Pacific Journal of Mathematics

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Volume 204 No. 2 June 2002

PACIFIC JOURNAL OF MATHEMATICS
Vol. 204, No. 2, 2002

DETERMINING THE POTENTIAL OF A STURM–LIOUVILLE OPERATOR FROM ITS DIRICHLET AND NEUMANN SPECTRA

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In this paper we consider the inverse spectral problem for the Sturm–Liouville Operator on the interval $[0,1]$. We show that given the Dirichlet and Neumann spectra of such an operator we find a generically uncountable family of potentials with these spectra.

1. Introduction.

We will consider this problem: Given the Dirichlet and Neumann spectra of the Sturm–Liouville Operator

$$-\frac{d^2}{dx^2} + q(x) \tag{1}$$

for a potential q in $C^3([0,1])$, determine q. Instead of finding a unique q we get a generically uncountable family of potentials that will have the given joint spectra. Borg [**1**] showed that if the gaps (see Figure 1) are all trivial then the potential $q(x)$ is 0. Levinson [**11**] showed that if given the spectra of (1) corresponding to the two sets of boundary conditions,

$$y(0)\cos\alpha + y'(0)\sin\alpha = 0, \quad y(1)\cos\beta + y'(1)\sin\beta = 0 \tag{2}$$

$$y(0)\cos\alpha + y'(0)\sin\alpha = 0, \quad y(1)\cos\gamma + y'(1)\sin\gamma = 0 \tag{3}$$

with $\sin(\gamma-\beta) \neq 0$, then $q(x)$ is uniquely determined. Notice that this theorem does not include the case of Dirichlet (boundary conditions $y(0) = y(1) = 0$) and Neumann (boundary conditions of $y'(0) = y'(1) = 0$) spectra. Borg [**1**], Levinson [**11**], Isaacson, McKean and Trubowitz [**8**] among others demonstrated that the spectrum given by one boundary condition does not determine the operator.

The dynamical behavior of solutions to Hill's Operator (the 1-D Schrödinger or Sturm-Liouville Operator with periodic potential) is determined by the properties of the associated Floquet discriminant function [**12**]. Its and Matveev [**10**], Gelfand [**5**], Gelfand and Levitan [**6**], McKean [**15**], Garnett [**4**], Trubowitz [**17**], and Buslaev and Faddeev [**2**] illustrate that for periodic potentials the periodic, anti-periodic, and Dirichlet spectra determine the potential.

We address our stated problem by applying the well understood periodic theory to a periodic extension of q. This approach to the problem was originally suggested by H. McKean [private communication]. To state the theorem we must first introduce some terminology. By $\{\mu_n\}$ (resp. $\{\nu_n\}$) we denote the Dirichlet (resp. Neumann) spectrum of the operator (1). Define an even periodic potential $Q(x)$ for which $\{4\mu_n, 4\nu_n\}$ comprise the periodic spectrum of the operator $-\frac{d^2}{dx^2} + Q(x)$. Let $\{\lambda_j\}$ denote the joint periodic and anti-periodic spectra of this operator. Note that the periodic spectrum determines the anti-periodic spectrum (see Proposition (6)). The advantage of an even potential is that its periodic and anti-periodic spectra are also its Dirichlet and Neumann spectra.

Theorem 1 (Main Theorem)**.** *We are given the Dirichlet* $\{\mu_n\}$*, and Neumann* $\{\nu_n\}$*, eigenvalues of* (1) *which satisfy the asymptotics*

$$\mu_n,\, \nu_n = n^2\pi^2 + \mathcal{O}\left(\frac{1}{n^2}\right).$$

The family of potentials, $q(x)$*, having the same Dirichlet and Neumann eigenvalues is of the form* $q(x) = \frac{1}{4}Q(\frac{1}{2}x) \quad x \in [0,1]$ (6) *where* $Q(x)$ *is an even potential of the form*

$$Q(x) = \lambda_0 + \sum_{n\geq 1} \lambda_{2n-1} + \lambda_{2n} - 2c_n(x), \tag{4}$$

with $c_n(x)$ *the* $W^{1,2}_{\text{per}}([0,1])$ (*the Sobolev space of differentiable functions with* $L^2([0,1])$ *first derivative*) *solution of*

$$\begin{aligned} c_{2n}(0) &= 4\mu_n \\ c_{2n-1}(0) &= \lambda_{4n-3} \quad \textit{or} \quad \lambda_{4n-2} \\ \frac{dc_n}{dx} &= \sqrt{(c_n - \lambda_{2n-1})(c_n - \lambda_{2n}) \prod_{k\neq n} \frac{(c_n - \lambda_{2k-1})(c_n - \lambda_{2k})}{(c_n - c_k)^2}}. \end{aligned} \tag{5}$$

