Pacific
Journal of
Mathematics

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Volume 204 No. 1 May 2002

PACIFIC JOURNAL OF MATHEMATICS
Vol. 204, No. 1, 2002

AN ASYMPTOTIC DIMENSION FOR METRIC SPACES, AND THE 0-TH NOVIKOV–SHUBIN INVARIANT

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A nonnegative number d_∞, called asymptotic dimension, is associated with any metric space. Such number detects the asymptotic properties of the space (being zero on bounded metric spaces), fulfills the properties of a dimension, and is invariant under rough isometries. It is then shown that for a class of open manifolds with bounded geometry the asymptotic dimension coincides with the 0-th Novikov–Shubin number α_0 defined in a previous paper [D. Guido, T. Isola, J. Funct. Analysis, 176 (2000)]. Thus the dimensional interpretation of α_0 given in the mentioned paper in the framework of noncommutative geometry is established on metrics grounds. Since the asymptotic dimension of a covering manifold coincides with the polynomial growth of its covering group, the stated equality generalises to open manifolds a result by Varopoulos.

0. Introduction.

In a recent paper [**14**], we extended the notion of Novikov-Shubin numbers to amenable open manifolds and showed that they have a dimensional interpretation in the framework of noncommutative geometry. Here we introduce an asymptotic dimension for metric spaces, which is an asymptotic counterpart of the Kolmogorov dimension [**15**], and show that for a class of open manifolds it coincides with the 0-th Novikov-Shubin number.

The dimension introduced by Kolmogorov and Tihomirov, also called box dimension, "corresponds to the possibility of characterizing the "massiveness" of sets in metric spaces by the help of the order of growth of the number of elements of their most economical ε-coverings, as $\varepsilon \to 0$" [**15**]. For non-totally bounded sets, denoting by $n(r,R)$ the minimum number of balls of radius r necessary to cover a ball of radius R (and given center), the box dimension is the "order of infinite" of $n(r,R)$ when $r \to 0$ (for big R, and often independently of R).

We then define the asymptotic dimension d_∞ as the "order of infinite" of $n(r,R)$ when $R \to \infty$ (for big r, and often independently of r),

$$d_\infty(X) = \lim_{r\to\infty} \lim_{R\to\infty} \frac{\log n(r,R)}{\log R},$$

show that d_∞ is a dimension, namely $d_\infty(X \cup Y) = \max(d_\infty(X), d_\infty(Y))$ and $d_\infty(X \times Y) \leq d_\infty(X) + d_\infty(Y)$, and prove that it is invariant under rough isometries.

Finally we show that the asymptotic dimension of an open manifold with C^∞-bounded geometry and satisfying an isoperimetric inequality introduced by Grigor'yan [8] coincides with the 0-th Novikov-Shubin number α_0 as defined in [14]. On the one hand this strengthens the dimensional interpretation given in [14], and on the other it shows that the generalised limit procedure used in the definition of α_0 does not affect the result. Moreover, the quasi-isometry invariance of the α_p proved in [14] becomes rough isometry invariance for the case of α_0.

Since the asymptotic dimension of a manifold with C^∞-bounded geometry may be computed in terms of its volume growth, the equality between α_0 and d_∞ may be seen as a generalization of the result of Varopoulos [21] for covering manifolds, namely the equality between α_0 and the growth of the covering group.

We recall that the Novikov-Shubin numbers [17] where introduced, after the definition by Atiyah [2] of the L^2-Betti numbers in terms of the von Neumann trace of the covering group, as finer invariants of the spectral behaviour of the p-Laplacian near zero, and where shown to be homotopy invariant by Gromov and Shubin [10]. It was observed by Roe [18] that, when the covering group is amenable, the von Neumann trace of an operator may be computed as an average of its integral kernel on the manifold w.r.t. a suitable exhaustion. Hence this procedure may be extended to manifolds admitting an amenable exhaustion. We show that, for manifolds satisfying Grigor'yan isoperimetric inequality, an amenable exhaustion exists and is given by concentric balls of increasing radius.

In [13] we showed that, from the operator algebraic point of view, the step from amenable coverings to amenable exhaustions corresponds to passing from a normal semifinite trace on a von Neumann algebra to a semicontinuous semifinite trace on a C*-algebra. The latter does not necessarily contain spectral projections, however, the spectral density function may be still defined, making use of the noncommutative Riemann integration developed in [14], hence Novikov-Shubin numbers are defined. Moreover, this definition coincides, at least for α_0, with the definition given in terms of the trace of the heat kernel, which does not require Riemann integration, and which is used here. More precisely,

$$\alpha_0(M) := 2 \limsup_{t \to \infty} \frac{\log(\tau(e^{-t\Delta_p}))}{\log 1/t},$$

where τ is the (semicontinuous semifinite) trace associated with an amenable exhaustion of M.

We now recall the dimensional interpretation of the Novikov-Shubin numbers given in [14] in the framework of noncommutative geometry.

On the one hand these numbers are defined in terms of the low frequency behaviour of the p-Laplacians, or the large time behaviour of the p-heat kernels, therefore they are the large scale counterpart of the spectral dimension, namely of the dimension as it is recovered by the Weyl asymptotics.

On the other hand, recall that in Alan Connes' noncommutative geometry [4], a nontrivial singular trace, associated to some power of the resolvent of the Dirac operator, plays the role of integration over the noncommutative space, and such a power is the dimension of the space.

This is analogous of what happens in geometric measure theory where a dimension is the unique exponent of the diameter of a ball which gives, via Hausdorff procedure, a (possibly) nontrivial measure on the space.

