{p} \), then these points (together with \( \hat{p} \)) lie on a finite curve in \( \hat{\phi}(\hat{\chi}) \). So suppose this is not the case. Since \( \hat{\phi}(\hat{\chi}) \) is an imbedded plane without boundary which is nowhere tangent to \( \hat{F} \), each leaf in the induced foliation of \( \hat{\phi}(\hat{\chi}) \) is an imbedded copy of \( R \) without boundary; in particular, each component of \( \hat{\phi}(\hat{\chi}) \cap \hat{L} \) has this property, where \( \hat{L} \) is the leaf of \( \hat{F} \) containing \( \hat{p} \).

We first consider the case where some component of \( \hat{\phi}(\hat{\chi}) \cap \hat{L} \) is not properly imbedded in \( \hat{L} \). It then follows that \( \hat{\phi}(\hat{\chi}) \cap \hat{L} \) limits on a curve in \( \hat{L} \) which, by the existence of the flow \( \hat{\psi} \) transverse to \( \hat{\phi}(\hat{\chi}) \), is an embedded loop. (For example, \( \hat{\phi}(\hat{\chi}) \cap \hat{L} \) cannot double back near itself with opposite orientation as in Figure 1.1. In addition, \( \hat{\phi}(\hat{\chi}) \cap \hat{L} \) cannot be fractal for the same reason.) This loop in \( \hat{L} \) bounds an open disk \( \hat{D} \) in \( \hat{L} \) which is not met by the limiting component of \( \hat{\phi}(\hat{\chi}) \cap \hat{L} \). With the appropriate choice of metric, \( \hat{\phi} \) is everywhere perpendicular to \( \hat{L} \) and, by assumption, \( \hat{\psi} \) is nowhere tangent to \( \hat{\phi} \). So at each point in \( \hat{L} \) the tangent vector to \( \hat{\psi} \) projects orthogonally onto a nonzero vector in \( \hat{L} \). Since the Euler characteristic of \( \hat{D} \) is one, some integral curve of the resulting vector field is transverse to \( \partial \hat{D} \). Furthermore, the curve \( \hat{\phi}(\hat{\chi}) \cap \hat{L} \) is transverse to this vector field on \( \hat{L} \) and limits on \( \partial \hat{D} \). So there exists a disk \( \hat{D}' \) in \( \hat{L} \) and containing \( \hat{D} \) whose boundary is everywhere transverse to the nonzero vector field on \( \hat{L} \) (see Figure 1.2). This is a contradiction since the Euler characteristic of \( \hat{D}' \) is one.

The following cannot occur:
The local picture is as follows, where the dashed lines indicate integral curves of the nonzero vector field on $\hat{L}$ obtained by projecting $\hat{\psi}$:
Figure 1.2. Modifying the composition of the integral curve from $x$ to $y$ with the curve in $\hat{\phi}(\hat{\chi}) \cap \hat{L}$ from $y$ to $x$, we get a loop transverse to the integral curves.

Now suppose that the components of $\hat{\phi}(\hat{\chi}) \cap \hat{L}$ are topologically closed in $\hat{L}$. Since $\hat{\phi}(\hat{\chi})$ has no boundary and $\hat{p}$ is an accumulation point of $\hat{\phi}(\hat{\chi})$, there is sequence $\{x_i\}$ of points in $\hat{\phi}(\hat{\chi}) \cap \hat{L}$ limiting on $\hat{p}$. Passing to a subsequence if necessary, we may assume that each $x_i$ is contained in a different component $E_i$ of $\hat{\phi}(\hat{\chi}) \cap \hat{L}$ and that $\hat{p} \notin E_i$ for all $i$. Now each $E_i$ is a leaf in the induced foliation of $\hat{\phi}(\hat{\chi})$. Since, by hypothesis, this foliation has finite branching, we may choose two such leaves $E_i$ and $E_j$ such that the corresponding points in the leaf space are joined by an imbedded Hausdorff 1-manifold. So there is a transverse arc from $E_i$ to $E_j$. We may modify the composition of this arc with a curve in $\hat{L}$ from $E_j$ to $E_i$ in order to obtain a loop transverse to $\hat{F}$, contradicting $F$ taut. (See Figure 1.3.)

Locally we have:
Figure 1.3. The dashed arc is transverse to $\hat{F}$.

So $\hat{\phi}(\hat{\chi})$ is an imbedded plane.

It is worth noting that any 2 leaves connected by a transverse arc are separated from each other by any leaf intersecting the interior of the arc. It follows that if $\hat{A}$ and $\hat{B}$ are nonseparable and $\hat{B}$ contains a dividing curve $\hat{\beta}$ for $\hat{\phi}$, then $\hat{\phi}(\hat{\beta})$ does not intersect $\hat{A}$.

We now define betweeness for branching leaves as follows:

Definition 1.4. Given a branching pair $(\hat{A}, \hat{B})$ in $\hat{F}$, suppose $\hat{C}$ is a leaf which is nonseparable from $\hat{A}$ and $\hat{B}$. We say $\hat{C}$ is between $\hat{A}$ and $\hat{B}$ with respect to the flow $\hat{\phi}$ (and write $\hat{A} - \hat{C} - \hat{B}$) if $\hat{C}$ contains a dividing curve $\hat{\chi}$ for $\hat{\phi}$ such that $\hat{A}$ and $\hat{B}$ lie on opposite sides of $\hat{\phi}(\hat{\chi})$.

Note that “betweenness” is dependent on our choice of the transverse flow $\hat{\phi}$. However, if $\hat{F}$ is a stable (unstable) Anosov foliation and we choose $\hat{\chi}$ to be the flow in the unstable (stable) direction, then this definition is equivalent to the usual definition ([Fe4]) of betweeness for branching leaves in an Anosov foliation.

Proposition 1.5. Given 3 nonseparable leaves $\hat{A}, \hat{B}$ and $\hat{C}$ of $\hat{F}$, if each contains a dividing curve, then precisely one of these leaves lies between the
other two. Further, for dividing curves \( \hat{\chi} \) and \( \hat{\chi}' \) in \( \hat{C} \), the leaves \( \hat{A} \) and \( \hat{B} \) are on opposite sides of \( \hat{\phi}(\hat{\chi}) \) if and only if \( \hat{A} \) and \( \hat{B} \) are on opposite sides of \( \hat{\phi}(\hat{\chi}') \).

