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REAL AND p-ADIC LIE ALGEBRA FUNCTORS ON THE CATEGORY OF TOPOLOGICAL GROUPS

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We define real and p-adic topological Lie algebra functors on the category of topological (Hausdorff) groups which extend the usual Lie algebra functors on the categories of real resp. p-adic Lie groups, and study their properties.

0. Introduction.

We define real and p-adic topological Lie algebra functors on the category of topological (Hausdorff) groups which extend the usual Lie algebra functors. The method is a refinement of ideas developed in Lashof [**20**], which allow the set $\mathrm{Hom}(\mathbb{R}, G)$ of one-parameter subgroups to be made into a real topological Lie algebra, for every topological group G whose identity component G_0 is a pro-Lie group. The main goals are the following.

1. First, we want to equip the set $\mathrm{Hom}(\mathbb{R}, G)$ of one-parameter subgroups of a topological group G with a topological Lie algebra structure by means of continuous homomorphisms from G into real Lie groups, for more general topological groups than those accessible by Lashof's original approach. We do this by assigning topological Lie algebras to arbitrary topological groups functorially in a first step—which is of independent interest—and then in a second step describe a class of groups for which $\mathrm{Hom}(\mathbb{R}, G)$ can be identified in a natural way with the associated Lie algebra, the class $\mathbb{CL}_{\mathbb{R}}$ of topological groups which "carry their real Lie algebra."

2. The second main goal of this article is the creation of tools for the investigation of totally disconnected, locally compact groups, a powerful structure theory of which was begun in [**35**]. We introduce p-adic Lie algebra functors $\Lambda^{\mathrm{loc}}_{\mathbb{Q}_p}$ and $\overline{\Lambda}^{\mathrm{loc}}_{\mathbb{Q}_p}$ which are well-adapted to the setting of totally disconnected, locally compact groups: They have the desirable property of describing "local properties" of such groups in the sense that the Lie algebras are unchanged (up to natural isomorphism) under passage to open subgroups, which form a basis of identity neighbourhoods here. We also introduce a so-called "p-adic component" of a topological group, which, as far as locally compact groups are concerned, is a p-adic analogue of the connected component.

3. The third goal is the investigation of properties of the real and p-adic topological Lie algebra functors, e.g., their behaviour on products, quotient maps, and embeddings. In particular, we want to point out differences between the real and p-adic cases. It is clear that p-adic Lie algebra functors cannot behave as nicely as their real counterparts. For example, any p-adic Lie algebra functor on a full subcategory of the category of Hausdorff groups subsuming all p-adic Lie groups, which extends the usual p-adic Lie algebra functor, has to be discontinuous (as the functor it extends).

4. The real and p-adic Lie algebra functors on the categories of real and p-adic Lie groups, respectively, allow many different extensions to topological Lie algebra functors on the category of Hausdorff groups. In fact, by very natural constructions, we shall obtain 15 real topological Lie algebra functors and 9 p-adic topological Lie algebra functors. Our forth goal is to show that all of these functors are distinct (i.e., not naturally isomorphic). To this end, we evaluate the various Lie algebra functors on several selected examples of topological groups. These explicit calculations are also intended as illustrations of the general abstract theory.

The structure of the article is as follows.

In Section 1, we assemble the category-theoretical formalism concerning categories of projective systems, as a prerequisite for our studies.

Many of the topological Lie algebras we shall be dealing with are projective limits of finite-dimensional Lie algebras and are therefore so-called "weakly complete" topological vector spaces. Important properties of such spaces are recalled in Section 2.

In Section 3, we describe how projective systems of Lie groups can be associated with topological groups in a functorial fashion. These projective systems are then used to define real and p-adic Lie algebra functors on the category of topological groups (Section 5).

In Section 4, we have a closer look at projective limits of Lie groups, and at pro-Lie groups. The material assembled in this section is mostly known, at least in special cases (e.g., for locally compact groups), but since we found it hard to find direct references for the general assertions, full proofs are given.

In Section 6, we study the behaviour of our topological Lie algebra functors on various classes of well-behaved topological groups. Restricted to the category of pro-Lie groups, all of the real Lie algebra functors defined in Section 5 turn out to be naturally isomorphic to Lashof's topological Lie algebra functor (Proposition 6.11). We also consider the behaviour of some of the real Lie algebra functors on the variety of Hausdorff groups generated by real Lie groups, and on the class of topological groups G which "carry their real Lie algebra" in the sense that $\mathrm{Hom}(\mathbb{R}, G)$ can be made a topological Lie algebra in a natural way. Most of the positive results established in this section are specific to the real case, and counterexamples in the p-adic setting are given.

Not all of the functors defined in this article preserve direct products (Example 7.5), but at least partial results concerning the behaviour on direct products are available (Section 7, also Corollary 10.4).

In Section 8, we compute the real Lie algebras associated with 9 selected examples of topological groups, and deduce that the 15 real Lie algebra functors defined in Section 5 are pairwise not naturally isomorphic. The p-adic Lie algebra functors are shown to be distinct in Section 12, after a more detailed discussion of the p-adic Lie algebra functors in Sections 9 through 11.

As a preliminary for our studies of the p-adic Lie algebra functors on the category of locally compact groups, we have a closer look at the topological $\mathbb{Z}_p$-Lie algebra functor $\mathrm{Hom}(\mathbb{Z}_p, \bullet)$ on the category of abelian topological groups (Section 9). In particular, given a continuous homomorphism $G \to H$ between locally compact groups, we determine which continuous homomorphisms $\mathbb{Z}_p \to H$ lift to continuous homomorphisms $\mathbb{Z}_p \to G$ (Corollary 9.3).

It is well-known that Lashof's topological Lie algebra functor maps quotient morphisms between locally compact groups to quotient morphisms of topological Lie algebras ([**18**], Lemma 1.3). In Section 10, we establish analogous results for our p-adic Lie algebra functors (Corollary 10.3, Corollary 10.7), and describe further properties of the p-adic Lie algebra functors specific to their behaviour on the category of locally compact groups.

We remark that the p-adic Lie algebra functors defined here cannot be used to equip the set $\Gamma(\mathbb{Q}_p, G)$ of germs at 0 of local p-adic one-parameter subgroups of G (which can be interpreted as the usual p-adic Lie algebra of G when G is a p-adic Lie group) with a p-adic Lie algebra structure for arbitrary totally disconnected, locally compact groups G, and not even for pro-p-adic Lie groups, as the examples show. However, whenever the local (or global) p-adic Lie algebra of a totally disconnected, locally compact group (see 5.7, 5.3) has finite dimension, it can be identified with a quotient of this set (Section 11). In the general case, such a quotient is still a dense subset of the Lie algebra (Corollary 10.2). Applications of the p-adic Lie algebra functors in the theory of locally compact groups will certainly be difficult. One obstacle is the absence of an analogue of the exponential function $\mathrm{Hom}(\mathbb{R}, G) \to G$, $X \mapsto X(1)$, which provides a link between a locally compact group and its real topological Lie algebra.

Additional examples and remarks are assembled in Section 13.

In the final section, we construct further real topological Lie algebra functors, which assign the correct topological Lie algebras to all Banach-Lie groups.

1. Categories of projective systems.

Although this article necessarily has to involve category-theoretical ideas, we try to keep the formalism at a minimum. In this section, we assemble the required terminology and preliminaries concerning projective systems; we content ourselves with aspects of the general theory (as developed in [**17**], Chapter IV, §1) which will be sufficient for our purposes. In Section 6, limits to arbitrary functors are needed in connection with continuity questions of functors; see [**21**] on the definition of these. See also [**11**] and [**13**] for additional information.

1.1. If $\mathbb{A}$ is a category, we define the category $\mathbb{A}^\pi$ of projective systems in $\mathbb{A}$, as follows. The objects of $\mathbb{A}^\pi$ are the projective systems in $\mathbb{A}$; recall that a *projective system over I in* $\mathbb{A}$ is a pair $\mathcal{S} = ((X_i)_{i\in I}, (\phi_{ij})_{i\leq j})$, where I is a directed set, each X_i is an object of $\mathbb{A}$, and $\phi_{ij} : X_j \to X_i$ is a morphism for $i \leq j$ such that $\phi_{ii} = \mathrm{id}_{X_i}$ for all $i \in I$ and $\phi_{ij} \circ \phi_{jk} = \phi_{ik}$ whenever $i \leq j \leq k$. Note that we do not fix a directed set I. If $\mathcal{S}$ is a projective system in $\mathbb{A}$ as above and $\mathcal{T} = ((Y_j)_{j\in J}, (\psi_{jk})_{j\leq k})$ is another projective system in $\mathbb{A}$, then a morphism $\mathcal{S} \to \mathcal{T}$ in $\mathbb{A}^\pi$ is a pair $(\sigma, (\eta_j)_{j\in J})$, where $\sigma\colon J \to I$ is an order preserving map and $(\eta_j)_{j\in J}$ is a family of morphisms $\eta_j : X_{\sigma(j)} \to Y_j$ such that $\psi_{jk} \circ \eta_k = \eta_j \circ \phi_{\sigma(j)\sigma(k)}$ for all $j \leq k$ in J; we then also say that the pair $(\sigma, (\eta_j)_{j\in J})$ is *compatible* with the directed systems $\mathcal{S}$ and $\mathcal{T}$. If $\mathcal{H} = ((H_k)_{k\in K}, (\chi_{kl})_{k\leq l})$ is another projective system in $\mathbb{A}$ and $(\tau, (\zeta_k)_{k\in K})\colon \mathcal{T} \to \mathcal{H}$ is a morphism, the composition $(\tau, (\zeta_k)_{k\in K}) \circ (\sigma, (\eta_j)_{j\in J})$ is defined as $(\sigma \circ \tau, (\zeta_k \circ \eta_{\tau(k)})_{k\in K})$.

1.2. If $\mathbb{B}$ is another category and $F\colon \mathbb{A} \to \mathbb{B}$ is a functor, we obtain a functor $F^\pi\colon \mathbb{A}^\pi \to \mathbb{B}^\pi$ via $((X_i)_{i\in I}, (\phi_{ij})_{i\leq j}) \mapsto ((F(X_i))_{i\in I}, (F(\phi_{ij}))_{i\leq j})$ on objects and $(\sigma, (\eta_j)_{j\in J}) \mapsto (\sigma, (F(\eta_j))_{j\in J})$ on morphisms.

1.3. A *cone over* $\mathcal{S} \in \mathrm{ob}\,\mathbb{A}^\pi$ is a pair $(X, (\phi_i)_{i\in I})$ of an object $X \in \mathrm{ob}\,\mathbb{A}$ and a family of morphisms $\phi_i : X \to X_i$ such that $\phi_{ij} \circ \phi_j = \phi_i$ whenever $i \leq j$. A cone $(X, (\phi_i)_{i\in I})$ is a *projective limit cone over* $\mathcal{S}$ if for every cone $(Y, (\psi_i)_{i\in I})$ over $\mathcal{S}$, there exists a unique morphism $\psi : Y \to X$ such that $\phi_i \circ \psi = \psi_i$ for all $i \in I$; then X is said to be a *projective limit* of $\mathcal{S}$.

1.4. Let $(X, (\phi_i)_{i\in I})$ be a cone over $\mathcal{S} = ((X_i)_{i\in I}, (\phi_{ij})_{i\leq j}) \in \mathrm{ob}\,\mathbb{A}^\pi$ and $(Y, (\psi_j)_{j\in J})$ be a projective limit cone over $\mathcal{T} = ((Y_j)_{j\in J}, (\psi_{jk})_{j\leq k}) \in \mathrm{ob}\,\mathbb{A}^\pi$; let $(\sigma, (\eta_j)_{j\in J})$ be a morphism $\mathcal{S} \to \mathcal{T}$. Then $(X, (\eta_j \circ \phi_{\sigma(j)})_{j\in J})$ is a cone over $\mathcal{T}$, which induces a morphism $\eta\colon X \to Y$ determined by $\psi_j \circ \eta = \eta_j \circ \phi_{\sigma(j)}$. If, in the preceding situation, $(X, (\phi_i)_{i\in I})$ is a projective limit cone over $\mathcal{S}$, we write $\varprojlim \eta_i := \eta$.

1.5. Now suppose that there exists a functional class which associates with every projective system $\mathcal{S}$ in $\mathbb{A}$ a projective limit cone, whose projective limit object we denote by $\varprojlim \mathcal{S}$. Then clearly $\varprojlim\colon \mathbb{A}^\pi \to \mathbb{A}$ is a functor, where $\varprojlim$ is defined on morphisms as described in the previous paragraph.[1]

1.6. In the category $\mathbb{TG}$ of topological (Hausdorff) groups and continuous homomorphisms, a projective limit cone over $\mathcal{S} = ((G_i)_{i\in I}, (\phi_{ij})_{j\geq i})$ is given by $G := \varprojlim G_i := \{(x_i)_{i\in I} \in \prod_{i\in I} G_i : (i \leq j \Rightarrow \phi_{ij}(x_j) = x_i)\}$, together with the morphisms $\phi_i := \mathrm{pr}_i|_G$. A similar construction works in the category of topological Lie algebras over a topological field K. Note that the sets underlying these projective limits are the projective limits of the sets involved; also note that the existence of the functional classes described above is guaranteed here by the preceding explicit description of projective limit cones. It is important for the constructions envisaged here that, unlike the conventions in [**33**] or [**20**], we do *not* require that the morphisms in a projective system of topological groups and the corresponding limit maps be quotient morphisms (i.e., surjective and open).

2. Weakly complete topological vector spaces.

Let $\mathbb{K}$ denote the field of real numbers, or a p-adic field. A locally convex topological vector space V over $\mathbb{K}$ is called *weakly complete* if it is complete and its topology coincides with the associated weak topology $\sigma(V, V')$ (see [**23**] on topological vector spaces over $\mathbb{Q}_p$). We need to recall some basic properties of weakly complete locally convex spaces (cf. [**14**], [**19**], [**30**]):

Proposition 2.1. *A locally convex space V over $\mathbb{K}$ is weakly complete if and only if V is isomorphic to $\mathbb{K}^I$ for some set I, if and only if V is isomorphic to a closed vector subspace of $\mathbb{K}^I$ for some set I. If V is weakly complete, the following holds:*

(a) *Any closed vector subspace V_1 of V is weakly complete, and there exists a closed vector subspace V_2 of V such that $V = V_1 \oplus V_2$ as a topological vector space.*
(b) *If also W is weakly complete and $f : W \to V$ is a morphism with dense image, then f splits, whence f is a quotient morphism. In particular, every continuous linear bijection between weakly complete topological vector spaces is an isomorphism of topological vector spaces.*

Finally, for every set I, the continuous linear functionals on $\mathbb{K}^I$ are precisely the linear combinations of point evaluations at the elements of I.

[1] Unlike [**21**] and part of the category-oriented literature, the author does not want to use a special model of set theory but stick to the standard von Neumann-Bernays-Gödel axioms. This accounts for some technicalities; e.g., it would not be enough to assume the mere *existence* of projective limit cones here.

3. Associated projective systems of Lie groups.

Let $\mathbb{K}$ be the field of real numbers, or a p-adic field. In this section, we define functors $\mathfrak{S}_{\mathbb{K}}$ which associate with every topological group a projective system of $\mathbb{K}$-Lie groups. It should be mentioned that $\mathbb{K}$-Lie groups are not assumed to be second countable here; e.g., every discrete group is a $\mathbb{K}$-Lie group. All $\mathbb{K}$-Lie groups are finite-dimensional, unless qualified as "Banach-Lie groups" or "direct limit Lie groups," which need not be finite-dimensional. Occasionally, real Lie groups will simply be referred to as "Lie groups," without specification of the field $\mathbb{R}$.

3.1. Let $G \in \mathbb{TG}$ (a shorthand for '$G \in \operatorname{ob} \mathbb{TG}$'). We consider the class Ω of morphisms (continuous homomorphisms) $f\colon G \to H_f$ into $\mathbb{K}$-Lie groups H_f such that f has dense image. Then Ω can be pre-ordered by declaring $f \leq g$ if and only if there exists a morphism $h : H_g \to H_f$ such that $f = h \circ g$. Note that $h =: \phi_{fg}$ is unique whenever it exists, because g has dense image. The class Ω is directed, since given $f, g \in \Omega$, the corestriction of $(f,g)\colon G \to H_f \times H_g$ to the closure of $\operatorname{im}(f,g)$ will be an upper bound for $\{f, g\}$. Consider f and g as equivalent if $f \leq g$ and $g \leq f$; let $I_{\mathbb{K}}(G)$ denote a set of representatives for the equivalence classes. Such a set exists, since the cardinalities of the groups H_f are bounded. Then $I_{\mathbb{K}}(G) \subseteq \Omega$ is a directed set, and $\mathfrak{S}_{\mathbb{K}}(G) := ((H_f)_{f \in I_{\mathbb{K}}(G)}, (\phi_{fg})_{f \leq g})$ is a projective system of $\mathbb{K}$-Lie groups.

3.2. To avoid set-theoretic difficulties, it is important to note that no choice of $I_{\mathbb{K}}(G)$ needs to be involved but $I_{\mathbb{K}}(G)$ can be determined unambiguously, as follows: We let $\Xi(G)$ be the set of all continuous homomorphisms $g\colon G \to (G/N, \mathcal{O}_g) =: Q_g$, where N is a closed normal subgroup of G and $\mathcal{O}_g$ a group topology on the abstract group G/N such that Q_g can be embedded into a $\mathbb{K}$-Lie group. If $Q_g = G/N$, equipped with the quotient topology, and Q_g is a $\mathbb{K}$-Lie group, we set $f_g := g$. Otherwise, the canonical completion H_g of Q_g in its left uniform structure (as defined in [**2**], Chapter II, §3.7, Definition 4), with canonical embedding $e_g\colon Q_g \hookrightarrow H_g$, can be equipped with a unique topological group structure such that $e_g : Q_g \hookrightarrow H_g$ becomes a homomorphism; then H_g is a $\mathbb{K}$-Lie group. We set $f_g := e_g \circ g\colon G \to H_g$. Then $I_{\mathbb{K}}(G) := \{f_g : g \in \Xi(G)\}$ satisfies our needs. Note that, by construction, the set $J_{\mathbb{K}}(G)$ of quotient morphisms $G \to G/N$ onto $\mathbb{K}$-Lie quotients of G is a subset of $I_{\mathbb{K}}(G)$.

3.3. Now suppose that $\Phi : G \to L$ is a morphism of topological groups. For every $f \in I_{\mathbb{K}}(L)$, there exists a unique $\sigma(f) \in I_{\mathbb{K}}(G)$ such that $\sigma(f)$ is equivalent to the corestriction h_f of $f \circ \Phi$ to the closure of its image. Let λ_f denote the inclusion map $\operatorname{im} h_f \hookrightarrow H_f$, and define $\mu_f := \lambda_f \circ \phi_{h_f \sigma(f)}$; thus $\mu_f \circ \sigma(f) = f \circ \Phi$. Then $\mathfrak{S}_{\mathbb{K}}(\Phi) := (\sigma, (\mu_f)_{f \in I_{\mathbb{K}}(L)})$ is compatible with the projective systems $\mathfrak{S}_{\mathbb{K}}(G) = ((H_f)_{f \in I_{\mathbb{K}}(G)}, (\phi_{fg})_{f \leq g})$ and $\mathfrak{S}_{\mathbb{K}}(L) =$

$((H'_f)_{f\in I_{\mathbb{K}}(L)}, (\phi'_{fg})_{f\le g})$; to check this, we have to verify that $\phi'_{fg}\circ\mu_g = \mu_f\circ\phi_{\sigma(f)\sigma(g)}$ for all $f\le g$ in $I_{\mathbb{K}}(L)$. Clearly $h_g\ge h_f$ via $\phi'_{fg}|_{\overline{\operatorname{im} g\circ\Phi}}^{\overline{\operatorname{im} f\circ\Phi}}$, hence also $\sigma(g)\ge\sigma(f)$; since $\sigma(g)$ has dense image, the assertion now follows from

$$\mu_f\circ\phi_{\sigma(f)\sigma(g)}\circ\sigma(g) = f\circ\Phi = \phi'_{fg}\circ g\circ\Phi = \phi'_{fg}\circ\mu_g\circ\sigma(g).$$

3.4. If $\Psi: L\to T$ is another morphism, define $\sigma'(f)$, h'_f, λ'_f, and μ'_f for $f\in I_{\mathbb{K}}(T)$ in analogy to the definitions of $\sigma(f)$, h_f, λ_f, and μ_f above.

Given $f\in I_{\mathbb{K}}(T)$, we obtain $f\circ\Psi = \mu'_f\circ\sigma'(f)$, where $\sigma'(f)\in I_{\mathbb{K}}(L)$. Now $\sigma'(f)\circ\Phi = \mu_{\sigma'(f)}\circ\sigma(\sigma'(f))$; hence $f\circ\Psi\circ\Phi = \omega_f\circ\sigma(\sigma'(f))$, where $\omega_f = \mu'_f\circ\mu_{\sigma'(f)}$. Note that $\sigma(\sigma'(f))\in I_{\mathbb{K}}(G)$ and that ω_f is an embedding. For $f\in I_{\mathbb{K}}(T)$, define $\sigma''(f)$, h''_f etc. as above for the morphism $\Psi\circ\Phi$. Since $\sigma(\sigma'(f))$ has dense image, we have $h''_f = \omega_f|^{\overline{\operatorname{im}\omega_f}}\circ\sigma(\sigma'(f))$. Thus $\sigma''(f) = \sigma(\sigma'(f))$, and $\phi''_{h''_f,\sigma''(f)} = \omega_f|^{\overline{\operatorname{im}\omega_f}}$. Now $\mu''_f = \lambda''_f\circ\phi_{h''_f,\sigma''_f} = \lambda''_f\circ\omega_f|^{\overline{\operatorname{im}\omega_f}} = \mu'_f\circ\mu_{\sigma'(f)}$ shows that $\mathfrak{S}_{\mathbb{K}}(\Psi)\circ\mathfrak{S}_{\mathbb{K}}(\Phi) = \mathfrak{S}_{\mathbb{K}}(\Psi\circ\Phi)$ indeed. Thus $\mathfrak{S}_{\mathbb{K}}: \mathbb{TG}\to(\mathbb{LIE}_{\mathbb{K}})^\pi$ is a functor; here $\mathbb{LIE}_{\mathbb{K}}$ denotes the category of $\mathbb{K}$-Lie groups and continuous homomorphisms.

4. Projective limits of Lie groups.

Again, let $\mathbb{K}=\mathbb{R}$ or $\mathbb{K}=\mathbb{Q}_p$ for some prime p. In this section, we give various characterizations of projective limits of $\mathbb{K}$-Lie groups, and give characterizations of pro-$\mathbb{K}$-Lie groups. We show that the forgetful functor from the category $\mathbb{GPL}_{\mathbb{K}}$ of $\mathbb{TG}$-projective limits of $\mathbb{K}$-Lie groups into the category $\mathbb{TG}$ has a left adjoint $(\ )_{\mathbb{K}}: \mathbb{TG}\to\mathbb{GPL}_{\mathbb{K}}$. All of the real topological Lie algebra functors constructed in the following shall coincide on $\mathbb{GPL}_{\mathbb{R}}$, and we shall see that they have particularly good properties on this class of topological groups.

Definition 4.1. Let $\mathbb{GPL}_{\mathbb{K}}$ be the full subcategory of $\mathbb{TG}$ whose objects are projective limits of $\mathbb{K}$-Lie groups, i.e., topological groups which can be made the projective limit (in $\mathbb{TG}$) of some projective system of $\mathbb{K}$-Lie groups. Let $\iota_{\mathbb{K}}: \mathbb{LIE}_{\mathbb{K}}\to\mathbb{GPL}_{\mathbb{K}}$ and $U_{\mathbb{K}}: \mathbb{GPL}_{\mathbb{K}}\to\mathbb{TG}$ be the respective inclusion functors (which forget the special type of group). Then $(\ )_{\mathbb{K}} := \varprojlim\circ\iota_{\mathbb{K}}^\pi\circ\mathfrak{S}_{\mathbb{K}}$ defines a functor $\mathbb{TG}\to\mathbb{GPL}_{\mathbb{K}}$, which associates with every Hausdorff group G a $\mathbb{TG}$-projective limit $G_{\mathbb{K}}$ of $\mathbb{K}$-Lie groups. We define a morphism $\eta_{\mathbb{K}}^G: G\to G_{\mathbb{K}}$ via $x\mapsto(f(x))_{f\in I_{\mathbb{K}}(G)}$.

The following fact will be used repeatedly:

Lemma 4.2. *Suppose that $(G,(\phi_i)_{i\in I})$ is a projective limit cone of a projective system $((G_i)_{i\in I},(\phi_{ij})_{i\le j})\in\mathbb{TG}^\pi$. Then $\{\phi_i^{-1}(V): i\in I,\ V\in\mathcal{N}_1(G_i)\}$ is a basis for the filter $\mathcal{N}_1(G)$ of identity neighbourhoods of G.*

Proof. It follows from the explicit description of projective limit cones given in 1.6 that the sets of the form $\bigcap_{i\in F}\phi_i^{-1}(V_i)$, where F is a finite subset of I and $V_i\in\mathcal{N}_1(G_i)$ for $i\in F$, form a basis for the filter $\mathcal{N}_1(G)$. Suppose that $U\in\mathcal{N}_1(G)$ is of this form. By directedness of I, there is $j\in I$ such that $j\geq i$ for all $i\in F$; then $\phi_j^{-1}(V)\subseteq U$, where $V:=\bigcap_{i\in F}\phi_{ij}^{-1}(V_i)\in\mathcal{N}_1(G_j)$. □

Lemma 4.3. *$\eta_{\mathbb{K}}^G$ has dense image in $G_{\mathbb{K}}$, for every $G\in\mathbb{TG}$.*

Proof. We have $G_{\mathbb{K}}\leq\prod_{f\in I_{\mathbb{K}}(G)}H_f$, with limit morphisms $\phi_f=\mathrm{pr}_f|_{G_{\mathbb{K}}}$ for $f\in I_{\mathbb{K}}(G)$. Let $y=(y_f)_{f\in I_{\mathbb{K}}(G)}\in G_{\mathbb{K}}$ and U be a neighbourhood of y in $G_{\mathbb{K}}$; it follows from Lemma 4.2 that there exists $g\colon G\to H_g$ in $I_{\mathbb{K}}(G)$ and $V\in\mathcal{N}_{y_g}(H_g)$ such that $\phi_g^{-1}(V)\subseteq U$. Since $g\in I_{\mathbb{K}}(G)$ has dense image, there is $x\in G$ such that $g(x)\in V$. Then $\phi_g(\eta_{\mathbb{K}}^G(x))=g(x)\in V$ entails $\eta_{\mathbb{K}}^G(x)\in\phi_g^{-1}(V)\subseteq U$. □

Proposition 4.4. *The functor $(\ )_{\mathbb{K}}\colon\mathbb{TG}\to\mathbb{GPL}_{\mathbb{K}}$ is a left adjoint for the forgetful functor $U_{\mathbb{K}}\colon\mathbb{GPL}_{\mathbb{K}}\to\mathbb{TG}$, with unit of adjunction $\eta_{\mathbb{K}}$. In other words, for every morphism $\Phi\colon G\to P$ into a projective limit P of $\mathbb{K}$-Lie groups, there exists a unique morphism $\overline{\Phi}\colon G_{\mathbb{K}}\to P$ such that $\overline{\Phi}\circ\eta_{\mathbb{K}}^G=\Phi$.*

Proof. Let $\Phi\colon G\to P$ be a morphism into a projective limit P of $\mathbb{K}$-Lie groups. Since $\eta_{\mathbb{K}}^G$ has dense image by Lemma 4.3, there is at most one morphism $\overline{\Phi}\colon G_{\mathbb{K}}\to P$ such that $\overline{\Phi}\circ\eta_{\mathbb{K}}^G=\Phi$. It only remains to prove the existence of $\overline{\Phi}$.