Therefore it is in this context that Novikov-Shubin numbers are interpreted as asymptotic spectral dimensions, since it was shown in [14] that the operator $\Delta_p^{-1/2}$, raised to the power α_p, is singularly traceable.

The identification of α_0 with d_∞ proved here puts on metric grounds the dimensional character of the 0-th Novikov-Shubin number.

Finally we study the relation of d_∞ with the notion of (metric) asymptotic dimension for cylindrical ends given by Davies [6]. Such definition is given in terms of the volume growth of the end, therefore when the end has bounded geometry Davies' asymptotic dimension coincides with ours. Indeed Davies requires the growth to be exactly polynomial, therefore, in contrast with ours, his dimension is not always defined. Davies also introduced a class of cylindrical ends called standard ends. We show that for standard ends with Davies' asymptotic dimension D the equality $d_\infty = D$ holds with no further assumptions. Making use of Davies standard ends one observes that d_∞ for a open manifold may take any value in $[1, \infty]$. Also, we discuss examples of standard ends where the growth is not exactly polynomial.

Some of the results contained in the present paper have been announced in several international conferences. In particular we would like to thank the Erwin Schrödinger Institute in Vienna, where the first draft of this paper was completed, and the organisers of the "Spectral Geometry Program" for their kind invitation.

1. An asymptotic dimension for metric spaces.

The main purpose of this section is to introduce an asymptotic dimension for metric spaces. Other notions of asymptotic dimension have been considered by Gromov [9] (see also the papers by Yu [23] and Dranishnikov [7]). Davies [6] proposed a definition in the case of cylindrical ends of a Riemannian manifold. We shall give a definition of asymptotic dimension in the setting of metric dimension theory, based on the (local) Kolmogorov dimension [15] and state its main properties. We compare our definition with Davies' in the next Section, and discuss its relations with Gromov's in Remark 1.18.

In the following, unless otherwise specified, (X, δ) will denote a metric space, $B_X(x, R)$ the open ball in X with centre x and radius R, $n_r(\Omega)$ the least number of open balls of radius r which cover $\Omega \subset X$, and $\nu_r(\Omega)$ the largest number of disjoint open balls of radius r centered in Ω.

Kolmogorov and Tihomirov [**15**] defined a dimension for totally bounded metric spaces X as $d_0(X) := \limsup_{r\to 0} \frac{\log n_r(X)}{\log(1/r)}$. A natural extension to all metric spaces is given by $d_0(X) = \lim_{R\to\infty} \limsup_{r\to 0} \frac{\log n_r(B_X(x,R))}{\log(1/r)}$. It can be shown that d_0 is independent of x, is a dimension, namely it satisfies the properties of Theorem 1.8, and is invariant under bi-Lipschitz mappings.

The following definition gives a natural asymptotic counterpart of the dimension of Kolmogorov-Tihomirov.

Definition 1.1. Let (X, δ) be a metric space. We call

$$d_\infty(X) := \lim_{r\to\infty} \limsup_{R\to\infty} \frac{\log n_r(B_X(x,R))}{\log R},$$

the *asymptotic dimension* of X.

Let us remark that, as $n_r(B_X(x, R))$ is nonincreasing in r, the function

$$r \mapsto \limsup_{R\to\infty} \frac{\log n_r(B_X(x,R))}{\log R}$$

is nonincreasing too, so the $\lim_{r\to\infty}$ exists.

Proposition 1.2. *$d_\infty(X)$ does not depend on x.*

Proof. Let $x, y \in X$, and set $\delta := \delta(x, y)$, so that $B(x, R) \subset B(y, R + \delta) \subset B(x, R + 2\delta)$. This implies,

$$\begin{aligned} \frac{\log n_r(B(x,R))}{\log R} &\leq \frac{\log n_r(B(y,R+\delta))}{\log(R+\delta)} \frac{\log(R+\delta)}{\log R} \\ &\leq \frac{\log n_r(B(x,R+2\delta))}{\log(R+2\delta)} \frac{\log(R+2\delta)}{\log R} \end{aligned}$$

so that, taking $\limsup_{R\to\infty}$ and then $\lim_{r\to\infty}$ we get the thesis. □

The following lemma is proved in [**15**]. For the sake of completeness, we include a proof.

Lemma 1.3. *$n_r(\Omega) \geq \nu_r(\Omega) \geq n_{2r}(\Omega)$.*

Proof. For the first inequality, let $B_X(x_i, r)$, $i = 1, \ldots, \nu_r(\Omega)$, be disjoint balls with centres in Ω. Then any r-ball of a covering of Ω may contain at most one of the x_i's. Indeed, if $B_X(x, r) \ni x_i, x_j$, then $B_X(x_i, r) \cap B_X(x_j, r) \supset \{x\} \neq \emptyset$, so that $x_i = x_j$.

As for the second inequality, we need to prove it only when ν_r is finite. Let us assume that $\{B(x_i,r)\}_{i=1}^{\nu_r(\Omega)}$ are disjoint balls centered in Ω and observe that, for any $y \in \Omega$, $\delta(y, \bigcup_{i=1}^{\nu_r(\Omega)} B(x_i,r)) < r$, otherwise $B(y,r)$ would be disjoint from $\bigcup_{i=1}^{\nu_r(\Omega)} B(x_i,r)$, contradicting the maximality of ν_r. So for all $y \in \Omega$ there is j s.t. $\delta(y, B(x_j,r)) < r$, that is $\Omega \subset \bigcup_{i=1}^{\nu_r(\Omega)} B(x_i,2r)$, which implies the thesis. □

Proposition 1.4.

$$d_\infty(X) = \lim_{r\to\infty} \limsup_{R\to\infty} \frac{\log \nu_r(B_X(x,R))}{\log R}.$$

Proof. Follows easily from Lemma 1.3. □

Definition 1.5. Let X, Y be metric spaces, $f : X \to Y$ is said to be a rough isometry if there are $a \geq 1$, $b, \varepsilon \geq 0$ s.t.

