ON THE STRUCTURE OF THE VALUE SEMIGROUP OF A VALUATION

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Let \( v \) be a valuation of the quotient field of a noetherian local domain \( R \). Assume that \( v \) is centered at \( R \). This paper studies the structure of the value semigroup of \( v \), \( S \). Ideals defining toric varieties can be defined from the graded algebra \( K[T] \) of cancellative commutative finitely generated semigroups such that \( T \cap (-T) = \{0\} \). The value semigroup of a valuation \( S \) need not be finitely generated but we prove that \( S \cap (-S) = \{0\} \) and so, the study in this paper can also be seen as a generalization to infinite dimension of that of toric varieties.

In this paper, we prove that \( K[S] \) can be regarded as a module over an infinitely dimensional polynomial ring \( A_v \). We show a minimal graded resolution of \( K[S] \) as \( A_v \)-module and we give an explicit method to obtain the syzygies of \( K[S] \) as \( A_v \)-module. Finally, it is shown that free resolutions of \( K[S] \) as \( A_v \)-module can be obtained from certain cell complexes related to the lattice associated to the kernel of the map \( A_v \rightarrow K[S] \).

1. Introduction.

Let \((R, m)\) be a noetherian local domain. Denote by \( F \) its quotient field and by \( K \) its residue field. A valuation of \( F \) centered at \( R \) (a valuation in the sequel) is a mapping \( v \) of the multiplicative group of \( F \) onto a totally ordered commutative group \( G \), such that the following conditions are satisfied:

1. \( v(xy) = v(x) + v(y) \);
2. \( v(x + y) \geq \min\{v(x), v(y)\} \);
3. \( v \) is nonnegative on \( R \) and strictly positive on \( m \).

\( G \) is called to be the value group of the valuation \( v \). The set \( S := \{v(f) | f \in R \setminus \{0\} \} \) is a commutative semigroup called the value semigroup of the valuation \( v \). Note that when \( \dim R = 2 \), the most known case, \( S \) contains a lot of information about \( v \). Our aim, in this paper, is to study the structure of \( S \) by extending methods of the toric geometry.

Section 2 of the paper provides some basic properties of the semigroup \( S \). \( S \) need not be finitely generated. However, it satisfies an interesting
property: $S$ is combinatorially finite, i.e., the number of decompositions of any element in $S$ as a finite sum of others in $S$ is finite. When $S$ is finitely generated, this property is equivalent to $S \cap (-S) = \{0\}$. There exists an extensive literature [5, 2, 1, 4], which studies the graded algebra $K[S]$ of cancellative commutative finitely generated semigroups $S$ such that $S \cap (-S) = \{0\}$ (this study includes ideals defining toric varieties). Therefore, we devote Section 3 to extend to the $S$-graded algebra $K[S]$, now $S$ being the value semigroup of a valuation, the ideas of the toric case. Essentially we use the fact that $S$ is combinatorially finite, so our study can also be seen as a generalization to infinite dimension of that of toric varieties.

Subsection 3.1 deals with $K[S]$ regarded as a module over an infinitely dimensional, in general, polynomial ring $A_v$. Both $K[S]$ and $A_v$ are $S$-graded. We construct a minimal graded resolution of $K[S]$ as $A_v$-module and prove that an explicit isomorphism can be given between the (finitely dimensional) vector space of degree $\alpha$ syzygies ($\alpha \in S$) and the vector space of augmented homology of a simplicial complex $\Delta_\alpha$ introduced in [3]. Furthermore, we give a combinatoric method, adapting the one in [4], that allows us to obtain $\bar{H}_i(\Delta_\alpha)$ explicitly from vector space complexes associated to directed graphs. These directed graphs are associated to partitions of certain finite subsets of a generating set of $S$. The most interesting situation arises when the partitions are induced by the value subsemigroup of $S$ defined by a subring $T$ of $R$ such that $v$ is also centered at $T$.

