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We justify the Aspinwall-Morrison calculation by connecting it to Gromov-Witten theory.

1. Introduction.

I. (For a good reference on the history of this problem see Sections 7.3.3 and 7.4.4 of [**6**].) One of the problems in the old and recent story of mirror symmetry has been the issue of multiple covers on a Calabi-Yau 3-fold X. In the pre Gromov-Witten era, this problem can be formulated in terms of topological field theories.

Let X be a Calabi-Yau threefold and $H_1, H_2, H_3 \in H^2(X)$. The corresponding 3-point correlator in the A-model of X is a path integral that can be expressed as follows:

$$\langle H_1, H_2, H_3\rangle = \int_X H_1H_2H_3 + \sum_{\beta\in H_2(X)} N_\beta(H_1, H_2, H_3)q^\beta. \tag{1}$$

The summation on the right is for all homology classes of rational curves on X. The parameter $q = (q_1, \dots , q_k)$ is a local coordinate on the Kähler moduli space of X. If $(d_1, \dots , d_k)$ are the coordinates of β with respect to an integral base of the Mori cone of X, then $q^\beta := q_1^{d_1} \cdots q_k^{d_k}$.

The path integral is not a well-defined notion, but more importantly, there is no rigorous definition of $N_\beta(H_1, H_2, H_3)$ in the framework of topological field theories. Let Z_i for $i = 1, 2, 3$ be a cycle whose fundamental class is Poincaré dual to H_i. Heuristically, the "invariant" $N_\beta(H_1, H_2, H_3)$ is described as the "number" of holomorphic maps in the set:

$$\{f : \mathbb{P}^1 \to X \mid f_*([\mathbb{P}^1]) = \beta, f(0) \in Z_1, f(1) \in Z_2, f(\infty) \in Z_3\}. \tag{2}$$

This is certainly not precise, for there may be infinitely many such maps. Let $C \subset X$ be a smooth rational curve. Fix an isomorphism $g : \mathbb{P}^1 \to C$. For any degree k multiple cover $f : \mathbb{P}^1 \to \mathbb{P}^1$ the composition $g \circ f : \mathbb{P}^1 \to C$ satisfies $(g \circ f)_*([\mathbb{P}^1]) = k[C]$. One would then naturally ask:

What is the contribution of the space of degree k multiple covers of C to the "invariant" $N_{k[C]}(H_1, H_2, H_3)$?

This is a question about the numbers $N_{k[C]}(H_1, H_2, H_3)$, hence it is not a well-defined one also. We will see how to make it precise in the framework of Gromov-Witten theory.

The answer was conjectured in [**5**] by looking at the classical example of a Calabi-Yau. If X is a quintic threefold then $H^2(X)$ is one dimensional. Let H be its generator. The 3-point correlator of the quintic can be calculated explicitly:

$$\langle H, H, H\rangle = 5 + \sum_{d=1}^{\infty} n_d d^3 \frac{q^d}{1-q^d}, \tag{3}$$

where n_d is the virtual number of degree d rational curves (instantons) in a generic quintic. The instanton number n_d agrees with the number of degree d rational curves in the quintic if every rational curve of degree d is smooth, isolated and with normal bundle $N = \mathcal{O}(-1) \oplus \mathcal{O}(-1)$. But there are 6-nodal rational plane quintic curves on a generic quintic threefold (see [**12**]), so a (pre Gromov-Witten) rigorous definition of the instanton numbers n_d does not exist either.

The last equation can be transformed as follows:

$$\langle H, H, H\rangle = 5 + \sum_{d=1}^{\infty} \left(\sum_{k|d} n_k k^3 \right) q^d. \tag{4}$$

By comparing it to Equation (1) we can see that:

$$N_d(H, H, H) = \sum_{k|d} n_k k^3. \tag{5}$$

This equation suggests that each degree k rational curve C in the quintic 3-fold X contributes by

$$k^3 = \int_C H \cdot \int_C H \cdot \int_C H \tag{6}$$

to $N_d(H, H, H)$ for any multiple d of k.