(i) $a^{-1}\delta_X(x_1,x_2) - b \leq \delta_Y(f(x_1), f(x_2)) \leq a\delta_X(x_1,x_2) + b$, for all $x_1, x_2 \in X$,

(ii) $\bigcup_{x\in X} B_Y(f(x),\varepsilon) = Y$.

Lemma 1.6 ([3], Proposition 4.3)**.** *If $f : X \to Y$ is a rough isometry, there is a rough isometry $f^- : Y \to X$, with constants a, b^-, ε^-, s.t.*

(i) $\delta_X(f^- \circ f(x), x) < c_X$, $x \in X$,

(ii) $\delta_Y(f \circ f^-(y), y) < c_Y$, $y \in Y$.

Theorem 1.7. *Let X, Y be metric spaces, and $f : X \to Y$ a rough isometry. Then $d_\infty(X) = d_\infty(Y)$.*

Proof. Let $x_0 \in X$, then for all $x \in B_X(x_0,r)$ we have $\delta_Y(f(x), f(x_0)) \leq a\delta_X(x,x_0) + b < ar + b$ so that $f(B_X(x_0,r)) \subset B_Y(f(x_0), ar+b)$. Then, with $n := n_r(B_Y(f(x_0), aR+b))$,

$$f(B_X(x_0,R)) \subset \bigcup_{j=1}^{n} B_Y(y_j,r),$$

which implies

$$\begin{aligned} f^- \circ f(B_X(x_0,R)) &\subset \bigcup_{j=1}^{n} f^-(B_Y(y_j,r)) \\ &\subset \bigcup_{j=1}^{n} B_X(f^-(y_j), ar+b^-). \end{aligned}$$

Let $x \in B_X(x_0,R)$, and j be s.t. $f^- \circ f(x) \in B_X(f^-(y_j), ar+b^-)$, then

$$\delta_X(x, f^-(y_j)) \leq \delta_X(x, f^- \circ f(x)) + \delta_X(f^- \circ f(x), f^-(y_j)) < c_X + ar + b^-,$$

so that

$$B_X(x_0,R)\subset\bigcup_{j=1}^n B_X(f^-(y_j),ar+b^-+c_X),$$

which implies $n_{ar+b^-+c_X}(B_X(x_0,R))\leq n_r(B_Y(f(x_0),aR+b))$.

Finally

$$\begin{aligned}d_\infty(X)&=\lim_{r\to\infty}\limsup_{R\to\infty}\frac{\log n_r(B_X(x_0,R))}{\log R}\\&=\lim_{r\to\infty}\limsup_{R\to\infty}\frac{\log n_{ar+b^-+c_X}(B_X(x_0,R))}{\log R}\\&\leq\lim_{r\to\infty}\limsup_{R\to\infty}\frac{\log n_r(B_Y(f(x_0),aR+b))}{\log R}\\&=\lim_{r\to\infty}\limsup_{R\to\infty}\frac{\log n_r(B_Y(f(x_0),R))}{\log R}\\&=d_\infty(Y)\end{aligned}$$

and exchanging the roles of X and Y we get the thesis. □

Theorem 1.8. *The set function d_∞ is a dimension, namely it satisfies:*

(i) *If $X\subset Y$ then $d_\infty(X)\leq d_\infty(Y)$.*
(ii) *If $X_1,X_2\subset X$ then $d_\infty(X_1\cup X_2)=\max\{d_\infty(X_1),d_\infty(X_2)\}$.*
(iii) *If X and Y are metric spaces, then $d_\infty(X\times Y)\leq d_\infty(X)+d_\infty(Y)$.*

Proof. (i) Let $x\in X$, then $B_X(x,R)\subset B_Y(x,R)$ and the claim follows easily.

(ii) By part (i), we get $d_\infty(X_1\cup X_2)\geq\max\{d_\infty(X_1),d_\infty(X_2)\}$. So we need to prove the converse inequality, only when both X_1 and X_2 have finite asymptotic dimension. Let $x_i\in X_i$, $i=1,2$, and set $\delta=\delta(x_1,x_2)$, $a=d_\infty(X_1)$, $b=d_\infty(X_2)$, with e.g., $a\leq b<\infty$. Then $B_{X_1\cup X_2}(x_1,R)\subset B_{X_1}(x_1,R)\cup B_{X_2}(x_2,R+\delta)$, therefore

$$n_r(B_{X_1\cup X_2}(x_1,R))\leq n_r(B_{X_1}(x_1,R))+n_r(B_{X_2}(x_2,R+\delta)).\tag{1.1}$$

Besides, $\forall\varepsilon>0\ \exists r_0>0$ s.t. $\forall r>r_0\ \exists R_0=R_0(\varepsilon,r)$ s.t. $\forall R>R_0$

$$n_r(B_{X_1}(x_1,R))\leq R^{a+\varepsilon}$$
$$n_r(B_{X_2}(x_2,R+\delta))\leq R^{b+\varepsilon},$$

hence, by inequality (1.1),

$$n_r(B_{X_1\cup X_2}(x_1,R))\leq R^{a+\varepsilon}+R^{b+\varepsilon}=R^{b+\varepsilon}(1+R^{a-b}).$$

Finally,

$$\frac{\log n_r(B_{X_1\cup X_2}(x_1,R))}{\log R}\leq b+\varepsilon+\frac{\log(1+R^{a-b})}{\log R}.$$

Taking the $\limsup_{R\to\infty}$ and then the $\lim_{r\to\infty}$ we get

$$d_\infty(X_1 \cup X_2) \le \max\{d_\infty(X_1), d_\infty(X_2)\} + \varepsilon$$

and the thesis follows by the arbitrariness of ε.