Subsection 3.2 is divided in two parts, their nexus being the fact that the kernel $I_0$ of the mapping $A_v \rightarrow K[S]$, which gives to $K[S]$ structure of $A_v$-module, is spanned by binomials. In 3.2.1, we characterize, by means of a graphic condition, when a set of binomials constitutes a minimal homogeneous generating set of $I_0$. On the other hand, $I_0$ is spanned by a set of binomials satisfying that the difference between their exponents is in a lattice $L$. $L$ is intimately related to the value group $G$ of the valuation (see the beginning of Section 3). In 3.2.2, we associate to $L$ an $A_v$-module $M_L$ and we show how suitable cell complexes on minimal generating sets of the $A_v$-module $M_L$ give rise to free resolutions of $M_L$, called cellular ones, and how some of these resolutions allow us to get free resolutions of $K[S]$ as $A_v$-module.

2. The value semigroup of a valuation.

Let $S$ be a commutative semigroup with a zero element. $S$ is said to be a cancellative semigroup if it satisfies a cancellative law, i.e., if $\alpha, \beta, \gamma \in S$ and $\alpha + \beta = \alpha + \gamma$ then $\beta = \gamma$. Associated to $S$, we can consider an abelian group $G(S)$ and a semigroup homomorphism $i : S \rightarrow G(S)$ satisfying the following universal property: If $H$ is a commutative group and $j : S \rightarrow H$ a semigroup homomorphism, then there exists a unique group homomorphism
Consider the functions \( l : S \to \mathbb{N} \cup \{\infty\} \) and \( t : S \to \mathbb{N} \cup \{\infty\} \) given by

\[
l(\alpha) := \sup \left\{ n \in \mathbb{N} \mid \alpha = \sum_{i=1}^{n} \alpha_i, \text{ where } \alpha_i \in S \setminus \{0\} \right\}
\]

and

\[
t(\alpha) := \text{card} \left\{ \{\alpha_i\}_{i=1}^{n} \text{ finite subset of } S \setminus \{0\} \mid \alpha = \sum_{i=1}^{n} \alpha_i \right\}.
\]

It is clear that if \( \alpha, \beta \in S \) then, \( l(\alpha + \beta) \geq l(\alpha) + l(\beta) \), and also that \( t(\alpha + \beta) \geq t(\alpha) + t(\beta) \).

**Definition 1.** A commutative semigroup with a zero element \( S \) is said to be combinatorially finite (C.F.) if \( t(\alpha) < \infty \) for each \( \alpha \in S \).

**Proposition 1.** Let \( S \) be a C.F. semigroup. Then the following statements hold.

i) For each \( \alpha \in S \), there is no infinite sequence \( \{\alpha_i\}_{i=1}^{\infty} \) of elements in \( S \setminus \{0\} \) such that \( \alpha - \sum_{i=1}^{n} \alpha_i \in S \) whenever \( n \geq 1 \).

ii) \( S \cap (-S) = \{0\} \), where \(-S = \{-x \in G(S) \mid x \in S\}\).

**Proof.**

i) If we had a sequence \( \{\alpha_i\}_{i=1}^{\infty} \) as above, then \( S \) would not be a C.F. semigroup since \( t(\alpha) \) would equal \( \infty \).

ii) Assume that there exists \( \alpha \in S \cap (-S), \alpha \neq 0 \). Write \( \alpha_i = \alpha \) if \( i \) is an even number and \( \alpha_i = -\alpha \) whenever \( \alpha \) is an odd number. Then, the sequence \( \{\alpha_i\}_{i=1}^{\infty} \) contradicts i).

**Corollary 1.** Assume that \( S \) is a C.F. semigroup, then:

i) \( t(\alpha) = 0 \) if, and only if, \( \alpha = 0 \).

ii) \( l(\alpha) = 0 \) if, and only if, \( \alpha = 0 \).

**Proof.** It is clear that \( t(\alpha) \) and \( l(\alpha) \) are not equal to 0 whenever \( \alpha \neq 0 \).

Conversely, \( t(0) \neq 0 \) (or \( l(0) \neq 0 \)) implies \( 0 = \sum_{i=1}^{n} \alpha_i, \alpha_i \in S \setminus \{0\}, n \geq 2 \) and therefore \( S \cap (-S) \neq \{0\} \) which contradicts Proposition 1.

