THE ESSENTIAL NORMS AND SPECTRA OF COMPOSITION OPERATORS ON $H^\infty$

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This paper gives a complete characterization of the spectra of composition operators acting on $H^\infty$ in the case that the symbol $\varphi$ has an interior fixed point. This is done after it proves that the essential norm of a composition operator acting on $H^\infty$ is either 1 or 0.

1. Introduction.

Throughout this paper, $D$ denotes the unit disk $\{z : |z| < 1\}$, $\varphi$ denotes an analytic self-map of $D$, $C_\varphi$ is the composition operator defined by $C_\varphi(f) = f \circ \varphi$, $\|C_\varphi\|_e$ and $\rho_e$ represent the essential norm and essential spectral radius of $C_\varphi$, respectively, and $H(D)$ is the space of analytic functions on $D$.

The essential norm of an operator is the distance from the operator to the space of compact operators. The essential norm of composition operator acting on $H^2$, the Hardy space of analytic functions $f$ on $D$ such that $\int_{-\pi}^{\pi} |f(e^{i\theta})|^2 d\theta < \infty$, was given by J. H. Shapiro in terms of the Nevanlinna counting function [Sh1]. The spectrum of composition operator on $H^2$ has also been studied extensively. Kamowitz was the first to investigate spectrum of composition operator whose symbol is not an inner function and has a fixed point in the disk. He proved in [Kam] that the spectrum of $C_\varphi$ on $H^2$ is the set $\{\lambda : |\lambda| \leq \rho_e\} \cup \{\varphi'(a)^n : n \in \mathbb{N}\} \cup \{1\}$ if $\varphi$ is analytic in a neighborhood of $D$, not an inner function and has an interior fixed point $a$. Then Cowen and MacCluer proved in [CM1] the same conclusion in the case that $\varphi$ is univalent, not an automorphism and has an interior fixed point. But a complete understanding of the spectrum of $C_\varphi$ on $H^2$ is still lacking. While much attention has been devoted to the study of $H^2$, the behavior of $C_\varphi$ acting on $H^\infty$, the space of bounded analytic functions on $D$, has barely been discussed. It is the purpose of this paper to investigate some properties of $C_\varphi$ acting on $H^\infty$.

There are two main results of this paper. One of the results, which is stated as Theorem 1, is that the essential norm of $C_\varphi$ acting on $H^\infty$ is either 1 or 0. This leads to the corollary that the essential spectral radius of $C_\varphi$ on $H^\infty$ is also 1 or 0. Then the two theorems described in the previous paragraph about the spectrum of $C_\varphi$ on $H^2$, together with this corollary,
suggest that if \( \varphi \) has an interior fixed point \( a \), the spectrum of \( C_\varphi \) on \( H^\infty \) is \( \overline{D} \) or the sequence \( \{0, 1, \varphi'(a)^n : n = 1, 2, \ldots\} \). This is the second main result, stated and proved below as Theorem 4. But unlike the theorems about the spectrum of \( C_\varphi \) acting on \( H^2 \) previously referred to, this does not require \( \varphi \) to be univalent or analytic in a neighborhood of \( \overline{D} \). It only requires \( \varphi \) to have an interior fixed point.

2. Essential norm.

The essential norm of \( C_\varphi \) on \( H^\infty \) is defined to be
\[
\|C_\varphi\|_e = \inf \{\|C_\varphi - K\| : K \text{ is compact operator on } H^\infty\}.
\]
Clearly \( C_\varphi \) is compact if and only if its essential norm is zero.

The next result shows that there is only one other possible value for the essential norm of a composition operator on \( H^\infty \).

Theorem 1. If \( C_\varphi \) is not compact on \( H^\infty \), then its essential norm is \( 1 \).

In order to prove this theorem, we need the following lemma, which was first proved by Schwartz [Sch].

Lemma 2. \( C_\varphi \) is compact on \( H^\infty \) if and only if \( \varphi(\overline{D}) \) is relatively compact in \( \overline{D} \).