(iii) By Theorem 1.7, we may endow $X \times Y$ with any metric roughly isometric to the product metric, e.g.,

$$\delta_{X\times Y}((x_1, y_1), (x_2, y_2)) = \max\{\delta_X(x_1, x_2), \delta_Y(y_1, y_2)\}. \tag{1.2}$$

Then, by $n_r(B_{X\times Y}((x, y), R)) \le n_r(B_X(x, R))\, n_r(B_Y(y, R))$, the thesis follows easily. □

Remark 1.9. (a) In part (ii) of the previous theorem we considered X_1 and X_2 as metric subspaces of X. If X is a Riemannian manifold and we endow the submanifolds X_1, X_2 with their geodesic metrics this property does not hold in general. A simple example is the following. Let $f(t) := (t\cos t, t\sin t)$, $g(t) := (-t\cos t, -t\sin t)$, $t \ge 0$ planar curves, and set X, Y for the closure in $\mathbb{R}^2$ of the two connected components of $\mathbb{R}^2 \setminus (G_f \cup G_g)$, where G_f, G_g are the graphs of f, g, and endow X, Y with the geodesic metric. Then X and Y are roughly-isometric to $[0, \infty)$ (see below) so that $d_\infty(X) = d_\infty(Y) = 1$, while $d_\infty(X \cup Y) = 2$.

(b) The choice of the lim sup in Definition 1.1 is the only one compatible with the classical dimensional inequality stated in Theorem 1.8 (iii).

In what follows we show that when X is equipped with a suitable measure, the asymptotic dimension may be recovered in terms of the volume asymptotics for balls of increasing radius. This is analogous to the fact that the local dimension may be recovered in terms of the volume asymptotics for balls of infinitesimal radius.

Definition 1.10. A Borel measure μ on (X, δ) is said to be uniformly bounded if there are functions β_1, β_2, s.t. $0 < \beta_1(r) \le \mu(B(x, r)) \le \beta_2(r)$, for all $x \in X$, $r > 0$.

That is

$$\beta_1(r) := \inf_{x\in X} \mu(B(x, r)) > 0, \quad \text{and} \quad \beta_2(r) := \sup_{x\in X} \mu(B(x, r)) < \infty.$$

Proposition 1.11. *If (X, δ) has a uniformly bounded measure, then every ball in X is totally bounded (so that if X is complete it is locally compact).*

Proof. Indeed, if there is a ball $B = B(x, R)$ which is not totally bounded, then there is $r > 0$ s.t. every r-net in B is infinite, so $n_r(B)$ is infinite, and $\nu_r(B)$ is infinite too. So that $\beta_2(R) \ge \mu(B) \ge \sum_{i=1}^{\nu_r(B)} \mu(B(x_i, r)) \ge \beta_1(r)\nu_r(B) = \infty$, which is absurd. □

Proposition 1.12. *If μ is a uniformly bounded Borel measure on X then*

$$d_\infty(X) = \limsup_{R\to\infty} \frac{\log \mu(B(x, R))}{\log R}.$$

Proof. As $\bigcup_{i=1}^{\nu_r(B(x,R))} B(x_i,r) \subset B(x,R+r) \subset \bigcup_{j=1}^{n_r(B(x,R+r))} B(y_j,r)$, we get

$$\begin{aligned}\beta_2(r)n_r(B(x,R+r)) &\geq \mu(B(x,R+r)) \\ &\geq \beta_1(r)\nu_r(B(x,R)) \geq \beta_1(r)n_{2r}(B(x,R)),\end{aligned}$$

by Lemma 1.3. So that

$$\beta_1(r/2) \leq \frac{\mu(B(x,R+r/2))}{n_r(B(x,R))}, \qquad \frac{\mu(B(x,R))}{n_r(B(x,R))} \leq \beta_2(r),$$

and the thesis follows easily. □

Let us now consider the particular case of complete Riemannian manifolds.

Definition 1.13. Let (M,g) be an n-dimensional complete Riemannian manifold. We say that M has bounded geometry if it has positive injectivity radius, sectional curvature bounded from above, and Ricci curvature bounded from below.

Lemma 1.14. *Let M be an n-dimensional complete Riemannian manifold with bounded geometry. Then the Riemannian volume is a uniformly bounded measure.*

Proof. We can assume, without loss of generality, that the sectional curvature is bounded from above by some positive constant c_1 and the Ricci curvature is bounded from below by $(n-1)c_2g$, with $c_2<0$. Then, denoting with $V_\delta(r)$ the volume of a ball of radius r in a manifold of constant sectional curvature equal to δ, we can set $\beta_1(r) := V_{c_1}(\min\{r,r_0\})$, and $\beta_2 := V_{c_2}(r)$, where $r_0 := \min\{\text{inj}(M), \frac{\pi}{\sqrt{c_1}}\}$, and $\text{inj}(M)$ is the injectivity radius of M. Then the result follows from ([**3**], p. 119 and 123). □

Proposition 1.15. *Let M,N be complete Riemannian manifolds.*

(i) *If M is noncompact, then $d_\infty(M) \geq 1$.*
(ii) *If M has bounded geometry, then*

$$d_\infty(M) = \limsup_{R\to\infty} \frac{\log \text{vol}\,(B_M(x,R))}{\log R}, \ x \in M.$$

