PARAHORIC FIXED SPACES IN UNRAMIFIED PRINCIPAL SERIES REPRESENTATIONS

Joshua M. Lansky

Let \( k \) be a non-archimedean locally compact field and let \( G \) be the set of \( k \)-points of a connected reductive group defined over \( k \). Let \( W \) be the relative Weyl group of \( G \), and let \( \mathcal{H}(G, B) \) be the Hecke algebra of \( G \) with respect to an Iwahori subgroup \( B \) of \( G \). We compute the effects of \( \mathcal{H}(G, B) \) and \( W \) on the \( B \)-fixed vectors of an unramified principal series representation \( I \) of \( G \). We use this computation to determine the dimension of the space of \( K \)-fixed vectors in \( I \), where \( K \) is a parahoric subgroup of \( G \).

1. Introduction.

Let \( G \) be a reductive group defined over a non-archimedean locally compact field \( k \) and let \( G = G(k) \). Let \( P \) be a minimal parabolic subgroup of \( G \) with Levi decomposition \( P = MN \), and let \( P^- = MN^- \) be the corresponding decomposition of the opposite parabolic \( P^- \). Let \( B \) be an Iwahori subgroup of \( G \) with an Iwahori decomposition with respect to \( P \) and \( M \), i.e.,

\[
B = (B \cap P)(B \cap M)(B \cap P^-).
\]

Denote by \( W \) the relative Weyl group of \( G \). Let \( \chi \) be an unramified character of \( M \) (i.e., \( \chi \) is trivial on \( M_0 \)). Since \( M \cong P/N \), \( \chi \) extends to a character of \( P \) which we will also denote by \( \chi \). Let \( \delta \) be the modulus character of \( P \). Define \( I(\chi) \) to be the unramified principal series representation of \( G \) induced by \( \chi \), i.e., the space of all locally constant functions \( G \rightarrow \mathbb{C} \) such that

\[
f(pg) = \chi^{\delta 1/2}(p)f(g) \text{ for all } p \text{ in } P, \ g \text{ in } G
\]
on which \( G \) acts by right translation. It is well-known that the space \( I(\chi)^B \) of \( B \)-fixed vectors in \( I(\chi) \) has dimension \( \dim I(\chi)^B = |W| \) [3, Prop. 2.1]. In this paper, we generalize this result to the fixed space \( I(\chi)^K \) where \( K \) is a parahoric subgroup of \( G \) containing \( B \).

Let \( A \) be a maximal split torus in \( M \) and let \( N \) be its normalizer in \( G \). If \( M_0 \) is the maximal compact subgroup of \( M \) and \( \tilde{W} = N/M_0 \), then we have a surjection \( \nu : \tilde{W} \rightarrow W = N/M \). Let \( K \) be a parahoric subgroup of \( G \) containing \( B \) and let \( W_K \) be the finite Coxeter subgroup of \( \tilde{W} \) such that \( K = BW_KB \) (see [4, §1]). We will prove the following:
**Theorem 1.1.** The dimension of $I(\chi)^K$ is $|W/\nu(W_K)|$.

As a Coxeter group, $W_K$ is generated by a canonical finite set $S$ of reflections. Thus

$$I(\chi)^K = \bigcap_{s \in S} I(\chi)^{(B,s)}.$$

In Section 3, we explicitly determine the effects of reflections $s \in S$ on $I(\chi)^B$ (Theorem 3.1) and as a corollary the actions of the generators of the Iwahori-Hecke algebra $\mathcal{H}(G, B)$ on $I(\chi)^B$ (Corollary 3.2). We then compute the subspaces $I(\chi)^{(B,s)}$ in terms of the usual basis of $I(\chi)$ as given in [3, Prop. 2.1]. Then in Section 4, we complete the proof of Theorem 1.1 by showing that the dimension of the intersection of the $I(\chi)^{(B,s)}$ is $|W/\nu(W_K)|$.

Let $\mathcal{H}(G, K)$ be the Hecke algebra of compactly supported functions $G \to \mathbb{C}$, bi-invariant by $K$. Let $E$ be a simple $\mathcal{H}(G, K)$-module. It is known that there is an irreducible admissible representation $V$ of $G$ such that $E$ is isomorphic as a $\mathcal{H}(G, K)$-module to the space $V^K$ of $K$-fixed vectors [1, 2.10]. Since $V^B \supset V^K = E \neq 0$, it follows from a well-known result that $V$ embeds inside some unramified principal series representation $I$ of $G$ so that $\dim E = \dim V^K \leq \dim I^K$. Thus Theorem 1.1 has the following corollary:

**Corollary 1.2.** If $K$ is a parahoric subgroup of $G$ and $E$ is a simple module over $\mathcal{H}(G, K)$, then

$$\dim E \leq |W/\nu(W_K)|.$$  

Moreover, this bound is sharp.

The sharpness of this bound is a result of the fact that there exist irreducible unramified principal series representations (see e.g., [2, Theorem 3.3]) and that for such a representation $I$, the $\mathcal{H}(G, K)$-module $I^K$ is simple [1, 2.10] and, by Theorem 1.1, of dimension $|W/\nu(W_K)|$.