Proof of Theorem 1. We know that \( C_\varphi \) is compact if and only if its essential norm is 0. The main argument here is that the essential norm must be 1 if \( C_\varphi \) is not compact. Since \( \|C_\varphi(f)\|_\infty = \sup_{z \in \overline{D}} |f \circ \varphi(z)| \leq \sup_{z \in \overline{D}} |f(z)| = \|f\|_\infty \cdot \|C_\varphi\| \leq 1 \), and hence \( \|C_\varphi\|_e \leq 1 \). It suffices to prove that \( \|C_\varphi\|_e \geq 1 \) if \( C_\varphi \) is not compact on \( H^\infty \).

Now assume \( C_\varphi \) is not compact on \( H^\infty \). By Lemma 2, \( \sup_{z \in \overline{D}} |\varphi(z)| = 1 \). There exists a sequence \( \{a_k\}_{k=1}^\infty \subset \overline{D} \) such that \( \varphi(a_k) \to e^{i\beta} \) as \( k \to \infty \) for some \( \beta \in \mathbb{R} \). Without loss of generality, let’s assume \( e^{i\beta} = 1 \). Let \( \{r_n\}_{n=1}^\infty \) be a nonnegative sequence increasing to 1, and
\[
\psi_n(z) = \frac{z - r_n}{1 - r_n \overline{z}}.
\]
Then \( \|\psi_n\|_\infty = 1 \), \( \psi_n \) fixes 1 and \(-1\) for all \( n \in \mathbb{N} \), and \( \psi_n(z) \to -1 \) as \( n \to \infty \) for all \( z \in \overline{D} \). Let \( K \) be a compact operator on \( H^\infty \). We want to show that \( \|C_\varphi - K\| \geq 1 \). Since \( K \) is compact and \( \|\psi_n\|_\infty = 1 \), there is a subsequence \( \{\psi_{n_j}\}_{j=1}^\infty \) and an \( f \in H^\infty \) such that \( \lim_{j \to \infty} \|K\psi_{n_j} - f\|_\infty = 0 \).

For \( \|C_\varphi - K\| \geq 1 \) to be true, it is enough to prove that \( \limsup_{j \to \infty} \|(C_\varphi - K)(\psi_{n_j})\|_\infty \geq 1 \). But
\[
\|(C_\varphi - K)(\psi_{n_j})\|_\infty \geq \|C_\varphi(\psi_{n_j}) - f\|_\infty - \|K\psi_{n_j} - f\|_\infty,
\]
and
which implies
\[ \limsup_{j \to \infty} \| (C_\varphi - K)(\psi_{n_j}) \|_\infty \geq \limsup_{j \to \infty} \| C_\varphi (\psi_{n_j}) - f \|_\infty. \]

It suffices to prove that \( \limsup_{j \to \infty} \| C_\varphi (\psi_{n_j}) - f \|_\infty \geq 1 \).