For a general Calabi-Yau X, the (pre Gromov-Witten) multiple cover formula can be formulated as follows:

Let $C \subset X$ be a smooth, rational curve such that $N_{C/X} = \mathcal{O}_C(-1) \oplus \mathcal{O}_C(-1)$. The contribution of degree k multiple covers of C in $N_{k[C]}(H_1, H_2, H_3)$ is:

$$\int_C H_1 \cdot \int_C H_2 \cdot \int_C H_3. \tag{7}$$

It was in this form that this formula was taken up by Aspinwall and Morrison in [**1**] and by Voisin in [**13**].

A rigorous definition of N_β and n_β requires a new conceptual framework which is now known as Gromov-Witten theory. Let X be a smooth, projective manifold and $\beta \in H_2(X)$. Let $\overline{M}_{0,n}(X,\beta)$ be the moduli stack of pointed, stable maps of degree β. Universal properties of $\overline{M}_{0,n}(X,\beta)$ imply the existence of several maps:

$$e = (e_1, e_2, \dots , e_n) : \overline{M}_{0,n}(X,\beta) \to X^n, \tag{8}$$
$$\pi_n : \overline{M}_{0,n}(X,\beta) \to \overline{M}_{0,n-1}(X,\beta)$$
$$\pi : \overline{M}_{0,n}(X,\beta) \to \overline{M}_{0,0}(X,\beta),\ \hat{\pi} : \overline{M}_{0,n}(X,\beta) \to \overline{M}_{0,n}.$$

The morphism e evaluates the pointed, stable map at the marked points, π_n forgets the last marked point and collapses the unstable components of the source curve, π forgets the marked points and $\hat{\pi}$ forgets the map and stabilizes the pointed source curve. The expected dimension of $\overline{M}_{0,n}(X,\beta)$ is dim $X + \int_\beta(-K_X) + n - 3$. The moduli stack of stable maps may have components of greater dimension. In this case, a Chow class of the expected dimension has been constructed. It plays the role of the fundamental class, hence it is called the virtual fundamental class and denoted by $[\overline{M}_{0,n}(X,\beta)]^{\mathrm{vir}}$ (see [**8**], [**3**]).

Let X be a Calabi-Yau threefold, $H_1, H_2, H_3 \in H^2(X)$ and β the homology class of a rational curve. In the Gromov-Witten setting the definition (1) of the 3 point correlator is made precise via

$$N_\beta(H_1,H_2,H_3) := \int_{[\overline{M}_{0,3}(X,\beta)]^{\mathrm{virt}}} e_1^*(H_1)e_2^*(H_2)e_3^*(H_3). \tag{9}$$

The expected dimension of $\overline{M}_{0,0}(X,\beta)$ is zero. Let:

$$N_\beta := \deg([\overline{M}_{0,0}(X,\beta)]^{\mathrm{virt}}). \tag{10}$$

By the divisor axiom:

$$N_\beta(H_1,H_2,H_3) = N_\beta \int_\beta H_1 \int_\beta H_2 \int_\beta H_3. \tag{11}$$

Let $C \subset X$ be a smooth rational curve with $N_{C/X} = \mathcal{O}_C(-1) \oplus \mathcal{O}_C(-1)$. The moduli space $\overline{M}_{0,0}(X, d[C])$ contains a component of positive dimension, namely $\overline{M}_{0,0}(C,d)$. The dimension of this component is $2d-2$. Consider the following diagram:

$$\begin{array}{ccc} \overline{M}_{0,1}(C,d) & \xrightarrow{e_1} & C \\ \big\downarrow{\scriptstyle\pi} & & \\ \overline{M}_{0,0}(C,d) & & . \end{array}$$

The sheaf:

$$V_d := R^1\pi_*(\mathcal{O}_C(-1) \oplus \mathcal{O}_C(-1)) \tag{12}$$

is locally free of rank $2d-2$. Let $\mathbb{E}_d$ be its top chern class. An assertion of Kontsevich in [**7**], which was proven by Behrend in [**2**], states that the part of $[\overline{M}_{0,0}(X,\beta)]^{\mathrm{virt}}$ supported in $\overline{M}_{0,0}(C,d)$ is Poincaré dual to $\mathbb{E}_d$. The multiple cover formula in this context says that:

$$\int_{\overline{M}_{0,0}(C,d)} \mathbb{E}_d = d^{-3}, \tag{13}$$

i.e., the curve C contributes by d^{-3} to to $N_{d[C]}$.