**Remark 1.3.** While Theorem 1.1 is needed to prove the sharpness in Corollary 1.2, the inequality itself can be proved by a simpler argument. Indeed, it is easily demonstrated that $\dim I(\chi)^K \leq |W/\nu(W_K)|$ by noting that

$$\dim I(\chi)^K \leq |P \backslash G/K|,$$

and

$$|P \backslash G/K| = |W/\nu(W_K)|.$$

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2. Preliminaries.

See [6] or [3, §1] as a reference for much of the material in this section. In the following, we let $k$ be a non-archimedean locally compact field. We denote by $G$ a connected reductive algebraic group defined over $k$ with group of
$k$-points $G$. Similarly, throughout this section, if $H$ is any algebraic group defined over $k$, we will denote its $k$-points by the corresponding non-bold letter $H$.

Let $P$ be a fixed minimal parabolic subgroup of $G$ containing a maximal split torus $A$ of $G$. Denote by $N$ the unipotent radical of $P$, and by $M$ the centralizer of $A$. Then $P$ has Levi decomposition $MN$. Let $\Phi'$ denote the set of roots of $G$ relative to $A$ and $\Phi'_{nd}$ the subset of non-divisible roots. Also, let $W$ be the relative Weyl group.

Denote by $B = B(G, k)$ the Bruhat-Tits building of $G$ over $k$ and by $A$ the apartment of $B$ stabilized by $A$. The normalizer $N$ of $A$ in $G$ is then the stabilizer of $A$ and the maximal compact subgroup $M_0$ of $M$ is the kernel of the map $N \to \text{Aut}(A)$. Let $\tilde{W} = N/M_0$. Denote by $\Phi_{aff}$ the canonical affine root system on $A$ and by $W_{aff}$ the corresponding affine Weyl group. Then $W_{aff}$ may be identified with a normal subgroup of $\tilde{W}$.

Fix a special point $x_0$ in $B$ and let $\Phi$ be the set of affine roots vanishing at $x_0$. Then $\Phi$ is a reduced root system, and we have a bijection between $\Phi$ and $\Phi'_{nd}$ corresponding to the choice of $x_0$. We let $\Phi^+$ be the subset of positive affine roots corresponding to $P$ and $\Delta$ the subset of simple roots.

Let $C$ be the unique chamber in $A$ containing $x_0$ with the property that every $\alpha$ in $\Phi^+$ takes positive values on $C$. Denote by $K_0$ the special maximal compact subgroup fixing $x_0$. Then $W = N/M_0$, which is the stabilizer of $x_0$ in $\tilde{W}$. We will identify these groups throughout. We denote by $\nu$ the surjection $\tilde{W} \to W$. The kernel of $\nu$ is the group of translations in $\tilde{W}$.

For each $\alpha$ in $\Phi^+$, denote by $N(\alpha)$ the pointwise stabilizer of the half-apartment $\{x \in A \mid \alpha(x) \geq 0\}$. We note that

$$B = M_0 \cdot \prod_{\alpha \in \Phi^+} N(\alpha) \cdot \prod_{\alpha \in \Phi^-} N(\alpha + 1).$$

Let $P_0 \subset P$ be the compact subgroup

$$P \cap K_0 = M_0 \cdot \prod_{\alpha \in \Phi^+} N(\alpha).$$

Let $\Phi = \bigcup \Phi_i$ be the decomposition of $\Phi$ into irreducible root systems. Denote by $\Delta$ the set containing the highest root $\tilde{\alpha}_i$ of $\Phi_i$ for each $i$. Let

$$\Delta_{aff} = \{\alpha \in \Phi_{aff} \mid \alpha \in \Delta \text{ or } \alpha = \tilde{\alpha} - 1 \text{ for some } \tilde{\alpha} \in \tilde{\Delta}\}.$$

For $\alpha$ in $\Delta_{aff}$, let $w_\alpha$ be the reflection in $\text{Aut}(A)$ through the vanishing hyperplane of $\alpha$. Then $S_{aff} = \{w_\alpha \mid \alpha \in \Delta_{aff}\}$ is a set of involutive generators for the Coxeter group $W_{aff}$.

For $\alpha$ in $\Phi$, let $a_\alpha$ be the translation $w_\alpha w_{\alpha - 1}$ on $A$. We note that

$$a_{-\alpha} = a_{\alpha}^{-1} \text{ for any } \alpha \text{ in } \Phi.$$
We let \( K \) be a fixed parahoric subgroup of \( G \) containing \( B \). Since the triple \((G, B, \mathcal{N})\) is a generalized Tits system (see [4, §1]), there exists a special subgroup \( W_K \) of \( \text{W}_{\text{aff}} \) such that \( K = BW_K B \); \( W_K \) is finite as \( K \) is compact. We denote by \( S \) the subset of \( S_{\text{aff}} \) generating \( W_K \).

