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We show that every link in S^3 is a sublink of an arithmetic link.

1. Introduction.

In this paper we show that arithmetic links play a central role in the Dehn surgery description of closed 3-manifolds. Let $L \subset S^3$ be a link of (one or more) circles. We prove that L is a sublink of an arithmetic link. Specifically:

Theorem 1. *Let $L \subset S^3$ be a link. Then L is a sublink of a link J such that $S^3 \setminus J$ is homeomorphic to $\mathbb{H}^3/\Gamma$, where Γ is a torsion-free subgroup of finite index in the Bianchi group $PSL_2(\mathbb{Z}[i])$.*

Since every closed, orientable 3-manifold can be obtained by Dehn surgery on a link in S^3 (see [**Li**]), we have:

Theorem 2. *Every closed, orientable 3-manifold can be obtained by Dehn surgery on an arithmetic link in S^3.*

While it is known that every closed orientable 3-manifold M contains an arithmetic link L (since the figure-eight knot complement is both arithmetic and universal), Theorem 2 asserts that L can be chosen so that $M \setminus L$ is homeomorphic to the complement of a link in S^3.

Recall that a link L in S^3 (resp. in M) is hyperbolic if $S^3 \setminus L$ (resp. $M \setminus L$) is homeomorphic to $\mathbb{H}^3/\Gamma$, where $\mathbb{H}^3$ is hyperbolic 3-space and Γ a discrete, torsion-free, finite covolume subgroup of $PSL_2(\mathbb{C})$. We say that L is arithmetic if Γ can be chosen commensurable with a Bianchi group $PSL_2(\mathcal{O}_m)$, where $\mathcal{O}_m$ is the integers of the imaginary quadratic number field $\mathbb{Q}(\sqrt{-m})$ (see [**R**] for a more general discussion). Finally, L is a sublink of J if it is a union of components of J.

2. Proof of Theorem 1.

Denote by L_1 the 6-circle link in Figure 1.

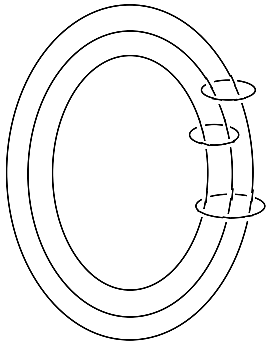

Figure 1.

We prove our result by showing that:

i) L_1 is an arithmetic link: $S^3 \setminus L_1 \cong \mathbb{H}^3/\Gamma_1$, where Γ_1 is a torsion-free subgroup of the Bianchi group $PSL_2(\mathbb{Z}[i])$.

ii) Every link L occurs as a sublink of a link J such that $S^3 \setminus J$ is a covering space of $S^3 \setminus L_1$.

Thus $S^3 \setminus J \cong \mathbb{H}^3/\Gamma$, where $\Gamma \subset \Gamma_1 \subset PSL_2(\mathbb{Z}[i])$, and so J is arithmetic and contains L as a sublink. We prove the arithmeticity of L_1 in **2.1**. Section **2.2** is devoted to proving property ii).

2.1. The link L_1 is arithmetic since $S^3 \setminus L_1$ is a 2-fold cover of $S^3 \setminus L_0$ where L_0 is the four component arithmetic link in Figure 2.

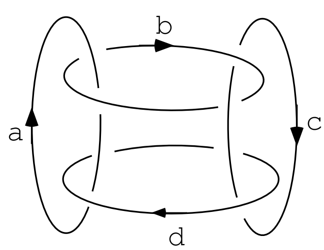

Figure 2.

Indeed, $S^3 \setminus L_0 \cong \mathbb{H}^3/\Gamma_0$ where

$$\Gamma_0 = \left\langle \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2i \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1-i & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1+i & 1 \end{bmatrix} \right\rangle \subset PSL_2(\mathbb{Z}[i])$$

(see [**Wi**] Example 3 for a detailed treatment), and $S^3 \setminus L_1$ is the 2-fold cover corresponding to the kernel of the map $\theta : \pi_1(S^3 \setminus L_0) \to \mathbb{Z}/2\mathbb{Z}$ given by $\theta(a) = \theta(c) = 1$ and $\theta(b) = \theta(d) = 0$.

A second proof of the arithmeticity of L_1 goes as follows (see [**R**]): Γ_1 is a subgroup of $PSL_2(\mathbb{Z}[i])$ if and only if $tr(\Gamma_1) = \{\mathrm{tr}\,(\gamma) \mid \gamma \in \Gamma_1\} \subset \mathbb{Z}[i]$, which is true if and only if, for a set of generators $\gamma_1, \dots, \gamma_n$ of Γ_1, the following traces are in $\mathbb{Z}[i]$: $\mathrm{tr}\,(\gamma_i)$ and $\mathrm{tr}\,(\gamma_i\gamma_j)$, $i < j$. We used SnapPea ([**W**]) to compute a matrix representation for $\pi_1(S^3 \setminus L_1)$ and verify that the above traces are indeed in $\mathbb{Z}[i]$.