In the above theorem we utilize the Trace Formula (4) for potentials q which are C^3 on all but a finite number of points. This formula says that such a q is determined by the periodic, anti-periodic and shifted Dirichlet eigenvalues of the operator $-\frac{d^2}{dx^2} + q(x)$ [**17**]. The shifted Dirichlet eigenvalues satisfy a first order ODE (5) and so are themselves determined by the Dirichlet eigenvalues of q.

Therefore the periodic and Dirichlet spectra of the Sturm-Liouville operator with periodic Q uniquely determine Q. It is at the step of passing from knowing the periodic, and only the half of the Dirichlet spectrum corresponding to periodic eigenvalues of the operator $-\frac{d^2}{dx^2} + Q(x)$ that we reach an ambiguity when we are given a choice as to the anti-periodic half of the Dirichlet spectrum.

2. Even potentials.

For an arbitrary potential $q(x) \in L^2_{\mathbb{R}}([0,1])$ we form an even periodic potential

$$Q(x) = \begin{cases} 4q(2x) & : \quad x \in [0,1/2] \\ 4q(2(1-x)) & : \quad x \in (1/2,1]\,. \end{cases} \tag{6}$$

Notice that for $q(x) \in C^3([0,1])$, $Q(x)$ is C^3 everywhere except at the point $\frac{1}{2}$ where it is only continuous.

Let $u(x)$ be a Dirichlet eigenfunction of the operator $-\frac{d^2}{dx^2} + q(x)$ corresponding to eigenvalue μ_j. Then

$$U(x) = \begin{cases} u(2x) & : \quad x \in [0,1/2] \\ -u(2(1-x)) & : \quad x \in (1/2,1] \end{cases} \tag{7}$$

is a Dirichlet eigenfunction of the operator $-\frac{d^2}{dx^2} + Q(x)$ with eigenvalue $4\mu_j$. Likewise if $v(x)$ is a Neumann eigenfunction with eigenvalue ν_j then

$$V(x) = \begin{cases} v(2x) & : \quad x \in [0,1/2] \\ v(2(1-x)) & : \quad x \in (1/2,1] \end{cases} \tag{8}$$

is a Neumann eigenfunction for the operator $-\frac{d^2}{dx^2} + Q(x)$ with eigenvalue $4\nu_j$.

Both $U(x)$ and $V(x)$ are also periodic eigenfunctions of the operator $-\frac{d^2}{dx^2} + Q(x)$. We conclude that the Dirichlet and Neumann spectra of $-\frac{d^2}{dx^2} + q(x)$ determine the periodic spectrum of $-\frac{d^2}{dx^2} + Q(x)$. Using the Counting Lemma (2) and Proposition (4) we see that the Dirichlet and Neumann eigenvalues paired with their respective $U(x)$ or $V(x)$ account for only the gaps which are given by the periodic spectrum.

Therefore we reduce the inverse problem to the case of a periodic potential. We use the monodromy matrix,

$$\begin{pmatrix} y_1(1,\lambda) & y_2(1,\lambda) \\ y_1'(1,\lambda) & y_2'(1,\lambda) \end{pmatrix} = M(\lambda), \tag{9}$$

where y_1 and y_2 are the two linearly independent fundamental solutions given by $y_1(0,\lambda) = y_2'(0,\lambda) = 1$ and $y_1'(0,\lambda) = y_2(0,\lambda) = 0$. This matrix describes the behavior of the solutions to Hill's operator on $\mathbb{R}$. For example periodic solutions to the differential equation correspond to unit eigenvalues of this matrix. Because of the initial values of y_1 and y_2 a Dirichlet eigenvalue μ corresponds to $y_2(1,\mu) = 0$ and a Neumann eigenvalue η corresponds to $y_1'(1,\eta) = 0$.

The periodic and anti-periodic eigenvalues are values of λ for which $M(\lambda)$ has respectively eigenvalues ± 1. In either case Equation (1) has a solution with period 2. $\Delta(\lambda)$ denotes the trace of $M(\lambda)$. Periodic (resp. anti-periodic) eigenvalues of $q(x)$ are roots of $\Delta - 2$ (resp. $\Delta + 2$).