The fact that \( \psi_n(z) \to -1 \) as \( n \to \infty \) for all \( z \in \mathbb{D} \) implies that \( \psi_n \circ \varphi(z) \to -1 \) as \( j \to \infty \), and hence \( \lim_{j \to \infty} | \psi_n \circ \varphi(z) - f(z) | = | -1 - f(z) | \) for all \( z \in \mathbb{D} \). If there is \( z_0 \in \mathbb{D} \) such that \( | -1 - f(z_0) | \geq 1 \), we have \( \| \psi_n \circ \varphi - f \|_\infty \geq | \psi_n \circ \varphi(z_0) - f(z_0) | \to | -1 - f(z_0) | \geq 1 \), which implies \( \limsup_{j \to \infty} \| \psi_n \circ \varphi - f \|_\infty \geq 1 \) as desired. Otherwise, \( | -1 - f(z) | < 1 \) for all \( z \in \mathbb{D} \). Then by the triangle inequality \( |1 - f(z)| > 1 \) for all \( z \in \mathbb{D} \). Consider the sequence \( \{ a_k \}_{k=1}^\infty \subset \mathbb{D} \) which was obtained at the beginning of the proof. We have \( \lim_{k \to \infty} \varphi(a_k) = 1 \) and \( \{ f(a_k) \}_{k=1}^\infty \) is bounded since \( f \in H^\infty \). Then there is a subsequence \( \{ f(a_{k_j}) \}_{j=1}^\infty \) converging to some \( \omega \in \mathbb{C} \). By re-indexing we may assume, without loss of generality that, \( \lim_{k \to \infty} f(a_k) = \omega \). Then by our assumption on \( f \), \( |1 - \omega| = \lim_{k \to \infty} |1 - f(a_k)| \geq 1 \). Since \( \psi_n \) is continuous, \( \psi_n(1) = 1 \) and \( \lim_{k \to \infty} \varphi(a_k) = 1 \), it follows that \( \lim_{k \to \infty} | \psi_n \circ \varphi(a_k) - f(a_k) | = |1 - \omega| \geq 1 \) for all \( n \). Then \( \| \psi_n \circ \varphi - f \|_\infty \geq \lim_{k \to \infty} | \psi_n \circ \varphi(a_k) - f(a_k) | \geq 1 \) for all \( j \). Hence \( \limsup_{j \to \infty} \| \psi_n \circ \varphi - f \|_\infty \geq 1 \) as desired.

If \( e^{i\beta} \neq 1 \), let \( \Psi_n(z) = e^{i\beta} \psi_n(e^{-i\beta} z) \). The same proof holds with \( \psi_n \) replaced by \( \Psi_n \), and the boundary points 1 and \( -1 \) replaced by \( e^{i\beta} \) and \( -e^{i\beta} \) respectively. This completes the proof of Theorem 1.

For the rest of this paper, \( \varphi_n \) will denote the \( n \)th iterate of \( \varphi \), i.e., \( \varphi_1 = \varphi \) and \( \varphi_n = \varphi \circ \varphi_{n-1} \) for \( n > 1 \).

**Definition** ([CM2, p. 150]). If \( T \) is a bounded linear operator on a Hilbert space, then the spectrum of the equivalence class in the Calkin algebra that contains \( T \) is called the essential spectrum of \( T \).

**Corollary 3.** The essential spectral radius of \( C_\varphi \) on \( H^\infty \) is either 1 or 0. If \( C_\varphi \) \((= C_\varphi^n)\) is compact for some \( n \geq 1 \), then \( \rho_e(C_\varphi) = 0 \). Otherwise \( \rho_e(C_\varphi) = 1 \).

**Proof.** The conclusion follows immediately from Theorem 1 and the formula that \( \rho_e(C_\varphi) = \lim_{n \to \infty} (\| C_\varphi^n \|_e)^{1/n} = \lim_{n \to \infty} (\| C_{\varphi^n} \|_e)^{1/n} \).

3. **Spectrum.**

For \( C_\varphi \) acting on \( H^\infty \), the spectrum \( \sigma(C_\varphi) \) is contained in \( \overline{\mathbb{D}} \). This is because the norm of \( C_\varphi \) acting on \( H^\infty \) is always 1, which implies the spectral radius \( \rho(C_\varphi) = \lim_{n \to \infty} \| C_{\varphi^n} \|_e^{1/n} = 1 \).

**Theorem 4.** If \( \varphi \) is not a constant, not an automorphism, and \( \varphi(a) = a \) for some \( a \in \mathbb{D} \), then
\[ \sigma(C_\varphi) = \overline{\mathbb{D}}, \text{ if } \| \varphi_n \|_\infty = 1 \text{ for all } n \in \mathbb{N} \]
and
\[ \sigma(C_\varphi) = \{ \varphi'(a)^k : k = 1, 2, \ldots \} \cup \{ 0, 1 \}, \text{ if } \|\varphi_n\|_\infty < 1 \text{ for some } n \in \mathbb{N}. \]

The proof of Theorem 4 will be given after some lemmas. Some ideas and approaches used in the proof are suggested by the work of Kamowitz in [Kam], and of Cowen and MacCluer in [CM1].