2.2. By the Alexander braiding theorem (see [**B-Z**]) any link can be realized as the closure of an n-braid (Figure 3).

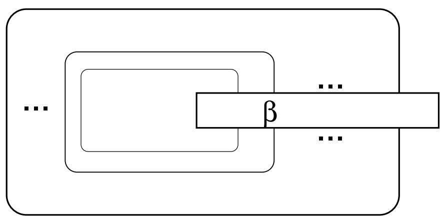

Figure 3.

Here $\beta = \alpha_{i_k}^{s_k} \cdots \alpha_{i_1}^{s_1}$ is a product of powers of the standard generators of the braid group B_n (Figure 4).

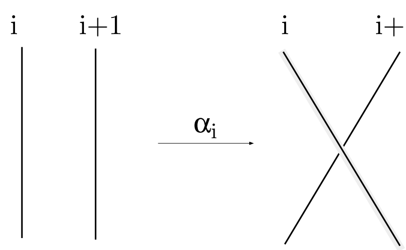

Figure 4.

We show that L is a sublink of a J such that $S^3 \setminus J$ is a cover of $S^3 \setminus L_1$ hence arithmetic. Before giving the construction of J in the general case, we first illustrate the process by treating the case when L is the trefoil knot.

2.2.1. The trefoil knot is the closure of the 2-strand braid $\beta = \alpha^3$ (Figure 5).

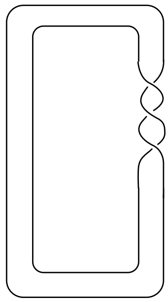

Figure 5.

Let X^1 be the 2-fold cover of $X \cong S^3 \setminus L_1$ branched over the circle c (see Figure 6. We draw only the braid part of the vertical components in order to save space). This cover is again a link complement since we are branching over an unknotted component of L_1.

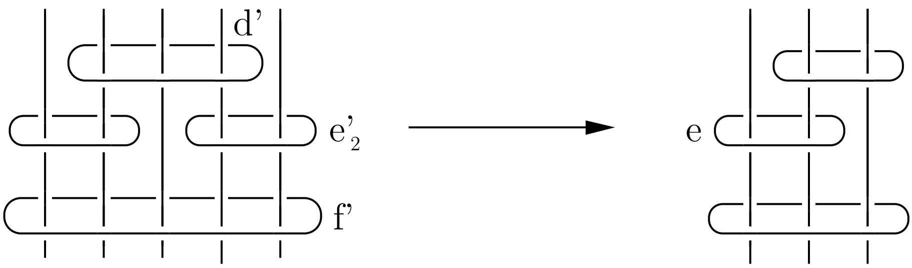

Figure 6.

Now we transform X^1 into $S^3 \setminus J$ by performing a 3/2 twist about the circle d' i.e., cutting along the disk D' bounded by d', twisting through 3π and regluing. This has the effect of α^3 on the circles b'_1 ,b'_2, changing them into the desired trefoil knot (Figure 7). The key point here is that $S^3 \setminus J$ is also a 2-fold cover of $S^3 \setminus L_1$. We now examine this point in greater detail.

2.2.2. We show (with notation as above) that any $n/2$ twist about d' transforms X^1 into another 2-fold cover of X. Since integer twists about d' are homeomorphisms, it suffices to consider the case of a $1/2$ twist. Note that D' 2-fold covers D, a disk bounded by circle d in X (Figure 8). Since the complement of an unknotted circle in S^3 is a solid torus, cutting along D' and D transforms X^1 and X into solid cylinders (minus circles and arcs): Y^1 2-fold covering Y. Now glue the two copies of D by the identity to get X, and note that there are two gluings of the D''s that give a cover of X: The identity which gives back X^1 and the order 2 automorphism of D' which yields the cover corresponding to the $1/2$ twist mentioned above.

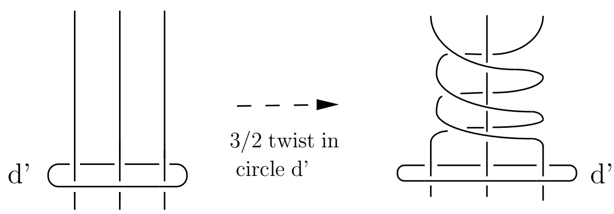

Figure 7.