Lemma 1. *If $Q(x)$ is an even potential and λ_j is both a Neumann and Dirichlet eigenvalue then $\Delta(\lambda_j) = \pm 2$ and $\Delta'(\lambda_j) = 0$.*

Proof. From the results above, $y_2(1, \lambda_j) = 0$ and $y_1'(1, \lambda_j) = 0$. Therefore $M(\lambda_j)$ is diagonal with determinant 1, so $\Delta(\lambda_j) = \pm 2$. To prove the statement about the derivative of Δ we will use a formula from [**12**],

$$\begin{aligned} \Delta'(\lambda) = \big(y_1(1,\lambda) - y_2'(1,\lambda)\big) \int_0^1 y_1(x,\lambda) y_2(x,\lambda) dx \\ - y_2(1,\lambda) \int_0^1 y_1^2(x,\lambda) dx + y_1'(1,\lambda) \int_0^1 y_2^2(x,\lambda) dx. \end{aligned} \tag{10}$$

Now notice that if $y_1(x, \lambda_j)$ is a solution of (1) then so is

$$y_1(1-x, \lambda_j) = y_1(1,\lambda_j) y_1(x,\lambda_j) + y_1'(1,\lambda_j) y_2(x,\lambda_j) \tag{11}$$

as $y_1(1-x, \lambda_j)$ will satisfy Equation (1) with $Q(1-x) = Q(x)$. Therefore, since λ_j is a Neumann eigenvalue we see that

$$y_1(1-x, \lambda_j) = y_1(1,\lambda_j) y_1(x,\lambda_j). \tag{12}$$

Setting $x = 1$ in the above equation we get $y_1(1, \lambda_j) = \pm 1$. The determinant of the monodromy matrix is 1, and because λ_j is a Dirichlet eigenvalue $y_2(1, \lambda_j) = 0$ so

$$y_2'(1,\lambda_j) = \frac{1}{y_1(1,\lambda_j)} = \pm 1. \tag{13}$$

We then substitute into (10)

$$y_2(1,\lambda_j) = y_1'(1,\lambda_j) = 0$$

and

$$y_1(1,\lambda_j) = y_2'(1,\lambda_j) = \pm 1$$

to get $\Delta'(\lambda_j) = 0$. ☐

Proposition 1. *λ_j is a periodic or anti-periodic eigenvalue of an even potential Q if and only if λ_j is a Neumann or Dirichlet eigenvalue of Q.*

Proof. Suppose λ_j is a periodic eigenvalue so $\Delta(\lambda_j) = 2$. We must show that $y_2(1, \lambda_j) = 0$ or $y_1'(1, \lambda_j) = 0$. Suppose λ_j is not a Neumann eigenvalue for Q; that is $y_1'(1, \lambda_j) \neq 0$.

From $\Delta(\lambda_j) = 2$ we see that the monodromy matrix is of the form

$$M(\lambda_j) = \begin{pmatrix} y_1(1,\lambda_j) & y_2(1,\lambda_j) \\ y_1'(1,\lambda_j) & 2 - y_1(1,\lambda_j) \end{pmatrix}, \tag{14}$$

as λ_j is a periodic eigenvalue.

If $Q(x)$ is even then $y_1(1-x, \lambda_j)$ is also a solution of

$$-\frac{d^2y}{dx^2} + (Q(x) - \lambda_j) y = 0. \tag{15}$$

Because y_1 and y_2 form a basis for the solutions to this equation we may write $y_1(1-x)$ as

$$y_1(1-x,\lambda_j) = y_1(1,\lambda_j)y_1(x,\lambda_j) - y_1'(1,\lambda_j)y_2(x,\lambda_j). \tag{16}$$

By setting $x = 1$ in this equation and its derivative we get the following two equations:

$$1 = y_1(1,\lambda_j)^2 - y_1'(1,\lambda_j)y_2(1,\lambda_j) \tag{17}$$

$$0 = y_1(1,\lambda_j)y_1'(1,\lambda_j) - y_1'(1,\lambda_j)y_2'(1,\lambda_j). \tag{18}$$

From (14) we have $y_2'(1,\lambda_j) = 2 - y_1(1,\lambda_j)$ and using this relation in (18) we get the equation

$$0 = 2y_1'(1,\lambda_j)(y_1(1,\lambda_j) - 1). \tag{19}$$

By the assumption that λ_j is not a Neumann eigenvalue we conclude that $y_1(1,\lambda_j) = 1$. Substituting this into Equation (17) we conclude that $y_2(1,\lambda_j) = 0$ and so λ_j is a Dirichlet eigenvalue.

