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MINIMAL SUBMANIFOLDS OF KÄHLER–EINSTEIN MANIFOLDS WITH EQUAL KÄHLER ANGLES

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We consider $F : M \to N$ a minimal submanifold M of real dimension $2n$, immersed into a Kähler–Einstein manifold N of complex dimension $2n$, and scalar curvature R. We assume that $n \geq 2$ and F has equal Kähler angles. Our main result is to prove that, if $n = 2$ and $R \neq 0$, then F is either a complex submanifold or a Lagrangian submanifold. We also prove that, if $n \geq 3$, M is compact and orientable, then: (A) If $R < 0$, then F is complex or Lagrangian; (B) If $R = 0$, the Kähler angle must be constant. We also study pluriminimal submanifolds with equal Kähler angles, and prove that, if they are not complex submanifolds, N must be Ricci-flat and there is a natural parallel homothetic isomorphism between TM and the normal bundle.

1. Introduction.

Let (N, J, g) be a Kähler manifold of complex dimension $2n$ and $F : M \to N$ an immersed submanifold of real dimension $2n$. We denote by ω the Kähler form of N, $\omega(X, Y) = g(JX, Y)$. On M we take the induced metric $g_M = F^*g$. N is Kähler-Einstein if its Ricci tensor is a multiple of the metric, $\text{Ricci}^N = Rg$. At each point $p \in M$, we identify $F^*\omega$ with a skew-symmetric operator of T_pM by using the musical isomorphism with respect to g_M, namely $g_M(F^*\omega(X), Y) = F^*\omega(X, Y)$. We take its polar decomposition

$$F^*\omega = \widetilde{g}J_\omega \tag{1.1}$$

where $J_\omega : T_pM \to T_pM$ is a (in fact unique) partial isometry with the same kernel $\mathcal{K}_\omega$ as of F^*w, and where $\widetilde{g}$ is the positive semidefinite operator $\widetilde{g} = |F^*\omega| = \sqrt{-(F^*\omega)^2}$. It turns out that $J_\omega : \mathcal{K}_\omega^\perp \to \mathcal{K}_\omega^\perp$ defines a complex structure on $\mathcal{K}_\omega^\perp$, the orthogonal complement of $\mathcal{K}_\omega$ in T_pM. Moreover, it is g_M-orthogonal. If we denote by Ω^0_{2k} the largest open set of M where $F^*\omega$ has constant rank $2k$, $0 \leq k \leq n$, then $\mathcal{K}_\omega^\perp$ is a smooth sub-vector bundle of TM on Ω^0_{2k}. Moreover, $\widetilde{g}$ and J_ω are both smooth on these open sets. The tensor $\widetilde{g}$ is continuous on all M and locally Lipschitz, for the map $P \to |P|$ is Lipschitz in the space of normal operators. Let $\{X_\alpha, Y_\alpha\}_{1 \leq \alpha \leq n}$

be a g_M-orthonormal basis of T_pM, that diagonalizes $F^*\omega$ at p, that is

$$F^*\omega = \bigoplus_{0\leq\alpha\leq n} \begin{bmatrix} 0 & -\cos\theta_\alpha \\ \cos\theta_\alpha & 0 \end{bmatrix}, \tag{1.2}$$

where $\cos\theta_1 \geq \cos\theta_2 \geq \cdots \geq \cos\theta_n \geq 0$. The angles $\{\theta_\alpha\}_{1\leq\alpha\leq n}$ are the *Kähler angles* of F at p. Thus, $\forall\alpha$, $F^*\omega(X_\alpha) = \cos\theta_\alpha Y_\alpha$, $F^*\omega(Y_\alpha) = -\cos\theta_\alpha X_\alpha$ and if $k \geq 1$, where $2k$ is the rank of $F^*\omega$ at p, $J_\omega X_\alpha = Y_\alpha$ $\forall\alpha \leq k$. The Weyl's perturbation theorem applied to the eigenvalues of the symmetric operator $|F^*\omega|$ shows that, ordering the $\cos\theta_\alpha$ in the above way, the map $p \to \cos\theta_\alpha(p)$ is locally Lipschitz on M, for each α. A *complex direction* of F is a real two-plane P of T_pM such that $dF(P)$ is a complex line of $T_{F(p)}N$, i.e., $JdF(P) \subset dF(P)$. Similarly, P is said to be a *Lagrangian direction* of F if ω vanishes on $dF(P)$, that is, $JdF(P)\perp dF(P)$. The immersion F has no complex directions iff $\cos\theta_\alpha < 1$ $\forall\alpha$. M is a complex submanifold iff $\cos\theta_\alpha = 1$ $\forall\alpha$, and is a Lagrangian submanifold iff $\cos\theta_\alpha = 0$ $\forall\alpha$. We say that F has *equal Kähler angles* if $\theta_\alpha = \theta$ $\forall\alpha$. Complex and Lagrangian submanifolds are examples of such case. If F is a complex submanifold, then J_ω is the complex structure induced by J of N. The Kähler angles are some functions that at each point p of M measure the deviation of the tangent plane T_pM of M from a complex or a Lagrangian subspace of $T_{F(p)}N$. This concept was introduced by Chern and Wolfson [**Ch-W**] for oriented surfaces, namely $F^*\omega = \cos\theta \,\mathrm{Vol}_M$. This $\cos\theta$ may have negative values and is smooth on all M. In our definition, for $n = 1$, we demanded $\cos\theta \geq 0$, that is, it is the modulus of the $\cos\theta$ given for surfaces. This may make our $\cos\theta$ do not be smooth. We have chosen this definition, because in higher dimensions we do not have a preferential orientation assigned to the real planes span$\{X_\alpha, Y_\alpha\}$.

Our main aim is to find conditions for a minimal submanifold F to be Lagrangian or complex, or M to be a Kähler manifold with respect to J_ω. A first result in this direction is due to Wolfson, for the case $n = 1$:

Theorem 1.1 ([**W**]). *If M is a real compact surface and N is a complex Kähler-Einstein surface with $R < 0$, anf if F is minimal with no complex points, then F is Lagrangian.*

Some results of [**S-V**] are a generalization of the above theorem to higher dimensions. In this paper we study the case of equal Kähler angles. Let us denote by $\nabla_X dF(Y) = \nabla dF(X,Y)$ the second fundamental form of F. It is a symmetric tensor and takes values in the normal bundle $NM = (dF(TM))^\perp$. F is minimal iff $\mathrm{trace}_{g_M}\nabla dF = 0$. Let $(\ )^\perp$ denote the orthogonal projection of $F^{-1}TN$ onto the normal bundle. If F is an immersion with no complex directions at p and $\{X_\alpha, Y_\alpha\}$ diagonalizes $F^*\omega$ at p, then $\{dF(Z_\alpha)$, $dF(Z_{\overline{\alpha}})$, $(JdF(Z_\alpha))^\perp$, $(JdF(Z_{\overline{\alpha}}))^\perp\}$ constitutes a complex basis of $T^c_{F(p)}N$, where

$$(1.3)\qquad Z_\alpha = \frac{X_\alpha - iY_\alpha}{2} = \text{“}\alpha\text{”}, \qquad Z_{\overline{\alpha}} = \overline{Z_\alpha} = \frac{X_\alpha + iY_\alpha}{2} = \text{“}\overline{\alpha}\text{”}$$

are complex vectors of the complexified tangent space of M at p. We extend to the complexified vector bundles the Riemannian tensor metric g_M (sometimes denoted by $\langle,\rangle$), the curvature tensors of M and N, and any other tensors that will occur, always by $\mathbb{C}$-multilinearity. On M the Ricci tensor of N can be described by the following expression ([**S-V**]): For $U, V \in T_{F(p)}N$,

$$(1.4)\qquad \mathrm{Ricci}^N(U,V) = \sum_{1\leq\mu\leq n} \frac{4}{\sin^2\theta_\mu} R^N(U, JV, dF(\mu), (JdF(\overline{\mu}))^\perp),$$

where R^N denotes the Riemannian curvature tensor of N. An application of Codazzi equation to the above expression proves that, if N is Kähler-Einstein with $R \neq 0$, Theorem 1.1 can be generalized to any dimension for totally geodesic immersions without complex directions ([**S-V**]).

We can also obtain the same conclusion to “broadly-pluriminimal” immersions for $n = 2$, and N Kähler-Einstein with negative Ricci tensor ([**S-V**]). A minimal immersion F is said to be *broadly-pluriminimal*, if, for each $p \in \Omega^0_{2k}$, with $k \geq 1$, F is pluriharmonic with respect to any g_M-orthogonal complex structure $\widetilde{J} = J_\omega \oplus J'$ on T_pM where J' is any g_M-orthogonal complex structure of $\mathcal{K}_\omega$ at p, that is, $(\nabla dF)^{(1,1)} = 0$. The (1,1)-part of ∇dF is just given by $(\nabla dF)^{(1,1)}(X,Y) = \frac{1}{2}\big(\nabla dF(X,Y) + \nabla dF(\widetilde{J}X, \widetilde{J}Y)\big)$ $\forall X, Y \in T_pM$. If $\mathcal{K}_\omega = 0$, this means that F is pluriharmonic with respect to the almost complex structure J_ω (see for example [**O-V**]). In this case, we say that F is pluriminimal in the usual sense, or simply *pluriminimal*. Pluriharmonic immersions are obviously minimal. If F has equal Kähler angles, then only Ω^0_{2n} is considered, where $\mathcal{K}_\omega = 0$ and $\widetilde{J} = J_\omega$. Products of minimal real surfaces of Kähler surfaces, totally geodesic submanifolds, minimal Lagrangian submanifolds, and complex submanifolds are examples of broadly-pluriminimal submanifolds. In Sections 2 and 3, using an isomorphism Φ from the tangent bundle of M into the normal bundle, we will see that pluriminimal immersions with equal Kähler angles immersed into Kähler-Einstein manifolds, and that are not complex submanifolds, can be interpretated as submanifolds with “*torsion free*” normal bundle. Moreover, they have constant Kähler angle, and only exist on Ricci-flat manifolds. In this case, Φ defines a parallel homothetic isomorphism between TM and NM.

For a minimal immersion F with no complex directions we consider the locally Lipschitz map, symmetric on the Kähler angles,

$$(1.5)\qquad \kappa = \sum_{1\leq\alpha\leq n} \log\left(\frac{1+\cos\theta_\alpha}{1-\cos\theta_\alpha}\right).$$

This map is smooth on each Ω^0_{2k}, nonnegative, and vanishes at Lagrangian points. It is an increasing map on each $\cos\theta_\alpha$. In [**S-V**] we have given an expression for $\triangle\kappa$ at a point $p_0 \in \Omega^0_{2k}$, which we prove in the appendix of this paper, namely,

(1.6)

$$\begin{aligned}\triangle\kappa = 4i\sum_{\beta}&\mathrm{Ricci}^N(JdF(\beta), dF(\overline{\beta}))\\ &+\sum_{\beta,\mu}\frac{32}{\sin^2\theta_\mu}\mathrm{Im}\left(R^N(dF(\beta), dF(\mu), dF(\overline{\beta}), JdF(\overline{\mu}) + i\cos\theta_\mu dF(\overline{\mu}))\right)\\ &-\sum_{\beta,\mu,\rho}\frac{64(\cos\theta_\mu+\cos\theta_\rho)}{\sin^2\theta_\mu\sin^2\theta_\rho}\\ &\cdot\mathrm{Re}\left(g(\nabla_\beta dF(\mu), JdF(\overline{\rho}))g(\nabla_{\overline{\beta}}dF(\rho), JdF(\overline{\mu}))\right)\\ &+\sum_{\beta,\mu,\rho}\frac{32(\cos\theta_\rho-\cos\theta_\mu)}{\sin^2\theta_\mu\sin^2\theta_\rho}\\ &\cdot\left(|g(\nabla_\beta dF(\mu), JdF(\rho))|^2 + |g(\nabla_{\overline{\beta}}dF(\mu), JdF(\rho))|^2\right)\\ &+\sum_{\beta,\mu,\rho}\frac{32(\cos\theta_\mu+\cos\theta_\rho)}{\sin^2\theta_\mu}\left(|\langle\nabla_\beta\mu,\rho\rangle|^2 + |\langle\nabla_{\overline{\beta}}\mu,\rho\rangle|^2\right),\end{aligned}$$

where $\{X_\alpha, Y_\alpha\}_{1\leq\alpha\leq n}$ is a g_M-orthonormal local frame of M, with $Y_\alpha = J_\omega X_\alpha$ for $\alpha \leq k$, $\{X_\alpha, Y_\alpha\}_{\alpha\geq k+1}$ any g_M-orthonormal frame of $\mathcal{K}_\omega$, and which at p_0 diagonalizes $F^*\omega$. For F pluriminimal on Ω^0_{2n} and N Kähler-Einstein, we can get the following very simple final expression on Ω^0_{2n} ([**S-V**])

$$\triangle\kappa = -2R\left(\sum_{1\leq\beta\leq n}\cos\theta_\beta\right). \tag{1.7}$$

If F has equal Kähler angles, then the expression of $\triangle\kappa$ given in (1.6) can also be substantially simplified. Minimal surfaces with constant curvature and constant Kähler angle in complex space forms have been classified in [**O**]. Conditions on the curvature of M, N, and/or constant equal Kähler angles lead to some conclusions in our case as well, as we show in the theorems below. Henceforth, we assume N is Kähler-Einstein. The expression for $\triangle\kappa$, where the Ricci tensor of N appears, and the Weitzenböck formula for $F^*\omega$, leading to an integral equation involving the scalar curvature R, some trigonometric functions of the common Kähler angle, and the gradient of its cosine (Proposition 4.2), are our tools to obtain the results of this paper. In Section 4 we prove our main results, namely:

Theorem 1.2. *Let F be a minimal immersion of a manifold M, into a Kähler-Einstein manifold N, with equal Kähler angles.*

(i) *If $n = 2$ and $R \neq 0$, then F is either a complex or a Lagrangian submanifold.*
(ii) *If $n \geq 3$, M is compact, orientable, $R < 0$, then F is either a complex or Lagrangian submanifold.*
(iii) *If $n \geq 3$, M is compact, orientable, $R = 0$, then the common Kähler angle must be constant.*

The conclusions in (i) and (ii) give a generalization of Theorem 1.1 to higher dimensions and equal Kähler angles. The case $n = 2$ is the most special, because, in this dimension, immersions with equal Kähler angles have harmonic $F^*\omega$, as we will see in Section 3. The cases $n = 3$ or 4 also have special properties. If the angle is constant we may allow $R > 0$:

Theorem 1.3. *Let F be minimal with constant equal Kähler angles, M compact, orientable, and $R \neq 0$. Then, F is either a complex or a Lagrangian submanifold.*

Theorem 1.4. *Let F be minimal with equal Kähler angles, and M compact, orientable, with nonnegative isotropic scalar curvature. If $n = 2$, 3 or 4, then one of the following cases holds:*

(i) *M is a complex submanifold of N.*
(ii) *M is a Lagrangian submanifold of N.*
(iii) *$R = 0$ and $\cos\theta = constant \neq 0, 1$, J_ω is a complex integrable structure, with (M, J_ω, g_M) a Kähler manifold.*

For any $n \geq 1$, any R, and constant equal Kähler angle, (i), (ii) *or* (iii) *hold as well.*

This theorem can be applied, for instance, to flat minimal tori on Calabi-Yau manifolds, or to spheres or products of S^2 with S^2 or with flat tori minimaly immersed into Kähler-Einstein manifolds with positive scalar curvature.

2. The morphism Φ.

We consider the following morphism of vector bundles

$$\begin{array}{rccc} \Phi: & TM & \to & NM \\ & X & \to & (JdF(X))^{\bot}. \end{array}$$

We easily verify that

$$\Phi(X) = JdF(X) - dF(F^*\omega(X)). \tag{2.1}$$

Both TM and NM are real vector bundles of the same dimension $2n$. F has no complex directions iff Φ is an isomorphism. In fact $\Phi(X) = 0$, iff $JdF(X) = dF(Y)$ for some Y, i.e., span $\{X, Y = \text{“}JX\text{”}\}$ is a complex direction of F. Assume there are no complex directions. Then,

$$\hat{g}(X,Y) = g_M(X,Y) - g_M(F^*\omega(X), F^*\omega(Y)) \tag{2.2}$$

defines a Riemannian metric on M. With this metric, $\Phi : (TM, \hat{g}) \to (NM, g)$ is an isomorphism of Riemannian vector bundles. Let us denote by ∇, $\hat{\nabla}$, $\nabla^{\perp}$, and ∇', respectively, the Levi-Civita connection of (M, g_M), the Levi-Civita connection of $(M, \hat{g})$, the usual connection of NM induced by the Levi-Civita connection of N, and the connection on TM that makes the isomorphism Φ parallel, namely $\nabla' = \Phi^{-1*}\nabla^{\perp}$. We will also denote by ∇ the Levi-Civita connection of N and the induced connection on $F^{-1}TN$, as well. Thus, if U is a smooth section of $NM \subset F^{-1}TN$, and X, Y are smooth vector fields on M, we have

$$\nabla^{\perp}_X U = (\nabla_X U)^{\perp} \qquad \Phi(\nabla'_X Y) = \nabla^{\perp}_X(\Phi(Y)).$$

The connections ∇ and $\hat{\nabla}$ have no torsion, because they are Levi-Civita, but ∇' may have nonzero torsion T'. Since both $\hat{\nabla}$ and ∇' are Riemannian connections of TM for the same Riemannian metric $\hat{g}$, then $T' = 0$ iff $\hat{\nabla} = \nabla'$ iff Φ is parallel. Note that, if F is Lagrangian, then $\Phi(X) = JdF(X) \in NM$, $J(NM) = dF(TM)$, and $\hat{g} = g_M$, $\hat{\nabla} = \nabla$. Therefore, $\nabla_X\Phi\ (Y) = \big(J\nabla_X dF(Y)\big)^{\perp} = 0$, that is, Φ is parallel, and so $\nabla' = \nabla$, as well. In the next section (Corollary 3.2), we will see a converse of this. We extend $\Phi : TM^c \to NM^c$ to the complexified spaces by $\mathbb{C}$-linearity.