**Proof.** By hypothesis, \( \hat{A}, \hat{B} \) and \( \hat{C} \) contain dividing curves \( \hat{\alpha}, \hat{\beta} \) and \( \hat{\chi} \) respectively. By Condition (2) for a dividing curve, there exists an \( r_{\alpha} \in R \), such that for almost every \( r \geq r_{\alpha} \), any leaf \( \hat{K} \) of \( F_0(r) \) sufficiently close to \( \hat{A} \cap P_0(r) \) has intersection number with \( \hat{\phi}(\hat{\alpha}) \) equal to 1. Further, there exists \( r_{\beta} \) and \( r_{\chi} \) with the analogous property. Consequently, we may choose \( r > \max\{r_{\alpha}, r_{\beta}, r_{\chi}\} \) so that any leaf \( \hat{K} \) of \( F_0(r) \) chosen sufficiently close to \( \hat{A} \cap P_0(r), \hat{B} \cap P_0(r) \) and \( \hat{C} \cap P_0(r) \) has intersection number with each of \( \hat{\phi}(\hat{\alpha}), \hat{\phi}(\hat{\beta}) \) and \( \hat{\phi}(\hat{\chi}) \) equal to 1. Such a leaf \( \hat{K} \) contains a finite arc \( \gamma \) whose endpoints \( \gamma(0) \) and \( \gamma(1) \) lie on opposite sides of \( \hat{\phi}(\hat{\alpha}), \hat{\phi}(\hat{\beta}) \) and \( \hat{\phi}(\hat{\chi}) \). Let \( x(y) \) be the first (last respectively) point along \( \gamma \) meeting \( \hat{\phi}(\hat{\alpha}) \cup \hat{\phi}(\hat{\beta}) \cup \hat{\phi}(\hat{\chi}) \). Since \( \hat{\phi}(\hat{\alpha}), \hat{\phi}(\hat{\beta}) \) and \( \hat{\phi}(\hat{\chi}) \) are pairwise disjoint, one of these planes contains neither \( x \) nor \( y \). Furthermore, \( x \) and \( y \) lie on opposite sides of this plane since \( x(y) \) is on the same side as \( \gamma(0) (\gamma(1) \) respectively). It follows that two of the planes \( \hat{\phi}(\hat{\alpha}), \hat{\phi}(\hat{\beta}) \) and \( \hat{\phi}(\hat{\chi}) \) lie on opposite sides of the third. In other words, precisely one of the leaves \( \hat{A}, \hat{B} \) and \( \hat{C} \) lies between the other two.

Now suppose that \( \hat{A} \) and \( \hat{B} \) are on opposite sides of \( \hat{\phi}(\hat{\chi}) \), and let \( \hat{\chi}' \) be another dividing curve in \( \hat{C} \). Since, \( \hat{\phi}(\hat{\beta}) \cap \hat{\phi}(\hat{\chi}) = \emptyset \), \( \hat{A} \) and \( \hat{C} \) are on the same side of \( \hat{\phi}(\hat{\beta}) \); that is, \( \hat{\phi}(\hat{\alpha}) \) and \( \hat{\phi}(\hat{\chi}') \) lie on the same side of \( \hat{\phi}(\hat{\beta}) \). Similarly, \( \hat{\phi}(\hat{\beta}) \) and \( \hat{\phi}(\hat{\chi}') \) lie on the same side of \( \hat{\phi}(\hat{\alpha}) \). So the plane \( \hat{\phi}(\hat{\chi}') \) is in the complementary component of \( \hat{\phi}(\hat{\alpha}) \cup \hat{\phi}(\hat{\beta}) \) that is bounded by both \( \hat{\phi}(\hat{\alpha}) \) and \( \hat{\phi}(\hat{\beta}) \). By the argument above, one of the planes \( \hat{\phi}(\hat{\alpha}), \hat{\phi}(\hat{\beta}) \) and \( \hat{\phi}(\hat{\chi}') \) lies between the other two. So \( \hat{\phi}(\hat{\alpha}) \) and \( \hat{\phi}(\hat{\beta}) \) are on opposite sides of \( \hat{\phi}(\hat{\chi}') \); that is, \( \hat{A} \) and \( \hat{B} \) are on opposite sides of \( \hat{\phi}(\hat{\chi}') \). \( \square \)

By Condition (1) for a dividing curve, \( \hat{\phi}(\hat{\alpha}) \) is a properly imbedded plane. Consequently, the orientation of any \( \hat{K} \) chosen, as in Condition (2), to have intersection number with \( \hat{\phi}(\hat{\alpha}) \) equal to 1 can be used to define the positive side of \( \hat{\phi}(\hat{\alpha}) \). We say \( \hat{B} < \hat{A} \) if \( \hat{B} \) is on the negative side of \( \hat{\phi}(\hat{\alpha}) \) and \( \hat{A} < \hat{B} \) if \( \hat{B} \) is on the positive side of \( \hat{\phi}(\hat{\alpha}) \). Since \( \hat{A} \) and \( \hat{B} \) are nonseparable, \( \hat{B} \cap \hat{\phi}(\hat{\alpha}) = \emptyset \). It follows that either \( \hat{A} < \hat{B} \) or \( \hat{B} < \hat{A} \). Suppose \( \hat{A} < \hat{B} \). There exists an \( r > \max\{r_{\alpha}, r_{\beta}\} \) such for any leaf \( \hat{K} \) in \( F_0(r) \) chosen sufficiently close to \( \hat{A} \cap P_0(r) \) and \( \hat{B} \cap P_0(r) \), \( \hat{K} \) meets \( \hat{\phi}(\hat{\alpha}) \) and \( \hat{\phi}(\hat{\beta}) \), each with intersection number equal to 1. So in this case, \( \hat{K} \) meets \( \hat{\phi}(\hat{\alpha}) \) transversely before meeting \( \hat{\phi}(\hat{\beta}) \). Consequently, \( \hat{A} \) is on the negative side of \( \hat{\phi}(\hat{\beta}) \). Clearly \( \leq \) is reflexive, antisymmetric and transitive, so we have the following:
**Proposition 1.6.** Let $F$ be a taut foliation with transverse flow $\phi$ such that every branching leaf of $\hat{F}$ contains a dividing curve for $\hat{\phi}$. Any collection of nonseparable leaves in $\hat{F}$ is totally ordered by $\leq$.