Suppose first that P is a $\mathbb{K}$-Lie group. Set $H_f:=\overline{\mathrm{im}\,\Phi}$ and $f:=\Phi|^{H_f}$; then there is a uniquely determined $g\in I_{\mathbb{K}}(G)$ and a uniquely determined isomorphism $\phi_{fg}\colon H_g\to H_f$ such that $f=\phi_{fg}\circ g$. If $\varepsilon\colon H_f\hookrightarrow P$ denotes the embedding, we set $\overline{\Phi}:=\varepsilon\circ\phi_{fg}\circ\phi_g$ (where $\phi_g\colon G_{\mathbb{K}}\to H_g$ is the respective limit map). Then $\overline{\Phi}\circ\eta_{\mathbb{K}}^G=\Phi$ indeed.

Now consider the general case where there exists a projective system $\mathcal{S}=((P_i)_{i\in I},(\psi_{ij})_{i\leq I})\in\mathbb{LIE}_{\mathbb{K}}^{\pi}$ of $\mathbb{K}$-Lie groups P_i, and morphisms $\psi_i\colon P\to P_i$ such that $(P,(\psi_i)_{i\in I})$ is a projective limit cone over $\mathcal{S}$ in $\mathbb{TG}$. We set $\Phi_i:=\psi_i\circ\Phi$ for $i\in I$; then $(G,(\Phi_i))$ is a cone over $\mathcal{S}$. For every $i\in I$, there is a unique morphism $\overline{\Phi_i}\colon G_{\mathbb{K}}\to P_i$ such that $\overline{\Phi_i}\circ\eta_{\mathbb{K}}^G=\Phi_i$ by the special case treated above. The uniqueness of the $\overline{\Phi_i}$'s implies that $(G_{\mathbb{K}},(\overline{\Phi_i})_{i\in I})$ is a cone over $\mathcal{S}$. Now $(P,(\psi_i))$ being a projective limit cone, we obtain a unique morphism $\overline{\Phi}\colon G_{\mathbb{K}}\to P$ such that $\psi_i\circ\overline{\Phi}=\overline{\Phi_i}$ for all $i\in I$. Then $\psi_i\circ(\overline{\Phi}\circ\eta_{\mathbb{K}}^G)=\overline{\Phi_i}\circ\eta_{\mathbb{K}}^G=\Phi_i=\psi_i\circ\Phi$ implies $\overline{\Phi}\circ\eta_{\mathbb{K}}^G=\Phi$. □

Remark 4.5. The preceding consideration shows in particular that for every morphism $\psi\colon G\to K$ of topological groups, there exists a unique morphism $\psi_{\mathbb{K}}\colon G_{\mathbb{K}}\to K_{\mathbb{K}}$ such that $\psi_{\mathbb{K}}\circ\eta_{\mathbb{K}}^G=\eta_{\mathbb{K}}^K\circ\psi$. Then $\eta_{\mathbb{K}}\colon\mathrm{id}_{\mathbb{TG}}\xrightarrow{\cdot}U_{\mathbb{K}}\circ(\ )_{\mathbb{K}}$ is a natural transformation.

Our next aim is to show that the class $\mathbb{GPL}_{\mathbb{K}}$ of projective limits of Lie groups is closed under the formation of cartesian products, closed subgroups, and also under the formation of Hausdorff quotients whenever these are complete.

Lemma 4.6. *Suppose that $\mathcal{S} = ((G_i)_{i\in I}, (\psi_{ij})_{i\le j})$ is a projective system of $\mathbb{K}$-Lie groups, $(G, (\psi_i))$ a projective limit cone over $\mathcal{S}$ in $\mathbb{TG}$, and K a closed subgroup of G. Then $(K, (\psi_i|_K^{K_i}))$ is a projective limit cone over the projective system $((K_i)_{i\in I}, (\psi_{ij}|_{K_j}^{K_i})_{i\le j}) =: \mathcal{K}$ of $\mathbb{K}$-Lie groups, where $K_i := \overline{\psi_i(K)} \le G_i$ for $i \in I$.*

Proof. We may assume that the projective limit cone $(G, (\psi_i)_{i\in I})$ is of the special form described in 1.6. It is easy to see that $\mathcal{K}$ is a projective system; its projective limit topological group specified in 1.6 is the subgroup $P = \{x \in G\colon (\forall i \in I)\ \psi_i(x) \in K_i\}$ of G. Then $K \le P$ clearly. It remains to show that $P \le K$. To this end, suppose that $x \in G\backslash K$. Since K is closed in G, we have $K = \bigcap_{U\in\mathcal{N}_1(G)} KU$; using Lemma 4.2, we deduce that there exists $i \in I$ and $V \in \mathcal{N}_1(G_i)$ such that $x \notin K\psi_i^{-1}(V)$. The latter set being a union of cosets of $\ker\psi_i$, we deduce that $\psi_i(x) \notin \psi_i(K\psi_i^{-1}(V)) = \psi_i(K)\psi_i(\psi_i^{-1}(V))$. Since $\operatorname{im}\psi_i$ is a subgroup of G_i and $\psi_i(\psi_i^{-1}(V)) = V \cap \operatorname{im}\psi_i$, we infer that $\psi_i(x) \notin \psi_i(K)V \supseteq \overline{\psi_i(K)} = K_i$, whence $x \notin P$. We have proved that $P \le K$: Hence $P = K$ indeed. □

Recall that a class $\mathcal{W}$ of Hausdorff groups is called a *variety of Hausdorff groups* if $\mathcal{W}$ is closed under the operations of forming subgroups (S), quotient groups with respect to closed normal subgroups ($\overline{\mathrm{Q}}$), arbitrary cartesian products (C) (and under passage to isomorphic topological groups), see [**16**]. If $\mathbb{A}$ is a class of Hausdorff groups, there is a smallest variety $\mathcal{V}(\mathbb{A})$ of Hausdorff groups such that $\mathbb{A}$ is a subclass of $\mathcal{V}(\mathbb{A})$. If, in addition to the operations above, (P) denotes the formation of finite cartesian products and ($\overline{\mathrm{S}}$) the formation of closed subgroups, it can be shown that $\mathcal{V}(\mathbb{A}) = \mathrm{SC}\overline{\mathrm{Q}}\,\overline{\mathrm{S}}\mathrm{P}(\mathbb{A})$, see [**4**], Theorem 2, also [**25**], Theorem 7. Since finite products, closed subgroups, and Hausdorff quotients of $\mathbb{K}$-Lie groups are $\mathbb{K}$-Lie groups, the variety of Hausdorff groups generated by the class of $\mathbb{K}$-Lie groups has the form

$$\mathcal{V}(\mathbb{LIE}_{\mathbb{K}}) = \mathrm{SC}(\mathbb{LIE}_{\mathbb{K}}) \tag{1}$$

(see [**16**], [**9**]).

Proposition 4.7. *For a Hausdorff group G, the following conditions are equivalent:*

(a) *$\eta_{\mathbb{K}}^G\colon G \to G_{\mathbb{K}}$ is an isomorphism of topological groups;*
(b) *G is a $\mathbb{TG}$-projective limit of $\mathbb{K}$-Lie groups;*
(c) *$G \in \mathcal{V}(\mathbb{LIE}_{\mathbb{K}})$, and G is complete.*

Furthermore, every $G \in \mathcal{V}(\mathbb{LIE}_{\mathbb{K}})$ admits a dense embedding into a complete Hausdorff group $\overline{G} \in \mathcal{V}(\mathbb{LIE}_{\mathbb{K}})$.

Proof. The implications "(a)⇒(b)" and "(b)⇒(c)" are trivial.

(c)⇒(b): If $G \in \mathcal{V}(\mathbb{LIE}_{\mathbb{K}})$ and G is complete, then G is isomorphic to a subgroup S of a product $P = \prod_{i\in I} H_i$ of $\mathbb{K}$-Lie groups H_i by Equation (1); we only need to show that S is a projective limit of $\mathbb{K}$-Lie groups. To this end, note that $P \cong \lim\limits_{\longleftarrow} \prod_{i\in F} H_i$ is the projective limit of the projective system of finite partial products $\prod_{i\in F} H_i$ and respective restriction maps; given a finite subset F of I, the limit map $\prod_{i\in I} H_i \to \prod_{i\in F} H_i =: P_F$ is the restriction map $\mathrm{pr}_F := (\mathrm{pr}_i)_{i\in F}$. Since G is assumed to be complete, S is closed in P. Now Lemma 4.6 shows that S is a projective limit of a projective system of closed subgroups of the partial products P_F: All of these are $\mathbb{K}$-Lie groups.

(b)⇒(a): Suppose that $(G, (\psi_i)_{i\in I})$ is a projective limit cone in $\mathbb{TG}$ over a projective system $((G_i)_{i\in I}, (\psi_{ij})_{i\leq j})$ of $\mathbb{K}$-Lie groups. By Proposition 4.4, for every $i \in I$ there is a morphism $\alpha_i\colon G_{\mathbb{K}} \to H_i$ such that $\alpha_i \circ \eta^G_{\mathbb{K}} = \psi_i$. Since $(\psi_i)_{i\in I}$ separates points on G, we deduce that $\eta^G_{\mathbb{K}}$ is injective. Let $\mathcal{O}$ be the topology on $G' := \operatorname{im} \eta^G_{\mathbb{K}}$ which makes the bijection $\eta^G_{\mathbb{K}}|^{G'}$ an isomorphism of topological groups. Then $\mathcal{O}$ is finer than the topology induced by $G_{\mathbb{K}}$, since $\eta^G_{\mathbb{K}}$ is continuous. On the other hand, the topology on G is the initial topology with respect to the family $(\psi_i)_{i\in I}$, whence $\mathcal{O}$ is the initial topology with respect to the family $(\alpha_i|_{G'})_{i\in I}$; since every α_i is continuous, we deduce that $\mathcal{O}$ coincides with the topology on G' induced by $G_{\mathbb{K}}$, whence $\eta^G_{\mathbb{K}}$ is an embedding. Now G being complete, G' is closed in $G_{\mathbb{K}}$; on the other hand, G' is dense in $G_{\mathbb{K}}$ by Lemma 4.3: Thus $G' = G_{\mathbb{K}}$ and $\eta^G_{\mathbb{K}}$ is an isomorphism of topological groups.

By Equation (1), every $G \in \mathcal{V}(\mathbb{LIE}_{\mathbb{K}})$ is isomorphic to a subgroup S of some cartesian product P of $\mathbb{K}$-Lie groups. Now the closure $\overline{S}$ of S in P is a complete group which contains S as a dense subgroup, and $\overline{S}$, being a subgroup of P, is a member of $\mathcal{V}(\mathbb{LIE}_{\mathbb{K}})$. □

Corollary 4.8. *$\mathbb{GPL}_{\mathbb{K}}$ is closed under the formation of closed subgroups and cartesian products; if $G \in \mathbb{GPL}_{\mathbb{K}}$ and N is a closed normal subgroup of G such that G/N is complete, then $G/N \in \mathbb{GPL}_{\mathbb{K}}$. In particular, $\mathbb{GPL}_{\mathbb{K}}$ is closed under the formation of $\mathbb{TG}$-projective limits.*

The remainder of this section is devoted to special projective limits of $\mathbb{K}$-Lie groups, the so-called *pro-$\mathbb{K}$-Lie groups*. A topological group G a *pro-$\mathbb{K}$-Lie group* if $J_{\mathbb{K}}(G)$ is cofinal in $I_{\mathbb{K}}(G)$ and $(G, (f)_{f\in J_{\mathbb{K}}(G)})$ is a projective limit cone over the projective system $((H_f)_{f\in J_{\mathbb{K}}(G)}, (\phi_{fg})_{f\leq g}) =: \mathcal{T}(G)$. Note that if the first condition is satisfied, the second is equivalent to $\eta^G_{\mathbb{K}}$ being a topological isomorphism. Let $\mathbb{PL}_{\mathbb{K}}$ be the full subcategory of $\mathbb{TG}$ whose objects are the pro-$\mathbb{K}$-Lie groups. For later use, we recall:

Lemma 4.9. *Let* $((G_i)_{i\in I},(q_{ij})_{i\leq j})$ *be a projective system in* $\mathbb{TG}$*, with projective limit cone* $(G,(q_i)_{i\in I})$*; let* $f\colon G\to H$ *be a morphism into a real Lie group* H*. Then there is* $i\in I$ *such that* $\ker q_i\leq \ker f$*; hence if* q_i *is a quotient map, there exists a morphism* $g\colon G_i\to H$ *such that* $f=g\circ q_i$.

Proof (cf. [**33**], p. 26). Let U be an identity neighbourhood in H which contains no nontrivial subgroup. Then all subgroups contained in the identity neighbourhood $f^{-1}(U)$ of G are subgroups of $\ker f$. It follows from Lemma 4.2 that there is $i\in I$ with $\ker q_i\subseteq U$: Then $\ker q_i\leq \ker f$. □

The definition of pro-$\mathbb{K}$-Lie groups given here is in accordance with that of a pro-$\mathbb{LIE}_{\mathbb{K}}$-group in the sense of [**13**], Definition 1.17, where pro-$\mathcal{P}$-groups are defined for arbitrary properties $\mathcal{P}$ of topological groups. Let us verify now that a Hausdorff group G is a pro-Lie group (i.e., a pro-$\mathbb{R}$-Lie group) if and only if it is an LP-group in the sense of Lashof [**20**], Definition 3.1, i.e., if and only if G is a projective limit of a projective system $((G_i)_{i\in I},(q_{ij})_{i\leq j})$ of (real) Lie groups such that all bonding maps q_{ij} are quotient morphisms and so are all limit maps $q_i\colon G\to G_i$.

Lemma 4.10. *Let* G *be a Hausdorff group. The following conditions are equivalent*:

(a) G *is a pro-Lie group*;
(b) G *is an LP-group*;
(c) $\mathrm{N}(G):=\{\ker f\colon f\in J_{\mathbb{R}}(G)\}$ *is a filter basis which converges to* 1 *in* G*, and* G *is complete.*

Proof. The implications "(a)⇒(b)" and "(b)⇒(c)" are trivial. If (c) holds, $J_{\mathbb{R}}(G)$ is a directed subset of $I_{\mathbb{R}}(G)$, whence we can form the projective limit $P:=\varprojlim_{f\in J_{\mathbb{R}}(G)} H_f$; let $\pi_f\colon P\to H_f$ be the limit morphism for $f\in J_{\mathbb{R}}(G)$. There is a unique morphism $\mu\colon G\to P$ such that $\pi_f\circ\mu=f$ for all $f\in J_{\mathbb{R}}(G)$. Since $\bigcap \mathrm{N}(G)=\{1\}$, the morphism μ is injective; every $f\in J_{\mathbb{R}}(G)$ being a quotient morphism, we deduce from $\mathrm{N}(G)\to 1$ that μ is an embedding with dense image (cf. [**9**], [**16**]). The isomorphic image being closed by completeness of G, we deduce that μ is a topological isomorphism. This implies that $(G,(f)_{f\in J_{\mathbb{R}}(G)})$ is a projective limit cone over the projective system $((H_f)_{f\in J_{\mathbb{R}}(G)},(\phi_{fg})_{f\leq g})$. We deduce from Lemma 4.9 that $J_{\mathbb{R}}(G)$ is cofinal in $I_{\mathbb{R}}(G)$: Thus G is a pro-Lie group. □

See also [**16**] and [**9**] for characterizations of *locally compact* pro-Lie and pro-p-adic Lie groups, and [**1**] for structural investigations of locally compact pro-Lie groups. We remark that if G is locally compact, $J_{\mathbb{K}}(G)$ is cofinal in $I_{\mathbb{K}}(G)$, as follows directly from [**3**], Chapter III, §8.2, Corollary 1 to Theorem 2. For later use, we recall that a topological group G is called a *residual* $\mathbb{K}$*-Lie group* if $\eta_{\mathbb{K}}^{G}$ is injective.

Note added in proof. At the time of writing of this article, it was an open question whether every projective limit of Lie groups is a pro-Lie group, and our results are formulated accordingly. In the meantime, an affirmative answer to this question was obtained by K. H. Hofmann and S. A. Morris [**15**].

5. Definition of the Lie algebra functors.

In this section, we define 15 real topological Lie algebra functors on the category of Hausdorff groups, and 9 p-adic topological Lie algebra functors. Any of these functors extends the real or p-adic Lie algebra functor on the category of real or p-adic Lie groups, respectively, and the functors are pairwise not naturally isomorphic, as shall be verified in Sections 8 and 12.

5.1. Recall that the Lie algebra functor on the category of real Lie groups is defined via $\mathcal{L}_{\mathbb{R}}(G) := \mathrm{Hom}(\mathbb{R}, G)$, canonically equipped with a real Lie algebra structure, and $\mathcal{L}_{\mathbb{R}}(\psi) := \mathrm{Hom}(\mathbb{R}, \psi)$ for morphisms $\psi\colon G \to H$ between Lie groups. We consider $\mathcal{L}_{\mathbb{R}}$ as a functor into the category of *topological* real Lie algebras, i.e., real Lie algebras equipped with a vector topology making the Lie bracket continuous.

5.2. Similarly, the p-adic Lie algebra of a p-adic Lie group G can be realized as $\mathcal{L}_{\mathbb{Q}_p}(G) := \Gamma(\mathbb{Q}_p, G)$, the set of function germs at 0 of local p-adic one-parameter subgroups of G, i.e., local homomorphisms $\mathbb{Q}_p \to G$. To see this, we may assume that G is a linear Lie group, in view of Ado's Theorem ([**3**], Chapter I, §7.3, Theorem 2) and [**3**], Chapter III, §4.2, Lemma 3 and Theorem 2. Then each local p-adic 1-parameter subgroup X of G satisfies the differential equation $X' = X'(0)X$ and hence, being analytic, is determined on a zero-neighbourhood by its derivative $X'(0)$; and conversely, every tangent vector at the identity element arises in this way. Here, a *local homomorphism* $\mathbb{Q}_p \to G$ is a continuous homomorphism $X : U \to G$, defined on an open subgroup U of $\mathbb{Q}_p$; its function germ at 0 is the equivalence class $[X]$ of all local homomorphisms Y such that $Y \sim X$, where $Y \sim X$ means that X and Y coincide on a neighbourhood of 0. In this picture, we have $\mathcal{L}_{\mathbb{Q}_p}(\psi) = \Gamma(\mathbb{Q}_p, \psi)$ for a morphism $\psi\colon G \to H$ between p-adic Lie groups, where $\Gamma(\mathbb{Q}_p, \psi)\colon \Gamma(\mathbb{Q}_p, G) \to \Gamma(\mathbb{Q}_p, H)$ is defined via $[X] \mapsto [\psi \circ X]$.

5.3. Now let $\mathbb{K}$ be $\mathbb{R}$ or $\mathbb{Q}_p$ for some prime p. Then $\mathrm{L}_{\mathbb{K}} := \varprojlim \circ (\mathcal{L}_{\mathbb{K}})^{\pi} \circ \mathfrak{S}_{\mathbb{K}}$ is a functor from $\mathbb{TG}$ into the category of topological $\mathbb{K}$-Lie algebras. For $G \in \mathbb{TG}$, the topological Lie algebra $\mathrm{L}_{\mathbb{K}}(G)$ is called the (*global*) $\mathbb{K}$-*Lie algebra of* G. Being a closed vector subspace of a product of finite-dimensional vector spaces, $\mathrm{L}_{\mathbb{K}}(G)$ is a weakly complete locally convex space, by Proposition 2.1. If G is a locally compact group, we can repeat the construction above with $I_{\mathbb{K}}(G)$ replaced by its cofinal subset $J_{\mathbb{K}}(G)$ and obtain a functor $\mathrm{L}'_{\mathbb{K}}$ on the category $\mathbb{LCG}$ of locally compact groups which is naturally isomorphic to $\mathrm{L}_{\mathbb{K}}|_{\mathbb{LCG}}$.

Note that $\Gamma(\mathbb{R},\bullet) := \mathrm{Hom}(\mathbb{R},\bullet) : \mathbb{TG} \to \mathbb{SET}$ and $\Gamma(\mathbb{Q}_p,\bullet) : \mathbb{TG} \to \mathbb{SET}$ (defined as above) are functors into the category of sets and maps. For $G \in \mathbb{TG}$, we define $r^G_{\mathbb{K}} \colon \Gamma(\mathbb{K}, G) \to \mathrm{L}_{\mathbb{K}}(G), X \mapsto (\Gamma(\mathbb{K}, f)(X))_{f\in I_{\mathbb{K}}(G)}$. Then we have:

Lemma 5.4. $r_{\mathbb{K}} : \Gamma(\mathbb{K},\bullet) \xrightarrow{\bullet} \mathrm{F}_{\mathbb{K}} \circ \mathrm{L}_{\mathbb{K}}$ *is a natural transformation, where* $\mathrm{F}_{\mathbb{K}}$ *is the forgetful functor from the category of topological* $\mathbb{K}$*-Lie algebras to the category of sets. In other words,* $r^H_{\mathbb{K}} \circ \Gamma(\mathbb{K},\psi) = \mathrm{L}_{\mathbb{K}}(\psi) \circ r^G_{\mathbb{K}}$*, for every morphism* $\psi \colon G \to H$*.*

Proof. In the proof, we use the notation for the case where $\mathbb{K} = \mathbb{Q}_p$; if $\mathbb{K} = \mathbb{R}$, the brackets "[" and "]" are to be ignored. Let $[X] \in \Gamma(\mathbb{K}, G)$. For $f \in I_{\mathbb{K}}(H)$, we let h_f be the corestriction of $f \circ \psi$ to the closure of its image. We let $\sigma(f) \in I_{\mathbb{K}}(G)$ be the unique representative which is equivalent to h_f, and $\lambda_f : \mathrm{im}\, h_f \to H_f$ be the inclusion map. Then the f-coordinate of $\mathrm{L}_{\mathbb{K}}(\psi)(r^G_{\mathbb{K}}([X]))$ is $[\lambda_f \circ \phi_{h_f\sigma(f)} \circ \sigma(f) \circ X] = [f \circ \psi \circ X]$, which coincides with the f-coordinate of $r^H_{\mathbb{K}}(\Gamma(\mathbb{K},\psi)([X]))$. Hence $\mathrm{L}_{\mathbb{K}}(\psi)(r^G_{\mathbb{K}}([X])) = r^H_{\mathbb{K}}(\Gamma(\mathbb{K},\psi)([X]))$ indeed. □

5.5. We can define further topological $\mathbb{K}$-Lie algebra functors as follows: Given a topological group G, we let $\Lambda_{\mathbb{K}}(G)$ be the $\mathbb{K}$-Lie subalgebra of $\mathrm{L}_{\mathbb{K}}(G)$ generated by the image of $r^G_{\mathbb{K}}$, and let $\overline{\Lambda}_{\mathbb{K}}(G)$ be its closure. For every morphism $\Psi \colon G_1 \to G_2$, the continuous Lie algebra homomorphism $\mathrm{L}_{\mathbb{K}}(\Psi)$ maps $\Lambda_{\mathbb{K}}(G_1)$ into $\Lambda_{\mathbb{K}}(G_2)$, as follows from the naturality of $r_{\mathbb{K}}$; the same holds for the closures. Thus by restriction and corestriction of the Lie algebra homomorphisms, we obtain functors $\Lambda_{\mathbb{K}}$ and $\overline{\Lambda}_{\mathbb{K}}$. These are linked by the natural transformations $i_{\mathbb{K}}, j_{\mathbb{K}}$ given by the inclusion morphisms

$$\Lambda_{\mathbb{K}}(G) \overset{i^G_{\mathbb{K}}}{\hookrightarrow} \overline{\Lambda}_{\mathbb{K}}(G) \overset{j^G_{\mathbb{K}}}{\hookrightarrow} \mathrm{L}_{\mathbb{K}}(G).$$

Note that every $\overline{\Lambda}_{\mathbb{K}}(G)$ is weakly complete, being a closed subspace of $\mathrm{L}_{\mathbb{K}}(G)$.

5.6. Since every morphism $\Psi : G \to H$ of topological groups maps the components (and path components) into each other, the assignments $G \mapsto \mathrm{L}^0_{\mathbb{R}}(G) := \mathrm{L}_{\mathbb{R}}(G_0)$ (where G_0 is the identity component of G) and $G \mapsto \mathrm{L}^{\mathrm{arc}}_{\mathbb{R}}(G) := \mathrm{L}_{\mathbb{R}}(G_{\mathrm{arc}})$ (where G_{arc} is the path-component of the identity in G) yield further real topological Lie algebra functors. Similarly, we define $\Lambda^0_{\mathbb{R}} := \Lambda_{\mathbb{R}} \circ (\bullet)_0$, $\overline{\Lambda}^0_{\mathbb{R}} := \overline{\Lambda}_{\mathbb{R}} \circ (\bullet)_0$, $\Lambda^{\mathrm{arc}}_{\mathbb{R}} := \Lambda_{\mathbb{R}} \circ (\bullet)_{\mathrm{arc}}$, and $\overline{\Lambda}^{\mathrm{arc}}_{\mathbb{R}} := \overline{\Lambda}_{\mathbb{R}} \circ (\bullet)_{\mathrm{arc}}$.

5.7. Note that the real Lie algebras defined in 5.6 are unaffected by passage to open subgroups; in contrast, the global real Lie algebra can change (Example 8.3). The same phenomenon occurs in the p-adic case: We shall encounter a totally disconnected, locally compact group G with an open, compact, normal subgroup U such that $\mathrm{L}_{\mathbb{Q}_p}(G) = \{0\}$ but $\mathrm{L}_{\mathbb{Q}_p}(U) \neq \{0\}$ (Example 12.2). In this setting, it would not make sense to pass to the component; still, it is possible to define Lie algebras which are unaffected by passage to open subgroups. To this end, we consider the set $\mathcal{U}(G)$ of open subgroups of a topological group G, which is directed via inverse inclusion; if $U, V \in \mathcal{U}(G)$ and $U \subseteq V$, we let $k_{VU} : U \hookrightarrow V$ denote the inclusion map. Then $\mathcal{R}(G) := ((U)_{U\in\mathcal{U}(G)}, (k_{VU})_{U\subseteq V}) \in \mathbb{TG}^{\pi}$. If $\Phi : G \to L$ is a morphism of topological groups, we set $\tau(U) := \Phi^{-1}(U)$ for $U \in \mathcal{U}(L)$, and $\zeta_U := \Phi|_{\tau(U)}^{U}$. We put $\mathcal{R}(\Phi) := (\tau, (\zeta_U)_{U\in\mathcal{U}(L)})$. It is easy to check that $\mathcal{R}\colon \mathbb{TG} \to \mathbb{TG}^{\pi}$ is a functor. We set

$$\Lambda_{\mathbb{K}}^{\mathrm{loc}} := \varprojlim \circ \Lambda_{\mathbb{K}}^{\pi} \circ \mathcal{R}, \quad \overline{\Lambda}_{\mathbb{K}}^{\mathrm{loc}} := \varprojlim \circ \overline{\Lambda}_{\mathbb{K}}^{\pi} \circ \mathcal{R}, \ \text{ and } \ \mathrm{L}_{\mathbb{K}}^{\mathrm{loc}} := \varprojlim \circ \mathrm{L}_{\mathbb{K}}^{\pi} \circ \mathcal{R}.$$

We call $\mathrm{L}_{\mathbb{K}}^{\mathrm{loc}}(G)$ the *local* $\mathbb{K}$*-Lie algebra of* G. Note that $\overline{\Lambda}_{\mathbb{K}}^{\mathrm{loc}}(G)$ and $\mathrm{L}_{\mathbb{K}}^{\mathrm{loc}}(G)$ are weakly complete locally convex spaces, for every Hausdorff group G. Clearly all maps $\Gamma(\mathbb{Q}_p, k_{GU})$ are bijections, and also the maps $\Gamma(\mathbb{R}, k_{GU})$, since every open subgroup U of G contains the path component of G. Then

$$(\Gamma(\mathbb{K}, G), (r_{\mathbb{K}}^{U} \circ \Gamma(\mathbb{K}, k_{GU})^{-1})_{U\in\mathcal{U}(G)})$$

is a cone over $\mathrm{L}_{\mathbb{K}}^{\pi}(\mathcal{R}(G))$, considered as a projective system in $\mathbb{SET}$, and hence induces a map $\rho_{\mathbb{K}}^{G}\colon \Gamma(\mathbb{K}, G) \to \mathrm{L}_{\mathbb{K}}^{\mathrm{loc}}(G)$.

