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THE SECOND COHOMOLOGY OF SMALL IRREDUCIBLE  
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## THE SECOND COHOMOLOGY OF SMALL IRREDUCIBLE MODULES FOR SIMPLE ALGEBRAIC GROUPS

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Let  $G$  be a connected, simply connected, quasisimple algebraic group over an algebraically closed field of characteristic  $p > 0$ , and let  $V$  be a rational  $G$ -module such that  $\dim V \leq p$ . According to a result of Jantzen,  $V$  is completely reducible, and  $H^1(G, V) = 0$ . In this paper we show that  $H^2(G, V) = 0$  unless some composition factor of  $V$  is a nontrivial Frobenius twist of the adjoint representation of  $G$ .

### 1. Introduction.

Let  $G$  be a quasisimple, connected, simply connected algebraic group over the algebraically closed field  $k$  of characteristic  $p > 0$ . By a  $G$ -module  $V$ , we always understand a rational  $G$ -module (one given by a morphism of algebraic groups  $G \rightarrow \mathrm{GL}(V)$ ). In this paper, we study the cohomology of a  $G$ -module  $V$  such that  $\dim V \leq p$ . By results of Jantzen [Jan96] one knows that  $V$  is semisimple and that  $H^1(G, V) = 0$ .

Recall that the Lie algebra  $\mathfrak{g}$  of  $G$  is a  $G$ -module via the adjoint action. Our main result is:

**Theorem A.** *Let  $V$  be a  $G$ -module with  $\dim V \leq p$ . Then  $H^2(G, V) \neq 0$  if and only if  $V$  has a composition factor isomorphic with a Frobenius twist  $\mathfrak{g}^{[d]}$  of  $\mathfrak{g}$  for some  $d \geq 1$ .*

Differentiating the representation of  $G$  on  $V$  gives a representation for the Lie algebra  $\mathfrak{g}$  on  $V$ . Assume that  $V^{\mathfrak{g}} = 0$ . Then the theorem says that  $H^2(G, V) = 0$ . For  $V$  of this sort, the vanishing of  $H^2$  is a consequence of the linkage principle for  $G$  together with results in Section 2 which give estimates for the dimensions of Weyl modules whose high weights are simultaneously in the low alcove and in the orbit  $W_p \bullet 0$ . In fact, the same argument shows that  $H^i(G, V)$  is 0 for all  $i \geq 1$ ; see Proposition 5.2. It was pointed out to me that an earlier version of this manuscript contained an overly complicated proof of this observation.

The crucial case for Theorem A is when  $V$  is simple, nontrivial and  $V^{\mathfrak{g}} = V$ . There is a unique  $d \geq 1$  such that the ‘‘Frobenius untwist’’  $V^{[-d]}$  is a  $G$ -module on which  $\mathfrak{g}$  acts nontrivially. We have already seen that

$H^i(G, V^{[-d]}) = 0$  for  $i = 1, 2$ , so Theorem A follows from the following two results (see 5.4). [We denote by  $h$  the Coxeter number of the group  $G$ .]

**Theorem B.** *Suppose that  $p \geq h$  and that  $W$  is a  $G$ -module for which  $H^i(G, W) = 0$  for  $i = 1, 2$ . Then  $H^2(G, W^{[d]}) \simeq \text{Hom}_G(\mathfrak{g}, W)$  for all  $d \geq 1$ .*

**Theorem C.** *If  $p > h$ ,  $\dim H^2(G, \mathfrak{g}^{[d]}) = 1$  for all  $d \geq 1$ . For any  $p$ , there is a  $d_0 \geq 1$  so that  $H^2(G, \mathfrak{g}^{[d]}) \neq 0$  for all  $d \geq d_0$ .*

Theorem B is proved in 5.3; it immediately implies the first assertion of Theorem C (see 5.5). We give a proof of the second assertion of Theorem C in Section 5.6.

We end the paper by applying the results of Section 2 to calculations of cohomology groups  $H^i(G_1, L)$ , where  $G_1$  is the Frobenius kernel, and  $L$  is a simple  $G_1$  module with  $\dim L \leq p$ ; see Proposition 6.

We make now the following remark concerning our hypothesis on  $G$ . Suppose that  $G$  is quasisimple, but not necessarily simply connected, and let  $\pi : G_{\text{sc}} \rightarrow G$  denote the isogeny from the corresponding simply connected covering group. Then any  $G$  representation  $V$  is also a  $G_{\text{sc}}$  representation, and the kernel of  $\pi$  is a diagonalizable group scheme. It follows that  $\pi$  induces an isomorphism  $H^i(G, V) \simeq H^i(G_{\text{sc}}, V)$  for each  $i \geq 0$ ; see [CPSvdK77, Remark (2.7)]. I thank W. van der Kallen for pointing this out to me. One may check using Lemma 4.1(A) and Proposition 5.1 that  $\text{Lie}(G_{\text{sc}})$  and  $\text{Lie}(G)$  are isomorphic simple  $G_{\text{sc}}$  representations whenever  $\dim G \leq p$ . Thus the conclusion of Theorem A remains true for  $G$ .

We conclude this introduction by remarking that the result of Jantzen [Jan96] cited above is one of several recent results studying the semisimplicity of low dimensional representations of groups in characteristic  $p$ . See [Ser94], [McN98], [McN99], [Gur99] and [McN00] for related work.

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## 2. Root systems.

**2.1.** We denote by  $R$  an indecomposable root system in its weight lattice  $X$  with simple roots  $S \subset R^+$ . For each  $\alpha \in S$ , there is a fundamental dominant weight  $\varpi_\alpha \in X$ ; the  $\varpi_\alpha$  form a  $\mathbb{Z}$  basis of  $X$ .

We write  $\alpha_0$  for the dominant short root, and  $\tilde{\alpha}$  for the dominant long root in  $R$  (these coincide in case there is only one root length).

The Coxeter number of  $R$  is given by

$$h - 1 = \sup_{\alpha \in R^+} \{\langle \rho, \alpha^\vee \rangle\} = \langle \rho, \alpha_0^\vee \rangle.$$

For  $m \in \mathbb{Z}$  and  $\alpha \in R$ , let  $s_{\alpha,m}$  denote the affine reflection of  $X_{\mathbb{R}} = X \otimes_{\mathbb{Z}} \mathbb{R}$  in the hyperplane  $H_{\alpha,m} = \{x \in X_{\mathbb{R}} : \langle x, \alpha^\vee \rangle = m\}$ .