Given a branching leaf $A \in \hat{F}$, let $E_-(A)$ denote the set consisting of $A$ and all leaves nonseparable from $A$ on the negative side.

We now show:

**Theorem 1.7.** Let $F$ be a taut foliation with transverse flow $\phi$ such that every branching leaf of $\hat{F}$ contains a dividing curve for $\hat{\phi}$. Given a branching leaf $A \in \hat{F}$, suppose $E_-(A)$ is order isomorphic to a subset of the integers. If there exists a covering translation $d$ fixing $\hat{A}$ such that $d(\hat{A}) < d(\hat{B})$ whenever $\hat{B} \in E_-(\hat{A})$ and $\hat{A} < \hat{B}$, then each leaf in $E_-(\hat{A})$ is left invariant by $d$. In particular, the quotient in $F$ of each leaf in $E_-(\hat{A})$ contains a loop freely homotopic to a loop in $A$.

**Proof.** Assume $E_-(\hat{A})$ is order isomorphic to a subset of the integers. We verify that any covering translation $d$ fixing $\hat{A}$ preserves the order in $E_-(\hat{A})$. By hypothesis, for any for $\hat{B} \in E_-(\hat{A})$, if $\hat{A} < \hat{B}$ then $d(\hat{A}) < d(\hat{B})$. So it suffices to show that $d$ preserves betweeness for elements of $E_-(\hat{A})$.

First note that $d$ preserves the foliation $\hat{F}$ and the transverse flow $\hat{\phi}$, but not necessarily the product foliation of $\hat{M}$ by the family $\{P_0(r) : r \in R\}$ of parallel imbedded planes. Now suppose $\hat{B}_1, \hat{B}_2$ and $\hat{B}_3$ are leaves in $E_-(\hat{A})$ such that $\hat{B}_1 - \hat{B}_2 - \hat{B}_3$ (i.e., $\hat{B}_2$ is between $\hat{B}_1$ and $\hat{B}_3$). Let $\beta_1, \beta_2$ and $\beta_3$ be dividing curves in $\hat{B}_1, \hat{B}_2$ and $\hat{B}_3$ respectively. We may choose a curve $\kappa$ in a leaf on the negative side of $\hat{A}$ so that the initial point of $\kappa$ flows, along an orbit of $\kappa$, into $\beta_1$ and the terminal point of $\kappa$ flows into $\beta_3$. Clearly the ends of $\kappa$ lie on opposite sides of $\hat{\phi}(\beta_2)$. Now the ends of $d(\kappa)$ flow into $d(\beta_1)$ and $d(\beta_3)$ respectively and lie on opposite sides of $d(\hat{\phi}(\beta_2))$. It follows that $d(\hat{B}_1)$ and $d(\hat{B}_2)$ lie on opposite sides of $d(\hat{\phi}(\beta_2))$. Since $d(\hat{B}_1), d(\hat{B}_2)$ and $d(\hat{B}_3)$ are nonseparable, they contain dividing curves $\beta'_1, \beta'_2$ and $\beta'_3$ respectively and, by Proposition 1.5, precisely one of these leaves lies between the other two. Now $d(\hat{B}_1)$ and $d(\hat{B}_2)$ lie on the same side of $\hat{\phi}(\beta'_2)$ since $\hat{\phi}(\beta'_3) \cap \hat{\phi}(\beta'_2) = \emptyset$. Likewise, $d(\hat{B}_2)$ and $d(\hat{B}_3)$ lie on the same side of $\hat{\phi}(\beta'_1)$. So $d(\hat{B}_1)$ and $d(\hat{B}_3)$ lie on opposite sides of $\hat{\phi}(\beta'_2)$; that is, $d(\hat{B}_1) - d(\hat{B}_2) - d(\hat{B}_3)$. Therefore $d$ preserves betweeness for elements of $E_-(\hat{A})$. \qed

In general, finiteness of branching is dependent on the transverse flow $\hat{\phi}$ since betweeness depends on this flow. However, for $\hat{F}$ a stable Anosov foliation, finite branching as defined in [Fe4] is equivalent to finite branching with respect to the flow $\hat{\phi}$ in the unstable direction. In this case, there is finite branching between any branching pair $(\hat{A}, \hat{B})$ [Fe4, Theorem C].
A critical step in Fenley’s proof is showing that $\hat{B}$ is left invariant under any covering translation fixing $\hat{A}$ [Fe4, Theorem D]. Each such covering translation fixes an orbit $\hat{\alpha}$ of the Anosov flow and satisfies the conditions of Theorem 1.7 above. (This latter follows since $\hat{\alpha}$ is a dividing curve and $\hat{\phi}(\hat{\alpha})$ covers a transversely orientable 2-manifold $\phi(\alpha)$; in particular, $\phi(\alpha)$ is a leaf in the unstable Anosov foliation of $M$ so the covering translation preserves the transverse orientation of $\hat{\phi}(\hat{\alpha})$.) So we have that [Fe4, Theorem C] also implies [Fe4, Theorem D].

2. Branched surfaces and branching in the universal cover.

In this section we use branched surfaces constructed from foliations. These branched surfaces are in the class of regular branched surfaces introduced by R. Williams [W] and are constructed according to a technique given in [Ch-Go]. We shall give a brief outline of this construction, including only those details necessary to understand the results which follow.

We begin with a foliation $F$, a flow $\phi$ transverse to $F$, and a generating set $\Delta = \{D_i\}$ of disjoint imbedded compact surfaces with boundary (which is finite if the ambient manifold $M$ is closed), satisfying the following general position requirements:

(i) each $D_i$ lies in a leaf of $F$ (hence is transverse to $\phi$)
(ii) every orbit of $\phi$ meets the interior of some element of $\Delta$ in forward and backward time
(iii) for every $i_0$, the set of points in $Bdy(D_{i_0})$ whose orbit under $\phi$ meets $\cup Bdy(D_i)$ before meeting $\cup \text{int}(D_i)$ is finite
(iv) any orbit of $\phi$ meets the boundary of at most two elements of $\Delta$.