Lemma 2.1. *If $\{X_\alpha, Y_\alpha\}$ is a diagonalizing g_M-orthonormal basis of $F^*\omega$ at p, then at p, and for each α, β*

$$\begin{aligned} \Phi(T'(Z_\alpha, Z_{\overline{\beta}})) &= i(\cos\theta_\alpha + \cos\theta_\beta)\nabla_{Z_\alpha} dF(Z_{\overline{\beta}}) \\ \Phi(T'(Z_\alpha, Z_\beta)) &= i(\cos\theta_\alpha - \cos\theta_\beta)\nabla_{Z_\alpha} dF(Z_\beta). \end{aligned}$$

Proof.

$$\begin{aligned} \Phi(\nabla'_X Y) &= \nabla^{\perp}_X(\Phi(Y)) = \big(\nabla_X(\Phi(Y))\big)^{\perp} \\ &= \big(\nabla_X(JdF(Y) - dF(F^*\omega(Y)))\big)^{\perp} \\ &= \big(J\nabla_X dF(Y) + JdF(\nabla_X Y) - \nabla_X dF(F^*\omega(Y))\big)^{\perp}. \end{aligned}$$

Therefore, using the symmetry of the ∇dF and the fact that ∇ is torsionless,

$$\begin{aligned} \Phi(T'(X,Y)) &= \Phi(\nabla'_X Y - \nabla'_Y X - [X,Y]) \\ &= -\nabla_X dF(F^*\omega(Y)) + \nabla_Y dF(F^*\omega(X)). \end{aligned} \tag{2.3}$$

The lemma follows now immediately. ☐

For each $U \in NM_p$, let us denote by $A^U : T_pM \to T_pM$ the symmetric operator $g_M(A^U(X), Y) = g(\nabla dF(X,Y), U)$. From Lemma 2.1 and (2.3) we have:

Proposition 2.1. *If F is an immersion without complex directions, then:*
(i) *Φ is parallel iff $F^*\omega$ anti-commutes with A^U, $\forall U \in NM$.*
(ii) *If F has equal Kähler angles, on Ω^0_{2n}, T' is of type $(1,1)$ with respect to J_ω.*
(iii) *On Ω^0_{2n}, F is pluriminimal iff T' is of type $(2,0)+(0,2)$ with respect to J_ω.*

Remark 1. If we call ω_{NM} the restriction of the Kähler form ω to the normal bundle NM, we see that, if $\{X_\alpha, Y_\alpha\}$ is a diagonalizing g_M-orthonormal basis of $F^*\omega$ at a point p, then $\left\{U_\alpha = \Phi\left(\frac{Y_\alpha}{\sin\theta_\alpha}\right), V_\alpha = \Phi\left(\frac{X_\alpha}{\sin\theta_\alpha}\right)\right\}$ is a diagonalizing g-orthonormal basis of ω_{NM}. Moreover, NM has the same Kähler angles as F. Let J_{NM} denote the complex structure on NM defined by this basis, that is, the one that comes from the polar decomposition of ω_{NM}. Then, $\Phi J_\omega = -J_{NM}\Phi$.

We should also remark the following:

Proposition 2.2. *If F is an immersion with parallel 2-form $F^*\omega$, then the Kähler angles are constant and, in particular, $M = \Omega^0_{2k}$ for some k. In this case, considering TM with the Levi-Civita connection ∇, $\mathcal{K}_\omega$ and $\mathcal{K}^\perp_\omega$ are parallel sub-vector bundles of TM, and $J_\omega \in C^\infty(\mathcal{K}^{\perp *}_\omega \otimes \mathcal{K}^\perp_\omega)$, $\widetilde{g}$, $\hat{g} \in C^\infty(\bigodot^2 T^*M)$ are parallel sections. Furthermore, $(X,Y,Z) \to g(\nabla_Z dF(X), JdF(Y))$ is symmetric on TM, and, if F has no complex directions, $\hat{\nabla} = \nabla$. Moreover, if $\cos\theta_{\alpha_1} > \cdots > \cos\theta_{\alpha_r}$ are the distinct eigenvalues of $F^*\omega$, the corresponding eigenspaces E_{α_t} define a smooth integrable distribution of TM whose integral submanifolds are totally geodesic submanifolds of M. The integral submanifolds of E_{α_r} are isotropic in N if $\cos\theta_{\alpha_r} = 0$, and the ones of E_{α_1} are complex submanifolds of N if $\cos\theta_{\alpha_1} = 1$. The other ones are Kähler manifolds with respect to J_ω, and F restricted to each one of them is an immersion of constant equal Kähler angles θ_{α_t} with respect to J.*

Proof. If X, Y are smooth vector fields on M and $Z \in T_pM$, an elementary computation gives

$$\nabla_Z F^*\omega(X,Y) = -g(\nabla_Z dF(X), JdF(Y)) + g(\nabla_Z dF(Y), JdF(X)), \tag{2.4}$$

which proves the symmetry of $(X,Y,Z) \to g(\nabla_Z dF(X), JdF(Y))$. From (2.2) we see that $\hat{g}$ is parallel. Consequently, away from complex directions, $\nabla = \hat{\nabla}$. If we parallel transport a diagonalizing orthonormal basis $\{X_\alpha, Y_\alpha\}$ of $F^*\omega$ at p_0 along geodesics, on a neighbourhood of p_0, since $F^*\omega$ is parallel we get a diagonalizing orthonormal frame on a whole neighbourhood with the property $\nabla X_\alpha(p_0) = \nabla Y_\alpha(p_0) = 0$. It also follows that $\cos\theta_\alpha$ remains constant along geodesics, so it is constant, and $J_\omega(X_\alpha) = Y_\alpha$ on a neighbourhood of p_0, with $\nabla J_\omega = 0$ at p_0, and so J_ω is parallel. Similarly we see that $\widetilde{g}$ is parallel. If we extend $F^*\omega$ to the complexified tangent space $T^c_{p_0}M$, then $F^*\omega(Z_\alpha) = i\cos\theta_\alpha Z_\alpha$, and $F^*\omega(Z_{\overline{\alpha}}) = -i\cos\theta_\alpha Z_{\overline{\alpha}}$. Obviously, the corresponding eigenspaces of $F^*\omega$, are parallel sub-vector bundles of T^cM. □

3. Immersions with equal Kähler angles.

If F has equal Kähler angles, then

$$F^*\omega = \cos\theta\, J_\omega \quad \text{and} \quad \hat{g} = \sin^2\theta\, g_M,$$

with $\cos\theta$ a locally Lipschitz map on M, smooth on the open set where it does not vanish, and $\Omega^0_{2k} = \emptyset\ \forall k \neq 0, n$. Note that $\sin^2\theta$ and $\cos^2\theta$ are smooth on all M. The set $\mathcal{L} = \cos\theta^{-1}(\{0\})$ is the set of Lagrangian points, for, at these points, the tangent space of M is a Lagrangian subspace of the tangent space of N. Similarly, we say that $\mathcal{C} = \cos\theta^{-1}(\{1\})$ is the set of complex points. If M has a fixed orientation, we can distinguish the set of *well-oriented* complex points from the *twisted* complex points, according J_ω defines the same or the opposite orientation. On the open set $\Omega^0_{2n} = \cos\theta^{-1}(\mathbb{R} \sim \{0\}) = M \sim \mathcal{L}$, J_ω defines a smooth almost complex structure g_M-orthogonal. On the open set $\cos\theta^{-1}(\mathbb{R} \sim \{1\}) = M \sim \mathcal{C}$, $\hat{g}$ is a smooth metric conformally equivalent to g_M. Thus, if $n \geq 2$, $\hat{\nabla} = \nabla$ iff θ is constant. Since the Kähler angles are equal, any smooth local orthonormal frame of the type $\{X_\alpha, Y_\alpha = J_\omega X_\alpha\}$ diagonalizes $F^*\omega$ on the whole set where it is defined. Differentiating $F^*\omega = \cos\theta J_\omega$, we get $\nabla_X F^*\omega = d\cos\theta(X) J_\omega +$ $\cos\theta \nabla_X J_\omega$, with J_ω orthogonal to $\nabla_X J_\omega$ with respect to the Hilbert-Schmidt inner product (because $\|J_\omega\|^2 = 2n$ is constant). Hence, considering $F^*\omega$ an operator on TM, on $\Omega^0_{2n} \cup \Omega^0_0$

$$\|\nabla F^*\omega\|^2 = 2n\|\nabla\cos\theta\|^2 + \cos^2\theta\|\nabla J_\omega\|^2. \tag{3.1}$$

Then, on Ω^0_{2n}, $\nabla F^*\omega = 0$ iff $\nabla J_\omega = 0$ and θ is constant. Note that $\|\nabla F^*\omega\|^2$, considering $F^*\omega$ an operator on TM, is twice the square norm when considering $F^*\omega$ a 2-form. From (2.3) we get, on $M \sim \mathcal{C}$,

$$\Phi(T'(X,Y)) = 2\cos\theta(\nabla dF)^{(1,1)}(J_\omega X, Y). \tag{3.2}$$

The right-hand side of (3.2) is defined to be zero at a Lagrangian point. Consequentely

Proposition 3.1. *If F is an immersion with equal Kähler angles and without complex points, then $T' = 0$, that is, $\nabla' = \hat{\nabla}$ iff Φ is parallel iff F is*

Lagrangian or pluriminimal. In particular, if F is minimal, Φ is parallel iff F is broadly-pluriminimal.

This also holds for $n = 1$, where pluriminimality condition coincides with minimality. In this case $\pm J_\omega$ is the natural (local) complex structure of the surface (the sign depends on the (local) chosen orientation). Let $\mathrm{Re}\,(u+iv) = u$, for $u, v \in NM$.

Proposition 3.2. *If F is any immersion with equal Kähler angles, then, away from complex and Lagrangian points,*

$$\Phi\left(\frac{1-n}{4}\nabla \log \sin^2\theta\right)$$
$$= \frac{4\cos\theta}{\sin^2\theta}\mathrm{Re}\left(i\sum_{\beta,\mu}\big(g(\nabla_{\overline{\mu}}dF(\mu), JdF(\beta)) - g(\nabla_{\overline{\mu}}dF(\beta), JdF(\mu))\big)\Phi(\overline{\beta})\right),$$

where $\nabla \log \sin^2\theta$ is the gradient with respect to g_M.

If F is a complex submanifold on a open set, then J_ω is the induced complex structure on M and ∇dF is of type $(2,0)$. Applying Proposition 2.2 on Ω_0^0, and Proposition 3.2 on open sets without complex and Lagrangian points, and noting that $\{\Phi(\beta), \Phi(\overline{\beta}) = \overline{\Phi(\beta)}\}_{1\leq\beta\leq n}$ multiplied by $\frac{\sqrt{2}}{\sin\theta}$ constitutes an unitary basis of NM^c, we immediately conclude:

Corollary 3.1. *If F is an immersion with equal Kähler angles, and $n \geq 2$, then θ is constant iff*

$$\sum_{\mu} g(\nabla_{\overline{\mu}}dF(\mu), JdF(\beta)) = \sum_{\mu} g(\nabla_{\overline{\mu}}dF(\beta), JdF(\mu)) \qquad \forall\beta. \tag{3.3}$$

Note that (3.3) is a sort of symmetry property, and the first term is just $\frac{n}{2}g(H, JdF(\beta))$, where $H = \frac{1}{2n}\mathrm{trace}_{g_M}\nabla dF = \frac{2}{n}\sum_\mu \nabla dF(\overline{\mu}, \mu)$ is the mean curvature of F.

Theorem 3.1. *If $n \geq 2$ and F is a pluriminimal immersion with equal Kähler angles then $\cos\theta =$ constant. Moreover, if it is not a complex submanifold, then $\nabla = \hat{\nabla} = \nabla'$, and N must be Ricci-flat. In particular, Φ defines a parallel homothetic isomorphism from (TM, g_M) onto (NM, g).*

Proof. On a neighbourhood of a non-complex point, from Proposition 3.1, $\hat{\nabla} = \nabla'$, and from Corollary 3.1, $\cos\theta$ is constant. Then $\hat{\nabla} = \nabla$, as well. So if F is not a complex submanifold, it has no complex points anywhere. Finally, (1.7) for pluriminimal immersions with $\kappa = constant$ gives $R = 0$. □

The above theorem and Proposition 3.1 lead to:

Corollary 3.2. *If F is a minimal immersion with equal Kähler angles, without complex points, $n \geq 2$, and $R \neq 0$, then F is Lagrangian iff Φ is parallel.*

To prove Proposition 3.2 we will need to relate the three connections of M, ∇, $\hat{\nabla}$ and ∇'. Let $\{e_1,\ldots,e_{2n}\} = \{X_\mu, Y_\mu = J_\omega X_\mu\}_{1\leq\mu\leq n}$ be a local g_M-orthonormal frame away from the Lagrangian and complex set, and $\partial_1,\ldots,\partial_{2n}$ a local frame of M defined by a coordinate chart. Set $g_{ij} = g_M(\partial_i,\partial_j)$, $\hat{g}_{ij} = \hat{g}(\partial_i,\partial_j) = \sin^2\theta g_{ij}$, and $e_s = \sum_i \lambda_{si}\partial_i$. The Christofel symbols are given by $2\hat{\Gamma}^k_{ij} = \sum_s \hat{g}^{ks}(\partial_i\hat{g}_{sj} + \partial_j\hat{g}_{is} - \partial_s\hat{g}_{ij}) = \delta_{kj}\partial_i\log\sin^2\theta + \delta_{ki}\partial_j\log\sin^2\theta - \sum_s g^{ks}g_{ij}\partial_s\log\sin^2\theta + 2\Gamma^k_{ij}$. Hence

$$\begin{aligned}\hat{\nabla}_{\partial_i}\partial_j - \nabla_{\partial_i}\partial_j &= \sum_k(\hat{\Gamma}^k_{ij} - \Gamma^k_{ij})\partial_k\\ &= \frac{1}{2}\big(\partial_i(\log\sin^2\theta)\partial_j + \partial_j(\log\sin^2\theta)\partial_i - g_{ij}\nabla(\log\sin^2\theta)\big).\end{aligned}$$

Since $\sum_{ij} g_{ij}\lambda_{si}\lambda_{sj} = 1$, $\sum_s \hat{\nabla}_{e_s}e_s - \nabla_{e_s}e_s = \sum_{sij}\lambda_{si}\lambda_{sj}(\hat{\nabla}_{\partial_i}\partial_j - \nabla_{\partial_i}\partial_j) = (1-n)\nabla\log\sin^2\theta$. Therefore,

$$\begin{aligned}(3.4)\quad \sum_\mu \hat{\nabla}_{\overline{\mu}}\mu - \nabla_{\overline{\mu}}\mu &= \frac{1}{4}\sum_\mu\Big(\hat{\nabla}_{X_\mu}X_\mu + \hat{\nabla}_{Y_\mu}Y_\mu - \nabla_{X_\mu}X_\mu - \nabla_{Y_\mu}Y_\mu\Big)\\ &\quad - i\big(\hat{\nabla}_{X_\mu}Y_\mu - \hat{\nabla}_{Y_\mu}X_\mu - \nabla_{X_\mu}Y_\mu + \nabla_{Y_\mu}X_\mu\big)\\ &= \frac{1}{4}\sum_s(\hat{\nabla}_{e_s}e_s - \nabla_{e_s}e_s) + \frac{i}{4}\sum_\mu\big([Y_\mu,X_\mu] - [Y_\mu,X_\mu]\big)\\ &= \frac{(1-n)}{4}\nabla\log\sin^2\theta.\end{aligned}$$

Set $S'(X,Y) = \nabla'_XY - \hat{\nabla}_XY$. Then $S'(X,Y) - S'(Y,X) = T'(X,Y)$. Similarly we get

$$(3.5)\qquad \textstyle\sum_\mu \nabla'_{\overline{\mu}}\mu - \hat{\nabla}_{\overline{\mu}}\mu = \frac{1}{4}\mathrm{trace}_{g_M}S' - \frac{i}{4}\sum_\mu T'(X_\mu,Y_\mu).$$

Lemma 3.1. $\forall X\in T_pM,\quad \sum_i \hat{g}(S'(e_i,e_i),X) = -\sum_i \hat{g}(T'(e_i,X),e_i)$.

Proof. We may assume that the local referencial ∂_i is $\hat{g}$-orthonormal at a fixed point p_0. On a neighbourhood of p_0, we define ${\Gamma'}^k_{ij}$ and ${S'}^k_{ij}$ as

$$\nabla'_{\partial_i}\partial_j = \sum_k {\Gamma'}^k_{ij}\partial_k,\quad S'(\partial_i,\partial_j) = \sum_k {S'}^k_{ij}\partial_k = \sum_k({\Gamma'}^k_{ij} - \hat{\Gamma}^k_{ij})\partial_k.$$

Then ${T'_{ij}}^k = {\Gamma'}^k_{ij} - {\Gamma'}^k_{ji}$, and, at p_0, ${\Gamma'}^k_{ij} = \hat{g}(\nabla'_{\partial_i}\partial_j,\partial_k)$, ${S'}^k_{ij} = \hat{g}(S'(\partial_i,\partial_j),\partial_k) = {\Gamma'}^k_{ij} - \hat{\Gamma}^k_{ij}$. ∇' is a Riemannian connection with respect to $\hat{g}$. Then

$$\partial_i\hat{g}_{jk}(p_0) = \hat{g}(\nabla'_{\partial_i}\partial_j,\partial_k) + \hat{g}(\partial_j,\nabla'_{\partial_i}\partial_k) = {\Gamma'}^k_{ij} + {\Gamma'}^j_{ik}.$$

Hence, at p_0

$$\begin{aligned}
2\hat{\Gamma}^k_{ij} &= \sum_s \hat{g}^{ks}(\partial_i \hat{g}_{sj} + \partial_j \hat{g}_{is} - \partial_s \hat{g}_{ij}) \\
&= {\Gamma'}^j_{ik} + {\Gamma'}^k_{ij} + {\Gamma'}^k_{ji} + {\Gamma'}^i_{jk} - {\Gamma'}^j_{ki} - {\Gamma'}^i_{kj} \\
&= ({\Gamma'}^k_{ij} + {\Gamma'}^k_{ji}) + ({\Gamma'}^j_{ik} - {\Gamma'}^j_{ki}) + ({\Gamma'}^i_{jk} - {\Gamma'}^i_{kj}) \\
&= ({\Gamma'}^k_{ij} + {\Gamma'}^k_{ji}) + {T'}^j_{ik} + {T'}^i_{jk}.
\end{aligned}$$

But ${\Gamma'}^k_{ij} + {\Gamma'}^k_{ji} = 2{\Gamma'}^k_{ij} + ({\Gamma'}^k_{ji} - {\Gamma'}^k_{ij}) = 2{\Gamma'}^k_{ij} + {T'}^k_{ji}$. Thus

$${S'}^k_{ij} = {\Gamma'}^k_{ij} - \hat{\Gamma}^k_{ij} = \frac{1}{2}({T'}^k_{ij} - {T'}^j_{ik} + {T'}^i_{kj}).$$

That is, at p_0, $\hat{g}(S'(\partial_i, \partial_j), \partial_k) = \frac{1}{2}\big(\hat{g}(T'(\partial_i, \partial_j), \partial_k) - \hat{g}(T'(\partial_i, \partial_k), \partial_j) + \hat{g}(T'(\partial_k, \partial_j), \partial_i)\big)$. We may assume that, at p_0, $\partial_i(p_0) = \frac{e_i}{\sin\theta}$, leading to the Lemma. $\square$

Proof of Proposition 3.2. Following the proof of Lemma 2.1, $\Phi(\nabla'_X\mu - \nabla_X\mu) = ((J - i\cos\theta)\nabla_X dF(\mu))^{\perp}$. Hence, from (3.4),

$$\begin{aligned}
\Phi\left(\frac{(1-n)}{4}\nabla \log \sin^2\theta\right) &= \Phi\left(\sum_\mu \hat{\nabla}_{\overline{\mu}}\mu - \nabla_{\overline{\mu}}\mu\right) \\
&= \left((J - i\cos\theta)\frac{nH}{2}\right)^{\perp} - \sum_\mu \Phi(\nabla'_{\overline{\mu}}\mu - \hat{\nabla}_{\overline{\mu}}\mu).
\end{aligned}$$

But, from (3.5), $\sum_\mu \Phi(\nabla'_{\overline{\mu}}\mu - \hat{\nabla}_{\overline{\mu}}\mu) = \frac{1}{4}\Phi(\mathrm{trace}_{g_M} S') - \frac{i}{4}\Phi(\sum_\mu T'(X_\mu, Y_\mu))$. The skew-symmetry of T' and (3.2) implies that

$$\Phi\left(\sum_\mu T'(X_\mu, Y_\mu)\right) = -2i\sum_\mu \Phi(T'(\mu, \overline{\mu})) = 4\cos\theta \nabla_\mu dF(\overline{\mu}) = 2n\cos\theta H.$$