For any \( w \) in \( W \), we denote by \( q(w) \) the index \([BW_B : B]\). Also for \( \alpha \) in \( \Phi_{\text{aff}} \), we let \( q_\alpha \) be the index \([N(\alpha - 1) : N(\alpha)]\). We note that \( q_{\alpha + 2} = q_\alpha \). Since (cf. [5, Cor. 2.7])

\[
Bw_\alpha B = N(\alpha)w_\alpha B \quad \text{for} \quad \alpha \in \Delta,
\]

\[
Bw_{\bar{\alpha} - 1} B = N(-\bar{\alpha} + 1)w_{\bar{\alpha} - 1} B \quad \text{for} \quad \bar{\alpha} \in \bar{\Delta},
\]

it follows that

\[ q(w_\alpha) = q_{\alpha + 1} \quad \text{for} \quad \alpha \in \Delta, \quad q(w_{\bar{\alpha} - 1}) = q_{\bar{\alpha} + 2} = q_\bar{\alpha} \quad \text{for} \quad \bar{\alpha} \in \bar{\Delta}. \]

If \( \alpha \in \Delta \), we denote by \( B_\alpha \) the group \( B \cap w_\alpha Bw_\alpha \), and if \( \bar{\alpha} \in \bar{\Delta} \), \( B_{\bar{\alpha} - 1} \) denotes the group \( B \cap w_{\bar{\alpha} - 1} Bw_{\bar{\alpha} - 1} \).

Let \( dx \) be the Haar measure on \( G \) for which \( B \) has volume 1. We denote by \( \mathcal{H}(G, B) \) the Iwahori-Hecke algebra of compactly supported functions \( G \to \mathbb{C} \) bi-invariant by \( B \). The product on \( \mathcal{H}(G, B) \) is given by convolution with respect to \( dx \). Fix an unramified character \( \chi \) of \( M \) and let \( \delta \) be the modulus character of \( P \). Denote by \( \text{Ind}_{B}^{G}(\chi \delta^{1/2}) \), i.e., the unramified principal series representation induced by \( \chi \) as described in Section 1. If \( x \) is an element of \( G \), we will denote the action of \( x \) on \( u \in \text{Ind}_{B}^{G}(\chi \delta^{1/2}) \) by \( u \mapsto x \cdot u \). Note that if \( w \in \bar{W} \) then the expression \( w \cdot u \) is well-defined for \( u \in \text{Ind}_{B}^{G}(\chi \delta^{1/2}) \) as \( w \) is determined modulo \( M_0 \subset B \). A function \( h \in \mathcal{H}(G, B) \) acts on \( \text{Ind}_{B}^{G}(\chi \delta^{1/2}) \) by the formula

\[ h \cdot u = \int_{G} (x \cdot u) h(x) \, dx, \]

where \( v \in \text{Ind}_{B}^{G}(\chi \delta^{1/2}) \).

Let \( C_{c}^{\infty}(G) \) be the space of locally constant, compactly supported functions \( G \to \mathbb{C} \). The map \( \mathcal{P}_{\chi} : C_{c}^{\infty}(G) \to \text{Ind}_{B}^{G}(\chi \delta^{1/2}) \) defined by

\[ \mathcal{P}_{\chi}(f)(g) = \int_{P} \chi^{-1} \delta^{1/2}(p) f(pg) \, dp \]

(where \( dp \) is the left Haar measure on \( P \) giving \( P_0 \) measure 1) is a \( G \)-equivariant surjection. The functions \( \phi_{w, \chi} = \mathcal{P}_{\chi}(\text{ch}_{BW_B}) \) \((w \in W)\) form a basis of the subspace of \( B \)-fixed vectors \( \text{Ind}_{B}^{G}(\chi \delta^{1/2}) \) [3, Prop. 2.1]. Concretely, for \( p \in P, w' \in W \) and \( b \in B, \phi_{w, \chi}(pw'b) \) equals \( \chi \delta^{1/2}(p) \) if \( w' = w \) and is zero otherwise.
The goal of this section is to compute the effect of $s \in S_{\text{aff}}$ on $I(\chi)^B$. This will be important for the proof in the following section since we will need to determine the space $I(\chi)^{B,s}$ of vectors in $I(\chi)^B$ fixed by $s$.

**Theorem 3.1.** Suppose that $w \in W$, $\alpha \in \Delta$ and $\tilde{\alpha} \in \tilde{\Delta}$. Then

$$w_\alpha \cdot \phi_{w,\chi} = \begin{cases} \chi \delta^{1/2}(a_w \tilde{\alpha}) \phi_{p w w_\alpha B, \chi} & \text{if } w_\alpha \in \Phi^+ \\ \phi_{w w_\alpha, \chi} + \chi \delta^{1/2}(a_w \tilde{\alpha}) \phi_{w w_\alpha, \chi} & \text{if } w_\alpha \in \Phi^- \\ \phi_{w w_\alpha, \chi} + \chi \delta^{1/2}(a_w \tilde{\alpha}) \phi_{w w_\alpha, \chi} & \text{if } w \tilde{\alpha} \in \Phi^+ \\ \phi_{w w_\alpha, \chi} + \chi \delta^{1/2}(a_w \tilde{\alpha}) \phi_{w w_\alpha, \chi} & \text{if } w \tilde{\alpha} \in \Phi^- \\ \end{cases}$$