Conversely, suppose that λ_j is a Dirichlet eigenvalue, $y_2(1,\lambda_j) = 0$. We must show that $\Delta(\lambda_j) = \pm 2$. Since $\det M(\lambda) = 1$ we have $y_2'(1,\lambda_j) = 1/y_1(1,\lambda_j)$. Substituting this into (18) we conclude that either $y_1'(1,\lambda_j) = 0$ in which case Lemma (1) completes the proof; otherwise, we get $y_1(1,\lambda_j) = \pm 1$, which implies that $\Delta(\lambda_j) = \pm 2$.

A similar argument may be made for the Neumann case with the additional feature that, when Q is an even potential, the lowest Neuman eigenvalue, ν_0, is equal to the lowest periodic eigenvalue, λ_0. $\square$

We introduce the picture of gaps and bands associated to $\Delta(Q,\lambda)$. The bands are the ranges of eigenvalues whose eigenfunctions are bounded (stable) on $\mathbb{R}$. That is the range of λ's for which the eigenvalues of the monodromy matrix are complex valued with modulus less than 1. These bands are clearly the intervals over which $|\Delta(\lambda)| < 2$. Correspondingly the intervals for which $|\Delta(\lambda)| > 2$ are called the gaps. These are intervals for which there exist unbounded (unstable) solutions to (1). Gap intervals may be trivial; i.e., they may collapse to a single point.

Finally we need Theorem 2 from [**17**].

Theorem 2 (Trace Formula)**.** *Let $q \in C^3[0,1]$ be a potential with Dirichlet eigenvalues μ_n and periodic, anti-periodic eigenvalues λ_j. Let $\mu_n(t)$, $n \geq 1$, be the unique periodic solution of the system*

$$\frac{d\mu_n}{dt} = \sqrt{(\mu_n - \lambda_{2n-1})(\mu_n - \lambda_{2n}) \prod_{k\neq n} \frac{(\mu_n - \lambda_{2k-1})(\mu_n - \lambda_{2k})}{(\mu_n - \mu_k)^2}}, \tag{20}$$

on the Riemann surface given by the equation

$$y_n = \sqrt{(\mu_n - \lambda_{2n-1})(\mu_n - \lambda_{2n})},$$

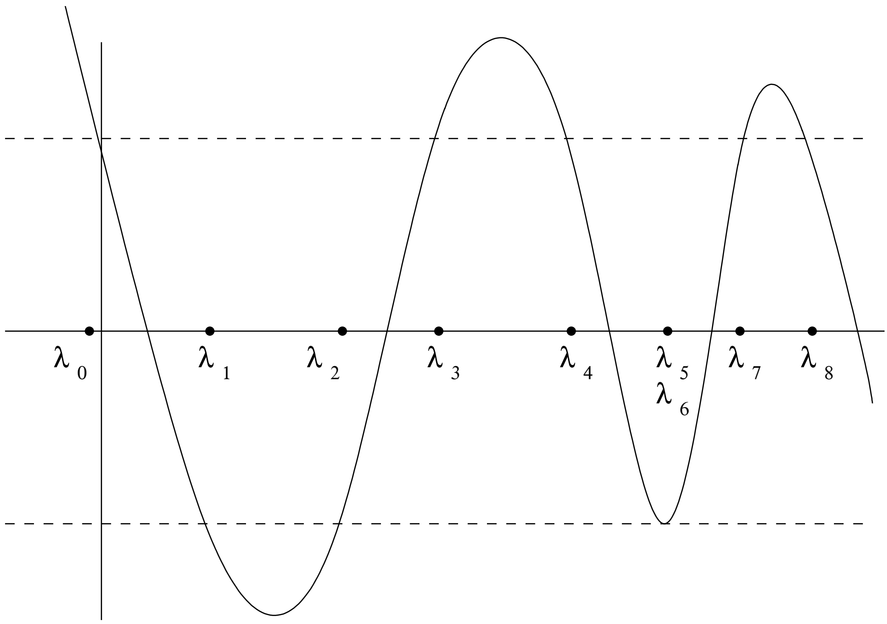

Figure 1. Gaps and Bands.