We cut $M$ open along the interior of each element of $\Delta$ to obtain a submanifold $M^*$ which can be imbedded in $M$ so that its boundary contains $\cup Bdy(D_i)$. This can be thought of as “blowing air” into the leaves of $F$ to create an air pocket at each element of the generating set. By requirement (ii) above, the restriction of $\phi$ to $M^*$ is a flow $\phi^*$ with the property that each orbit is homeomorphic to the unit interval. We next form a quotient space by identifying points that lie on the same orbit of $\phi^*$. That is, we take the quotient $M^*/\sim$, where $x \sim y$ if $x$ and $y$ lie on the same interval orbit of $\phi^*$. We may think of this as enlarging the components of $M - M^*$ until each interval orbit of $\phi^*$ is contracted to a point in $M$. The imbedded copy of the resulting quotient space is the branched surface $W$; it is called the branched surface corresponding to $(F, \phi, \Delta)$. By its construction, $W$ is transverse to $\phi$. The general position requirements for $\Delta$ imply that $W$ is a connected 2-dimensional complex with a set of charts defining local orientation preserving diffeomorphisms onto one of the models in the figure below,
and such that the transition maps are smooth and preserve transverse orientation. (Each local model projects horizontally onto a vertical model of $\mathbb{R}^2$ so has a smooth structure induced by $T\mathbb{R}^2$ which pulls back to $W$.)

Figure 2.1.

So each branched surface obtained in the above fashion is a 2-manifold except on a small subset $\mu$ called the maw. The set $\mu$ is a 1-manifold except at isolated points called crossings where it intersects itself transversely. (There are only finitely many of these points when $M$ is closed.) Each component of $W - \mu$ is called a sector of $W$.

Note that if we thicken $W$ in the transverse direction to recover the interval orbits of $\phi^s$, we retrieve $M^*$ which, for that reason, we shall henceforth call $N(W)$, the neighborhood of $W$. The interval orbit of $\phi^s$ whose quotient is a point $x$ in $W$ will be referred to as the fiber of $N(W)$ over $x$. (See Figure 2.2.)
fiber over $x$
A foliation $F$ clearly gives rise to a foliation of $N(W)$ with leaves transverse to the fibers of $N(W)$, which we shall also denote by $F$. Each leaf meeting the boundary of $N(W)$ contains the entire boundary component (these leaves can be thought of as leaves of the original foliation with air blown into them). That is, the leaves meeting the boundary of $N(W)$ are precisely the (cut-open) leaves of the original foliation containing elements of $\Delta$.

Throughout, $\pi : N(W) \to W$ will denote the quotient map which identifies points in the same fiber. We say the image $x$ of a point under this map is the projection of that point. Accordingly, we say points in preimage of $x$ lie over $x$.

In what follows, $W$ will be a branched surface constructed from a foliation $F$ of a closed manifold $M$, and $\hat{W}$ will be the lift of $W$ to the universal cover $\hat{M}$. In other words, if $W$ corresponds to $(F, \Delta, \phi)$, then $\hat{W}$ is the branched surface corresponding to $(\hat{F}, \hat{\Delta}, \hat{\phi})$, where $\hat{\Delta}$ is the lift of $\Delta$ and $\hat{\phi}$ is the lift of $\phi$ to the universal cover. The covering map $\rho_M$ of $M$ by $\hat{M}$ acts on $N(\hat{W})$ and induces a covering map $\rho : \hat{W} \to W$ such that $\pi \circ \rho_M = \rho \circ \hat{\pi}$. (Details are given in [Sh].)

Suppose leaves $\hat{A}$ and $\hat{B}$ of $\hat{F}$ are nonseparable on their negative sides; that is, there is a 1-parameter family $\{\hat{K}_n\}$ of leaves, parameterized by $n \in I$ (where $I$ is the real interval $(0,1)$) which lie on the negative side of $\hat{A}$ and $\hat{B}$ and monotonically approach $(\hat{A}, \hat{B})$ in the leaf space as $n \to 1$. (We assume points in the leaf space are ordered according to the transverse orientation of $\hat{F}$.) When one considers what this behavior might yield in a branched surface, one is eventually led to the following type of picture:
More precisely, if $\hat{F}$ contains a pair of leaves $\hat{A}$ and $\hat{B}$ that are nonseparable on their negative sides, then there is an arc $\hat{\beta}$ in $\hat{W}$ nowhere tangent to the maw $\hat{\mu}$ (and containing no crossings of $\hat{\mu}$) with its ends branching into the negative sides of $\hat{\pi}(\hat{A})$ and $\hat{\pi}(\hat{B})$ respectively [Go-Sh]. Such an arc $\hat{\beta}$ can be chosen to lie in $\hat{\pi}(\hat{K})$ for some leaf $\hat{K}$ of $\hat{F}$ and is said to be a branching arc linking $\hat{A}$ and $\hat{B}$. (The branching arc $\hat{\beta}$ covers a “branching arc” $\beta$ in $W$.) If there are no sectors branching into the positive side of $\hat{\pi}(\hat{K})$ which meet the interior of $\hat{\beta}$, then $\hat{\beta}$ and $\beta$ are said to be simple branching arcs. For example, $\hat{\beta}$ in Figure 2.3 is a simple branching arc linking $\hat{A}$ and $\hat{B}$.

It is easy to see that any simple branching arc $\hat{\beta}$ in $\hat{W}$ corresponds to an arc in an element of $\hat{\Delta}$ (generating $\hat{W}$) with endpoints in the boundary of that generating surface. For such a branching arc, the leaf $\hat{K}$ may be chosen so that it contains a curve $\hat{\kappa}$ in $\partial N(\hat{W})$ which lies over $\hat{\beta}$ and begins in the interior of some fiber. When we collapse the components of $\hat{M} - N(\hat{W})$ to retrieve the original foliation $\hat{F}$ of $\hat{M}$, the image of $\hat{\kappa}$ is contained in an element $\hat{D}$ of $\hat{\Delta}$, and its endpoints in $\partial \hat{D}$ flow into $\hat{A}$ and $\hat{B}$ respectively before meeting $\hat{\Delta}$ again. See Figure 2.4.
Conversely, if for some branching arc $\hat{\beta}$ with endpoints $\hat{\beta}(0)$ and $\hat{\beta}(1)$ respectively, the component $\hat{C}$ of $\partial N(\hat{W})$ meeting the interior of the fiber over $\hat{\beta}(0)$ also meets the fiber over $\hat{\beta}(1)$, then $\hat{\beta}$ is simple.

