THE ESSENTIAL NORMS AND SPECTRA OF COMPOSITION OPERATORS ON $H^\infty$

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This paper gives a complete characterization of the spectra of composition operators acting on $H^\infty$ in the case that the symbol $\varphi$ has an interior fixed point. This is done after it proves that the essential norm of a composition operator acting on $H^\infty$ is either 1 or 0.

1. Introduction.

Throughout this paper, $\mathbb{D}$ denotes the unit disk $\{z : |z| < 1\}$, $\varphi$ denotes an analytic self-map of $\mathbb{D}$, $C_\varphi$ is the composition operator defined by $C_\varphi(f) = f \circ \varphi$, $\|C_\varphi\|_e$ and $\rho_e$ represent the essential norm and essential spectral radius of $C_\varphi$ respectively, and $H(\mathbb{D})$ is the space of analytic functions on $\mathbb{D}$.

The essential norm of an operator is the distance from the operator to the space of compact operators. The essential norm of composition operator acting on $H^2$, the Hardy space of analytic functions $f$ on $\mathbb{D}$ such that $\int_{-\pi}^{\pi} |f(e^{i\theta})|^2 d\theta < \infty$, was given by J. H. Shapiro in terms of the Nevanlinna counting function $[\text{Sh1}]$. The spectrum of composition operator on $H^2$ has also been studied extensively. Kamowitz was the first to investigate spectrum of composition operator whose symbol is not an inner function and has a fixed point in the disk. He proved in $[\text{Kam}]$ that the spectrum of $C_\varphi$ on $H^2$ is the set $\{\lambda : |\lambda| \leq \rho_e\} \cup \{\varphi'(a)^n : n \in \mathbb{N}\} \cup \{1\}$ if $\varphi$ is analytic in a neighborhood of $\mathbb{D}$, not an inner function and has an interior fixed point $a$. Then Cowen and MacCluer proved in $[\text{CM1}]$ the same conclusion in the case that $\varphi$ is univalent, not an automorphism and has an interior fixed point. But a complete understanding of the spectrum of $C_\varphi$ on $H^2$ is still lacking. While much attention has been devoted to the study of $H^2$, the behavior of $C_\varphi$ acting on $H^\infty$, the space of bounded analytic functions on $\mathbb{D}$, has barely been discussed. It is the purpose of this paper to investigate some properties of $C_\varphi$ acting on $H^\infty$.

There are two main results of this paper. One of the results, which is stated as Theorem 1, is that the essential norm of $C_\varphi$ acting on $H^\infty$ is either 1 or 0. This leads to the corollary that the essential spectral radius of $C_\varphi$ on $H^\infty$ is also 1 or 0. Then the two theorems described in the previous paragraph about the spectrum of $C_\varphi$ on $H^2$, together with this corollary,
suggest that if \( \varphi \) has an interior fixed point \( a \), the spectrum of \( C_\varphi \) on \( H^\infty \) is \( \bar{\mathbb{D}} \) or the sequence \( \{0, 1, \varphi'(a)^n : n = 1, 2, \ldots\} \). This is the second main result, stated and proved below as Theorem 4. But unlike the theorems about the spectrum of \( C_\varphi \) acting on \( H^2 \) previously referred to, this does not require \( \varphi \) to be univalent or analytic in a neighborhood of \( \mathbb{D} \). It only requires \( \varphi \) to have an interior fixed point.

2. Essential norm.

The essential norm of \( C_\varphi \) on \( H^\infty \) is defined to be

\[
\| C_\varphi \|_e = \inf \{ \| C_\varphi - K \| : K \text{ is compact operator on } H^\infty \}.
\]

Clearly \( C_\varphi \) is compact if and only if its essential norm is zero.

The next result shows that there is only one other possible value for the essential norm of a composition operator on \( H^\infty \).

**Theorem 1.** If \( C_\varphi \) is not compact on \( H^\infty \), then its essential norm is 1.

In order to prove this theorem, we need the following lemma, which was first proved by Schwartz [Sch].

**Lemma 2.** \( C_\varphi \) is compact on \( H^\infty \) if and only if \( \varphi(\mathbb{D}) \) is relatively compact in \( \mathbb{D} \).