$$w_{\tilde{\alpha}^{-1}} \cdot \phi_{w,\chi} = \begin{cases} \chi \delta^{1/2}(a_w \tilde{\alpha}) \phi_{p w w_\alpha B, \chi} & \text{if } w \tilde{\alpha} \in \Phi^+ \\ \phi_{w w_\alpha, \chi} + \chi \delta^{1/2}(a_w \tilde{\alpha}) \phi_{w w_\alpha, \chi} & \text{if } w \tilde{\alpha} \in \Phi^- \\ \phi_{w w_\alpha, \chi} + \chi \delta^{1/2}(a_w \tilde{\alpha}) \phi_{w w_\alpha, \chi} & \text{if } w \tilde{\alpha} \in \Phi^+ \\ \phi_{w w_\alpha, \chi} + \chi \delta^{1/2}(a_w \tilde{\alpha}) \phi_{w w_\alpha, \chi} & \text{if } w \tilde{\alpha} \in \Phi^- \\ \end{cases}$$

**Proof.** For any $s$ in $S_{\text{aff}}$, $g \in G$,

$$(s \cdot \phi_{w,\chi})(g) = \phi_{w,\chi}(gs).$$

The Iwasawa decomposition enables us to write $g = p' w' b'$ for some $p'$ in $P$, $w'$ in $W$, and $b'$ in $B$. We will evaluate $\phi_{w,\chi}(gs) = \phi_{w,\chi}(p' w' b' s)$ by determining the double coset in which $p' w' b'$ lies.

We first consider $s = w_\alpha$ for $\alpha \in \Delta$. Now if $w' \alpha \in \Phi^+$ then by (1)

$$p' w' b' w_\alpha \subset p' w' B w_\alpha$$

$$= p' w' N(\alpha) w_\alpha B$$

$$= p' N(w' \alpha) w' w_\alpha B$$

$$\subset (p' N) w' w_\alpha B.$$  

Since $\chi \delta^{1/2}$ is trivial on $N$, it follows that $\phi_{w,\chi}(p' w' b' w_\alpha)$ equals $\chi \delta^{1/2}(p')$ if $w = w' w_\alpha$ and 0 otherwise.

If, on the other hand, $w' \alpha \in \Phi^-$ then suppose first that $b' \in B_\alpha$. Then

$$p' w' b' w_\alpha \subset p' w' b' w_\alpha B = p' w' w_\alpha B$$

since $w_\alpha B_\alpha w_\alpha \subset B$. Thus $\phi_{w,\chi}(p' w' b' w_\alpha)$ equals $\chi \delta^{1/2}(p')$ if $w = w' w_\alpha$ and 0 otherwise.

Lastly, suppose that $w' \alpha \in \Phi^-$ and $b' \in B - B_\alpha$. It is easily deduced from $w' \alpha \in \Phi^-$ that

$$P w' B w_\alpha B = P w' w_\alpha B \cup P w' B.$$  

Moreover, one can show that $p' w' b' w_\alpha \in P w' B$ if and only if $b'$ is an element of $B - B_\alpha$. Thus $p' w' b' w_\alpha = p w' b$ for some $p \in P$, $b \in B$. Since

$$p^{-1} p' = w' b w_\alpha b'^{-1} w'^{-1} \in P \cap K_0 = P_0$$

and since $\chi \delta^{1/2}$ is trivial on $P_0$, we have that $\chi \delta^{1/2}(p) = \chi \delta^{1/2}(p')$. Therefore, $\phi_{w,\chi}(p' w' b' w_\alpha)$ equals $\chi \delta^{1/2}(p')$ if $w = w'$ and 0 otherwise.
Note that \( w'\alpha \in \Phi^\pm \) if and only if \( w'w_\alpha \alpha = -w'\alpha \in \Phi^\mp \). Using this, we assemble the preceding cases to obtain that

\[
(w_\alpha \cdot \phi_{w,\chi})(p'u'b') = \begin{cases} 
\chi^{\delta/2}(p') & \text{if } w_\alpha \in \Phi^+, w' = ww_\alpha, b' \in B_\alpha \\
\chi^{\delta/2}(p') & \text{if } w_\alpha \in \Phi^-, w' = ww_\alpha \\
\chi^{\delta/2}(p') & \text{if } w_\alpha \in \Phi^-, w' = w, b' \in B - B_\alpha \\
0 & \text{otherwise.}
\end{cases}
\]

This immediately implies the first result of the theorem.

