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Dedicated to Professor Mitsuyoshi Kato for his 60th Birthday

Using representations of fundamental groups, we introduce the concept of twisted linking numbers. If the corresponding representation is trivial, the twisted linking number coincides with the linking number. In the case of a nontrivial representation, however, this is not necessarily true, which implies that the twisted linking number can detect the nontriviality of an embedding of \mathbb{S}^1 . An example is given for which the linking number is trivial but the twisted linking number is not.

1. Introduction.

A linking number is an invariant of an oriented 2-component link. Linking numbers are frequently used to study Alexander polynomials of knots and links, and coverings of certain spaces such as the complement of a knot or the complement of a link. There are several definitions of linking numbers, corresponding to the various contexts within which they are studied: The theories regarding fundamental groups, the Gauss integral, coverings of complements of knots, diagrams, etc. For example, there are eight equivalent definitions in [7]. In the context of knot complements, linking number is defined in terms of the fundamental group of the knot complement and its abelianization. Since the abelianization gives only a 1-dimensional representation, the following is a natural question: If we consider nontrivial representations instead of the 1-dimensional abelian representation of the abelianization, how do we define the extension of a linking number? If it can be defined, what does this extended concept represent?

The purpose of this paper is to introduce the notion of linking numbers through representations of fundamental groups. For the sake of generality, we consider links with r components ($r \geq 1$). Let L be an oriented, non-split link in \mathbb{S}^3 . We use the Wirtinger presentation as the presentation of $\pi L = \pi_1(\mathbb{S}^3 \setminus L)$. Let K be an embedding of \mathbb{S}^1 in $\mathbb{S}^3 \setminus L$. We denote the corresponding element in πL by k . k naturally inherits its orientation from K . Then k is uniquely determined up to conjugation; i.e., the presentation of k depends on the base point in $\mathbb{S}^3 \setminus L$. Next, let ψ be a representation of πL .

Using πL and ψ , we construct a $\mathcal{K}\psi(\pi L)$ module V_ψ , where \mathcal{K} is a rational polynomial field. When the twisted Alexander polynomial of L (see [5], [8]) does not vanish, we show that V_ψ is generated by one element (Theorem 1). Therefore, if we fix a generator, as a coefficient of the generator, any v in V_ψ can be expressed by an element of $\mathcal{K}\psi(\pi L)$. Here we apply the “*derivation*” to k , whose consequence is an element of V_ψ . Hence it is also expressed by a matrix over \mathcal{K} . After some calculation, we can show that it is explicitly represented by a more simpler matrix determined by k and the choice of the generator. We define $lk_\psi(K, L)$ in \mathcal{K} by its determinant. It is remarkable that the definition of lk_ψ itself does not need the condition that the twisted Alexander polynomial does not vanish. It can be shown that $lk_\psi(K, L)$ coincides with $lk(K, L)$ if ψ is trivial. However, when ψ is nontrivial, these two quantities do not necessarily coincide. We give such an example here.

2. Preliminaries.

2.1. Groups and Presentations. In this paper, we use the term ‘groups’ strictly in reference to finitely generated and finitely presented groups.

Let us begin by giving a brief summary of the points of combinatorial group theory [6] relevant to the present study. Let \mathfrak{F}_s be a free group generated by x_1, \dots, x_s , and let \mathfrak{R} be a normal subgroup of \mathfrak{F}_s generated by $r_1, \dots, r_t \in \mathfrak{F}_s$. The elements of \mathfrak{R} are of the form $\prod_{k=1}^l u_k r_{i_k}^{\varepsilon_k} u_k^{-1}$, where $u_k \in \mathfrak{F}_s$ and $\varepsilon_k = \pm 1$, and referred to as the *consequences* of r_1, \dots, r_t . \mathfrak{R} itself is also referred to as the *consequence* of r_1, \dots, r_t . \mathfrak{F}_s and \mathfrak{R} uniquely determine the quotient group $\mathfrak{G} = \mathfrak{F}_s/\mathfrak{R}$. Then, let ϕ be the associated homomorphism from \mathfrak{F}_s to \mathfrak{G} :

$$(1) \quad \phi : \mathfrak{F}_s = \langle x_1, \dots, x_s \rangle \rightarrow \mathfrak{G} = \langle x_1, \dots, x_s \mid r_1, \dots, r_t \rangle.$$

We call $\langle x_1, \dots, x_s \mid r_1, \dots, r_t \rangle$ a *presentation* of \mathfrak{G} . The set r_1, \dots, r_t is called a set of *relators*, and the equations $r_i = 1$ are called *defining relations* for \mathfrak{G} .