Let  $l > h$  be an integer. The affine Weyl group  $W_l$  is the group of affine transformations of  $X_{\mathbb{R}}$  generated by all  $s_{\alpha,ln}$  for  $n \in \mathbb{Z}$ . According to [Bou72, Ch. VI, §2.1, Prop. 1]  $W_l$  is isomorphic to the semidirect product of  $W$  (the finite Weyl group) with  $l\mathbb{Z}R$ . The normalizer of  $W_l$  in the full affine transformation group of  $X_{\mathbb{R}}$  contains all translations by  $lX$ , hence  $W_l$  is a normal subgroup of  $\widehat{W}_l$ , the semidirect product of  $W$  and  $lX$ . Moreover,  $\widehat{W}_l/W_l \simeq lX/l\mathbb{Z}R \simeq X/\mathbb{Z}R$  is the fundamental group of  $R$ , which we will denote by  $\pi$ .

Let  $\rho = \frac{1}{2} \sum_{\alpha \in S} \alpha$ . We always consider the dot action of  $\widehat{W}_l$  (also of  $W$  and  $W_l$ ) on  $X$ : For  $w \in \widehat{W}_l$  and  $\lambda \in X$ , this is given by  $w \bullet \lambda = w(\lambda + \rho) - \rho$ .

The closure of the subset  $C_l$  of  $X_{\mathbb{R}}$  given by

$$C_l = \{\lambda \in X_{\mathbb{R}} \mid 0 < \langle \lambda + \rho, \alpha^\vee \rangle < l \text{ for each } \alpha \in R^+\}$$

is a fundamental domain for the dot action of  $W_l$  on  $X$ ; the conjugates of  $C_l$  under  $W_l$  are known as alcoves, and  $C_l$  is the lowest alcove. Since  $\widehat{W}_l$  normalizes  $W_l$ , [Bou72, Ch. VI, §2.1] shows that  $\widehat{W}_l$  permutes the alcoves.

Let  $\Omega$  be the stabilizer in  $\widehat{W}_l$  of  $C$ . Since  $W_l$  permutes the alcoves simply transitively, one deduces that  $\widehat{W}_l$  is the semidirect product of  $\Omega$  and  $W_l$ . Thus  $\Omega \simeq \widehat{W}_l/W_l \simeq \pi$ .

Since  $l > h$ , the intersection  $C_l \cap X^+$  is nonempty. [Note that if  $l \leq h$  had been allowed, we would have  $C_l \cap X^+ = \{0\}$  in case  $l = h$ , and  $C_l \cap X^+ = \emptyset$  if  $l < h$ .] It is then clear that  $\widehat{W}_l \bullet 0 \cap C_l = \{\omega \bullet 0 \mid \omega \in \Omega\}$ .

**2.2.** Let  $I$  index the simple roots  $S = \{\alpha_i\}$ , write  $\alpha_0^\vee = \sum_{i \in I} n_i \alpha_i^\vee$ , and put  $J = \{i \in I \mid n_i = 1\}$ . A dominant weight  $0 \neq \varpi \in X$  is *minuscule* if whenever  $\lambda \leq \varpi$  and  $\lambda$  is a dominant weight, then  $\varpi = \lambda$ . According to [Bou72, Ch. VI, Exerc. 23, 24],  $\varpi$  is minuscule just in case  $\varpi = \varpi_i$  for some  $i \in J$ .

For  $i \in I \cup \{0\}$ , let  $S_i = S \setminus \{\alpha_i\}$  (so  $S_0 = S$ ). Write  $R_i \subset R$  for the root subsystem determined by  $S_i$ , and  $W_i$  for the parabolic subgroup of  $W$  associated with  $R_i$ . Let  $w_i \in W_i$  be the unique element which makes all positive roots in  $R_i$  negative.

For  $x \in X$ , let  $t(x)$  denote the affine translation by  $x$ ; for  $i \in J$ , let  $\gamma_i = t(l\varpi_i)w_0w_i \in \widehat{W}_l$ . Note that  $\gamma_i$  represents  $\varpi_i \in X/\mathbb{Z}R \simeq lX/l\mathbb{Z}R \simeq \widehat{W}_l/W_l$ .

Applying [Bou72, Ch. VI, §2.2 Prop. 6 and Cor.] one obtains:

**Proposition.**

- (a) *Each non-0 coset of  $\mathbb{Z}R$  in  $X$  is uniquely represented by a minuscule weight. In particular,  $|\pi| = |J| + 1$ .*

(b) The nonidentity elements of  $\Omega$  are precisely the  $\gamma_i$  for  $i \in J$ . We have

$$\widehat{W}_l \bullet 0 \cap C_l = \{0\} \cup \{\gamma_i \bullet 0 = (l - h)\varpi_i \mid i \in J\}.$$

**2.3.** For a dominant weight  $\lambda$ , let

$$(1) \quad d(\lambda) = \prod_{\alpha > 0} \frac{\langle \lambda + \rho, \alpha^\vee \rangle}{\langle \rho, \alpha^\vee \rangle}$$

be the value of Weyl’s degree formula at  $\lambda$ .

**Proposition.** Let  $\lambda = (l - h)\varpi_i$  for some  $i \in J$ .

- (a)  $d(\lambda) \geq \binom{l-1}{l-h}$ , with equality if and only if  $h - 1 = \ell(w_0 w_i)$ .
- (b) If  $l - h \geq 2$  and  $h \geq 3$ , then  $d(\lambda) > l$ .