5.8. There are further interesting subgroups of a topological group G which can be associated in a functorial fashion, for instance the subgroup $c_{\mathbb{R}}(G)$ of G generated by the set $R_{\mathbb{R}}(G) := \{X(1)\colon X \in \mathrm{Hom}(\mathbb{R}, G)\}$ of group elements which lie on one-parameter subgroups of G, and the subgroups $c_{\mathbb{Q}_p}(G)$ of G generated by the set $R_{\mathbb{Q}_p}(G) := \{X(1)\colon X \in \mathrm{Hom}(\mathbb{Z}_p, G)\}$ of elements which lie on p-adic local one-parameter subgroups of G. We call $R_{\mathbb{K}}(G)$ the set of $\mathbb{K}$*-reachable elements of* G, and $c_{\mathbb{K}}(G)$ the *proper* $\mathbb{K}$*-component of* G. The $\mathbb{K}$*-component of* G is defined as $C_{\mathbb{K}}(G) := \overline{c_{\mathbb{K}}(G)}$, the closure of the proper $\mathbb{K}$-component. The (proper) $\mathbb{Q}_p$-component of G is also called its (*proper*) *p-adic component*. Let us say that G is $\mathbb{K}$*-connected*, respectively, *properly* $\mathbb{K}$*-connected*, if it coincides with its $\mathbb{K}$-component and proper $\mathbb{K}$-component, respectively. It is plain that every morphism $\psi\colon G \to H$ between topological groups maps $c_{\mathbb{K}}(G)$ into $c_{\mathbb{K}}(H)$, whence we can make $c_{\mathbb{K}}$ a functor by defining $c_{\mathbb{K}}(\psi) := \psi|_{c_{\mathbb{K}}(G)}^{c_{\mathbb{K}}(H)}$ on morphisms.

Then $\mathrm{L}_{\mathbb{K}}^{\exp} := \mathrm{L}_{\mathbb{K}} \circ c_{\mathbb{K}}$, $\Lambda_{\mathbb{K}}^{\exp} := \Lambda_{\mathbb{K}} \circ c_{\mathbb{K}}$, and $\overline{\Lambda}_{\mathbb{K}}^{\exp} := \overline{\Lambda}_{\mathbb{K}} \circ c_{\mathbb{K}}$ define topological $\mathbb{K}$-Lie algebra functors. Further functors might be obtained by replacing $c_{\mathbb{K}}$ by $C_{\mathbb{R}}$ in the preceding definitions, but these would be naturally isomorphic to the ones just defined, by the following proposition: We therefore refrain from introducing extra terminology.

Proposition 5.9. *Let G be a Hausdorff group, and D be a dense subgroup of G; let $\varepsilon\colon D \hookrightarrow G$ denote the embedding. Then* $\mathrm{L}_{\mathbb{K}}(\varepsilon)\colon \mathrm{L}_{\mathbb{K}}(D) \to \mathrm{L}_{\mathbb{K}}(G)$ *is an isomorphism of topological Lie algebras.*

Proof. Lie groups being complete, every morphism $f \in I_{\mathbb{K}}(D)$ extends to a morphism $\bar{f} : G \to H_f$ with dense image; conversely, the restriction $g|_D$ of $g \in I_{\mathbb{K}}(G)$ to D is a morphism with dense image, and g is uniquely determined by $g|_D$. Hence $\mathfrak{S}_{\mathbb{K}}(\varepsilon)$ is an isomorphism, which implies that so is $\mathrm{L}_{\mathbb{K}}(\varepsilon)$. □

Remark 5.10. It is plain from the above that $c_{\mathbb{K}}(G)$ is a strongly characteristic subgroup of G, i.e., $\psi(c_{\mathbb{K}}(G)) \subseteq c_{\mathbb{K}}(G)$ for every continuous endomorphism ψ of G. Hence every simple topological group G has trivial proper $\mathbb{K}$-component or is properly $\mathbb{K}$-connected. For example, $\mathrm{PSL}_n(\mathbb{Q}_p)$ is properly $\mathbb{Q}_p$-connected, for all $n \in \mathbb{N}$. If S is a simple topological group which contains an element of order p, then S is properly $\mathbb{Q}_p$-connected. For example, $\mathrm{PSL}_2(\mathbb{Q}_p)$ is properly $\mathbb{Q}_2$-connected, for every prime p. Note that $C_{\mathbb{R}}(G) = G_0$ whenever G_0 is a pro-Lie group ([**20**], Theorem 3.5), hence in particular whenever G a pro-Lie group ([**20**], Lemma 3.9) or locally compact [**38**]. It should also be mentioned that the set $\mathrm{Hom}(\mathbb{Q}_p, G)$ of global p-adic one-parameter subgroups of a p-adic Lie group tends to be very small in general; e.g., $\mathrm{Hom}(\mathbb{Q}_p, \mathbb{Z}_p) = \{0\}$, since every morphism $\mathbb{Q}_p \to \mathbb{Q}_p$ is linear. Interesting information on global p-adic one-parameter subgroups of linear groups can be found in [**28**].

It is important to note that all of the functors we have constructed assign the proper $\mathbb{K}$-Lie algebras to $\mathbb{K}$-Lie groups (up to natural isomorphism). This follows from the following proposition:

Proposition 5.11. *Let* $\mathbb{K}, \mathbb{L} \in \{\mathbb{R}\} \cup \{\mathbb{Q}_p : p \text{ prime}\}$. *Then*

$$\Lambda_{\mathbb{L}}(G) = \overline{\Lambda}_{\mathbb{L}}(G) = \mathrm{L}_{\mathbb{L}}(G) \cong \begin{cases} \mathcal{L}_{\mathbb{K}}(G) & \text{if } \mathbb{K} = \mathbb{L} \\ \{0\} & \text{else,} \end{cases}$$

for every $\mathbb{K}$-Lie group G.

Proof. Assume $\mathbb{K} = \mathbb{L}$ first. Then $\mathrm{id} : G \to G$ is an upper bound for $I_{\mathbb{K}}(G)$, hence $\mathrm{L}_{\mathbb{K}}(G) \cong \mathcal{L}_{\mathbb{K}}(G)$; noting that $r^G_{\mathbb{K}}$ is a bijection here, we obtain $\Lambda_{\mathbb{K}}(G) = \overline{\Lambda}_{\mathbb{K}}(G) = \mathrm{L}_{\mathbb{K}}(G)$. If $\mathbb{K} \neq \mathbb{L}$ and $f \in J_{\mathbb{L}}(G)$, the group H_f will be discrete, since f has open kernel by [**3**], Chapter III, §8.1, Proposition 1. Thus $\mathrm{L}_{\mathbb{L}}(G) \cong \mathrm{L}'_{\mathbb{L}}(G) = \{0\}$. □

Section 6 makes it rather clear that $\Lambda^{\exp}_{\mathbb{R}}$ and $\overline{\Lambda}^{\exp}_{\mathbb{R}}$ are the most reasonable real topological Lie algebra functors; in the setting of totally disconnected, locally compact groups, $\Lambda^{\mathrm{loc}}_{\mathbb{Q}_p}$ and $\overline{\Lambda}^{\mathrm{loc}}_{\mathbb{Q}_p}$ seem to be the p-adic topological Lie algebra functors with the most reasonable properties. Figure 1 elucidates the relations between the various functors defined above. In this figure, the arrows describe some of the natural transformations between the corresponding functors which are rather obvious from the above construction. For the sake of completeness, we give some more explanations, for the left diagram, say. The horizontal arrows in rows 1, 2, 3, and 5 (counted from top to bottom) designate the natural transformations $i_{\mathbb{R}} \circ c_{\mathbb{R}}$, $j_{\mathbb{R}} \circ c_{\mathbb{R}}$, $i_{\mathbb{R}} \circ (\,)_{\mathrm{arc}}$, $j_{\mathbb{R}} \circ (\,)_{\mathrm{arc}}$, $i_{\mathbb{R}} \circ (\,)_0$, $j_{\mathbb{R}} \circ (\,)_0$, $i_{\mathbb{R}}$, and $j_{\mathbb{R}}$, respectively. The horizontal arrows in the 4th row are the natural transformations

$$G \mapsto \varprojlim_{U \in \mathcal{U}(G)} i_{\mathbb{R}}^U \quad \text{and} \quad G \mapsto \varprojlim_{U \in \mathcal{U}(G)} j_{\mathbb{R}}^U,$$

respectively. The first and second vertical arrow in the first column depict the natural transformations given by $G \mapsto \Lambda_{\mathbb{R}}(c_{\mathbb{R}}(G) \hookrightarrow G_{\mathrm{arc}})$ and $G \mapsto \Lambda_{\mathbb{R}}(G_{\mathrm{arc}} \hookrightarrow G_0)$, respectively. The third natural transformation is defined as follows: Given a topological group G, we have $G_0 \leq U$ for every $U \in \mathcal{U}(G)$; if $\varepsilon_U : G_0 \to U$ denotes the embedding, the cone $(\Lambda_{\mathbb{R}}^0(G), (\Lambda_{\mathbb{R}}(\varepsilon_U))_{U \in \mathcal{U}(G)})$ induces the natural morphism $\Lambda_{\mathbb{R}}^0(G) \to \Lambda_{\mathbb{R}}^{\mathrm{loc}}(G)$. The lower-most arrow in the first column denotes the natural transform which associates with $G \in \mathbb{TG}$ the limit map $\Lambda_{\mathbb{R}}^{\mathrm{loc}}(G) = \varprojlim \Lambda_{\mathbb{R}}(U) \to \Lambda_{\mathbb{R}}(G)$. The vertical arrows in the middle column and right column are defined analogously. Note that all horizontal arrows correspond to embeddings of topological Lie algebras. All vertical arrows in the first column correspond to surjective morphisms of topological Lie algebras, except for the arrow $\Lambda_{\mathbb{R}}^0 \to \Lambda_{\mathbb{R}}^{\mathrm{loc}}$, which corresponds to morphisms with dense image: These assertions follow easily from the naturality of $r_{\mathbb{K}}$. Thus all vertical arrows in the second column correspond to quotient morphisms of topological Lie algebras, by the preceding observation and Proposition 2.1.

$$\begin{array}{ccccccccccc}
\Lambda_{\mathbb{R}}^{\exp} & \hookrightarrow & \overline{\Lambda}_{\mathbb{R}}^{\exp} & \hookrightarrow & \mathrm{L}_{\mathbb{R}}^{\exp} & & & & & & \\
\downarrow & & \downarrow & & \downarrow & & & & & & \\
\Lambda_{\mathbb{R}}^{\mathrm{arc}} & \hookrightarrow & \overline{\Lambda}_{\mathbb{R}}^{\mathrm{arc}} & \hookrightarrow & \mathrm{L}_{\mathbb{R}}^{\mathrm{arc}} & & & & & & \\
\downarrow & & \downarrow & & \downarrow & & & & & & \\
\Lambda_{\mathbb{R}}^{0} & \hookrightarrow & \overline{\Lambda}_{\mathbb{R}}^{0} & \hookrightarrow & \mathrm{L}_{\mathbb{R}}^{0} & \qquad & \Lambda_{\mathbb{Q}_p}^{\exp} & \hookrightarrow & \overline{\Lambda}_{\mathbb{Q}_p}^{\exp} & \hookrightarrow & \mathrm{L}_{\mathbb{Q}_p}^{\exp} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\Lambda_{\mathbb{R}}^{\mathrm{loc}} & \hookrightarrow & \overline{\Lambda}_{\mathbb{R}}^{\mathrm{loc}} & \hookrightarrow & \mathrm{L}_{\mathbb{R}}^{\mathrm{loc}} & & \Lambda_{\mathbb{Q}_p} & \hookrightarrow & \overline{\Lambda}_{\mathbb{Q}_p} & \hookrightarrow & \mathrm{L}_{\mathbb{Q}_p} \\
\downarrow & & \downarrow & & \downarrow & & \uparrow & & \uparrow & & \uparrow \\
\Lambda_{\mathbb{R}} & \hookrightarrow & \overline{\Lambda}_{\mathbb{R}} & \hookrightarrow & \mathrm{L}_{\mathbb{R}} & & \Lambda_{\mathbb{Q}_p}^{\mathrm{loc}} & \hookrightarrow & \overline{\Lambda}_{\mathbb{Q}_p}^{\mathrm{loc}} & \hookrightarrow & \mathrm{L}_{\mathbb{Q}_p}^{\mathrm{loc}}
\end{array}$$

Figure 1. The various Lie algebra functors and natural transformations between them.

Remark 5.12. The reader might wonder whether it is possible to associate K-Lie algebras to a topological group G for arbitrary local fields K. This is indeed *not* possible by the ideas presented here. The reason for this is that the analytic structure on such a group is an extra information in general, which is not encoded in the topological group structure. For example, take $K = \mathbb{F}_2(\!(X)\!)$ and $G := (K,+) \cong \mathbb{Z}(2)^{(\mathbb{N})} \times \mathbb{Z}(2)^{\mathbb{N}}$ (where $\mathbb{Z}(2)$ denotes the cyclic group of order 2). Then $G \cong G^n$ for every $n \in \mathbb{N}$, and we cannot hope to recover the Lie algebra K from the topological group G.

6. Behaviour of the Lie algebra functors on special classes of topological groups.

In this section, we have a closer look at several classes of topological groups, and study the behaviour of the Lie algebra functors on these. In particular, we describe and study a class $\mathbb{CL}_{\mathbb{K}}$ of topological groups whose Lie algebras $\Lambda_{\mathbb{K}}^{\exp}(G)$ can be used to make $\Gamma(\mathbb{K}\,, G)$ a topological Lie algebra $\lambda_{\mathbb{K}}(G)$ (where $\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{Q}_p$). In doing so, we achieve the first goal formulated in the Introduction: We use one of our topological Lie algebra functors to turn the set $\mathrm{Hom}(\mathbb{R}, G)$ of one-parameter subgroups into a topological Lie algebra, for rather general topological groups G.

In the real case, various positive results can be obtained. We show that $F \circ \lambda_{\mathbb{R}}$ is a continuous functor, where F is the forgetful functor from the category of topological real Lie algebras to the category of abstract real Lie algebras. On the subcategory $\mathbb{GPL}_{\mathbb{R}}$ of $\mathbb{CL}_{\mathbb{R}}$, all of the 15 real topological Lie algebra functors defined above coincide (up to natural isomorphism), and these functors are continuous on $\mathbb{GPL}_{\mathbb{R}}$. It is also shown that the functor $\mathrm{L}_{\mathbb{R}}|_{\mathcal{V}(\mathbb{LIE}_{\mathbb{R}})}$ maps embeddings to embeddings.

Most of the results obtained in the real case do not have analogues in the p-adic case, as the examples show. Several differences between the real and p-adic cases are recorded (in accordance with the third goal formulated in the Introduction). The most severe difference is not specific to the p-adic Lie algebra functors defined here: Any p-adic Lie algebra functor which extends the usual p-adic Lie algebra functor is discontinuous.

At the end of this section, we briefly clarify the relation of our construction to Lashof's Lie algebra functor, and to other classical constructions.

6.1. Categories of well-behaved topological groups. In Section 4, we already encountered the class $\mathbb{GPL}_{\mathbb{R}}$ of projective limits of Lie groups and its subclass $\mathbb{PL}_{\mathbb{R}}$ of pro-Lie groups (as well as p-adic analogues). It turns out that $\mathrm{Hom}(\mathbb{R}, G)$ can be made a Lie algebra for more general classes of topological groups, which we introduce now.

Definition 6.1. We say that a topological group G *carries its* $\mathbb{K}$*-Lie algebra* if $s_{\mathbb{K}}^G := r_{\mathbb{K}}^{c_{\mathbb{K}}(G)}|^{\Lambda_{\mathbb{K}}^{\exp}(G)}$ is a bijection; we use this bijection to make $\Gamma(\mathbb{K}, G) = \Gamma(\mathbb{K}\,, c_{\mathbb{K}}(G))$ a topological Lie algebra $\lambda_{\mathbb{K}}(G)$. We let $\mathbb{CL}_{\mathbb{K}}$ be the full subcategory of $\mathbb{TG}$ whose objects are the groups which carry their $\mathbb{K}$-Lie algebras. If $\psi\colon G \to H$ is a morphism in $\mathbb{CL}_{\mathbb{K}}$, we set $\lambda_{\mathbb{K}}(\psi) := \Gamma(\mathbb{K}, \psi)$; it is immediate from Lemma 5.4 that $\lambda_{\mathbb{K}}(\psi)$ is a continuous Lie algebra homomorphism, that $\lambda_{\mathbb{K}}$ is a functor, and that $s_{\mathbb{K}} : \lambda_{\mathbb{K}} \xrightarrow{\bullet} \Lambda^{\exp}_{\mathbb{K}}|_{\mathbb{CL}_{\mathbb{K}}}$ is a natural isomorphism. We call a residual (real) Lie group G *special* if $r_{\mathbb{R}}^{c_{\mathbb{R}}(G)}$ has image $\Lambda^{\exp}_{\mathbb{R}}(G)$, and let $\mathbb{SRL}_{\mathbb{R}}$ be the full subcategory of $\mathbb{TG}$ whose objects are the special residual Lie groups. Finally, we let $\mathbb{PL}^0_{\mathbb{R}}$ be the full subcategory of $\mathbb{TG}$ whose objects are those Hausdorff groups whose identity components are pro-Lie groups (the class of groups dealt with in Lashof [**20**]).

Let us have a look at examples now, and clarify the relations between the various classes of topological groups defined so far. The knowledge of the Lie algebras of some of these simple examples will be exploited later when we determine the Lie algebras of more complicated topological groups. Several basic differences between the real and p-adic cases will already become apparent.

Recall that $\mathrm{Hom}(\mathbb{R}, A)_{\text{c.o.}}$, equipped with compact-open topology and pointwise addition, is a real topological vector space, for every abelian topological group A ([**14**], p. 334 ff); furthermore, $\mathrm{Hom}(\mathbb{R}, \bullet)_{\text{c.o.}}$ is a functor from the category $\mathbb{TAB}$ of abelian topological Hausdorff groups into the category of (abelian) topological Lie algebras over $\mathbb{R}$. If V is a topological vector space, we let V' denote its topological dual space (the space of continuous linear functionals), and V^* its algebraic dual space, the space of all linear functionals on V.

Proposition 6.2. *Let $\mathbb{K}, \mathbb{L} \in \{\mathbb{Q}_p : p \text{ prime}\} \cup \{\mathbb{R}\}$. Then the following holds:*

(a) *Every $\mathbb{K}$-Lie group G carries its $\mathbb{L}$-Lie algebra.*
(b) *If $G_1, \ldots, G_n$ are Hausdorff groups which carry their $\mathbb{K}$-Lie algebras, then so does their cartesian product $G = \prod_{i=1}^n G_i$.*
(c) *There is a sequence $(G_n)_{n\in\mathbb{N}}$ of finite groups whose cartesian product G does not carry its p-adic Lie algebra for any prime p.*
(d) *The natural map $u^A_{\mathbb{R}} : \mathrm{Hom}(\mathbb{R}, A)_{\text{c.o.}} \to \Lambda_{\mathbb{R}}(A)$ induced by $r^A_{\mathbb{R}}$ is a surjective morphism of abelian topological real Lie algebras over $\mathbb{R}$, for every abelian topological group A. The mapping $u^A_{\mathbb{R}}$ is injective if A is a residual Lie group, which holds if and only if the dual group $\widehat{A}$ separates points on A. Whenever $u^A_{\mathbb{R}}$ is injective, A carries its real Lie algebra. If A is a locally compact abelian group, then $u^A_{\mathbb{R}}$ is an isomorphism of topological vector spaces.*
(e) *Every real locally convex space V is a special residual Lie group, and $\Lambda_{\mathbb{R}}(V) \cong V_w$, where V_w denotes V, equipped with its weak topology $\sigma(V, V')$. Furthermore, $\overline{\Lambda}_{\mathbb{R}}(V) = \mathrm{L}_{\mathbb{R}}(V) \cong (V')^*$, equipped with the weak $*$-topology. The locally convex space V is a projective limit of Lie*

groups if and only if it is weakly complete, in which case it is a pro-Lie group.

(f) *If G is a locally compact group such that G/G_0 is compact, then G is a pro-Lie group. In particular, every compact group is a pro-Lie group, and every connected locally compact group is a pro-Lie group. Also every locally compact abelian group is a pro-Lie group.*

(g) $\mathbb{PL}_\mathbb{R} \subseteq \mathbb{GPL}_\mathbb{R} \subset \mathbb{SRL}_\mathbb{R} \subset \mathbb{CL}_\mathbb{R}$ *holds, and* $\mathbb{PL}_\mathbb{R} \subset \mathbb{PL}^0_\mathbb{R} \subset \mathbb{CL}_\mathbb{R}$*; here* $\Lambda_\mathbb{R}(G) = \overline{\Lambda}_\mathbb{R}(G) = \mathrm{L}_\mathbb{R}(G)$*, for every* $G \in \mathbb{GPL}_\mathbb{R}$*, and the mapping* $r^G_\mathbb{R}$ *is a bijection. In contrast,* $\mathbb{PL}_{\mathbb{Q}_p} \not\subseteq \mathbb{CL}_{\mathbb{Q}_p}$*, for every prime p, and the mapping* $r^G_{\mathbb{Q}_p}$ *neither needs to be injective nor surjective for a pro-p-adic Lie group G, not even if G is compact.*

Proof. (a) If p, q are primes, and G is a p-adic Lie group, then G carries its q-adic Lie algebra. To see this, assume $p = q$ first. Since $c_{\mathbb{Q}_p}(G) = C_{\mathbb{Q}_p}(G)$ is a p-adic Lie group, we may assume that G is $\mathbb{Q}_p$-connected. Now $\mathrm{id}\colon G \to G$ is an upper bound for $I_{\mathbb{Q}_p}(G)$, whence $\mathrm{L}_{\mathbb{Q}_p}(G) = \Lambda_{\mathbb{Q}_p}(G) \cong \mathcal{L}_{\mathbb{Q}_p}(G)$, where $\mathcal{L}_{\mathbb{Q}_p}(G) = \Gamma(\mathbb{Q}_p, G)$ as a set and the preceding isomorphism is $r^G_{\mathbb{Q}_p}$ as a map. If $p \neq q$, we have $\Gamma(\mathbb{Q}_q, G) = \{[1]\}$ since every morphism $\mathbb{Z}_q \to G$ is locally constant. Since also every morphism from the p-adic Lie group $C_{\mathbb{Q}_q}(G)$ into q-adic Lie groups has open kernel, we have $\Lambda^{\exp}_{\mathbb{Q}_q}(G) = \Lambda_{\mathbb{Q}_q}(c_{\mathbb{Q}_q}(G)) \cong \Lambda_{\mathbb{Q}_q}(C_{\mathbb{Q}_q}(G)) = \{0\}$ (Proposition 5.11). The remaining cases can be proved along the same lines.

(b) It suffices to show that a direct product G of two Hausdorff groups G_1 and G_2 carries its $\mathbb{K}$-Lie algebra if G_1 and G_2 do so. The notation in the following proof is adapted to the p-adic case; if $\mathbb{K} = \mathbb{R}$, all brackets "[" and "]" are to be ignored. Since $c_\mathbb{K}(G_1 \times G_2) = c_\mathbb{K}(G_1) \times c_\mathbb{K}(G_2)$, we may assume that G_1 and G_2 are properly $\mathbb{K}$-connected. For $i \in \{1, 2\}$, we let $\mathrm{pr}_i : G_1 \times G_2 \to G_i$ and $\mathrm{Pr}_i : \Gamma(\mathbb{K}, G_1) \times \Gamma(\mathbb{K}, G_2) \to \Gamma(\mathbb{K}, G_i)$ denote the respective canonical projections, and $\varepsilon_i : G_i \hookrightarrow G_1 \times G_2$ the canonical embedding $x \mapsto (x, 1)$, resp., $y \mapsto (1, y)$. It is plain that the mapping

$$h := (\Gamma(\mathbb{K}, \mathrm{pr}_1), \Gamma(\mathbb{K}, \mathrm{pr}_2))\colon\ \begin{array}{rcl} \Gamma(\mathbb{K}, G) & \to & \Gamma(\mathbb{K}, G_1) \times \Gamma(\mathbb{K}, G_2), \\ {[X]} & \mapsto & ([\mathrm{pr}_1 \circ X], [\mathrm{pr}_2 \circ X]) \end{array}$$

is a bijection, with inverse $([X_1], [X_2]) \mapsto [(X_1, X_2)]$ (where we choose representatives X_1 and X_2 with a common domain of definition in the p-adic case). Note that $\mathrm{L}_\mathbb{K}(\mathrm{pr}_i) \circ r^G_\mathbb{K} \circ h^{-1} = r^{G_i}_\mathbb{K} \circ \Gamma(\mathbb{K}, \mathrm{pr}_i) \circ h^{-1} = r^{G_i}_\mathbb{K} \circ \mathrm{Pr}_i$ for $i \in \{1, 2\}$. Here $r^{G_1}_\mathbb{K}$ and $r^{G_2}_\mathbb{K}$ are injective, since G_1 and G_2 are properly $\mathbb{K}$-connected and carry their $\mathbb{K}$-Lie algebras: We therefore deduce that $(\mathrm{L}_\mathbb{K}(\mathrm{pr}_1), \mathrm{L}_\mathbb{K}(\mathrm{pr}_2)) \circ r^G_\mathbb{K}$ is injective, whence so is $r^G_\mathbb{K}$. We use the bijection h to transport the $\mathbb{K}$-Lie algebra structure of $\lambda_\mathbb{K}(\mathbb{K}, G_1) \times \lambda_\mathbb{K}(\mathbb{K}, G_2)$ (which is $\Gamma(\mathbb{K}, G_1) \times \Gamma(\mathbb{K}, G_2)$ as a set) to $\Gamma(\mathbb{K}, G)$. If we can show that this structure makes $r^G_\mathbb{K}$ a Lie algebra homomorphism, its image will be a Lie algebra, hence coincide with $\Lambda_\mathbb{K}(G)$. To this end, suppose that $f \in I_\mathbb{K}(G)$.

Let $\phi_f \colon \mathrm{L}_{\mathbb{K}}(G) \to \mathcal{L}_{\mathbb{K}}(H_f)$ denote the restriction of the canonical projection $\prod_{g \in I_{\mathbb{K}}(G)} \mathcal{L}_{\mathbb{K}}(H_g) \to \mathcal{L}_{\mathbb{K}}(H_f)$. Then

$$\begin{aligned}(\phi_f \circ r_{\mathbb{K}}^G \circ h^{-1})([X_1],[X_2]) &= [f \circ (X_1, X_2)] \\ &= [(f \circ \varepsilon_1 \circ X_1) \cdot (f \circ \varepsilon_2 \circ X_2)] \\ &= [f \circ \varepsilon_1 \circ X_1] + [f \circ \varepsilon_2 \circ X_2],\end{aligned}$$

where the dot in the second line denotes the pointwise product, and the last equality holds by the Trotter Product Formula (see [**3**], §3.3, Proposition 4), since $\operatorname{im} f \circ \varepsilon_1$ and $\operatorname{im} f \circ \varepsilon_2$ centralize each other. Thus

$$\phi_f \circ r_{\mathbb{K}}^G \circ h^{-1} = \phi_f \circ \mathrm{L}_{\mathbb{K}}(\varepsilon_1) \circ r_{\mathbb{K}}^{G_1} \circ \mathrm{Pr}_1 + \phi_f \circ \mathrm{L}_{\mathbb{K}}(\varepsilon_2) \circ r_{\mathbb{K}}^{G_2} \circ \mathrm{Pr}_2$$

is a Lie algebra homomorphism $\lambda_{\mathbb{K}}(G_1) \times \lambda_{\mathbb{K}}(G_2) \to \mathcal{L}_{\mathbb{K}}(H_f)$, being a sum of two Lie algebra homomorphisms whose images centralize each other. Since f was arbitrary, we conclude that $r_{\mathbb{K}}^G \circ h^{-1}$ is a Lie algebra homomorphism into the projective limit Lie algebra $\mathrm{L}_{\mathbb{K}}(G)$. Hence so is $r_{\mathbb{K}}^G$.