2.2. Tietze Transformations. A group \mathfrak{G} can have many presentations. Tietze showed that if \mathfrak{G} has two presentations, one can be transformed into the other by a finite sequence of Tietze transformations [6]. More precisely, he showed that if we are given a presentation

$$\mathfrak{G} = \langle x_1, \dots, x_s \mid r_1, \dots, r_t \rangle,$$

then any other presentation for \mathfrak{G} can be obtained by using a finite number of the following transformations successively:

T₁: If a word r is r_1, \dots, r_t , then r is added to the defining relators in \mathfrak{G} . The resulting presentation is

$$\langle x_1, \dots, x_s \mid r_1, \dots, r_t, r \rangle.$$

T_2 : If y is any word in x_1, \dots, x_s , then x is added to the generating symbols in \mathfrak{G} and the word xy^{-1} is added to the relators in \mathfrak{G} . The resulting presentation is

$$\langle x_1, \dots, x_s, x \mid r_1, \dots, r_t, xy^{-1} \rangle.$$

The transformations T_1, T_1^{-1}, T_2 and T_2^{-1} are called *Tietze transformations*, where T_1^{-1} and T_2^{-1} are the inverse operations of T_1 and T_2 , respectively.

2.3. Free Differential Calculus. We now review the theory of free differential calculus [3], [4].

Let \mathfrak{G} be a group and let $\mathbb{Z}\mathfrak{G}$ be the group ring of \mathfrak{G} over the integers. A *derivation* D in $\mathbb{Z}\mathfrak{G}$ is defined as a mapping of $\mathbb{Z}\mathfrak{G}$ into itself with the properties

$$\begin{aligned} D(u + v) &= Du + uDv, & u, v \in \mathbb{Z}\mathfrak{G}, \\ D(u \cdot v) &= Du \cdot v^\varepsilon + u \cdot Dv, \end{aligned}$$

where v^ε is the sum of the coefficients of v . This definition implies that

$$\begin{aligned} D(a) &= 0, & a \in \mathbb{Z}, \\ D(g^{-1}) &= -g^{-1}Dg, & g \in \mathfrak{G}, \\ D(g^n) &= (1 + g + g^2 + \dots + g^{n-1})Dg. \end{aligned}$$

The derivations in $\mathbb{Z}\mathfrak{G}$ form a right $\mathbb{Z}\mathfrak{G}$ module, where addition and right multiplication by v are defined by

$$\begin{aligned} (D_1 + D_2)u &= D_1u + D_2u, & u \in \mathbb{Z}\mathfrak{G}, \\ (D \cdot v)u &= Du \cdot v. \end{aligned}$$

In particular, when \mathfrak{G} is the free group \mathfrak{F}_s , all derivations are explicitly determined in the following sense. For each generator x_j of \mathfrak{F}_s , derivations $D_{x_j} = \partial/\partial x_j$ are *partial derivations* if they satisfy

$$\frac{\partial x_i}{\partial x_j} = \delta_{ij}.$$

Fox showed that the partial derivations form a basis of the module of derivations [3]. That is, every derivation $D : \mathbb{Z}\mathfrak{F}_s \rightarrow \mathbb{Z}\mathfrak{F}_s$ is given by

$$D = \sum_{j=1}^s \frac{\partial}{\partial x_j} Dx_j,$$

where the Dx_j are the prescribed elements in $\mathbb{Z}\mathfrak{F}_s$. One can easily check that the map $D_\varepsilon(w) = w - w^\varepsilon$ is also a derivation. Hence, for any element $w \in \mathfrak{F}_s$, we have the following formula:

$$(2) \quad w - 1 = \sum_{j=1}^s \frac{\partial w}{\partial x_j} (x_j - 1).$$

3. Alexander modules associated with representations.

Let \mathfrak{G} be a group, and let ϕ be the associated homomorphism presented in (1). We extend ϕ to a linear map on the group ring. (For simplicity we also denote this extension by ϕ .) Let J be the $t \times s$ matrix over $\mathbb{Z}\mathfrak{G}$ defined by

$$J = \left(\phi \left(\frac{\partial r_i}{\partial x_j} \right) \right) = \begin{pmatrix} \phi \left(\frac{\partial r_1}{\partial x_1} \right) & \cdots & \phi \left(\frac{\partial r_1}{\partial x_s} \right) \\ \vdots & & \vdots \\ \phi \left(\frac{\partial r_t}{\partial x_1} \right) & \cdots & \phi \left(\frac{\partial r_t}{\partial x_s} \right) \end{pmatrix}.$$

J is called the *Jacobian*.