Certain branching arcs in $\hat{W}$ (and $W$) can be regarded as essentially the same. More precisely, the boundary of each sector $S$ is contained in the maw and can be partitioned by crossings into disjoint open arcs. Any two arcs whose interiors lie in $S$ are “equivalent” if the initial points of the arcs both lie in the interior of $S$ or in the same interval of the partition, and if the same condition holds for their terminal points. Any branching arc in $\hat{W}$ ($W$) is a composition of finitely many arcs contained in sectors of $\hat{W}$.
(W respectively). Accordingly, 2 branching arcs are equivalent if they are a composition of equivalent arcs. Indeed, if \( \hat{\beta} \) is a branching arc linking nonseparable leaves \( \hat{A} \) and \( \hat{B} \), and \( \hat{\beta}' \) is equivalent to \( \hat{\beta} \), then \( \hat{\beta}' \) is also a branching arc linking \( \hat{A} \) and \( \hat{B} \).

Now for each element of the finite set \( \Delta \) (generating \( W \)), the corresponding component of \( \partial (M - W) \) contains a finite number of sectors of \( W \). It follows that the number of simple branching arcs in \( W \) is finite up to equivalence.

We shall now use branched surfaces to prove Theorems 2.1 and 2.3.

**Theorem 2.1.** Let \( F \) be a taut foliation with transverse flow \( \phi \) such that every branching leaf of \( \hat{F} \) contains a dividing curve for \( \hat{\phi} \). Given a branching leaf \( \hat{A} \in \hat{F} \), suppose \( E_-(\hat{A}) \) is order isomorphic to the integers. There exists a branched surface \( W \) carrying \( F \) such that some simple branching arc in \( W \) has at least 3 lifts linking distinct branching pairs in \( E_-(\hat{A}) \) if and only if there are at most finitely many leaves of \( F \) that lift to a leaf in \( E_-(\hat{A}) \); in other words, the set \( E_-(\hat{A}) \) is finite up to the action of covering translations.

**Proof.** Suppose there exists a branched surface \( W \) as above. Since \( E_-(\hat{A}) \) is order isomorphic to the set \( Z \) of integers, we may index the leaves in \( E_-(\hat{A}) \) so that \( E_-(\hat{A}) = \{ \hat{A}_i : i \in Z, \hat{A}_0 = \hat{A} \} \). It suffices to show there is a covering translation \( d \) that maps some leaf \( \hat{A}_i \in E_-(\hat{A}) \) onto another leaf \( \hat{A}_{i+N} \) in \( E_-(\hat{A}) \) such that \( d(\hat{A}_{i+N}) \neq \hat{A}_i \). Then since by Theorem 1.7 the maps \( d \) and \( d^{-1} \) preserve betweeness, \( \hat{A}_{i+j+kN} = d^k(\hat{A}_{i+j}) \) for every \( 0 \leq j \leq N \) and \( k \in Z \); that is, \( \rho_M(\hat{A}_{i+j+kN}) = \rho_M(\hat{A}_{i+j}) \).

By assumption, there is a simple branching arc \( \beta \) in \( W \) with 3 lifts \( \hat{\beta}_i \), \( \hat{\beta}_j \) and \( \hat{\beta}_k \) linking distinct branching pairs \( (\hat{A}_i, \hat{A}_{i+a}), (\hat{A}_j, \hat{A}_{j+b}) \) and \( (\hat{A}_k, \hat{A}_{k+c}) \) in \( E_-(\hat{A}) \) respectively, such that \( ab > 0 \). (By definition, \( \hat{\beta}_i \), \( \hat{\beta}_j \) and \( \hat{\beta}_k \) are contained in the projection of leaves \( \hat{K}_i \), \( \hat{K}_j \) and \( \hat{K}_k \) respectively into \( \hat{W} \).)

In particular, some covering translate of \( \hat{\beta}_j \) is equal to \( \hat{\beta}_i \). So there exists a covering translation \( d \) such that \( \hat{\beta}_j \) links \( d(\hat{A}_j) = \hat{A}'_j \) and \( d(\hat{A}_{j+b}) = \hat{A}'_{j+b} \). Furthermore, \( \hat{A}'_j \) cannot cut the fiber over \( \hat{\beta}_j(1) \) (since \( \hat{A}'_{j+b} \) meets this fiber).

So since the leaf \( \hat{A}'_j \) is on the positive side of \( \hat{K}_i \), the nonseparable points in the leaf space representing \( \hat{A}'_j \) and \( \hat{A}'_{j+b} \) cannot lie below those representing \( \hat{A}_i \) and \( \hat{A}_{i+a} \). Similarly, \( \hat{A}'_j \) and \( \hat{A}'_{j+b} \) cannot lie above \( \hat{A}_i \) and \( \hat{A}_{i+a} \) in the leaf space. It follows that \( \hat{A}_i = \hat{A}'_j \) and \( \hat{A}_{i+a} = \hat{A}'_{j+b} \). That is, \( d(\hat{A}_j) = \hat{A}_i \) and \( d(\hat{A}_{j+b}) = \hat{A}_{i+a} \). Since \( ab > 0 \), \( (\hat{A}_i) \neq \hat{A}_j \). It follows that the set \( E_-(\hat{A}) \) is finite up to the action of covering translations.

Conversely, suppose that \( E_-(\hat{A}) \) is finite up to the action of covering translations. Let \( W \) be a branched surface carrying \( F \) and let \( \hat{W} \) be its lift to the universal cover. For each \( i \in Z \), let \( \hat{\beta}_i \) be a branching arc in \( \hat{W} \).
linking $\hat{A}_i$ and $\hat{A}_{i+1}$ which is contained in $\tilde{\pi}(\hat{K}_i)$ for some leaf $\hat{K}_i$ of $\hat{F}$. We choose $\hat{\beta}_i$ to be simple whenever possible. Substituting an equivalent arc if necessary, we may assume $\beta_i = \rho(\hat{\beta}_i)$ is not a loop. We may also guarantee that if $(\hat{A}_j, \hat{A}_{j+1})$ is a covering translate of the branching pair $(\hat{A}_i, \hat{A}_{i+1})$, (i.e., $d(\hat{A}_i) = \hat{A}_j$ and $d(\hat{A}_{i+1}) = \hat{A}_{j+1}$ for some covering translation $d$), then $\beta_i = \beta_j$. So, up to the action of covering translations, the set $\{\hat{\beta}_i : i \in \mathbb{Z}\}$ is finite.