whose initial values $\mu_n(0) = \mu_n$ *the* n^{th} *Dirichlet eigenvalue and for which the initial velocities are prescribed by choosing the signature of the radical* $\sqrt{\Delta^2(\mu_n) - 4}$ *such that*

$$\sqrt{\Delta(\mu_n)^2 - 4} = 2y_2'(1, \mu_n) - \Delta(\mu_n). \tag{21}$$

Then,

$$q(t) = \lambda_0 + \sum_{n \geq 1} \lambda_{2n-1} + \lambda_{2n} - 2\mu_n(t). \tag{22}$$

The proof in [**17**] is given for potentials in $C^3_{\mathrm{per}}([0,1])$. For the purposes of this paper we wish to apply this theorem to the potential $Q(x)$ which is C^3 for every point except 0, $\frac{1}{2}$ and 1 where Q is not continuously differentiable. To show that the theorem still holds in this case we will demonstrate that the trace formula is still well-defined.

We have from [**16**] the estimate

$$\mu_n = n^2\pi^2 - \int_0^1 \cos(2\pi n x) q(x) dx + \mathcal{O}\left(\frac{1}{n^2}\right) \tag{23}$$

for $q(x) \in W^{2,2}([0,1])$. In fact our $q(x)$ is twice differentiable for all but one point. Below we give an argument for a bound on the $\cos(2\pi n x)$ inner product above. This same technique shows that the $\mathcal{O}\left(\frac{1}{n^2}\right)$ term above will remain of the same order. Compute an estimate of the integral above

$$\begin{aligned}\int_0^1 &\cos(2\pi nx)q(x)dx \\ &= q(x)\left.\frac{\sin(2\pi nx)}{2\pi n}\right|_0^1 \\ &\quad - \int_0^{\frac{1}{2}} q'(x)\frac{\sin(2\pi nx)}{2\pi n}dx - \int_{\frac{1}{2}}^1 q'(x)\frac{\sin(2\pi nx)}{2\pi n}dx.\end{aligned} \tag{24}$$

Integrating by parts a second time we get

$$\begin{aligned}\int_0^1 &\cos(2\pi nx)q(x)dx \\ &= q'(x)\left.\frac{\cos(2\pi nx)}{4\pi^2 n^2}\right|_0^{\frac{1}{2}} + q'(x)\left.\frac{\cos(2\pi nx)}{4\pi^2 n^2}\right|_{\frac{1}{2}}^1 \\ &\quad - \int_0^{\frac{1}{2}} q''(x)\frac{\cos(2\pi nx)}{4\pi^2 n^2}dx - \int_{\frac{1}{2}}^1 q''(x)\frac{\cos(2\pi nx)}{4\pi^2 n^2}dx.\end{aligned} \tag{25}$$

So we see that $\mu_n = n^2\pi^2 + \mathcal{O}(1/n^2)$ for the $Q(x)$ we are concerned with. There was nothing special about the points where differentiability failed so the shifted Dirichlet eigenvalues will have the same asymptotics as well. The periodic and anti-periodic spectra satisfy the same asymptotics as they are the Dirichlet and Neumann spectra of the even potential. The sum we are concerned with is

$$\sum_{n\geq 1} \left|\lambda_{2n-1} + \lambda_{2n} - 2\mu_n(t)\right|. \tag{26}$$

By the analysis above each term satisfies the asymptotics $n^2\pi^2 + \mathcal{O}(1/n^2)$ and so the sum converges absolutely.

Proposition 2. *If $\mu_n(0) = \lambda_{2n-1}$ or λ_{2n} for all n then the function $q(x)$ determined by (22) is even.*

Proof. As a consequence of the trace formula it will suffice to show that $\mu_n(x) = \mu_n(1-x)$. Differentiating this function with respect to x we get

$$\begin{aligned}&\frac{d}{dx}\mu_n(1-x) \\ &= -\sqrt{(\mu_n - \lambda_{2n-1})(\mu_n - \lambda_{2n})\prod_{k\neq n}\frac{(\mu_n - \lambda_{2k-1})(\mu_n - \lambda_{2k})}{(\mu_n - \mu_k)^2}}.\end{aligned} \tag{27}$$

From the periodicity of the original solutions the initial conditions which determine $\mu_n(1-x)$ are the same as the ones for $\mu_n(x)$ specifically $\mu_n(0) = \mu_n(1) = \lambda_{2n-1}$ or λ_{2n}. There exists a solution of Equation (27) which is periodic and does not pause at the endpoints of the interval $[\lambda_{2n-1}, \lambda_{2n}]$. However from the ambiguity of the choice of sign this solution must be identical to that of the original equation which is also periodic and does not pause at the endpoints of the interval. □

3. Eigenvalues.

The following proposition examines further the relationship between the Dirichlet and Neumann spectra, and the gaps.