Now, let V be a $\mathbb{Z}\mathfrak{G}$ module whose generators are dx_1, \dots, dx_s and defining relations are given by J :

$$V = \left\langle dx_1, \dots, dx_s \mid \sum_{j=1}^s \lambda \left(\phi \left(\frac{\partial r_i}{\partial x_j} \right) \right) dx_j = 0 \ (i = 1, \dots, t) \right\rangle,$$

where λ denotes an action on V . V is called the *Alexander module*.

Next, let \mathfrak{H} be a group and let ψ be the homomorphism from \mathfrak{G} to \mathfrak{H} . We extend ψ to a \mathbb{Z} linear map (and denote this extension by ψ). Let V_ψ be the $\mathbb{Z}\mathfrak{H}$ module defined by

$$V_\psi = \left\langle dx_1, \dots, dx_s \mid \sum_{j=1}^s \lambda_\psi \left(\psi \phi \left(\frac{\partial r_i}{\partial x_j} \right) \right) dx_j = 0 \ (i = 1, \dots, t) \right\rangle,$$

where λ_ψ denotes an action on V_ψ . We call V_ψ the *Alexander module associated with the representation ψ* . Note that ψ naturally induces a linear map $\tilde{\psi}$ from V to V_ψ by

$$\tilde{\psi}(\lambda(g)dx_i) = \lambda_\psi(\psi(g))dx_i.$$

Therefore we have the following commutative diagram:

$$\begin{array}{ccc} GL(V) & \xrightarrow{\tilde{\psi}} & GL(V_\psi) \\ \lambda \uparrow & & \uparrow \lambda_\psi \\ \mathfrak{G} & \xrightarrow{\psi} & \psi(\mathfrak{G}) \end{array}.$$

When there is no danger of confusion, we omit λ and λ_ψ .

Formally, each relation in V is derived as follows. Let d be a derivation in $\mathbb{Z}\mathfrak{G}_s$. For an element w in \mathfrak{F}_s , its derivation by d is presented by $d(w) = \sum_{j=1}^s \frac{\partial w}{\partial x_j} dx_j$. Here we define an element of V by regarding dx_1, \dots, dx_s as generators and considering the ϕ images of each coefficient of $d(w)$. For convenience, we denote such an element by $\phi d(w)$. Then, each relation is defined by the relation $\phi d(r_i) = 0$.

Indeed, V is a homology of the universal covering of the space X satisfying $\pi_1 X = \mathfrak{G}$. The covering transformation is $\pi_1 X$ itself. For a closed path x in X whose corresponding element in $\pi_1 X$ is $\phi(w)$, $\phi d(w)$ is a lift of x . For more details, see [2], [7].

Let \mathcal{K} be a commutative field such that there exists a natural inclusion or projection from \mathbb{Z} to \mathcal{K} . We extend V to a $\mathcal{K}\mathfrak{G}$ module by the map $\mathbb{Z} \rightarrow \mathcal{K}$. Similarly, we extend V_ψ to a $\mathcal{K}\mathfrak{H}$ module, and \mathbb{Z} linear maps ϕ and ψ to \mathcal{K} linear maps. Below, we consider a rational polynomial field $\mathbb{K}(t_1^{\pm 1}, \dots, t_r^{\pm 1})$ for \mathcal{K} , where \mathbb{K} represents \mathbb{C} , \mathbb{R} or a finite field \mathbb{F} .

Since the Jacobian depends on the choice of the presentation of \mathfrak{G} , let us consider the operations to which the Jacobian matrices are subject as induced by the Tietze transformations [9]. These operations are of the following three general types (and any operation belonging to one of the following three types is an operation induced by a Tietze transformation):

- J_0 : Permutation of rows or columns.
- J_1 : Adjoinment to J of a new row that is a left $\mathbb{Z}\mathfrak{G}$ combination of the rows of J .
- J_2 : Adjoinment to J of a new row and a new column satisfying the condition that the entry in their intersection is $1 = 1 \cdot e \in \mathbb{Z}\mathfrak{G}$ (e is the identity element of \mathfrak{G}) and the remaining entries in the new column are all zero:

$$J \rightarrow \begin{pmatrix} J & 0 \\ * & 1 \end{pmatrix}.$$

We consider two matrices over $\mathbb{Z}\mathfrak{G}$ to be *equivalent* if one can be obtained from the other by a finite sequence of operations of the types J_0, J_1, J_2, J_1^{-1} and J_2^{-1} , where J_1^{-1} and J_2^{-1} are the inverse operations of types J_1 and J_2 , respectively. It is well-known that the Jacobians of \mathfrak{G} are all equivalent.