Thus, $\sum_\mu \Phi(\nabla'_{\overline{\mu}}\mu - \hat{\nabla}_{\overline{\mu}}\mu) = \frac{1}{4}\Phi(\mathrm{trace}_{g_M} S') - \frac{ni}{2}\cos\theta H$. Therefore,

$$\Phi\left(\frac{(1-n)}{4}\nabla \log \sin^2\theta\right) = \frac{1}{4}\Big(2n(JH)^{\perp} - \Phi(\mathrm{Trace}_{g_M} S')\Big). \tag{3.6}$$

Using Lemma 3.1, (3.2), and $\Phi(\mu) = JdF(\mu) - i\cos\theta dF(\mu)$, we have

$$
\begin{aligned}
&\Phi(\mathrm{Trace}_{g_M} S') \\
&= \sum_{j,k} \hat{g}\left(S'(e_j, e_j), \frac{e_k}{\sin\theta}\right) \Phi\left(\frac{e_k}{\sin\theta}\right) \\
&= \sum_{j,k} -\hat{g}\left(T'\left(e_j, \frac{e_k}{\sin\theta}\right), e_j\right) \Phi\left(\frac{e_k}{\sin\theta}\right) \\
&= \frac{-4}{\sin^2\theta} \sum_{\mu,\beta} \big((\hat{g}(T'(\mu,\beta),\overline{\mu}) + \hat{g}(T'(\overline{\mu},\beta),\mu))\Phi(\overline{\beta}) \\
&\quad + (\hat{g}(T'(\mu,\overline{\beta}),\overline{\mu}) + \hat{g}(T'(\overline{\mu},\overline{\beta}),\mu))\Phi(\beta)\big) \\
&= -\frac{4}{\sin^2\theta} \sum_{\mu,\beta} \big(g(\Phi(T'(\overline{\mu},\beta)),\Phi(\mu))\Phi(\overline{\beta}) + g(\Phi(T'(\mu,\overline{\beta})),\Phi(\overline{\mu}))\Phi(\beta)\big) \\
&= \frac{8i\cos\theta}{\sin^2\theta} \sum_{\mu,\beta} \big(g(\nabla_{\overline{\mu}} dF(\beta), JdF(\mu))\Phi(\overline{\beta}) - g(\nabla_{\mu} dF(\overline{\beta}), JdF(\overline{\mu}))\Phi(\beta)\big).
\end{aligned}
$$

Writing $(JH)^\perp$ in terms of $\Phi(\beta)$ and $\Phi(\overline{\beta})$,

$$
\begin{aligned}
2n(JH)^\perp &= \sum_{\beta} \frac{4n}{\sin^2\theta}\big(g(JH,\Phi(\beta))\Phi(\overline{\beta}) + g(JH,\Phi(\overline{\beta}))\Phi(\beta)\big) \\
&= \sum_{\beta,\mu} \frac{8i\cos\theta}{\sin^2\theta}\big(g(\nabla_{\overline{\mu}} dF(\mu), JdF(\beta))\Phi(\overline{\beta}) \\
&\quad - g(\nabla_{\overline{\mu}} dF(\mu), JdF(\overline{\beta}))\Phi(\beta)\big),
\end{aligned}
$$

and substituing these equations into (3.6), we prove Proposition 3.2. □

3.1. The Weitzenböck formula for $F^*\omega$. For simplicity let us use the notation

$$g_X YZ = g(\nabla_X dF(Y), JdF(Z)).$$

We also observe that, from

$$\forall\mu \quad \frac{i}{2}\cos\theta = F^*\omega(\mu,\overline{\mu}), \tag{3.7}$$

valid on an open set, and from (2.4), we obtain $\forall\mu$

$$
\begin{aligned}
\frac{i}{2} d\cos\theta(X) &= d(F^*\omega(\mu,\overline{\mu}))(X) \\
&= \nabla_X F^*\omega(\mu,\overline{\mu}) + F^*\omega(\nabla_X\mu,\overline{\mu}) + F^*\omega(\mu,\nabla_X\overline{\mu}) \\
&= -g_X\mu\overline{\mu} + g_X\overline{\mu}\mu + 2(\langle\nabla_X\mu,\overline{\mu}\rangle + \langle\nabla_X\overline{\mu},\mu\rangle)F^*\omega(\mu,\overline{\mu}) \\
&= -g_X\mu\overline{\mu} + g_X\overline{\mu}\mu \qquad \text{(no sumation over } \mu).
\end{aligned} \tag{3.8}
$$

Then (3.3) is equivalent to $g(\nabla_X dF(\mu), JdF(\overline{\mu})) = g(\nabla_X dF(\overline{\mu}), JdF(\mu))$, $\forall \mu$ (or some μ). From $J_\omega Z_\alpha = iZ_\alpha$, $J_\omega Z_{\overline{\alpha}} = -iZ_{\overline{\alpha}}$ and the fact that J_ω is g_M-orthogonal, we get, on Ω^0_{2n}, $\forall \alpha, \beta$, and $\forall v \in TM$

$$\langle \nabla_v J_\omega(\alpha), \beta\rangle = 2i\langle \nabla_v \alpha, \beta\rangle, \quad \langle \nabla_v J_\omega(\alpha), \overline{\beta}\rangle = 0. \tag{3.9}$$

Recall that, if ξ is a $r+1$-form on M, $r \geq 0$, with values on a vector bundle E over M with a connection ∇^E, then $\delta\xi$, the divergence of ξ, is the r-form on M with values on E given by

$$\delta\xi(u_1, \ldots, u_r) = -\sum_s \nabla^E_{e_s} \xi(e_s, u_1, \ldots, u_r),$$

where $e_1, \ldots, e_m$ is an orthonormal basis of T_pM, $u_i \in T_pM$, and $\nabla^E \xi$ is the covariant derivative of ξ on $\bigwedge^{r+1} T^*M \otimes E$. Thus, δ is the formal adjoint of d on forms (cf. [**E-L**]). Note that $\delta F^*\omega(X) = \langle \delta F^*\omega, X\rangle$, $\forall X \in T_pM$, considering on the left-hand side $F^*\omega$ a (closed) 2-form on M and on the right-hand side an endomorphism of TM.

Proposition 3.3. *Let F be an immersion with equal Kähler angles and $\nabla \cos\theta$ denote the gradient with respect to g_M. On Ω^0_{2n}, and considering $F^*\omega$ an endomorphism of TM.*

$$\delta F^*\omega = (n-2)J_\omega(\nabla \cos\theta), \quad \cos\theta(\delta J_\omega) = (n-1)J_\omega(\nabla\cos\theta).$$

Thus,

(i) *For $n = 1$, $\delta J_\omega = 0$ (obviously!), and $\delta F^*\omega = 0$ iff θ is constant.*
(ii) *For $n = 2$, $\delta F^*\omega = 0$. Moreover, $\delta J_\omega = 0$ iff θ is constant.*
(iii) *For $n \neq 1, 2$, $\delta F^*\omega = 0$ iff $\delta J_\omega = 0$ iff θ is constant.*

In particular, if $n \geq 2$ and $(\Omega^0_{2n}, J_\omega, g)$ is Kähler, then θ is constant.

Proof. Considering $F^*\omega$ a 2-form on M, using the symmetry of ∇dF and (2.4), if $X \in T_pM$,

$$\begin{aligned}
\delta(F^*\omega)(X) &= \sum_\mu -2\nabla_\mu F^*\omega(\overline{\mu}, X) - 2\nabla_{\overline{\mu}} F^*\omega(\mu, X)\\
&= \sum_\mu 2g_\mu \overline{\mu} X - 2g_\mu X\overline{\mu} + 2g_{\overline{\mu}}\mu X - 2g_{\overline{\mu}} X\mu\\
&= 2\sum_\mu (-g_X \mu\overline{\mu} + g_X \overline{\mu}\mu) - 4\sum_\mu (g_{\overline{\mu}} X\mu - g_{\overline{\mu}}\mu X).
\end{aligned}$$

From (3.8), $\frac{ni}{2} d\cos\theta(X) = \sum_\mu -g_X\mu\overline{\mu} + g_X\overline{\mu}\mu$. Therefore,

$$\delta(F^*\omega)(X) = ni\, d\cos\theta(X) - 4\sum_\mu \nabla_{\overline{\mu}} F^*\omega(\mu, X). \tag{3.10}$$

Since $F^*\omega$ is of type $(1,1)$ with respect to J_ω, and $\forall Z \in T^c_pM$, $\forall \mu,\beta$, $\langle \nabla_Z\beta,\mu\rangle = -\langle \beta, \nabla_Z\mu\rangle$, we get using (3.9)

$$\begin{aligned}\text{(3.11)}\qquad \nabla_Z F^*\omega(\mu,\beta) &= d(F^*\omega(\mu,\beta))(Z) - F^*\omega(\nabla_Z\mu,\beta) - F^*\omega(\mu,\nabla_Z\beta)\\ &= 2i\cos\theta\langle\nabla_Z\mu,\beta\rangle = \cos\theta\langle\nabla_Z J_\omega(\mu),\beta\rangle.\end{aligned}$$

Note that, since $J_\omega^2 = -\mathrm{Id}$, $\nabla_X J_\omega(J_\omega Y) = -J_\omega(\nabla_X J_\omega(Y))$, $\forall X, Y \in T_pM$. So

$$\begin{aligned} 4\sum_\mu \nabla_{\overline{\mu}} J_\omega(\mu) &= \sum_\mu \nabla_{X_\mu}J_\omega(X_\mu) + \nabla_{Y_\mu}J_\omega(Y_\mu) + i\nabla_{Y_\mu}J_\omega(X_\mu) - i\nabla_{X_\mu}J_\omega(Y_\mu)\\ &= -\delta J_\omega + i\sum_\mu(-\nabla_{X_\mu}J_\omega(J_\omega X_\mu) - \nabla_{Y_\mu}J_\omega(J_\omega Y_\mu))\\ &= -\big(\delta J_\omega + iJ_\omega(\delta J_\omega)\big).\end{aligned}$$

Hence, from (3.11), and since J_ω is g_M-orthogonal, $\forall\beta$

$$\sum_\mu \nabla_{\overline{\mu}} F^*\omega(\mu,\beta) = -\frac{\cos\theta}{4}\langle\delta J_\omega + iJ_\omega(\delta J_\omega),\beta\rangle = -\frac{\cos\theta}{2}\langle\delta J_\omega,\beta\rangle.$$

Moreover, $id\cos\theta(\beta) = d\cos\theta(J_\omega\beta) = \langle\nabla\cos\theta, J_\omega\beta\rangle = -\langle J_\omega(\nabla\cos\theta),\beta\rangle$. From (3.10), $\delta F^*\omega(\beta) = \langle -nJ_\omega(\nabla\cos\theta) + 2\cos\theta\,\delta J_\omega,\ \beta\rangle$. Thus, if we consider $F^*\omega$ an endomorphism of TM, and since $\langle,\rangle$, J_ω, and $F^*\omega$ are real operators,

$$\text{(3.12)}\qquad \delta F^*\omega = -nJ_\omega(\nabla\cos\theta) + 2\cos\theta\,\delta J_\omega.$$

On the other hand, $F^*\omega = \cos\theta J_\omega$. Then, an elementary computation gives

$$\text{(3.13)}\qquad \delta F^*\omega = -J_\omega(\nabla\cos\theta) + \cos\theta\,\delta J_\omega.$$

Comparing (3.12) with (3.13) we get the Proposition. □

If we apply the Weitzenböck formula to the 2-form $F^*\omega$, for an immersion $F: M\to N$ we get (see e.g., [**E-L**] (1.32))

$$\text{(3.14)}\qquad \frac{1}{2}\triangle\|F^*\omega\|^2 = -\langle\triangle F^*\omega, F^*\omega\rangle + \|\nabla F^*\omega\|^2 + \langle SF^*\omega, F^*\omega\rangle,$$

where $\langle,\rangle$ denotes the Hilbert-Schmidt inner product for 2-forms, and S is the Ricci operator of $\bigwedge^2 T^*M$. We note that we use the the sign convention $\triangle\phi = +\mathrm{Trace}_{g_M}\mathrm{Hess}\,\phi$, for ϕ a smooth real map on M. This sign is opposite to the one of [**E-L**], but here we use the same sign as in [**E-L**] for the Laplacian of forms $\triangle = d\delta + \delta d$. If $\overline{R}$ denotes the curvature tensor of $\bigwedge^2 T^*M$, and $X,Y,u,v \in T_pM$, $\xi\in\bigwedge^2 T^*_pM$, then

$$\begin{aligned}\overline{R}(X,Y)\xi\ (u,v) &= -\xi(R^M(X,Y)u,\ v) - \xi(u,\ R^M(X,Y)v),\\ SF^*\omega(X,Y) &= \sum_{1\le i\le 2n} -\overline{R}(e_i,X)F^*\omega\ (e_i,Y) + \overline{R}(e_i,Y)F^*\omega\ (e_i,X),\end{aligned}$$

where R^M denotes the curvature tensor of M. In general, we use the following sign convention for curvature tensors: $R^M(X,Y)Z = -\nabla_X\nabla_Y Z + \nabla_Y\nabla_X Z + \nabla_{[X,Y]}Z$. Then, $R^M(X,Y,Z,W) = g_M(R^M(X,Y)Z,W)$. It is straightforward to prove:

Lemma 3.2. *If $\{X_\alpha, Y_\alpha\}$ is a diagonalizing orthonormal basis of $F^*\omega$ at p,*

$$\begin{aligned}\langle SF^*\omega, F^*\omega\rangle &= \sum_\mu 4\cos^2\theta_\mu \mathrm{Ricci}^M(\mu,\overline{\mu}) + \sum_{\mu,\rho} 8\cos\theta_\mu\cos\theta_\rho R^M(\rho,\overline{\rho},\mu,\overline{\mu})\\ &= \sum_{\mu,\rho} 4(\cos\theta_\mu+\cos\theta_\rho)^2 R^M(\rho,\mu,\overline{\rho},\overline{\mu})\\ &\quad + 4(\cos\theta_\mu-\cos\theta_\rho)^2 R^M(\overline{\rho},\mu,\rho,\overline{\mu}).\end{aligned}$$

In particular, if F has equal Kähler angles at p, then, at p,

$$\langle SF^*\omega, F^*\omega\rangle = 16\cos^2\theta \sum_{\rho,\mu} R^M(\rho,\mu,\overline{\rho},\overline{\mu}).$$

Moreover, if (M, J_ω, g_M) is Kähler in a neighbourhood of p, then $\langle SF^*\omega, F^*\omega\rangle = 0$.

We recall the concept of nonnegative *isotropic sectional curvature,* for M with dimension ≥ 4, defined by Micallef and Moore in [**Mi-Mo**]. Let

$$K_{\mathrm{isot}}(\sigma) = \frac{R^M(z,w,\overline{z},\overline{w})}{||z\wedge w||^2},$$

where $\sigma = \mathrm{span}_{\mathbb{C}}\{z,w\}$ is a totally isotropic complex two-plane in T^cM, that is, $u\in\sigma \Rightarrow g_M(u,u)=0$, and where $R^M(x,y,u,v)$ is extend to the complexified tangent space by $\mathbb{C}$-multilinearity. The curvature of M is said to be nonnegative (resp. positive) on totally isotropic two-planes at p, if $K_{\mathrm{isot}}(\sigma)\geq 0$ (resp. >0) whenever $\sigma\subset T^c_pM$ is a totally isotropic two-plane over p. If M is compact, simply connected and has positive isotropic sectional curvature everywhere, then M is homeomorphic to a sphere ([**Mi-Mo**]). If $n\geq 1$, T^{2n} is the flat torus, and S^2 is the euclidean sphere of $\mathbb{R}^3$, then $S^2\times T^{2n}$, $S^2\times S^2$, $S^2\times S^2\times T^{2n}$ and the complex projective space $\mathbb{C}\mathrm{P}^n$ have isotropic sectional curvature ≥ 0 but not >0. If $\{X_\alpha, Y_\alpha\}$ is any orthonormal basis of T_pM, and "μ" denotes Z_μ as in (1.3), the expression

$$S_{\mathrm{isot}}(\{Z_\alpha\}_{1\leq\alpha\leq n}) = \sum_{\rho\neq\mu} K_{\mathrm{isot}}(\mathrm{span}_{\mathbb{C}}\{\rho,\mu\}) = 4\sum_{\rho,\mu} R^M(\rho,\mu,\overline{\rho},\overline{\mu}) \tag{3.15}$$

is a hermitian trace of the curvature of M restricted to the maximal totally isotropic subspace $\mathrm{span}_{\mathbb{C}}\{Z_1,\ldots,Z_n\}$ of T^cM. To require it to be ≥ 0, for all maximal totally isotropic subspaces — and we will say that M has nonnegative *isotropic scalar curvature* — seems, for $n\geq 2$, to be strictly weaker than to have nonnegative isotropic sectional curvature. We also note that, any other metric conformaly equivalent to the flat metric g_0 on the 2n-torus with nonnegative isotropic scalar curvature is homothetically equivalent to g_0, hence flat. In fact, in general, if $\hat{g} = e^{\phi} g_M$ is a conformaly equivalent metric on M, then, for each g_M-orthonormal basis $\{X_\alpha, Y_\alpha\}$, $\hat{S}_{\rm isot}(\{\hat{Z}_\alpha\}) = e^{-\phi} S_{\rm isot}(\{Z_\alpha\}) - (n-1)e^{-2\phi}(2\triangle\phi + (n-1)\|\nabla\phi\|^2)$, where $\hat{Z}_\alpha$ are defined by the $\hat{g}$-orthonormal basis $\{e^{-\frac{\phi}{2}}X_\alpha, e^{-\frac{\phi}{2}}Y_\alpha\}$. To require $2\triangle\phi + (n-1)\|\nabla\phi\|^2 \leq 0$, implies, in case of M compact, ϕ constant. We observe that, if $\dim_{\mathbb{R}} M \geq 6$, then $S_{\rm isot} \equiv 0$ does not imply M to be flat, but $K_{\rm isot} \equiv 0$ implies so. We also note that, if $\dim_{\mathbb{R}}(T_pM) = 4$, the set of curvature tensors at p, with zero isotropic sectional curvature is a vector space of dimension 9.