The multiple cover formula in this form was proven by Kontsevich [**7**], Lian-Liu-Yau [**9**], Manin [**10**] and Pandharipande [**11**].

By the divisor property, the multiple cover formula (13) follows from:

$$\int_{\overline{M}_{0,3}(C,d)} e_1^*(h)e_2^*(h)e_3^*(h)\pi^*(\mathbb{E}_d) = 1. \tag{14}$$

The instanton numbers n_γ are defined inductively by:

$$N_\beta = \sum_{\beta=k\gamma} n_\gamma k^{-3}. \tag{15}$$

The point of this introduction is that the Aspinwall-Morrison calculation deals with concepts and questions that were not well defined at the time. Hence their calculation, although useful and convincing, is incomplete. The purpose of this paper is to relate the two calculations, hence justifying the Aspinwall-Morrison calculation and closing this historic chapter in the subject.

Here is the relation between the two formulations of the multiple cover formula for the quintic threefold:

$$N_d(H,H,H) = d^3 N_d = d^3 \sum_{k|d} n_k \left(\frac{k}{d}\right)^3 = \sum_{k|d} n_k k^3. \tag{16}$$

II. We now review the Aspinwall-Morrison calculation. Let X be a Calabi-Yau threefold X and a $C \subset X$ a smooth, rational curve such that $N_{C/X} = \mathcal{O}_C(-1) \oplus \mathcal{O}_C(-1)$. Let:

$$N_d(C) := \{f : \mathbb{P}^1 \to X \mid f(\mathbb{P}^1) = C, \deg f = d\} \tag{17}$$

be the space of parameterized maps from $\mathbb{P}^1$ to X. Since C is isolated, $N_d(C)$ is a component of the space of all maps from $\mathbb{P}^1$ to X.

At a moduli point $[f]$, the tangent space and the obstruction space are given respectively by $H^0(f^*(T_X))$ and $H^1(f^*(T_X))$, i.e., locally $N_d(C)$ is given by dim $H^1(f^*(T_X))$ equations in the tangent space. The virtual dimension is:

$$\dim H^0(f^*(T_X)) - \dim H^1(f^*(T_X)) = 3. \tag{18}$$

The space $N_d(C)$ compactifies to $\overline{N}_d(C) = \mathbb{P}^{2d+1}$. Let $\overline{\Gamma}$ be the compactification of the universal graph $\Gamma \subset N_d(C) \times \mathbb{P}^1 \times C$ and H the hyperplane class in $\overline{N}_d(C)$.

The dimension of $H^1(f^*(T_X))$ is $2d-2$ for any f. These vector spaces fit together to form a bundle $\mathcal{U}_d$ over $N_d(C)$. Let p_i be the i-th projection on $\overline{N}_d(C) \times \mathbb{P}^1 \times C$. The bundle $\mathcal{U}_d$ extends to:

$$U_d := R^1 p_{1*}(p_3^*(T_X|C)|_{\overline{\Gamma}}) \tag{19}$$

over $\overline{N}_d(C)$. Aspinwall and Morrison showed that $U_d = \mathcal{O}(-1)^{\oplus 2d-2}$. Based primarily on considerations from topological field theories, they asserted that the cycle corresponding to the degree d multiple covers of C is Poincaré dual to $c_{\text{top}}(U_d) = H^{2d-2}$. We will see that this is consistent with the notion of the virtual fundamental class.