(iii) *If M,N have bounded geometry, and admit asymptotic dimension in a strong sense, that is $d_\infty(M) = \lim_{R\to\infty} \frac{\log \text{vol}\,(B_M(x,R))}{\log R}$, $x\in M$, and analogously for N, then*

$$d_\infty(M\times N) = d_\infty(M) + d_\infty(N).$$

Proof. (i) Let us fix $x_0 \in M$ and $R>0$, and consider $x_R \in M$ s.t. $\delta(x_0,x_R) = R$, which exists because M is not compact, and let $\gamma : [0,1] \to M$ be a minimizing geodesics between $\gamma(0) = x_0$, $\gamma(1) = x_R$. Clearly $\gamma([0,1)) \subset$

$B_M(x_0, R)$, hence, if $x_1, \dots, x_k$ are the centres of a minimal covering by r-balls of $\gamma([0,1))$, we have $k \leq n_r(B_M(x_0, R))$. Then

$$R = \text{length}(\gamma) \leq \sum_{i=1}^{k} \text{length}(\gamma \cap B_M(x_i, r)) \leq 2rk,$$

namely $n_r(B_M(x_0, R)) \geq \frac{R}{2r}$, from which the thesis follows.

(ii) The result follows from Lemma 1.14 and Proposition 1.12.

(iii) As in the proof of Theorem 1.8 (iii), we may endow $M \times N$ with the metric (1.2). Then $\text{vol}\,(B_{M\times N}((x,y), R)) = \text{vol}\,(B_M(x,R))\,\text{vol}\,(B_N(y,R))$, and we get

$$\begin{aligned} d_\infty(M \times N) &= \lim_{R\to\infty} \frac{\log\, \text{vol}\,(B_{M\times N}((x,y),R))}{\log R} \\ &= \lim_{R\to\infty} \frac{\log\, \text{vol}\,(B_M(x,R))}{\log R} + \lim_{R\to\infty} \frac{\log\, \text{vol}\,(B_N(y,R))}{\log R} \\ &= d_\infty(M) + d_\infty(N). \end{aligned}$$

□

Remark 1.16. (a) Conditions under which the inequality in Theorem 1.8 (iii) becomes an equality are often studied in the case of (local) dimension theory (cf. [**1**, **19**]). The previous Proposition gives such a condition for the asymptotic dimension.

(b) As the asymptotic dimension is invariant under rough isometries, it is natural to substitute the continuous space with a coarse graining, which destroys the local structure, but preserves the large scale structure. Then (cf. [**3**], Theorem 4.9) if M is a complete Riemannian manifold with Ricci curvature bounded from below, M is roughly isometric to any of its discretizations, endowed with the combinatorial metric. Therefore M has the same asymptotic dimension of any of its discretizations. In particular, when M has a discrete group of isometries Γ with a compact quotient, the asymptotic dimension of the manifold coincides with the asymptotic dimension of the group, hence with its growth (cf. [**12**]). Therefore, by a result of Varopoulos [**21**], it coincides with the 0-th Novikov-Shubin invariant. We will generalise this result in Section 3.

Let us conclude this Section with some examples. Other examples are contained in the next Section.

Example 1.17.

(i) $\mathbb{R}^n$ has asymptotic dimension n.

(ii) Set $X := \cup_{n\in\mathbb{Z}}\{(x,y) \in \mathbb{R}^2 : \delta((x,y),(n,0)) < \frac{1}{4}\}$, endowed with the Euclidean metric, then $d_0(X) = 2$, $d_\infty(X) = 1$.

(iii) Set $X = \mathbb{Z}$ with the usual distance, then $d_0(X) = 0$, and $d_\infty(X) = 1$.

(iv) Let X be the unit ball in an infinite dimensional Banach space. Then $d_0(X) = +\infty$ while $d_\infty(X) = 0$.

(v) Let X be the $\mathbb{Z}^\infty$-lattice determined by an orthonormal base in an infinite dimensional Hilbert space. Then $d_0(X) = 0$ while $d_\infty(X) = \infty$.

Remark 1.18. M. Gromov introduced a notion of "large scale dimension" for metric spaces: The asymptotic dimension of X is the smallest integer n such that, for any $r > 0$, there is a cover $\mathcal{U} = \{U_i\}$ of X such that the diameters of the sets U_i are bounded, and no ball of radius r meets more than $n+1$ of them.

Our asymptotic dimension can be very different from Gromov's. For example hyperbolic space H_n has finite Gromov dimension, but $d_\infty(H_n) = \infty$. Conversely, one can find a sequence of cylindrical ends with fixed d_∞ and arbitrarily large Gromov dimension (cf. Corollary 2.4).

The two notions however, coincide on some very special spaces, such as cartesian products of $\mathbb{R}^n$ and a compact set with the product metric. Moreover both dimensions are in a sense "coarse", since they are invariant under rough isometries.

Finally we remark that Gromov dimension is an asymptotic topological dimension, since it is a coarse analogue of the Lebesgue covering dimension, according to Dranishnikov [**7**]. Ours instead is an asymptotic metric dimension. Indeed it is an asymptotic counterpart of the Kolmogorov-Tihomirov metric dimension, and is a dimension in the context of noncommutative geometric measure theory [**12**, **14**].