(c) Choose a bijection $\mathbb{N} \to \mathbb{P}$, $i \mapsto p_i$ onto the set $\mathbb{P}$ of all primes; given $i \in \mathbb{N}$, set $G_i := \mathbb{Z}(p_1^i) \times \cdots \times \mathbb{Z}(p_i^i)$, where $\mathbb{Z}(m)$ denotes the cyclic group of order $m \in \mathbb{N}$. For any prime p, we can embed $\mathbb{Z}_p$ into $G := \prod_{i \in \mathbb{N}} G_i$: Thus $\Gamma(\mathbb{Q}_p, G) \neq [0]$. Being compact and totally disconnected, G is a pro-p-adic Lie group. Fix $p \in \mathbb{P}$, $p = p_j$, say. It is easy to see that $c_{\mathbb{Q}_p}(G) = C_{\mathbb{Q}_p}(G) = \prod_{i \geq j} \mathbb{Z}(p^i)$. Now every $f \in J_{\mathbb{Q}_p}(C_{\mathbb{Q}_p}(G))$ has open kernel, whence H_f is discrete; therefore $\mathrm{L}_{\mathbb{Q}_p}^{\exp}(G) = \{0\}$ and $r_{\mathbb{Q}_p}^{C_{\mathbb{Q}_p}(G)}$ is not injective, whence G does not carry its p-adic Lie algebra. Indeed, let U be an open, compact Campbell-Hausdorff subgroup in H_f, i.e., an open compact subgroup isomorphic to a $\mathbb{Z}_p$-submodule of the Lie algebra of H_f, equipped with the Campbell-Hausdorff multiplication (cf. [**3**], §4.2, Lemma 3 and Theorem 2). Then U is torsion-free. Now $V := f^{-1}(U)$ is an open subgroup of $C_{\mathbb{Q}_p}(G)$; since $\operatorname{tor}(C_{\mathbb{Q}_p}(G))$ contains $\bigoplus_{i \geq j} \mathbb{Z}(p^i)$, it is dense in $C_{\mathbb{Q}_p}(G)$, whence $V \cap \operatorname{tor} C_{\mathbb{Q}_p}(G) \leq \ker f$ is dense in V. Thus $V \leq \ker f$, using that $\ker f$ is closed.

(d) Let A be an abelian topological group, and $f \in I_{\mathbb{R}}(A)$. Then H_f is an abelian Lie group; the compact-open topology on $\operatorname{Hom}(\mathbb{R}, H_f) = \mathcal{L}_{\mathbb{R}}(H_f)$ is the Hausdorff vector topology. All mappings $\operatorname{Hom}(\mathbb{R}, f)_{\text{c.o.}} : \operatorname{Hom}(\mathbb{R}, A)_{\text{c.o.}} \to \operatorname{Hom}(\mathbb{R}, H_f)_{\text{c.o.}}$ being morphisms of topological vector spaces, so is $u_{\mathbb{R}}^A : \operatorname{Hom}(\mathbb{R}, A)_{\text{c.o.}} \to \Lambda_{\mathbb{R}}(A)$, $X \mapsto (\operatorname{Hom}(\mathbb{R}, f)(X))_{f \in I_{\mathbb{R}}(A)}$. Thus $\operatorname{im} u_{\mathbb{R}}^A = \operatorname{im} r_{\mathbb{R}}^A = \Lambda_{\mathbb{R}}(A)$ is an abelian topological Lie algebra indeed. Clearly A is a residual Lie group if the characters separate points on A. Conversely, if A is a residual Lie group and $f \in I_{\mathbb{R}}(A)$, then H_f is an abelian Lie group and hence is a locally compact, abelian group, whose characters separate points on H_f: We easily deduce that the characters of A separate points on A. Now if A is a locally compact abelian group, then the characters separate points on A, whence A is a residual Lie group. By the preceding, $u^A_{\mathbb{R}}$ is a continuous linear bijection. By [**14**], Theorem 7.66 (i), the topological vector space $\mathrm{Hom}(\mathbb{R}, G)_{\text{c.o.}}$ is weakly complete; since also $\overline{\Lambda}_{\mathbb{R}}(A)$ is weakly complete, we deduce from Proposition 2.1 that $u^A_{\mathbb{R}}$ is an isomorphism of topological vector spaces, and $\Lambda_{\mathbb{R}}(A) = \overline{\Lambda}_{\mathbb{R}}(A)$.

(e) Let V be a real locally convex vector space; then $V \to \mathrm{Hom}(\mathbb{R}, V)_{\text{c.o.}}$, $v \mapsto (t \mapsto tv)$ is an isomorphism of topological vector spaces, with inverse $\exp_V : \mathrm{Hom}(\mathbb{R}, V)_{\text{c.o.}} \to V$, $X \mapsto X(1)$. The Hahn-Banach Theorem entails that V is a residual Lie group; we deduce with Part (d) that $V \cong \Lambda_{\mathbb{R}}(V)$ as a vector space. Next, we use that V is connected and locally path connected. If $f : V \to H_f$ is a morphism in $I_{\mathbb{R}}(V)$, the density of $f(V)$ in H_f implies that H_f is connected. Let $\widetilde{H}_f$ be a universal covering Lie group of H_f, with universal covering morphism $c\colon \widetilde{H}_f \to H_f$; there exists a unique morphism $g\colon V \to \widetilde{H}_f$ such that $c \circ g = f$. Note that H_f is abelian, since f has dense image. Being an abelian simply connected Lie group, $\widetilde{H}_f$ is isomorphic to $\mathbb{R}^n$ for some n. This implies that g is linear, and now $h := g|^{\mathrm{im}\, g}$ is a linear map onto the finite dimensional real vector space $\mathrm{im}\, h$; note that any such map is a quotient morphism. We have shown that the set K of linear quotient maps onto finite-dimensional real vector spaces is cofinal in $I_{\mathbb{R}}(V)$. If $\phi_f \colon \mathrm{L}_{\mathbb{R}}(V) \to \mathcal{L}_{\mathbb{R}}(H_f)$ denotes the limit map for $f \in I_{\mathbb{R}}(V)$, we have

$$\exp_{H_f} \circ \phi_f \circ r^V_{\mathbb{R}} \circ \exp_V^{-1} = f \tag{2}$$

for $f \in K$, where $\exp_{H_f}$ is an isomorphism; the topology on $\Lambda_{\mathbb{R}}(V)$ is the initial topology with respect to the family $(\phi_f|_{\Lambda_{\mathbb{R}}(V)})_{f \in K}$. We easily deduce that the topology on V which makes $s^V_{\mathbb{R}} \circ \exp_V^{-1} = r^V_{\mathbb{R}}|^{\Lambda_{\mathbb{R}}(V)} \circ \exp_V^{-1}$ an isomorphism of topological vector spaces coincides with the weak topology on V. It follows from Equation (2) that the mappings $\phi_f \circ r^V_{\mathbb{R}}$ are surjective, for every $f \in K$; this implies that $\Lambda_{\mathbb{R}}(V) = \mathrm{im}\, r^V_{\mathbb{R}}$ is dense in $\mathrm{L}_{\mathbb{R}}(V)$, whence $\overline{\Lambda}_{\mathbb{R}}(V) = \mathrm{L}_{\mathbb{R}}(V)$. Since $(V')^*$ is the completion of V_w, we obtain $\mathrm{L}_{\mathbb{R}}(V) \cong (V')^*$. It remains to observe that if $\eta^V_{\mathbb{R}}$ is an isomorphism, then V is isomorphic to a subspace of a product of finite-dimensional spaces and therefore is weakly complete. The converse is clear.

(f) See [**24**], p. 175, and [**14**], Corollary 7.54.

(g) Every pro-Lie group $G \in \mathbb{PL}_{\mathbb{R}}$ is a projective limit of Lie groups. Let G be a $\mathbb{TG}$-projective limit of Lie groups now; then $(G, (f)_{f \in I_{\mathbb{R}}(G)})$ is a projective limit cone over $\mathfrak{S}_{\mathbb{R}}(G)$, as follows from Proposition 4.7 (a). Given $(X_f)_{f \in I_{\mathbb{R}}(G)} \in \mathrm{L}_{\mathbb{R}}(G)$, we have $X_f = \mathcal{L}_{\mathbb{R}}(\phi_{fg}).X_g = \phi_{fg} \circ X_g$ for all $f, g \in I_{\mathbb{R}}(G)$ such that $f \leq g$: Therefore $(\mathbb{R}, (X_f)_{f \in I_{\mathbb{R}}(G)})$ is a cone over $\mathfrak{S}_{\mathbb{R}}(G)$, which induces a unique morphism $X : \mathbb{R} \to G$ such that $f \circ X = X_f$ for all $f \in I_{\mathbb{R}}(G)$. Thus $r^G_{\mathbb{R}} : \mathrm{Hom}(\mathbb{R}, G) \to \mathrm{L}_{\mathbb{R}}(G)$ is a bijection, from which we deduce that $\Lambda_{\mathbb{R}}(G) = \overline{\Lambda}_{\mathbb{R}}(G) = \mathrm{L}_{\mathbb{R}}(G)$ indeed. Note that G is a residual Lie group, since $I_{\mathbb{R}}(G)$ separates points on G. Since $C_{\mathbb{R}}(G)$, being a closed subgroup of G, is a projective limit of Lie groups (Lemma 4.6), $r_{\mathbb{R}}^{C_{\mathbb{R}}(G)}$ is a bijection. Noting that the natural map $\mathrm{L}_{\mathbb{R}}(c_{\mathbb{R}}(G)) \to \mathrm{L}_{\mathbb{R}}(C_{\mathbb{R}}(G))$ is an isomorphism of topological Lie algebras and using the naturality of $r_{\mathbb{R}}$, we deduce that $r_{\mathbb{R}}^{c_{\mathbb{R}}(G)}$ is a bijection. Thus G carries its real Lie algebra and is a special residual Lie group. Let G be a special residual Lie group now. Then $c_{\mathbb{R}}(G)$ is a residual Lie group as well: Thus $I_{\mathbb{R}}(c_{\mathbb{R}}(G))$ separates points on $c_{\mathbb{R}}(G)$, which implies that $r_{\mathbb{R}}^{c_{\mathbb{R}}(G)}$ is injective. By the definition of special residual Lie groups, $r_{\mathbb{R}}^{c_{\mathbb{R}}(G)}$ has image $\Lambda_{\mathbb{R}}(G)$. Thus G carries its real Lie algebra. Thus $\mathbb{PL}_{\mathbb{R}} \subseteq \mathbb{GPL}_{\mathbb{R}} \subseteq \mathbb{SRL}_{\mathbb{R}} \subseteq \mathbb{CL}_{\mathbb{R}}$, and then obviously also $\mathbb{PL}^0_{\mathbb{R}} \subseteq \mathbb{CL}_{\mathbb{R}}$; [**20**], Lemma 3.9 shows $\mathbb{PL}_{\mathbb{R}} \subseteq \mathbb{PL}^0_{\mathbb{R}}$. Let us show now that the containments just proved are strict. We consider the simple p-adic Lie group $S := \mathrm{PSL}_2(\mathbb{Q}_p)$ (for any prime p), and set $G := \mathbb{R} \times S$. Then G is not a residual Lie group, since every morphism into a real Lie group has S in its kernel. However, G carries its real Lie algebra, since $c_{\mathbb{R}}(G) = \mathbb{R}$. Thus $\mathbb{SRL}_{\mathbb{R}} \subset \mathbb{CL}_{\mathbb{R}}$. Furthermore, $G \in \mathbb{PL}^0_{\mathbb{R}}$ but $G \notin \mathbb{PL}_{\mathbb{R}}$; thus $\mathbb{PL}_{\mathbb{R}} \subset \mathbb{PL}^0_{\mathbb{R}}$. Next, consider any locally convex real vector space V which is not complete. Then $V \in \mathbb{SRL}_{\mathbb{R}} \subseteq \mathbb{CL}_{\mathbb{R}}$ but $V \notin \mathbb{GPL}_{\mathbb{R}}$ and $V \notin \mathbb{PL}^0_{\mathbb{R}}$, by (e). Thus $\mathbb{GPL}_{\mathbb{R}} \subset \mathbb{SRL}_{\mathbb{R}}$ and $\mathbb{PL}^0_{\mathbb{R}} \subset \mathbb{CL}_{\mathbb{R}}$.

The group G constructed in the proof of (c) is a totally disconnected, compact group and therefore pro-finite, thus a pro-p-adic Lie group for every prime p. However, as observed above, it does not carry its p-adic Lie algebra: The mapping $r^G_{\mathbb{Q}_p}$ is not injective for this group. The Hausdorff group G constructed in Example 7.5 below is a compact, totally disconnected group, and hence a pro-p-adic Lie group; the mapping $r^G_{\mathbb{Q}_p}$ is not surjective for this group. □

6.2. Behaviour of the real Lie algebra functors on these categories. In this subsection, we study the behaviour of the real topological Lie algebra functors on the categories $\mathbb{GPL}_{\mathbb{R}}$, $\mathbb{PL}^0_{\mathbb{R}}$, $\mathbb{CL}_{\mathbb{R}}$, and on the variety $\mathcal{V}(\mathbb{LIE}_{\mathbb{R}})$ of Hausdorff groups generated by the class of real Lie groups.

Let us have a closer look at the category $\mathbb{GPL}_{\mathbb{R}}$ of $\mathbb{TG}$-projective limits of Lie groups first. We show that $\lambda_{\mathbb{R}}|_{\mathbb{GPL}_{\mathbb{R}}}$ is a continuous functor,[2] and we show that all of the 15 real topological Lie algebra functors defined above coincide on $\mathbb{GPL}_{\mathbb{R}}$ (up to natural isomorphism).

Proposition 6.3. *For every $\mathbb{TG}$-projective limit G of real Lie groups, we have* $\lambda_{\mathbb{R}}(G) \cong \Lambda^{\exp}_{\mathbb{R}}(G) = \overline{\Lambda}^{\exp}_{\mathbb{R}}(G) = \mathrm{L}^{\exp}_{\mathbb{R}}(G) \cong \Lambda^{\mathrm{arc}}_{\mathbb{R}}(G) = \overline{\Lambda}^{\mathrm{arc}}_{\mathbb{R}}(G) = \mathrm{L}^{\mathrm{arc}}_{\mathbb{R}}(G) \cong \Lambda^0_{\mathbb{R}}(G) = \overline{\Lambda}^0_{\mathbb{R}}(G) = \mathrm{L}^0_{\mathbb{R}}(G) \cong \Lambda^{\mathrm{loc}}_{\mathbb{R}}(G) = \overline{\Lambda}^{\mathrm{loc}}_{\mathbb{R}}(G) = \mathrm{L}^{\mathrm{loc}}_{\mathbb{R}}(G) \cong \Lambda_{\mathbb{R}}(G) = \overline{\Lambda}_{\mathbb{R}}(G) = \mathrm{L}_{\mathbb{R}}(G)$. *Here, the first isomorphism is $s^G_{\mathbb{R}}$, and the other*

[2] Recall that a functor is called *continuous* if it respects limits of small diagrams.

isomorphisms are the natural morphisms described in Figure 1. *The real topological Lie algebra functor $\lambda_{\mathbb{R}}|_{\mathbb{GPL}_{\mathbb{R}}}$ is continuous and extends the real topological Lie algebra functor $\mathcal{L}_{\mathbb{R}}$ on the category of real Lie groups; up to natural isomorphism, it is uniquely determined by these properties. In particular, $\lambda_{\mathbb{R}}|_{\mathbb{GPL}_{\mathbb{R}}}$ respects direct products and projective limits.*

Proof. The subgroups G_0, $C_{\mathbb{R}}(G)$, and $\overline{G_{\mathrm{arc}}}$ of G, as well as any open subgroup U of G, are closed subgroups of G and therefore are $\mathbb{TG}$-projective limits of Lie groups (Corollary 4.8). We have $\mathrm{Hom}(\mathbb{R}, c_{\mathbb{R}}(G)) = \mathrm{Hom}(\mathbb{R}, G_{\mathrm{arc}}) = \mathrm{Hom}(\mathbb{R}, C_{\mathbb{R}}(G)) = \mathrm{Hom}(\mathbb{R}, G_0) = \mathrm{Hom}(\mathbb{R}, U) = \mathrm{Hom}(\mathbb{R}, G)$; using that $r_{\mathbb{R}}^P$ is a bijection for every projective limit of Lie groups P, the naturality of $r_{\mathbb{R}}$, Proposition 5.9, and Proposition 2.1, we easily deduce that all of the natural morphisms between the topological Lie algebras associated with G are isomorphisms of topological Lie algebras.

Now suppose that S is a small category (i.e., a category whose class of objects is a set), and $F : S \to \mathbb{GPL}_{\mathbb{R}}$ a functor; let us write $G_i := F(i)$ and $F_\phi := F(\phi) : G_i \to G_j$ for $i, j \in \mathrm{ob}\, S$, $\phi \in \mathrm{Hom}(i, j)$. Suppose that $G \in \mathbb{GPL}_{\mathbb{R}}$ is a limit in $\mathbb{GPL}_{\mathbb{R}}$ of the functor F, with limit maps $\pi_i : G \to G_i$ for $i \in \mathrm{ob}\, S$ (cf. [**21**], Chapter III, §4).[3] We claim that $(\lambda_{\mathbb{R}}(G), (\lambda_{\mathbb{R}}(\pi_i))_{i \in \mathrm{ob}\, S})$ is a limit cone over the topological Lie algebra functor $\lambda_{\mathbb{R}} \circ F$. To see this, note that there is a unique continuous Lie algebra homomorphism $\Phi \colon \lambda_{\mathbb{R}}(G) \to \lim \lambda_{\mathbb{R}}(G_i) := \{(X_i)_{i \in \mathrm{ob}\, S} \in \prod_{i \in \mathrm{ob}\, S} \lambda_{\mathbb{R}}(G_i) \colon \lambda_{\mathbb{R}}(F_\phi)(X_i) = X_j \text{ for all } i, j \in \mathrm{ob}\, S \text{ and } \phi \in \mathrm{Hom}(i, j)\}$ such that $\Pi_j \circ \Phi = \lambda_{\mathbb{R}}(\pi_j)$ for all $j \in \mathrm{ob}\, S$, where $\Pi_j : \lim \lambda_{\mathbb{R}}(G_i) \to \lambda_{\mathbb{R}}(G_j)$, $(X_i) \mapsto X_j$ is the limit map. Now, for every $(X_i)_{i \in \mathrm{ob}\, S} \in \lim \lambda_{\mathbb{R}}(G_i)$, we obtain a cone $(\mathbb{R}, (X_i)_{i \in \mathrm{ob}\, S})$ over F; by the universal property of G, there is a unique continuous homomorphism $X \colon \mathbb{R} \to G$ such that $\lambda_{\mathbb{R}}(\pi_i)(X) = \pi_i \circ X = X_i$ for all $i \in \mathrm{ob}\, S$. The latter condition being equivalent to $\Phi(X) = (X_i)_{i \in \mathrm{ob}\, S}$, we deduce that Φ is a bijection. As any continuous linear bijection between weakly complete spaces, Φ is a topological isomorphism. $\square$

As a special case, we obtain:

Corollary 6.4. $\Lambda_{\mathbb{R}}^{\exp}(G) = \overline{\Lambda}_{\mathbb{R}}^{\exp}(G) = \mathrm{L}_{\mathbb{R}}^{\exp}(G) \cong \Lambda_{\mathbb{R}}^{\mathrm{arc}}(G) = \overline{\Lambda}_{\mathbb{R}}^{\mathrm{arc}}(G) = \mathrm{L}_{\mathbb{R}}^{\mathrm{arc}}(G) \cong \Lambda_{\mathbb{R}}^{0}(G) = \overline{\Lambda}_{\mathbb{R}}^{0}(G) = \mathrm{L}_{\mathbb{R}}^{0}(G)$ *holds, for every $G \in \mathbb{PL}_{\mathbb{R}}^0$.*

On the variety of Hausdorff groups $\mathcal{V}(\mathbb{LIE}_{\mathbb{R}})$, the functor $\mathrm{L}_{\mathbb{R}}$ respects embeddings:

[3]We mention that G is a $\mathbb{TG}$-limit of the functor F here, as follows from Corollary 4.8; the reader can therefore think of G as the topological group

$$\left\{ (x_i)_{i \in \mathrm{ob}\, S} \in \prod_{i \in \mathrm{ob}\, S} G_i \colon\ F_\phi(x_i) = x_j \text{ for all } i, j \in \mathrm{ob}\, S \text{ and } \phi \in \mathrm{Hom}(i, j) \right\},$$

with limit maps $\pi_i \colon G \to G_i$, $(x_j)_{j \in \mathrm{ob}\, S} \mapsto x_i$. The proof does not make use of this additional information.

Proposition 6.5. *The real topological Lie algebra functor* $\mathrm{L}_{\mathbb{R}}|_{\mathcal{V}(\mathbb{LIE}_{\mathbb{R}})}$ *has the following properties, and is uniquely determined by these (up to natural isomorphism):*

(a) *The functor* $\mathrm{L}_{\mathbb{R}}|_{\mathbb{LIE}_{\mathbb{R}}}$ *is naturally isomorphic to the real Lie algebra functor* $\mathcal{L}_{\mathbb{R}}$ *on the category of real Lie groups;*
(b) $\mathrm{L}_{\mathbb{R}}|_{\mathbb{GPL}_{\mathbb{R}}}$ *is a continuous functor;*
(c) *If* $G, H \in \mathcal{V}(\mathbb{LIE}_{\mathbb{R}})$ *and* $\varepsilon : G \to H$ *is an embedding of topological groups, then* $\mathrm{L}_{\mathbb{R}}(\varepsilon) : \mathrm{L}_{\mathbb{R}}(G) \to \mathrm{L}_{\mathbb{R}}(H)$ *is an embedding of topological Lie algebras. If* ε *has dense image, then* $\mathrm{L}_{\mathbb{R}}(\varepsilon)$ *is a topological isomorphism.*

Proof. (a) and (b) were established in Proposition 5.11 and Proposition 6.3, respectively. Now suppose that $G, H \in \mathcal{V}(\mathbb{LIE}_{\mathbb{R}})$ and suppose that $\varepsilon\colon G \to H$ is a topological embedding. By Proposition 4.7, there exist dense embeddings $\varepsilon_G\colon G \to \overline{G}$ and $\varepsilon_H\colon H \to \overline{H}$ into certain $\overline{G}, \overline{H} \in \mathbb{GPL}_{\mathbb{R}}$. There exists a unique continuous homomorphism $\overline{\varepsilon}\colon \overline{G} \to \overline{H}$ such that $\overline{\varepsilon} \circ \varepsilon_G = \varepsilon_H \circ \varepsilon$; it is easy to see that $\overline{\varepsilon}$ is an embedding of topological groups since ε is so. It remains to note that $\mathrm{L}_{\mathbb{R}}(\varepsilon_H) \circ \mathrm{L}_{\mathbb{R}}(\varepsilon) = \mathrm{L}_{\mathbb{R}}(\overline{\varepsilon}) \circ \mathrm{L}_{\mathbb{R}}(\varepsilon_G)$, where $\mathrm{L}_{\mathbb{R}}(\varepsilon_H)$ and $\mathrm{L}_{\mathbb{R}}(\varepsilon_G)$ are isomorphisms of topological Lie algebras by Proposition 5.9 and $\mathrm{L}_{\mathbb{R}}(\overline{\varepsilon}) = r_{\mathbb{R}}^{\overline{H}} \circ \mathrm{Hom}(\mathbb{R}, \overline{\varepsilon}) \circ (r_{\mathbb{R}}^{\overline{G}})^{-1}$ is an injective continuous linear map between weakly complete topological vector spaces and therefore a topological embedding; this implies that so is $\mathrm{L}_{\mathbb{R}}(\varepsilon)$. If ε has dense image, $\mathrm{L}_{\mathbb{R}}(\varepsilon)$ is an isomorphism by Proposition 5.9. Thus (c) is verified.

The functor $\mathrm{L}_{\mathbb{R}}|_{\mathcal{V}(\mathbb{LIE}_{\mathbb{R}})}$ is determined up to natural isomorphism by properties (a), (b), and (c), since every $G \in \mathcal{V}(\mathbb{LIE}_{\mathbb{R}})$ is a dense subgroup of a projective limit of Lie groups by Proposition 4.7. □

Although it is not clear whether $\lambda_{\mathbb{R}}$ has the good properties described in Propositions 6.3 and 6.5 on all of $\mathbb{CL}_{\mathbb{R}}$, we can obtain analogous results if we consider $\lambda_{\mathbb{R}}$ as an abstract Lie algebra functor (forgetting the topology on the topological Lie algebras):

Proposition 6.6. *The functor* $F \circ \lambda_{\mathbb{R}}$ *from* $\mathbb{CL}_{\mathbb{R}}$ *into the category of real Lie algebras and Lie algebra homomorphisms is continuous and maps embeddings to injective Lie algebra homomorphisms; here* F *denotes the forgetful functor from the category of real topological Lie algebras into the category of real Lie algebras.*

Proof. The continuity of $\lambda_{\mathbb{R}}$ can be shown along the lines of the proof of Proposition 6.3; the only difference is that since $\lambda_{\mathbb{R}}(G) \cong \Lambda_{\mathbb{R}}(G)$ need not be weakly complete if $G \in \mathbb{CL}_{\mathbb{R}}$ is a $\mathbb{CL}_{\mathbb{R}}$-limit of groups $G_i \in \mathbb{CL}_{\mathbb{R}}$, we cannot deduce that the bijective continuous Lie algebra homomorphism $\Phi\colon \lambda_{\mathbb{R}}(G) \to \lim \lambda_{\mathbb{R}}(G_i)$ is an isomorphism of *topological* Lie algebras.[4] Still, Φ

[4]The author does not know whether this pathology really occurs.

is an isomorphism of Lie algebras. Note that $\mathbb{CL}_{\mathbb{R}}$-limits need not coincide with $\mathbb{TG}$-limits a priori, but this does not matter in the proof.[5] It is obvious that $\lambda_{\mathbb{R}} = \mathrm{Hom}(\mathbb{R}, \bullet)$ maps injective homomorphisms to injective Lie algebra homomorphisms. □

Remark 6.7. We have observed above that $\lambda_{\mathbb{R}}|_{\mathbb{GPL}_{\mathbb{R}}}$ is the unique continuous functor on the category $\mathbb{GPL}_{\mathbb{R}}$ of $\mathbb{TG}$-projective limits of Lie groups which extends the Lie algebra functor $\mathcal{L}_{\mathbb{R}}$ on the category of Lie groups; put otherwise, it is the right Kan extension of $\mathcal{L}_{\mathbb{R}}$ to $\mathbb{GPL}_{\mathbb{R}}$ (cf. [**21**], [**13**], [**17**]). It was already noted in Hofmann [**13**] that $\lambda_{\mathbb{R}}|_{\mathbb{PL}_{\mathbb{R}}}$ is the right Kan extension of $\mathcal{L}_{\mathbb{R}}$ to the category $\mathbb{PL}_{\mathbb{R}}$ of pro-Lie groups.

None of the Lie algebra functors introduced here respects *direct* limits.