Proposition 3. *Suppose q is a potential on $[0,1]$ and $[\lambda_{2j-1}, \lambda_{2j}]$ is a gap. In other words $|\Delta(\lambda)| \geq 2$ for all λ in $[\lambda_{2j-1}, \lambda_{2j}]$, then there is a μ and η in $[\lambda_{2j-1}, \lambda_{2j}]$ such that μ is a Dirichlet eigenvalue and η is a Neumann eigenvalue.*

Proof. We will prove this by showing that $y_1'(1,\lambda)$ and $y_2(1,\lambda)$ switch sign from the left of λ_{2j-1} to the right of λ_{2j}.

We follow Magnus and Winkler for this proof [**12**].

Combining Formula (10) into one integral and shortening notation via $\eta_1 = y_1(1, \lambda_{2j-1})$, $\eta_1' = y_1'(1, \lambda_{2j-1})$ etc., we get the equation

$$\Delta'(\lambda_{2j-1}) = \int_0^1 \left((\eta_1 - \eta_2') \, y_1 y_2 - \eta_2 y_1^2 + \eta_1' y_2^2 \right) dx. \tag{28}$$

We also need the formula

$$\begin{aligned} \Delta^2 - 4 &= (\eta_1 + \eta_2')^2 - 4 \left(\eta_1 \eta_2' - \eta_1' \eta_2 \right) \\ &= (\eta_1 - \eta_2')^2 + 4\eta_1' \eta_2. \end{aligned} \tag{29}$$

Note that $\eta_1' \neq 0$ to the left of λ_{2j-1} and to the right of λ_{2j}, where $|\Delta(\lambda)| < 2$. So by adding and subtracting $(\Delta^2 - 4) \int_0^1 y_1^2 dx / 4\eta_1'$ from (28) we get

$$\begin{aligned} &\int_0^1 \Bigg((\eta_1 - \eta_2') \, y_1 y_2 - \eta_2 y_1^2 + \eta_1' y_2^2 \\ &\qquad + \frac{(\eta_1 - \eta_2')^2 \, y_1^2}{4\eta_1'} + \frac{4\eta_1' \eta_2 y_1^2}{4\eta_1'} - \frac{\Delta^2 - 4}{4\eta_1'} y_1^2 \Bigg) dx \\ &= \operatorname{sign}(\eta_1') \int_0^1 \left(\left(\sqrt{|\eta_1'|} y_2 + \frac{\eta_1 - \eta_2'}{2\sqrt{|\eta_1'|}} \operatorname{sign}(\eta_1') y_1 \right)^2 - \frac{\Delta^2 - 4}{4|\eta_1'|} y_1^2 \right). \end{aligned} \tag{30}$$

As $|\Delta(\lambda)| < 2$ in the regions being considered, the integrand is a positive number. Yet $\Delta'(\lambda)$ switches sign once in $[\lambda_{2j-1}, \lambda_{2j}]$. This implies that η_1' switches sign as needed. A similar proof will show that η_2 also switches sign. □

A corollary of Formula (30) is that if $\Delta'(\lambda) = 0$ then $|\Delta(\lambda)| \geq 2$.

4. Counting Lemma.

How many Dirichlet and Neumann eigenvalues are in each gap? To answer this question we use the Counting Lemma from Pöschel and Trubowitz [**16**].

Lemma 2 (Counting Lemma: Dirichlet Eigenvalues). *Let* $q \in L^2_{\mathbb{R}}([0,1])$ *and let* $N > 2e^{\|q\|}$ *be an integer. Then* $y_2(1,\lambda)$ *has exactly* N *roots, counted with multiplicities, in the open half plane*

$$Re\,(\lambda) < \left(N + \frac{1}{2}\right)^2 \pi^2 \tag{31}$$

and for each $n > N$*, exactly one simple root in the egg shaped region*

$$|\sqrt{\lambda} - n\pi| < \frac{\pi}{2}. \tag{32}$$

There are no other roots.