**Remark.** Statement ii) in Proposition 1 allows us to prove the existence of a function \( h : S \to \mathbb{N} \) satisfying \( h(\alpha + \beta) = h(\alpha) + h(\beta) \) for \( \alpha, \beta \in S \) and \( h(0) = 0 \) if, and only if, \( \alpha = 0 \). When \( S \) is finitely generated, the above condition implies \( S \) combinatorially finite. As a consequence, both the existence of \( h \) and statements i) and ii) in Proposition 1 can be taken as a definition of C.F. finitely generated semigroup.

On the other hand, we can not interchange the functions \( t \) and \( l \) in Definition 1, since although \( t(\alpha) < \infty \) for all \( \alpha \in S \) holds whenever \( S \) be a
The value semigroup 

\[ \text{Theorem 1. Let } S \text{ be a valuation of the residue field of } R. \text{ Then:} \]

i) The groups \( G(S) \) and \( G \) are equal. Therefore \( G(S) \) is ordered.

ii) \( S \) is a cancellative ordered commutative semigroup which is torsion free.

iii) \( S \) is a C.F. semigroup.

Proof.

i) \( G \) contains \( S \) and, since \( F \) is the quotient field of \( R \), we have \( G \subseteq G(S) \). Therefore \( G = G(S) \).

ii) Denote by \( R_v = \{ f \in F \setminus \{0\} \mid v(f) \geq 0 \} \) the valuation ring of \( v \). \( R_v \) is a local ring and \( m_v := \{ f \in F \setminus \{0\} \mid v(f) > 0 \} \) is its maximal ideal. Let \( f \in F \setminus \{0\} \) be such that \( v(f) \neq 0 \). Then \( v(f) \) (or \( v(1/f) \)) > 0, so \( f \) (or \( 1/f \)) \( \in m_v \) and thus \( f^p \) (or \( 1/f^p \)) \( \in m_v \) whenever \( p \in N \setminus \{0\} \). As a consequence, \( v(f) \neq 0 \) implies \( v(f^p) \neq 0 \). This proves that \( G \) is a torsion-free group.

iii) Recall that the Krull dimension of \( R_v \) is usually called the rank of \( v \) \( (rk_v) \) and that a \( v \)-ideal of \( R \) is the intersection of \( R \) with an ideal of \( R_v \). \( R \) is a noetherian ring, therefore \( rk_v < \infty \) (see [6, App. 2]) and each \( v \)-ideal \( a \) is spanned by finitely many elements in \( R \), i.e., \( a = \langle h_1, h_2, \ldots, h_r \rangle, h_i \in R \) \( (1 \leq i \leq r) \). If \( \alpha = \min \{v(h_i) \mid i = 1, 2, \ldots, r \} \), then it is straightforward that \( a = P_\alpha := \{ f \in R \mid v(f) \geq \alpha \} \). So, the family \( F = \{ P_\alpha \}_{\alpha \in S} \) consists of all \( v \)-ideals of \( R \).

To prove that \( S \) is C.F., we first assume that \( rk_v = 1 \). Then \( F \) forms a simple infinite descending chain under inclusion [6, Lemma 3, App. 3] and therefore, the elements in \( S \) form a simple infinite ascending chain under the ordering in \( S \). So \( S \) is C.F. Now, apply induction on the rank of \( v \) and assume that \( S \) is not C.F. Then, we can express \( \alpha = \alpha_{i1} + \alpha_{i2}, \alpha, \alpha_{i1}, \alpha_{i2} \in S \) and the sets \( \{ \alpha_{i1} \} \in_{i=1}^\infty \) and \( \{ \alpha_{i2} \} \in_{i=1}^\infty \) are infinite. \( S \) is well-ordered since the set of \( v \)-ideals so is [6, App.3]. Consequently, rearranging the sets \( \{ \alpha_{i1} \} \in_{i=1}^\infty \) and \( \{ \alpha_{i2} \} \in_{i=1}^\infty \), we obtain that one of them constitutes a simple infinite descending chain. To show that this fact is not possible, we only need to observe that \( v \) can be written \( v = u \circ w \), where \( u \) is of rank \( rk_v - 1 \) and \( w \) is a rank one valuation of the residue field of \( u \) and then, apply induction and the
corollary of [6, App. 3], which asserts that if $b_2 \subset b_1$ are two consecutive $\nu$-ideals, then the $\nu$-ideals $a$ such that $b_2 \subset a \subset b_1$ are either finite in number or form a simple descending infinite sequence.