For some $i \in \mathbb{Z}$, there exists infinitely many covering translates of $\hat{\beta}_i$ linking distinct branching pairs in $E_-(\hat{A})$. Suppose that $\hat{\beta}_i$ is not simple. Let $\hat{C}_i$ be the component of $\partial N(\hat{W})$ meeting the interior of the fiber over $\hat{\beta}_i(0)$. It follows that $\hat{C}_i$ does not meet the fiber over $\hat{\beta}_i(1)$. If the leaf of $\hat{F}$ containing $\hat{C}_i$ does not equal $\hat{K}_i$, then it lies above $\hat{K}_i$; in either case, this leaf lies below $(\hat{A}_i, \hat{A}_{i+1})$ in the leaf space, hence meets the fiber over $\hat{\beta}_i(1)$. Now $\hat{C}_i$ corresponds to an element $\hat{D}_i$ of $\hat{\Delta}$. In particular, enlarging $\hat{D}_i \in \hat{\Delta}$ corresponds to splitting $N(\hat{W})$ open further along a portion of the leaf containing $\hat{C}_i$. We may therefore enlarge $\hat{D}_i$ so that $\hat{C}_i$ cuts the fiber over $\hat{\beta}_i(1)$; we then obtain a curve $\hat{\kappa}_i$ from the fiber over $\hat{\beta}_i(0)$ to the fiber over $\hat{\beta}_i(1)$ that is contained in the outwardly oriented section of $\hat{C}_i$. See Figure 2.5. When we enlarge $\hat{D}_i$ as above to include an additional portion of the leaf containing it, it is important to ensure that the corresponding portion of the quotient leaf in $N(W)$ does not meet the fibers over $\beta_i(0)$ and $\beta_i(1)$ more than once. This guarantees that there is no covering translate of $\hat{D}_i$ cutting an orbit of $\hat{\phi}$ from an end of $\hat{\kappa}_i$ into $\hat{A}_i$ or $\hat{A}_{i+1}$. In particular, we ensure that in the process of modifying $\hat{\Delta}$ (and hence $\Delta$) so that $\hat{C}_i$ meets the fibers over both $\hat{\beta}_i(0)$ and $\hat{\beta}_i(1)$, a covering translate of $\hat{C}_i$ does not destroy this property.
\[ \Pi(\hat{A}_i) \]
Figure 2.5.

Clearly when we modify $\hat{W}$ in this manner to ensure $\beta_i$ is simple, we ensure all covering translates of $\beta_i$ are simple. So since there exists infinitely many covering translates of $\beta_i$ linking distinct branching pairs in $E_-(\hat{A})$, we guarantee $\hat{W}$ has the desired property. □

If $F$ is Anosov, then for every leaf $\hat{A}$ in the universal cover the set $E_-(\hat{A})$ is order isomorphic to a subset of the integers [Fe4, Theorem C]. By [Fe4, Theorem F], the conditions in Theorem 2.1 are satisfied by all Anosov foliations.

We now show the following:

**Theorem 2.2.** Let $F$ be a taut foliation with transverse flow $\phi$ such that each branching leaf of $\hat{F}$ contains a dividing curve for $\hat{\phi}$. For every leaf $\hat{A}$ in $\hat{F}$, the set $E_-(\hat{A})$ is order isomorphic to a subset of the rational numbers $Q$.

*Proof.* We begin by taking a countable cover of $\hat{M}$ by foliation boxes for $\hat{F}$. We may associate any $\hat{C} \in E_-(\hat{A})$ with one of the boxes $U_C$ which it intersects. Now for every $\hat{C}' \in E_-(\hat{A})$ such that $\hat{C}' \neq \hat{C}$, $U_C \cap \hat{C}' = \emptyset$; in particular, $U_C \neq U_{C'}$. Using Proposition 1.6, it follows that $E_-(\hat{A})$ is a countable totally ordered set, hence it is order isomorphic to a subset of $Q$. □

So for every $\hat{B} \in E_-(\hat{A})$, the subset of leaves in $E_-(\hat{A})$ between $\hat{A}$ and $\hat{B}$ is order isomorphic to a subset of $Q$. Now consider the submanifold $N(\hat{W})$ obtained during the construction of $\hat{W}$ from $(\hat{F}, \hat{\phi}, \hat{A})$ and let $I_A$ and $I_B$ be fibers of $N(\hat{W})$ through dividing curves $\hat{\alpha}$ and $\hat{\beta}$ in $\hat{A}$ and $\hat{B}$ respectively. We may choose an arc $\hat{\gamma}_1$ in some leaf $\hat{K}_1$ on the negative side of $\hat{A}$ which begins in $I_A$, ends in $I_B$ and whose interior lies between $\hat{\phi}(\hat{\alpha})$ and $\hat{\phi}(\hat{\beta})$. Continuing inductively, we may choose a curve $\hat{\gamma}_{n+1}$ in a leaf $\hat{K}_{n+1}$ between $\hat{K}_n$ and $\hat{A}$ with the property that any fiber through an element of $E_-(\hat{A})$ which is met by $\hat{\gamma}_n$ is also met by $\hat{\gamma}_{n+1}$. In this manner we obtain a sequence of arcs $\{\hat{\gamma}_n\}$. By carefully choosing the leaves in the sequence $\{\hat{K}_n\}$, we may guarantee that these leaves converge monotonically to $\hat{A}$ in the leaf space. We may also ensure that the interior of each arc $\hat{\gamma}_n$ in this sequence lies between $\hat{\phi}(\hat{\alpha})$ and $\hat{\phi}(\hat{\beta})$ and that if it meets fibers through some $\hat{C} \in E_-(\hat{A})$ at points $x$ and $y$, then each point of $\hat{\gamma}_n$ between $x$ and $y$ intersects a fiber through $\hat{C}$. Now since $E_-(\hat{A})$ is order isomorphic to a subset of the rational numbers $Q$, it is countable. So we may assume $\{\hat{\gamma}_n\}$ has been chosen so that for every $\hat{C} \in E_-(\hat{A})$, there is a positive integer $n$ such that $\hat{\gamma}_n$ meets a fiber through $\hat{C}$.
Now if some $\hat{\gamma}_n$ meets fibers through a leaf $\hat{C} \in E_-(\hat{A})$, there is a first point and a last point along $\hat{\gamma}_n$ for which this is the case. So if $\hat{C} \neq \hat{A}$ or $\hat{B}$, it follows that $\hat{\gamma}_n$ yields a curve $\hat{\pi}(\hat{\gamma}_n)$ in $\hat{W}$ with the image $\hat{\pi}(\hat{C})$ of $\hat{C}$ branching from it, as shown in Figure 2.6. That is, $\hat{\pi}(\hat{\gamma}_n)$ contains 2 arcs (indicated by the dashed segments), each with an end branching into the negative side of $\hat{\pi}(\hat{C})$ whose interiors do not intersect $\hat{\pi}(\hat{C})$. Clearly $\hat{\pi}(\hat{C}')$ for any other $\hat{C}' \in E_-(\hat{A})$ does not meet $\hat{\pi}(\hat{\gamma}_n)$ between these two arcs.
Bracing in Taut Foliations