*Proof.* For  $1 \leq k \leq h - 1$ , let  $e(k)$  be the number of  $\alpha \in R^+ \setminus R_i^+$  with  $\langle \rho, \alpha^\vee \rangle = k$ . The argument in the remark on p. 520-521 of [Ser94] (following Prop. 6) shows that  $e(k) \geq 1$  for each  $1 \leq k \leq h - 1$ . Thus, we have

$$d(\lambda) = \prod_{k=1}^{h-1} \left( \frac{l - h + k}{k} \right)^{e(k)} \geq \prod_{k=1}^{h-1} \frac{l - h + k}{k} = \binom{l - 1}{l - h}.$$

If  $\ell(w_0 w_i) = |R^+| - |R_i^+| = h - 1$ , then  $e(k) = 1$  for each  $1 \leq k \leq h - 1$  and equality holds. This proves (a).

For (b), note that under the given hypothesis we have  $l \geq 5$ . Since  $\binom{l-1}{l-h} \geq \binom{l-1}{2} > l$  for all such  $l$ , (b) follows immediately.  $\square$

**Remark.** Using the table in the proof of Proposition 2.4 below, it is straightforward to verify that equality holds in (a) if and only if either  $R = A_r$  and  $i \in \{1, r\}$  or  $R = C_r$  and  $i = 1$ . (Since  $B_2 = C_2$ , the latter case includes  $B_2$  and  $i = 2$ .)

**2.4.** In the following, let me emphasize the standing assumption  $l > h$ .

**Proposition.** If  $0 \neq \lambda \in \widehat{W}_l \bullet 0 \cap C$  and  $d(\lambda) < l$  then  $d(\lambda) = \ell - 1$  and  $(R, \lambda)$  is listed in the following table. If the rank of  $R$  is  $\geq 2$ , then  $l = h + 1$ .

$R$	$l$	$\lambda$
$A_1$	any	$(l - 2)\varpi_1$
$A_{l-2}$		$\varpi_1, \varpi_{l-2}$
$B_2$	$l = 5$	$\varpi_2$
$C_{(l-1)/2}$	$l$ odd	$\varpi_1$

*Proof.* The rank 1 situation leads to the item listed in the table. When the rank is at least 2, one applies Proposition 2.3 to obtain  $l = h + 1$ , whence  $\lambda = \varpi_i$  for some  $i \in J$ ; i.e.,  $\lambda$  is minuscule.

We handle the minuscule cases by classification. For each indecomposable root system  $R$  for which  $J \neq \emptyset$ , we list in the following table the Coxeter number, the set  $J$ , and the value  $d(\varpi_i)$  for each  $i \in J$ . The simple roots are indexed as in the tables in [Bou72, Planche I-X]; the data recorded here, with the exception of the values  $d(\varpi_i)$ , may be verified by inspecting those tables as well. The values  $d(\varpi_i)$  are well-known (and can anyway be computed from the formula, or by representation theoretic arguments).

Type of $R$	$h$	$J$	$d(\varpi_i), i \in J$
$A_r$	$r + 1$	$\{1, 2, \dots, r\}$	$\binom{r+1}{i}$
$B_r, r \geq 2$	$2r$	$\{r\}$	$2^r$
$C_r, r \geq 2$	$2r$	$\{1\}$	$2r$
$D_r, r \geq 4$	$2r - 2$	$\{1, r - 1, r\}$	$2r, 2^{r-1}, 2^{r-1}$ respectively
$E_6$	12	$\{1, 6\}$	27, 27
$E_7$	18	$\{7\}$	56

From this table, one can list all pairs  $(R, \lambda)$  for which  $R$  has Coxeter number  $l - 1$  and  $\lambda$  is minuscule. It is a simple matter to see that  $d(\lambda) < l$  only when  $(R, \lambda)$  is as claimed.  $\square$

### 3. The algebraic groups.

**3.1.** Let  $k$  be an algebraically closed field of characteristic  $p > 0$ , and let  $G$  be a connected, simply connected semisimple algebraic  $k$ -group. The non-0 weights of a maximal torus  $T \leq G$  on  $\mathfrak{g} = \text{Lie}(G)$  form a root system  $R$  of rank  $r = \dim T$  in the character group  $X = X^*(T)$ . Since  $G$  is simply connected,  $X$  identifies with the full weight lattice of  $R$  as in Section 2. We fix a choice of simple roots  $S$  and positive roots  $R^+$ . The dominant weights are denoted  $X^+$ . The group  $G$  is assumed to be *quasisimple*; i.e., the root system  $R$  is indecomposable.

**3.2.** For each dominant weight  $\lambda \in X^+$ , the space of global sections of the corresponding line bundle on the flag variety affords an indecomposable rational  $G$ -module  $H^0(\lambda)$  with simple socle. The modules  $L(\lambda) = \text{soc } H^0(\lambda)$  comprise all of the simple rational modules for  $G$  (and are pairwise non-isomorphic).

The character of each  $H^0(\lambda)$  is the same as in characteristic 0; hence in particular  $\dim_k H^0(\lambda)$  is given by the Weyl degree formula, whose value at  $\lambda$  we denote  $d(\lambda)$  as in 2.3.

**3.3.** Any dominant  $\lambda$  may be written as a finite sum  $\sum_{i \geq 0} p^i \lambda_i$  with each  $\lambda_i$  a *restricted* weight. Recall that a dominant weight  $\mu$  is restricted if  $\langle \mu, \alpha^\vee \rangle < p$  for all simple roots  $\alpha$ . Steinberg’s tensor product theorem says:

$$L(\lambda) \simeq L(\lambda_0) \otimes L(\lambda_1)^{[1]} \otimes L(\lambda_2)^{[2]} \otimes \dots$$

where for a  $G$ -module  $V$ ,  $V^{[m]}$  stands for the  $m$ -th Frobenius twist of  $V$ .

For  $d \geq 1$ , let  $G_d$  be the  $d$ -th Frobenius kernel of  $G$ . Let  $V$  be a rational  $G$ -module and  $m \geq 1$ . If there is a rational  $G$  module  $W$  with  $W^{[m]} \simeq V$ , we regard  $W$  as the Frobenius *untwist*  $W = V^{[-m]}$  of  $V$ . Now regard  $V$  as a module for  $G_d$ . Since  $G_d$  is a normal subgroup scheme,  $G$  acts on  $V^{G_d}$ ; since  $G_d$  acts trivially on this  $G$ -module, there is an untwisted rational  $G$ -module  $(V^{G_d})^{[-d]}$ . It follows that there is an untwist  $H^i(G_d, V)^{[-d]}$  for all  $i \geq 0$ .