Recall that, for an immersion with equal Kähler angles, $F^*\omega$ is parallel iff θ is constant and if $\cos\theta \neq 0$, (M, J_ω, g_M) is a Kähler manifold. We are going to see that an extra condition on the scalar isotropic curvature of M may imply itself that the Kähler angle is constant and/or $\nabla J_\omega = 0$. From Proposition 3.3, for any $n \geq 1$, on $\Omega^0_{2n} \cup \Omega^0_0$

$$\|\delta F^*\omega\|^2 = (n-2)^2 \|\nabla \cos\theta\|^2. \tag{3.16}$$

In particular, if $n \neq 2$, $\|\nabla\cos\theta\|^2$ is smoothly extended to all M, and from (3.1) we get that $\cos^2\theta\|\nabla J_\omega\|^2$ is also smooth. Observe that $\|\delta F^*\omega\|^2$ has the same value considering $\delta F^*\omega$ a vector or a 1-form, but considering $F^*\omega$ a 2-form (as in (3.14)) $\|\nabla F^*\omega\|^2$ is half of the square norm when considering $F^*\omega$ an operator of TM (as in (3.1)). For $n = 2$, $F^*\omega$ is co-closed, and so it is a harmonic 2-form. In fact, since F has equal Kähler angles, $F^*\omega = \cos\theta(X^1_* \wedge Y^1_* + X^2_* \wedge Y^2_*)$, and so $*F^*\omega = \pm F^*\omega$, where $*$ is the Hodge star-operator of (M, g), for a fixed local orientation of M, and the $\pm$ sign depends on the orientation of the diagonalizing basis. In particular, $F^*\omega$ is co-closed (on $\Omega^0_0 \cup \Omega^0_n$ and so on all M). From harmonicity of $F^*\omega$ we may conclude that if the set of Lagrangian points has non-empty interior, or more generaly, if $F^*\omega$ as a zero of infinite order, then F is Lagrangian (see e.g., [**E-L**] (1.27), (1.28)). For any $n \geq 2$, integrating (3.14) on M, using Stokes, (3.16) and (3.1), and the fact that $\int_M \langle \triangle F^*\omega, F^*\omega\rangle {\rm Vol}_M = \int_M \|\delta F^*\omega\|^2 {\rm Vol}_M$, we have

$$\begin{aligned} 0 = &\int_M \left((n-(n-2)^2)\|\nabla\cos\theta\|^2 + \frac{1}{2}\cos^2\theta\|\nabla J_\omega\|^2 \right) {\rm Vol}_M \\ &+ \int_M \langle SF^*\omega, F^*\omega\rangle {\rm Vol}_M. \end{aligned} \tag{3.17}$$

The first integrand is smooth on M, for all n (for $n = 2$ it gives half of (3.1)). The factor $n - (n-2)^2$ is respectively, > 0, $= 0$, < 0, according $n = 2$ or 3, $n = 4$, and $n \geq 5$. If M has nonnegative isotropic scalar curvature, $\langle SF^*\omega, F^*\omega\rangle \geq 0$, by Lemma 3.2. Recall from Proposition 3.3 that $(\Omega^0_{2n}, J_\omega, g)$ to be Kähler is a sufficient condition to conclude θ is constant. Then we conclude:

Proposition 3.4. *Let F be a non-Lagrangian immersion with equal Kähler angles of a compact orientable M with nonnegative isotropic scalar curvature into a Kähler manifold N. If $n = 2$, 3 or 4, then θ is constant and (M, J_ω, g_M) is a Kähler manifold. For any $n \geq 1$ and θ constant, $F^*\omega$ is parallel, i.e., (M, J_ω, g_M) is a Kähler manifold.*

4. Minimal immersions with equal Kähler angles.

Let us assume that $F : M \to N$ is minimal with equal Kähler angles. On a open set of $M \sim \mathcal{L}$ where a orthonormal frame $\{X_\alpha, Y_\alpha = J_\omega(X_\alpha)\}$ is defined, we have from (3.11) and (2.4), for any p, $Z \in T_pM$ and μ, γ,

$$2\cos\theta\langle\nabla_Z\mu,\gamma\rangle = -i\nabla_Z F^*\omega(\mu,\gamma) = ig_Z\mu\gamma - ig_Z\gamma\mu. \tag{4.1}$$

Note that $F^*\omega(\nabla_Z\mu,\overline{\gamma}) = i\cos\theta\langle\nabla_Z\mu,\overline{\gamma}\rangle = -i\cos\theta\langle\mu,\nabla_Z\overline{\gamma}\rangle = -F^*\omega(\mu, \nabla_Z\overline{\gamma})$. Hence, if $\mu\neq\gamma$, $\nabla_Z F^*\omega(\mu,\overline{\gamma}) = d(F^*\omega(\mu,\overline{\gamma}))(Z) = 0$. Thus

$$g_Z\mu\overline{\gamma} = g_Z\overline{\gamma}\mu, \quad \forall\mu\neq\gamma. \tag{4.2}$$

From (3.8), for each μ,

$$-\frac{i}{2}d\cos\theta(Z) = -\nabla_Z F^*\omega(\mu,\overline{\mu}) = g_Z\mu\overline{\mu} - g_Z\overline{\mu}\mu \text{ (no sumation over } \mu). \tag{4.3}$$

From (1.6), on $M \sim (\mathcal{L}\cup\mathcal{C})$

$$\triangle\kappa = 4i\sum_\beta \mathrm{Ricci}^N(JdF(\beta), dF(\overline{\beta}))$$

$$+\frac{32}{\sin^2\theta}\sum_{\beta,\mu}\mathrm{Im}\left(R^N(dF(\beta),dF(\mu),dF(\overline{\beta}),JdF(\overline{\mu})+i\cos\theta dF(\overline{\mu}))\right) \tag{4.4}$$

$$-\frac{128\cos\theta}{\sin^4\theta}\sum_{\beta,\mu,\rho}\mathrm{Re}\left(g_{\beta\mu\overline{\rho}}\,g_{\overline{\beta}\rho\overline{\mu}}\right) \tag{4.5}$$

$$+\frac{64\cos\theta}{\sin^2\theta}\sum_{\beta,\mu,\rho}\left(|\langle\nabla_\beta\mu,\rho\rangle|^2 + |\langle\nabla_{\overline{\beta}}\mu,\rho\rangle|^2\right), \tag{4.6}$$

where now $\kappa = n\log\left(\frac{1+\cos\theta}{1-\cos\theta}\right)$. Since $R(X,Y,Z,JW)$ is skew-symmetric on (X,Y) and symmetric on (Z,W), $\sum_{\mu,\beta} R^N(dF(\beta),dF(\mu),dF(\overline{\beta}),JdF(\overline{\mu})) = 0$. Then, from the Gauss equation and minimality of F,

$$\begin{aligned}(4.4) \quad &= \sum_{\beta,\mu}\frac{32}{\sin^2\theta}\mathrm{Im}\left(i\cos\theta R^N(dF(\beta),dF(\mu),dF(\overline{\beta}),dF(\overline{\mu}))\right)\\ &= \frac{32\cos\theta}{\sin^2\theta}\sum_{\beta,\mu}R^M(\beta,\mu,\overline{\beta},\overline{\mu}) + g(\nabla dF(\beta,\overline{\mu}),\nabla dF(\mu,\overline{\beta})).\end{aligned}$$

Using the unitary basis $\{\frac{\sqrt{2}}{\sin\theta}\Phi(\rho), \frac{\sqrt{2}}{\sin\theta}\Phi(\overline{\rho})\}$ of the normal bundle,

$$\begin{aligned}(4.7) \qquad & \frac{32\cos\theta}{\sin^2\theta}\sum_{\beta,\mu} g(\nabla dF(\beta,\overline{\mu}), \nabla dF(\mu,\overline{\beta})) \\ &= \frac{64\cos\theta}{\sin^4\theta}\sum_{\beta,\mu,\rho}(|g_{\beta}\overline{\mu}\rho|^2 + |g_{\beta}\overline{\mu\rho}|^2) \\ &= \frac{64\cos\theta}{\sin^4\theta}\sum_{\beta,\mu,\rho}(|g_{\beta}\overline{\rho}\mu|^2 + |g_{\overline{\mu}}\beta\overline{\rho}|^2) = \frac{128\cos\theta}{\sin^4\theta}\sum_{\beta,\mu,\rho}|g_{\beta}\overline{\rho}\mu|^2.\end{aligned}$$

From (4.2) and (4.3),

$$\begin{aligned}&\sum_{\beta,\mu,\rho}\mathrm{Re}\left(g_{\beta}\mu\overline{\rho}\, g_{\overline{\beta}}\rho\overline{\mu}\right) = \sum_{\beta,\mu}\sum_{\rho\neq\mu}|g_{\beta}\overline{\rho}\mu|^2 + \sum_{\beta,\mu}\mathrm{Re}\left(g_{\beta}\mu\overline{\mu}\, g_{\overline{\beta}}\mu\overline{\mu}\right) \\ &= \sum_{\beta,\mu,\rho}|g_{\beta}\overline{\rho}\mu|^2 - \sum_{\beta,\mu}|g_{\beta}\overline{\mu}\mu|^2 + \sum_{\beta,\mu}\mathrm{Re}\left(g_{\beta}\mu\overline{\mu}\, g_{\overline{\beta}}\mu\overline{\mu}\right) \\ &= \sum_{\beta,\mu,\rho}|g_{\beta}\overline{\rho}\mu|^2 - \sum_{\beta,\mu}\mathrm{Re}\left(\frac{i}{2}d\cos\theta(\beta) g_{\overline{\beta}}\mu\overline{\mu}\right),\end{aligned}$$

so

$$(4.7) + (4.5) = \frac{128\cos\theta}{\sin^4\theta}\sum_{\beta,\mu} Re\left(\frac{i}{2}d\cos\theta(\beta) g_{\overline{\beta}}\mu\overline{\mu}\right).$$

On the other hand, Proposition 3.2 and minimality of F gives,

$$-\sum_{\beta,\mu}\frac{4\cos\theta}{\sin^2\theta}\mathrm{Re}\big(i g_{\beta}\overline{\mu}\mu\cdot\overline{\beta}\big) = \frac{1-n}{4}\nabla\log\sin^2\theta = \frac{(n-1)\cos\theta}{2\sin^2\theta}\nabla\cos\theta.$$

Consequentely,

$$\begin{aligned}&\frac{128\cos\theta}{\sin^4\theta}\sum_{\beta,\mu}\mathrm{Re}\left(\frac{i}{2}d\cos\theta(\beta) g_{\overline{\beta}}\mu\overline{\mu}\right) \\ &= \frac{128\cos\theta}{\sin^4\theta}\sum_{\beta,\mu}\mathrm{Re}\left(-\frac{i}{2}d\cos\theta(\overline{\beta}) g_{\beta}\overline{\mu}\mu\right) \\ &= -\frac{64\cos\theta}{\sin^4\theta}d\cos\theta\left(\mathrm{Re}\left(\sum_{\beta,\mu} i g_{\beta}\overline{\mu}\mu\cdot\overline{\beta}\right)\right) = \frac{8(n-1)\cos\theta}{\sin^4\theta}\|\nabla\cos\theta\|^2.\end{aligned}$$

That is,

$$(4.8) \qquad (4.7) + (4.5) = \frac{8(n-1)\cos\theta}{\sin^4\theta}\|\nabla\cos\theta\|^2.$$

Using (3.9),

$$\|\nabla J_\omega\|^2 = \sum_\beta 4\langle \nabla_\beta J_\omega,\ \nabla_{\overline{\beta}} J_\omega\rangle \tag{4.9}$$
$$= \sum_\beta \sum_{\mu,\rho} 16\big(|\langle \nabla_\beta J_\omega(\mu),\rho\rangle|^2 + |\langle \nabla_\beta J_\omega(\overline{\mu}),\overline{\rho}\rangle|^2\big)$$
$$= 64 \sum_{\beta,\mu,\rho} \big(|\langle \nabla_\beta \mu,\rho\rangle|^2 + |\langle \nabla_{\overline{\beta}}\mu,\rho\rangle|^2\big).$$

Thus we see that (4.6) $= \frac{\cos\theta}{\sin^2\theta}\|\nabla J_\omega\|^2$. So we have obtained the following formula:

Proposition 4.1. *If N is Kähler-Einstein with Ricci tensor* $\mathrm{Ricci}^N = Rg$, *and F is a minimal immersion with equal Kähler angles, on an open set without complex and Lagrangian points,*

$$\triangle\kappa = \cos\theta\bigg(-2nR + \frac{32}{\sin^2\theta}\sum_{\beta,\mu} R^M(\beta,\mu,\overline{\beta},\overline{\mu}) + \frac{1}{\sin^2\theta}\|\nabla J_\omega\|^2 + \frac{8(n-1)}{\sin^4\theta}\|\nabla\cos\theta\|^2\bigg). \tag{4.10}$$

Note that if $n = 1$ we get the expression of Wolfson [**W**], $\triangle\kappa = -2R\cos\theta$.

Proposition 4.2. *If N is Kähler-Einstein with Ricci tensor* $\mathrm{Ricci}^N = Rg$, *and F is a minimal immersion with equal Kähler angles, then:*

(i) *If $n = 2$,*

$$R\sin^2\theta\cos^2\theta = 0. \tag{4.11}$$

(ii) *If $n \geq 3$, then $\|\nabla|\sin\theta|\|^2$ can be smoothly extended to all M. Moreover, if M is compact and orientable,*

$$\int_M nR\sin^2\theta\cos^2\theta\,\mathrm{Vol}_M = \int_M \big((n-2)^2\|\nabla\cos\theta\|^2 + 2(n-2)\|\nabla|\sin\theta|\ \|^2\big)\,\mathrm{Vol}_M. \tag{4.12}$$

Proof. Multiplying (4.10) by $\sin^2\theta\cos\theta$, we get, on $M \sim \mathcal{C}\cup\mathcal{L}$, and using Lemma 3.2,

$$\sin^2\theta\cos\theta\triangle\kappa = -2n\sin^2\theta\cos^2\theta R + 2\langle SF^*\omega, F^*\omega\rangle + \cos^2\theta\|\nabla J_\omega\|^2 + \frac{8(n-1)\cos^2\theta}{\sin^2\theta}\|\nabla\cos\theta\|^2.$$

On the other hand, $\kappa = n\log\left(\frac{1+\cos\theta}{1-\cos\theta}\right)$, and so, $\triangle\kappa = \frac{2n}{\sin^2\theta}\triangle\cos\theta + \frac{4n\cos\theta}{\sin^4\theta}\|\nabla\cos\theta\|^2$. Hence,

$$
\begin{aligned}
(4.13)\qquad & 2n\cos\theta\triangle\cos\theta + \frac{4n\cos^2\theta}{\sin^2\theta}\|\nabla\cos\theta\|^2 \\
&= -2n\sin^2\theta\cos^2\theta R + 2\langle SF^*\omega, F^*\omega\rangle \\
&\quad + \cos^2\theta\|\nabla J_\omega\|^2 + \frac{8(n-1)\cos^2\theta}{\sin^2\theta}\|\nabla\cos\theta\|^2.
\end{aligned}
$$

Recall that, from (3.1), and considering $F^*\omega$ a 2-form, $\|\nabla F^*\omega\|^2 = \frac{1}{2}\cos^2\theta\|\nabla J_\omega\|^2 + n\|\nabla\cos\theta\|^2$. Since $\triangle\cos^2\theta = 2\cos\theta\triangle\cos\theta + 2\|\nabla\cos\theta\|^2$, substituting this into (4.13), we have

$$
\begin{aligned}
(4.14)\qquad n\triangle\cos^2\theta &= -2n\sin^2\theta\cos^2\theta R + 2\langle SF^*\omega, F^*\omega\rangle \\
&\quad + 2\|\nabla F^*\omega\|^2 + \frac{4(n-2)\cos^2\theta}{\sin^2\theta}\|\nabla\cos\theta\|^2
\end{aligned}
$$

and, for $n = 2$,

$$
(4.15)\qquad n\triangle\cos^2\theta = -2n\sin^2\theta\cos^2\theta R + 2\langle SF^*\omega, F^*\omega\rangle + 2\|\nabla F^*\omega\|^2.
$$

Let us now suppose that $n \geq 3$. The sign of $\sin\theta$ is not determined, because we have chosen the interval where θ is such that $\cos\theta \geq 0$. Nevertheless we have $|\sin\theta| = \sqrt{1-\cos\theta}\sqrt{1+\cos\theta}$. This map is continuous, smooth on $M \sim \mathcal{C}\cup\mathcal{L}$ but could be not Lipschitz near complex points. The last term of (4.14) is given by

$$
\begin{aligned}
\frac{4(n-2)\cos^2\theta}{\sin^2\theta}\|\nabla\cos\theta\|^2 &= (n-2)\frac{\|\nabla\cos^2\theta\|^2}{\sin^2\theta} \\
&= (n-2)\frac{\|\nabla\sin^2\theta\|^2}{\sin^2\theta} = 4(n-2)\|\nabla|\sin\theta|\|^2.
\end{aligned}
$$

Then (4.14) is the equation

$$
\begin{aligned}
(4.16)\qquad n\triangle\cos^2\theta &= -2n\sin^2\theta\cos^2\theta R + 2\langle SF^*\omega, F^*\omega\rangle \\
&\quad + 2\|\nabla F^*\omega\|^2 + 4(n-2)\|\nabla|\sin\theta|\|^2.
\end{aligned}
$$

Clearly, (4.16) is valid on $\Omega^0_{2n} \sim \mathcal{C}$ and also on Ω^0_0 and at interior points of $\mathcal{C}$. From smoothness over all M of all terms but the last, and the fact that the remaining set is a set of Lagrangian and complex points with no interior, formula (4.16) is valid on all M, extending smoothly and nonnegatively $\|\nabla|\sin\theta|\,\|^2$. Integrating it over M, and using (3.17) and (3.1), we have

$$
\begin{aligned}
\int_M 2nR\sin^2\theta\cos^2\theta\,\mathrm{Vol}_M = \int_M &\Big(-2(n-(n-2)^2)+2n\Big)\|\nabla\cos\theta\|^2 \\
&+ 4(n-2)\|\nabla|\sin\theta|\|^2\,\mathrm{Vol}_M,
\end{aligned}
$$

leading to (4.12). If $n = 2$, we see that (4.15) is also valid at Lagrangian and complex points. In fact all terms of (4.15) vanish at interior points of the Lagrangian and complex sets (see Lemma 3.2 and (3.1)). Since they are smooth on all M, they must vanish at boundary points of $\mathcal{C}$ and of $\mathcal{L}$. Thus, the above equation is valid on all M, with or without complex or Lagrangian points. Now, (4.11) follows from (4.15), and use of (3.14) with $\|F^*\omega\|^2 = n\cos^2\theta$ and $\triangle F^*\omega = 0$. □

Proof of Theorem 1.2 *and Theorem* 1.3. If $n = 2$ and $R \neq 0$, (4.11) implies $\sin^2\theta\cos^2\theta = 0$. Hence F is either Lagrangian or a complex submanifold. If $n \geq 3$, and M is compact and oriented, the right-hand side of (4.12) is nonnegative, while the left-hand side is non-positive for $R < 0$. Then, $\sin^2\theta\cos^2\theta = 0$ must hold on all M, that is, F is either Lagrangian or complex. If $R = 0$, the right-hand side of (4.12) must vanish. Then, for $n \geq 3$, $\cos\theta$ must be constant, and we have proved Theorem 1.2. If $\cos\theta$ is constant, the right-hand side of (4.12) vanishes. Hence, if $R \neq 0$, F is either complex or Lagrangian, and Theorem 1.3 is proved. □

Proof of Theorem 1.4. If M is not Lagrangian, under the curvature condition on M, by Proposition 3.4, for $n = 2$, 3 or 4, (M, J_ω, g_M) is a Kähler manifold and $\cos\theta$ is constant. So, if M is not a complex submanifold, by (4.11), or (4.12), $R = 0$. In general, if $n \geq 1$ and θ is constant, Proposition 3.4 also applies. □

Under the conditions of Proposition 3.4, if M is homeomorphic to a 4 or 6 dimensional sphere, immersed into a Kähler-Einstein manifold, with equal Kähler angles and with nonnegative isotropic scalar curvature, then it must be Lagrangian, for it is well-known that such manifolds cannot carry a Kähler structure. Obviously, any Riemannian manifold M with strictly positive isotropic scalar curvature cannot carry any Kähler structure, and so the same conclusion must hold. No minimality is required to conclude this.