Since we would like to consider \mathfrak{H} to be $GL(\mathcal{K}^n)$ and ψ to be a representation of \mathfrak{G} to $GL(\mathcal{K}^n)$, we define equivalence classes of matrices over $\Lambda = \text{Mat}(\mathcal{K}^n)$ in terms of operations for the following types. (We use “ A ” to represent an arbitrary such matrix.):

- t_0 : Permutation of the i -th block and the j -th block rows (or columns).
- t_1 : Adjoinment to A of a new row that is a Λ combination of the rows of A .
- t_2 : Adjoinment to A of a new row and a new column satisfying the conditions that the entry in their intersection $1 = 1 \cdot E_n$ (E_n is the identity matrix) and the remaining entries in the new column are all 0:

$$A \rightarrow \begin{pmatrix} A & 0 \\ * & 1 \end{pmatrix}.$$

- t_3 : Addition, to the i -th row of A , of a Λ combination of the rows in the complementary set.

t₄: Change of the basis for the representation.

Similarly to the case of the Jacobians, we consider two matrices over Λ , A and B , to be equivalent if one can be obtained from the other by a finite sequence of operations of types t_0, \dots, t_4 , and $t_1^{-1}, t_2^{-1}, t_3^{-1}$, where t_1^{-1}, t_2^{-1} and t_3^{-1} are the inverse operations of types t_1, t_2 and t_3 , respectively. We denote that matrices A and B are equivalent by $A \sim B$. When we would like to emphasize that A is transformed to B by t_* , we write $A \stackrel{t_*}{\sim} B$.

It is clear that if two groups \mathfrak{G}_1 and \mathfrak{G}_2 are isomorphic, then the presentation matrices of V_1 and V_2 are equivalent. Furthermore, if ψ_1 and ψ_2 are equivalent representations, the presentation matrices of $V_{1\psi_1}$ and $V_{2\psi_2}$ are also equivalent.

4. Main results.

In order to establish our main results, we need some preliminary discussion.

Lemma 1. *Let A be a 2×2 matrix whose entries are in Λ . Then, A is nonsingular as a $2n \times 2n$ matrix if and only if*

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \stackrel{t_0, t_3}{\sim} \begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix},$$

where D_1 and D_2 are in $GL(\mathcal{K}^n)$.

Proof. Demonstrating the ‘if’ part is trivial, and we therefore present only the proof for the ‘only if’ part. If an entry in A is a nonsingular matrix, we may assume that A_{11} is nonsingular by an operation of type t_0 . It is easy to see that

$$\begin{pmatrix} E_n & 0 \\ -A_{21}A_{11}^{-1} & E_n \end{pmatrix} \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ 0 & -A_{21}A_{11}^{-1}A_{12} + A_{22} \end{pmatrix}.$$

Since A is nonsingular, $-A_{21}A_{11}^{-1}A_{12} + A_{22}$ is also nonsingular. Then, considering t_3 , we observe

$$A \sim \begin{pmatrix} A_{11} & 0 \\ 0 & -A_{21}A_{11}^{-1}A_{12} + A_{22} \end{pmatrix}.$$

If all entries in A are singular, then there exist P in $GL(\mathcal{K}^n)$ and Q in $GL(\mathbb{K}^n)$ such that

$$PQ^{-1}A_{11}Q = \begin{pmatrix} E_k & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix},$$

where $k = \text{rank}A_{11}$. Note that we can choose matrices P and Q in such a way that P consists of multiplications of matrices which induce an operation

of type t_3 and Q is a permutation matrix. For simplicity, we denote the matrix

$$\begin{pmatrix} Q^{-1} & 0 \\ 0 & E_n \end{pmatrix} A \begin{pmatrix} Q & 0 \\ 0 & E_n \end{pmatrix}$$

by A_Q . Then, for an appropriate matrix P_1 , we have

$$\begin{aligned} A' &= \begin{pmatrix} E_n & 0 \\ P_1 & E_n \end{pmatrix} \begin{pmatrix} P & 0 \\ 0 & E_n \end{pmatrix} A_Q \\ &= \begin{pmatrix} E_n & 0 \\ P_1 & E_n \end{pmatrix} \left(\begin{array}{cc|c} E_k & \vdots & PQ^{-1}A_{12} \\ \dots & 0 & \\ \hline A_{21}Q & & A_{22} \end{array} \right) \\ &\stackrel{t_3}{=} \left(\begin{array}{cc|c} E_k & \vdots & A'_{12} \\ \dots & 0 & \\ \hline 0 & A'_{21} & A'_{22} \end{array} \right). \end{aligned}$$