2. Asymptotic dimension of some cylindrical ends.

In this Section we want to compare our work with a work of Davies. In [**6**] he defines the asymptotic dimension of cylindrical ends of a Riemannian manifold M as follows. Let $E \subset M$ be homeomorphic to $(1, \infty) \times A$, where A is a compact Riemannian manifold. Set $\partial E := \{1\} \times A$, $E_r := \{x \in E : \delta(x, \partial E) < r\}$, where δ is the restriction of the metric in M. Then E has asymptotic dimension D if there is a positive constant c s.t.

$$c^{-1} r^D \leq \mathrm{vol}\,(E_r) \leq c r^D, \tag{2.1}$$

for all $r \geq 1$. Davies does not assume bounded geometry for E. If one does, the two definitions coincide, more precisely if an asymptotic dimension *à la* Davies exists, it coincides with ours.

Proposition 2.1. *With the above notation, if the volume form on E is a uniformly bounded measure* (*as in Definition* 1.10), *or in particular if E has bounded geometry* (*as in Definition* 1.13), *and there is D as in* (2.1), *then $d_\infty(E) = D$.*

Proof. Choose $o \in E$, and set $\delta := \delta(o, \partial E)$, $\Delta :=$ diam$\,(\partial E)$. Then it is easy to prove that $E_{R-\delta-\Delta} \subset B_E(o, R) \subset E_{R+\delta}$.

Then $c^{-1}(R-\delta-\Delta)^D \leq \operatorname{vol}(B_E(o,R)) \leq c(R+\delta)^D$, and from Proposition 1.12 the thesis follows. $\square$

Motivated by ([6], Example 16), let us set the following:

Definition 2.2. E is a standard end of local dimension N if it is homeomorphic to $(1,\infty)\times A$, endowed with the metric $ds^2 = dx^2 + f(x)^2 d\omega^2$, and with the volume form $d\operatorname{vol} = f(x)^{N-1} dx d\operatorname{vol}_\omega$, where (A,ω) is an $(N-1)$-dimensional compact Riemannian manifold, and f is an increasing smooth function.

Proposition 2.3. *The volume form on a standard end E is a uniformly bounded measure. Therefore, if E satisfies equation (2.1), we get $d_\infty(E) = D$.*

Proof. It is easy to show that, for $(x_0,p_0)\in E$, $r < x_0-1$,

$$[x_0 - r/2, x_0 + r/2] \times B_A\left(p_0, \frac{r/2}{f(x_0+r/2)}\right) \subset B_E((x_0,p_0),r) \subset [x_0-r, x_0+r] \times B_A\left(p_0, \frac{r}{f(x_0-r)}\right).$$

So that, with $V_X(x,r) := \operatorname{vol}(B_X(x,r))$,

$$\begin{aligned}&\int_{x_0-r/2}^{x_0+r/2} f(x)^{N-1}dx\ V_A\left(p_0, \frac{r/2}{f(x_0+r/2)}\right)\\ &\leq V_E((x_0,p_0),r)\\ &\leq \int_{x_0-r}^{x_0+r} f(x)^{N-1}dx\ V_A\left(p_0, \frac{r}{f(x_0-r)}\right)\end{aligned}$$

which implies

$$\begin{aligned}&rf(x_0-r/2)^{N-1}\ V_A\left(p_0, \frac{r/2}{f(x_0+r/2)}\right)\\ &\leq V_E((x_0,p_0),r)\\ &\leq 2rf(x_0+r)^{N-1}\ V_A\left(p_0, \frac{r}{f(x_0-r)}\right).\end{aligned}$$

As for $x_0\to\infty$, $V_A\left(p_0, \frac{r}{f(x_0-r)}\right) \sim c\left(\frac{r}{f(x_0-r)}\right)^{N-1}$, and the same holds for $V_A\left(p_0, \frac{r/2}{f(x_0+r/2)}\right)$, we get the thesis. $\square$

Corollary 2.4. *Let E be the standard end given by $E := (1,\infty)\times S^{N-1}$, endowed with the metric $ds^2 = dr^2 + r^{2(D-1)/(N-1)}d\omega^2$, and with the volume form $d\operatorname{vol} = r^{D-1}drd^{N-1}\omega$ ([6], Example 16). Then $d_\infty(E) = D$.*

Remark 2.5. Observe that $d_\infty(M)$ makes sense for any metric space, hence for any cylindrical end, while Davies' asymptotic dimension does not. Indeed let $E := (1,\infty)\times S^1$, endowed with the metric $ds^2 = dr^2 + f(r)^2 d\omega^2$, and with the volume form $d\,\mathrm{vol} = f(r)drd\omega$, where $f(r) := \frac{d}{dr}(r^2\log r)$. Then $d_\infty(E) = 2$, but $\mathrm{vol}\,(E_r)$ does not satisfy one of the inequalities in (2.1).

Before closing this section we observe that the notion of standard end allows us to construct an example which shows that we could obtain quite different results if we used lim inf instead of lim sup in the definition of the asymptotic dimension. It makes use of the following function

$$f(x) = \begin{cases} \sqrt{x} & x\in[1,a_1] \\ 2+b_{n-1}+c_{n-1}+(x-a_{2n-1}) & x\in[a_{2n-1},a_{2n}] \\ 2+b_{n-1}+c_n+\sqrt{x-a_{2n}+1} & x\in[a_{2n},a_{2n+1}] \end{cases}$$

where $a_0 := 0$, $a_n - a_{n-1} := 2^{2^n}$, $b_n := \sum_{k=1}^n \sqrt{2^{2^{2k+1}}+1}$, $c_n := \sum_{k=1}^n (2^{2^{2k}}-1)$, $n\geq 1$.