Remark 6.8. Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{Q}_p$. Consider the strict directed system $\mathrm{SL}_1(\mathbb{K}) \hookrightarrow \mathrm{SL}_2(\mathbb{K}) \hookrightarrow \cdots$, with embeddings $A \mapsto \mathrm{diag}(A, \mathbf{1})$; the direct limit topological group of this directed system has underlying group $\mathrm{SL}_\infty(\mathbb{K}) \subseteq \mathbf{1} + \mathbb{K}^{(\mathbb{N}\times\mathbb{N})} \subseteq \mathbb{K}^{\mathbb{N}\times\mathbb{N}}$, the group of $\mathbb{N}\times\mathbb{N}$-matrices A differing from the identity matrix at only finitely many positions, with determinant $\det(A) = 1$; here $\mathrm{SL}_n(\mathbb{K}) \hookrightarrow \mathrm{SL}_\infty(\mathbb{K})$ via $A \mapsto \mathrm{diag}(A, \mathbf{1})$ (cf. [**8**]). It follows from the fact that $\mathrm{SL}_n(\mathbb{K})$ is simple for n in some cofinal subset of $\mathbb{N}$ that $\mathrm{SL}_\infty(\mathbb{K})$ is a simple group. Since $\dim_{\mathbb{K}} \mathrm{sl}_n(\mathbb{K}) \to \infty$ as $n \to \infty$, we easily deduce that every morphism from $\mathrm{SL}_\infty(\mathbb{K})$ into a (finite-dimensional) $\mathbb{K}$-Lie group is trivial. Thus $\mathrm{L}_{\mathbb{K}}(\mathrm{SL}_\infty(\mathbb{K})) = \{0\}$. On the other hand, $\varinjlim \mathrm{sl}_n(\mathbb{K}) \cong \mathrm{sl}_\infty(\mathbb{K})$ is a nontrivial Lie algebra. We remark that it is still possible to identify $\mathrm{sl}_\infty(\mathbb{K})$ with $\Gamma(\mathbb{K}, \mathrm{SL}_\infty(\mathbb{K}))$ here, as a special case of [**8**], Section 5 and [**8**], Corollary 8.4.

6.3. The contrasting *p*-adic case. In Proposition 6.2 (g), we already encountered a difference between the real and p-adic cases. In this subsection, we describe further important differences.

The most important difference concerns continuity properties of the functors. In contrast to Remark 6.7, any p-adic Lie algebra functor which extends the usual p-adic Lie algebra functor $\mathcal{L}_{\mathbb{Q}_p}$ has to be discontinuous, since already $\mathcal{L}_{\mathbb{Q}_p}$ does not respect a certain $\mathbb{TG}$-projective limit.

Example 6.9. We have $\mathcal{L}_{\mathbb{Q}_p}(\mathbb{Z}_p) \cong \mathbb{Q}_p$, where $\mathbb{Z}_p = \varprojlim \mathbb{Z}(p^n)$. However $\mathcal{L}_{\mathbb{Q}_p}(\mathbb{Z}(p^n)) = \{0\}$ for every $n \in \mathbb{N}$, whence $\varprojlim \mathcal{L}_{\mathbb{Q}_p}(\mathbb{Z}(p^n)) = \{0\}$.

In contrast to Proposition 6.5, the p-adic Lie algebra functors constructed here do not respect embeddings, not even of pro-finite groups.

Example 6.10. Let G be the topological group defined in the proof of Proposition 6.2 (c); as observed there, there exists a subgroup Z of G isomorphic to $\mathbb{Z}_p$. Let $\varepsilon\colon Z \hookrightarrow G$ be the embedding. We have $\mathrm{L}_{\mathbb{Q}_p}(Z) \cong \mathbb{Q}_p$ and $\mathrm{L}_{\mathbb{Q}_p}(G) = \{0\}$, whence $\mathrm{L}_{\mathbb{Q}_p}(\varepsilon) = 0$.

[5]The author does not know whether this pathology really occurs.

Thus $\mathrm{L}_{\mathbb{Q}_p}$ does not map embeddings to injective Lie algebra homomorphisms in general, not even embeddings of pro-finite groups. Neither do the other p-adic Lie algebra functors defined in this paper (by the same counterexample). In particular, unlike the real case, if $\psi\colon H \to G$ is a morphism between locally compact groups and $\zeta\colon \ker\psi \hookrightarrow H$, then $\operatorname{im}\mathrm{L}_{\mathbb{Q}_p}(\zeta)$ can be a proper subset of $\ker \mathrm{L}_{\mathbb{Q}_p}(\psi)$, i.e., $\mathrm{L}_{\mathbb{Q}_p}$ is not left exact (and the same problem occurs for the other p-adic Lie algebra functors).

Despite these problems, the p-adic Lie algebra functors do have specific good properties on the category of locally compact groups, as we shall see in Section 10 below. For example, we shall see that both $\mathrm{L}_{\mathbb{Q}_p}$ and $\mathrm{L}^{\mathrm{loc}}_{\mathbb{Q}_p}$ map quotient morphisms between locally compact groups to quotient morphisms (Corollary 10.3, Corollary 10.7).

6.4. Relation to Lashof's functor and other classical constructions. Let us clarify the relation to Lashof's construction now. If G is a pro-Lie group, the Lie algebra of G in Lashof's sense is defined in [**20**] only up to an isomorphism of topological Lie algebras, depending on the choice of projective system. Using the projective system $\mathcal{T}(G)$ defined after Corollary 4.8, we obtain a canonical realization,

$$\ell_{\mathbb{R}}(G) := \lim_{\longleftarrow\, f\in J_{\mathbb{R}}(G)} \mathcal{L}_{\mathbb{R}}(H_f).$$

If $\psi\colon G \to K$ is a morphism between pro-Lie groups, Lashof associates with it the unique continuous Lie algebra homomorphism $\ell_{\mathbb{R}}(\psi)\colon \ell_{\mathbb{R}}(G) \to \ell_{\mathbb{R}}(K)$ such that $\psi\circ\exp_G = \exp_H \circ\, \ell_{\mathbb{R}}(\psi)$, where the exponential function of a pro-Lie group G is defined via $\exp_G := \lim_{\longleftarrow\, f\in J_{\mathbb{R}}(G)} \exp_{H_f}$ ([**20**], Theorem 3.11); here $\exp_{H_f}\colon \mathcal{L}_{\mathbb{R}}(H_f) \to H_f$, $X \mapsto X(1)$. More generally, if $G \in \mathbb{PL}^0_{\mathbb{R}}$, Lashof associates the topological Lie algebra $\ell^0_{\mathbb{R}}(G) := \ell_{\mathbb{R}}(G_0)$ with G; then $\ell^0_{\mathbb{R}}$ becomes a functor if we define $\ell^0_{\mathbb{R}}(\psi) := \ell_{\mathbb{R}}(\psi_0)$ for morphisms $\psi\colon G \to K$ in $\mathbb{PL}^0_{\mathbb{R}}$, where $\psi_0 := \psi|^{K_0}_{G_0}$.

Proposition 6.11. *There is a natural isomorphism* $\Theta\colon \mathrm{L}_{\mathbb{R}}|_{\mathbb{PL}_{\mathbb{R}}} \overset{\bullet}{\to} \ell_{\mathbb{R}}$ *and a natural isomorphism* $\Theta^0\colon \mathrm{L}^0_{\mathbb{R}}|_{\mathbb{PL}^0_{\mathbb{R}}} \overset{\bullet}{\to} \ell^0_{\mathbb{R}}$.

Proof. If G is a pro-Lie group, we deduce from the cofinality of $J_{\mathbb{R}}(G)$ in $I_{\mathbb{R}}(G)$ that the natural map

$$\Theta^G\colon \lim_{\longleftarrow\, f\in I_{\mathbb{R}}(G)} \mathcal{L}_{\mathbb{R}}(H_f) \to \lim_{\longleftarrow\, f\in J_{\mathbb{R}}(G)} \mathcal{L}_{\mathbb{R}}(H_f), \quad (X_f)_{f\in I_{\mathbb{R}}(G)} \mapsto (X_f)_{f\in J_{\mathbb{R}}(G)}$$

is an isomorphism of topological Lie algebras. Note that, if $Y = (Y_f)_{f\in J_{\mathbb{R}}(G)}$ is an element of $\ell_{\mathbb{R}}(G)$, we have $f(\exp_G(Y)) = \exp_{H_f}(Y_f) = Y_f(1)$ for $f \in J_{\mathbb{R}}(G)$; in particular, given $X \in \operatorname{Hom}(\mathbb{R}, G)$ we have $f(\exp_G(\Theta^G(r^G_{\mathbb{R}}(X)))) = f(X(1))$ for all $f \in J_{\mathbb{R}}(G)$, which implies $X(1) = \exp_G(\Theta^G(r^G_{\mathbb{R}}(X)))$. Now suppose that $\psi : G \to K$ is a morphism between pro-Lie groups. Given $X \in \mathrm{Hom}(\mathbb{R}, G)$, we have

$$\begin{aligned}(f \circ \exp_K \circ \Theta^K \circ \mathrm{L}_{\mathbb{R}}(\psi) \circ r^G_{\mathbb{R}})(X) &= (f \circ \exp_K \circ \Theta^K \circ r^K_{\mathbb{R}})(\psi \circ X) \\ &= (f \circ \psi \circ X)(1) \\ &= (f \circ \psi)((\exp_G \circ \Theta^G \circ r^G_{\mathbb{R}})(X))\end{aligned}$$

for all $f \in J_{\mathbb{R}}(K)$. Since $J_{\mathbb{R}}(K)$ separates points on K and $r^G_{\mathbb{R}}$ is bijective, we deduce that $\exp_K \circ \Theta^K \circ \mathrm{L}_{\mathbb{R}}(\psi) \circ (\Theta^G)^{-1} = \psi \circ \exp_G$. Now $\Theta^K \circ \mathrm{L}_{\mathbb{R}}(\psi) \circ (\Theta^G)^{-1}$ being a continuous Lie algebra homomorphism, we deduce from the defining property of $\ell_{\mathbb{R}}(\psi)$ that $\ell_{\mathbb{R}}(\psi) = \Theta^K \circ \mathrm{L}_{\mathbb{R}}(\psi) \circ (\Theta^G)^{-1}$, i.e., $\ell_{\mathbb{R}}(\psi) \circ \Theta^G = \Theta^K \circ \mathrm{L}_{\mathbb{R}}(\psi)$. We have proved that Θ^G is a natural isomorphism. It is plain that $(\Theta^0)^G := \Theta^{G_0}$ defines a natural isomorphism $\Theta^0 \colon \mathrm{L}_{\mathbb{R}}|_{\mathbb{PL}^0_{\mathbb{R}}} \stackrel{\bullet}{\to} \ell^0_{\mathbb{R}}$. $\square$

Remark 6.12. The usual Lie algebra functor on the category $\mathbb{LCG}$ of locally compact groups and continuous homomorphisms coincides with $\lambda_{\mathbb{R}}|_{\mathbb{LCG}}$; thus the Lie algebra of the locally compact group G is the topological Lie algebra $\lambda_{\mathbb{R}}(G) \cong \ell^0_{\mathbb{R}}(G) \cong \Lambda^0_{\mathbb{R}}(G)$, with underlying set $\mathrm{Hom}(\mathbb{R}, G)$, the set of one-parameter subgroups of G.

Remark 6.13. Note that it is possible to identify the topological Lie algebra $\mathrm{L}_{\mathbb{R}}(G)$ with $\mathrm{Hom}(\mathbb{R}, G_{\mathbb{R}})$ as a set, for arbitrary $G \in \mathbb{TG}$. To see this, note that $\mathfrak{S}_{\mathbb{R}}(\eta_{\mathbb{R}})$ is an isomorphism, whence $\mathrm{L}_{\mathbb{R}}(\eta^G_{\mathbb{R}})$ is an isomorphism of topological Lie algebras. Now $G_{\mathbb{R}}$ being a projective limit of Lie groups, $s^{G_{\mathbb{R}}} : \lambda_{\mathbb{R}}(G_{\mathbb{R}}) \to \Lambda_{\mathbb{R}}(G_{\mathbb{R}}) = \mathrm{L}_{\mathbb{R}}(G_{\mathbb{R}})$ is an isomorphism of topological Lie algebras as well. Thus $\mathrm{L}_{\mathbb{R}}(G) \cong \lambda_{\mathbb{R}}(G_{\mathbb{R}})$ via the isomorphism $d^G := (s^{G_{\mathbb{R}}})^{-1} \circ \mathrm{L}_{\mathbb{R}}(\eta^G_{\mathbb{R}})$, and it is easy to see that $d \colon \mathrm{L}_{\mathbb{R}} \stackrel{\bullet}{\to} \lambda_{\mathbb{R}} \circ (\ )_{\mathbb{R}}$ is a natural isomorphism. In particular, the argument shows that $\mathrm{L}_{\mathbb{R}} \cong \mathrm{L}_{\mathbb{R}}|_{\mathbb{GPL}_{\mathbb{R}}} \circ (\ )_{\mathbb{R}}$. However:

Remark 6.14. If G is an arbitrary Hausdorff group, it is not clear at all how the set $\mathrm{Hom}(\mathbb{R}, G)$ of one-parameter subgroups of G might be turned into a Lie algebra. A strategy different from the one pursued here is to try to make $\mathrm{Hom}(\mathbb{R}, G)$ a Lie algebra directly by means of the Trotter Product Formula and the Commutator Formula: E.g., this works for the Banach-Lie groups, and for countable strict direct limits of finite-dimensional Lie groups [**8**]; see [**5**] for a very general approach along these lines.[6] In general, the groups which can be given a Lie algebra in this way need not be objects of $\mathbb{CL}_{\mathbb{R}}$, and also the Lie algebras we associate with them will not be the Lie algebras one would like to obtain in general. For example, let $\mathcal{H}$ be an infinite-dimensional separable Hilbert space, and $G := \mathrm{GL}(\mathcal{H}) = B(\mathcal{H})^\times$; by [**29**], Theorem 12.37, we have $c_{\mathbb{R}}(G) = G$. Then G carries a natural Banach-Lie group structure (cf. Section 14); the Banach-Lie algebra of the Banach-Lie

[6] We should also mention the "quotient approach" to Lie groups here [**37**].

group G is $B(\mathcal{H})$, a Lie algebra all of whose finite-dimensional representations are trivial, since any proper Lie algebra ideal of $B(\mathcal{H})$ has infinite codimension ([**27**], Theorem IV A). Since $G = c_{\mathbb{R}}(G)$, we deduce in view of the naturality of the exponential function that every morphism from G into (finite-dimensional) Lie groups is trivial. Thus $\mathrm{L}_{\mathbb{R}}^{\exp}(G) = \mathrm{L}_{\mathbb{R}}(G) = \{0\}$. It is certainly a drawback of the Lie algebra functors defined up to now that they do not associate the desired Lie algebras with Banach-Lie groups in general. In Section 14, we briefly outline how topological Lie algebra functors on $\mathbb{TG}$ can be defined which associate the correct topological Lie algebras to Banach-Lie groups.

7. Preservation of direct products.

Throughout this section, let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{Q}_p$ for some prime p. As observed in the preceding section, $\lambda_{\mathbb{R}} \cong \Lambda_{\mathbb{R}}^{\exp}|_{\mathbb{CL}_{\mathbb{R}}}$, considered as a functor into the category of (abstract) real Lie algebras, preserves limits, and hence preserves direct products in particular. In this section, we address the problem whether the functors defined on all of $\mathbb{TG}$ preserve direct products.

Theorem 7.1. *The functors* $\Lambda_{\mathbb{K}}$ *and* $\overline{\Lambda}_{\mathbb{K}}$ *respect finite direct products. Moreover, if* $(G_i)_{i\in I}$ *is a family of Hausdorff groups and* G *its cartesian product, with canonical projections* pr_i, *then the map* $\nabla := (\overline{\Lambda}_{\mathbb{K}}(\mathrm{pr}_i))_{i\in I} : \overline{\Lambda}_{\mathbb{K}}(G) \to \prod_{i\in I} \overline{\Lambda}_{\mathbb{K}}(G_i)$ *is surjective and open. If* $\mathbb{K} = \mathbb{R}$, *then* ∇ *is an isomorphism of topological Lie algebras. Provided it respects the direct products of all finite subfamilies occurring,*[7] *the analogous assertions hold for the functor* $\mathrm{L}_{\mathbb{K}}$.

Proof. In the following, we use the notation corresponding to the case $\mathbb{K} = \mathbb{Q}_p$; if $\mathbb{K} = \mathbb{R}$, the brackets "[" and "]" are to be ignored. Let G_1, G_2 be topological groups, and G be their direct product. Then we have, for $i \in \{1, 2\}$, the canonical inclusion morphism $\varepsilon_i : G_i \to G$, and the canonical projections $\mathrm{pr}_i : G \to G_i$. The canonical projection $\mathrm{L}_{\mathbb{K}}(G_1) \times \mathrm{L}_{\mathbb{K}}(G_2) \to \mathrm{L}_{\mathbb{K}}(G_i)$ will be denoted by Pr_i. We consider the following continuous linear maps:

$$\phi := (\mathrm{L}_{\mathbb{K}}(\mathrm{pr}_1), \mathrm{L}_{\mathbb{K}}(\mathrm{pr}_2)) : \mathrm{L}_{\mathbb{K}}(G) \to \mathrm{L}_{\mathbb{K}}(G_1) \times \mathrm{L}_{\mathbb{K}}(G_2), \qquad \text{and}$$
$$\psi := \mathrm{L}_{\mathbb{K}}(\varepsilon_1) \circ \mathrm{Pr}_1 + \mathrm{L}_{\mathbb{K}}(\varepsilon_2) \circ \mathrm{Pr}_2 : \mathrm{L}_{\mathbb{K}}(G_1) \times \mathrm{L}_{\mathbb{K}}(G_2) \to \mathrm{L}_{\mathbb{K}}(G).$$

Clearly ϕ is a morphism of topological $\mathbb{K}$-Lie algebras. To see that ψ is a Lie algebra homomorphism as well, suppose that $X \in \mathrm{L}_{\mathbb{K}}(G_1)$, $Y \in \mathrm{L}_{\mathbb{K}}(G_2)$, and $f \in I_{\mathbb{K}}(G)$. Then X is of the form $([X_g])_{g\in I_{\mathbb{K}}(G_1)}$, say. Set $H_1 := \overline{\mathrm{im}\, f \circ \varepsilon_1}$, and $h_1 := (f \circ \varepsilon_1)|^{H_1}$; denote the inclusion $H_1 \hookrightarrow H_f$ by μ_1. To ease notation, let us assume that $h_1 \in I_{\mathbb{K}}(G_1)$. Then the f-coordinate of $\mathrm{L}_{\mathbb{K}}(\varepsilon_1).X$ is given by $[\mu_1 \circ X_{h_1}]$. With analogous notation, the f-coordinate of $\mathrm{L}_{\mathbb{K}}(\varepsilon_2).Y$ is

[7] The author doesn't know whether $\mathrm{L}_{\mathbb{K}}$ respects finite direct products in general.

$[\mu_2 \circ Y_{h_2}]$. Note that H_1 and H_2 centralize each other, since so do the images of ε_1 and ε_2. Hence, by the Commutator Formula, [**3**], §3.3, Proposition 4, the Lie bracket of $[\mu_1 \circ X_{h_1}]$ and $[\mu_2 \circ Y_{h_2}]$ vanishes. Thus the images of $\mathrm{L}_{\mathbb{K}}(\varepsilon_1) \circ \mathrm{Pr}_1$ and $\mathrm{L}_{\mathbb{K}}(\varepsilon_2) \circ \mathrm{Pr}_2$ centralize each other, whence ψ is a Lie algebra homomorphism.

Now let $[X] \in \Gamma(\mathbb{K}, G)$ be given, and $f \in I_{\mathbb{K}}(G)$. In view of the naturality of $r_{\mathbb{K}}$, the f-coordinate of $(\psi \circ \phi \circ r_{\mathbb{K}}^G)([X])$ is given by

$$\begin{aligned}&[f \circ \varepsilon_1 \circ \mathrm{pr}_1 \circ X] + [f \circ \varepsilon_2 \circ \mathrm{pr}_2 \circ X] \\ &= [(f \circ \varepsilon_1 \circ \mathrm{pr}_1 \circ X) \cdot (f \circ \varepsilon_2 \circ \mathrm{pr}_2 \circ X)] = [f \circ X],\end{aligned}$$

where the second equality is obtained from the Trotter Product Formula [**3**], Chapter III, §4.3, Proposition 4, again using that $\operatorname{im} \varepsilon_1$ and $\operatorname{im} \varepsilon_2$ centralize each other. Thus $(\psi \circ \phi)(r_G([X])) = r_G([X])$ for all $[X]$, whence the restriction of $\psi \circ \phi$ to $\Lambda_{\mathbb{K}}(G)$, and indeed to $\overline{\Lambda}_{\mathbb{K}}(G)$, is the identity. Similarly, one recognizes that the restriction of $\phi \circ \psi$ to $\Lambda_{\mathbb{K}}(G_1) \times \Lambda_{\mathbb{K}}(G_2)$ is the identity, and so is its restriction to $\overline{\Lambda}_{\mathbb{K}}(G_1) \times \overline{\Lambda}_{\mathbb{K}}(G_2)$. This proves the assertions on finite products.

Suppose now that $(G_i)_{i \in I}$ is a family of Hausdorff groups, with direct product G. For every finite subset F of I, we set $\mathrm{pr}_F := (\mathrm{pr}_i)_{i \in F}$ and $\mathrm{Pr}_F := (\mathrm{Pr}_i)_{i \in F}$, where $\mathrm{Pr}_i : \prod_{j \in I} \overline{\Lambda}_{\mathbb{K}}(G_j) \to \overline{\Lambda}_{\mathbb{K}}(G_i)$ is the canonical projection. Furthermore, we let $\varepsilon_F : \prod_{i \in F} G_i \to G$ denote the canonical inclusion, and we set $\gamma := (\overline{\Lambda}_{\mathbb{K}}(\mathrm{pr}'_i))_{i \in F}$, where pr'_i is the canonical projection $\prod_{j \in F} G_j \to G_i$. Then the diagram

$$\begin{array}{ccccc}\overline{\Lambda}_{\mathbb{K}}(\prod_{i \in F} G_i) & \xrightarrow{\overline{\Lambda}_{\mathbb{K}}(\varepsilon_F)} & \overline{\Lambda}_{\mathbb{K}}(G) & \xrightarrow{\nabla} & \prod_{i \in I} \overline{\Lambda}_{\mathbb{K}}(G_i) \\ & \searrow \mathrm{id} & \downarrow \overline{\Lambda}_{\mathbb{K}}(\mathrm{pr}_F) & & \downarrow \mathrm{Pr}_F \\ & & \overline{\Lambda}_{\mathbb{K}}(\prod_{i \in F} G_i) & \xrightarrow{\gamma} & \prod_{i \in F} \overline{\Lambda}_{\mathbb{K}}(G_i)\end{array}$$

commutes. Since γ is an isomorphism by the above, we conclude that $\mathrm{Pr}_F(\operatorname{im} \nabla) = \prod_{i \in F} \overline{\Lambda}_{\mathbb{K}}(G_i)$. Since F was arbitrary, $\operatorname{im} \nabla$ is dense in the product $\prod_{i \in I} \overline{\Lambda}_{\mathbb{K}}(G_i)$. By Proposition 2.1, ∇ is open and surjective.

Now assume that $\mathbb{K} = \mathbb{R}$. If $0 \neq X := (X_f)_{f \in I_{\mathbb{R}}(G)} \in \overline{\Lambda}_{\mathbb{R}}(G)$, then there is $g \in I_{\mathbb{R}}(G)$ such that $X_g \neq 0$. We now use that $\prod_{i \in I} G_i \cong \varprojlim_{F \in \Phi(I)} \prod_{i \in F} G_i$ in a natural way, where $\Phi(I)$ is the directed set of finite subsets of I. By Lemma 4.9, there is $F \in \Phi(I)$ and a morphism $h : P := \prod_{i \in F} G_i \to H_g$ such that $g = h \circ \mathrm{pr}_F$. There is $k \in I_{\mathbb{R}}(P)$ such that $k \sim h$, and it is easy to see that the k-coordinate of $\overline{\Lambda}_{\mathbb{R}}(\mathrm{pr}_F)(X)$ is nonzero. This implies the injectivity of ∇ in the real case. The remainder is plain. □

Remark 7.2. Note that $C_{\mathbb{K}}(G) = \prod_{i \in I} C_{\mathbb{K}}(G_i)$, $G_{\mathrm{arc}} = \prod_{i \in I} (G_i)_{\mathrm{arc}}$, and $G_0 = \prod_{i \in I} (G_i)_0$ in the situation of Theorem 7.1. This allows us to deduce the analogues of the theorem for the functors $\Lambda_{\mathbb{K}}^{\exp}$ and $\overline{\Lambda}_{\mathbb{K}}^{\exp}$, $\Lambda_{\mathbb{R}}^{\mathrm{arc}}$ and $\overline{\Lambda}_{\mathbb{R}}^{\mathrm{arc}}$, as well as for the functors $\Lambda_{\mathbb{R}}^{0}$ and $\overline{\Lambda}_{\mathbb{R}}^{0}$.

Theorem 7.3. *Let $G, H \in \mathbb{TG}$ such that $\mathrm{L}_{\mathbb{K}}(U \times V) \cong \mathrm{L}_{\mathbb{K}}(U) \times \mathrm{L}_{\mathbb{K}}(V)$ naturally for all $U \in \mathcal{U}(G)$, $V \in \mathcal{U}(H)$. Then $\mathrm{L}_{\mathbb{K}}^{\mathrm{loc}}(G \times H) \cong \mathrm{L}_{\mathbb{K}}^{\mathrm{loc}}(G) \times \mathrm{L}_{\mathbb{K}}^{\mathrm{loc}}(H)$. If $(G_i)_{i \in I}$ is a family of Hausdorff groups, with direct product $G := \prod_{i \in I} G_i$, such that $\mathrm{L}_{\mathbb{K}}(\prod_{i \in F} U_i) \cong \prod_{i \in F} \mathrm{L}_{\mathbb{K}}(U_i)$ naturally for all finite subsets F of I and $U_i \in \mathcal{U}(G_i)$, then $(\mathrm{L}_{\mathbb{K}}^{\mathrm{loc}}(\mathrm{pr}_i))_{i \in I} \colon \mathrm{L}_{\mathbb{K}}^{\mathrm{loc}}(G) \to \prod_{i \in I} \mathrm{L}_{\mathbb{K}}^{\mathrm{loc}}(G_i)$ is a quotient morphism of topological $\mathbb{K}$-Lie algebras.*

Proof. The assertions are immediate consequences of Theorem 7.1. For example, since $\{U \times V : U \in \mathcal{U}(G), V \in \mathcal{U}(H)\}$ is cofinal in $\mathcal{U}(G \times H)$, it is easy to see that $\mathrm{L}_{\mathbb{K}}^{\mathrm{loc}}(G \times H) = \varprojlim \mathrm{L}_{\mathbb{K}}(U \times V) \cong \varprojlim \mathrm{L}_{\mathbb{K}}(U) \times \varprojlim \mathrm{L}_{\mathbb{K}}(V) = \mathrm{L}_{\mathbb{K}}^{\mathrm{loc}}(G) \times \mathrm{L}_{\mathbb{K}}^{\mathrm{loc}}(H)$. □

Problem 7.4. The author does not know whether the mapping ∇ defined in Theorem 7.1 will be injective in general in the case where $\mathbb{K} = \mathbb{Q}_p$. The same applies to the corresponding mappings in Theorem 7.3 and Corollary 10.4 below.

Example 7.5. The following example shows that the functor $\Lambda_{\mathbb{Q}_p}$ does not preserve infinite direct products in general. We note first that a local p-adic one-parameter subgroup $X : U \to \mathbb{Q}_p$ of $(\mathbb{Q}_p, +)$, defined on an open subgroup U of $\mathbb{Q}_p$, is uniquely determined by its function germ at 0. To see this, suppose that W is an open subgroup of $\mathbb{Q}_p$ contained in U. Since $\mathbb{Q}_p$ is a $\mathbb{Q}$-vector space, we have unique divisibility. Noting that $U \subseteq \bigcup_{n \in \mathbb{N}} p^{-n} W$, the claim follows.