As an observation, Proposition 3.4 should be compared with the following lemma:

Lemma 4.1. *Let F be an immersion with equal Kähler angles, and $n \geq 2$. If $\cos\theta$ is constant then:*

(i) *$(A, B, C) \to g_A BC$ is symmetric whenever A, B, and C are not all of the same type.*
(ii) $\langle \nabla_{\overline{\beta}}\mu, \gamma\rangle = 0,\ \forall \beta, \mu, \gamma$.
(iii) *$F^*\omega$ is an harmonic 2-form of constant norm.*
(iv) $32\sum_{\beta,\mu} R^M(\beta, \mu, \overline{\beta}, \overline{\mu}) = -64\sum_{\beta,\mu,\rho} |\langle\nabla_\beta\mu, \rho\rangle|^2 = -\|\nabla J_\omega\|^2 \leq 0$ *(only in the case* $\cos\theta \neq 0$*).*

Proof. Since $\cos\theta$ is constant, we obtain (4.3) = 0. This, together (4.2), and the symmetry of ∇dF, proves (i). But (i) and (4.1) imply (ii). (iii) comes from (3.16) and that $\|F^*\omega\|^2 = n\cos^2\theta$. Now we prove (iv). Since $F^*\omega$ is harmonic, from Weitzenböck formula (3.14) we conclude $\langle SF^*\omega, F^*\omega\rangle = -\|\nabla F^*\omega\|^2$. Lemma 3.2 and (3.1) (but considering $F^*\omega$ a 2-form) gives (iv). □

Remark 2. If N is a Kähler manifold of constant holomorphic sectional curvature equal to K (and so $R = 2(2n+1)K$), and the isotropic scalar curvature of M satisfies $S_{\text{isot}} \geq c$, c a constant, we get by Gauss equation, that $c \leq \frac{n(n-1)K}{4}$. Thus, nonnegative isotropic scalar curvature on M is a possible condition for $K \geq 0$. In the case $K = 0$, that is, N is the flat complex torus, for $n \geq 2$, F must be totally geodesic, and so M is flat. We also note that $S_{\text{isot}} > \frac{nR}{4}$ is not a possible condition if $K > 0$. Such a condition, when possible (and so N cannot be of constant holomorphic sectional curvature), could lead to some conclusion by applying the maximum principle to (4.10) at a maximum point of κ.

Example. Let (N, I, J, g) be an hyper-Kähler manifold of real dimension 8. Thus, I and J are two g-orthogonal complex structures on N, such that $IJ = -JI$ and $\nabla I = \nabla J = 0$, where ∇ is the Levi-Civita connection relative to g. It is known that such manifolds are Ricci-flat ([**B**]). Set $K = IJ$. For each ν, ϕ, we take "$\nu\phi$" $= (\cos\nu, \sin\nu\cos\phi, \sin\nu\sin\phi) \in S^2$, and define $J_{\nu\phi} = \cos\nu I + \sin\nu\cos\phi J + \sin\nu\sin\phi K$. These $J_{\nu\phi}$ are the complex structures on N compatible with its hyper-Kähler structure, that is, they are g-orthogonal and $\nabla J_{\nu\phi} = 0$.

Two of such complex structures, $J_{\nu\phi}$ and $J_{\mu\rho}$, anti-commute at a point p iff $J_{\nu\phi}(X)$ and $J_{\mu\rho}(X)$ are orthogonal for some nonzero $X \in T_pN$, iff $\nu\phi$ and $\mu\rho$ are orthogonal in $\mathbb{R}^3$. Thus, they anti-commute at a point p iff they anti-commute everywhere. If that is the case $J_{\nu\phi} \circ J_{\mu\rho} = J_{\sigma\epsilon}$, where $\{\nu\phi, \mu\rho, \sigma\epsilon\}$ is a direct orthonormal basis of $\mathbb{R}^3$. For each unit vector $X \in T_pN$, set $H_X = \text{span}\{X, IX, JX, KX\} = \text{span}\{X, J_{\nu\phi}(X), J_{\mu\rho}(X), J_{\sigma\epsilon}(X)\}$, for any orthonormal basis $\{\nu\phi, \mu\rho, \sigma\epsilon\}$. If $Y \in H_X^{\perp}$ is another unit vector, then $H_X \perp H_Y$. Let $\omega_{\nu\phi}$ be the Kähler form of $(N, J_{\nu\phi}, g)$. Let E be a 4-dimensional vector subspace of T_pN. We first note that $E = H_X$ for some $X \in E$, iff $J_{\nu\phi}(E) \subset E$ for any ν, ϕ. If that is the case, then E is not a Lagrangian subspace with respect to any complex structure $J_{\mu\rho}$. In general, E contains a $J_{\nu\phi}$-complex line for some $\nu\phi$ iff $\dim(E \cap H_X) \geq 2$ for some $X \in E$. If that is the case, and if E is a Lagrangian subspace of T_pN with respect to $J_{\mu\rho}$, then $\nu\phi \perp \mu\rho$. Furthermore, if E is a $J_{\nu\phi}$-complex subspace, then E is $J_{\mu\rho}$-Lagrangian iff there exist an orthonormal basis $\{X, J_{\nu\phi}X, Y, J_{\nu\phi}Y\}$ of E with $H_X \perp H_Y$. To see this, let us suppose E is $J_{\nu\phi}$-complex subspace and $J_{\mu\rho}$-Lagrangian. We take $\{X, J_{\nu\phi}X, Y, J_{\nu\phi}Y\}$ an ortonormal basis of E.

Then $Y \in \mathrm{span}\{X, J_{\nu\phi}X, J_{\mu\rho}X\}^{\perp}$. So $Y = tJ_{\sigma\epsilon}X + \widetilde{Y}$, for some $t \in \mathbb{R}$ and $\widetilde{Y} \in H_X^{\perp}$, and where $\{\nu\phi, \mu\rho, \sigma\epsilon\}$ is an ortonormal basis of $\mathbb{R}^3$. As $E \neq H_X$, $\widetilde{Y} \neq 0$. From $0 = \langle J_{\mu\rho}Y, J_{\nu\phi}X\rangle$, we get $t = 0$. Thus, $Y \in H_X^{\perp}$. We observe that, in general, $J_{\mu\rho}$-Lagrangian subspaces do not need to be $J_{\nu\phi}$-complex, as for example $E = \{X, J_{\nu\phi}X, Y, J_{\sigma\epsilon}Y\}$, with $Y \in H_X^{\perp}$, that contains two orthogonal complex lines for different complex strutures.

Any $J_{\nu\phi}$-complex submanifold $F : M \to N$ of real dimension 4, is for each $\mu\rho$, a minimal submanifold of $(N, J_{\mu\rho}, g)$ with equal Kähler angles. Moreover, if for each point $p \in M$, there exist an orthonormal basis $\{X, J_{\nu\phi}X, Y, J_{\nu\phi}Y\}$ of T_pM with $H_X \perp H_Y$, the Kähler angle is constant, given by $\cos\theta = |\langle\nu\phi, \mu\rho\rangle|$, where $\langle,\rangle$ is the inner product of $\mathbb{R}^3$, and $\pm J_{\nu\phi}$ is the complex structure of M which comes from polar decomposition of $\omega_{\mu\rho}$ restricted to M. In fact, such an orthonormal basis of T_pM diagonalizes $\omega_{\mu\rho}$ restricted to M. Next proposition is an application of Proposition 3.4 or Theorem 1.4, for 4-dimensional submanifolds of N, where ω_I is the Kähler form of (N, I, g):

Proposition 4.3. *Let $F : M \to N$ be a minimal immersion of a compact, oriented 4-dimensional submanifold with nonnegative isotropic scalar curvature, and such that $\forall \nu\phi \in S^2$, F has equal Kähler angles with respect to $J_{\nu\phi}$. If $\exists p \in M$ and $\exists X \in T_pM$, unit vector, such that $\dim(T_pM \cap H_X) \geq 2$, then there exists $\nu\phi \in S^2$ such that M is a $J_{\nu\phi}$-complex submanifold. Furthermore, if $J_{\nu\phi} = I$ then $F : M \to (N, I, g)$ is obviously pluriminimal. If $J_{\nu\phi} \neq I$ but $T_pM \cap H_X^{\perp} \neq \{0\}$, then $F^*\omega_I = \cos\nu J_{\nu\phi}$, and if F is not I-Lagrangian, $F : M \to (N, I, g)$ is still pluriminimal.*

Note that, if $T_pM = H_X$, then $J_{\nu\phi}$ can be chosen equal to I. The first conclusion of this result should be compared with a result of Wolfson [**W**], for M a real minimal surface and N a Ricci-flat K3 surface. In the latter case, there is only one Kähler angle, $\forall X \ \dim(T_pM \cap H_X) = 2$ is automatically satisfied, and the isotropic scalar curvature is always zero.

Proof. From the assumption, $\dim(T_pM \cap H_X) \geq 2$, we may take a unit vector $Z \in T_pM \cap H_X$ such that $Z \perp X$. Then, $Z = J_{\nu\phi}(X)$ for some $\nu\phi$. Thus, span $\{X, J_{\nu\phi}(X)\} \subset T_pM$. This implies $F^*\omega_{\nu\phi}(X, J_{\nu\phi}(X)) = 1$. As the Kähler angles are equal, $\cos\theta_{\nu\phi} = 1$ at p. Applying Proposition 3.4 to $F : M \to (N, J_{\nu\phi}, g)$, $F^*\omega_{\nu\phi} = \cos\theta_{\nu\phi} J_{\omega_{\nu\phi}}$ with $\cos\theta_{\nu\phi}$ constant. Then $\cos\theta_{\nu\phi} = 1$ everywhere. That is, M is a $J_{\nu\phi}$-complex submanifold. Moreover, from the second assumption, $T_pM \cap H_X^{\perp} \neq \{0\}$, we may take a unit vector $Y \in T_pM \cap H_X^{\perp}$. Then $\{X, J_{\nu\phi}X, Y, J_{\nu\phi}Y\}$ constitutes an orthonormal basis of T_pM, that diagonalizes $F^*\omega_I$, and $F^*\omega_I = \cos\nu J_{\nu\phi}$. This means that ν or $\nu + \pi$ is the constant Kähler angle of $F : M \to (N, I, g)$, and, since M is a $J_{\nu\phi}$-complex submanifold, it is pluriharmonic with respect to $\pm J_{\nu\phi}$, and so, if $\cos\nu \neq 0$, it is pluriminimal as an immersion into (N, I, g). □

5. Appendix: The computation of $\triangle\kappa$.

We prove (1.6) for F minimal and away from complex directions. First, we compute some derivative formulas of a determinant, which we will need.

Lemma 5.1. *Let $A : M \to \mathcal{M}_{m\times m}(\mathbb{C})$ be a smooth map of matrices $p \to A(p) = [A_1, \ldots, A_m]$, where $A_i(p)$ is a column vector of $\mathbb{C}^m$ and M is a Riemannian manifold with its Levi-Civita connection ∇. Assume that, at p_0, $A(p_0)$ is a diagonal matrix $D = D(\lambda_1, \ldots, \lambda_m)$. Then, at p_0*

$$d\ (\det A)(Z) = \sum_{1\le j\le m}\left(\prod_{k\neq j}\lambda_k\right) dA^j_j(Z),$$

$$\begin{aligned}\mathrm{Hess}\,(\det A)(Z,W) &= \nabla d(\det A)(Z,W)\\ &= \sum_{1\le j,k\le m}\left(\prod_{s\neq j,k}\lambda_s\right)\det\begin{bmatrix} dA^j_j(Z) & dA^k_j(Z)\\ dA^j_k(W) & dA^k_k(W)\end{bmatrix}\\ &\quad+\sum_{1\le j\le m}\left(\prod_{s\neq j}\lambda_s\right)\mathrm{Hess}\ A^j_j(Z,W).\end{aligned}$$

In particular, if $e_1, \ldots, e_r$ is an orthonormal basis of $T_{p_0}M$, then, at p_0,

$$\begin{aligned}\triangle(\det\ A) &= \mathrm{Trace\ Hess}\ (\det\ A)\\ &= \sum_{1\le\alpha\le r}\ \sum_{1\le j,k\le m}\left(\prod_{s\neq j,k}\lambda_s\right)\det\begin{bmatrix} dA^j_j(e_\alpha) & dA^k_j(e_\alpha)\\ dA^j_k(e_\alpha) & dA^k_k(e_\alpha)\end{bmatrix}\\ &\quad+\sum_{1\le j\le m}\left(\prod_{s\neq j}\lambda_s\right)\triangle A^j_j.\end{aligned}$$

On each Ω^0_{2k}, the complex structure J_ω and the sub-vector bundle $\mathcal{K}^\perp_\omega$ are smooth. Moreover, J_ω is g_M-orthogonal. Thus, for each $p_0 \in \Omega^0_{2k}$, there exists a locally g_M-orthonormal frame of $\mathcal{K}^\perp_\omega$ defined on a neighbourhood of p_0, of the form $X_1, J_\omega X_1, \ldots, X_k, J_\omega X_k$. We enlarge this frame to a g_M-orthonormal local frame on M, on a neighbourhood of p_0:

(5.1) $\qquad X_1, Y_1 = J_\omega X_1, \ldots, X_k\ , Y_k = J_\omega X_k,\ X_{k+1}, Y_{k+1}, \ldots, X_n, Y_n$

where $X_{k+1}, Y_{k+1}, \ldots X_n, Y_n$ is any g_M-orthonormal frame of $\mathcal{K}_\omega$, and which at p_0 is a diagonalizing basis of $F^*\omega$. Note that in general, without some restrictive conditions, it is not possible to get smooth diagonalizing g_M-orthonormal frames in a whole neighbourhood of a point p_0. We use the notations in Section 3.1. We define a local complex structure on a neighbourhood of $p_0 \in \Omega^0_{2k}$ as $\widetilde{J} = J_\omega \oplus J'$, where J_ω is defined only on $\mathcal{K}^\perp_\omega$, and

J' is the local complex structure on $\mathcal{K}_\omega$, defined on a neighbourhood of p_0 by

$$J'Z_\alpha = iZ_\alpha, \quad J'Z_{\overline{\alpha}} = -iZ_{\overline{\alpha}},\ \forall\alpha \geq k+1. \tag{5.2}$$

Thus, the vectors Z_α are of type (1,0) with respect to $\widetilde{J}$, for $\forall\alpha$. Since $\widetilde{J}$ is g_M-orthogonal, then, $\forall\alpha,\beta$, on a neighbourhood of p_0,

$$\langle\nabla_Z\widetilde{J}(\alpha),\beta\rangle = 2i\langle\nabla_Z\alpha,\beta\rangle = -\langle\alpha,\nabla_Z\widetilde{J}(\beta)\rangle, \quad \langle\nabla_Z\widetilde{J}(\alpha),\overline{\beta}\rangle = 0. \tag{5.3}$$

Note that $F^*\omega$ and $\widetilde{g}$, where $\widetilde{g}$ is given in (1.1), are both of type $(1,1)$ with respect to $\widetilde{J}$, and have the same kernel $\mathcal{K}_\omega$. They are related by $\widetilde{g}(X,Y) = F^*\omega(X,J_\omega(Y))$. Set $\widetilde{g}_{AB} = \widetilde{g}(A,B)$, and define $\overline{\overline{B}} = B$, $\forall A,B \in \{1,\ldots,n,\overline{1},\ldots,\overline{n}\}$, and set $\epsilon_\alpha = +1$, $\epsilon_{\overline{\alpha}} = -1$, $\forall 1\leq\alpha\leq n$. Let $\forall 1\leq\alpha,\beta\leq n$ and $A,B\in\{1,\ldots,n,\overline{1},\ldots,\overline{n}\}$, $C\in\{1,\ldots,n\}\cup\{\overline{k+1},\ldots,\overline{n}\}$. Then

$$\begin{cases} F^*\omega(\alpha,C) = g(JdF(\alpha),dF(C)) = 0 & \forall p \text{ near } p_0\\ F^*\omega(\alpha,\overline{\beta}) = g(JdF(\alpha),dF(\overline{\beta})) = \frac{i}{2}\delta_{\alpha\beta}\cos\theta_\alpha & \text{at } p_0\\ \widetilde{g}_{AB} = i\epsilon_B F^*\omega(A,B) = i\epsilon_B g(JdF(A),dF(B)) & \forall p \text{ near } p_0\\ \widetilde{g}_{\alpha C} = \widetilde{g}_{\overline{\alpha}\overline{C}} = 0 & \forall p \text{ near } p_0\\ \widetilde{g}_{\alpha\overline{\beta}} = \widetilde{g}_{\overline{\alpha}\beta} = \frac{1}{2}\delta_{\alpha\beta}\cos\theta_\alpha & \text{at } p_0. \end{cases} \tag{5.4}$$

At a point p_0, with Kähler angles θ_α, $g_M \pm \widetilde{g}$ is represented in the unitary basis $\{\sqrt{2}\alpha,\sqrt{2}\overline{\alpha}\}$, by the diagonal matrix $g_M\pm\widetilde{g} = D(1\pm\cos\theta_1,\ldots,1\pm\cos\theta_n,1\pm\cos\theta_1,\ldots,1\pm\cos\theta_n)$, and so

$$\det(g_M\pm\widetilde{g}) = \prod_{1\leq\alpha\leq n}(1\pm\cos\theta_\alpha)^2. \tag{5.5}$$

If p_0 is a point without complex directions, $\cos\theta_\alpha \neq 1$, $\forall\alpha\in\{1,\ldots,n\}$, then $\widetilde{g} < g_M$. Thus, on a neighbourhood of p_0, we may consider the map κ.

$$\kappa = \frac{1}{2}\log\left(\frac{\det(g_M+\widetilde{g})}{\det(g_M-\widetilde{g})}\right) = \sum_{1\leq\alpha\leq n}\log\left(\frac{1+\cos\theta_\alpha}{1-\cos\theta_\alpha}\right). \tag{5.6}$$

This map is continuous away from the complex directions, and smooth on each Ω^0_{2k}. We wish to compute $\triangle\kappa$ on Ω^0_{2k}.