**Proof of Theorem 1.** We know that \( C_\varphi \) is compact if and only if its essential norm is 0. The main argument here is that the essential norm must be 1 if \( C_\varphi \) is not compact. Since \( \| C_\varphi(f) \|_\infty = \sup_{z \in \mathbb{D}} |f \circ \varphi(z)| \leq \sup_{z \in \mathbb{D}} |f(z)| = \| f \|_\infty \cdot \| C_\varphi \| \leq 1 \), and hence \( \| C_\varphi \|_e \leq 1 \). It suffices to prove that \( \| C_\varphi \|_e \geq 1 \) if \( C_\varphi \) is not compact on \( H^\infty \).

Now assume \( C_\varphi \) is not compact on \( H^\infty \). By Lemma 2, \( \sup_{z \in \mathbb{D}} |\varphi(z)| = 1 \). There exists a sequence \( \{a_k\}_{k=1}^\infty \subset \mathbb{D} \) such that \( \varphi(a_k) \to e^{i\beta} \) as \( k \to \infty \) for some \( \beta \in \mathbb{R} \). Without loss of generality, let’s assume \( e^{i\beta} = 1 \). Let \( \{r_n\}_{n=1}^\infty \) be a nonnegative sequence increasing to 1, and

\[
\psi_n(z) = \frac{z - r_n}{1 - r_n z}.
\]

Then \( \| \psi_n \|_\infty = 1 \), \( \psi_n \) fixes 1 and \(-1\) for all \( n \in \mathbb{N} \), and \( \psi_n(z) \to -1 \) as \( n \to \infty \) for all \( z \in \mathbb{D} \). Let \( K \) be a compact operator on \( H^\infty \). We want to show that \( \| C_\varphi - K \| \geq 1 \). Since \( K \) is compact and \( \| \psi_n \|_\infty = 1 \), there is a subsequence \( \{\psi_{n_j}\}_{j=1}^\infty \) and an \( f \in H^\infty \) such that \( \lim_{j \to \infty} \| K \psi_{n_j} - f \|_\infty = 0 \). For \( \| C_\varphi - K \| \geq 1 \) to be true, it is enough to prove that \( \lim \sup_{j \to \infty} \| (C_\varphi - K)(\psi_{n_j}) \|_\infty \geq 1 \). But

\[
\|(C_\varphi - K)(\psi_{n_j})\|_\infty \geq \|C_\varphi(\psi_{n_j}) - f\|_\infty - \|K\psi_{n_j} - f\|_\infty.
\]
which implies
\[
\limsup_{j \to \infty} \| (C_{\varphi} - K)(\psi_{n_j}) \|_\infty \geq \limsup_{j \to \infty} \| C_{\varphi}(\psi_{n_j}) - f \|_\infty.
\]
It suffices to prove that \(\limsup_{j \to \infty} \| C_{\varphi}(\psi_{n_j}) - f \|_\infty \geq 1\).

The fact that \(\psi_n(z) \to -1\) as \(n \to \infty\) for all \(z \in \mathbb{D}\) implies that \(\psi_n \circ \varphi(z) \to -1\) as \(j \to \infty\), and hence \(\lim_{j \to \infty} |\psi_n \circ \varphi(z) - f(z)| = |1 - f(z)|\) for all \(z \in \mathbb{D}\). If there is \(z_0 \in \mathbb{D}\) such that \(|1 - f(z_0)| \geq 1\), we have \(\| \psi_n \circ \varphi - f \|_\infty \geq |\psi_n \circ \varphi(z_0) - f(z_0)| \to |1 - f(z_0)| \geq 1\), which implies \(\limsup_{j \to \infty} \| \psi_n \circ \varphi - f \|_\infty \geq 1\) as desired. Otherwise, \(|1 - f(z)| < 1\) for all \(z \in \mathbb{D}\). Then by the triangle inequality \(|1 - f(z)| > 1\) for all \(z \in \mathbb{D}\). Consider the sequence \(\{a_k\}_{k=1}^\infty \subset \mathbb{D}\) which was obtained at the beginning of the proof. We have \(\lim_{k \to \infty} \varphi(a_k) = 1\) and \(\{f(a_k)\}_{k=1}^\infty\) is bounded since \(f \in H^\infty\).