Now consider the compact, totally disconnected group $G := \mathbb{Z}_p^{\mathbb{N}}$. By Proposition 5.11 and Theorem 7.1, the map $\psi := (\mathrm{L}_{\mathbb{Q}_p}(\mathrm{pr}_n))_{n \in \mathbb{N}} \colon \mathrm{L}_{\mathbb{Q}_p}(G) \to \mathrm{L}_{\mathbb{Q}_p}(\mathbb{Z}_p)^{\mathbb{N}}$ is surjective; here $\mathrm{L}_{\mathbb{Q}_p}(\mathbb{Z}_p) = \Lambda_{\mathbb{Q}_p}(\mathbb{Z}_p) \cong \mathbb{Q}_p$. Since $r_{\mathbb{Q}_p}^{\mathbb{Z}_p}$ is a bijection, we may consider ψ as a map $\mathrm{L}_{\mathbb{Q}_p}(G) \to \Gamma(\mathbb{Q}_p, \mathbb{Z}_p)^{\mathbb{N}}$. We set $\theta := \psi|_{\Lambda_{\mathbb{Q}_p}(G)}$. The mapping $\delta \colon \Gamma(\mathbb{Q}_p, \mathbb{Z}_p) \to \mathbb{Q}_p$, $[X] \mapsto X'(0)$ is an isomorphism of abelian topological $\mathbb{Q}_p$-Lie algebras; we put $\chi := \delta^{\mathbb{N}} \circ \theta$. We claim that $\chi(\operatorname{im} r_{\mathbb{Q}_p}^G) = \ell^\infty$, the space of bounded sequences of p-adic numbers; then also $\chi(\Lambda_{\mathbb{Q}_p}(G)) = \ell^\infty$, since χ is a homomorphism of abelian $\mathbb{Q}_p$-Lie algebras, whence $\chi(\Lambda_{\mathbb{Q}_p}(G)) \neq \mathbb{Q}_p^{\mathbb{N}}$ indeed, i.e., $\operatorname{im} \theta \neq \Lambda_{\mathbb{Q}_p}(\mathbb{Z}_p)^{\mathbb{N}}$. Note that G will also provide an example of a group such that $r_{\mathbb{Q}_p}^G$ is not surjective, once we have verified the claim. One part is clear: If $z = (z_n)_{n \in \mathbb{N}} \in \ell^\infty$, there is $k \in \mathbb{Z}$ such that $z_n \in p^k \mathbb{Z}_p$ for all $n \in \mathbb{N}$; then $Z \colon p^{-k} \mathbb{Z}_p \to G$, $t \mapsto t(z_n)$ is a local p-adic one-parameter subgroup of G such that $\chi(r_{\mathbb{Q}_p}^G([Z])) = (z_n)_{n \in \mathbb{N}}$. Thus $\ell^\infty \leq \operatorname{im} \theta$ indeed. Now suppose that $X \colon p^k \mathbb{Z}_p \to G$ is any local p-adic one-parameter subgroup of G; put $X_n := \mathrm{pr}_n \circ X$. Then $X_n(t) = X_n'(0) t \in \mathbb{Z}_p$

for all $t \in p^k\mathbb{Z}_p$, as follows from the above discussion: Thus $X_n'(0) \in p^{-k}\mathbb{Z}_p$ for all $n \in \mathbb{N}$. This implies that $\chi(\operatorname{im} r^G_{\mathbb{Q}_p}) \subseteq \ell^\infty$.

8. The real Lie algebra functors are all different.

In this section, we show that the real topological Lie algebra functors $\Lambda^{\exp}_{\mathbb{R}}$, $\overline{\Lambda}^{\exp}_{\mathbb{R}}$, $\mathrm{L}^{\exp}_{\mathbb{R}}$, $\Lambda^{\mathrm{arc}}_{\mathbb{R}}$, $\overline{\Lambda}^{\mathrm{arc}}_{\mathbb{R}}$, $\mathrm{L}^{\mathrm{arc}}_{\mathbb{R}}$, $\Lambda^{0}_{\mathbb{R}}$, $\overline{\Lambda}^{0}_{\mathbb{R}}$, $\mathrm{L}^{0}_{\mathbb{R}}$, $\Lambda_{\mathbb{R}}$, $\overline{\Lambda}_{\mathbb{R}}$, $\mathrm{L}_{\mathbb{R}}$, $\Lambda^{\mathrm{loc}}_{\mathbb{R}}$, $\overline{\Lambda}^{\mathrm{loc}}_{\mathbb{R}}$, $\mathrm{L}^{\mathrm{loc}}_{\mathbb{R}}$ are pairwise not naturally isomorphic, by explicit calculation of the real Lie algebras for nine selected examples.

8.1. Let $G_1 := \mathbb{Q}$, equipped with the topology induced by $\mathbb{R}$. Then $c_{\mathbb{R}}(\mathbb{Q}) = \mathbb{Q}_{\mathrm{arc}} = \mathbb{Q}_0 = \{0\}$, and $U = \mathbb{Q}$ for every open subgroup U of $\mathbb{Q}$. Using that $\mathbb{Q}$ is dense in $\mathbb{R}$, we deduce from Proposition 5.9 that $\mathrm{L}^{\mathrm{loc}}_{\mathbb{R}}(\mathbb{Q}) \cong \mathrm{L}_{\mathbb{R}}(\mathbb{Q}) \cong \mathbb{R}$. However, all of $\Lambda^{\exp}_{\mathbb{R}}(\mathbb{Q})$, $\overline{\Lambda}^{\exp}_{\mathbb{R}}(\mathbb{Q})$, $\mathrm{L}^{\exp}_{\mathbb{R}}(\mathbb{Q})$, $\Lambda^{\mathrm{arc}}_{\mathbb{R}}(\mathbb{Q})$, $\overline{\Lambda}^{\mathrm{arc}}_{\mathbb{R}}(\mathbb{Q})$, $\mathrm{L}^{\mathrm{arc}}_{\mathbb{R}}(\mathbb{Q})$, $\Lambda^{0}_{\mathbb{R}}(\mathbb{Q})$, $\overline{\Lambda}^{0}_{\mathbb{R}}(\mathbb{Q})$, $\mathrm{L}^{0}_{\mathbb{R}}(\mathbb{Q})$, $\Lambda^{\mathrm{loc}}_{\mathbb{R}}(\mathbb{Q})$, $\overline{\Lambda}^{\mathrm{loc}}_{\mathbb{R}}(\mathbb{Q})$, $\Lambda_{\mathbb{R}}(\mathbb{Q})$, and $\overline{\Lambda}_{\mathbb{R}}(\mathbb{Q})$ are the zero algebra $\{0\}$.

8.2. Let $G_2 = F$ be a connected, locally connected, dense, proper subfield F of $\mathbb{C}$ (see [**6**]). The path component P of F being an ideal, either F is totally pathwise disconnected or F is path-connected. The latter cannot be true since it would entail that P is an analytic subgroup of $\mathbb{C}$ and hence a vector subspace: Thus F is totally pathwise disconnected. Note that $\mathcal{U}(F) = \{F\}$ by connectedness. We easily deduce that $\mathrm{L}^{0}_{\mathbb{R}}(F) \cong \mathrm{L}^{\mathrm{loc}}_{\mathbb{R}}(F) \cong \mathrm{L}_{\mathbb{R}}(F) \cong \mathrm{L}_{\mathbb{R}}(\mathbb{C}) \cong \mathbb{R}^2$, whereas all of $\Lambda^{\exp}_{\mathbb{R}}(F)$, $\overline{\Lambda}^{\exp}_{\mathbb{R}}(F)$, $\mathrm{L}^{\exp}_{\mathbb{R}}(F)$, $\Lambda^{\mathrm{arc}}_{\mathbb{R}}(F)$, $\overline{\Lambda}^{\mathrm{arc}}_{\mathbb{R}}(F)$, $\mathrm{L}^{\mathrm{arc}}_{\mathbb{R}}(F)$, $\Lambda^{0}_{\mathbb{R}}(F)$, $\overline{\Lambda}^{0}_{\mathbb{R}}(F)$, $\Lambda^{\mathrm{loc}}_{\mathbb{R}}(F)$, $\overline{\Lambda}^{\mathrm{loc}}_{\mathbb{R}}(F)$, $\Lambda_{\mathbb{R}}(F)$, and $\overline{\Lambda}_{\mathbb{R}}(F)$ are the zero algebra.

8.3. Let I be a set such that $\operatorname{card}(I) \geq 2^{\aleph_0}$, and consider the locally compact semidirect product $G_3 := G := \mathbb{T}^I \rtimes \operatorname{Sym}(I)$, where the group $\operatorname{Sym}(I)$ of all permutations of I is equipped with the the discrete topology and acts on the product $\mathbb{T}^I$ by permuting indices. Then $\mathbb{T}^I \leq \ker f$, for every $f \in I_{\mathbb{R}}(G)$, cf. [**9**], proof of Theorem 5.2 (a). Hence H_f is discrete for all $f \in J_{\mathbb{R}}(G)$, and $\Lambda_{\mathbb{R}}(G) = \overline{\Lambda}_{\mathbb{R}}(G) = \mathrm{L}_{\mathbb{R}}(G) = \{0\}$. On the other hand, we have $c_{\mathbb{R}}(G) = G_{\mathrm{arc}} = G_0 = \mathbb{T}^I$, and $\mathbb{T}^I$ is an upper bound for $\mathcal{U}(G)$ with respect to inverse inclusion. We easily deduce that all of $\Lambda^{\exp}_{\mathbb{R}}(G)$, $\overline{\Lambda}^{\exp}_{\mathbb{R}}(G)$, $\mathrm{L}^{\exp}_{\mathbb{R}}(G)$, $\Lambda^{\mathrm{arc}}_{\mathbb{R}}(G)$, $\overline{\Lambda}^{\mathrm{arc}}_{\mathbb{R}}(G)$, $\mathrm{L}^{\mathrm{arc}}_{\mathbb{R}}(G)$, $\Lambda^{0}_{\mathbb{R}}(G)$, $\overline{\Lambda}^{0}_{\mathbb{R}}(G)$, $\mathrm{L}^{0}_{\mathbb{R}}$, $\Lambda^{\mathrm{loc}}_{\mathbb{R}}(G)$, $\overline{\Lambda}^{\mathrm{loc}}_{\mathbb{R}}(G)$, and $\mathrm{L}^{\mathrm{loc}}_{\mathbb{R}}(G)$ are isomorphic to $\mathbb{R}^I$.

8.4. Let $G_4 := V$ be a real locally convex vector space of dimension $\aleph_0$. Then $V \cong \Lambda_{\mathbb{R}}(V)$ as an abstract vector space by Proposition 6.2 (e), whence $\mathrm{L}_{\mathbb{R}}(V)$ is an infinite-dimensional vector space. By a Baire argument, the completely metrizable topological vector space $\mathbb{R}^{\mathbb{N}}$ must have uncountable dimension: The same then holds for every infinite-dimensional weakly complete space. Thus all of $\overline{\Lambda}^{\exp}_{\mathbb{R}}(V)$, $\mathrm{L}^{\exp}_{\mathbb{R}}(V)$, $\overline{\Lambda}^{\mathrm{arc}}_{\mathbb{R}}(V)$, $\mathrm{L}^{\mathrm{arc}}_{\mathbb{R}}(V)$, $\overline{\Lambda}^{0}_{\mathbb{R}}(V)$, $\mathrm{L}^{0}_{\mathbb{R}}(V)$,

$\overline{\Lambda}^{\mathrm{loc}}_{\mathbb{R}}(V)$, $\mathrm{L}^{\mathrm{loc}}_{\mathbb{R}}(V)$, $\overline{\Lambda}_{\mathbb{R}}(V)$, and $\mathrm{L}_{\mathbb{R}}(V)$ have uncountable dimension, whereas $\Lambda_{\mathbb{R}}(V) = \Lambda^{\exp}_{\mathbb{R}}(V) = \Lambda^{\mathrm{arc}}_{\mathbb{R}}(V) = \Lambda^{0}_{\mathbb{R}}(V) \cong \Lambda^{\mathrm{loc}}_{\mathbb{R}}(V)$ have countable dimension.

8.5. Let $G_5 := G$ be the additive subgroup of the Banach space $V := L^1(\mathbb{R})$ consisting of the classes of those Lebesgue integrable functions which take integer values almost everywhere; equip G with the norm-topology. Let $\varepsilon : G \to V$ be the embedding. It is well-known that G is a contractible, locally contractible group, and that every one-parameter subgroup of G is trivial. Let K be the set of linear surjective maps $f : V \to H_f$ in $I_{\mathbb{R}}(V)$ such that H_f is a finite-dimensional vector space; as observed in the proof of Proposition 6.2 (e), K is a cofinal subset of $I_{\mathbb{R}}(V)$. Similarly, let κ be the set of surjective morphisms $f : G \to H_f$ in $I_{\mathbb{R}}(G)$ onto finite-dimensional vector spaces H_f. We claim that κ is cofinal in $I_{\mathbb{R}}(G)$. To see this, let $f\colon G \to H_f$ be in $I_{\mathbb{R}}(G)$; then H_f is an abelian, connected Lie group, and its universal covering Lie group $\widetilde{H}_f =: W$ is an abelian, simply connected Lie group and therefore is a finite-dimensional real vector space. If $c\colon W \to H_f$ is a covering morphism, there exists a unique morphism $\widetilde{f} : G \to W$ such that $c \circ \widetilde{f} = f$. Then $\operatorname{im} \widetilde{f}$ is a path-connected subgroup of W and hence is an analytic subgroup. Thus $\operatorname{im} \widetilde{f}$ is a vector subspace W_f of W; it remains to observe that there is $g \in \kappa$ such that $g \sim \widetilde{f}|^{W_f}$, and $g \geq f$. Note that if $f \in K$, then $f|_G = f \circ \varepsilon$ is surjective as well, so that $f \circ \varepsilon$ is equivalent to an element of κ. To see this, note that $f(G)$, being a path-connected subgroup of the vector space H_f, is a vector subspace. Using that f is a continuous linear map, we deduce that $f(G) = f(\overline{\mathbb{Q}G}) = f(V) = H_f$. Here f is uniquely determined by $f|_G$, since $\operatorname{span} G$ is dense in V. Conversely, every morphism $f\colon G \to W$ in κ extends to a morphism $g\colon V \to W$. Indeed, since W is a torsion-free divisible group, f has a unique extension to a homomorphism $h\colon \mathbb{Q}G \to W$. If U is a convex, symmetric zero-neighbourhood in W and $r > 0$ such that $f(B_r(0) \cap G) \subseteq U$, where $B_r(0)$ denotes the open ball with radius r around 0 in $L^1(\mathbb{R})$, then $h(B_r(0) \cap \mathbb{Q}G) \subseteq 2U$. To see this, suppose that $\phi \in G$ and $n, m \in \mathbb{N}$ such that $\frac{n}{m}\phi \in B_r(0)$; we may assume $n = 1$ without loss of generality. Then $h(\frac{1}{m}\phi) = \frac{1}{m}f(\phi)$. If k is the smallest natural number such that $kr > \|\phi\|_1$, we find k disjoint measurable subsets $X_1, \ldots, X_k$ of $\mathbb{R}$ such that $\phi = \sum_{j=1}^{k} \phi\, \mathbf{1}_{X_j}$, where $\|\phi\, \mathbf{1}_{X_j}\|_1 < r$. We have $f(\phi\, \mathbf{1}_{X_j}) \in U$ for all j, and thus $f(\phi) \in kU$, whence $h(\frac{1}{m}\phi) \in \frac{k}{m}U$. It remains to note that $\|\frac{1}{m}\phi\|_1 < r$ and $rk \leq \|\phi\|_1 + r$, whence

$$\frac{k}{m} \leq \frac{\|\phi\|_1 + r}{rm} = \frac{\|\phi\|_1}{rm} + \frac{1}{m} < 2;$$

thus $h(\frac{1}{m}\phi) \in 2U$ indeed. We deduce that h is continuous; $\mathbb{Q}G$ being a dense subgroup of V, there exists a unique extension of h to a morphism

$g\colon V \to W$, and g is linear. The preceding observations imply that $\mathfrak{S}_{\mathbb{R}}(\varepsilon)$ is an isomorphism of projective systems, whence $\mathrm{L}_{\mathbb{R}}(\varepsilon) : \mathrm{L}_{\mathbb{R}}(G) \to \mathrm{L}_{\mathbb{R}}(V)$ is an isomorphism of topological Lie algebras. By Proposition 6.2, we have $\mathrm{L}_{\mathbb{R}}(V) \cong (V')^* \cong L^\infty(\mathbb{R})^*$; in particular, $\mathrm{L}_{\mathbb{R}}(G) \neq \{0\}$. We deduce that $\{0\} \neq \mathrm{L}^{\mathrm{arc}}_{\mathbb{R}}(G) = \mathrm{L}^0_{\mathbb{R}}(G) \cong \mathrm{L}^{\mathrm{loc}}_{\mathbb{R}}(G) \cong \mathrm{L}_{\mathbb{R}}(G)$, whereas $\mathrm{Hom}(\mathbb{R}, G) = \{0\}$ entails that all of $\Lambda^{\exp}_{\mathbb{R}}(G)$, $\overline{\Lambda}^{\exp}_{\mathbb{R}}(G)$, $\mathrm{L}^{\exp}_{\mathbb{R}}(G)$, $\Lambda^{\mathrm{arc}}_{\mathbb{R}}(G)$, $\overline{\Lambda}^{\mathrm{arc}}_{\mathbb{R}}(G)$, $\Lambda^0_{\mathbb{R}}(G)$, $\overline{\Lambda}^0_{\mathbb{R}}(G)$, $\Lambda^{\mathrm{loc}}_{\mathbb{R}}(G)$, $\overline{\Lambda}^{\mathrm{loc}}_{\mathbb{R}}(G)$, $\Lambda_{\mathbb{R}}(G)$, and $\overline{\Lambda}_{\mathbb{R}}(G)$ are the zero algebra.

8.6. Again, let F be a connected, locally connected, dense, totally pathwise disconnected subfield of $\mathbb{C}$. Since F is locally connected, the identity component $M := (F^\times)_0$ of $F^\times = F\backslash\{0\}$ is an open subset of F. The closure of M in $\mathbb{C}^\times$ is an open subgroup of the connected group $\mathbb{C}^\times$ and hence is all of $\mathbb{C}^\times$. Then $A := \mathbb{C}\rtimes\mathbb{C}^\times$ (where $\mathbb{C}^\times$ acts on $\mathbb{C}$ via multiplication) is a connected locally compact group; we consider $B := F\rtimes M$ as a dense subgroup of A. Let π be an irreducible continuous unitary representation of A on an infinite-dimensional complex Hilbert space $\mathcal{H}$; for example, we can take the representation π on $\mathcal{H} = L^2(\mathbb{C})$ given by

$$\pi(b, a)(f)(z) := |a| \cdot e^{i\,\mathrm{Re}(bz)} \cdot f(az)$$

for $(b, a) \in A$, $z \in \mathbb{C}$, $f \in L^2(\mathbb{C})$. Then also $\tau := \pi|_B$ is irreducible. We define $G_6 := G := \mathcal{H}\,{}_\tau\!\rtimes B$. If $f : G \to H_f$ is in $I_{\mathbb{R}}(G)$, we deduce as in the preceding example that there is a morphism $g\colon \mathcal{H} \to \mathbb{R}^n$ for some n such that $\ker g \leq \ker f$; hence $\ker f \cap \mathcal{H}$ contains a nontrivial vector subspace. Now the largest vector subspace V of $\mathcal{H}$ contained in $\ker f \cap \mathcal{H}$ is invariant under all automorphisms of the topological group $\mathcal{H}$ which leave $\ker f \cap \mathcal{H}$ invariant. Since $\ker f \cap \mathcal{H}$ is a normal subgroup of G, we deduce that V is invariant under the action of B. The irreducibility assumption yields $V = \mathcal{H}$, whence $\mathcal{H} \leq \ker f$. Thus, if $q : G \to B$ denotes the quotient morphism, $\mathfrak{S}_{\mathbb{R}}(q)$ is an isomorphism. Note that $\mathcal{U}(G) = \{G\}$, since G is connected. Thus $\mathrm{L}^{\mathrm{loc}}_{\mathbb{R}}(G) \cong \mathrm{L}^0_{\mathbb{R}}(G) = \mathrm{L}_{\mathbb{R}}(G) \cong \mathrm{L}_{\mathbb{R}}(B) \cong \mathrm{L}_{\mathbb{R}}(A)$ is a nontrivial finite-dimensional Lie algebra, $\Lambda^0_{\mathbb{R}}(G)$, $\overline{\Lambda}^0_{\mathbb{R}}(G)$, $\Lambda^{\mathrm{loc}}_{\mathbb{R}}$, $\overline{\Lambda}^{\mathrm{loc}}_{\mathbb{R}}(G)$, $\Lambda_{\mathbb{R}}(G)$, and $\overline{\Lambda}_{\mathbb{R}}(G)$ is the zero algebra, whereas $\Lambda^{\exp}_{\mathbb{R}}(G)$, $\overline{\Lambda}^{\exp}_{\mathbb{R}}(G)$, $\mathrm{L}^{\exp}_{\mathbb{R}}(G)$ $\Lambda^{\mathrm{arc}}_{\mathbb{R}}(G)$, $\overline{\Lambda}^{\mathrm{arc}}_{\mathbb{R}}(G)$, and $\mathrm{L}^{\mathrm{arc}}_{\mathbb{R}}(G)$ are infinite-dimensional abelian Lie algebras.

8.7. Let M be any connected locally compact group which admits an irreducible continuous unitary representation π on an infinite-dimensional Hilbert space $\mathcal{H}$ (see **8.6**). By [**32**], there exists a quotient morphism $q\colon T \to M$ from a totally disconnected topological group T onto M; then $\tau := \pi \circ q$ is an irreducible continuous unitary representation of T on $\mathcal{H}$. We claim that $G := G_7 := \mathcal{H}\,{}_\tau\!\rtimes T$ is an example of a topological group satisfying $\Lambda^{\mathrm{loc}}_{\mathbb{R}}(G) \neq \Lambda^0_{\mathbb{R}}(G)$. To see this, suppose that U is an open subgroup of G. Since $U \cap \mathcal{H}$ is an open subgroup of $\mathcal{H}$, we have $U \cap \mathcal{H} = \mathcal{H}$. This entails that $U = \mathcal{H}\,{}_{\tau|_V}\!\rtimes V$, where $V := U \cap T$ is an open subgroup of T. Now $q(V)$ being

	$\overline{\Lambda}^{\mathrm{exp}}_{\mathbb{R}}$	$\mathrm{L}^{\mathrm{exp}}_{\mathbb{R}}$	$\Lambda^{\mathrm{arc}}_{\mathbb{R}}$	$\overline{\Lambda}^{\mathrm{arc}}_{\mathbb{R}}$	$\mathrm{L}^{\mathrm{arc}}_{\mathbb{R}}$	$\Lambda^{0}_{\mathbb{R}}$	$\overline{\Lambda}^{0}_{\mathbb{R}}$	$\mathrm{L}^{0}_{\mathbb{R}}$	$\Lambda^{\mathrm{loc}}_{\mathbb{R}}$	$\overline{\Lambda}^{\mathrm{loc}}_{\mathbb{R}}$	$\mathrm{L}^{\mathrm{loc}}_{\mathbb{R}}$	$\Lambda_{\mathbb{R}}$	$\overline{\Lambda}_{\mathbb{R}}$	$\mathrm{L}_{\mathbb{R}}$
$\Lambda^{\mathrm{exp}}_{\mathbb{R}}$	4	4	8	4	4	6	4	2	6	4	1	3	3	1
$\overline{\Lambda}^{\mathrm{exp}}_{\mathbb{R}}$		9	4	8	5	4	6	2	4	6	1	3	3	1
$\mathrm{L}^{\mathrm{exp}}_{\mathbb{R}}$			4	8	5	4	6	2	4	6	1	3	3	1
$\Lambda^{\mathrm{arc}}_{\mathbb{R}}$				4	4	6	4	2	6	4	1	3	3	1
$\overline{\Lambda}^{\mathrm{arc}}_{\mathbb{R}}$					5	4	6	2	4	6	1	3	3	1
$\mathrm{L}^{\mathrm{arc}}_{\mathbb{R}}$						4	5	2	4	5	1	3	3	1
$\Lambda^{0}_{\mathbb{R}}$							4	2	7	4	1	3	3	1
$\overline{\Lambda}^{0}_{\mathbb{R}}$								2	4	7	1	3	3	1
$\mathrm{L}^{0}_{\mathbb{R}}$									2	2	1	2	2	1
$\Lambda^{\mathrm{loc}}_{\mathbb{R}}$										4	1	3	3	1
$\overline{\Lambda}^{\mathrm{loc}}_{\mathbb{R}}$											1	3	3	1
$\mathrm{L}^{\mathrm{loc}}_{\mathbb{R}}$												1	1	3
$\Lambda_{\mathbb{R}}$													4	1
$\overline{\Lambda}_{\mathbb{R}}$														1

Figure 2. The entry i indicates that the abstract Lie algebras associated with G_i by the functors indexing the respective row and column are not isomorphic.

an open subgroup of M, the connectedness of M implies that $q(V) = M$. Thus $\tau|_V$ is an irreducible continuous unitary representation of V on the infinite-dimensional complex Hilbert space $\mathcal{H}$. As in **8.6**, we deduce that $\mathcal{H} \subseteq \ker f$, for every $f \in I_{\mathbb{R}}(U)$. On the other hand, $\mathrm{im}(X) \subseteq \mathcal{H}$ for every one-parameter subgroup $X : \mathbb{R} \to U$ of $U = \mathcal{H}_{\tau|_V} \rtimes V$, the group V being totally disconnected. We deduce that $\Lambda_{\mathbb{R}}(U) = \{0\}$. Thus $\Lambda^{\mathrm{loc}}_{\mathbb{R}}(G) = \{0\}$. In contrast, we have $\Lambda^{0}_{\mathbb{R}}(G) = \Lambda_{\mathbb{R}}(\mathcal{H}) \cong \mathcal{H}_w$ by Proposition 6.2 (e), which is an infinite-dimensional abelian topological Lie algebra.