\[ \hat{\Pi}(A) \]

\[ \hat{\Pi}({\hat{C}}) \]

\[ \hat{\Pi}(\gamma_n) \]
For every \( n \), the subset \( \Sigma_n = \{ \hat{C} \in E_-(\hat{A}) : \hat{\pi}(\hat{C}) \cap \hat{\pi}(\gamma_n) \neq \emptyset \} \) is finite. For suppose that, on the contrary, \( \Sigma_n \) is infinite for some \( n \). Then there exists infinitely many branching of \( \hat{W} \) along \( \hat{\pi}(\gamma_n) \) and these branchings accumulate in the lift of some evenly covered neighborhood \( U \) of \( W \). It follows that \( U \) contains infinitely many branching of \( W \) along some component of \( \pi(\gamma_n) \cap U \), which is impossible under our construction of \( W \).

Now given \( \hat{C} \) in \( E_-(\hat{A}) \) between \( \hat{A} \) and \( \hat{B} \), there exists an integer \( N_C > 0 \) such that \( \hat{C} \in \Sigma_n \) for all \( n > N_C \) (see Figure 2.7). Moreover, for every \( n \) there is a homeomorphism from \( \hat{\pi}(\gamma_n) \) onto \([0, 1]\). These homeomorphisms can be chosen so that for any \( \hat{C} \in \Sigma_n \), the mapping on \( \hat{\pi}(\gamma_n) \) takes \( \hat{\pi}(\hat{C}) \cap \hat{\pi}(\gamma_n) \) to the same subinterval for all \( n > N_C \).
In particular, we have:

**Theorem 2.3.** Let $F$ be a taut foliation with transverse flow $\phi$ such that each branching leaf of $\hat{F}$ contains a dividing curve for $\hat{\phi}$. For any positively branching pair $\hat{A}$ and $\hat{B}$ in $\hat{F}$, there is an order preserving map $f$ assigning each element of $E_-(\hat{A})$ between $\hat{A}$ and $\hat{B}$ to a connected component of $[0, 1] - \Sigma, \Sigma$ a Cantor set, and such that $f(\hat{A}) = \{0\}$ and $f(\hat{B}) = \{1\}$. Moreover, for any finite subset $\{\hat{C}_1, \ldots, \hat{C}_n\}$ of leaves between $\hat{A}$ and $\hat{B}$, there is an imbedded copy $\hat{\gamma}$ of $[0, 1]$ in $\hat{W}$ intersecting each $\hat{\pi}(\hat{C}_i)$ in $f(\hat{C}_i)$ with the property that for every $\hat{C}' \in E_-(\hat{A})$, either $\hat{\pi}(\hat{C}') \cap \hat{\gamma} = \emptyset$ or $\hat{\pi}(\hat{C}') \cap \hat{\gamma} = f(\hat{C}')$.

According to Theorem 1.7, $E_-(\hat{A})$ is order isomorphic to a subset of the integers only when every $\hat{B} \in E_-(\hat{A})$ is left invariant by certain covering translations fixing $\hat{A}$. To check that the latter is the case, it is sufficient to verify that the corresponding surface $\hat{\pi}(\hat{B})$ in the combinatorial object $\hat{W}$ is left invariant by the corresponding covering translation of $\hat{W}$. Before proving this (in Proposition 2.5) we review the relationship between covering translations of $\hat{M}$ and “covering translations of $\hat{W}$”.

**Definition 2.4.** A diffeomorphism $D_{\hat{W}}$ of $\hat{W}$ is called a covering translation of $\hat{W}$ if for some covering map $p : \hat{W} \to W$, $p \circ D_{\hat{W}} = p$.

Suppose $D_{\hat{M}}$ is a covering translation of $\hat{M}$. Then $D_{\hat{M}}$ preserves the transverse flow $\hat{\phi}$ (i.e., maps orbits to orbits). Since the generating set for $\hat{W}$ is the lift of the generating set for $W$, it is also preserved by $D_{\hat{M}}$. Therefore, $D_{\hat{M}}$ maps each fiber of $N(\hat{W})$ onto another fiber. Consequently, when we collapse fibers of $N(\hat{W})$, $D_{\hat{M}}$ induces a covering translation $D_{\hat{W}}$ of $\hat{W}$; in other words, on $N(\hat{W})$, $\hat{\pi} \circ D_{\hat{M}} = D_{\hat{W}} \circ \hat{\pi}$.

Conversely, suppose $D_{\hat{W}}$ is a nontrivial covering translation of $\hat{W}$. For any $\hat{x}$ in $\hat{W}$, if $D_{\hat{W}}(\hat{x}) = \hat{y}$, then $x = p(\hat{x}) = p \circ D_{\hat{W}}(\hat{x}) = p(\hat{y}) = y$. We may define a homeomorphism from the fiber over $\hat{x}$ onto the fiber over $\hat{y}$ which covers the identity map on the fiber over $x$ (under the covering map $p_M$ of $M$ corresponding to $p$). In this way, $D_{\hat{W}}$ gives rise to a covering translation $D_{\hat{M}}$ of the manifold $N(\hat{W})$ and, after we collapse the complement of $N(\hat{W})$ in $\hat{M}$, $D_{\hat{M}}$ is a covering translation of $\hat{M}$. In particular, $\hat{\pi} \circ D_{\hat{M}} = D_{\hat{W}} \circ \hat{\pi}$.