Consider now two  $G$ -modules  $V_1$  and  $V_2$ , and form  $W = V_1 \otimes V_2^{[d]}$ . The Frobenius kernel  $G_d$  acts trivially on  $V_2^{[d]}$ , so that

$$(1) \quad H^i(G_d, W)^{[-d]} \simeq H^i(G_d, V_1)^{[-d]} \otimes V_2$$

as  $G$ -modules for every  $i \geq 0$ .

**3.4.** Let  $W_p \leq \widehat{W}_p$  be as in Section 2 (for  $l = p$ ), let  $C = C_p \cap X^+$  denote the dominant weights in the lowest alcove, and let  $\bar{C} = \bar{C}_p \cap X^+$  ( $\bar{C}_p$  is the closure in  $X_{\mathbb{R}}$ ).

**Proposition.** *Let  $\lambda \in X^+$ .*

- (a) *If  $H^i(G, L(\lambda)) \neq 0$  for some  $i \geq 0$ , then  $\lambda \in W_p \bullet 0$ .*
- (b) *If  $H^i(G_1, L(\lambda)) \neq 0$  for some  $i \geq 0$ , then  $\lambda \in \widehat{W}_p \bullet 0$ .*
- (c)  *$H^i(G, H^0(\lambda)) = 0$  for all  $i > 0$ .*
- (d) *If  $\lambda \in \bar{C}$ , then  $L(\lambda) = H^0(\lambda)$ ; in particular,  $\dim L(\lambda) = d(\lambda)$ .*

*Proof.* (a) follows from the *linkage principle* for  $G$  [Jan87, Cor. II.6.17], and (b) from the linkage principle for  $G_1$  [Jan87, Lemma II.9.16]. (c) follows from [Jan87, II.4.12]. (d) follows from [Jan87, II.6.13, II.5.10]. □

### 4. The Lie algebra and the cohomology of $G_1$ .

We want to describe explicitly the cohomology  $H^*(G_1, k)$  in degree  $\leq 2$ . For this, we need some information on the Lie algebra  $\mathfrak{g}$ .

**4.1.** Recall that the prime  $p$  is *bad* [=not good] for the indecomposable root system  $R$  if one of the following holds:  $p = 2$  and  $R$  is not of type  $A_r$ ;  $p = 3$  and  $R$  is of type  $G_2, F_4$ , or  $E_r$ ;  $p = 5$  and  $R$  is of type  $E_8$ .

The prime  $p$  is *very good* if it is not bad, and in case  $R = A_r$ , if also  $p$  does not divide  $r + 1$ . Notice that if  $p > h$ , then  $p$  is very good.

Application of the summary in [Hum95, 0.13] yields the following:

**Lemma A.** *Assume that  $p$  is very good. Then  $\mathfrak{g}$  is a simple Lie algebra. The adjoint  $G$ -module is simple, self-dual, and isomorphic with  $L(\tilde{\alpha})$  where  $\tilde{\alpha}$  is the dominant long root.*

**Lemma B.** *Assume that  $p \geq h$ . If  $W$  is any  $G$ -module, then  $\text{Hom}_G(\mathfrak{g}, W^{[d]}) = 0$  for  $d \geq 1$ .*

*Proof.* When  $p > h$  this follows since by the previous lemma  $\mathfrak{g}$  is a simple  $\mathfrak{g}$ -module with restricted highest weight. When  $p = h$ , we have  $R = A_{p-1}$ . Since  $G$  is simply connected, we have  $\mathfrak{g} = \mathfrak{sl}_p$ . Thus  $\mathfrak{g}$  is an indecomposable  $G$ -module with unique simple quotient  $L(\tilde{\alpha})$ , and the lemma follows.  $\square$

**4.2.** Let  $B$  be a Borel subgroup of  $G$ , and let  $\mathfrak{u}$  be the nilradical of  $\text{Lie}(B)$ . Regarding  $\mathfrak{u}^*$  as a  $B$ -module, we get a vector bundle on  $G/B$  which we also write as  $\mathfrak{u}^*$ . According to [AJ84, 3.8], the formal character of the  $G$ -module  $H^0(G/B, \mathfrak{u}^*)$  is  $\chi(\tilde{\alpha}) = \text{ch}(\mathfrak{g}^*)$ .

Let  $\mathcal{N} \subset \mathfrak{g}$  be the nilpotent cone. There is by [AJ84, 3.9] an injective homomorphism of graded algebras  $k[\mathcal{N}] \rightarrow H^0(G/B, \text{Su}^*)$ .

**Lemma.** *For simply connected, quasisimple algebraic groups  $G$ ,  $\mathfrak{g}^* \simeq k[\mathcal{N}]_1 \simeq H^0(G/B, \mathfrak{u}^*)$ .*

*Proof.* Let  $I(\mathcal{N}) \triangleleft k[\mathfrak{g}] = S\mathfrak{g}^*$  be the (homogeneous) defining ideal of the variety  $\mathcal{N}$ . We need to show that  $I(\mathcal{N})_1 = 0$ . If not, then  $\mathcal{N} \subset V \subset \mathfrak{g}$  for some proper  $G$ -submodule  $V$ . A look at the summary in [Hum95, 0.13] shows that, since  $G$  is simply connected, the only  $G$ -submodules of  $\mathfrak{g}$  have dimension 0 or 1. On the other hand, by [Hum95, Theorem 6.19], the variety  $\mathcal{N}$  has codimension  $\text{rank}(G)$  in  $\mathfrak{g}$  and so clearly can't be contained in a 1 dimensional linear subspace!  $\square$

**Remarks.**

- (1) Here is a fancier result which implies the lemma if we assume that the prime  $p$  is good for  $G$ . Since  $G$  is simply connected and  $p$  is good, the Springer resolution

$$\varphi : \tilde{\mathcal{N}} = G \times^B \mathfrak{u} \rightarrow \mathcal{N}$$

given by  $(g, X) \mapsto \text{Ad}(g)(X)$  is a *desingularization*, hence in particular a birational map; see [Hum95, Theorem 6.3 and Theorem 6.20]. Again since  $G$  is simply connected and  $p$  is good, the variety  $\mathcal{N}$  is normal ([Hum95, Theorem 4.24]). Standard arguments then yield an isomorphism of graded algebras  $k[\mathcal{N}] \xrightarrow[\simeq]{\varphi^*} \Gamma(\tilde{\mathcal{N}}, \mathcal{O}_{\tilde{\mathcal{N}}})$ . Finally, the projection  $\tilde{\mathcal{N}} \rightarrow G/B$  is an affine morphism, so that  $\Gamma(\tilde{\mathcal{N}}, \mathcal{O}_{\tilde{\mathcal{N}}}) = H^0(G/B, \text{Su}^*)$  as a graded algebra.