We now prove the second formula by calculating \( w_{\tilde{\alpha}-1} \cdot \phi_{w,\chi} \) for \( \tilde{\alpha} \in \tilde{\Delta} \). Assume first that \( w'\tilde{\alpha} \in \Phi^- \). Then by (2)

\[
p'w'b'w_{\tilde{\alpha}-1} \in p'w'Bw_{\tilde{\alpha}-1}B = p'w'N(-\tilde{\alpha} + 1)w_{\tilde{\alpha}-1}B = p'N(-w'\tilde{\alpha} + 1)w'\tilde{\alpha}a_{\tilde{\alpha}}B \subseteq (p'a_{w'\tilde{\alpha}}N)w'w_{\tilde{\alpha}}B.
\]

Since \( \chi \) is trivial on \( N \), it follows that \( \phi_{w,\chi}(p'w'b'w_{\tilde{\alpha}-1}) \) equals \( \chi^{\delta/2}(p'a_{-w'\tilde{\alpha}}) \) if \( w = w'w_{\tilde{\alpha}} \) and 0 otherwise.

Now suppose that \( w'\tilde{\alpha} \in \Phi^+ \) and that \( b' \in B_{\tilde{\alpha}-1} \). Then

\[
p'w'b'w_{\tilde{\alpha}-1} \in p'w'b'w_{\tilde{\alpha}-1}B = p'w'w_{\tilde{\alpha}-1}B = (p'a_{-w'\tilde{\alpha}})w'w_{\tilde{\alpha}}B
\]

since \( w_{\tilde{\alpha}-1}B_{\tilde{\alpha}-1}w_{\tilde{\alpha}-1} \subseteq B \). It follows that \( \phi_{w,\chi}(p'w'b'w_{\tilde{\alpha}-1}) \) is equal to \( \chi^{\delta/2}(p'a_{-w'\tilde{\alpha}}) \) if \( w = w'w_{\tilde{\alpha}} \) and 0 otherwise.

Finally, suppose that \( b' \in B - B_{\tilde{\alpha}-1} \). As before, it can be shown that

\[
Pw'Bw_{\tilde{\alpha}-1}B = Pw'w_{\tilde{\alpha}}B \cup Pw'B,
\]

and furthermore that \( p'w'b'w_{\tilde{\alpha}-1} \in Pw'B \) if and only if \( b' \) is an element of \( B - B_{\tilde{\alpha}-1} \). Hence \( p'w'b'w_{\tilde{\alpha}-1} = pw'b \) for some \( p \in P, b \in B \). It is easily shown that this forces \( p^{-1}p' \in NP_0 \) so that \( \chi^{\delta/2}(p') = \chi^{\delta/2}(p) \). Thus \( \phi_{w,\chi}(p'w'b'w_{\tilde{\alpha}-1}) \) equals \( \chi^{\delta/2}(p') \) if \( w = w' \) and 0 otherwise.

Noting that \( w'\tilde{\alpha} \in \Phi^\pm \) if and only if \( w'w_{\tilde{\alpha}} = -w'\tilde{\alpha} \in \Phi^\mp \), we obtain

\[
(w_\alpha \cdot \phi_{w,\chi})(p'u'b') = \begin{cases} 
\chi^{\delta/2}(a_{w_{\tilde{\alpha}}})\chi^{\delta/2}(p') & \text{if } w_{\tilde{\alpha}} \in \Phi^-, w' = ww_{\tilde{\alpha}}, b' \in B_{\tilde{\alpha}-1} \\
\chi^{\delta/2}(a_{w_{\tilde{\alpha}}})\chi^{\delta/2}(p') & \text{if } w_{\tilde{\alpha}} \in \Phi^+, w' = ww_{\tilde{\alpha}} \\
\chi^{\delta/2}(p') & \text{if } w_{\tilde{\alpha}} \in \Phi^+, w' = w, b' \in B - B_{\tilde{\alpha}-1} \\
0 & \text{otherwise.}
\end{cases}
\]

The second result follows.

Theorem 3.1 has the following corollary giving the action of \( \text{ch}_{B_b} \) for \( s \) in \( S_{\text{aff}} \).
Corollary 3.2. Suppose that $w \in W$, $\alpha \in \Delta$ and $\tilde{\alpha} \in \tilde{\Delta}$. Then
\[
\text{ch}_{Bw_{\alpha}}B \cdot \phi_{w, \chi} = \begin{cases} 
\phi_{ww_{\alpha}, \chi} & \text{if } w\alpha \in \Phi^+ \\
q_{\alpha+1}\phi_{ww_{\alpha}, \chi} + (q_{\alpha+1} - 1)\phi_{w, \chi} & \text{if } w\alpha \in \Phi^-,
\end{cases}
\]

\[
\text{ch}_{Bw_{\tilde{\alpha}-1}}B \cdot \phi_{w, \chi} = \begin{cases} 
\chi^1/2(a_{\tilde{\alpha}})\phi_{ww_{\tilde{\alpha}}, \chi} & \text{if } w\tilde{\alpha} \in \Phi^- \\
\chi^1/2(a_{\tilde{\alpha}})q_{\tilde{\alpha}}\phi_{ww_{\tilde{\alpha}}, \chi} + (q_{\tilde{\alpha}} - 1)\phi_{w, \chi} & \text{if } w\tilde{\alpha} \in \Phi^+.
\end{cases}
\]