Then there is a subsequence \(\{f(a_{k_j})\}_{j=1}^\infty\) converging to some \(\omega \in \mathbb{C}\). By re-indexing we may assume, without loss of generality that, \(\lim_{k \to \infty} f(a_k) = \omega\).

Then by our assumption on \(f\), \(|1 - \omega| = \lim_{k \to \infty} |1 - f(a_k)| \geq 1\). Since \(\psi_n\) is continuous, \(\psi_n(1) = 1\) and \(\lim_{k \to \infty} \varphi(a_{k_j}) = 1\), it follows that \(\lim_{k \to \infty} |\psi_n \circ \varphi(a_{k_j}) - f(a_{k_j})| = |1 - \omega| \geq 1\) for all \(n\). Then \(\| \psi_n \circ \varphi - f \|_\infty \geq \lim_{k \to \infty} |\psi_{n_j} \circ \varphi(a_{k_j}) - f(a_{k_j})| \geq 1\) for all \(j\). Hence \(\limsup_{j \to \infty} \| \psi_n \circ \varphi - f \|_\infty \geq 1\) as desired.

If \(e^{i\beta} \neq 1\), let \(\Psi_n(z) = e^{i\beta} \psi_n(e^{-i\beta} z)\). The same proof holds with \(\psi_n\) replaced by \(\Psi_n\), and the boundary points \(1\) and \(-1\) replaced by \(e^{i\beta}\) and \(-e^{i\beta}\) respectively. This completes the proof of Theorem 1.

For the rest of this paper, \(\varphi_n\) will denote the \(n\)th iterate of \(\varphi\), i.e., \(\varphi_1 = \varphi\) and \(\varphi_n = \varphi \circ \varphi_{n-1}\) for \(n > 1\).

**Definition** ([CM2, p. 150]). If \(T\) is a bounded linear operator on a Hilbert space, then the spectrum of the equivalence class in the Calkin algebra that contains \(T\) is called the essential spectrum of \(T\).

**Corollary 3.** The essential spectral radius of \(C_{\varphi}\) on \(H^\infty\) is either 1 or 0. If \(C_{\varphi_n} (= C_{\varphi}^n)\) is compact for some \(n \geq 1\), then \(\rho_e(C_{\varphi}) = 0\). Otherwise \(\rho_e(C_{\varphi}) = 1\).

**Proof.** The conclusion follows immediately from Theorem 1 and the formula that \(\rho_e(C_{\varphi}) = \lim_{n \to \infty} (\|C_{\varphi}^n\|_e)^{1/n} = \lim_{n \to \infty} (\|C_{\varphi_n}^n\|_e)^{1/n}\).

**3. Spectrum.**

For \(C_{\varphi}\) acting on \(H^\infty\), the spectrum \(\sigma(C_{\varphi})\) is contained in \(\overline{\mathbb{D}}\). This is because the norm of \(C_{\varphi}\) acting on \(H^\infty\) is always 1, which implies the spectral radius \(\rho(C_{\varphi}) = \lim_{n \to \infty} \| C_{\varphi_n} \|^{1/n} = 1\).

**Theorem 4.** If \(\varphi\) is not a constant, not an automorphism, and \(\varphi(a) = a\) for some \(a \in \mathbb{D}\), then
\[
\sigma(C_{\varphi}) = \overline{\mathbb{D}}, \text{ if } \|\varphi_n\|_\infty = 1 \text{ for all } n \in \mathbb{N}
\]
and
\[ \sigma(C_\varphi) = \{ \varphi'(a)^k : k = 1, 2, \ldots \} \cup \{0, 1\}, \text{ if } \|\varphi_n\|_\infty < 1 \text{ for some } n \in \mathbb{N}. \]

The proof of Theorem 4 will be given after some lemmas. Some ideas and approaches used in the proof are suggested by the work of Kamowitz in [Kam], and that of Cowen and MacCluer in [CM1].