In the sequel, $S$ will denote the value semigroup of a valuation. An element $\alpha \in S$ is said to be irreducible if $l(\alpha) = 1$. Then, we can state the following:

**Corollary 2.** The semigroup $S$ is generated by its irreducible elements. This set need not be finite.

**Proof.** The first statement is clear since $S$ is C.F. Now consider a valuation $v$ centered at a regular 2-dimensional noetherian local ring. Assume that the rank and the rational rank of $v$ equal 1 and that the transcendence degree of $v$ is 0. Finally, suppose that the value group of $v$ is not isomorphic to $\mathbb{Z}$, then, $S$ has an infinite minimal system of generators. These generators are exactly the irreducible elements of $S$ which concludes the proof.

3. The semigroup algebra of a valuation.

Let $v$ be a valuation. Denote by $S$ its value semigroup. The semigroup algebra of $v$ is the semigroup $K$-algebra associated to $S$ and it will be denoted by $K[S]$. $K[S]$ is the $S$-graded $K$-algebra $K[S] = \bigoplus_{\alpha \in S} (K[S])_\alpha$, $(K[S])_\alpha := K\alpha$.

Denote by $\Lambda$ a minimal set of generators of $S$ as semigroup. For instance, we can think of $\Lambda$ as the set of irreducible elements in $S$. $\Lambda$ is, in general, an infinite set. For a set $T$, write $T^{(\Lambda)} = \bigoplus_{\lambda \in \Lambda} T_\lambda$ where $T_\lambda = T$. Consider the mapping $\psi : \mathbb{Z}^{(\Lambda)} \to G(S)$ given by $\psi(\epsilon_\lambda) = \lambda$, $\{\epsilon_\lambda\}_{\lambda \in \Lambda}$ being the standard basis of the $\mathbb{Z}$-module $\mathbb{Z}^{(\Lambda)}$. The ordering in $G(S)$ gives to $\mathbb{Z}^{(\Lambda)}$ an structure of lattice. The kernel of $\psi$, $L$, is a sublattice of $\mathbb{Z}^{(\Lambda)}$ whose intersection with $\mathbb{N}^{(\Lambda)}$ is the origin 0. This can be easily deduced from the fact that $S \cap (-S) = \{0\}$. The morphism $\psi$ induces a surjective $\bar{K}$-algebra homomorphism $\phi_0 : K[\mathbb{N}^{(\Lambda)}] \to K[S]$ which allows to regard $K[S]$ as a $K[\mathbb{N}^{(\Lambda)}]$-module. We shall use two approaches to study the semigroup algebra of $v$. Firstly, we shall construct a minimal free resolution of the $K[\mathbb{N}^{(\Lambda)}]$-module $K[S]$ and we shall study its syzygy modules by means of a concrete simplicial complex and secondly, we shall obtain minimal free resolutions of the former module from certain type of cell complexes on the lattice module $M_L = K[\mathbb{N}^{(\Lambda)} + L] \subseteq K[\mathbb{Z}^{(\Lambda)}]$. In particular, we shall get a more explicit free resolution of $K[S]$.

3.1. Syzygies of the semigroup algebra.

3.1.1. For a start, we state a basic result for the development of this subsection. It holds for semigroups $S$ satisfying $l(\alpha) < \infty$ for all nonzero
element $\alpha \in S$. Thus, we can use it in our case: $S$ is the value semigroup of a valuation. Let $A$ be an $S$-graded ring $A = \bigoplus_{\alpha \in S} A_\alpha$ and $M = \bigoplus_{\alpha \in S} M_\alpha$ an $S$-graded $A$-module.

**Proposition 2** (Graded Nakayama’s Lemma). Let $A$ and $M$ be as above. Denote by $m = \bigoplus_{\alpha \in S, \alpha \neq 0} A_\alpha$ the irrelevant ideal of $A$. If $mM = M$, then $M = 0$.

**Proof.** If $M \neq 0$, then there exists an element $\beta \in S$ such that the degree $\beta$ homogeneous component of $M$, $M_{\beta}$, does not vanish. Now $M_{\beta} = (mM)_{\beta}$ proves that $\beta$ can be written $\beta = \delta + \gamma$; $\delta, \gamma \in S$ and $M_{\gamma} \neq 0$. Iterating, we conclude that $l(\beta)$ is not finite, which is a contradiction.