Lemma 5.2. *At $p_0\in\Omega^0_{2k}$, without complex directions and for $Z,W\in T_{p_0}M$,*

$$d(\det(g_M\pm\widetilde{g}))(Z) = \pm 4\sum_{1\leq\mu\leq n}\frac{\prod_{1\leq\alpha\leq n}(1\pm\cos\theta_\alpha)^2}{(1\pm\cos\theta_\mu)}d\widetilde{g}_{\mu\overline{\mu}}(Z),$$

$$\begin{aligned}
&\mathrm{Hess}(\det(g_M \pm \widetilde{g}))(Z,W)\\
&= 16\left(\prod_{1\leq\alpha\leq n}(1\pm\cos\theta_\alpha)^2\right)\sum_{\mu,\rho}\frac{1}{(1\pm\cos\theta_\mu)(1\pm\cos\theta_\rho)}d\widetilde{g}_{\mu\overline{\mu}}(Z)d\widetilde{g}_{\rho\overline{\rho}}(W)\\
&\quad - 8\left(\prod_{1\leq\alpha\leq n}(1\pm\cos\theta_\alpha)^2\right)\sum_{\mu,\rho}\frac{1}{(1\pm\cos\theta_\mu)(1\pm\cos\theta_\rho)}d\widetilde{g}_{\mu\overline{\rho}}(W)d\widetilde{g}_{\rho\overline{\mu}}(Z)\\
&\pm 4\left(\prod_{1\leq\alpha\leq n}(1\pm\cos\theta_\alpha)^2\right)\sum_{\mu}\frac{1}{(1\pm\cos\theta_\mu)}\mathrm{Hess}\,\widetilde{g}_{\mu\overline{\mu}}(Z,W).
\end{aligned}$$

Proof. Using the unitary basis $\{\sqrt{2}\alpha, \sqrt{2}\overline{\alpha}\}$ of T^c_pM, for p near p_0, $g_M \pm \widetilde{g}$ is represented by the matrix

$$\begin{aligned}
g_M \pm \widetilde{g} &= \begin{bmatrix} g_M \pm \widetilde{g}(\sqrt{2}\alpha, \sqrt{2}\overline{\gamma}) & g_M \pm \widetilde{g}(\sqrt{2}\alpha, \sqrt{2}\gamma) \\ g_M \pm \widetilde{g}(\sqrt{2}\overline{\alpha}, \sqrt{2}\overline{\gamma}) & g_M \pm \widetilde{g}(\sqrt{2}\overline{\alpha}, \sqrt{2}\gamma) \end{bmatrix}\\
&= \begin{bmatrix} \delta_{\alpha\gamma} \pm 2\widetilde{g}_{\alpha\overline{\gamma}} & 0 \\ 0 & \delta_{\alpha\gamma} \pm 2\widetilde{g}_{\overline{\alpha}\gamma} \end{bmatrix}
\end{aligned}$$

that at p_0 is the diagonal matrix $D(1\pm\cos\theta_1,\ldots,1\pm\cos\theta_n,1\pm\cos\theta_1,\ldots,1\pm\cos\theta_n)$. The lemma follows as a simple application of Lemma 5.1, and noting that $\widetilde{g}_{\mu\overline{\rho}} = \widetilde{g}_{\overline{\rho}\mu}$. ☐

On Ω^0_{2k},

$$\begin{aligned}
2\triangle\kappa &= \triangle\log(\det(g_M+\widetilde{g})) - \triangle\log(\det(g_M-\widetilde{g}))\\
&= \frac{\triangle(\det(g_M+\widetilde{g}))}{\det(g_M+\widetilde{g})} - \frac{\|d(\det(g_M+\widetilde{g}))\|^2}{(\det(g_M+\widetilde{g}))^2}\\
&\quad - \frac{\triangle(\det(g_M-\widetilde{g}))}{\det(g_M-\widetilde{g})} + \frac{\|d(\det(g_M-\widetilde{g}))\|^2}{(\det(g_M-\widetilde{g}))^2}.
\end{aligned}$$

From the above lemma and

$$\|d(\det(g_M\pm\widetilde{g}))\|^2 = 4\sum_\beta d(\det(g_M\pm\widetilde{g}))(\beta)d(\det(g_M\pm\widetilde{g}))(\overline{\beta})$$

$$\triangle\det(g_M\pm\widetilde{g}) = 4\sum_\beta \mathrm{Hess}(\det(g_M\pm\widetilde{g}))(\beta,\overline{\beta})$$

we have at p_0,

(5.7)

$$2\triangle\kappa = \sum_{\beta,\mu,\rho}\frac{64(\cos\theta_\mu+\cos\theta_\rho)}{\sin^2\theta_\mu\sin^2\theta_\rho}d\widetilde{g}_{\mu\overline{\rho}}(\overline{\beta})d\widetilde{g}_{\rho\overline{\mu}}(\beta) + \sum_{\beta,\mu}\frac{32}{\sin^2\theta_\mu}\mathrm{Hess}\widetilde{g}_{\mu\overline{\mu}}(\beta,\overline{\beta}).$$

Recalling (2.4), and $d(F^*\omega(X,Y))(Z) = \nabla_Z F^*\omega(X,Y) + F^*\omega(\nabla_Z X,Y) + F^*\omega(X,\nabla_Z Y)$, using (5.4), we obtain:

Lemma 5.3. *$\forall p$ near $p_0 \in \Omega^0_{2k}$, $Z \in T^c_p M$, and $\mu, \gamma \in \{1, \ldots, n\}$*

$$d\widetilde{g}_{\mu\overline{\gamma}}(Z) = ig_Z\mu\overline{\gamma} - ig_Z\overline{\gamma}\mu + 2\sum_{\rho}\left(\langle\nabla_Z\mu, \overline{\rho}\rangle\widetilde{g}_{\rho\overline{\gamma}} + \langle\nabla_Z\overline{\gamma}, \rho\rangle\widetilde{g}_{\mu\overline{\rho}}\right)$$

$$0 = d\widetilde{g}_{\mu\gamma}(Z) = -ig_Z\mu\gamma + ig_Z\gamma\mu + 2\sum_{\rho}\left(\langle\nabla_Z\mu, \rho\rangle\widetilde{g}_{\overline{\rho}\gamma} - \langle\nabla_Z\gamma, \rho\rangle\widetilde{g}_{\mu\overline{\rho}}\right).$$

In particular, at p_0

$$d\widetilde{g}_{\mu\overline{\gamma}}(Z) = ig_Z\mu\overline{\gamma} - ig_Z\overline{\gamma}\mu - (\cos\theta_\mu - \cos\theta_\gamma)\langle\nabla_Z\mu, \overline{\gamma}\rangle$$

$$0 = d\widetilde{g}_{\mu\gamma}(Z) = -ig_Z\mu\gamma + ig_Z\gamma\mu + (\cos\theta_\mu + \cos\theta_\gamma)\langle\nabla_Z\mu, \gamma\rangle.$$

Lemma 5.4. *If F is minimal and $p_0 \in \Omega^0_{2k}$ is a point without complex directions, then for each $\mu \in \{1, \ldots, n\}$*

$$\begin{aligned}
\sum_{1\leq\beta\leq n} \mathrm{Hess}\,\widetilde{g}_{\mu\overline{\mu}}(\beta, \overline{\beta}) &= \sum_{1\leq\beta\leq n} d\Big(d\widetilde{g}_{\mu\overline{\mu}}(\beta)\Big)(\overline{\beta}) - d\widetilde{g}_{\mu\overline{\mu}}(\nabla_{\overline{\beta}}\beta) \\
&= \sum_{1\leq\beta\leq n} iR^N(dF(\beta), dF(\overline{\beta}), dF(\mu), JdF(\overline{\mu}) + i\cos\theta_\mu dF(\overline{\mu})) \\
&\quad + 2\mathrm{Im}\left(R^N(dF(\beta), dF(\mu), dF(\overline{\beta}), JdF(\overline{\mu}) + i\cos\theta_\mu dF(\overline{\mu}))\right) \\
&\quad + 2\sum_{1\leq\rho\leq n} \frac{(\cos\theta_\rho - \cos\theta_\mu)}{\sin^2\theta_\rho}\left(|g_\beta\mu\rho|^2 + |g_\beta\overline{\mu}\overline{\rho}|^2\right) \\
&\quad - 2\sum_{1\leq\rho\leq n} \frac{(\cos\theta_\rho + \cos\theta_\mu)}{\sin^2\theta_\rho}\left(|g_\beta\mu\overline{\rho}|^2 + |g_\beta\overline{\mu}\rho|^2\right) \\
&\quad + \sum_{1\leq\rho\leq n} -2i\langle\nabla_\mu\beta, \overline{\rho}\rangle g_{\overline{\beta}}\rho\overline{\mu} - 2i\langle\nabla_\mu\beta, \rho\rangle g_{\overline{\beta}}\overline{\rho}\overline{\mu} - 2i\langle\nabla_\mu\overline{\beta}, \overline{\rho}\rangle g_\rho\beta\overline{\mu} \\
&\quad + \sum_{1\leq\rho\leq n} 2i\langle\nabla_{\overline{\beta}}\mu, \overline{\rho}\rangle g_\beta\rho\overline{\mu} - 2i\langle\nabla_\mu\overline{\beta}, \rho\rangle g_{\overline{\rho}}\beta\overline{\mu} + 2i\langle\nabla_{\overline{\beta}}\mu, \rho\rangle g_{\overline{\rho}}\beta\overline{\mu} \\
&\quad + \sum_{1\leq\rho\leq n} 2i\langle\nabla_{\overline{\mu}}\beta, \overline{\rho}\rangle g_{\overline{\beta}}\rho\mu + 2i\langle\nabla_{\overline{\mu}}\beta, \rho\rangle g_{\overline{\beta}}\overline{\rho}\mu + 2i\langle\nabla_{\overline{\mu}}\overline{\beta}, \overline{\rho}\rangle g_\rho\beta\mu \\
&\quad + \sum_{1\leq\rho\leq n} -2i\langle\nabla_{\overline{\beta}}\overline{\mu}, \overline{\rho}\rangle g_\rho\beta\mu + 2i\langle\nabla_{\overline{\mu}}\overline{\beta}, \rho\rangle g_{\overline{\rho}}\beta\mu - 2i\langle\nabla_{\overline{\beta}}\overline{\mu}, \rho\rangle g_{\overline{\rho}}\beta\mu \\
&\quad + \sum_{1\leq\rho\leq n} 2i\langle\nabla_{\overline{\beta}}\overline{\mu}, \overline{\rho}\rangle g_\beta\mu\rho + 2i\langle\nabla_{\overline{\beta}}\overline{\mu}, \rho\rangle g_\beta\mu\overline{\rho} - 2i\langle\nabla_{\overline{\beta}}\mu, \rho\rangle g_\beta\overline{\mu}\overline{\rho} \\
&\quad + \sum_{1\leq\rho\leq n} -2i\langle\nabla_{\overline{\beta}}\mu, \overline{\rho}\rangle g_\beta\overline{\mu}\rho + 2i\langle\nabla_\beta\mu, \overline{\rho}\rangle g_{\overline{\beta}}\rho\overline{\mu} - 2i\langle\nabla_\beta\mu, \overline{\rho}\rangle g_{\overline{\beta}}\overline{\mu}\rho \\
&\quad + \sum_{1\leq\rho\leq n} 2i\langle\nabla_\beta\overline{\mu}, \rho\rangle g_{\overline{\beta}}\mu\overline{\rho} - 2i\langle\nabla_\beta\overline{\mu}, \rho\rangle g_{\overline{\beta}}\overline{\rho}\mu \\
&\quad - 2\sum_{1\leq\rho\leq n} (\cos\theta_\mu - \cos\theta_\rho)\left(|\langle\nabla_\beta\mu, \overline{\rho}\rangle|^2 + |\langle\nabla_{\overline{\beta}}\mu, \overline{\rho}\rangle|^2\right).
\end{aligned}$$

Proof. We denote by $\nabla_X \nabla_Y dF$ the covariant derivative of $\nabla_Y dF$ in $T^*M \otimes F^{-1}TN$, and by $\overline{R}(X,Y)\xi$, the curvature tensor of this connection, namely $(\overline{R}(X,Y)\xi)(Z) = R^N(dF(X), dF(Y))\xi(Z) - \xi(R^M(X,Y)Z)$. From Lemma 5.3, for p on a neighbourhood of p_0,

$$d\widetilde{g}_{\mu\overline{\mu}}(\beta) = ig(\nabla_\beta dF(\mu), JdF(\overline{\mu})) - ig(\nabla_\beta dF(\overline{\mu}), JdF(\mu)) \\ + 2\sum_\rho \big(\langle \nabla_\beta \mu, \overline{\rho}\rangle \widetilde{g}_{\rho\overline{\mu}} + \langle \nabla_\beta \overline{\mu}, \rho\rangle \widetilde{g}_{\mu\overline{\rho}}\big).$$

Then at p_0,

$$\begin{aligned}
&d(d\widetilde{g}_{\mu\overline{\mu}}(\beta))(\overline{\beta}) \\
&= ig\big(\nabla_{\overline{\beta}}(\nabla_\beta dF(\mu)), JdF(\overline{\mu})\big) + ig\big(\nabla_\beta dF(\mu), \nabla_{\overline{\beta}}(JdF(\overline{\mu}))\big) \\
&\quad - ig\big(\nabla_{\overline{\beta}}(\nabla_\beta dF(\overline{\mu})), JdF(\mu)\big) - ig\big(\nabla_\beta dF(\overline{\mu}), \nabla_{\overline{\beta}}(JdF(\mu))\big) \\
&\quad + 2\sum_\rho \big(\nabla_{\overline{\beta}}(\langle \nabla_\beta \mu, \overline{\rho}\rangle)\widetilde{g}_{\rho\overline{\mu}} + \nabla_{\overline{\beta}}(\langle \nabla_\beta \overline{\mu}, \rho\rangle)\widetilde{g}_{\mu\overline{\rho}}\big) \\
&\quad + \sum_\rho 2\langle \nabla_\beta \mu, \overline{\rho}\rangle d\widetilde{g}_{\rho\overline{\mu}}(\overline{\beta}) + 2\langle \nabla_\beta \overline{\mu}, \rho\rangle d\widetilde{g}_{\mu\overline{\rho}}(\overline{\beta}) \qquad (5.8) \\
&= ig\big(\nabla_{\overline{\beta}}(\nabla_\beta dF(\mu)), JdF(\overline{\mu})\big) + ig\big(\nabla_\beta dF(\mu), J\nabla_{\overline{\beta}} dF(\overline{\mu})\big) \\
&\quad + ig\big(\nabla_\beta dF(\mu), JdF(\nabla_{\overline{\beta}}\overline{\mu})\big) - ig\big(\nabla_{\overline{\beta}}(\nabla_\beta dF(\overline{\mu})), JdF(\mu)\big) \\
&\quad - ig\big(\nabla_\beta dF(\overline{\mu}), J\nabla_{\overline{\beta}} dF(\mu)\big) - ig\big(\nabla_\beta dF(\overline{\mu}), JdF(\nabla_{\overline{\beta}}\mu)\big) \\
&\quad + \cos\theta_\mu\big(\nabla_{\overline{\beta}}(\langle \nabla_\beta \mu, \overline{\mu}\rangle) + \nabla_{\overline{\beta}}(\langle \mu, \nabla_\beta \overline{\mu}\rangle)\big) + (5.8) \\
&= ig\big(\nabla_{\overline{\beta}}(\nabla_\beta dF(\mu)), JdF(\overline{\mu})\big) \qquad (5.9) \\
&\quad + ig\big(\nabla_\beta dF(\mu), J\nabla_{\overline{\beta}} dF(\overline{\mu})\big) \\
&\quad + \sum_\rho 2i\langle \nabla_{\overline{\beta}}\overline{\mu}, \rho\rangle g_{\beta\mu\overline{\rho}} + 2i\langle \nabla_{\overline{\beta}}\overline{\mu}, \overline{\rho}\rangle g_{\beta\mu\rho} \\
&\quad - ig\big(\nabla_{\overline{\beta}}(\nabla_\beta dF(\overline{\mu})), JdF(\mu)\big) \qquad (5.10) \\
&\quad - ig\big(\nabla_\beta dF(\overline{\mu}), J\nabla_{\overline{\beta}} dF(\mu)\big) \\
&\quad + \sum_\rho -2i\langle \nabla_{\overline{\beta}}\mu, \rho\rangle g_{\beta\overline{\mu}\overline{\rho}} - 2i\langle \nabla_{\overline{\beta}}\mu, \overline{\rho}\rangle g_{\beta\overline{\mu}\rho} \\
&\quad + \cos\theta_\mu\big(\nabla_{\overline{\beta}}(\langle \nabla_\beta \mu, \overline{\mu}\rangle) + \nabla_{\overline{\beta}}(\langle \mu, \nabla_\beta \overline{\mu}\rangle)\big) \qquad (5.11) \\
&\quad + (5.8).
\end{aligned}$$

The term (5.11) vanishes because $\langle \nabla_\beta \mu, \overline{\mu}\rangle = -\langle \mu, \nabla_\beta \overline{\mu}\rangle$ on a neighbourhood of p_0. Minimality of F implies

$$\begin{aligned}
&\sum_\beta \nabla_{\overline{\beta}}\big(\nabla_\beta dF(\mu)\big)\\
&= \sum_\beta \nabla_{\overline{\beta}}\big(\nabla_\mu dF(\beta)\big) = \sum_\beta \nabla_{\overline{\beta}}\nabla_\mu dF(\beta) + \nabla_\mu dF(\nabla_{\overline{\beta}}\beta)\\
&= \sum_\beta \nabla_\mu \nabla_{\overline{\beta}} dF(\beta) - \nabla_{[\mu,\overline{\beta}]} dF(\beta) + (\overline{R}(\mu,\overline{\beta})dF)(\beta) + \nabla_\mu dF(\nabla_{\overline{\beta}}\beta)\\
&= \sum_\beta \nabla_\mu(\nabla_{\overline{\beta}} dF(\beta)) - \nabla_{\overline{\beta}} dF(\nabla_\mu \beta) - \nabla_{[\mu,\overline{\beta}]} dF(\beta)\\
&\quad + R^N(dF(\mu), dF(\overline{\beta}))dF(\beta) - dF(R^M(\mu,\overline{\beta})\beta) + \nabla_\mu dF(\nabla_{\overline{\beta}}\beta)\\
&= \sum_\beta \sum_\rho -2\langle \nabla_\mu \beta, \overline{\rho}\rangle \nabla_{\overline{\beta}} dF(\rho) + \sum_\rho -2\langle \nabla_\mu\beta, \rho\rangle \nabla_{\overline{\beta}} dF(\overline{\rho})\\
&\quad - \sum_\rho (2\langle \nabla_\mu \overline{\beta}, \overline{\rho}\rangle - 2\langle \nabla_{\overline{\beta}}\mu, \overline{\rho}\rangle)\nabla_\rho dF(\beta)\\
&\quad - \sum_\rho (2\langle \nabla_\mu \overline{\beta}, \rho\rangle - 2\langle \nabla_{\overline{\beta}}\mu, \rho\rangle)\nabla_{\overline{\rho}} dF(\beta)\\
&\quad + R^N(dF(\mu), dF(\overline{\beta}))dF(\beta) - dF(R^M(\mu,\overline{\beta})\beta)\\
&\quad + \sum_\rho 2\langle \nabla_{\overline{\beta}}\beta, \overline{\rho}\rangle \nabla_\mu dF(\rho) + \sum_\rho 2\langle \nabla_{\overline{\beta}}\beta, \rho\rangle \nabla_\mu dF(\overline{\rho}).
\end{aligned}$$

Hence

$$\begin{aligned}
\sum_\beta (5.9) = \sum_\beta\ & iR^N(dF(\mu), dF(\overline{\beta}), dF(\beta), JdF(\overline{\mu})) - \cos\theta_\mu R^M(\mu, \overline{\beta}, \beta, \overline{\mu})\\
&+ \sum_{\beta\rho} -2i\langle \nabla_\mu\beta, \overline{\rho}\rangle g_{\overline{\beta}\rho\overline{\mu}} - 2i\langle \nabla_\mu \beta, \rho\rangle g_{\overline{\beta}\overline{\rho}\overline{\mu}}\\
&+ \sum_{\beta\rho} 2i(-\langle \nabla_\mu \overline{\beta}, \overline{\rho}\rangle + \langle \nabla_{\overline{\beta}}\mu, \overline{\rho}\rangle) g_{\rho\beta\overline{\mu}}\\
&+ \sum_{\beta\rho} 2i(-\langle \nabla_\mu \overline{\beta}, \rho\rangle + \langle \nabla_{\overline{\beta}}\mu, \rho\rangle) g_{\overline{\rho}\beta\overline{\mu}}\\
&+ \sum_{\beta\rho} 2i\langle \nabla_{\overline{\beta}}\beta, \overline{\rho}\rangle g_{\mu\rho\overline{\mu}} + 2i\langle \nabla_{\overline{\beta}}\beta, \rho\rangle g_{\mu\overline{\rho}\overline{\mu}}.
\end{aligned}$$