The following lemma, Lemma 5, follows immediately from Koenigs' Theorem [Koe] (see also [Sh2, Chapter 6]), if we can show that the Koenigs' function \( \xi \) of \( \varphi \) is in \( H^\infty \) under the assumption of the Lemma. Since \( \xi \circ \varphi = \varphi'(a)\xi \), which implies \( \xi \circ \varphi_n = \varphi'(a)^n\xi \), and hence \( \xi = \varphi'(a)^{-n}\xi(\varphi_n) \) for all \( n \in \mathbb{N} \), we conclude that \( \xi \) is in \( H^\infty \) under the assumption that \( \|\varphi_n\|_\infty < 1 \) for some \( n \in \mathbb{N} \).

**Lemma 5.** For \( \varphi \) as in Theorem 4, suppose \( \|\varphi_n\|_\infty < 1 \) for some \( n \in \mathbb{N} \). If \( \varphi'(a) \neq 0 \), then \( \{ \varphi'(a)^n, n = 0, 1, 2, \ldots \} \) is the set of eigenvalues of \( C_\varphi \) on \( H^\infty \). If \( \varphi'(a) = 0 \), the only eigenvalue of \( C_\varphi \) is 1.

**Lemma 6.** Let \( \varphi \) be the same as in Theorem 4 and suppose \( \|\varphi_n\|_\infty < 1 \) for some \( n \in \mathbb{N} \). Then for \( C_\varphi \) on \( H^\infty \),
\[ \sigma(C_\varphi) = \{ \varphi'(a)^k : k = 1, 2, \ldots \} \cup \{0, 1\}. \]

**Proof.** Under the hypothesis that \( \|\varphi_n\|_\infty < 1 \), \( C_{\varphi_n} = C_\varphi^m \) is compact, which implies that \( C_\varphi \) is not invertible and hence 0 is in the spectrum. Also, since \( C_\varphi^m \) is compact, \( C_\varphi \) is a Riesz operator. So its nonzero spectrum consists of eigenvalues [Köhn, p. 19-21], and the result follows from Lemma 5.

**Lemma 7.** Suppose \( \varphi(0) = 0 \). Then \( H_m = z^m H^\infty \) is an invariant subspace of \( C_\varphi \) and \( \sigma(C_m) \subset \sigma(C_\varphi) \) where \( C_m = C_\varphi | H_m \).

**Proof.** It's easy to see that \( H_m \) is invariant under \( C_\varphi \). Since \( \varphi(0) = 0 \), \( \varphi(z) = z\phi(z) \) for some \( \phi \in H^\infty \). Then if \( f \in H^\infty \), \( C_\varphi(z^m f) = \varphi^m(f \circ \varphi) \in H_m \).

Suppose \( \lambda \) is in the spectrum of \( C_m \). If \( \lambda \) is an eigenvalue of \( C_m \), it must be an eigenvalue of \( C_\varphi \) and hence in the spectrum of \( C_\varphi \). If \( \lambda \) is not an eigenvalue of \( C_m \), then \( C_m - \lambda I \) is one-one. But it is not invertible, and hence not onto. So there exists \( f \in H_m \) with \( f \notin (C_m - \lambda I)(H_m) \). If we can show that \( C_\varphi - \lambda I \) on \( H^\infty \) is not onto, then it will follow that \( \lambda \notin \sigma(C_\varphi) \) and the conclusion holds. Suppose to the contrary \( C_\varphi - \lambda I \) is onto. Then \( f \in (C_\varphi - \lambda I)(H^\infty) \) and there is \( g \in H^\infty \) with \( g \notin H_m \) such that \( (C_\varphi - \lambda I)g = f \). Let \( g = g_1 + g_2 \), where \( g_1 \in \text{span}(1, z, z^2, \ldots, z^{m-1}) \) and \( g_2 \in H_m \). We have \( g_1 \neq 0 \) since \( g \notin H_m \). Let \( f_1 = (C_\varphi - \lambda I)g_1 = f - (C_\varphi - \lambda I)g_2 \). Then \( f_1 \in H_m \) since \( f, g_2 \in H_m \).