- (2) On the other hand, if  $G = PGL_r$ , and  $p|r$ , one can find a linear form on  $\mathfrak{g}$  that vanishes on  $\mathcal{N}$ , hence there can be no isomorphism  $k[\mathcal{N}]_1 \rightarrow H^0(G/B, \mathfrak{u}^*)$  (compare formal characters). So the lemma can fail when  $G$  is not simply connected. [Note that  $\varphi$  is not birational in this example. One can show that there is a  $G_{sc}$ -isomorphism  $\psi : \tilde{\mathcal{N}}_{sc} \rightarrow \tilde{\mathcal{N}}$  (using some obvious notations). We get therefore a commuting

diagram:

$$\begin{array}{ccc}
 \tilde{\mathcal{N}} & \xrightarrow{\varphi_{\text{sc}} \circ \psi^{-1}} & \mathcal{N}_{\text{sc}} \\
 & \searrow \varphi & \downarrow \gamma \\
 & & \mathcal{N}
 \end{array}$$

The map  $\varphi_{\text{sc}} \circ \psi^{-1}$  is birational. Since  $\gamma^*k(\mathcal{N}) \subset k(\mathcal{N}_{\text{sc}})$  is a proper purely inseparable extension, so too is  $\varphi^*k(\mathcal{N}) \subset k(\tilde{\mathcal{N}})$ .

**Proposition.**

- (1) If  $p \neq 2$  or if  $R$  is not of type  $C_r$ , then  $H^1(G_1, k) = 0$ .
- (2) Assume that  $p \geq h$ . Then  $H^2(G_1, k)^{[-1]} \simeq \mathfrak{g}^*$  as  $G$ -modules.

*Proof.* For (1) see [Jan87, Lemma II.12.1]. For (2), first suppose that  $p > h$ . By [AJ84, 3.7, 3.9], there is a  $G$ -equivariant isomorphism of graded rings  $k[\mathcal{N}]' \simeq H^*(G_1, k)^{[-1]}$  where  $k[\mathcal{N}]'$  is again the graded coordinate ring of  $\mathcal{N}$ , but with the linear functions on  $\mathfrak{g}$  given degree 2. The claim now follows from the lemma.

When  $p = h$ , apply [AJ84, Cor. 6.3] to see that  $H^2(G_1, k)^{[-1]} \simeq H^0(G/B, \mathfrak{u}^*)$ ; the claim follows again from the lemma in this case.  $\square$

**5. Low dimensional modules for  $G$ .**

**5.1.** We recall first some facts about low dimensional modules established in [Jan96] and [Ser94].

**Proposition.** Let  $L$  be a simple nontrivial restricted  $G$  module with highest weight  $\lambda$ . Suppose that  $\dim L \leq p$ .

- (a)  $\lambda \in \bar{C}$ .
- (b)  $\lambda \in C$  if and only if  $\dim_k L < p$ .
- (c)  $h \leq p$ . If moreover  $\dim L < p$ , then  $h < p$ .
- (d) If  $R$  is not of type  $A$  and  $\dim L = p$ , then  $h < p$ . If  $p = h$  and  $\dim L = p$ , then  $R = A_{p-1}$  and  $\lambda = \varpi_i$  with  $i \in \{1, p-1\}$ .

*Proof.* (a) follows from [Jan96, Lemma 1.4], and (b) from [Jan96, 1.6], see also [Ser94]. For (c), note first that (a) implies  $\dim L = d(\lambda)$  by Proposition 3.4(d). If  $\lambda \in \bar{C} \setminus C$ , then (a) and (b) imply that  $\dim L = p$ , whence  $p = h$  follows from Weyl’s degree formula. (c) now follows since  $C$  is empty if  $p < h$  and  $C = \{0\}$  if  $p = h$ .

In [Jan96, 1.6], Jantzen made a list of all simple restricted modules for  $G$  with dimension  $p$ . Inspecting that list yields (d).  $\square$

**5.2. Vanishing results when  $\mathfrak{g}$  acts nontrivially.** Let  $L$  be a simple module for  $G$ .

**Proposition.** *If  $G_1$  (equivalently:  $\mathfrak{g}$ ) acts nontrivially on  $L$  and  $\dim L \leq p$ , then  $H^i(G, L) = 0$  for all  $i \geq 0$ .*

*Proof.* Write the highest weight of  $L$  as  $\lambda = \mu_1 + p\mu_2$  with  $\mu_1$  restricted. Since  $L^{\mathfrak{g}} = 0$ , we have  $\mu_1 \neq 0$ . Since  $p \geq \dim L \geq \dim L(\mu_1)$ , Proposition 5.1 implies that  $\mu_1 \in \bar{C}$  and that  $h \leq p$ . We have in particular that  $L(\mu_1) = H^0(\mu_1)$ , hence the proposition will follow from Proposition 3.4 if we show that  $\mu_2$  is 0.