Similarly

$$
\begin{aligned}
-\sum_{\beta}(5.10) = &\sum_{\beta} iR^N(dF(\overline{\mu}), dF(\overline{\beta}), dF(\beta), JdF(\mu)) + \cos\theta_\mu R^M(\overline{\mu}, \overline{\beta}, \beta, \mu)\\
&+\sum_{\beta\rho} -2i\langle \nabla_{\overline{\mu}}\beta, \overline{\rho}\rangle g_{\overline{\beta}}\rho\mu - 2i\langle \nabla_{\overline{\mu}}\beta, \rho\rangle g_{\overline{\beta}}\overline{\rho}\mu\\
&+\sum_{\beta\rho} 2i(-\langle \nabla_{\overline{\mu}}\overline{\beta}, \overline{\rho}\rangle + \langle \nabla_{\overline{\beta}}\overline{\mu}, \overline{\rho}\rangle) g_{\rho}\beta\mu\\
&+\sum_{\beta\rho} 2i(-\langle \nabla_{\overline{\mu}}\overline{\beta}, \rho\rangle + \langle \nabla_{\overline{\beta}}\overline{\mu}, \rho\rangle) g_{\overline{\rho}}\beta\mu\\
&+\sum_{\beta\rho} 2i\langle \nabla_{\overline{\beta}}\beta, \overline{\rho}\rangle g_{\overline{\mu}}\rho\mu + 2i\langle \nabla_{\overline{\beta}}\beta, \rho\rangle g_{\overline{\mu}}\overline{\rho}\mu.
\end{aligned}
$$

Using Bianchi identity,

$$
\begin{aligned}
&iR^N(dF(\mu), dF(\overline{\beta}), dF(\beta), JdF(\overline{\mu}))\\
&\quad - iR^N(dF(\overline{\mu}), dF(\overline{\beta}), dF(\beta), JdF(\mu))\\
&= -iR^N(dF(\beta), dF(\mu), dF(\overline{\beta}), JdF(\overline{\mu}))\\
&\quad - iR^N(dF(\overline{\beta}), dF(\beta), dF(\mu), JdF(\overline{\mu}))\\
&\quad - iR^N(dF(\overline{\mu}), dF(\overline{\beta}), dF(\beta), JdF(\mu))\\
&= iR^N(dF(\beta), dF(\overline{\beta}), dF(\mu), JdF(\overline{\mu}))\\
&\quad + 2\mathrm{Im}\big(R^N(dF(\beta), dF(\mu), dF(\overline{\beta}), JdF(\overline{\mu}))\big),
\end{aligned}
$$

and by Gauss equation, and minimality of F,

$$
\begin{aligned}
&\sum_{\beta} -R^M(\mu, \overline{\beta}, \beta, \overline{\mu}) - R^M(\overline{\mu}, \overline{\beta}, \beta, \mu)\\
&= \sum_{\beta} R^M(\beta, \mu, \overline{\beta}, \overline{\mu}) + R^M(\overline{\beta}, \beta, \mu, \overline{\mu}) - R^M(\overline{\mu}, \overline{\beta}, \beta, \mu)\\
&= \sum_{\beta} -R^M(\beta, \overline{\beta}, \mu, \overline{\mu}) + 2R^M(\beta, \mu, \overline{\beta}, \overline{\mu})\\
&= \sum_{\beta} -R^N(dF(\beta), dF(\overline{\beta}), dF(\mu), dF(\overline{\mu}))\\
&\quad - g\big(\nabla_\beta dF(\mu), \nabla_{\overline{\beta}} dF(\overline{\mu})\big) + g\big(\nabla_\beta dF(\overline{\mu}), \nabla_{\overline{\beta}} dF(\mu)\big)\\
&\quad + 2R^N(dF(\beta), dF(\mu), dF(\overline{\beta}), dF(\overline{\mu}))\\
&\quad + 2g\big(\nabla_\beta dF(\overline{\beta}), \nabla_\mu dF(\overline{\mu})\big) - 2g\big(\nabla_\beta dF(\overline{\mu}), \nabla_\mu dF(\overline{\beta})\big)
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_\beta -R^N(dF(\beta), dF(\overline{\beta}), dF(\mu), dF(\overline{\mu})) \\
&\quad + 2R^N(dF(\beta), dF(\mu), dF(\overline{\beta}), dF(\overline{\mu})) \\
&\quad - g\big(\nabla_\beta dF(\mu), \nabla_{\overline{\beta}} dF(\overline{\mu})\big) - g\big(\nabla_\beta dF(\overline{\mu}), \nabla_\mu dF(\overline{\beta})\big).
\end{aligned}
$$

Note that $R^N(dF(\beta), dF(\mu), dF(\overline{\beta}), dF(\overline{\mu})) = \mathrm{Im}\big(iR^N(dF(\beta), dF(\mu), dF(\overline{\beta}), dF(\overline{\mu}))\big)$, since it is real. Therefore,

$$
\begin{aligned}
&\sum_\beta d\big(d\widetilde{g}_{\mu\overline{\mu}}(\beta)\big)(\overline{\beta}) \\
&= \sum_\beta \; iR^N(dF(\beta), dF(\overline{\beta}), dF(\mu), JdF(\overline{\mu}) + i\cos\theta_\mu dF(\overline{\mu})) \\
&\quad + 2\mathrm{Im}\big(R^N(dF(\beta), dF(\mu), dF(\overline{\beta}), JdF(\overline{\mu}) + i\cos\theta_\mu dF(\overline{\mu}))\big) \\
&\quad - \cos\theta_\mu\, g\big(\nabla_\beta dF(\mu), \nabla_{\overline{\beta}} dF(\overline{\mu})\big) - \cos\theta_\mu\, g\big(\nabla_\beta dF(\overline{\mu}), \nabla_\mu dF(\overline{\beta})\big) \qquad (5.12)\\
&\quad + \sum_\rho -2i\langle \nabla_\mu \beta, \overline{\rho}\rangle g_{\overline{\beta}\rho\overline{\mu}} - 2i\langle \nabla_\mu \beta, \rho\rangle g_{\overline{\beta}\overline{\rho}\overline{\mu}} \\
&\quad + \sum_\rho 2i(-\langle \nabla_\mu \overline{\beta}, \overline{\rho}\rangle + \langle \nabla_{\overline{\beta}} \mu, \overline{\rho}\rangle) g_{\rho\beta\overline{\mu}} \\
&\quad + \sum_\rho 2i(-\langle \nabla_\mu \overline{\beta}, \rho\rangle + \langle \nabla_{\overline{\beta}} \mu, \rho\rangle) g_{\overline{\rho}\beta\overline{\mu}} \\
&\quad + \sum_\rho 2i\langle \nabla_{\overline{\beta}} \beta, \overline{\rho}\rangle g_{\mu\rho\overline{\mu}} + 2i\langle \nabla_{\overline{\beta}} \beta, \rho\rangle g_{\mu\overline{\rho}\overline{\mu}} \qquad (5.13)\\
&\quad + \sum_\rho 2i\langle \nabla_{\overline{\mu}} \beta, \overline{\rho}\rangle g_{\overline{\beta}\rho\mu} + 2i\langle \nabla_{\overline{\mu}} \beta, \rho\rangle g_{\overline{\beta}\overline{\rho}\mu} \\
&\quad + \sum_\rho 2i(\langle \nabla_{\overline{\mu}} \overline{\beta}, \overline{\rho}\rangle - \langle \nabla_{\overline{\beta}} \overline{\mu}, \overline{\rho}\rangle) g_{\rho\beta\mu} + 2i(\langle \nabla_{\overline{\mu}} \overline{\beta}, \rho\rangle - \langle \nabla_{\overline{\beta}} \overline{\mu}, \rho\rangle) g_{\overline{\rho}\beta\mu} \\
&\quad + \sum_\rho -2i\langle \nabla_{\overline{\beta}} \beta, \overline{\rho}\rangle g_{\overline{\mu}\rho\mu} - 2i\langle \nabla_{\overline{\beta}} \beta, \rho\rangle g_{\overline{\mu}\overline{\rho}\mu} \qquad (5.14)\\
&\quad + ig\big(\nabla_\beta dF(\mu), J\nabla_{\overline{\beta}} dF(\overline{\mu})\big) - ig\big(\nabla_\beta dF(\overline{\mu}), J\nabla_{\overline{\beta}} dF(\mu)\big) \qquad (5.15)\\
&\quad + \sum_\rho 2i\langle \nabla_{\overline{\beta}} \overline{\mu}, \rho\rangle g_{\beta\mu\overline{\rho}} + 2i\langle \nabla_{\overline{\beta}} \overline{\mu}, \overline{\rho}\rangle g_{\beta\mu\rho} \\
&\qquad \sum_\rho -2i\langle \nabla_{\overline{\beta}} \mu, \rho\rangle g_{\beta\overline{\mu}\overline{\rho}} - 2i\langle \nabla_{\overline{\beta}} \mu, \overline{\rho}\rangle g_{\beta\overline{\mu}\rho} \;+\; (5.8).
\end{aligned}
$$

Using the unitary basis $\left\{\frac{\sqrt{2}}{\sin\theta_\rho}\Phi(\rho), \frac{\sqrt{2}}{\sin\theta_\rho}\Phi(\overline{\rho})\right\}$ of the normal bundle, and (2.1)

$$
\begin{aligned}
&(5.12) + (5.15)\\
&= -\sum_{\beta,\rho} \frac{2\cos\theta_\mu}{\sin^2\theta_\rho}\big(|g_\beta\mu\rho|^2 + |g_\beta\mu\overline{\rho}|^2\big) - \sum_{\beta,\rho} \frac{2\cos\theta_\mu}{\sin^2\theta_\rho}\big(|g_\beta\overline{\mu}\rho|^2 + |g_\beta\overline{\mu}\overline{\rho}|^2\big)\\
&\quad - \sum_{\beta,\rho} \frac{2\cos\theta_\rho}{\sin^2\theta_\rho}\big(|g_\beta\mu\overline{\rho}|^2 - |g_\beta\mu\rho|^2\big) + \sum_{\beta,\rho} \frac{2\cos\theta_\rho}{\sin^2\theta_\rho}\big(|g_\beta\overline{\mu}\overline{\rho}|^2 - |g_\beta\overline{\mu}\rho|^2\big)\\
&= 2\sum_{\beta,\rho} \frac{(\cos\theta_\rho - \cos\theta_\mu)}{\sin^2\theta_\rho}\ |g_\beta\mu\rho|^2 - 2\sum_{\beta,\rho} \frac{(\cos\theta_\rho + \cos\theta_\mu)}{\sin^2\theta_\rho}\ |g_\beta\mu\overline{\rho}|^2\\
&\quad - 2\sum_{\beta,\rho} \frac{(\cos\theta_\rho + \cos\theta_\mu)}{\sin^2\theta_\rho}\ |g_\beta\overline{\mu}\rho|^2 + 2\sum_{\beta,\rho} \frac{(\cos\theta_\rho - \cos\theta_\mu)}{\sin^2\theta_\rho}\ |g_\beta\overline{\mu}\overline{\rho}|^2.
\end{aligned}
$$

Applying Lemma 5.3 we have

$$
\begin{aligned}
d\widetilde{g}_{\mu\overline{\mu}}\left(\nabla_{\overline{\beta}}\beta\right) &= \sum_\rho 2\left\langle \nabla_{\overline{\beta}}\beta, \overline{\rho}\right\rangle d\widetilde{g}_{\mu\overline{\mu}}(\rho) + \sum_\rho 2\left\langle \nabla_{\overline{\beta}}\beta, \rho\right\rangle d\widetilde{g}_{\mu\overline{\mu}}(\overline{\rho})\\
&= 2i\sum_\rho \Bigg(\left\langle \nabla_{\overline{\beta}}\beta, \overline{\rho}\right\rangle g_\rho\mu\overline{\mu} - \left\langle \nabla_{\overline{\beta}}\beta, \overline{\rho}\right\rangle g_\rho\overline{\mu}\mu\\
&\qquad + \left\langle \nabla_{\overline{\beta}}\beta, \rho\right\rangle g_{\overline{\rho}}\mu\overline{\mu} - \left\langle \nabla_{\overline{\beta}}\beta, \rho\right\rangle g_{\overline{\rho}}\overline{\mu}\mu\Bigg)\\
&= (5.13) + (5.14).
\end{aligned}
$$

Finally

$$
\begin{aligned}
(5.8) &= \sum_\rho 2\left\langle \nabla_\beta\mu, \overline{\rho}\right\rangle \left(ig_{\overline{\beta}}\rho\overline{\mu} - ig_{\overline{\beta}}\overline{\mu}\rho\right)\\
&\quad - \sum_\rho 2\left\langle \nabla_\beta\mu, \overline{\rho}\right\rangle (\cos\theta_\rho - \cos\theta_\mu)\left\langle \nabla_{\overline{\beta}}\rho, \overline{\mu}\right\rangle\\
&\quad + \sum_\rho 2\langle \nabla_\beta\overline{\mu}, \rho\rangle \left(ig_{\overline{\beta}}\mu\overline{\rho} - ig_{\overline{\beta}}\overline{\rho}\mu\right)\\
&\quad - \sum_\rho 2\langle \nabla_\beta\overline{\mu}, \rho\rangle(\cos\theta_\mu - \cos\theta_\rho)\left\langle \nabla_{\overline{\beta}}\mu, \overline{\rho}\right\rangle\\
&= \sum_\rho 2i\langle \nabla_\beta\mu, \overline{\rho}\rangle g_{\overline{\beta}}\rho\overline{\mu} - 2i\langle \nabla_\beta\mu, \overline{\rho}\rangle g_{\overline{\beta}}\overline{\mu}\rho\\
&\quad + \sum_\rho 2i\langle \nabla_\beta\overline{\mu}, \rho\rangle g_{\overline{\beta}}\mu\overline{\rho} - 2i\langle \nabla_\beta\overline{\mu}, \rho\rangle g_{\overline{\beta}}\overline{\rho}\mu\\
&\quad - 2\sum_\rho(\cos\theta_\mu - \cos\theta_\rho)\big(|\langle \nabla_\beta\mu, \overline{\rho}\rangle|^2 + |\langle \nabla_{\overline{\beta}}\mu, \overline{\rho}\rangle|^2\big).
\end{aligned}
$$

These expressions lead to the expression of the lemma. □

Finally, we have:

Proposition 5.1. *If F is minimal without complex directions, then for each $0 \leq k \leq 2n$ at each $p_0 \in \Omega^0_{2k}$,*

$$\begin{aligned}
\triangle\kappa = 4i\sum_{\beta} &\mathrm{Ricci}^N(JdF(\beta), dF(\overline{\beta}))\\
&+\sum_{\beta,\mu}\frac{32}{\sin^2\theta_\mu}\mathrm{Im}\big(R^N(dF(\beta), dF(\mu), dF(\overline{\beta}), JdF(\overline{\mu}) + i\cos\theta_\mu dF(\overline{\mu}))\big)\\
&-\sum_{\beta,\mu,\rho}\frac{64(\cos\theta_\mu+\cos\theta_\rho)}{\sin^2\theta_\mu\sin^2\theta_\rho}\mathrm{Re}(g_{\beta\mu\overline{\rho}}g_{\overline{\beta}\rho\overline{\mu}})\\
&+\sum_{\beta,\mu,\rho}\frac{32(\cos\theta_\rho-\cos\theta_\mu)}{\sin^2\theta_\mu\sin^2\theta_\rho}\,(|g_{\beta\mu\rho}|^2+|g_{\overline{\beta}\mu\rho}|^2)\\
&+\sum_{\beta,\mu,\rho}\frac{32(\cos\theta_\mu+\cos\theta_\rho)}{\sin^2\theta_\mu}\,\Big(|\langle\nabla_\beta\mu,\rho\rangle|^2+|\langle\nabla_{\overline{\beta}}\mu,\rho\rangle|^2\Big).
\end{aligned}$$

Proof. From (5.7) and Lemma 5.4 we get

$$\begin{aligned}
&2\triangle\kappa\\
&=\sum_{\beta,\mu,\rho}\frac{64(\cos\theta_\mu+\cos\theta_\rho)}{\sin^2\theta_\mu\sin^2\theta_\rho}d\widetilde{g}_{\mu\overline{\rho}}(\overline{\beta})d\widetilde{g}_{\rho\overline{\mu}}(\beta)\\
&\quad+\sum_{\beta,\mu}\frac{32i}{\sin^2\theta_\mu}R^N(dF(\beta), dF(\overline{\beta}), dF(\mu), JdF(\overline{\mu})+i\cos\theta_\mu dF(\overline{\mu}))\\
&\quad+\sum_{\beta,\mu}\frac{64}{\sin^2\theta_\mu}\mathrm{Im}\big(R^N(dF(\beta), dF(\mu), dF(\overline{\beta}), JdF(\overline{\mu})\\
&\qquad+i\cos\theta_\mu dF(\overline{\mu}))\big)\\
&\quad+\sum_{\beta,\mu,\rho}\frac{64(\cos\theta_\rho-\cos\theta_\mu)}{\sin^2\theta_\mu\sin^2\theta_\rho}\,(|g_{\beta\mu\rho}|^2+|g_{\beta\overline{\mu}\overline{\rho}}|^2)\\
&\quad-\sum_{\beta,\mu,\rho}\frac{64(\cos\theta_\rho+\cos\theta_\mu)}{\sin^2\theta_\mu\sin^2\theta_\rho}\,(|g_{\beta\mu\overline{\rho}}|^2+|g_{\beta\overline{\mu}\rho}|^2)
\end{aligned}$$

(5.16)

$$+\sum_{\beta,\mu,\rho}-\tfrac{64i}{\sin^2\theta_\mu}\langle\nabla_\mu\beta,\overline{\rho}\rangle g_{\overline{\beta}\rho\overline{\mu}}-\tfrac{64i}{\sin^2\theta_\mu}\langle\nabla_\mu\beta,\rho\rangle g_{\overline{\beta}\overline{\rho}\overline{\mu}}-\tfrac{64i}{\sin^2\theta_\mu}\langle\nabla_\mu\overline{\beta},\overline{\rho}\rangle g_{\rho\beta\overline{\mu}}$$

(5.17)

$$+\sum_{\beta,\mu,\rho}\tfrac{64i}{\sin^2\theta_\mu}\left\langle\nabla_{\overline{\beta}}\mu,\overline{\rho}\right\rangle g_{\beta\rho\overline{\mu}}-\tfrac{64i}{\sin^2\theta_\mu}\langle\nabla_\mu\overline{\beta},\rho\rangle g_{\overline{\rho}\beta\overline{\mu}}+\tfrac{64i}{\sin^2\theta_\mu}\left\langle\nabla_{\overline{\beta}}\mu,\rho\right\rangle g_{\overline{\rho}\beta\overline{\mu}}$$

(5.18)

$$+\sum_{\beta,\mu,\rho} \tfrac{64i}{\sin^2\theta_\mu}\langle\nabla_{\overline{\mu}}\beta,\overline{\rho}\rangle\, g_{\overline{\beta}}\rho\mu + \tfrac{64i}{\sin^2\theta_\mu}\langle\nabla_{\overline{\mu}}\beta,\rho\rangle g_{\overline{\beta}}\overline{\rho}\mu + \tfrac{64i}{\sin^2\theta_\mu}\langle\nabla_{\overline{\mu}}\overline{\beta},\overline{\rho}\rangle g_\rho\beta\mu$$

(5.19)

$$+\sum_{\beta,\mu,\rho} -\tfrac{64i}{\sin^2\theta_\mu}\left\langle\nabla_{\overline{\beta}}\overline{\mu},\overline{\rho}\right\rangle g_\rho\beta\mu + \tfrac{64i}{\sin^2\theta_\mu}\langle\nabla_{\overline{\mu}}\overline{\beta},\rho\rangle g_{\overline{\rho}}\beta\mu - \tfrac{64i}{\sin^2\theta_\mu}\left\langle\nabla_{\overline{\beta}}\overline{\mu},\rho\right\rangle g_{\overline{\rho}}\beta\mu$$