Proposition 2.6. *Let M be the Riemannian manifold obtained as a C^∞ regularization of $C\cup_\varphi E$, where $C := \{(x,y,z)\in\mathbb{R}^3 : (x-1)^2+y^2+z^2 = 1,\ x\leq 1\}$, with the Euclidean metric, $E := [1,\infty)\times S^1$, endowed with the metric $ds^2 = dx^2 + f(x)^2 d\omega^2$, and with the volume form $d\,\mathrm{vol} = f(x)dxd\omega$, where φ is the identification of $\{y^2+z^2 = 1,\ x = 1\}$ with $\{1\}\times S^1$. Then the volume form is a uniformly bounded measure, $d_\infty(M)\geq 2$ but $\underline{d}_\infty(M)\leq 3/2$, where $\underline{d}_\infty(M) := \lim_{r\to\infty}\liminf_{R\to\infty}\frac{\log n_r(B_M(x,R))}{\log R}$.*

Proof. Set $o := (0,0,0)\in M$, then it is easy to see that, for $n\to\infty$, $a_n\sim 2^{2^n}$, $b_n\sim c_n\sim 2^{2^{2n}}$, and

$$\mathrm{area}\,(B_M(o,a_{2n}))\sim\frac{1}{2}a_{2n}^2$$
$$\mathrm{area}\,(B_M(o,a_{2n-1}))\sim\frac{5}{3}a_{2n-1}^{3/2}$$

so that, calculating the limit of $\frac{\log\,\mathrm{area}\,(B_M(o,R))}{\log R}$ on the sequence $R = a_{2n}$ we get 2, while on the sequence $R = a_{2n-1}$ we get $3/2$. The thesis follows easily, using Proposition 1.12. $\square$

3. The asymptotic dimension and the 0-th Novikov-Shubin invariant.

In this Section we show that, for a class of open manifolds of bounded geometry, the asymptotic dimension coincides with the 0-th Novikov-Shubin invariant defined in [**14**]. In all this Section M denotes a manifold of C^∞-bounded geometry, i.e., M has positive injectivity radius, and the curvature tensor is bounded together with all its covariant derivatives. We assume moreover that M satisfies:

Assumption 3.1. There are $A,\ C,\ C' > 0$ s.t. for all $x \in M$, $r > 0$,

$$V(x, 2r) \leq AV(x, r) \tag{3.1}$$

$$\frac{C}{V(x, \sqrt{r})} \leq p_r(x, x) \leq \frac{C'}{V(x, \sqrt{r})} \tag{3.2}$$

where $V(x, r) := \operatorname{vol}(B(x, r))$ and $p_t(x, y)$ is the heat kernel on M.

Remark 3.2.

(i) Inequality (3.1) is introduced in [**5**] and called the volume doubling property.

(ii) A result of Coulhon-Grigor'yan ([**5**], Corollary 7.3) ([**8**], Proposition 5.2) states that the assumption above is equivalent to the following isoperimetric inequality introduced in [**5**]. There are $\alpha,\ \beta > 0$ s.t. for all $x \in M$, $r > 0$, and all regions $U \subset B(x, r)$,

$$\lambda_1(U) \geq \frac{\alpha}{r^2}\left(\frac{V(x, r)}{\operatorname{vol}(U)}\right)^{\beta},$$

where $\lambda_1(U)$ is the first Dirichlet eigenvalue of Δ in U.

(iii) Assumption 3.1 is satisfied by all manifolds with positive Ricci curvature [**16**], and covering manifolds whose group of deck transformations has polynomial growth [**20**].

Proposition 3.3. *Let M be a complete Riemannian manifold of C^∞-bounded geometry, satisfying Assumption* 3.1.

Then $d_\infty(M) = \limsup_{t\to\infty} \frac{-2\log p_t(x,x)}{\log t}$, *for any $x \in M$.*

Proof. Follows from Proposition 1.15 and estimates (3.2). □

Remark 3.4. The previous result shows that there are some connections between the asymptotic dimension of a manifold and the notion of dimension at infinity for semigroups (in our case the heat kernel semigroup) considered by Varopoulos (see [**22**]).

The volume doubling property is a weak form of polynomial growth condition, and still guarantees the finiteness of the asymptotic dimension (for manifolds of bounded geometry).

Proposition 3.5. *Let M be a complete Riemannian manifold of C^∞-bounded geometry, and suppose the volume doubling property* (3.1) *holds. Then M has finite asymptotic dimension.*

Proof. Let $R > 1$, and $n \in \mathbb{N}$ be s.t. $2^{n-1} < R \leq 2^n$. Then $V(x, R) \leq V(x, 2^n) \leq A^n V(x, 1)$, so that

$$1 \leq \frac{V(x, R)}{V(x, 1)} \leq A^n \leq AR^{\log_2 A}.$$

Therefore $d_\infty(M) = \limsup_{R\to\infty} \frac{\log V(x,R)}{\log R} \leq \log_2 A$. □

Definition 3.6 ([**18**])**.** A regular exhaustion $\mathcal{K}$ is an increasing sequence $\{K_n\}$ of compact subsets of M, whose union is M, and such that, for any $r > 0$

$$\lim_{n\to\infty} \frac{\mathrm{vol}\,(\mathrm{Pen}^+(K_n, r))}{\mathrm{vol}\,(\mathrm{Pen}^-(K_n, r))} = 1,$$

where we set $\mathrm{Pen}^+(K, r) := \{x \in M : \delta(x, K) \leq r\}$, and $\mathrm{Pen}^-(K, r) :=$ the closure of $M \setminus \mathrm{Pen}^+(M \setminus K, r)$.