If  $\dim L = p$ , Steinberg’s tensor product theorem gives  $\mu_2 = 0$ . If  $\dim L < p$  then 5.1 shows that  $p < h$  and  $\mu_1 \in C$ . If  $H^i(G, L) \neq 0$  for some  $i$ , then  $\lambda \in W_p \bullet 0$  by the linkage principle, whence  $\mu_1 \in W \bullet 0 + pX = \widehat{W}_p \bullet 0$ . Now Proposition 2.4 applies; it shows that  $\dim L(\mu_1) = p - 1$  whence we have  $\mu_2 = 0$  by another application of Steinberg’s theorem.  $\square$

**5.3. Second cohomology.** Here we prove our main tool for describing second cohomology; first we require the following:

**Lemma.** *Let  $E_2^{p,q} \implies H^{p+q}$  be a convergent, first quadrant spectral sequence.*

- (1) *If  $E_2^{0,1} = E_2^{1,1} = E_2^{0,2} = 0$ , then  $H^2 \simeq E_2^{2,0}$ .*
- (2) *If  $E_2^{1,0} = E_2^{2,0} = E_2^{0,2} = 0$ , then  $H^2 \simeq E_2^{0,2}$ .*

*Proof.* We verify (1), the argument for (2) is the same. We must show that  $E_\infty^{2,0} \simeq E_2^{2,0}$ ; first note that  $E_3^{2,0}$  is the cohomology of the sequence

$$E_2^{0,1} \rightarrow E_2^{2,0} \rightarrow E_2^{4,-1}$$

from which we get  $E_3^{2,0} \simeq E_2^{2,0}$ . For any first quadrant spectral sequence one has (by similar reasoning) that  $E_a^{2,0} \simeq E_{a+1}^{2,0}$  for  $a > 2$ , so we get the desired isomorphism.  $\square$

**Theorem.** *Suppose that  $p \geq h$ . Let  $V$  be a  $G$ -module for which  $H^i(G, V) = 0$  for  $i = 1, 2$ , and let  $d \geq 1$ .*

- (1)  *$H^1(G, V^{[d]}) = 0$ , and*
- (2)  *$H^2(G, V^{[d]}) \simeq \text{Hom}_G(\mathfrak{g}, V)$ .*

*Proof.* The Frobenius kernel  $G_1$  is a normal subgroup of  $G$ ; thus there is a Lyndon-Hochschild-Serre spectral sequence computing  $H^i(G, V^{[d]})$  which in view of 3.3 (1) has the form

$$E_2^{s,t} = H^s(G, H^t(G_1, V^{[d]})^{[-1]}) = H^s(G, H^t(G_1, k)^{[-1]} \otimes V^{[d-1]}).$$

If  $t = 1$ ,  $E_2^{s,t} = 0$  by Proposition 4.2(1).

There is an exact sequence of the form [Jan87, I.4.1(4)]

$$0 \rightarrow E_2^{1,0} \rightarrow H^1(G, V^{[d]}) \rightarrow E_2^{0,1} = 0.$$

Thus  $H^1(G, V^{[d]}) \simeq E_2^{1,0} \simeq H^1(G, V^{[d-1]})$ . We get now (1) by induction on  $d$ .

Proposition 4.2(2) shows now that  $H^2(G_1, k) \simeq \mathfrak{g}^*$ . Thus, the only possible non-0  $E_2$  terms of total degree 2 are

$$\begin{aligned} E_2^{0,2} &= H^0(G, \mathfrak{g}^* \otimes V^{[d-1]}) = \text{Hom}_G(\mathfrak{g}, V^{[d-1]}) \\ E_2^{2,0} &= H^2(G, V^{[d-1]}). \end{aligned}$$

For  $d > 1$ , we apply Lemma 4.1(B) to see that  $E_2^{0,2} = 0$  whence  $H^2(G, V^{[d]}) \simeq E_2^{2,0} = H^2(G, V^{[d-1]})$  by part (1) of the lemma; thus (2) will follow provided it holds for  $d = 1$ . In that case, we have  $E_2^{2,0} = 0$  by assumption, and the result just proved in part (1) shows that  $E_2^{1,0} = 0$ . Thus part (2) of the lemma applies; it shows that  $H^2(G, V^{[1]}) \simeq E_2^{0,2} = \text{Hom}_G(\mathfrak{g}, V)$  as desired.  $\square$

**5.4. The second cohomology of small modules.** Let  $L = L(\lambda)$  be a simple  $G$ -module, and suppose that  $\dim L \leq p$ . Proposition 5.2 showed that the vanishing of cohomology for  $L$  is a consequence of the linkage principle when  $\lambda \notin pX$ . However, if  $\lambda \in pZR$ ,  $\lambda$  is linked to 0, so the linkage principle does not yield vanishing. The following result shows that, despite the linkage of  $\lambda$  and 0 in this case, the second cohomology is usually 0.

**Theorem.** *Let  $L$  be a simple  $G$ -module with  $\dim L \leq p$ . If  $H^2(G, L) \neq 0$ , then  $L \simeq \mathfrak{g}^{[d]}$  for some  $d \geq 1$ .*

*Proof.* Let  $L'$  be such that  $L \simeq (L')^{[d]}$  for  $d \geq 0$ , and such that  $\mathfrak{g}$  acts nontrivially on  $L'$ . We have by 5.1 that  $p \geq h$ . Also, we have by Proposition 5.2 that  $H^i(G, L') = 0$  for  $i \geq 1$ . If  $d = 0$ , we are done. If  $d > 1$ , then Theorem 5.3 applies, and we get that

$$H^2(G, L) \simeq \text{Hom}_G(\mathfrak{g}, L').$$

We get by Proposition 5.1 that  $p > h$  unless  $R = A_{p-1}$  and  $L' = L(\varpi_i)$  with  $i \in \{1, p-1\}$ . If  $p > h$ , then  $\mathfrak{g}$  is a simple  $G$ -module by Lemma 4.1(A). So if  $\text{Hom}_G(\mathfrak{g}, L') \neq 0$  then  $L' \simeq \mathfrak{g}$  whence  $L \simeq \mathfrak{g}^{[d]}$  as claimed.