**Proposition 2.5.** A branching leaf $\hat{A}$ is left invariant under a covering translation $D_{\hat{M}}$ of $M$ if and only if $\hat{\pi}(\hat{A})$ is left invariant under the corresponding covering translation $D_{\hat{W}}$ of $\hat{W}$.
Proof. Without loss of generality, assume \( \hat{A} \) is a positively branching leaf. Clearly, if \( \hat{A} \) is left invariant under a covering translation \( D_{\hat{M}} \) of \( \hat{M} \), then \( \hat{\pi}(\hat{A}) \) is left invariant under the induced covering translation \( D_{\hat{W}} \) of \( \hat{W} \). So suppose \( \hat{A} \) is not left invariant by \( D_{\hat{M}} \) yet the induced translation \( D_{\hat{W}} \) leaves \( \hat{\pi}(\hat{A}) \) invariant. Then there exists a positively branching leaf \( \hat{A}' \neq \hat{A} \) such that \( D_{\hat{M}}(\hat{A}) = \hat{A}' \) and \( \hat{\pi}(\hat{A}) = \hat{\pi}(\hat{A}') \). It follows that \( \hat{A} \) and \( \hat{A}' \) are separable (since fibers meeting both leaves are contained in orbits of the transverse flow). Without loss of generality, assume \( \hat{A}' \) is on the negative side of \( \hat{A} \). By assumption, \( \hat{A} \) is a positively branching leaf, so there exists a leaf \( \hat{B} \in E_-(\hat{A}) \) and a leaf \( \hat{K} \) on the negative side of \( \hat{A} \) and \( \hat{B} \) which meets a fiber through \( \hat{A} \) and a fiber through \( \hat{B} \). Since \( \hat{A}' \) and \( \hat{A} \) meet the same fibers of \( N(\hat{W}) \), \( \hat{A}' \) cannot meet any fiber over \( \hat{B} \). So for \( \hat{K} \) sufficiently close to \( \hat{A} \) and \( \hat{B} \), the leaf \( \hat{A}' \) is on the negative side of \( \hat{K} \), and \( \hat{\pi}(\hat{A}') \) branches away from \( \hat{\pi}(\hat{K}) \cap \hat{\pi}(\hat{A}) \). This means \( \hat{\pi}(\hat{A}) \) contains all sectors whose boundaries include the segment of the maw at this branching (see Figure 2.8), a contradiction since \( F \) taut implies \( \hat{\pi} \) is injective on \( \hat{A} \). \( \square \)
Figure 2.8.

References


Guidelines for Authors

Authors may submit manuscripts at pjm.math.berkeley.edu/about/journal/submissions.html and choose an editor at that time. Exceptionally, a paper may be submitted in hard copy to one of the editors; authors should keep a copy.

By submitting a manuscript you assert that it is original and is not under consideration for publication elsewhere. Instructions on manuscript preparation are provided below. For further information, visit the web address above or write to pacific@math.berkeley.edu or to Pacific Journal of Mathematics, University of California, Los Angeles, CA 90095–1555. Correspondence by email is requested for convenience and speed.

Manuscripts must be in English, French or German. A brief abstract of about 150 words or less in English must be included. The abstract should be self-contained and not make any reference to the bibliography. Also required are keywords and subject classification for the article, and, for each author, postal address, affiliation (if appropriate) and email address if available. A home-page URL is optional.

Authors are encouraged to use LATEX, but papers in other varieties of TEx, and exceptionally in other formats, are acceptable. At submission time only a PDF file is required; follow the instructions at the web address above. Carefully preserve all relevant files, such as LATEX sources and individual files for each figure; you will be asked to submit them upon acceptance of the paper.

Bibliographical references should be listed alphabetically at the end of the paper. All references in the bibliography should be cited in the text. Use of BibTEx is preferred but not required. Any bibliographical citation style may be used but tags will be converted to the house format (see a current issue for examples).

Figures, whether prepared electronically or hand-drawn, must be of publication quality. Figures prepared electronically should be submitted in Encapsulated PostScript (EPS) or in a form that can be converted to EPS, such as GnuPlot, Maple or Mathematica. Many drawing tools such as Adobe Illustrator and Aldus FreeHand can produce EPS output. Figures containing bitmaps should be generated at the highest possible resolution. If there is doubt whether a particular figure is in an acceptable format, the authors should check with production by sending an email to pacific@math.berkeley.edu.

Each figure should be captioned and numbered, so that it can float. Small figures occupying no more than three lines of vertical space can be kept in the text (“the curve looks like this:”). It is acceptable to submit a manuscript will all figures at the end, if their placement is specified in the text by means of comments such as “Place Figure 1 here”. The same considerations apply to tables, which should be used sparingly.

Forced line breaks or page breaks should not be inserted in the document. There is no point in your trying to optimize line and page breaks in the original manuscript. The manuscript will be reformatted to use the journal’s preferred fonts and layout.

Page proofs will be made available to authors (or to the designated corresponding author) at a Web site in PDF format. Failure to acknowledge the receipt of proofs or to return corrections within the requested deadline may cause publication to be postponed.
Crossing number of alternating knots in $\mathbb{S} \times I$

COLIN ADAMS, THOMAS FLEMING, MICHAEL LEVIN AND ARI M. TURNER

Choix des signes pour la formalité de M. Kontsevich

D. ARNAL, D. MANCHON AND M. MASMOUDI

The domain algebra of a C*-semigroup

WILLIAM ARVESON

On boundary avoiding selections and some extension theorems

STOYU BAROV AND JAN J. DIJKSTRA

Renormalization of certain integrals defining triple product L-functions

JENNIFER E. BEINEKE

Lagrangian mappings of the first open Whitney umbrella

I.A. BOGAEVSKI AND G. ISHIKAWA

Lagrangian sections and holomorphic $U(1)$-connections

JINGYI CHEN

Polynomials with general $C^2$-fibers are variables

SH. KALIMAN

A Combinatorial approach to the quantification of Lie algebras

V.K. KHARCHENKO

Branching in the universal cover of taut foliations

SANDRA SHIELDS