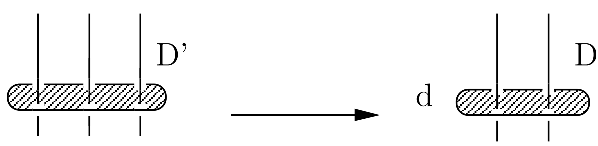

Figure 8.

2.2.3. Given L the closed n-braid corresponding to $\beta = \alpha_{i_k}^{s_k} \cdots \alpha_{i_1}^{s_1}$, we obtain J by the construction in steps 1–4 below.

1) Let X^1 be the 2-fold cover of $X \cong S^3 \setminus L_1$ branched over the circle c.
2) For $r > 1$, let X^r be the 2-fold cover of X^{r-1} branched over the rightmost preimage of the circle a.

The cover X^r is a link complement (since branched over an unknotted circle in X^{r-1}) containing 2^r unknotted, unlinked preimages of the circle b. Choose r so that $2^r > n$.

3) Let $\widetilde{X}$ be the k-fold cyclic cover of X^r branched over the preimage of circle f (see Figure 9).

4) Transform $\widetilde{X}$ into $S^3 \setminus J$ by twists corresponding to $\beta = \alpha_{i_k}^{s_k} \cdots \alpha_{i_1}^{s_1}$ in the appropriate preimages of circles d and e.

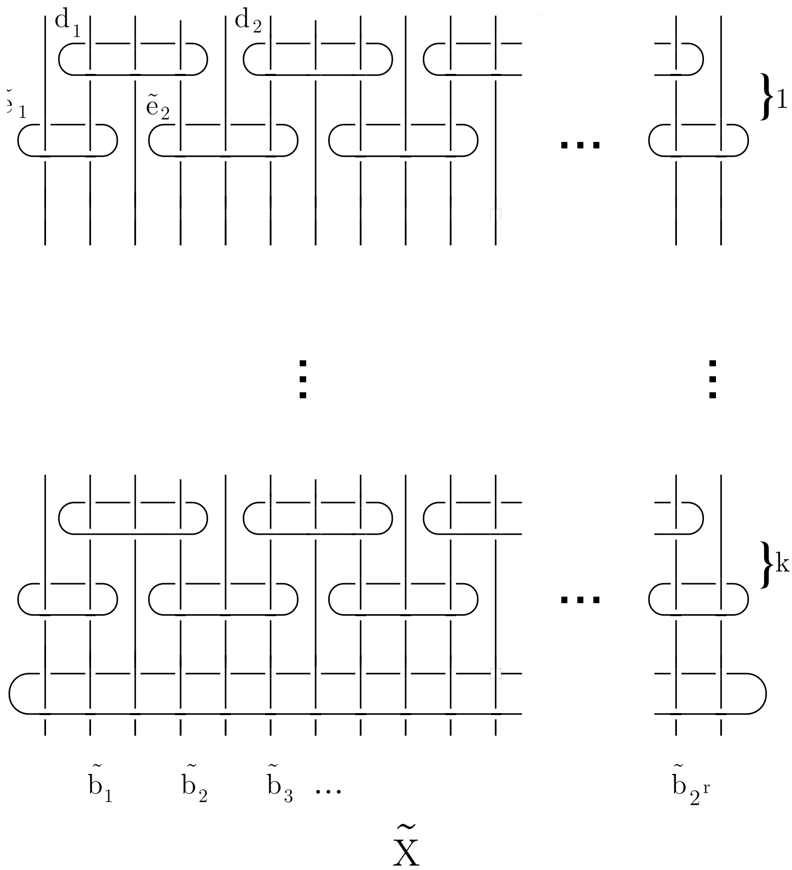

Figure 9.

Consider the n left-most preimages $\widetilde{b}_1, \ldots, \widetilde{b}_n$ in $\widetilde{X}$ of circle b. Adjacent circles are linked by preimages of d and e: $\widetilde{b}_1$,$\widetilde{b}_2$ by $\widetilde{d}_1$; $\widetilde{b}_2$,$\widetilde{b}_3$ by $\widetilde{e}_2$ and so forth. This pattern is repeated in k blocks from top to bottom in Figure 9. Now, for each of the k factors $\alpha_{i_j}^{s_j}$ in β perform a $s_j/2$ twist in the $\widetilde{d}$ or $\widetilde{e}$ linking $\widetilde{b}_{i_j}$,$\widetilde{b}_{i_j+1}$ in the j-th block. This changes the circles $\widetilde{b}_1, \ldots, \widetilde{b}_n$ into L. Finally, $S^3 \setminus J$ is a $(2^r k)$-fold cover of $S^3 \setminus L_1$ as explained in **2.2.2**.

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