In the remaining case, one must just note that weight considerations yield  $\text{Hom}_G(\mathfrak{g}, L(\varpi_i)) = 0$  for  $i = 1, p-1$ , whence  $H^2(G, L) = 0$ .  $\square$

**5.5. The second cohomology of twists of the adjoint module.** The first assertion of Theorem C of the introduction follows from the following:

**Proposition.** *Assume that  $p > h$ . Then  $H^1(G, \mathfrak{g}^{[d]}) = 0$  and  $H^2(G, \mathfrak{g}^{[d]}) \simeq \text{End}_G(\mathfrak{g})$  has dimension 1 for  $d \geq 1$ .*

*Proof.* Since  $p > h$ , Lemma 4.1(A) shows that  $\mathfrak{g}$  is the simple module with highest weight  $\tilde{\alpha}$ . It follows that  $\mathfrak{g} = H^0(\tilde{\alpha})$ , and thus that  $H^i(G, \mathfrak{g}) = 0$  for  $i \geq 1$  by Proposition 3.4. The proposition now follows from Theorem 5.3.  $\square$

**Remark.** Note that  $\dim \mathfrak{g} > h$  (in fact,  $\dim \mathfrak{g} = (h + 1)r$  where  $r$  is the rank of  $G$ ). So we get also: If  $\dim \mathfrak{g} \leq p$ , then  $\dim H^2(G, \mathfrak{g}^{[d]}) = 1$  for  $d \geq 1$ .

**5.6. A second proof.** Here we give a second proof of the non-vanishing of  $H^2$  for twists of the adjoint module; the result proved here verifies the remaining assertion of Theorem C of the introduction. We have included the argument since it offers some “explanation” for the non-vanishing.

The group  $G$  arises by base change from a split reductive group scheme  $\mathbf{G}$  over  $\mathbb{Z}$ . Let  $\mathbb{Z}_p$  be the complete ring of  $p$ -adic integers, and let  $\mathbb{Q}_p$  be its field of quotients. For any finite field extension  $F$  of  $\mathbb{Q}_p$ , let  $\mathfrak{o}$  denote the integers in  $F$ . The residue field  $\mathfrak{o}/\mathfrak{m}$  may be identified with the extension  $\mathbb{F}_q$  of  $\mathbb{F}_p$ .

Let  $K$  denote the group of points  $\mathbf{G}(\mathfrak{o})$  regarded as a subgroup of  $\mathbf{G}(F)$ . Since  $\mathbf{G}$  is smooth, the reduction homomorphism  $K \rightarrow \mathbf{G}(\mathbb{F}_q)$  is surjective (see [Tit79, 3.4.4]).

For  $n \geq 1$ , let  $K_n \subset K$  be the kernel of the map  $K \rightarrow \mathbf{G}(\mathfrak{o}/\mathfrak{m}^n)$ . Note that  $K/K_1 = \mathbf{G}(\mathbb{F}_q)$  acts by conjugation on each quotient  $K_n/K_{n+1}$ .

**Proposition.** (a) *There is for each  $m \geq 1$  a canonical isomorphism  $K_m/K_{m+1} \simeq \mathfrak{g}_{\mathbb{F}_q}$  as representations for  $\mathbf{G}(\mathbb{F}_q)$ , where  $\mathfrak{g}_{\mathbb{F}_q}$  is the Lie algebra of  $\mathbf{G}_{\mathbb{F}_q}$ .*

(b) *If  $H^2(\mathbf{G}(\mathbb{F}_q), \mathfrak{g}_{\mathbb{F}_q}) = 0$ , the exact sequence of groups*

$$1 \rightarrow K_1 \rightarrow K \rightarrow \mathbf{G}(\mathbb{F}_q) \rightarrow 1$$

*splits.*

(c) *There is a  $p$ -power  $q_0$ , depending only on the root system  $R$  of  $G$ , such that  $H^2(\mathbf{G}(\mathbb{F}_q), \mathfrak{g}_{\mathbb{F}_q}) \neq 0$  whenever  $q \geq q_0$ .*

(d) *There is an integer  $a_0 \geq 1$  such that  $H^2(G, \mathfrak{g}^{[a]}) \neq 0$  whenever  $a \geq a_0$ .*

*Proof.* (a) Follows from [DG70, II.§4.3]. (b) Since  $K_1$  is a pro- $p$  group [PR94, Lemma 3.8], this follows from [Ser67, Lemma 3].

(c) Choose a  $\mathbb{Q}_p$  vectorspace  $V$  and a nontrivial faithful  $\mathbb{Q}_p$ -rational representation  $\mathbf{G}_{\mathbb{Q}_p} \rightarrow \mathrm{GL}(V)$ . For each extension  $F$  of  $\mathbb{Q}_p$  with integers  $\mathfrak{o}$ , the group  $K = \mathbf{G}(\mathfrak{o})$  is a subgroup of (the group of  $F$ -points of)  $\mathrm{GL}(V_F)$ . If  $H^2(\mathbf{G}(\mathbb{F}_q), \mathfrak{g}_{\mathbb{F}_q}) = 0$ , the sequence in (b) is split and  $V_F$  is a nontrivial  $F[\mathbf{G}(\mathbb{F}_q)]$ -module.

Since  $F$  has characteristic 0, it is well-known that the minimal dimension of a nontrivial  $F[\mathbf{G}(\mathbb{F}_q)]$  module is bounded below by the value  $f(q)$  of a polynomial  $f \in \mathbb{Q}[x]$ , depending only on  $G$ , for which  $f(q) \rightarrow \infty$  as  $q \rightarrow \infty$ . We may choose  $q_0$  such that  $f(q) > \dim_{\mathbb{Q}_p} V$  for each  $q > q_0$ , and (c) follows at once.

(d) now follows from (c) and [CPSvdK77, Cor. 6.9]. □

### 6. Small simple modules for $G_1$ .

Combining results of [KLT99] with the results recorded in 2.4, we obtain some explicit results on  $G_1$  cohomology of low dimensional simple modules:

**Proposition.** *Let  $L$  be a nontrivial simple  $G_1$  module with  $\dim L \leq p$ . Assume for some  $i \geq 0$  that  $H^i(G_1, L) \neq 0$ . Then  $\dim L = p - 1$ . Moreover, there is a quadruple  $(R, \lambda, i(0), V)$  in the following table for which  $R$  is the root system of  $G$ ,  $\lambda$  the high weight of  $L$ ,  $i \geq i(0)$  and  $H^{i(0)}(G_1, L)^{[-1]} \simeq V$  as  $G$ -modules.*

$R$	$\lambda$	$i(0)$	$H^{i(0)}(G_1, L)^{[-1]}$
$A_1$	$(p - 2)\varpi_1$	1	$L(\varpi_1)$
$A_{p-2}$	$\varpi_1, \varpi_{p-2}$	$p - 2$	$L(\lambda)$
$C_{(p-1)/2}$ $p$ odd	$\varpi_1$	$p - 2$	$L(\lambda)$

*Proof.* By [Jan87, Prop. II.3.14],  $L = \text{res}_{G_1}^G L(\lambda)$  for some restricted dominant weight  $0 \neq \lambda$ . Thus  $L(\lambda)$  is a restricted, simple  $G$  module with dimension  $\leq p$ . It follows from Proposition 5.1 that  $h \leq p$ , that  $\lambda \in \bar{C}$ , and that  $L = H^0(\lambda)$  as modules for  $G$ .