The following lemma, Lemma 5, follows immediately from Koenigs' Theorem [Koe] (see also [Sh2, Chapter 6]), if we can show that the Koenigs' function \( \xi \) of \( \varphi \) is in \( H^\infty \) under the assumption of the Lemma. Since \( \xi \circ \varphi = \varphi'(a)\xi \), which implies \( \xi \circ \varphi_n = \varphi'(a)^n\xi \), and hence \( \xi = \varphi'(a)^{-n}\xi(\varphi_n) \) for all \( n \in \mathbb{N} \), we conclude that \( \xi \) is in \( H^\infty \) under the assumption that \( \|\varphi_n\|_\infty < 1 \) for some \( n \in \mathbb{N} \).

**Lemma 5.** For \( \varphi \) as in Theorem 4, suppose \( \|\varphi_n\|_\infty < 1 \) for some \( n \in \mathbb{N} \). If \( \varphi'(a) \neq 0 \), then \( \{ \varphi'(a)^n, n = 0, 1, 2, \ldots \} \) is the set of eigenvalues of \( C_\varphi \) on \( H^\infty \). If \( \varphi'(a) = 0 \), the only eigenvalue of \( C_\varphi \) is 1.

**Lemma 6.** Let \( \varphi \) be the same as in Theorem 4 and suppose \( \|\varphi_n\|_\infty < 1 \) for some \( n \in \mathbb{N} \). Then for \( C_\varphi \) on \( H^\infty \),
\[ \sigma(C_\varphi) = \{ \varphi'(a)^k : k = 1, 2, \ldots \} \cup \{ 0, 1 \}. \]

**Proof.** Under the hypothesis that \( \|\varphi_n\|_\infty < 1 \), \( C_\varphi = C_\varphi^n \) is compact, which implies that \( C_\varphi \) is not invertible and hence 0 is in the spectrum. Also, since \( C_\varphi^m \) is compact, \( C_\varphi \) is a Riesz operator. So its nonzero spectrum consists of eigenvalues [Köhn, p. 19-21], and the result follows from Lemma 5.

**Lemma 7.** Suppose \( \varphi(0) = 0 \). Then \( H_m = z^m H^\infty \) is an invariant subspace of \( C_\varphi \) and \( \sigma(C_m) \subset \sigma(C_\varphi) \) where \( C_m = C_\varphi|_{H_m} \).

**Proof.** It’s easy to see that \( H_m \) is invariant under \( C_\varphi \). Since \( \varphi(0) = 0 \), \( \varphi(z) = z\phi(z) \) for some \( \phi \in H^\infty \). Then if \( f \in H^\infty \), \( C_\varphi(z^mf) = \varphi^m(f \circ \varphi) \in H_m \).

Suppose \( \lambda \) is in the spectrum of \( C_m \). If \( \lambda \) is an eigenvalue of \( C_m \), it must be an eigenvalue of \( C_\varphi \) and hence in the spectrum of \( C_\varphi \). If \( \lambda \) is not an eigenvalue of \( C_m \), then \( C_m - \lambda I \) is one-one. But it is not invertible, and hence not onto. So there exists \( f \in H_m \) with \( f \notin (C_m - \lambda I)(H_m) \). If we can show that \( C_\varphi - \lambda I \) on \( H^\infty \) is not onto, then it will follow that \( \lambda \in \sigma(C_\varphi) \) and the conclusion holds. Suppose to the contrary \( C_\varphi - \lambda I \) is onto. Then \( f \in (C_\varphi - \lambda I)(H^\infty) \) and there is \( g \in H^\infty \) with \( g \notin H_m \) such that \( (C_\varphi - \lambda I)g = f \). Let \( g = g_1 + g_2 \), where \( g_1 \in \text{span}(1, z, z^2, \ldots, z^{m-1}) \) and \( g_2 \in H_m \). We have \( g_1 \neq 0 \) since \( g \notin H_m \). Let \( f_1 = (C_\varphi - \lambda I)g_1 = f - (C_\varphi - \lambda I)g_2 \). Then \( f_1 \in H_m \) since \( f, g_2 \in H_m \).