Let $H_i \in H^2(X)$ for $i = 1, 2, 3$ and Z_i their Poincaré duals. The space:

$$\{f \in N_d(C) \mid f(0) = 0\} \tag{20}$$

extends to a linear subspace of $\overline{N}_d(C)$. Therefore:

$$\begin{aligned} &\#\{f \in N_d(C) \mid f(0) = 0, f(1) = 1, f(\infty) = \infty\} \\ &= \int_{\overline{N}_d(C)} H \cdot H \cdot H \cdot c_{\text{top}} U_d = 1. \end{aligned} \tag{21}$$

It follows that the contribution of $N_d(C)$ to:

$$\#\{f : \mathbb{P}^1 \to X \mid f_*[\mathbb{P}^1] = d[C], f(0) \in Z_1, f(1) \in Z_2, f(\infty) \in Z_3\} \tag{22}$$

is

$$\int_C H_1 \cdot \int_C H_2 \cdot \int_C H_3. \tag{23}$$

We emphasize that the multiple cover formula in this approach follows from:

$$\int_{\overline{N}_d(C)} H \cdot H \cdot H \cdot c_{\text{top}} U_d = \int_{\overline{N}_d(C)} H^{2d+1} = 1. \tag{24}$$

III. The purpose of this note is to establish a connection between the Aspinwall-Morrison calculation and Gromov-Witten theory. The main result is the following:

Proposition 1.0.1. *There exists a birational morphism:*

$$\alpha : \overline{M}_{0,3}(C, d) \to \overline{N}_d(C) \tag{25}$$

such that:

1) $\alpha_*(e_i^*(h)) = H$ *for* $i = 1, 2, 3$.
2) $\alpha_*(e_1^*(h) e_2^*(h) e_3^*(h)) = H^3$.
3) $\alpha_*(e_1^*(h) e_2^*(h) e_3^*(h) \pi^*(\mathbb{E}_d)) = H^{2d+1}$.

This proposition implies that Equations (15) and (25) are equivalent, hence connecting the Aspinwall-Morrison calculation to the Gromov-Witten theory.

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2. Relation of the Aspinwall-Morrison formula with Gromov-Witten invariants.

The space of nonparameterized degree d maps $f : \mathbb{P}^1 \to \mathbb{P}^n$ has two particular compactifications that have been employed successfully especially in proving mirror theorems for projective spaces: The nonlinear sigma model (or the graph space):

$$M_d^n := \overline{M}_{0,0}(\mathbb{P}^n \times \mathbb{P}^1, (d,1)) \tag{26}$$

and the linear sigma model:

$$N_d^n := \mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^1}(d))). \tag{27}$$

Elements of N_d^n are $(n+1)$-tuples $[P_0, \ldots , P_n]$ of degree d polynomials in two variables w_0, w_1. The linear sigma model N_d^n is a projective space via the identification $[P_0, \ldots] = [\sum_i a_i w_0^i w_1^{d-i}, \ldots] = [a_0, \ldots , a_d, \ldots]$. Note that $N_d^1 = \overline{N}_d(C)$ for $C \simeq \mathbb{P}^1$. Let H be the hyperplane class in N_d^n.

There exists a birational morphism $\phi : M_d^n \to N_d^n$. We describe this morphism set-theoretically. Let $(C', f) \in M_d^n$. There is a unique component C_0 of C' that is mapped with degree 1 to $\mathbb{P}^1$. Let $C_1, \ldots , C_r$ be the irreducible components of the rest of the curve and $q_i = [c_i, d_i]$ the nodes of C' on C_0. Let d_i be the degree of the map $p_2 \circ f : C' \to \mathbb{P}^n$ on C_i for $i = 0, 1, \ldots , r$. Let $R(w_0, w_1) = \prod_{i=1}^r (c_i w_1 - d_i w_0)^{d_i}$. If the restriction of the map $p_2 \circ f$ is given by $[Q_0, \ldots , Q_n]$ then:

$$\phi(C', f) := [RQ_0, \ldots , RQ_n]. \tag{28}$$

A proof of the fact that ϕ is a morphism is given by J. Li in [**9**].