Also by the assumption that \( C_\varphi - \lambda I \) is onto, for each function \( z^i, i = 1, 2, \ldots, m - 1 \), there exists \( h_i \in H^\infty \) such that \( (C_\varphi - \lambda I)h_i(z) = z^i \). Let \( h_i = k_i + l_i \) where \( k_i \in \text{span}(1, z, z^2, \ldots, z^{m-1}) \) and \( l_i \in H_m \). The next
step is to show that \( g_1, k_0, k_1, \ldots, k_{m-1} \) are linearly independent. Suppose \( \beta g_1 + \sum_{i=0}^{m-1} \alpha_i k_i = 0 \) for some \( \beta \) and \( \alpha_i, i = 0, 1, \ldots, m - 1 \). Then \( \beta g_1 + \sum_{i=0}^{m-1} \alpha_i h_i = (\beta g_1 + \sum_{i=0}^{m-1} \alpha_i k_i) + \sum_{i=0}^{m-1} \alpha_i l_i = \sum_{i=0}^{m-1} \alpha_i l_i \in H_m \). So \((C_\varphi - \lambda I)(\beta g_1(z) + \sum_{i=0}^{m-1} \alpha_i h_i(z)) = \beta f_1(z) + \sum_{i=0}^{m-1} \alpha_i z^i \in H_m \). Since \( f_1 \in H_m \), we have \( \alpha_i = 0 \) for \( i = 0, 1, \ldots, m - 1 \). Then it follows that \( \beta = 0 \). So \( g_1, k_0, k_1, \ldots, k_{m-1} \) are linearly independent. But this is impossible since \( \{g_1, k_0, k_1, \ldots, k_{m-1}\} \subset \text{span}(1, z, \ldots, z^{m-1}) \), which is only \( m \) dimensional. So \( C_\varphi - \lambda I \) is not onto, and hence \( \lambda \) is in the spectrum of \( C_\varphi \).

**Definition.** We say the sequence of points \( \{z_k\}_{k=0}^M \) is an iteration sequence for \( \varphi \) if \( \varphi(z_k) = z_{k+1} \) for \( K \leq k < M \) where \( K \geq -\infty \) and \( M \leq \infty \).

**Lemma 8** ([CM2, p. 292, Lemma 7.34]). If \( \varphi \) is not an automorphism and \( \varphi(0) = 0 \), then given \( 0 < r < 1 \), there exists \( M_r \) with \( 1 \leq M_r < \infty \) such that if \( \{z_k\}_{k=0}^\infty \) is an iteration sequence with \( |z_k| \geq r \) for some \( l \geq 0 \) and if \( \{w_k\}_{k=0}^l \) is arbitrary, there is \( h \in H^\infty \) such that \( h(z_k) = w_k \) for \( -K \leq k \leq l \) and \( \|h\|_{\infty} \leq M_r \sup \{|w_k| : -K \leq k \leq l\} \).

**Lemma 9** ([CM2, p. 293, Lemma 7.35]). For \( \varphi \) in Lemma 8 and \( \{z_k\} \) any iteration sequence, there exists \( c \) such that \( |z_{k+1}|/|z_k| \leq c \) whenever \( |z_k| \leq 0.5 \).

**Proof of Theorem 4.** The statement in the case that \( \|\varphi_n\|_{\infty} < 1 \) for some \( n \in \mathbb{N} \) is the result of Lemma 6. Now suppose \( \|\varphi_n\|_{\infty} = 1 \) for all \( n \in \mathbb{N} \). We want to prove \( \sigma(C_\varphi) = \mathbb{D} \). If the interior fixed point \( a \neq 0 \), let \( \tau(z) = (a - z)/(1 - az) \) and \( \psi = \tau \circ \varphi \circ \tau \). Then \( \tau^{-1} = \tau \), \( \psi(0) = 0 \) and \( C_\psi = C_\tau \circ C_\varphi \circ C_\tau = C_\tau \circ C_\varphi \circ C_\tau^{-1} \). \( C_\varphi \) and \( C_\psi \) are similar and hence have the same spectrum. So without loss of generality, we can assume \( \varphi(0) = 0 \).