Also by the assumption that \( C_\varphi - \lambda I \) is onto, for each function \( z^i, i = 1, 2, \ldots, m - 1 \), there exists \( h_i \in H^\infty \) such that \( (C_\varphi - \lambda I)h_i(z) = z^i \). Let \( h_i = k_i + l_i \) where \( k_i \in \text{span}(1, z, z^2, \ldots, z^{m-1}) \) and \( l_i \in H_m \). The next
We say the sequence of points \( (z_k)_{k=1}^M \) is an iteration sequence for \( \varphi \) if \( \varphi(z_k) = z_{k+1} \) for \( K \leq k < M \) where \( K \geq -\infty \) and \( M \leq \infty \).

**Lemma 8** ([CM2, p. 292, Lemma 7.34]). If \( \varphi \) is not an automorphism and \( \varphi(0) = 0 \), then given \( 0 < r < 1 \), there exists \( M_r \) with \( 1 \leq M_r < \infty \) such that if \( \{z_k\}_K^\infty \) is an iteration sequence with \( |z_l| \geq r \) for some \( l \geq 0 \) and if \( \{w_k\}_K^L \) is arbitrary, there is \( h \in H^\infty \) such that \( h(z_k) = w_k \) for \( -K \leq k \leq L \) and \( \|h\|_{\infty} \leq M_r \sup \{|w_k| : -K \leq k \leq L\} \).

**Lemma 9** ([CM2, p. 293, Lemma 7.35]). For \( \varphi \) in Lemma 8 and \( \{z_k\} \) any iteration sequence, there exists \( c < 1 \) such that \( |z_{k+1}|/|z_k| \leq c \) whenever \( |z_k| \leq 0.5 \).

**Proof of Theorem 4.** The statement in the case that \( \|\varphi_n\|_{\infty} < 1 \) for some \( n \in \mathbb{N} \) is the result of Lemma 6. Now suppose \( \|\varphi_n\|_{\infty} = 1 \) for all \( n \in \mathbb{N} \). We want to prove \( \sigma(C_\varphi) = \mathbb{D} \). If the interior fixed point \( a \neq 0 \), let \( \tau(z) = (a - z)/(1 - az) \) and \( \psi = \tau \circ \varphi \circ \tau \). Then \( \tau^{-1} = \tau \), \( \psi(0) = 0 \) and \( C_\psi = C_\tau \circ C_\varphi \circ C_\tau = C_\tau \circ C_\varphi \circ C_\tau^{-1} \). \( C_\varphi \) and \( C_\psi \) are similar and hence have the same spectrum. So without loss of generality, we can assume \( \varphi(0) = 0 \).

Since \( \sigma(C_\varphi) \subset \mathbb{D} \) and \( \sigma(C_\varphi) \) is closed, it suffices to prove \( \mathbb{D} - \{0\} \subset \sigma(C_\varphi) \). Let \( 0 \neq \lambda \in \mathbb{D} \), \( H_m = z^m H^\infty \), and \( C_m = C_\varphi|_{H_m} \). By Lemma 7, it suffices to prove that \( \lambda \) is in the spectrum of \( C_m \) for some positive integer \( m \). Since \( C_m - \lambda I \) is not onto if and only if \( (C_m - \lambda I)^* \) is not bounded from below, it is enough to find a positive integer \( m \) with \( (C_m - \lambda I)^* \) not bounded from below.

Let \( M \) be the constant \( M_r \) in Lemma 8 corresponding to \( r = 0.25 \) and suppose we have an iteration sequence \( \{z_k\}_K^\infty \) with \( K \geq 0 \) and \( |z_0| > 0.5 \) (this sequence will be determined later on). Let \( n = \max \{k : |z_k| \geq 0.25\} \).