By the assumption, the $n \times (n - i)$ matrix A'_{21} must be of full rank. Next, we apply t_3 twice to each row of A' . That is, for appropriate matrices P_2 and P_3 , we have

$$\begin{aligned} A'' &= \begin{pmatrix} E_n & 0 \\ P_3 & E_n \end{pmatrix} \begin{pmatrix} E_n & P_2 \\ 0 & E_n \end{pmatrix} A' \\ &\stackrel{t_3}{=} \begin{pmatrix} E_n & 0 \\ P_3 & E_n \end{pmatrix} \left(\begin{array}{cc|c} E_k & & A''_{12} \\ & E_{n-k} & \\ \hline 0 & A'_{21} & A'_{22} \end{array} \right) \\ &= \begin{pmatrix} E_n & 0 \\ P_3 & E_n \end{pmatrix} \begin{pmatrix} E_n & A''_{12} \\ A''_{21} & A'_{22} \end{pmatrix} \\ &\stackrel{t_3}{=} \begin{pmatrix} E_n & A''_{12} \\ 0 & A''_{22} \end{pmatrix}. \end{aligned}$$

Also by the assumption, A'' and A''_{22} must be nonsingular. Now, we summarize the above equations:

$$\begin{aligned}
 A'' &= \begin{pmatrix} E_n & 0 \\ P_3 & E_n \end{pmatrix} \begin{pmatrix} E_n & P_2 \\ 0 & E_n \end{pmatrix} \begin{pmatrix} E_n & 0 \\ P_1 & E_n \end{pmatrix} \begin{pmatrix} P & 0 \\ 0 & E_n \end{pmatrix} A_Q \\
 &= \begin{pmatrix} P & 0 \\ 0 & E_n \end{pmatrix} \begin{pmatrix} E_n & 0 \\ P_3 P & E_n \end{pmatrix} \begin{pmatrix} E_n & P^{-1} P_2 \\ 0 & E_n \end{pmatrix} \begin{pmatrix} E_n & 0 \\ P_1 P & E_n \end{pmatrix} A_Q, \\
 \therefore &\begin{pmatrix} E_n & 0 \\ P_3 P & E_n \end{pmatrix} \begin{pmatrix} E_n & P^{-1} P_2 \\ 0 & E_n \end{pmatrix} \begin{pmatrix} E_n & 0 \\ P_1 P & E_n \end{pmatrix} A_Q \\
 &= \begin{pmatrix} P^{-1} & 0 \\ 0 & E_n \end{pmatrix} A'' \\
 &= \begin{pmatrix} P^{-1} & P^{-1} A''_{12} \\ 0 & A''_{22} \end{pmatrix}.
 \end{aligned}$$

Therefore, by considering t_3 , we have

$$A_Q \stackrel{t_3}{\sim} \begin{pmatrix} P^{-1} & 0 \\ 0 & A''_{22} \end{pmatrix}.$$

Finally, we obtain

$$A \stackrel{t_3}{\sim} \begin{pmatrix} QP^{-1}Q^{-1} & 0 \\ 0 & A''_{22} \end{pmatrix}.$$

This completes the proof of the lemma. \square

We extend the statement of Lemma 1 from 2×2 matrices to $m \times m$ matrices.

Proposition 1. *Let A be an $m \times m$ matrix whose entries are in Λ . Then, A is nonsingular as an $mn \times mn$ matrix if and only if*

$$A \stackrel{t_0, t_3}{\sim} \begin{pmatrix} D_1 & & 0 \\ & \ddots & \\ 0 & & D_m \end{pmatrix},$$

where D_1, \dots, D_m are in $GL(\mathcal{K}^n)$.

Proof. This proof is similar to that of Lemma 1. Suppose that operations of the types applied there succeed through the $(i-1)$ -th row and fail at the i -th row:

$$A \stackrel{t_0, t_3}{\sim} \begin{pmatrix} A_{11} & & 0 & * \\ & \ddots & & \vdots \\ 0 & & A_{i-1, i-1} & * \\ 0 & \dots & 0 & B \end{pmatrix}, \quad B = \begin{pmatrix} A_{ii} & \dots & A_{im} \\ \vdots & & \vdots \\ A_{mi} & \dots & A_{mm} \end{pmatrix},$$

where $A_{11}, \dots, A_{i-1, i-1}$ are nonsingular, and each entry in B is singular. Then it is enough into show that A_{ii} can be transformed into a nonsingular

matrix by an operation of type t_3 . First, set $k = \text{rank}(A_{ii}) < n$. Then, from the nonsingularity of A , we can increase the rank of A_{ii} by at least one by using some A_{ji} ($j = i + 1, \dots, n$), as in the proof of Lemma 1. Repeating this process at most $n - k$ times, we have a nonsingular matrix in the (i, i) -th entry of A . This concludes the proof. \square