Proof. We prove the first formula in the case $w\alpha \in \Phi^-$. The other cases are handled similarly. For $g \in G$ we have
\[
(ch_{Bw_{\alpha}}B \cdot \phi_{w, \chi})(g) = \int_{G} \phi_{w, \chi}(gx) \text{ch}_{Bw_{\alpha}}B(x) dx = \int_{Bw_{\alpha}}B \phi_{w, \chi}(gx) dx = \sum_n \phi_{w, \chi}(gnw_{\alpha}) = \sum_n (w_{\alpha} \cdot \phi_{w, \chi})(gn),
\]
where $n$ ranges over a set of representatives in $N(\alpha)$ for $N(\alpha)/N(\alpha + 1)$.

If $g \in Pww_{\alpha}B$ then so is $gn$ for each of the $q_{w_{\alpha}} = q_{\alpha+1}$ representatives $n$. On the other hand, if $g \in PwB$, then $gn \in Pw(B - B_{\alpha})$ for precisely $q_{\alpha+1} - 1$ of the representatives $n$. Thus
\[
(ch_{Bw_{\alpha}}B \cdot \phi_{w, \chi})(g) = \sum_n (w_{\alpha} \cdot \phi_{w, \chi})(gn) = \sum_n \left[ \phi_{ww_{\alpha}, \chi}(gn) + \text{ch}_{Pw(B - B_{\alpha})}(gn)\phi_{w, \chi}(gn) \right] = q_{\alpha+1}\phi_{ww_{\alpha}, \chi}(g) + (q_{\alpha+1} - 1)\phi_{w, \chi}(g).
\]

The following corollary of Theorem 3.1 gives a basis for $I(\chi)^{(B,s)}$, $s \in S_{aff}$.

Corollary 3.3. Suppose $\alpha \in \Delta$ and $\tilde{\alpha} \in \tilde{\Delta}$. Then
(i) $\{ \phi_{w, \chi} + \phi_{ww_{\alpha}, \chi} \mid w \in W, w\alpha \in \Phi^+ \}$ is a basis for the fixed space $I(\chi)^{(B, w_{\alpha})}$.
(ii) $\{ \phi_{w, \chi} + \chi^1/2(a_{\tilde{\alpha}})\phi_{ww_{\tilde{\alpha}}, \chi} \mid w \in W, w\tilde{\alpha} \in \Phi^+ \}$ is a basis for the fixed space $I(\chi)^{(B, w_{\tilde{\alpha}-1})}$.

Proof. Let $s \in S_{aff}$. Note that
\[
s \cdot I(\chi)^{B} \cap I(\chi)^{B} = I(\chi)^{Bs} \cap I(\chi)^{B} = I(\chi)^{(sBs,B)} = I(\chi)^{(B,s)}.\]
Thus $I(\chi)^{(B,s)}$ is precisely the set of vectors in $I(\chi)^B$ sent to $I(\chi)^B$ by $s$. It is clear from Theorem 3.1 that if $s = w_\alpha$ this set is spanned by
\[ \{ \phi_{w,\chi} + \phi_{ww_\alpha,\chi} \mid w \in W, w\alpha \in \Phi^+ \}, \]
and if $s = w_{\tilde{\alpha}-1}$ this set is spanned by
\[ \{ \phi_{w,\chi} + \chi^{1/2}(a_{w\tilde{\alpha}})\phi_{ww_{\tilde{\alpha}},\chi} \mid w \in W, w\tilde{\alpha} \in \Phi^+ \}. \]

4. Proof of Theorem 1.1.

We now prove that the dimension of
\[ I(\chi)^K = I(\chi)^{BW_K} = \bigcap_{s \in S} I(\chi)^{(B,s)} \]
is equal to $\vert W/\nu(W_K) \vert$.

Suppose that $f = \sum_{w \in W} c(w)\phi_{w,\chi}$ is a vector in $I(\chi)^B$ with the $c(w) \in \mathbb{C}$. Then it is easily deduced from Corollary 3.3 that $f \in \bigcap_{s \in S} I(\chi)^{(B,s)}$ if and only if for all $w \in W$, \begin{align*}
(3) & \quad c(ww_\alpha) = c(w) \text{ for all } \alpha \in \Delta \text{ with } w_\alpha \in S \\
(4) & \quad c(ww_\alpha) = \chi^{1/2}(a_{w\tilde{\alpha}})c(w) \text{ for all } \tilde{\alpha} \in \tilde{\Delta} \text{ with } w_{\tilde{\alpha}-1} \in S.
\end{align*}

Let $V$ be the space of functions $c : W \to \mathbb{C}$ satisfying (3) and (4). Then $\dim I(\chi)^K = \dim V$. Since $\nu(w_{\beta-1}) = \nu(w_{\beta}) = w_{\beta}$ for all $\beta \in \Phi$, it follows that $c(w)$ determines $c(ww')$ for all $w' \in \langle \nu(s) \mid s \in S \rangle = \nu(W_K)$ so \[ \dim V \leq \vert W/\nu(W_K) \vert. \]

We will prove that $\dim V = \vert W/\nu(W_K) \vert$.