8.8. Let H be the Heisenberg group associated with $\mathbb{R}$; thus $H := \mathbb{R} \times \mathbb{R} \times \mathbb{T}$ as a topological space, with multiplication given by $(x, y, z)(x', y', z') := (x + x', y + y', zz'e^{iyx'})$. Let π be the Schrödinger representation of H on $\mathcal{H} := L^2(\mathbb{R})$; then π is an irreducible, continuous, unitary representation, see [**26**], Lemma A.VIII.2; the torus acts via scalar multiplication. Let $A := G_5$ be as in **8.5**, and $f : A \to \mathbb{R}$ be a surjective morphism; we make the topological space $K := A \times A \times \mathbb{T}$ a topological group by setting

$(x,y,z)(x',y',z') := (x+x', y+y', zz'e^{if(y)f(x')})$. Then $\psi := f \times f \times \mathrm{id}_{\mathbb{T}}$, $K \to H$ is a surjective morphism, whence $\tau := \pi \circ \psi$ is an irreducible, continuous, unitary representation of K. We set $G_8 := G := \mathcal{H}\,{}_\tau\!\rtimes K$; then $G_{\mathrm{arc}} = G$, and $c_{\mathbb{R}}(G) = \mathcal{H} \rtimes \mathbb{T}$. As above, we find that every morphism from G into a Lie group has $\mathcal{H}$ in its kernel: Thus $\Lambda^{\mathrm{arc}}_{\mathbb{R}}(G) \cong \Lambda^{\mathrm{arc}}_{\mathbb{R}}(K) = \Lambda_{\mathbb{R}}(K)$. If $h : \mathbb{T} \hookrightarrow K$ is the embedding, $\mathrm{Hom}(\mathbb{R}, h) : \mathrm{Hom}(\mathbb{R}, \mathbb{T}) \to \mathrm{Hom}(\mathbb{R}, K)$ is a bijection. Since $r^{\mathbb{T}}_{\mathbb{R}}$ is a bijection onto $\Lambda_{\mathbb{R}}(\mathbb{T}) \cong \mathbb{R}$, we easily deduce that $\Lambda_{\mathbb{R}}(h)$ maps $\Lambda_{\mathbb{R}}(\mathbb{T})$ onto $\Lambda_{\mathbb{R}}(K)$, whence the latter is at most one-dimensional. Let F be a finite-dimensional complex vector subspace of $\mathcal{H}$ now; then F and $F^{\perp}$ are invariant under the multiplication action of $\mathbb{T}$, and hence are normal subgroups of $\mathcal{H} \rtimes \mathbb{T}$; we obtain a quotient map $q : \mathcal{H} \rtimes \mathbb{T} \to (\mathcal{H} \rtimes \mathbb{T})/F^{\perp} \cong F \rtimes \mathbb{T}$. It is easy to see that $\Lambda_{\mathbb{R}}(q)$ is surjective, whence $\Lambda^{\exp}_{\mathbb{R}}(G) = \Lambda_{\mathbb{R}}(\mathcal{H} \rtimes \mathbb{T})$ is infinite-dimensional. Thus $\overline{\Lambda}^{\exp}_{\mathbb{R}}(G)$ is infinite-dimensional as well, whereas $\overline{\Lambda}^{\mathrm{arc}}_{\mathbb{R}}(G)$ has finite dimension.

8.9. Let $\xi : \mathbb{R} \to \mathbb{T}^2$ be a morphism with dense image $G_9 := \mathrm{im}\,\xi$. Then $G = c_{\mathbb{R}}(G)$ and $\Lambda^{\exp}_{\mathbb{R}}(G) = \overline{\Lambda}^{\exp}_{\mathbb{R}}(G) = \Lambda_{\mathbb{R}}(G) \cong \mathbb{R}$, whereas $\mathrm{L}^{\exp}_{\mathbb{R}}(G) = \mathrm{L}_{\mathbb{R}}(G) \cong \mathrm{L}_{\mathbb{R}}(\mathbb{T}^2) \cong \mathbb{R}^2$.

The information assembled in the preceding examples suffices to conclude that the Lie algebra functors $\Lambda^{\exp}_{\mathbb{R}}$, $\overline{\Lambda}^{\exp}_{\mathbb{R}}$, $\mathrm{L}^{\exp}_{\mathbb{R}}$, $\Lambda^{\mathrm{arc}}_{\mathbb{R}}$, $\overline{\Lambda}^{\mathrm{arc}}_{\mathbb{R}}$, $\mathrm{L}^{\mathrm{arc}}_{\mathbb{R}}$, $\Lambda^{0}_{\mathbb{R}}$, $\overline{\Lambda}^{0}_{\mathbb{R}}$, $\mathrm{L}^{0}_{\mathbb{R}}$, $\Lambda^{\mathrm{loc}}_{\mathbb{R}}$, $\overline{\Lambda}^{\mathrm{loc}}_{\mathbb{R}}$, $\mathrm{L}^{\mathrm{loc}}_{\mathbb{R}}$, $\Lambda_{\mathbb{R}}$, $\overline{\Lambda}_{\mathbb{R}}$, and $\mathrm{L}_{\mathbb{R}}$ are pairwise not naturally isomorphic, see Figure 2. Indeed, we have shown more: The functors are not even pairwise naturally isomorphic when considered as functors into the category of abstract Lie algebras.

9. $\mathbb{Z}_p$-Lie algebras of abelian topological groups.

It is our next goal to study the specific behaviour of the p-adic Lie algebra functors on the category of locally compact groups (Section 10 below). As a prerequisite, in the present section we assemble information concerning the natural $\mathbb{Z}_p$-Lie algebras of abelian topological groups.

On the category $\mathbb{TAB}$ of abelian topological groups, we can consider the functor $\mathrm{Hom}(\mathbb{Q}_p, \bullet)$ into the category of topological abelian $\mathbb{Q}_p$-Lie algebras, and the functor $\mathrm{Hom}(\mathbb{Z}_p, \bullet)$ into the category of abelian topological $\mathbb{Z}_p$-Lie algebras (the Hom-groups being equipped with the compact-open topology, addition being defined pointwise, and multiplication via $(rX)(t) := X(rt)$). To see that given $G \in \mathbb{TAB}$, the topological abelian group $\mathrm{Hom}(\mathbb{Q}_p, G)$ is a topological $\mathbb{Q}_p$-vector space, recall that if V is a $\mathbb{Q}_p$-vector space and $\mathcal{V}$ a filter in V, then $\mathcal{V}$ is the neighbourhood filter at 0 of a $\mathbb{Q}_p$-topological vector space topology on V if and only if:

(a) For every $U \in \mathcal{V}$, there is $W \in \mathcal{V}$ such that $W + W \subseteq U$;
(b) Every $U \in \mathcal{V}$ is absorbing, that is, for every $x \in V$, there is $n \in \mathbb{N}$ such that $p^m x \in U$ for all $m \geq n$;

(c) For every $U \in \mathcal{V}$, there exists a balanced set $W \in \mathcal{V}$ (i.e., a set satisfying $\mathbb{Z}_p W = W$) such that $W \subseteq U$;
(d) For every $U \in \mathcal{V}$, there exists $W \in \mathcal{V}$ such that $W \subseteq pU$.

The sets $\lfloor K, U \rfloor := \{X \in \mathrm{Hom}(\mathbb{Q}_p, G) : X(K) \subseteq U\}$, where K ranges through the compact subsets of $\mathbb{Q}_p$ and U through the identity neighbourhoods in G, generate a filter $\mathcal{V}$. Since $\mathrm{Hom}(\mathbb{Q}_p, G)$ is a topological group, $\mathcal{V}$ satisfies Condition (a). Now given K and U, note that $K' := \mathbb{Z}_p K$ is balanced, hence so is $\lfloor K', U \rfloor \subseteq \lfloor K, U \rfloor$. For every $X \in \mathrm{Hom}(\mathbb{Q}_p, G)$, the open subset $X^{-1}(U)$ in $\mathbb{Q}_p$ contains a ball $p^m \mathbb{Z}_p$ for some m. Furthermore, there is $n \in \mathbb{Z}$ such that $K \subseteq p^n \mathbb{Z}_p$. Then $p^k X \in \lfloor K, U \rfloor$ holds, for every $k \geq m - n$, whence the latter set is absorbing. Note that Condition (d) is satisfied since $\lfloor p^{-1} K, U \rfloor \subseteq p \lfloor K, U \rfloor$. Hence $\mathrm{Hom}(\mathbb{Q}_p, G)$ is a topological vector space over $\mathbb{Q}_p$. The proof that $\mathrm{Hom}(\mathbb{Z}_p, G)$ is a topological $\mathbb{Z}_p$-module is similar: We have to check Conditions (a), (b), and (c) above.

The restriction map $\mathrm{Hom}(\mathbb{Q}_p, G) \to \mathrm{Hom}(\mathbb{Z}_p, G)$ is a morphism of topological $\mathbb{Z}_p$-modules. Furthermore, we have $\mathbb{Q}_p$-linear mappings

$$\mathrm{Hom}(\mathbb{Q}_p, G) \xrightarrow{[.]^G_{\mathbb{Q}_p}} \Gamma(\mathbb{Q}_p, G) \xrightarrow{t^G_{\mathbb{Q}_p}} \Lambda_{\mathbb{Q}_p}(G),$$

where we define addition and scalar multiplication on $\Gamma(\mathbb{Q}_p, G)$ analogously to the definitions for $\mathrm{Hom}(\mathbb{Q}_p, G)$, and set $t^G_{\mathbb{Q}_p} := r^G_{\mathbb{Q}_p}|^{\Lambda_{\mathbb{Q}_p}(G)}$; here $[.]^G_{\mathbb{Q}_p}$ denotes the mapping which assigns the germ $[X]$ to a morphism $X \colon \mathbb{Q}_p \to G$. The map $[.]^G_{\mathbb{Q}_p}$ need not be surjective: E.g., $\mathrm{Hom}(\mathbb{Q}_p, \mathbb{Z}_p) = \{0\}$ but $\Gamma(\mathbb{Q}_p, \mathbb{Z}_p) \cong \mathbb{Q}_p$; it also does not have to be injective in general. For instance, the discrete Prüfer p-group $G := \mathbb{Z}(p^\infty)$ is isomorphic to $\mathbb{Q}_p/\mathbb{Z}_p$, whence $\mathrm{Hom}(\mathbb{Q}_p, G) \neq \{0\}$; however, $\Gamma(\mathbb{Q}_p, G) = \{0\}$ since G is discrete. The mapping $t^G_{\mathbb{Q}_p}$ need not be injective (as the group constructed in the proof of Proposition 6.2 (c) shows), but it is always surjective in the abelian setting (since $r^G_{\mathbb{Q}_p}$ is linear).

Given an abelian topological group G, we define $\exp^G_p : \mathrm{Hom}(\mathbb{Z}_p, G) \to G$ via $X \mapsto X(1)$; note that $\exp^G_p$ is a continuous homomorphism, since $\mathrm{Hom}(\mathbb{Z}_p, G)$ is equipped with the compact-open topology, which is finer than the topology of pointwise convergence. Let F denote the forgetful functor from the category of topological $\mathbb{Z}_p$-modules to the category of abelian topological groups. Plainly $\exp_p \colon \mathrm{id} \stackrel{\bullet}{\to} F \circ \mathrm{Hom}(\mathbb{Z}_p, \bullet)$ is a natural transformation.

Let us have a closer look at the functor $\mathrm{Hom}(\mathbb{Z}_p, \bullet)$ now.

Theorem 9.1. *The functor* $\mathrm{Hom}(\mathbb{Z}_p, \bullet)$ *has the following properties:*

(a) *If G is an abelian topological group, H is a closed subgroup of G, and $\alpha : H \hookrightarrow G$ is the inclusion morphism, then* $\mathrm{Hom}(\mathbb{Z}_p, \alpha)$ *is an isomorphism onto its image* $\{X \in \mathrm{Hom}(\mathbb{Z}_p, G) \colon \mathrm{im}\, X \leq H\}$.

(b) *For every morphism* $\beta\colon G\to H$ *between abelian topological groups, we have* $\ker(\mathrm{Hom}(\mathbb{Z}_p,\beta))=\mathrm{Hom}(\mathbb{Z}_p,\ker(\beta))$.

(c) *If* G *and* H *are compact abelian groups and* $\gamma:G\to H$ *is a quotient morphism, then* $\mathrm{Hom}(\mathbb{Z}_p,\gamma):\mathrm{Hom}(\mathbb{Z}_p,G)\to\mathrm{Hom}(\mathbb{Z}_p,H)$ *is a surjective morphism of topological* $\mathbb{Z}_p$*-modules.*

(d) *If* $(G_i)_{i\in I}$ *is a family of abelian topological groups, with direct product* G*, then* $(\mathrm{Hom}(\mathbb{Z}_p,\mathrm{pr}_i))_{i\in I}:\mathrm{Hom}(\mathbb{Z}_p,G)\to\prod_{i\in I}\mathrm{Hom}(\mathbb{Z}_p,G_i)$ *is an isomorphism of topological* $\mathbb{Z}_p$*-modules. Similarly,* $\mathrm{Hom}(\mathbb{Z}_p,\bullet)$ *respects projective limits.*

Proof. (cf. [**14**], Proposition 7.38).

(a) Let $\alpha:H\hookrightarrow G$ denote the inclusion morphism. It is plain that $\mathrm{Hom}(\mathbb{Z}_p,\alpha)\colon\mathrm{Hom}(\mathbb{Z}_p,H)\to\mathrm{Hom}(\mathbb{Z}_p,G)$ is a continuous injective homomorphism with image $\{X\in\mathrm{Hom}(\mathbb{Z}_p,G)\colon \operatorname{im}X\le H\}$, whence we may identify $\operatorname{im}(\mathrm{Hom}(\mathbb{Z}_p,\alpha))$ with $\mathrm{Hom}(\mathbb{Z}_p,H)$ as an abstract abelian group. If U is an open identity neighbourhood in H, there exists an open identity neighbourhood V in G such that $H\cap V=U$. Then $\lfloor\mathbb{Z}_p,V\rfloor_H=\lfloor\mathbb{Z}_p,U\rfloor_G\cap\mathrm{Hom}(\mathbb{Z}_p,H)$ proves that $\mathrm{Hom}(\mathbb{Z}_p,\alpha)$ is open onto its image, i.e., an embedding of topological $\mathbb{Z}_p$-modules.

(b) The sets clearly coincide; now use (a).

(c) The discrete Prüfer p-group $\mathbb{Z}(p^\infty)$ is divisible, whence it is an injective object in the abelian category of discrete abelian groups ([**7**], Theorem 21.1; cf. [**11**] for the terminology). In view of Pontryagin duality, the dual group $\mathbb{Z}(p^\infty)^\wedge\cong\mathbb{Z}_p$ is a projective object in the abelian category of compact abelian groups: This is what (c) asserts.

(d) The proof of (d) is analogous to the real case treated in [**14**], Proposition 7.38; we shall not need it in the following. □

The following question arises naturally:

Problem 9.2. If G, H are compact abelian groups and $\gamma:G\to H$ is a quotient morphism, we have seen in Theorem 9.1 (c) that $\mathrm{Hom}(\mathbb{Z}_p,\gamma):\mathrm{Hom}(\mathbb{Z}_p,G)\to\mathrm{Hom}(\mathbb{Z}_p,H)$ is a continuous surjective homomorphism of $\mathbb{Z}_p$-modules. Is it true in general that $\mathrm{Hom}(\mathbb{Z}_p,\gamma)$ is also open, hence a quotient morphism?

We conclude this section with a number of applications of Theorem 9.1 (c). If G is a locally compact group, let $P(G)$ be the set of periodic elements of G; recall that an element $g\in G$ is called *periodic* if the subgroup generated by g is relatively compact. If G is a totally disconnected, locally compact group, then $P(G)$ is closed in G, see [**36**].

Corollary 9.3. *Let* $\psi:G\to H$ *be a morphism between locally compact groups, and* $Y:\mathbb{Z}_p\to H$ *be a morphism. Then there exists a morphism* $X\colon\mathbb{Z}_p\to G$ *such that* $\psi\circ X=Y$ *if and only if* $\psi^{-1}(\{Y(1)\})\cap P(G)\neq\emptyset$.

Proof. Suppose that X exists; then clearly $X(1) \in P(G)$ and $X(1) \in \psi^{-1}(\{Y(1)\})$. Conversely, if there is $x \in \psi^{-1}(\{Y(1)\}) \cap P(G)$, we let A be the closed subgroup of G generated by x. Then A is a compact abelian group, and $Y(\mathbb{Z}) \leq B := \psi(A)$. Using the density of $\mathbb{Z}$ in $\mathbb{Z}_p$ and the fact that B is compact, we deduce that $Y(\mathbb{Z}_p) \subseteq B$. By Theorem 9.1 (c), there is $X \in \mathrm{Hom}(\mathbb{Z}_p, A)$ such that $\psi|_A^B \circ X = Y|^B$. Thus X, considered as a morphism $\mathbb{Z}_p \to G$, satisfies $\psi \circ X = Y$. □

Corollary 9.4. *Let $\psi : G \to H$ be a morphism between locally compact groups. Then $\psi(R_{\mathbb{Q}_p}(G)) \subseteq R_{\mathbb{Q}_p}(H)$; an element $y \in H$ is contained in $\psi(R_{\mathbb{Q}_p}(G))$ if and only if $y \in R_{\mathbb{Q}_p}(H)$ and $\phi^{-1}(\{y\}) \cap P(G) \neq \emptyset$.*

Proof. Since $\psi \circ X \in \mathrm{Hom}(\mathbb{Z}_p, H)$ for every $X \in \mathrm{Hom}(\mathbb{Z}_p, G)$, it is obvious that $\psi(R_{\mathbb{Q}_p}(G)) \subseteq R_{\mathbb{Q}_p}(H)$. The remainder follows immediately from Corollary 9.3. □

As a special case, we obtain:

Corollary 9.5. *Let $q : G \to Q$ be a quotient morphism between compact groups. Then $q(R_{\mathbb{Q}_p}(G)) = R_{\mathbb{Q}_p}(Q)$, $q(c_{\mathbb{Q}_p}(G)) = c_{\mathbb{Q}_p}(Q)$, and $q(C_{\mathbb{Q}_p}(G)) = C_{\mathbb{Q}_p}(Q)$.*

10. $\mathbb{Q}_p$-Lie algebras of locally compact groups.

When restricted to the category of locally compact groups, the p-adic Lie algebra functors behave much better than they do on the whole category of Hausdorff groups. In this section, we describe various results specific to the locally compact case.

The investigation of the p-adic Lie algebras $\mathrm{L}_{\mathbb{Q}_p}(G)$ for locally compact groups G can be reduced to the case where G is totally disconnected, since every morphism $f : G \to H_f$ into a p-adic Lie group H_f factors through the totally disconnected group G/G_0, whence $\mathrm{L}_{\mathbb{Q}_p}(\pi) : \mathrm{L}_{\mathbb{Q}_p}(G) \to \mathrm{L}_{\mathbb{Q}_p}(G/G_0)$ is an isomorphism (here $\pi : G \to G/G_0$ denotes the canonical quotient morphism).

Proposition 10.1. *Let G and H be totally disconnected, locally compact groups, and $q : G \to H$ be a quotient morphism. Then $\Gamma(\mathbb{Q}_p, q) : \Gamma(\mathbb{Q}_p, G) \to \Gamma(\mathbb{Q}_p, H)$ is surjective. The same conclusion holds if G and H are arbitrary compact groups.*

Proof. Let U be a compact open subgroup of G; then $q(U)$ is a compact open subgroup of H. If $[Y] \in \Gamma(\mathbb{Q}_p, H)$, we may assume that the representative Y is chosen such that $\mathrm{im}\, Y \subseteq q(U)$ and such that Y has domain of definition $p^n\mathbb{Z}_p$ for some $n \in \mathbb{Z}$. Then $Y_1 := Y(p^n \bullet) : \mathbb{Z}_p \to H$ is a morphism with image in $q(U)$, and by Corollary 9.3, there is a morphism $X_1 : \mathbb{Z}_p \to G$ such that $q \circ X_1 = Y_1$. Then $q \circ X_1(p^{-n} \bullet) = Y$. □

We abbreviate $\mathbb{K} := \mathbb{Q}_p$ for the rest of this section.

Corollary 10.2. *If G is a totally disconnected, locally compact group, or if G is a compact group, then* $\operatorname{im} r_{\mathbb{K}}^G$ *is dense in* $\mathrm{L}_{\mathbb{K}}(G)$*, i.e.,* $\overline{\Lambda}_{\mathbb{K}}(G) = \mathrm{L}_{\mathbb{K}}(G)$*. Also,* $\operatorname{im} \rho_{\mathbb{K}}^G$ *is dense in* $\mathrm{L}_{\mathbb{K}}^{\mathrm{loc}}(G)$*.*

Proof. Suppose that $X = (X_f)_{f \in I_{\mathbb{K}}(G)} \in \mathrm{L}_{\mathbb{K}}(G)$, and suppose that U is a neighbourhood of X in $\mathrm{L}_{\mathbb{K}}(G)$. We may assume that U is of the form $\mathrm{L}_{\mathbb{K}}(G) \cap \prod_{f \in I_{\mathbb{K}}(G)} U_f$, where U_f is open in $\mathcal{L}_{\mathbb{K}}(H_f)$ and $U_f = \mathcal{L}_{\mathbb{K}}(H_f)$ unless $f \in F$ for some finite subset F of $I_{\mathbb{K}}(G)$. Let g be an upper bound for F in $I_{\mathbb{K}}(G)$. Since G is locally compact, $J_{\mathbb{K}}(G)$ is cofinal in $I_{\mathbb{K}}(G)$. We may therefore assume that g is a quotient morphism. By Proposition 10.1, there exists $Y \in \Gamma(\mathbb{K}, G)$ such that $\Gamma(\mathbb{K}, g)(Y) = X_g$. Then $r_{\mathbb{K}}^G(Y) \in U$, which proves that $\operatorname{im} r_{\mathbb{K}}^G$ is dense in $\mathrm{L}_{\mathbb{K}}(G)$.

Now suppose that $X = (X_U)_{U \in \mathcal{U}(G)} \in \mathrm{L}_{\mathbb{K}}^{\mathrm{loc}}(G)$ and that V is a neighbourhood of X in $\mathrm{L}_{\mathbb{K}}^{\mathrm{loc}}(G)$; without loss of generality $V = \mathrm{L}_{\mathbb{K}}^{\mathrm{loc}} \cap \prod_{U \in \mathcal{U}(G)} V_U$, where each V_U is open in $\mathrm{L}_{\mathbb{K}}(U)$ and $V_U = \mathrm{L}_{\mathbb{K}}(U)$ unless $U \in F$ for a finite subset F of $\mathcal{U}(G)$. Then $W := \bigcap F$ is an upper bound for F. Since $r_{\mathbb{K}}^W$ has dense image, there exists $Y \in \Gamma(\mathbb{K}, W) = \Gamma(\mathbb{K}, G)$ such that $r_{\mathbb{K}}^W(Y) \in \bigcap_{U \in F} \mathrm{L}_{\mathbb{K}}(k_{UW})^{-1}(V_U)$. Then $\rho_{\mathbb{K}}^G(Y) \in V$. □

The following result is the p-adic analogue of [**18**], Lemma 1.3:

Corollary 10.3. *Suppose that G, H are locally compact groups and $q\colon G \to H$ is a quotient morphism. Then* $\mathrm{L}_{\mathbb{K}}(q) : \mathrm{L}_{\mathbb{K}}(G) \to \mathrm{L}_{\mathbb{K}}(H)$ *is a quotient morphism.*

Proof. Set $G' := G/G_0$, $H' := H/H_0$, and let $g : G \to G'$, $h : H \to H'$ denote the canonical quotient morphisms; there is a unique quotient morphism $q' : G' \to H'$ such that $q' \circ g = h \circ q$. Since $\mathrm{L}_{\mathbb{K}}(g)$ and $\mathrm{L}_{\mathbb{K}}(h)$ are isomorphisms, we only need to show that $\mathrm{L}_{\mathbb{K}}(q')$ is a quotient morphism, i.e., we may assume that G and H are totally disconnected. By naturality of $r_{\mathbb{K}}$ and Proposition 10.1, we conclude that $\operatorname{im} \mathrm{L}_{\mathbb{K}}(q) \supseteq \operatorname{im} \mathrm{L}_{\mathbb{K}}(q) \circ r_{\mathbb{K}}^G = \operatorname{im} r_{\mathbb{K}}^H \circ \Gamma(\mathbb{K}, q) = \operatorname{im} r_{\mathbb{K}}$, which is dense in $\mathrm{L}_{\mathbb{K}}(H)$ by Corollary 10.2. Now Proposition 2.1 applies. □

Corollary 10.4. *Suppose that $G_1, \ldots, G_n$ are locally compact groups, and G their direct product. Then* $\mathrm{L}_{\mathbb{K}}(G) \cong \mathrm{L}_{\mathbb{K}}(G_1) \times \cdots \times \mathrm{L}_{\mathbb{K}}(G_n)$*. If $(G_i)_{i \in I}$ is a family of locally compact groups, almost all of which are compact, with cartesian product G, then* $\nabla := (\mathrm{L}_{\mathbb{K}}(\mathrm{pr}_i))_{i \in I} : \mathrm{L}_{\mathbb{K}}(G) \to \prod_{i \in I} \mathrm{L}_{\mathbb{K}}(G_i)$ *is a quotient morphism.*

Proof. Reduce to the disconnected case; now the hypotheses of Theorem 7.1 are satisfied in view of Corollary 10.2. □

Corollary 10.5. *Suppose that U, G are locally compact groups and that $\alpha\colon U \to G$ is an open embedding. Then* $\mathrm{L}_{\mathbb{K}}(\alpha)$ *is a quotient morphism.*

Proof. We may assume that U and G are totally disconnected. Since $\alpha(U)$ is open in G, we deduce that $\Gamma(\mathbb{K}, \alpha)$ is a bijection. Now proceed as in the proof of Corollary 10.3. □

Corollary 10.6. *Suppose that G is locally compact, and U an open subgroup of G. Then the canonical morphism $\pi_U : \mathrm{L}^{\mathrm{loc}}_{\mathbb{K}}(G) \to \mathrm{L}_{\mathbb{K}}(U)$ is a quotient map.*

Proof. We can reduce to the totally disconnected case. Now $\pi_U \circ \rho^G_{\mathbb{K}} = r^U_{\mathbb{K}}$ shows that $\operatorname{im} r^U_{\mathbb{K}} \subseteq \operatorname{im} \pi_U$, whence $\operatorname{im} \pi_U$ is dense in $\mathrm{L}_{\mathbb{K}}(U)$. The claim follows from Proposition 2.1. □

Corollary 10.7. *The analogues of Corollary* 10.3 *and Corollary* 10.4 *hold for the functor $\mathrm{L}^{\mathrm{loc}}_{\mathbb{K}}$. An analogue of Corollary* 10.4 *holds for the functor $\mathrm{L}^{\exp}_{\mathbb{K}}$; also an analogue of Corollary* 10.3 *holds for this functor, with the restriction that all groups involved have to be totally disconnected.*

Proof. Let G and H be locally compact groups and $q\colon G \to H$ be a quotient morphism. Then $q|^U_{q^{-1}(U)}$ is a quotient morphism, for every $U \in \mathcal{U}(H)$, whence $\mathrm{L}_{\mathbb{K}}(k_{U,q^{-1}(U)}) : \mathrm{L}_{\mathbb{K}}(q^{-1}(U)) \to \mathrm{L}_{\mathbb{K}}(U)$ (where $k_{U,q^{-1}(U)}$ is defined as in the definition of $\mathcal{R}$) is a quotient morphism of topological Lie algebras, by Corollary 10.3. From the preceding and Corollary 10.6, we deduce that $\mathrm{L}^{\mathrm{loc}}_{\mathbb{K}}(q)$ has dense image; now Proposition 2.1 shows that $\mathrm{L}^{\mathrm{loc}}_{\mathbb{K}}(q)$ is a quotient morphism. In the case where G and H are totally disconnected, we know from Proposition 10.1 that $\Gamma(K, q)$ is surjective; then clearly $\Gamma(\mathbb{K}, c_{\mathbb{K}}(q))$ is surjective as well. As in the proof of Corollary 10.3, this implies that $\mathrm{L}^{\exp}_{\mathbb{K}}(q) = \mathrm{L}_{\mathbb{K}}(c_{\mathbb{K}}(q))$ is a quotient map. The analogue of Corollary 10.4 for the functor $\mathrm{L}^{\mathrm{loc}}_{\mathbb{K}}$ follows immediately from Corollary 10.4 and Theorem 7.3. The analogue for the functor $\mathrm{L}^{\exp}_{\mathbb{K}}$ follows from Corollary 10.4 and Remark 7.2. □

11. Groups of finite *p*-adic dimension.

If G is a totally disconnected, locally compact group, then $\mathrm{L}^{\mathrm{loc}}_{\mathbb{Q}_p}(G)$ is a weakly complete topological Lie algebra and hence is isomorphic to $\mathbb{Q}_p^I$ for a set I, where $\operatorname{card}(I) = \dim_{\mathbb{Q}_p}(\mathrm{L}^{\mathrm{loc}}_{\mathbb{Q}_p}(G)')$ is uniquely determined. We call $\operatorname{card}(I)$ the (*local*) *p-adic dimension of* G. The *global p-adic dimension of* G is the dimension of $\mathrm{L}_{\mathbb{Q}_p}(G)'$. In the following, we show that for every locally compact group G of finite p-adic dimension, the set $\Gamma(\mathbb{Q}_p, G)$ of germs at 0 of local p-adic one-parameter subgroups, *modulo suitable equivalence relations*, can be identified with the local p-adic Lie algebra of G.