Here, as \mathfrak{G} we take the fundamental group of the complement of the link L with r components in \mathbb{S}^3 :

$$\mathfrak{G} = \pi L = \langle x_1, \dots, x_s \mid r_1, \dots, r_{s-1} \rangle.$$

Then, let ρ be an irreducible representation of πL :

$$\rho : \pi L \rightarrow GL(\mathbb{K}^n).$$

Note that $\mathbb{K}\rho(\pi L)$ is isomorphic to the full matrix algebra $Mat(\mathbb{K}^n)$ if and only if ρ is irreducible. Because πL is described by the Wirtinger presentation, there exists a natural abelianization homomorphism α from πL to $\pi L/\pi L'$:

$$\alpha : \pi L \rightarrow \pi L/\pi L' \cong \langle t_1, \dots, t_r \mid t_i t_j = t_j t_i \ (i, j = 1, \dots, r) \rangle.$$

Next, let θ be a linear map from $\mathbb{K}\pi L$ to $\mathcal{K} = \mathbb{K}(t_1^{\pm 1}, \dots, t_r^{\pm 1})$ that factors through α :

$$\begin{array}{ccc} \pi L & \xrightarrow{\alpha} & \pi L/\pi L' \\ & \searrow \theta & \downarrow \\ & & \mathcal{K}. \end{array}$$

We use the same symbol t_i in $\pi L/\pi L'$ and \mathcal{K} for convenience. Let ψ be a representation of πL ,

$$\psi : \pi L \rightarrow GL(\mathcal{K}^n),$$

defined by $\psi(x_i) = \theta(x_i)\rho(x_i)$. Then the irreducibility of ρ implies

$$\mathcal{K}\psi(\pi L) = Mat(\mathcal{K}^n).$$

Therefore we can apply Proposition 1 to the presentation matrix of V_ψ .

Let us recall the twisted Alexander polynomial [5], [8]. Then, let $R_{\hat{k}}$ be an $(s - 1) \times (s - 1)$ submatrix of presentation matrix of V obtained by removing the k -th column. Let $\psi(R_{\hat{k}})$ be an $(s - 1) \times (s - 1)$ matrix whose (i, j) -th entry is the ψ image of the (i, j) -th entry of $R_{\hat{k}}$. Hence, as a matrix, we can see that $\psi(R_{\hat{k}})$ is an $n(s - 1) \times n(s - 1)$ matrix. Then, the twisted Alexander polynomial $\Delta_{\pi L, \psi} = \Delta_{\pi L, \psi}(t_1, \dots, t_r)$ is defined by

$$\Delta_{\pi L, \psi} = \frac{\det \psi(R_{\hat{k}})}{\det \psi(1 - x_k)}.$$

In our setting, the denominator of $\Delta_{\pi L, \psi}$ is essentially the characteristic polynomial of the nonsingular matrix $\rho(x_k)$ because we use the Wirtinger

presentation. This definition is independent of the choice of the index k and the presentation of πL up to a factor $\pm t_1^{e_1} \dots t_r^{e_r}$, where e_1, \dots, e_r are integers.

Proposition 2. $\Delta_{\pi L, \psi}$ does not vanish if and only if

$$\psi(R_{\hat{k}}) \stackrel{t_0, t_3}{\sim} \begin{pmatrix} D_1 & & 0 \\ & D_2 & \\ 0 & & \ddots \end{pmatrix},$$

where D_1, D_2, \dots are nonsingular matrices.

Proof. $\Delta_{\pi L, \psi} \neq 0$ if and only if $\det \psi(R_{\hat{k}}) \neq 0$ for any k , which implies that $\psi(R_{\hat{k}})$ is nonsingular. The conclusion follows from Proposition 1. \square

In the remainder of this paper, we assume that $\Delta_{\pi L, \psi}$ does not vanish. By Proposition 2, we have

$$(3) \quad \left(\psi \phi \left(\frac{\partial r_i}{\partial x_j} \right) \right) \stackrel{t_0, t_3}{\sim} \begin{pmatrix} A_1 & & & B_1 \\ & \ddots & & \\ & & A_{s-1} & B_{s-1} \end{pmatrix},$$

where $B_i \in Mat(\mathcal{K}^n) (i = 1, \dots, s - 1)$. Because (3) represents a transformation consisting of operations of types t_0 and t_3 , we obtain

$$V_\psi \cong \langle dx_1, \dots, dx_s \mid A_i dx_i + B_i dx_s = 0 (i = 1, \dots, s - 1) \rangle.$$