Remark 4.1. We note that if $W_K \subset W$ (i.e., if $K \subset K_0$) then it is clear that $\dim V = \dim I(\chi)^K = \vert W/\nu(W_K) \vert$ since in this case only the relations in (3) appear.

Since $W_K$ is finite, it contains no non-trivial translations so $\nu$ is injective on $W_K$. Thus $\nu(W_K) \cong W_K$, and $\nu(W_K)$ is generated as a Coxeter group by $\nu(S)$. We will denote the element of $W_K$ corresponding to $t \in \nu(S)$ by $\nu^{-1}(t)$. Define recursively a function $[ \ ]$ from the set of finite sequences of elements of $\nu(S)$ to $\text{aff}$. Let $t_1, \ldots, t_n \in \nu(S)$. For the empty sequence $\emptyset$, let $[\emptyset] = e$. Define
\[ [t_1] = \begin{cases} e & \text{if } \nu^{-1}(t_1) = w_\alpha, \alpha \in \Delta \\
a_{\tilde{\alpha}} & \text{if } \nu^{-1}(t_1) = w_{\tilde{\alpha}-1}, \tilde{\alpha} \in \tilde{\Delta}, \end{cases} \]
and then set
\[ [t_1, \ldots, t_n] = \begin{cases} [t_1, \ldots, t_{n-1}] & \text{if } \nu^{-1}(t_n) = w_\alpha, \alpha \in \Delta \\
[t_1, \ldots, t_{n-1}]a_{t_1 \cdots t_{n-1}\tilde{\alpha}} & \text{if } \nu^{-1}(t_n) = w_{\tilde{\alpha}-1}, \tilde{\alpha} \in \tilde{\Delta}. \end{cases} \]
It follows easily from the definition of \([ \ ] \) that
\[
[t_1, \ldots, t_k](t_1 \cdots t_k)[t_{k+1}, \ldots, t_n](t_1 \cdots t_k)^{-1} = [t_1, \ldots, t_n].
\] (5)

We claim that the element \([t_1, \ldots, t_n] \) of \(W_{\text{aff}}\) depends only on the product \(t_1 \cdots t_n\) and not on the particular sequence \(t_1, \ldots, t_n\).

**Lemma 4.2.** Let \(t_1, \ldots, t_n, u_1, \ldots, u_m\) be elements of \(\nu(S)\) such that
\[
[t_1, \ldots, t_n] = [u_1, \ldots, u_m].
\]
Then \([t_1, \ldots, t_n] = [u_1, \ldots, u_m]\).

**Proof.** Since \((\nu(W_K), \nu(S))\) is a Coxeter group, the word \(t_1 \cdots t_n\) obtainable from \(u_1 \cdots u_m\) via the basic Coxeter group relations among the elements of \(\nu(S)\), i.e., those of the form \((tu)^{m(t,u)} = e\), where \(t, u \in \nu(S)\) and \(m(t,u)\) is some number in \(\{1, 2, 3, 4, 6\}\) (see e.g. [5, 1.6]). Therefore, it suffices to show that \([ \ ] \) remains unchanged when a subsequence of consecutive terms in a sequence \(t_1, \ldots, t_n\) is deleted according to such a relation. In fact, due to (5) one need only show that
\[
[t, u, t, u, \ldots, t, u] = [\emptyset] = e
\]
for each basic relation \((tu)^{m(t,u)} = e\) among the elements of \(\nu(S)\).

It is clear that (6) holds if \(\nu^{-1}(t), \nu^{-1}(u) \in W\). Therefore we shall consider only those relations which involve some reflection \(t \in \nu(S)\) such that \(\nu^{-1}(t) \notin W\). Such a \(t\) is necessarily of the form \(w_{\tilde{\alpha}} = \nu(w_{\tilde{\alpha}-1})\) for some \(\tilde{\alpha} \in \tilde{\Delta}\). The basic relations involving \(w_{\tilde{\alpha}}\) are of the form
\[
(w_{\tilde{\alpha}}u)^m = e
\]
where \(u \in \nu(S)\) and \(m \in \{1, 2, 3, 4\}\). (It is never the case that \(m = 6\).)