Suppose that  $H^i(G_1, L) \neq 0$  for some  $i$ . By the linkage principle for  $G_1$  (Proposition 3.4(b)), we must have  $\lambda \in \widehat{W}_p \bullet 0$ , hence  $\lambda \in C$ . This implies that  $h < p$ . Proposition 2.2 shows that  $\lambda = (p - h)\varpi_i = w_0 w_i \bullet 0 + p\varpi_i$  for some  $i \in J$ , and Proposition 2.3 yields  $\dim L = p - 1$  and lists the possible pairs  $(R, \lambda)$ .

For  $h < p$ , Kumar, Lauritzen and Thomsen [KLT99, Theorem 8] have extended a result of Andersen and Jantzen [AJ84, 3.7]; this result implies in particular that the minimal degree for which  $H^*(G_1, L)$  is non-0 is  $\ell(w_0 w_i)$ , and that

$$H^{\ell(w_0 w_i)}(G_1, L)^{[-1]} \simeq H^0(\varpi_i).$$

It is straightforward to compute for each pair  $(R, \lambda)$  the length  $\ell(w_0 w_i)$ ; one gets in this way the result. □

**Remark.** The Theorem implies the fact (used by Jantzen in the proof of [Jan96, Lemma 1.7]) that  $H^1(G_1, L) = 0$  for all simple  $G_1$  modules  $L$  with  $\dim L \leq p$ . The argument used by Jantzen there relied on the calculations of  $H^1$  carried out in [Jan91].

### References

[AJ84] H.H. Andersen and J.C. Jantzen, *Cohomology of induced representations for algebraic groups*, Math. Ann., **269** (1984), 487-525, MR 86g:20057, Zbl 0529.20027.

- [Bou72] N. Bourbaki, *Groupes et algèbres de Lie, Chapitres 4, 5, 6*, Hermann, Paris, 1972, MR 58 #28083a, Zbl 0249.22001.
- [CPSvdK77] E. Cline, B. Parshall, L. Scott and W. van der Kallen, *Rational and generic cohomology*, Invent. Math., **39** (1977), 143-163, MR 55 #12737, Zbl 0346.20031.
- [DG70] M. Demazure and P. Gabriel, *Groupes algébriques. Tome I: Géométrie algébrique, généralités, groupes commutatifs*, Masson & Cie, Éditeur, Paris, 1970, Avec un appendice 'Corps de classes local' par Michiel Hazewinkel, MR 46 #1800, Zbl 0203.23401.
- [Gur99] R.M. Guralnick, *Small representations are completely reducible*, J. Algebra, **220**(2) (1999), 531-541, MR 2000m:20018, Zbl 0941.20001.
- [Hum95] J.E. Humphreys, *Conjugacy classes in semisimple algebraic groups*, Math. Surveys and Monographs, **43**, Amer. Math. Soc., 1995, MR 97i:20057, Zbl 0834.20048.
- [Jan87] J.C. Jantzen, *Representations of algebraic groups*, Pure and Applied Mathematics, **131**, Academic Press, Orlando, FL, 1987, MR 89c:20001, Zbl 0654.20039.
- [Jan91] ———, *First cohomology groups for classical Lie algebras*, Representation Theory of Finite Groups and Finite Dimensional Algebras (Bielefeld) (G.O. Michler and C.M. Ringel, eds.), Progr. in Math., **95**, Birkhäuser, Boston, 1991, 289-315, MR 92e:17024, Zbl 0749.17020.
- [Jan96] ———, *Low dimensional representations of reductive groups are semisimple*, Algebraic Groups and Related Subjects; a Volume in Honour of R. W. Richardson (G.I. Lehrer et al., ed.), Austral. Math. Soc. Lect. Ser., Cambridge Univ. Press, Cambridge, 1996, 255-266, MR 99g:20079, Zbl 0877.20029.
- [KLT99] S. Kumar, N. Lauritzen and J.F. Thomsen, *Frobenius splitting of cotangent bundles of flag varieties*, Invent. Math., **136** (1999), 603-621, MR 2000g:20088, Zbl 0959.14031.
- [McN98] G.J. McNinch, *Dimensional criteria for semisimplicity of representations*, Proc. London Math. Soc., **76**(3) (1998), 95-149, MR 99b:20076, Zbl 0891.20032.
- [McN99] ———, *Semisimple modules for finite groups of Lie type*, J. London Math. Soc. **60** (1999), no. 2, 771-792, MR 2001k:20096, Zbl 0961.20014.
- [McN00] ———, *Semisimplicity of exterior powers of simple representations of groups*, J. Alg., **225** (2000), 646-666, MR 2001c:20016, Zbl 0971.20005.
- [PR94] V. Platonov and A. Rapinchuk, *Algebraic groups and number theory*, Pure and Applied Mathematics, **139**, Academic Press, 1994, English translation, MR 95b:11039, Zbl 0841.20046.
- [Ser67] J.P. Serre, *Local class field theory*, Algebraic Number Theory (Proc. Instructional Conf., Brighton, 1965), Thompson, Washington, D.C., 1967, 128-161, MR 36 #3753.
- [Ser94] ———, *Sur la semi-simplicité des produits tensoriels de représentations de groupes*, Invent. Math., **116** (1994), 513-530, MR 94m:20091, Zbl 0816.20014.
- [Tit79] J. Tits, *Reductive groups over local fields*, **XXXIII** (1), Proc. Sympos. Pure Math., Amer. Math. Soc., (1979), 29-69, MR 80h:20064, Zbl 0415.20035.

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