Next, we determine each B_i . From (2), we have

$$\psi \phi(w) - 1 = \sum_{j=1}^s \psi \phi \left(\frac{\partial w}{\partial x_j} \right) (\psi \phi(x_j) - 1).$$

If we choose the relator r_i as w , then

$$\sum_{j=1}^s \psi \phi \left(\frac{\partial r_i}{\partial x_j} \right) (\psi \phi(x_j) - 1) = 0.$$

These equations show that the presentation matrix has the property that the sum of each row multiplied by $\psi \phi(x_j) - 1$ vanishes. Furthermore, it is obvious that operations of types t_0 and t_3 do not change this property. Hence, we obtain

$$A_i(\psi \phi(x_i) - 1) + B_i(\psi \phi(x_s) - 1) = 0.$$

Since we use the Wirtinger presentation as a presentation of πL , the matrix $\psi \phi(x_s)$ does not have an eigenvalue 1. This implies that $\psi \phi(x_s) - 1$ is invertible, and we have

$$B_i = -A_i(\psi \phi(x_i) - 1)(\psi \phi(x_s) - 1)^{-1}.$$

Then, since each A_i in $(\psi\phi(\frac{\partial r_i}{\partial x_j}))$ is a nonsingular matrix, we obtain

$$V_\psi \cong \langle dx_1, \dots, dx_s \mid dx_i - (\psi\phi(x_i) - 1)(\psi\phi(x_s) - 1)^{-1}dx_s = 0 \\ (i = 1, \dots, s - 1) \rangle.$$

Therefore, for any element v in V_ψ , we can write $v = v_s dx_s$, where $v_s \in Mat(\mathcal{K}^n)$. This means that V_ψ is generated by dx_s .

Summarizing the above discussion, we have the following theorem.

Theorem 1. *For any $i = 1, \dots, s$, V_ψ is generated by dx_i :*

$$V_\psi = \mathcal{K}\psi(\pi L)dx_i.$$

Now, we fix the index of the generator of V_ψ , denoted by s . Let c be an element of \mathfrak{F}_s described by $c = x_{i_1}x_{i_2}x_{i_3} \dots$. Then

$$\begin{aligned} \tilde{\psi}\phi(d(c)) &= dx_{i_1} + \psi\phi(x_{i_1})dx_{i_2} + \psi\phi(x_{i_1}x_{i_2})d(x_{i_3} \dots) \\ &= (\psi\phi(x_{i_1}) - 1)(\psi\phi(x_s) - 1)^{-1}dx_s \\ &\quad + \psi\phi(x_{i_1})(\psi\phi(x_{i_2}) - 1)(\psi\phi(x_s) - 1)^{-1}dx_s + \dots \\ &= (\psi\phi(x_{i_1})\psi\phi(x_{i_2}) - 1)(\psi\phi(x_s) - 1)^{-1}dx_s \\ &\quad + \psi\phi(x_{i_1})\psi\phi(x_{i_2})d(x_{i_3} \dots) \\ &= \dots \\ &= (\psi\phi(c) - 1)(\psi\phi(x_s) - 1)^{-1}dx_s. \end{aligned}$$

Similarly, for any element $c \in \mathfrak{F}_s$, we have

$$(4) \quad \tilde{\psi}\phi(d(c)) = (\psi\phi(c) - 1)(\psi\phi(x_s) - 1)^{-1}dx_s.$$

This calculation leads us to the following definitions.

Definition 1. Let k be the element of πL corresponding to K for a suitable base point, and let c be an element of \mathfrak{F}_s satisfying $k = \phi(c)$. Then we define $Lk_\psi(K, L)$ in $Mat(\mathcal{K}^n)$ and $lk_\psi(K, L)$ in \mathcal{K} by

$$Lk_\psi(K, L) = (\psi\phi(c) - 1)(\psi\phi(x_s) - 1)^{-1}, \\ lk_\psi(K, L) = \det(Lk_\psi(K, L)).$$

Remark 1. The condition such that $\Delta_{\pi L, \psi}$ does not vanish leads V_ψ to be 1-dimensional, and furthermore if $c \in \mathfrak{F}_s$, $\tilde{\psi}\phi(d(c))$ can be explicitly described by (4). Conversely if we accept the definition of Lk_ψ and lk_ψ as Definition 1, we need not the condition.