The first step in connecting the Aspinwall-Morrison calculation to Gromov-Witten invariants is showing that M_d^n and N_d^n are birational models for $\overline{M}_{0,3}(\mathbb{P}^n, d)$.

Lemma 2.0.1. *There exists a birational map* $\psi : \overline{M}_{0,3}(\mathbb{P}^n, d) \to M_d^n$.

Proof. Consider the following diagram:

$$\begin{array}{ccc} \overline{M}_{0,4}(\mathbb{P}^n, d) & \xrightarrow{(\hat{\pi}, e_4)} & \overline{M}_{0,4} \times \mathbb{P}^n \\ \Big\downarrow{\scriptstyle \pi_4} & & \\ \overline{M}_{0,3}(\mathbb{P}^n, d). & & \end{array}$$

Since $\overline{M}_{0,4} \simeq \mathbb{P}^1$ and e_4 is stable in the fibers of π_4, the above diagram exhibits a stable family of maps of degree $(1,d)$ parametrized by $\overline{M}_{0,3}(\mathbb{P}^n,d)$. Universal properties of M^n_d yield a morphism:

$$\psi : \overline{M}_{0,3}(\mathbb{P}^n,d) \to M^n_d. \tag{29}$$

The map ψ is an isomorphism in the smooth locus, hence it is a birational map. □

Let $\pi_4 : \overline{M}_{0,4} \to \overline{M}_{0,3} = \{pt\}$ be the map that forgets the last marked point and σ_i be the section of the i-th marked point for $i = 1,2,3$. Choose coordinates on $\overline{M}_{0,4} \simeq \mathbb{P}^1$ such that the images of these three sections are respectively $0 = [1,0], \infty = [0,1], 1 = [1,1]$. Let

$$\alpha := \phi \circ \psi : \overline{M}_{0,3}(\mathbb{P}^n,d) \to N^n_d. \tag{30}$$

Proposition 2.0.2. *Let h be the hyperplane class of $\mathbb{P}^n$.*

1) $\alpha_*(e_i^*(h)) = H$ *for* $i = 1,2,3$.
2) $\alpha_*(e_1^*(h)e_2^*(h)e_3^*(h)) = H^3$.

Proof. Let

$$\nu_1 : N_d - -- > \mathbb{P}^n \tag{31}$$

be a rational map defined by

$$\nu_1([P_0,P_1,\dots,P_n]) = [P_0(1,0),P_1(1,0),\dots,P_n(1,0)]. \tag{32}$$

This map is defined in the complement U of a codimension $n+1$ linear subspace $P(W_1)$ of N^n_d. Clearly $\nu_1^*(h) = H$ on U. The preimage $D_{1,23}$ of $P(W_1)$ in $\overline{M}_{0,3}(\mathbb{P}^n,d)$ is a sum of d boundary divisors $D(\{x_1\},\{x_2,x_3\},d_1,d_2)$ with $d_1 > 0$ and $d_1 + d_2 = d$. The evaluation map e_1 over U factors through the rational map ν_1. It follows that

$$e_1^*(h) = \alpha^*(H) + D_1, \tag{33}$$

where D_1 is a divisor supported in $D_{1,23}$.[1] Using the evaluations at 1 and ∞ on N^n_d, we obtain:

$$e_2^*(h) = \alpha^*(H) + D_2 \tag{34}$$

and

$$e_3^*(h) = \alpha^*(H) + D_3, \tag{35}$$

where D_2 is a divisor supported in $D_{2,13}$ and D_3 is supported in $D_{3,12}$.

The ψ-image of $D(\{x_1\},\{x_2,x_3\},d_1,d_2)$ does not detect the movement of the marking x_1 along its incident component, hence it is a codimension

[1] It can be shown that $D_1 = -\sum_{d_1} d_1 D(\{x_1\},\{x_2,x_3\},d_1,d-d_1)$ but this is not important in this paper.