Proposition 3.7. *Let M be an open manifolds of C^∞-bounded geometry and satisfying Assumption 3.1.*

(i) *There is $\gamma > 0$ s.t. for any $x,\ y \in M$, $r > 0$, if $B(x, r) \cap B(y, r) \neq \emptyset$, then*

$$\gamma^{-1} \leq \frac{V(x, r)}{V(y, r)} \leq \gamma.$$

(ii) *There is a sequence $n_k \in \mathbb{N}$ s.t. $\{B(x, n_k)\}$ is a regular exhaustion of M.*

Proof. (i) The inequality easily follows by a result of Grigor'yan ([**8**], Proposition 5.2), where it is shown that Assumption 3.1 implies the existence of a constant γ such that

$$\gamma^{-1} \left(\frac{R}{r}\right)^{\alpha_1} \leq \frac{V(x, R)}{V(y, r)} \leq \gamma \left(\frac{R}{r}\right)^{\alpha_2}$$

for some positive constants $\alpha_1,\ \alpha_2$, for any $R \geq r$, and $B(x, R)\cap B(y, r) \neq \emptyset$.

(ii) The statement follows from the fact that the volume doubling property implies subexponential (volume) growth, so that the result is contained in ([**18**], Proposition 6.2). □

Recall from [**13**] that the C*-algebra $\mathcal{A}$ of almost local operators on M is the norm closure of the finite propagation operators on $L^2(M, d\,\mathrm{vol}\,)$. Then:

Proposition 3.8 ([**13**])**.** *There is on $\mathcal{A}$ a lower semicontinuous semifinite trace $Tr_{\mathcal{K}}$, which, on the heat semigroup, is given by the following formula,*

$$Tr_{\mathcal{K}}(\mathrm{e}^{-t\Delta}) = \mathrm{Lim}_\omega \frac{\int_{K_n} tr(p_t(x, x)) d\,\mathrm{vol}\,(x)}{\mathrm{vol}\,(K_n)},$$

where Lim_ω is a generalized limit.

Remark 3.9.

(i) The above formula for the trace was considered by J. Roe in [**18**]. However, this formula does not describe a semicontinuous trace on the C*-algebra of almost local operators. Therefore we introduced a semicontinuous semifinite regularization in [**13**].

(ii) L^2-Betti numbers for open manifolds have been introduced in [**18**], where it is shown that the 0-th L^2-Betti number of a noncompact manifold is zero. For this reason it does not appear in the formula for α_0 below.

By means of $Tr_{\mathcal{K}}$ we defined the 0-th Novikov-Shubin invariant as

$$\alpha_0(M, \mathcal{K}) := 2 \limsup_{t\to\infty} \frac{\log Tr_{\mathcal{K}}(\mathrm{e}^{-t\Delta})}{\log 1/t}.$$

Theorem 3.10. *Let M be an open manifold of C^∞-bounded geometry and satisfying Assumption 3.1, endowed with the regular exhaustion $\mathcal{K}$ given by Proposition 3.7 (ii). Then the asymptotic dimension of M coincides with the 0-th Novikov-Shubin invariant, namely $d_\infty(M) = \alpha_0(M, \mathcal{K})$. In particular α_0 is independent of the limit procedure* Lim_ω.

Proof. First, from Equation (3.2) and Proposition 3.7 (i), we get

$$\begin{aligned}
\frac{C\gamma^{-1}}{V(o,\sqrt{t})} &\leq \frac{\int_{B(o,r)} \frac{C}{V(x,\sqrt{t})} d\,\mathrm{vol}\,(x)}{V(o,r)} \\
&\leq \frac{\int_{B(o,r)} p_t(x,x) d\,\mathrm{vol}\,(x)}{V(o,r)} \\
&\leq \frac{\int_{B(o,r)} \frac{C'}{V(x,\sqrt{t})} d\,\mathrm{vol}\,(x)}{V(o,r)} \leq \frac{C'\gamma}{V(o,\sqrt{t})}
\end{aligned}$$

therefore, by Proposition 3.8 we have,

$$\frac{C\gamma^{-1}}{V(o,\sqrt{t})} \leq Tr_{\mathcal{K}}(\mathrm{e}^{-t\Delta}) \leq \frac{C'\gamma}{V(o,\sqrt{t})}$$

hence, finally,

$$\begin{aligned}
d_\infty(M) &= 2 \limsup_{t\to\infty} \frac{\log(V(o,t))}{2\log t} \\
&= 2 \limsup_{t\to\infty} \frac{\log(C'\gamma V(o,\sqrt{t})^{-1})}{\log \frac{1}{t}} \\
&\leq \alpha_0(M,\mathcal{K}) \equiv 2 \limsup_{t\to\infty} \frac{\log Tr_{\mathcal{K}}(\mathrm{e}^{-t\Delta})}{\log \frac{1}{t}} \\
&\leq 2 \limsup_{t\to\infty} \frac{\log(C\gamma^{-1} V(o,\sqrt{t})^{-1})}{\log \frac{1}{t}} \\
&= 2 \limsup_{t\to\infty} \frac{\log(V(o,t))}{2\log t} = d_\infty(M).
\end{aligned}$$

The thesis follows. $\square$

Acknowledgement. We would like to thank D. Burghelea, M. Farber and L. Friedlander for conversations. We also thank I. Chavel, E.B. Davies, A. Grigor'yan, P. Li and L. Saloff-Coste for having suggested useful references on heat kernel estimates, and the referee for drawing our attention to the notion of Gromov asymptotic dimension.

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Received April 10, 2000 and revised October 16, 2000.

Dipartimento di Matematica
Università della Basilicata
I–85100 Potenza, Italy

Universita' di Roma "Tor Vergata"
I–00133 Roma
Italy
E-mail address: guido@mat.uniroma2.it

Dipartimento di Matematica
Università di Roma "Tor Vergata"
I–00133 Roma, Italy
E-mail address: isola@mat.uniroma2.it