Remark 2. If we choose another base point of k , or if we choose another index such that the component to which the index belongs is the same component to which s belongs, then Lk_ψ changes to $P_1(\psi\phi(c) - 1)P_1^{-1}P_2(\psi\phi(x_s) - 1)P_2^{-1}$, where $P_1, P_2 \in GL(\mathbb{K}^n)$. If we take a representation of \mathfrak{G} equivalent to ρ , then Lk_ψ also changes to $Q(\psi\phi(c) - 1)(\psi\phi(x_s) - 1)Q^{-1}$, where $Q \in GL(\mathbb{K}^n)$. We thus obtain the following theorem.

Theorem 2. *The definition of lk_ψ does not depend on the choice of the base point. Furthermore, if we choose representations ρ and θ satisfying all $\rho(x_i)$ to be conjugate with each other and all $\theta(x_i)$ to be the same monomial t , then lk_ψ does not depend on the choice of the index s . Therefore, in this case, lk_ψ only depends on the group \mathfrak{G} and its representation ρ (or ψ).*

The condition stated in Theorem 2 obviously holds when ρ is trivial and $t_i = t$ for $i = 1, \dots, r$.

Proposition 3. *If ρ is trivial and $t = t_1 = \dots = t_r$, then we have*

$$lk_\psi(K, L) = \frac{t^n - 1}{t - 1},$$

where n is the exponent of k .

Proof. If ρ is trivial, we have $\psi\phi(x_1) = \dots = \psi\phi(x_s) = t$. Let $c = \prod_{j=1}^m x_{i_j}^{\varepsilon_j}$ ($\varepsilon_j = \pm 1$) be the element of \mathfrak{F}_s satisfying $k = \phi(c)$. Then $\psi(k) = \psi\phi(c) = t^n$, where $n = \sum_{j=1}^m \varepsilon_j$. This concludes the proof. \square

Back to the definition of the linking number [7], we can easily see that n appeared in the proof of Proposition 3 is equal to the linking number. Hence, in this case, we obtain the following corollary.

Corollary 1. *Under the assumption of Proposition 3, we have*

$$lk_\psi(K, L) \xrightarrow{t \rightarrow 1} \begin{cases} lk(K, L), & \text{if } \mathbb{K} = \mathbb{C} \text{ or } \mathbb{R}, \\ lk(K, L) \bmod p, & \text{if } \mathbb{K} = \mathbb{F}_p. \end{cases}$$

Thus in this case lk_ψ is equal to the linking number.

5. Example.

Let W be Whitehead's link. Then πW is generated by three generators x , y and z , and its presentation is described by

$$\pi W = \langle x, y, z \mid zyz^{-1}yzy^{-1}z^{-1}x^{-1}, xzxz^{-1}x^{-1}zy^{-1}z^{-1} \rangle.$$

Let K be an embedding of S^1 in $S^3 \setminus W$ which is parallel to one component, as illustrated in Figure 1.

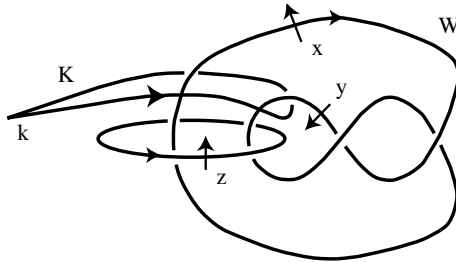


Figure 1. Whitehead's link.

We consider a representation ρ to $SL_2(\mathbb{F}_5)$ such that the image of each generator belongs to the conjugacy class of $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. This representation is described by

$$\rho : \pi W \rightarrow SL_2(\mathbb{F}_5)$$

$$\rho(x) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \rho(y) = \begin{pmatrix} 0 & 1 \\ 4 & 2 \end{pmatrix}, \quad \rho(z) = \begin{pmatrix} 4 & 1 \\ 1 & 3 \end{pmatrix}.$$

Such a representation onto $SL_2(\mathbb{F}_5)$ is unique up to inner automorphisms. We set $t = t_1 = t_2$ and choose the index x as a generator of V_ψ . Then,

$$\begin{aligned} Lk_\psi(K, W) &= \left\{ \psi(y^{-1}x) - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\} \left\{ \psi(x) - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}^{-1} \\ &= \left\{ \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\} \left\{ t \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}^{-1} \\ &= \frac{1}{(t-1)^2} \begin{pmatrix} t-1 & -1 \\ t-1 & -t \end{pmatrix}, \end{aligned}$$

and

$$lk_\psi(K, W) = \frac{4}{(t-1)^2}.$$

On the other hand, the linking number $lk(K, W)$ is 0. This demonstrates that K is a nontrivial embedding in $\mathbb{S}^3 \setminus W$.

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