First consider the case \(m = 1\). Here \(u\) must equal \(w_{\tilde{\alpha}}\) so (6) holds as
\[
[w_{\tilde{\alpha}}, w_{\tilde{\alpha}}] = a_{\tilde{\alpha}}a_{w_{\tilde{\alpha}}} = a_{\tilde{\alpha}}a_{\tilde{\alpha}} = e.
\]

Now suppose that \(m > 1\) and \(\nu^{-1}(u) \in W\) in (7). Then
\[
[w_{\tilde{\alpha}}, u, \ldots, w_{\tilde{\alpha}}, u] = a_{\tilde{\alpha}} \ldots a_{(w_{\tilde{\alpha}}u)^{m-1}_{\tilde{\alpha}}}
\]
Since \(w_{\tilde{\alpha}}u\) is a rotation of order \(m\), \(\tilde{\alpha} + \ldots + (w_{\tilde{\alpha}}u)^{m-1}_{\tilde{\alpha}} = 0\) so (6) holds as
\[
a_{\tilde{\alpha}} \ldots a_{(w_{\tilde{\alpha}}u)^{m-1}_{\tilde{\alpha}}} = e.
\]
Finally, suppose \(m > 1\) and \(\nu^{-1}(u) \notin W\) in (7). In this case, it follows that \(m = 2\) and \(u = w_{\tilde{\beta}}\) for some \(\tilde{\beta} \in \tilde{\Delta}\). Then \(w_{\tilde{\beta}}(\tilde{\alpha}) = \tilde{\alpha}\) and \(w_{\tilde{\alpha}}(\tilde{\beta}) = \tilde{\beta}\). It follows that (6) holds again as
\[
[w_{\tilde{\alpha}}, w_{\tilde{\beta}}, w_{\tilde{\alpha}}, w_{\tilde{\beta}}, \ldots] = a_{\tilde{\alpha}}a_{w_{\tilde{\alpha}}}a_{w_{\tilde{\beta}}}a_{w_{\tilde{\alpha}}}a_{w_{\tilde{\alpha}}} = a_{\tilde{\alpha}}a_{\tilde{\beta}}a_{\tilde{\alpha}}a_{\tilde{\beta}} = e.
\]
Let \( t_1, \ldots, t_n \in \nu(S) \). Since \([t_1, \ldots, t_n]\) depends only on the product \( t_1 \cdots t_n \), \([\cdot]\) gives a function \( \nu(W_K) \to \text{Waff} \), which we will also denote by \([\cdot]\). Explicitly, for \( w \in \nu(W_K), [w] = [t_1, \ldots, t_n] \) for any \( t_1, \ldots, t_n \in \nu(S) \) with \( w = t_1 \cdots t_n \). Note that \([\cdot]\) is a 1-cocycle from \( \nu(W_K) \) to the group of translations in \( \text{Waff} \).

**Proposition 4.3.** The space \( V \) of functions \( W \to \mathbb{C} \) satisfying (3) and (4) has dimension \( |W/\nu(W_K)| \).

**Proof.** Let \( R \) be a set of representatives for the left cosets of \( \nu(W_K) \) in \( W \). For each \( \sigma \in R \), define the function \( c_\sigma : W \to \mathbb{C} \) by setting

\[
c_\sigma(w) = \begin{cases} 
\chi \delta^{1/2}([w']) & \text{if } w = \sigma w' \in \sigma \nu(W_K) \\
0 & \text{if } w \notin \sigma \nu(W_K).
\end{cases}
\]

The \( c_\sigma \) are clearly linearly independent and are \( |W/\nu(W_K)| \) in number. It suffices then to show that the \( c_\sigma \) are in \( V \).

Fix \( \sigma \in R \). Let \( \alpha \) be an element of \( \Delta \) such that \( w_\alpha \in S \). If \( w \notin \sigma \nu(W_K) \) then \( ww_\alpha \notin \sigma \nu(W_K) \) so

\[
c_\sigma(w) = 0 = c_\sigma(ww_\alpha).
\]

If \( w = \sigma w' \in \sigma \nu(W_K) \) then

\[
c_\sigma(ww_\alpha) = c_\sigma(\sigma w' w_\alpha) = \chi \delta^{1/2}([w'w_\alpha]) = \chi \delta^{1/2}([w']) = c_\sigma(w).
\]

Thus (3) holds for \( c_\sigma \).

Now let \( \alpha \) be an element of \( \bar{\Delta} \) such that \( w_{\bar{\alpha}}-1 \in S \). As before, if \( w \notin \sigma \nu(W_K) \) then

\[
c_\sigma(w) = 0 = \chi \delta^{1/2}(a_{w_{\bar{\alpha}}}c_\sigma(ww_{\bar{\alpha}})).
\]

And if \( w = \sigma w' \in \sigma \nu(W_K) \) then

\[
c_\sigma(ww_{\bar{\alpha}}) = \begin{align*}
c_\sigma(\sigma w' w_{\bar{\alpha}}) \\
= \chi \delta^{1/2}([w'w_{\bar{\alpha}}]) \\
= \chi \delta^{1/2}([w']a_{w_{\bar{\alpha}}}) \\
= \chi \delta^{1/2}([w'])\chi \delta^{1/2}(a_{w'_{\bar{\alpha}}}) \\
= \chi \delta^{1/2}(a_{w'_{\bar{\alpha}}})c_\sigma(w).
\end{align*}
\]

Thus \( c_\sigma \) satisfies (4) and lies in \( V \). \( \square \)

It follows that \( \dim I(\chi)^K = \dim V = |W/\nu(W_K)| \).
References


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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF ROCHESTER
ROCHESTER, NEW YORK 14627
E-mail address: lansky@math.rochester.edu