Since \( \sigma(C_\varphi) \subset \mathbb{D} \) and \( \sigma(C_\varphi) \) is closed, it suffices to prove \( \mathbb{D} - \{0\} \subset \sigma(C_\varphi) \). Let \( 0 \neq \lambda \in \mathbb{D} \), \( H_m = z^m H^\infty \), and \( C_m = C_\varphi |H_m \). By Lemma 7, it suffices to prove that \( \lambda \) is in the spectrum of \( C_m \) for some positive integer \( m \). Since \( C_m - \lambda I \) is not onto if and only if \( (C_m - \lambda I)^* \) is not bounded from below, it is enough to find a positive integer \( m \) with \( (C_m - \lambda I)^* \) not bounded from below.

Let \( M \) be the constant \( M_r \) in Lemma 8 corresponding to \( r=0.25 \) and suppose we have an iteration sequence \( \{z_k\}_{k=0}^\infty \) with \( K \geq 0 \) and \( |z_0| > 0.5 \) (this sequence will be determined later on). Let \( n = \max\{k : |z_k| \geq 0.25\} \). Then \( n \geq 0 \) and \( |z_k| < 0.25 < 0.5 \) for all \( k \geq n + 1 \). By Lemma 9, there is a number \( c_1 < 1 \) so that \( |z_{k+1}| \leq c_1 |z_k| \) whenever \( k \geq n + 1 \). If \( z_n \leq 0.5 \), the inequality also holds for \( k = n \). If \( |z_n| > 0.5 \), then \( |z_{n+1}| < 0.25 \), we have \( |z_{n+1}| < 0.5 |z_n| \). Let \( c = \max\{c_1, 0.5\} \). Then \( |z_{k+1}| \leq c |z_k| \) for all \( k \geq n \). It follows that \( |z_k| \leq c^{k-n} |z_n| \) whenever \( k \geq n \). Since \( c < 1 \), there exists a positive integer \( m \) such that \( c^m/|\lambda| < 1/(2M+1) < 1 \). For this \( m \), \( (C_m - \lambda I)^* \) is not bounded from below. To see that, let \( \varepsilon > 0 \) and we will construct a bounded linear functional \( \Lambda \) on \( H_m \) with \( \|(C_m - \lambda I)^* \Lambda \|/\|\Lambda\| < \varepsilon \).
Let’s define $L_\lambda$ by $L_\lambda(f) = \sum_{k=-K}^\infty \lambda^{-k} f(z_k)$ for $f \in H_m$. We will see that $L_\lambda$ is well-defined and indeed it is bounded.

For $f \in H_m$, $z^{-m} f(z)$ is analytic and $\|f\|_\infty = \|z^{-m} f(z)\|_\infty$. So

$$|f(z_k)| = |z_k|^m |z_k^{-m} f(z_k)| \leq |z_k|^m \|z^{-m} f(z)\|_\infty = |z_k|^m \|f\|_\infty.$$  

Then

$$\sum_{k=-K}^\infty |\lambda|^{-k} |f(z_k)| \leq \|f\|_\infty \sum_{k=-K}^\infty |\lambda|^{-k} |z_k|^m = \|f\|_\infty \sum_{k=-K}^n |\lambda|^{-k} |z_k|^m + \sum_{k=n+1}^\infty |\lambda|^{-k} |z_k|^m.$$  

(1)

For $k > n$, $|z_k| \leq e^{k-n} |z_n|$ and so $|z_k|^m \leq (e^n)^{k-n} |z_n|^m$. We see that

$$\sum_{k=n+1}^\infty |\lambda|^{-k} |z_k|^m \leq \sum_{k=n+1}^\infty \frac{(e^n)^{k-n} |z_n|^m}{|\lambda|^{k-n} |\lambda|^n} = |z_n|^m \sum_{k=n+1}^\infty \left(\frac{e^n}{|\lambda|}\right)^{k-n} < \infty.$$  

It follows that $\sum_{k=-K}^\infty \lambda^{-k} f(z_k)$ converges. Hence $L_\lambda$ is well-defined and by (1) it is bounded.