Proposition 11.1. *Suppose that G is a totally disconnected, locally compact group of finite p-adic dimension. Then $\rho^G_{\mathbb{Q}_p} : \Gamma(\mathbb{Q}_p, G) \to \mathrm{L}^{\mathrm{loc}}_{\mathbb{Q}_p}(G)$ is* *surjective. If G is a locally compact, totally disconnected group of finite global p-adic dimension, the map $r^G_{\mathbb{Q}_p}\colon \Gamma(\mathbb{Q}_p, G) \to \mathrm{L}_{\mathbb{Q}_p}(G)$ is surjective.*

Proof. We abbreviate $\mathbb{K} := \mathbb{Q}_p$. Assume that G has finite p-adic dimension; the case of finite global p-adic dimension can be treated similarly. Since the finite-dimensional Lie algebra $\mathrm{L}^{\mathrm{loc}}_{\mathbb{K}}(G)$ is the projective limit of the Lie algebras $\mathrm{L}_{\mathbb{K}}(U)$ and the limit maps $\pi_U : \mathrm{L}^{\mathrm{loc}}_{\mathbb{K}}(G) \to \mathrm{L}_{\mathbb{K}}(U)$ are surjective by Corollary 10.6 for all $U \in \mathcal{U}(G)$, there exists an open subgroup W of G such that the canonical morphism $\pi_W : \mathrm{L}^{\mathrm{loc}}_{\mathbb{K}}(G) \to \mathrm{L}_{\mathbb{K}}(W)$ is an isomorphism. Similarly, since $\mathrm{L}_{\mathbb{K}}(W)$ is finite-dimensional, there exists $f \in I_{\mathbb{K}}(W)$, $f : W \to H$ such that $\mathrm{L}_{\mathbb{K}}(f)$ is injective. By the cofinality of $J_{\mathbb{K}}(W)$ in $I_{\mathbb{K}}(W)$, we may assume that f is a quotient morphism: Then $\mathrm{L}_{\mathbb{K}}(f)$ is an isomorphism, using Corollary 10.3. Since H is a p-adic Lie group, $r^H_{\mathbb{K}}$ is surjective. Using Proposition 10.1 and the naturality of $r_{\mathbb{K}}$, we deduce that $r^W_{\mathbb{K}}$ is surjective. Now $\pi_W \circ \rho^G_{\mathbb{K}} = r^W_{\mathbb{K}}$ shows that $\rho^G_{\mathbb{K}}$ is surjective. □

Remark 11.2. If G is a locally compact group of finite p-adic dimension, we use the surjection $\rho^G_{\mathbb{Q}_p}$ to define a Lie algebra structure on the set $\Gamma(\mathbb{Q}_p, G)^\sim := \Gamma(\mathbb{Q}_p, G)/\ker \rho^G_{\mathbb{Q}_p}$. We write $[[X]]$ for the equivalence class of $[X]$. Let $m^G_{\mathbb{Q}_p}$ denote the map obtained by factoring $\rho^G_{\mathbb{Q}_p}$ through $\Gamma(\mathbb{Q}_p, G)^\sim$; then $m^G_{\mathbb{Q}_p}$ is an isomorphism of Lie algebras. If $\psi\colon G \to H$ is a morphism, where G and H are locally compact groups of finite p-adic dimension, then $\Gamma(\mathbb{Q}_p, \psi)^\sim : \Gamma(\mathbb{Q}_p, G)^\sim \to \Gamma(\mathbb{Q}_p, H)^\sim, [[X]] \mapsto [[\phi \circ X]]$ is well-defined and makes $\Gamma(\mathbb{Q}_p, \bullet)^\sim$ a functor from the category $\mathbb{DISC}_{\mathbb{Q}_p}$ of totally disconnected, locally compact groups of finite p-adic dimension into the category of (finite-dimensional) p-adic Lie algebras; $m_{\mathbb{Q}_p}\colon \Gamma(\mathbb{Q}_p, \bullet)^\sim \to \mathrm{L}^{\mathrm{loc}}_{\mathbb{Q}_p}|_{\mathbb{DISC}_{\mathbb{Q}_p}}$ is a natural isomorphism.

Problem 11.3. Is there a topological group G (preferably locally compact and totally disconnected), such that $\operatorname{im} r^G_{\mathbb{Q}_p}$ is a proper subset of $\Lambda_{\mathbb{Q}_p}(G)$? If not, we could always identify $\Lambda_{\mathbb{Q}_p}(G)$ with $\Gamma(\mathbb{Q}_p, G)$, modulo a suitable equivalence relation. The analogous question in the real case seems to be open as well.

12. The p-adic Lie algebra functors are all different.

The p-adic Lie algebra functors $\Lambda^{\exp}_{\mathbb{Q}_p}$, $\overline{\Lambda}^{\exp}_{\mathbb{Q}_p}$, $\mathrm{L}^{\exp}_{\mathbb{Q}_p}$, $\Lambda_{\mathbb{Q}_p}$, $\overline{\Lambda}_{\mathbb{Q}_p}$, $\mathrm{L}_{\mathbb{Q}_p}$, $\mathrm{L}^{\mathrm{loc}}_{\mathbb{Q}_p}$, $\Lambda^{\mathrm{loc}}_{\mathbb{Q}_p}$, and $\overline{\Lambda}^{\mathrm{loc}}_{\mathbb{Q}_p}$ are pairwise not naturally isomorphic (and nor even are the composites of the forgetful functor to abstract p-adic Lie algebras with these functors). To see this, we compute the p-adic Lie algebras of five examples.

12.1. Let $G_1 := V := \mathbb{Q}_p^{(\mathbb{N})}$, equipped with any locally convex topology. By Section 9, we have a continuous $\mathbb{Q}_p$-linear map $t^V_{\mathbb{Q}_p} \circ [.]^V_{\mathbb{Q}_p} : \mathrm{Hom}(\mathbb{Q}_p, V) \to$

$\Lambda_{\mathbb{Q}_p}(V)$. It is easy to see that every $\kappa \in \mathrm{Hom}(\mathbb{Q}_p, V)$ is $\mathbb{Q}_p$-linear, and that the mapping $\exp_V : \mathrm{Hom}(\mathbb{Q}_p, V) \to V$, $\kappa \mapsto \kappa(1)$ is an isomorphism of topological vector spaces, with inverse $v \mapsto (z \mapsto zv)$. Thus $\psi := t^V_{\mathbb{Q}_p} \circ [.]^V_{\mathbb{Q}_p} \circ \exp_V^{-1}$ is a linear map. By the argument given in Example 7.5, $[.]^V_{\mathbb{Q}_p}$ is an isomorphism, and $t^V_{\mathbb{Q}_p}$ is surjective as a linear map into an abelian p-adic Lie algebra whose image generates the latter. We deduce that ψ is surjective. If $v \in V$ is a nonzero element, the p-adic Hahn-Banach Theorem (cf. [**23**]) shows that there exists a continuous linear functional $g : V \to \mathbb{Q}_p$ such that $g(v) \neq 0$. Now the quotient map $f : V \to V/\ker g$ is an element of $I_{\mathbb{Q}_p}(V)$, and the f-coordinate of $\psi(v)$ is the germ at 0 of the mapping $\mathbb{Q}_p \to V/\ker g$, $z \mapsto z(v + \ker g)$, a nonzero element of $\mathcal{L}_{\mathbb{Q}_p}(V/\ker g)$. We have proved that ψ is an isomorphism of vector spaces. Thus $\Lambda_{\mathbb{Q}_p}(V)$ has countable, infinite dimension; this implies that $\overline{\Lambda}_{\mathbb{Q}_p}(V)$ and $\mathrm{L}_{\mathbb{Q}_p}(V)$ have uncountable dimension. Note that $V = c_{\mathbb{Q}_p}(V)$; thus $\mathrm{L}^{\exp}_{\mathbb{Q}_p}(V) = \mathrm{L}_{\mathbb{Q}_p}(V)$, $\Lambda^{\exp}_{\mathbb{Q}_p}(V) = \Lambda_{\mathbb{Q}_p}(V)$ and $\overline{\Lambda}^{\exp}_{\mathbb{Q}_p}(V) = \overline{\Lambda}_{\mathbb{Q}_p}(V)$. It is easy to see that $\Lambda^{\mathrm{loc}}_{\mathbb{Q}_p}(V) \cong \Lambda_{\mathbb{Q}_p}(V)$, whence this Lie algebra has countable, infinite dimension; thus $\overline{\Lambda}^{\mathrm{loc}}_{\mathbb{Q}_p}(V)$ and $\mathrm{L}^{\mathrm{loc}}_{\mathbb{Q}_p}(V)$ have uncountable dimension.

12.2. Let I be a set of cardinality $\geq 2^{\aleph_0}$ and consider the locally compact semidirect product $G_2 := G := \mathbb{Z}_p^I \rtimes \mathrm{Sym}(I)$ and its open, compact, normal subgroup $U := \mathbb{Z}_p^I$. Then every morphism from G into a p-adic Lie group has U in its kernel, see [**9**], proof of Theorem 5.2, whence every p-adic Lie quotient of G is discrete. Hence $\mathrm{L}_{\mathbb{Q}_p}(G) = \{0\}$ and therefore also $\Lambda_{\mathbb{Q}_p}(G) = \overline{\Lambda}_{\mathbb{Q}_p}(G) = \{0\}$. Now the open, normal subgroup U of G has a nontrivial p-adic Lie algebra: By Theorem 7.1, we have a quotient morphism of topological Lie algebras $\nabla : \overline{\Lambda}_{\mathbb{Q}_p}(U) \to \mathbb{Q}_p^I$. We deduce from Corollary 10.6 that $\mathrm{L}^{\mathrm{loc}}_{\mathbb{Q}_p}(G)$ is an infinite-dimensional Lie algebra as well. Now G being locally compact and totally disconnected, we have $\overline{\Lambda}^{\mathrm{loc}}_{\mathbb{Q}_p}(G) = \mathrm{L}^{\mathrm{loc}}_{\mathbb{Q}_p}(G)$ as a consequence of Corollary 10.2. This in turn implies that $\Lambda^{\mathrm{loc}}_{\mathbb{Q}_p}(G)$ has infinite dimension. The proper $\mathbb{Q}_p$-component $c_{\mathbb{Q}_p}(G)$ contains U and all p-torsion elements of $\mathrm{Sym}(I)$; since in the proof of [**9**], Theorem 5.2, the permutation ϕ can always be assumed to have order p, every morphism from $c_{\mathbb{Q}_p}(G) = C_{\mathbb{Q}_p}(G)$ into a p-adic Lie group has U in its kernel, so that $\Lambda^{\exp}_{\mathbb{Q}_p}(G) = \Lambda^{\exp}_{\mathbb{Q}_p}(G) = \overline{\Lambda}^{\exp}_{\mathbb{Q}_p}(G) = \{0\}$.

12.3. Let $G_3 := G := \mathbb{Q}$, equipped with the topology induced by $\mathbb{Q}_p$. Then $\Gamma(\mathbb{Q}_p, U) = \{[0]\}$ for every open subgroup U of G (using that there is no nontrivial continuous homomorphism from $\mathbb{Z}_p$ into a countable torsion-free group), whence $\Lambda_{\mathbb{Q}_p}(G) = \overline{\Lambda}_{\mathbb{Q}_p}(G)$, $\Lambda^{\mathrm{loc}}_{\mathbb{Q}_p}(G)$ and $\overline{\Lambda}^{\mathrm{loc}}_{\mathbb{Q}_p}(G)$ are the zero-algebra.

	$\overline{\Lambda}^{\exp}_{\mathbb{Q}_p}$	$\mathrm{L}^{\exp}_{\mathbb{Q}_p}$	$\Lambda_{\mathbb{Q}_p}$	$\overline{\Lambda}_{\mathbb{Q}_p}$	$\mathrm{L}_{\mathbb{Q}_p}$	$\Lambda^{\mathrm{loc}}_{\mathbb{Q}_p}$	$\overline{\Lambda}^{\mathrm{loc}}_{\mathbb{Q}_p}$	$\mathrm{L}^{\mathrm{loc}}_{\mathbb{Q}_p}$
$\Lambda^{\exp}_{\mathbb{Q}_p}$	1	1	4	1	1	2	1	1
$\overline{\Lambda}^{\exp}_{\mathbb{Q}_p}$		5	1	4	3	1	2	2
$\mathrm{L}^{\exp}_{\mathbb{Q}_p}$			1	4	3	1	2	2
$\Lambda_{\mathbb{Q}_p}$				1	1	2	1	1
$\overline{\Lambda}_{\mathbb{Q}_p}$					3	1	2	2
$\mathrm{L}_{\mathbb{Q}_p}$						1	2	2
$\Lambda^{\mathrm{loc}}_{\mathbb{Q}_p}$							1	1
$\overline{\Lambda}^{\mathrm{loc}}_{\mathbb{Q}_p}$								3

Figure 3. The entry i indicates that the abstract Lie algebras associated with G_i by the functors indexing the respective row and column are not isomorphic.

Also $c_{\mathbb{Q}_p}(G) = \{0\}$, whence $\Lambda^{\exp}_{\mathbb{Q}_p}(G) = \overline{\Lambda}^{\exp}_{\mathbb{Q}_p}(G) = \mathrm{L}^{\exp}_{\mathbb{Q}_p}(G) = \{0\}$. On the other hand, $\mathrm{L}_{\mathbb{Q}_p}(G) \cong \mathrm{L}_{\mathbb{Q}_p}(\mathbb{Q}_p) \cong \mathbb{Q}_p$, and $\mathrm{L}^{\mathrm{loc}}_{\mathbb{Q}_p}(G) \cong \mathrm{L}^{\mathrm{loc}}_{\mathbb{Q}_p}(\mathbb{Q}_p) \cong \mathbb{Q}_p$.

12.4. Set $U := \mathbb{Q}_p^{\mathbb{Z}}$ and $G_4 := G := U \times \mathbb{Z}$, where $\mathbb{Z}$ acts by the shift action. Then $c_{\mathbb{Q}_p}(G) \leq U$, since every morphism $\mathbb{Z}_p \to \mathbb{Z}$ is trivial, and then clearly $c_{\mathbb{Q}_p}(G) = U$. Thus $\overline{\Lambda}^{\exp}_{\mathbb{Q}_p}(G) = \overline{\Lambda}_{\mathbb{Q}_p}(U)$ has $(\mathbb{Q}_p)^{\mathbb{Z}}$ as a quotient by Theorem 7.1; we deduce that both $\overline{\Lambda}^{\exp}_{\mathbb{Q}_p}(G)$ and $\Lambda^{\exp}_{\mathbb{Q}_p}(G)$ have infinite dimension. Now let $f \in I_{\mathbb{Q}_p}(G)$; then $K := \overline{f(U)}$ is an abelian p-adic Lie group, of dimension d, say. It follows from [**3**], §4.2, Lemma 3 and Theorem 2 that K has an open subgroup K_0 isomorphic to $\mathbb{Z}_p^d$. Then $V := f^{-1}(K_0) \cap U$ is an open subgroup in U; we therefore find some $n \in \mathbb{N}$ such that $W := (\mathbb{Q}_p)^{\{n,n+1,\dots\}} \leq V$. Now it is easy to see that every continuous homomorphism $\mathbb{Q}_p \to \mathbb{Q}_p$ is linear, whence every morphism $\mathbb{Q}_p \to \mathbb{Z}_p$ vanishes. W being a $\mathbb{Q}_p$-vector space, we easily deduce that $f|_W$, which maps into $K_0 \cong \mathbb{Z}_p^d$, vanishes. Thus $W \leq \ker f$; using the shift action, we find that $U \cap \ker f$ is dense in U and therefore that $U \leq \ker f$. Thus f factors through $G/U \cong \mathbb{Z}$, which implies $\mathrm{L}_{\mathbb{Q}_p}(G) \cong \mathrm{L}_{\mathbb{Q}_p}(\mathbb{Z}) = \{0\}$. Then $\Lambda_{\mathbb{Q}_p}(G) = \overline{\Lambda}_{\mathbb{Q}_p}(G) = \{0\}$ as well.

12.5. Consider $G_5 := G := \mathrm{PSL}_p(\mathbb{Q})$ as a dense subgroup of $\mathrm{PSL}_p(\mathbb{Q}_p)$ and equip it with the induced topology. Since G is countable, $R_{\mathbb{Q}_p}(G) = \mathrm{tor}_p(G)$ is the subset of p-torsion elements of G; in particular, every $X \in \mathrm{Hom}(\mathbb{Z}_p, G)$ has open kernel. It is evident that G contains an element of order p; the simplicity of G yields $c_{\mathbb{Q}_p}(G) = G$. Thus $\overline{\Lambda}^{\exp}_{\mathbb{Q}_p}(G) = \Lambda^{\exp}_{\mathbb{Q}_p}(G) = \Lambda_{\mathbb{Q}_p}(G) =$

$\{0\}$, whereas $\mathrm{L}^{\exp}_{\mathbb{Q}_p}(G) = \mathrm{L}_{\mathbb{Q}_p}(G) \cong \mathrm{L}_{\mathbb{Q}_p}(\mathrm{PSL}_p(\mathbb{Q}_p)) \cong \mathcal{L}_{\mathbb{Q}_p}(\mathrm{PSL}_p(\mathbb{Q}_p))$ are nonzero Lie algebras.

Figure 3 will help the reader verify that these examples suffice to show that the functors $\Lambda^{\exp}_{\mathbb{Q}_p}$, $\overline{\Lambda}^{\exp}_{\mathbb{Q}_p}$, $\mathrm{L}^{\exp}_{\mathbb{Q}_p}$, $\Lambda_{\mathbb{Q}_p}$, $\overline{\Lambda}_{\mathbb{Q}_p}$, $\mathrm{L}_{\mathbb{Q}_p}$, $\Lambda^{\mathrm{loc}}_{\mathbb{Q}_p}$, $\overline{\Lambda}^{\mathrm{loc}}_{\mathbb{Q}_p}$, and $\mathrm{L}^{\mathrm{loc}}_{\mathbb{Q}_p}$ are pairwise not naturally isomorphic.

13. Further examples and remarks.

Remark 13.1. We remark that continuous linear isomorphisms between subspaces of weakly complete spaces (as encountered above) need not be isomorphisms of topological vector spaces. To see this, note that Proposition 2.1 entails that the closed graph theorem holds for linear mappings between weakly complete spaces. Now the topological dual space $(\mathbb{R}^{\mathbb{N}})' \cong \mathbb{R}^{(\mathbb{N})}$ has countable dimension, whereas the algebraic dual $(\mathbb{R}^{\mathbb{N}})^*$ has uncountable dimension. Therefore there exists a discontinuous linear functional α on $\mathbb{R}^{\mathbb{N}}$. By reason of dimension, there exists an isomorphism of vector spaces $\kappa\colon \ker\alpha \to \mathbb{R}^{\{2,3,\dots\}}$. Pick an element $x_0 \in \mathbb{R}^{\mathbb{N}}$ such that $\alpha(x_0) = 1$ and define $\psi\colon \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}}$ via $\psi(x) := (\alpha(x), \kappa(x - \alpha(x)x_0))$. Then ψ is a discontinuous linear automorphism of $\mathbb{R}^{\mathbb{N}}$. By the above, the graph V of ψ is not closed in $W := \mathbb{R}^{\mathbb{N}} \times \mathbb{R}^{\mathbb{N}}$. Let $\mathrm{pr}_1\colon W \to \mathbb{R}^{\mathbb{N}}$ be the projection $W \to \mathbb{R}^{\mathbb{N}}$ onto the first coordinate. Then $\theta := \mathrm{pr}_1|_V$ is a continuous linear bijection onto the weakly complete space $\mathbb{R}^{\mathbb{N}}$. However, θ is not a topological isomorphism since V, not being closed in W, cannot be complete.

Remark 13.2. As in the preceding remark, let ψ be a discontinuous linear automorphism of $\mathbb{R}^{\mathbb{N}}$, equipped with the product topology. Then $\mathfrak{g}_1 := \mathbb{R}^{\mathbb{N}}$, together with the restriction maps $\mathrm{pr}_F := (\mathrm{pr}_i)_{i\in F}$ for finite subsets F of $\mathbb{N}$, is the projective limit topological Lie algebra of the projective system of finite-dimensional abelian Lie algebras $\mathbb{R}^F$, together with the restriction maps. Let $\mathfrak{g}_2 := \mathbb{R}^{\mathbb{N}}$, equipped with the topology which makes ψ a homeomorphism onto the cartesian product $\mathbb{R}^{\mathbb{N}}$. Then $\mathfrak{g}_2$, together with the mappings $\mathrm{pr}_F \circ \psi$, is another projective limit of the projective system above. Both $\mathfrak{g}_1$ and $\mathfrak{g}_2$ have the same underlying abstract Lie algebra, but their topologies are different. Thus [**20**], Theorem 2.10 is incorrect.[8]

Remark 13.3. Let $p_1, \dots, p_n$ be distinct primes, and G_i be a p_i-adic Lie group for $i = 1, \dots, n$. By Proposition 6.2 (a) and (b), $G := \prod_{i=1}^n G_i$ carries its p-adic Lie algebra $\Lambda^{\exp}_{\mathbb{Q}_p}(G) \cong \Lambda_{\mathbb{Q}_p}(G)$, for every prime p. If we choose Campbell-Hausdorff groups H_p in $\Lambda_{\mathbb{Q}_p}(G)$ (almost all of which are trivial),

[8]The error seems to be located in line 10 of the “proof” of the cited theorem. It does not seem to make any sense to say that the underlying [abstract] Lie algebra is linearly compact, as there is no topology on an abstract Lie algebra.

the group G will be locally isomorphic to $\prod_{p\in\mathbb{P}} H_p$. Hence we can reconstruct G locally from the family $\Lambda(G) := (\Lambda_{\mathbb{Q}_p}(G))_{p\in\mathbb{P}}$: the object $\Lambda(G)$ completely determines the local structure of G. Of course, one cannot obtain such an easy description of the local structure for more general classes of groups.

14. Extension of the Banach-Lie algebra functor.

In this final section, we construct topological Lie algebra functors on the category of Hausdorff groups which even assign the desired topological Lie algebras to all real Banach-Lie groups.

In the following, $\mathbb{BLIE}_{\mathbb{R}}$ denotes the full subcategory of $\mathbb{TG}$ whose objects are (possibly infinite-dimensional) real Lie groups in the sense of [**3**], Chapter III, §8.1, Definition 1; in order to distinguish these from the finite-dimensional Lie groups considered before, we follow the custom of calling such Lie groups *Banach-Lie groups*. To every real Banach-Lie group G can be associated a complete normable topological real Lie algebra $\mathcal{L}(G)$ in a functorial fashion (cf. [**3**]). We let $\exp_G\colon \mathcal{L}(G)\to G$ denote the exponential function of G ([**3**], Chapter III, §6.4, Theorem 4).

Recall that a real Banach-Lie group H is called an *analytic subgroup* of a real Banach-Lie group G if the group underlying H is a subgroup of G, the inclusion map $\varepsilon\colon H\to G$ is continuous (hence real analytic), and $\mathfrak{h} := \operatorname{im}\mathcal{L}(\varepsilon)$ is a closed Lie subalgebra of $\mathcal{L}(G)$; then there is an identity neighbourhood W in H such that $\varepsilon|_W$ is a topological embedding, and $\mathcal{L}(\varepsilon)|^{\mathfrak{h}}$ is an isomorphism of topological Lie algebras, which we use to identify $\mathcal{L}(H)$ with $\mathfrak{h}$ (cf. [**22**]). The following lemma is essential:

Lemma 14.1. *Suppose that G and H are real Banach-Lie groups, and suppose that $f : G\to H$ is a continuous homomorphism. Then there exists an analytic subgroup K of H, with Lie algebra $\mathfrak{k} := \overline{\operatorname{im}\mathcal{L}(f)}$, such that K contains* $\operatorname{im} f$ *as a dense subgroup, and such that the homomorphism $f|^K\colon G\to K$ is continuous. K is uniquely determined by these properties.*

Proof. See [**10**], Corollary 24.6 for the existence of K; its uniqueness is obvious. □

In view of Lemma 14.1, we can proceed as in Sections 3 and 5, with the only difference that instead of replacing continuous homomorphisms into Lie groups by their corestrictions to the closure of the image (which was our general philosophy in Section 3), we replace them by their corestrictions to the analytic subgroups provided by Lemma 14.1 now. Thus, given a topological group G, we let Ω denote the class of continuous homomorphisms $G\to H_f$ into Banach-Lie groups H_f such that f has dense image; we pre-order this class and define an equivalence relation as in Section 3, and determine a set $S(G)$ of representatives of the equivalence classes in the same way in which we determined $I_{\mathbb{R}}(G)$ in Section 3. Given $f \leq g$ in $S(G)$, we have a unique continuous homomorphism $\phi_{fg}\colon H_g \to H_f$ such that $\phi_{fg} \circ g = f$; in this way, we obtain a projective system $\mathcal{B}(G) := ((H_f)_{f\in S(G)}, (\phi_{fg})_{f\leq g})$ of Banach-Lie groups. If $\Phi : G \to L$ is a morphism of topological groups, we define $\mathcal{B}(\Phi)\colon \mathcal{B}(G) \to \mathcal{B}(H)$ along the lines of the definition of $\mathfrak{S}_{\mathbb{R}}(\Phi)$ in Section 3. We define $\mathrm{L}_{\mathrm{Ban}} := \varprojlim \circ (\mathcal{L})^{\pi} \circ \mathcal{B}$: then, as in the finite-dimensional case discussed above, one verifies that $\mathrm{L}_{\mathrm{Ban}}$ is a real topological Lie algebra functor which extends the Lie algebra functor $\mathcal{L}$ on the category $\mathbb{BLIE}_{\mathbb{R}}$ of real Banach-Lie groups. Clearly, we might define further real topological Lie algebra functors Λ_{Ban}, $\overline{\Lambda}_{\mathrm{Ban}}$, $\mathrm{L}^0_{\mathrm{Ban}}$, Λ^0_{Ban}, $\overline{\Lambda}^0_{\mathrm{Ban}}$, $\mathrm{L}^{\mathrm{arc}}_{\mathrm{Ban}}$, $\Lambda^{\mathrm{arc}}_{\mathrm{Ban}}$, $\overline{\Lambda}^{\mathrm{arc}}_{\mathrm{Ban}}$, $\mathrm{L}^{\mathrm{loc}}_{\mathrm{Ban}}$, $\Lambda^{\mathrm{loc}}_{\mathrm{Ban}}$, $\overline{\Lambda}^{\mathrm{loc}}_{\mathrm{Ban}}$, $\mathrm{L}^{\mathrm{exp}}_{\mathrm{Ban}}$, $\Lambda^{\mathrm{exp}}_{\mathrm{Ban}}$, $\overline{\Lambda}^{\mathrm{exp}}_{\mathrm{Ban}}$ now, in analogy to the definitions of $\Lambda_{\mathbb{R}}, \ldots, \overline{\Lambda}^{\mathrm{exp}}_{\mathbb{R}}$ in Section 5; any of these functors extends the Banach-Lie algebra functor $\mathcal{L}$, in contrast to the functors defined earlier. We have to pay a price: The topological Lie algebras defined in this section cease to be weakly complete in general (infinite-dimensional Banach spaces not being weakly complete), so that we lose a property which was so useful in our discussion of the functors defined in Section 5.

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Technische Universität Darmstadt
Fachbereich Mathematik, AG 5
Schlossgartenstr. 7
64289 Darmstadt, Germany
E-mail address: gloeckner@mathematik.tu-darmstadt.de

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