An analogous result is true for Neumann eigenvalues. The necessary tools are available in [**16**]. For completeness we will state the lemma here:

Lemma 3 (Counting Lemma: Neumann Eigenvalues). *Let* $q \in L^2_{\mathbb{R}}([0,1])$ *and let* $N > 2e^{\|q\|}$ *be an integer. Then* $y_1'(1,\lambda)$ *has exactly* $N+1$ *roots, counted with multiplicities, in the open half plane*

$$Re\,(\lambda) < \left(N + \frac{1}{2}\right)^2 \pi^2 \tag{33}$$

and for each $n > N$*, exactly one simple root in the egg shaped region*

$$|\sqrt{\lambda} - n\pi| < \frac{\pi}{2}. \tag{34}$$

There are no other roots.

The "extra" Neumann eigenvalue in the half plane corresponds to the "ground state" of the Neumann problem. For general potentials, this eigenvalue is less than or equal to λ_0, the first periodic eigenvalue; but, when q is even it is pinned at λ_0.

For the periodic and anti-periodic spectra we shift the potential until it is an even potential, then the Dirichlet and Neumann spectra form the periodic and anti-periodic spectra. Therefore we get the analogous result for the periodic and anti-periodic spectra (the periodic and anti-periodic spectra are invariant under translation of the potential q and the average value of q is invariant under translation).

Proposition 4. *For periodic* $q \in L^2_{\mathbb{R}}([0,1])$ *there is one and only one Dirichlet eigenvalue within each gap.*

Proof. Choose N satisfying the hypothesis of the Counting Lemma. By Proposition 3 there is at least one Dirichlet eigenvalue in each gap. There are N Dirichlet eigenvalues in the region $\mathrm{Re}(\lambda) < (N+\frac{1}{2})^2\pi^2$. Therefore in the same region there is one and only one Dirichlet eigenvalue within each gap.

With $n > N$ for the intervals $|\sqrt{\lambda} - n\pi| < \frac{\pi}{2}$ there is one gap. Within this same region there is one Dirichlet eigenvalue. In these zones there is one and only one Dirichlet eigenvalue within each gap. □

Proposition 5. *For periodic* $q \in L^2_{\mathbb{R}}([0,1])$ *there is one and only one Neumann eigenvalue within each gap. There is one and only one Neumann eigenvalue within the interval* $(-\infty, \lambda_0]$.

The proof of this proposition follows the one above.

Proposition 6. Δ *is determined by the periodic eigenvalues of* q.

Proof. The periodic eigenvalues are the roots of $\Delta - 2$, they are real since they are eigenvalues of a self-adjoint operator . Therefore we have

$$\Delta - 2 = C\left(\prod_{n=1}^{\infty} \frac{(\lambda_{2n-1} - \lambda)(\lambda_{2n} - \lambda)}{n^4\pi^4}\right)(\lambda - \lambda_0), \tag{35}$$

(see [**13**]) where $\{\lambda_i\}$ are the periodic eigenvalues and C is a constant, provided that this product converges. C is determined by the asymptotic condition on the roots of $\Delta^2 - 4$,

$$\lambda_{2n-1}, \lambda_{2n} = n^2\pi^2 + \int_0^1 q(x)dx + \mathcal{O}(n^{-2}) \tag{36}$$

for $q \in C^3([0,1])$. Without loss of generality we may take $\int_0^1 q(x)dx = 0$.

We first show that the product converges uniformly. From Markushevich ([**13**]) we have that $\prod\limits_{i=1}^{\infty}(1 - \frac{\lambda}{\lambda_i})$ converges uniformly if and only if $\sum\limits_{i=1}^{\infty} \frac{\lambda}{\lambda_i}$ is uniformly convergent.

Choose N such that $|\lambda_{2n-1} - n^2\pi^2| < \delta$ and $|\lambda_{2n} - n^2\pi^2| < \delta$ for all $n > N$ and that

$$\sum_{i=N+1}^{\infty} \frac{1}{i^2\pi^2} < \frac{\delta}{4}. \tag{37}$$

Consider

$$\sum_{i=1}^{\infty}\left|\frac{\lambda}{\lambda_i}\right| = |\lambda|\sum_{i=1}^{\infty}\left|\frac{1}{\lambda_i}\right| = |\lambda|\left(\sum_{j=1}^{\infty}\left|\frac{1}{\lambda_{2j-1}}\right| + \sum_{j=1}^{\infty}\left|\frac{1}{\lambda_{2j}}\right|\right). \tag{38}$$

We examine the tail of this series,

$$\sum_{i=N+1}^{\infty}\left|\frac{1}{\lambda_{2i-1}}\right|+\sum_{i=N+1}^{\infty}\left|\frac{1}{\lambda_{2i}}\right|\le\sum_{i=N+1}^{\infty}\frac{2}{i^2\pi^2-\delta}\le 4\sum_{i=N+1}^{\infty}\frac{1}{i^2\pi^2}. \tag{39}$$

Which is the estimate we need.