2 cycle in M_d^n. It follows that $\psi_*(D_1) = 0$. Similarly $\psi_*(D_2) = 0$ and $\psi_*(D_3) = 0$. Both ψ and ϕ are birational hence by the projection formula:

$$\alpha_*(e_i^*(h)) = H \tag{36}$$

for $i = 1, 2, 3$.

Let $D' \in D_{1,23}, D'' \in D_{2,13}, D''' \in D_{3,12}$ be irreducible boundary divisors. The intersection of any two of them either is 0 or its image is a codimension 4 cycle in M_d^n. It follows that:

$$\psi_*(D'D'') = \psi_*(D'D''') = \psi_*(D''D''') = 0. \tag{37}$$

Notice also that:

$$D'D''D''' = 0. \tag{38}$$

The projection formula yields:

$$\begin{aligned}\psi_*(e_1^*(h)e_2^*(h)e_3^*(h)) &= \psi_*\left(\prod_i(\psi^*(\phi^*(H)) + D_i)\right) \\ &= \prod_i(\phi^*(H)) = \phi^*(H^3).\end{aligned} \tag{39}$$

The lemma follows from the fact that ϕ is a birational map. □

We now return to the case $n = 1$.

Let $\rho : M_d^1 \to \overline{M}_{0,0}(C, d)$ be the natural morphism. The composition:

$$\rho \circ \psi : \overline{M}_{0,3}(C, d) \to \overline{M}_{0,0}(C, d) \tag{40}$$

is the map π that forgets the 3 marked points and stabilizes the source curve. Recall Kontsevich's obstruction bundle V_d on $\overline{M}_{0,0}(C, d)$. Its fiber is $H^1(C', f^*(\mathcal{O}(-1) \oplus \mathcal{O}(-1)))$. Its top chern class is $\mathbb{E}_d$. We are now ready to exhibit the connection between the Aspinwall-Morrison calculation and Gromov-Witten invariants.

Proposition 2.0.3. $\alpha_*\left(e_1^*(h)e_2^*(h)e_3^*(h)\pi^*(\mathbb{E}_d)\right) = H^{2d+1}$.

Proof. Let E_d be the top chern class of the bundle $\rho^*(V_d)$ on M_d^1. Recall from part II of the introduction that H^{2d-2} is the top chern class of the Aspinwall-Morrison obstruction bundle U_d on N_d^1. It is shown in [**9**] that $\phi_*(E_d) = H^{2d-2}$. On the other hand $\psi^*(E_d) = \pi^*(\mathbb{E}_d)$. But ψ is birational, hence by the projection formula $\psi_*(\pi^*(\mathbb{E}_d)) = E_d$.

We compute:

$$\begin{aligned}(41)\qquad \alpha_*\left(\prod_i e_i^*(h)\mathbb{E}_d\right) &= \alpha_*\left(\prod_i e_i^*(h)\psi^*(E_d)\right)\\ &= \phi_*\left(\psi_*\left(\prod_i e_i^*(h)\right)E_d\right)\\ &= \phi_*(\phi^*(H^3)E_d) = H^3\phi_*(E_d)\\ &= H^3H^{2d-2} = H^{2d+1}.\end{aligned}$$

The proposition is proven. □

The last proposition yields:

$$\begin{aligned}(42)\qquad \int_{\overline{M}_{0,3}(C,d)}\prod_{i=1}^3 e_i^*(h)\mathbb{E}_d &= \int_{\overline{N}_d(C)}\alpha_*\left(\prod_{i=1}^3 e_i^*(h)\psi^*(E_d)\right)\\ &= \int_{\overline{N}_d(C)} H^{2d+1} = 1,\end{aligned}$$

i.e., the Aspinwall-Morrison calculation is a pushforward of Kontsevich's calculation from $\overline{M}_{0,3}(C,d)$ to the projective space $\overline{N}_d(C)$.

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