Then \( n \geq 0 \) and \( |z_k| < 0.25 < 0.5 \) for all \( k \geq n + 1 \). By Lemma 9, there is a number \( c_1 < 1 \) so that \( |z_{k+1}| \leq c_1 |z_k| \) whenever \( k \geq n + 1 \). If \( z_n \leq 0.5 \), the inequality also holds for \( k = n \). If \( |z_n| > 0.5 \), since \( |z_{n+1}| < 0.25 \), we have \( |z_{n+1}| < 0.5 |z_n| \). Let \( c = \max \{c_1, 0.5\} \). Then \( |z_{k+1}| \leq c |z_k| \) for all \( k \geq n \).

It follows that \( |z_k| \leq c^{k-n} |z_n| \) whenever \( k \geq n \). Since \( c < 1 \), there exists a positive integer \( m \) such that \( c^m/|\lambda| < 1/(2M+1) < 1 \). For this \( m \), \( (C_m - \lambda I)^* \) is not bounded from below. To see that, let \( \varepsilon > 0 \) and we will construct a bounded linear functional \( L_\lambda \) on \( H_m \) with \( \|(C_m - \lambda I)^* L_\lambda\|/\|L_\lambda\| < \varepsilon \).
Let’s define $L_\lambda$ by $L_\lambda(f) = \sum_{k=-K}^{\infty} \lambda^{-k} f(z_k)$ for $f \in H_m$. We will see that $L_\lambda$ is well-defined and indeed it is bounded.

For $f \in H_m$, $z^{-m} f(z)$ is analytic and $\|f\|_\infty = \|z^{-m} f(z)\|_\infty$. So

$$|f(z_k)| = |z_k|^m |z_k^{-m} f(z_k)| \leq |z_k|^m \|z^{-m} f(z)\|_\infty = |z_k|^m \|f\|_\infty.$$

Then

$$\sum_{k=-K}^{\infty} |\lambda|^{-k} |f(z_k)| \leq \|f\|_\infty \sum_{k=-K}^{\infty} |\lambda|^{-k} |z_k|^m$$

$$= \|f\|_\infty \left( \sum_{k=-K}^{n} |\lambda|^{-k} |z_k|^m + \sum_{k=n+1}^{\infty} |\lambda|^{-k} |z_k|^m \right).$$

For $k > n$, $|z_k| \leq e^{k-n} |z_n|$ and so $|z_k|^m \leq (e^{m})^{k-n} |z_n|^m$. We see that

$$\sum_{k=n+1}^{\infty} |\lambda|^{-k} |z_k|^m \leq \sum_{k=n+1}^{\infty} \left( \frac{(e^{m})^{k-n}}{|\lambda|^{k-n}} \right) |z_n|^m = |z_n|^m \sum_{k=n+1}^{\infty} \left( \frac{(e^{m})^{k-n}}{|\lambda|^m} \right) < \infty.$$

It follows that $\sum_{k=-K}^{\infty} \lambda^{-k} f(z_k)$ converges. Hence $L_\lambda$ is well-defined and by (1) it is bounded.

Now let’s estimate $\| (C_m - \lambda I)^* L_\lambda \|$. For $f \in H_m$,

$$\langle f, (C_m - \lambda I)^* L_\lambda \rangle = \langle (C_m - \lambda I) f, L_\lambda \rangle$$

$$= \langle f \circ \varphi - \lambda f, L_\lambda \rangle$$

$$= \sum_{k=-K}^{\infty} \lambda^{-k} (f \circ \varphi(z_k) - \lambda f(z_k))$$

$$= \sum_{k=-K}^{\infty} \lambda^{-k} f(z_k+1) - \lambda^{-(k-1)} f(z_k))$$

$$= -\lambda^{K+1} f(z_{-K}).$$

Then

$$\| (C_m - \lambda I)^* L_\lambda \| = \sup_{0 \neq f \in H_m} \frac{|\langle f, (C_m - \lambda I)^* L_\lambda \rangle|}{\|f\|_\infty}$$