Now let’s estimate $\frac{\|(C_m - \lambda I)^* L_\lambda\|}{\|L_\lambda\|}$. For $f \in H_m$,

$$\langle f, (C_m - \lambda I)^* L_\lambda \rangle = \langle (C_m - \lambda I) f, L_\lambda \rangle = \langle f \circ \varphi - \lambda f, L_\lambda \rangle = \sum_{k=-K}^\infty \lambda^{-k} (f \circ \varphi(z_k) - \lambda f(z_k)) = \sum_{k=-K}^\infty (\lambda^{-k} f(z_{k+1}) - \lambda^{-k-1} f(z_k)) = -\lambda^{K+1} f(z_{-K}).$$  

Then

$$\|(C_m - \lambda I)^* L_\lambda\| = \sup_{0 \neq f \in H_m} \frac{|\langle f, (C_m - \lambda I)^* L_\lambda \rangle|}{\|f\|_\infty} = \|\lambda|^{K+1}|f(z_{-K})|\|f\|_\infty \leq |\lambda|^{K+1}.$$  

We also need a lower bound for $\|L_\lambda\|$. If we apply Lemma 8 to the iteration sequence $\{z_k\}_K^\infty$ with $r=0.25$, we can find a function $h \in H^\infty$ with $\|h\|_\infty \leq M$, $|h(z_k)| = 1$ and $\lambda^{-k} z_k^m h(z_k) > 0$ for $-K \leq k \leq n$. Let $g(z) = z^m h(z) \in$
$H_m$. Then $\|g\|_\infty \leq M$ and

$$L_\lambda(g) = \sum_{k=-K}^{\infty} \lambda^{-k} z_k^m h(z_k)$$

$$= \sum_{k=-K}^{n-1} |\lambda|^{-k} |z_k|^m + |\lambda|^{-n} |z_n|^m + \sum_{k=n+1}^{\infty} \lambda^{-k} z_k^m h(z_k).$$

By the estimate in (2) and because $M \geq 1$ and $c^m/|\lambda| < 1/(2M + 1)$ from the choice of $m$, we have

$$\left| \sum_{k=n+1}^{\infty} \lambda^{-k} z_k^m h(z_k) \right| \leq \sum_{k=n+1}^{\infty} |\lambda|^{-k} |z_k|^m |h(z_k)|$$

$$\leq M |\lambda|^{-n} |z_n|^m \sum_{k=n+1}^{\infty} \left( \frac{c^m}{|\lambda|} \right)^{k-n}$$

$$= M |\lambda|^{-n} |z_n|^m \frac{c^m/|\lambda|}{1 - c^m/|\lambda|}$$

$$\leq M |\lambda|^{-n} |z_n|^m \frac{2M+1}{1 - 1/(2M+1)}$$

$$= \frac{1}{2} |\lambda|^{-n} |z_n|^m.$$

This shows that

$$|L_\lambda(g)| \geq \sum_{k=-K}^{n-1} |\lambda|^{-k} |z_k|^m + |\lambda|^{-n} |z_n|^m - \left| \sum_{k=n+1}^{\infty} \lambda^{-k} z_k^m h(z_k) \right|$$

$$\geq \sum_{k=-K}^{n-1} |\lambda|^{-k} |z_k|^m + \frac{1}{2} |\lambda|^{-n} |z_n|^m \geq \frac{1}{2} |z_0|^m.$$

Then

$$\|L_\lambda\| \geq \frac{|L_\lambda(g)|}{\|g\|_\infty} \geq \frac{|z_0|^m}{2M} \geq \frac{(0.5)^m}{2M}.$$

It follows that

$$\| (C_m - \lambda I)^* L_\lambda \| \leq \frac{2M |\lambda|^{K+1}}{0.5^m}.$$

Since $|\lambda| < 1$, this is less than $\varepsilon$ if we choose $K$ sufficiently large. For the chosen $K$, we can determine the iteration sequence $\{z_k\}_{-K}^{\infty}$. Since $\|\varphi_K\|_\infty = 1$ by assumption, there exists $w \in \mathbb{D}$ with $|\varphi_K(w)| > 0.5$. Let $z_{-K} = w$ and $z_{k+1} = \varphi(z_k)$ for $k > -K$. Then $|z_0| = |\varphi_K(z_{-K})| > 0.5$. The above calculation follows, thus $(C_m - \lambda I)^*$ is not bounded from below as desired.
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References


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