Finally to combine this with our problem we have the infinite product

$$\prod_{i=1}^{\infty}\frac{\lambda_{2n-1}\lambda_{2n}}{n^4\pi^4}\left(1-\frac{\lambda}{\lambda_{2n}}\right)\left(1-\frac{\lambda}{\lambda_{2n-1}}\right). \tag{40}$$

The term we have factored out of each part of the product is a constant in λ and therefore our conclusion is that the original product converges uniformly. □

This proposition says that Δ is determined by the periodic spectrum.

5. Proof of the main theorem.

If we are given the Dirichlet and Neumann spectra with appropriate asymptotic conditions for an $C^3([0,1])$ potential q on $[0,1]$ we begin the solution of the inverse problem by first extending q to an even potential $Q(x)$ on $[0,1]$. $Q(x)$ is $C^3([0,1])$ at all but one point. As described in Section 1 the Dirichlet and Neumann spectra, $\{\mu_n,\nu_n\}$ give the periodic spectrum, $\{4\mu_n,4\nu_n,4\nu_0\}$, of the operator with potential $Q(x)$. The eigenvalue $4\nu_0$ is the first periodic eigenvalue of the operator with potential $Q(x)$. The corresponding eigenfunctions remain Dirichlet and Neumann eigenfunctions.

For an even potential we have shown that for each pair of endpoints of a gap one is Dirichlet and the other is Neumann. We have also shown that these are all of the Dirichlet and Neumann spectra. The endpoints of a gap are either a pair of periodic or of anti-periodic eigenvalues of Q. Therefore the Dirichlet and Neumann eigenvalues of $-\frac{d^2}{dx^2}+q(x)$ give the periodic half of the Dirichlet and Neumann spectra.

This periodic spectrum, $\{4\mu_n,4\nu_n,4\nu_0\}$ determines $\Delta(\lambda)$ by Proposition (6). From $\Delta(\lambda)$ we find the anti-periodic spectrum as the roots of $\Delta(\lambda)+2$. For each pair of anti-periodic eigenvalues one must be Dirichlet and the other Neumann by Propositions (4) and (1). The choice we make as to which anti-periodic eigenvalue of a given pair are to be a Dirichlet eigenvalue is where the ambiguity in the problem arises. That is we do not get a determined Dirichlet spectrum; potentially, one half of the spectrum is known only up to a sequence of pairs from which it may be chosen.

The Dirichlet spectrum once chosen specifies the initial conditions for the ODEs found previously (20). The solutions to these ODEs and the periodic and anti-periodic spectrum are inserted into the trace formula (22) giving an expression for $Q(x)$. An admissible $q(x)$ for the stated inverse problem is the first half of $Q(x)$ appropriately scaled to be a function on $[0, 1]$, explicitly $q(x) = \frac{1}{4}Q(\frac{1}{2}x)$ $x \in [0, 1]$.

What Dirichlet and Neumann spectra would lead to only a finite number of possibilities for $q(x)$ determined by the method discussed above? One interesting method used to address this question utilizes theta functions and other tools from algebraic geometry, constructing $q(x)$ as a ratio of theta functions ([**10**] and [**7**]). Hochstadt [**7**] showed that if the gaps (see Figure 1) are all trivial then the potential $q(x)$ is 0. Hochstadt went on to show that if only one of the gaps does not vanish then $q(x)$ is an elliptic function. He finished with a proof that if only a finite number of the instability intervals were nontrivial then $q(x)$ is a C^∞ function. In these cases there are only finitely many $q(x)$ solving the inverse problem.

This work was supported by a scholarship from the ARCS Foundation and by an NSF VIGRE Grant #DMS9977116. I would like to thank Nick Ercolani and Leonid Friedlander for introducing me to this problem and for many helpful conversations. I would also like to thank Doug Pickrell for his comments.

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Received October 20, 2000 and revised December 15, 2000.

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