$$= \sup_{0 \neq f \in H_m} \frac{|\lambda^{K+1} f(z_{-K})|}{\|f\|_\infty}$$

$$\leq |\lambda|^{K+1}.$$

We also need a lower bound for $\|L_\lambda\|$. If we apply Lemma 8 to the iteration sequence $\{z_k\}_{K}^{\infty}$ with $r=0.25$, we can find a function $h \in H^\infty$ with $\|h\|_\infty \leq M$, $|h(z_k)| = 1$ and $\lambda^{-k} z_k^m h(z_k) > 0$ for $-K \leq k \leq n$. Let $g(z) = z^m h(z) \in H^\infty$. We see that
$H_m$. Then $\|g\|_\infty \leq M$ and

$$L_\lambda(g) = \sum_{k=-K}^{\infty} \lambda^{-k} z_k^m h(z_k)$$

$$= \sum_{k=-K}^{n-1} |\lambda|^{-k} |z_k|^m + |\lambda|^{-n} |z_n|^m + \sum_{k=n+1}^{\infty} \lambda^{-k} z_k^m h(z_k).$$

By the estimate in (2) and because $M \geq 1$ and $c^m/|\lambda| < 1/(2M + 1)$ from the choice of $m$, we have

$$\left| \sum_{k=n+1}^{\infty} \lambda^{-k} z_k^m h(z_k) \right| \leq \sum_{k=n+1}^{\infty} |\lambda|^{-k} |z_k|^m |h(z_k)|$$

$$\leq M |\lambda|^{-n} |z_n|^m \sum_{k=n+1}^{\infty} \left( \frac{c^m}{|\lambda|} \right)^{k-n}$$

$$= M |\lambda|^{-n} |z_n|^m \frac{c^m}{|\lambda| - c^m}$$

$$\leq M |\lambda|^{-n} |z_n|^m \frac{2M + 1}{1 - \frac{1}{2M + 1}}$$

$$= \frac{1}{2} |\lambda|^{-n} |z_n|^m.$$

This shows that

$$|L_\lambda(g)| \geq \sum_{k=-K}^{n-1} |\lambda|^{-k} |z_k|^m + |\lambda|^{-n} |z_n|^m - \left| \sum_{k=n+1}^{\infty} \lambda^{-k} z_k^m h(z_k) \right|$$

$$\geq \sum_{k=-K}^{n-1} |\lambda|^{-k} |z_k|^m + \frac{1}{2} |\lambda|^{-n} |z_n|^m \geq \frac{1}{2} |z_0|^m.$$

Then

$$\|L_\lambda\| \geq \frac{|L_\lambda(g)|}{\|g\|_\infty} \geq \frac{|z_0|^m}{2M} \geq \frac{(0.5)^m}{2M}.$$}

It follows that

$$\frac{||(C_m - \lambda I)^* L_\lambda||}{\|L_\lambda\|} \leq 2M |\lambda|^{K+1} \frac{0.5^m}{0.5^m}.$$}

Since $|\lambda| < 1$, this is less than $\varepsilon$ if we choose $K$ sufficiently large. For the chosen $K$, we can determine the iteration sequence $\{z_k\}_{-K}^{\infty}$. Since $\|\varphi_K\|_\infty = 1$ by assumption, there exists $w \in \mathbb{D}$ with $|\varphi_K(w)| > 0.5$. Let $z_{-K} = w$ and $z_{k+1} = \varphi(z_k)$ for $k > -K$. Then $|z_0| = |\varphi_K(z_{-K})| > 0.5$. The above calculation follows, thus $(C_m - \lambda I)^*$ is not bounded from below as desired.
Acknowledgement. I would like to thank my advisor Professor Wayne Smith for his valuable suggestions and patient guidance in my writing this paper.

References


Received March 1, 2000 and revised October 2, 2000.

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