(5.20)

$$+\sum_{\beta,\mu,\rho} \tfrac{64i}{\sin^2\theta_\mu}\left\langle\nabla_{\overline{\beta}}\overline{\mu},\overline{\rho}\right\rangle g_\beta\mu\rho + \tfrac{64i}{\sin^2\theta_\mu}\left\langle\nabla_{\overline{\beta}}\overline{\mu},\rho\right\rangle g_\beta\mu\overline{\rho} - \tfrac{64i}{\sin^2\theta_\mu}\left\langle\nabla_{\overline{\beta}}\mu,\rho\right\rangle g_\beta\overline{\mu\rho}$$

(5.21)

$$+\sum_{\beta,\mu,\rho} -\tfrac{64i}{\sin^2\theta_\mu}\left\langle\nabla_{\overline{\beta}}\mu,\overline{\rho}\right\rangle g_\beta\overline{\mu}\rho + \tfrac{64i}{\sin^2\theta_\mu}\langle\nabla_\beta\mu,\overline{\rho}\rangle g_{\overline{\beta}}\rho\overline{\mu} - \tfrac{64i}{\sin^2\theta_\mu}\langle\nabla_\beta\mu,\overline{\rho}\rangle g_{\overline{\beta}}\overline{\mu}\rho$$

(5.22)

$$+\sum_{\beta,\mu,\rho} \frac{64i}{\sin^2\theta_\mu}\langle\nabla_\beta\overline{\mu},\rho\rangle g_{\overline{\beta}}\mu\overline{\rho} - \frac{64i}{\sin^2\theta_\mu}\langle\nabla_\beta\overline{\mu},\rho\rangle g_{\overline{\beta}}\overline{\rho}\mu$$

$$-\sum_{\beta,\mu,\rho}\frac{64(\cos\theta_\mu-\cos\theta_\rho)}{\sin^2\theta_\mu}\left(\left|\left\langle\nabla_\beta\mu,\overline{\rho}\right\rangle\right|^2+\left|\left\langle\nabla_{\overline{\beta}}\mu,\overline{\rho}\right\rangle\right|^2\right).$$

Interchanging ρ with β in the first term of (5.16) (that we named by (5.16)(1), and similarly to other equations), we see that (5.16)(1)+(5.17)(2) = 0. Interchanging ρ with β in (5.18)(1), we get (5.18)(1)+(5.19)(2) = 0. In (5.16)(2), $\langle\nabla_\mu\beta,\rho\rangle$ is skew-symmetric on ρ and β, and $g_{\overline{\beta}}\overline{\rho\mu}$ is symmetric on ρ and β. Hence (5.16)(2) = 0. Similarly (5.16)(3) = (5.18)(2) = (5.18)(3) = 0. If we interchange ρ with μ in (5.17)(1),

$$(5.17)(1)+(5.20)(2) = -\sum_{\beta,\mu,\rho}\frac{64i(\sin^2\theta_\mu-\sin^2\theta_\rho)}{\sin^2\theta_\mu\sin^2\theta_\rho}\left\langle\nabla_{\overline{\beta}}\overline{\mu},\rho\right\rangle g_\beta\mu\overline{\rho}.$$

Interchanging ρ with μ in (5.17)(3), we get

$$(5.17)(3)+(5.20)(3) = -\sum_{\beta,\mu,\rho}\frac{64i(\sin^2\theta_\mu+\sin^2\theta_\rho)}{\sin^2\theta_\mu\sin^2\theta_\rho}\left\langle\nabla_{\overline{\beta}}\mu,\rho\right\rangle g_\beta\overline{\mu\rho}.$$

Interchanging ρ with μ in (5.19)(1), we get

$$(5.19)(1)+(5.20)(1) = \sum_{\beta,\mu,\rho}\frac{64i(\sin^2\theta_\mu+\sin^2\theta_\rho)}{\sin^2\theta_\mu\sin^2\theta_\rho}\left\langle\nabla_{\overline{\beta}}\overline{\mu},\overline{\rho}\right\rangle g_\beta\mu\rho.$$

Interchanging ρ with μ in (5.19)(3), we get

$$(5.19)(3)+(5.21)(1) = \sum_{\beta,\mu,\rho}\frac{64i(\sin^2\theta_\mu-\sin^2\theta_\rho)}{\sin^2\theta_\mu\sin^2\theta_\rho}\left\langle\nabla_{\overline{\beta}}\mu,\overline{\rho}\right\rangle g_\beta\overline{\mu}\rho.$$

Interchanging ρ with μ in (5.21)(2),

$$(5.21)(2) + (5.22)(1) = \sum_{\beta,\mu,\rho} \frac{64i(-\sin^2\theta_\mu + \sin^2\theta_\rho)}{\sin^2\theta_\mu \sin^2\theta_\rho} \langle \nabla_{\overline{\beta}}\overline{\mu}, \rho\rangle\, g_{\overline{\beta}}\mu\overline{\rho}.$$

Interchanging ρ with μ in (5.22)(2), we obtain

$$(5.22)(2) + (5.21)(3) = \sum_{\beta,\mu,\rho} \frac{64i(\sin^2\theta_\mu - \sin^2\theta_\rho)}{\sin^2\theta_\mu \sin^2\theta_\rho} \langle \nabla_{\beta}\mu, \overline{\rho}\rangle\, g_{\overline{\beta}}\overline{\mu}\rho.$$

Therefore,

$$
\begin{aligned}
& 2\triangle\kappa \\
(5.23)\quad & = \sum_{\beta,\mu,\rho} \frac{64(\cos\theta_\mu + \cos\theta_\rho)}{\sin^2\theta_\mu \sin^2\theta_\rho} d\widetilde{g}_{\mu\overline{\rho}}(\overline{\beta}) d\widetilde{g}_{\rho\overline{\mu}}(\beta) \\
(5.24)\quad & \quad + \sum_{\beta,\mu} \frac{32i}{\sin^2\theta_\mu} R^N(dF(\beta), dF(\overline{\beta}), dF(\mu), JdF(\overline{\mu}) + i\cos\theta_\mu dF(\overline{\mu})) \\
& \quad + \sum_{\beta,\mu} \frac{64}{\sin^2\theta_\mu} \mathrm{Im}\Big(R^N(dF(\beta), dF(\mu), dF(\overline{\beta}), JdF(\overline{\mu}) \\
& \quad + i\cos\theta_\mu dF(\overline{\mu}))\Big) \\
(5.25)\quad & \quad + \sum_{\beta,\mu,\rho} \frac{64(\cos\theta_\rho - \cos\theta_\mu)}{\sin^2\theta_\mu \sin^2\theta_\rho} |g_\beta\mu\rho|^2 \\
(5.26)\quad & \quad - \sum_{\beta,\mu,\rho} \frac{64(\cos\theta_\rho + \cos\theta_\mu)}{\sin^2\theta_\mu \sin^2\theta_\rho} |g_\beta\mu\overline{\rho}|^2 \\
(5.27)\quad & \quad - \sum_{\beta,\mu,\rho} \frac{64(\cos\theta_\rho + \cos\theta_\mu)}{\sin^2\theta_\mu \sin^2\theta_\rho} |g_\beta\overline{\mu}\rho|^2 \\
(5.28)\quad & \quad + \sum_{\beta,\mu,\rho} \frac{64(\cos\theta_\rho - \cos\theta_\mu)}{\sin^2\theta_\mu \sin^2\theta_\rho} |g_\beta\overline{\mu}\overline{\rho}|^2 \\
(5.29)\quad & \quad - \sum_{\beta,\mu,\rho} \frac{64i(\sin^2\theta_\mu - \sin^2\theta_\rho)}{\sin^2\theta_\mu \sin^2\theta_\rho} \left\langle \nabla_{\overline{\beta}}\overline{\mu}, \rho\right\rangle g_\beta\mu\overline{\rho} \\
(5.30)\quad & \quad - \sum_{\beta,\mu,\rho} \frac{64i(\sin^2\theta_\mu + \sin^2\theta_\rho)}{\sin^2\theta_\mu \sin^2\theta_\rho} \left\langle \nabla_{\overline{\beta}}\mu, \rho\right\rangle g_\beta\overline{\mu}\overline{\rho} \\
(5.31)\quad & \quad + \sum_{\beta,\mu,\rho} \frac{64i(\sin^2\theta_\mu + \sin^2\theta_\rho)}{\sin^2\theta_\mu \sin^2\theta_\rho} \left\langle \nabla_{\overline{\beta}}\overline{\mu}, \overline{\rho}\right\rangle g_\beta\mu\rho \\
(5.32)\quad & \quad + \sum_{\beta,\mu,\rho} \frac{64i(\sin^2\theta_\mu - \sin^2\theta_\rho)}{\sin^2\theta_\mu \sin^2\theta_\rho} \left\langle \nabla_{\overline{\beta}}\mu, \overline{\rho}\right\rangle g_\beta\overline{\mu}\rho
\end{aligned}
$$

$$(5.33)\qquad + \sum_{\beta,\mu,\rho} \frac{64i(-\sin^2\theta_\mu + \sin^2\theta_\rho)}{\sin^2\theta_\mu \sin^2\theta_\rho} \langle \nabla_\beta \overline{\mu}, \rho \rangle \, g_{\overline{\beta}} \mu \overline{\rho}$$

$$(5.34)\qquad + \sum_{\beta,\mu,\rho} \frac{64i(\sin^2\theta_\mu - \sin^2\theta_\rho)}{\sin^2\theta_\mu \sin^2\theta_\rho} \langle \nabla_\beta \mu, \overline{\rho} \rangle \, g_{\overline{\beta}} \overline{\mu} \rho$$

$$(5.35)\qquad - \sum_{\beta,\mu,\rho} \frac{64(\cos\theta_\mu - \cos\theta_\rho)}{\sin^2\theta_\mu} \left(\left| \left\langle \nabla_\beta \mu, \overline{\rho} \right\rangle \right|^2 + \left| \left\langle \nabla_{\overline{\beta}} \mu, \overline{\rho} \right\rangle \right|^2 \right).$$

By Lemma 5.3,

$$\begin{aligned}
(5.23) = & \sum_{\beta,\mu,\rho} \frac{64(\cos\theta_\mu + \cos\theta_\rho)}{\sin^2\theta_\mu \sin^2\theta_\rho} \\
& \cdot \left(i g_{\overline{\beta}} \mu \overline{\rho} - i g_{\overline{\beta}} \overline{\rho} \mu - (\cos\theta_\mu - \cos\theta_\rho) \left\langle \nabla_{\overline{\beta}} \mu, \overline{\rho} \right\rangle \right) \cdot \\
& \cdot \left(i g_\beta \rho \overline{\mu} - i g_\beta \overline{\mu} \rho - (\cos\theta_\rho - \cos\theta_\mu) \langle \nabla_\beta \rho, \overline{\mu} \rangle \right) \\
= & - \sum_{\beta,\mu,\rho} \frac{64(\cos\theta_\mu + \cos\theta_\rho)}{\sin^2\theta_\mu \sin^2\theta_\rho} g_{\overline{\beta}} \mu \overline{\rho} g_\beta \rho \overline{\mu}
\end{aligned}$$

$$(5.36)\qquad + \sum_{\beta,\mu,\rho} \frac{64(\cos\theta_\mu + \cos\theta_\rho)}{\sin^2\theta_\mu \sin^2\theta_\rho} |g_\beta \overline{\mu} \rho|^2$$

$$(5.37)\qquad + \sum_{\beta,\mu,\rho} \frac{64i(\cos^2\theta_\mu - \cos^2\theta_\rho)}{\sin^2\theta_\mu \sin^2\theta_\rho} g_{\overline{\beta}} \mu \overline{\rho} \langle \nabla_\beta \rho, \overline{\mu} \rangle$$

$$(5.38)\qquad + \sum_{\beta,\mu,\rho} \frac{64(\cos\theta_\mu + \cos\theta_\rho)}{\sin^2\theta_\mu \sin^2\theta_\rho} |g_\beta \rho \overline{\mu}|^2$$

$$- \sum_{\beta,\mu,\rho} \frac{64(\cos\theta_\mu + \cos\theta_\rho)}{\sin^2\theta_\mu \sin^2\theta_\rho} g_\beta \overline{\mu} \rho g_{\overline{\beta}} \overline{\rho} \mu$$

$$(5.39)\qquad - \sum_{\beta,\mu,\rho} \frac{64i(\cos^2\theta_\mu - \cos^2\theta_\rho)}{\sin^2\theta_\mu \sin^2\theta_\rho} \langle \nabla_\beta \rho, \overline{\mu} \rangle \, g_{\overline{\beta}} \overline{\rho} \mu$$

$$(5.40)\qquad - \sum_{\beta,\mu,\rho} \frac{64i(\cos^2\theta_\mu - \cos^2\theta_\rho)}{\sin^2\theta_\mu \sin^2\theta_\rho} \left\langle \nabla_{\overline{\beta}} \mu, \overline{\rho} \right\rangle g_\beta \rho \overline{\mu}$$

$$(5.41)\qquad + \sum_{\beta,\mu,\rho} \frac{64i(\cos^2\theta_\mu - \cos^2\theta_\rho)}{\sin^2\theta_\mu \sin^2\theta_\rho} \left\langle \nabla_{\overline{\beta}} \mu, \overline{\rho} \right\rangle g_\beta \overline{\mu} \rho$$

$$(5.42)\qquad + \sum_{\beta,\mu,\rho} \frac{64(\cos^2\theta_\mu - \cos^2\theta_\rho)}{\sin^2\theta_\mu \sin^2\theta_\rho} (\cos\theta_\rho - \cos\theta_\mu) \left\langle \nabla_{\overline{\beta}} \mu, \overline{\rho} \right\rangle \langle \nabla_\beta \rho, \overline{\mu} \rangle .$$

Immediately we have, (5.27) + (5.36) = (5.32) + (5.41) = (5.33) + (5.37) = 0, and interchanging μ with ρ in (5.26), (5.34) and in (5.40), we get, (5.26)+

(5.38) = (5.29) + (5.40) = (5.34) + (5.39) = 0. Note that

$$\sum_{\mu,\rho} \frac{(\cos\theta_\mu - \cos\theta_\rho)}{\sin^2\theta_\mu} |\langle \nabla_\beta \mu, \overline{\rho}\rangle|^2 = \sum_{\mu,\rho} \frac{(\cos\theta_\rho - \cos\theta_\mu)}{\sin^2\theta_\rho} \left|\left\langle \nabla_{\overline{\beta}} \mu, \overline{\rho}\right\rangle\right|^2.$$

Hence (5.35) + (5.42) = 0. Then,

$$2\triangle\kappa$$

$$= \sum_{\beta,\mu} \frac{32i}{\sin^2\theta_\mu} R^N(dF(\beta), dF(\overline{\beta}), dF(\mu), JdF(\overline{\mu}) + i\cos\theta_\mu dF(\overline{\mu})) \tag{5.43}$$

$$+ \sum_{\beta,\mu} \frac{64}{\sin^2\theta_\mu} \mathrm{Im}\big(R^N(dF(\beta), dF(\mu), dF(\overline{\beta}), JdF(\overline{\mu}) + i\cos\theta_\mu dF(\overline{\mu}))\big)$$

$$+ \sum_{\beta,\mu,\rho} -\frac{64(\cos\theta_\mu + \cos\theta_\rho)}{\sin^2\theta_\mu \sin^2\theta_\rho} (g_{\overline{\beta}\mu\overline{\rho}} g_{\beta\rho\overline{\mu}} + g_{\beta\overline{\mu}\rho} g_{\overline{\beta}\overline{\rho}\mu}) \tag{5.44}$$

$$+ \sum_{\beta,\mu,\rho} \frac{64(\cos\theta_\rho - \cos\theta_\mu)}{\sin^2\theta_\mu \sin^2\theta_\rho} (|g_{\beta\mu\rho}|^2 + |g_{\overline{\beta}\mu\rho}|^2)$$

$$- \sum_{\beta,\mu,\rho} \frac{64i(\sin^2\theta_\mu + \sin^2\theta_\rho)}{\sin^2\theta_\mu \sin^2\theta_\rho} \left\langle \nabla_{\overline{\beta}} \mu, \rho\right\rangle g_{\beta\overline{\mu}\overline{\rho}} \tag{5.45}$$

$$+ \sum_{\beta,\mu,\rho} \frac{64i(\sin^2\theta_\mu + \sin^2\theta_\rho)}{\sin^2\theta_\mu \sin^2\theta_\rho} \left\langle \nabla_{\overline{\beta}} \overline{\mu}, \overline{\rho}\right\rangle g_{\beta\mu\rho}. \tag{5.46}$$

Using Lemma 5.3, and interchanging ρ by μ when necessary,

$$(5.45) + (5.46)$$

$$= \sum_{\beta,\mu,\rho} -\frac{64i}{\sin^2\theta_\rho} \left\langle \nabla_{\overline{\beta}} \mu, \rho\right\rangle g_{\beta\overline{\mu}\overline{\rho}} - \frac{64i}{\sin^2\theta_\mu} \left\langle \nabla_{\overline{\beta}} \mu, \rho\right\rangle g_{\beta\overline{\mu}\overline{\rho}}$$

$$+ \frac{64i}{\sin^2\theta_\mu} \left\langle \nabla_{\overline{\beta}} \overline{\mu}, \overline{\rho}\right\rangle g_{\beta\mu\rho} + \frac{64i}{\sin^2\theta_\rho} \left\langle \nabla_{\overline{\beta}} \overline{\mu}, \overline{\rho}\right\rangle g_{\beta\mu\rho}$$

$$= \sum_{\beta,\mu,\rho} \frac{-64i}{\sin^2\theta_\mu} \left\langle \nabla_{\overline{\beta}} \mu, \rho\right\rangle \big(g_{\beta\overline{\mu}\overline{\rho}} - g_{\beta\overline{\rho}\overline{\mu}}\big)$$

$$+ \sum_{\beta,\mu,\rho} \frac{64i}{\sin^2\theta_\mu} \left\langle \nabla_{\overline{\beta}} \overline{\mu}, \overline{\rho}\right\rangle \big(g_{\beta\mu\rho} - g_{\beta\rho\mu}\big)$$

$$= \sum_{\beta,\mu,\rho} \frac{64}{\sin^2\theta_\mu} \left\langle \nabla_{\overline{\beta}} \mu, \rho\right\rangle (\cos\theta_\mu + \cos\theta_\rho) \langle \nabla_\beta \overline{\mu}, \overline{\rho}\rangle$$

$$+ \frac{64}{\sin^2\theta_\mu} \left\langle \nabla_{\overline{\beta}} \overline{\mu}, \overline{\rho}\right\rangle (\cos\theta_\mu + \cos\theta_\rho) \langle \nabla_\beta \mu, \rho\rangle$$

$$= \sum_{\beta,\mu,\rho} \frac{64(\cos\theta_\mu + \cos\theta_\rho)}{\sin^2\theta_\mu} \left(\left| \left\langle \nabla_\beta \mu, \rho \right\rangle \right|^2 + \left| \left\langle \nabla_{\overline{\beta}} \mu, \rho \right\rangle \right|^2 \right).$$

Obviously

$$(5.44) = \sum_{\beta,\mu,\rho} \frac{-128(\cos\theta_\mu + \cos\theta_\rho)}{\sin^2\theta_\mu \sin^2\theta_\rho} \mathrm{Re}(g_\beta \mu\overline{\rho} g_{\overline{\beta}} \rho\overline{\mu}).$$

From (1.4), (2.1), and the J-invariance of Ricci,

$$(5.43) = 8i \sum_\beta \mathrm{Ricci}^N(JdF(\beta), dF(\overline{\beta})),$$

and the expression of the Proposition follows. □

After completion and posting of this work in the e-print archive (with no. math.DG/0002050) my attention was drawn to a related paper by A.Ghigi [**G**], published in the meantime, which contains the same result as ours for the case $R \neq 0$ and $n = 2$, but proved in a different way.

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