Now consider the $K$-algebra $K[N(\Lambda)]$ which, for the sake of simplicity, will be expressed as a polynomial ring $K\{X_\lambda|\lambda \in \Lambda\}$ with, possibly, infinitely many indeterminates and it will be denoted by $A_v$. $A_v$ is $S$-graded if we give degree $\lambda \in S$ to the indeterminate $X_\lambda$ and so, we can express $A_v = \bigoplus_{\alpha \in S}(A_v)_\alpha$, where $(A_v)_\alpha$ denotes the homogeneous component of degree $\alpha$ of $A_v$. $(A_v)_\alpha$ is a $K$-vector space. Note that, for any semigroup $S$, we have that $S$ is C.F. if, and only if, $\dim_K (A_v)_\alpha < \infty$ and $l(\alpha) < \infty$ for all $\alpha \in S$. Denote by $M_v$ the irrelevant ideal of $A_v$ and by $I_0$ the kernel of $\phi_0$. $I_0$ is a homogeneous ideal of $A_v$. Let $B$ be a minimal homogeneous generating set of $I_0$ and denote by $B_\alpha$ the set of elements in $B$ of degree $\alpha$. Applying Proposition 2, it is straightforward to deduce that the set of classes in $I_0/M_vI_0$ of the elements of $B_\alpha$ is a basis of the vector space of the homogeneous component of degree $\alpha$ of $I_0/M_vI_0$. $B_\alpha$ is a finite set since $(A_v)_\alpha$ is a finite-dimensional vector space. Set $B_\alpha = \{Q_1, Q_2, \ldots, Q_{d(\alpha)}\}$ and $L_1 := \bigoplus_{\alpha \in S}(A_v)^{d(\alpha)}$. If $\phi_{1,\alpha} : (A_v)^{d(\alpha)} \rightarrow A_v$ is the $A_v$-module homomorphism given by $\phi_{1,\alpha}(a_1, a_2, \ldots, a_{d(\alpha)}) = \sum_{i=1}^{d(\alpha)} a_i Q_i$, then we have the $A_v$-module homomorphism $\phi_1 : L_1 \rightarrow A_v$, $\phi_1 = \sum_{\alpha \in S} \phi_{1,\alpha}$. We give degree $\alpha$ to the generators of $(A_v)^{d(\alpha)}$, thus $L_1$ is an $S$-graded free $A_v$-module and $\phi_1$ a homogeneous homomorphism of degree 0. Repeating this procedure for each syzygy module $I_i := \ker \phi_i$, we get a minimal free resolution of the $S$-graded $A_v$-module $K[S]$:

$$\cdots \rightarrow L_i \xrightarrow{\phi_i} L_{i-1} \rightarrow \cdots \rightarrow L_1 \xrightarrow{\phi_1} A_v \rightarrow K[S] \rightarrow 0.$$  

Tensoring by $K$, we note that there exists a homogeneous degree 0 isomorphism of $S$-graded $A_v$-modules between the $i$-th Tor module $\text{Tor}_i^A_v(K[S], K)$ and $L_i \otimes_{A_v} K$, $i \geq 0$.

On the other hand, we can consider a generalized Koszul complex as follows:

$$\cdots \rightarrow \bigwedge^p A_v^{(\Lambda)} \xrightarrow{d_p} \bigwedge^{p-1} A_v^{(\Lambda)} \rightarrow \cdots \rightarrow A_v^{(\Lambda)} \xrightarrow{d_1} \bigwedge^1 A_v \xrightarrow{d_0} K \rightarrow 0,$$
$d_0$ is the natural obvious epimorphism and if $\{e_\lambda\}_{\lambda \in \Lambda}$ is the standard basis of the $A_v$-module $A_v^{(\Lambda)}$, then we have
\[ d_p(e_J) = \sum_{r=1}^p (-1)^r X_{\lambda_r} e_J \setminus \{\lambda_r\}, \]
where $e_J = e_{\lambda_1} \wedge e_{\lambda_2} \wedge \cdots \wedge e_{\lambda_p}$ whenever $J = \{\lambda_1, \lambda_2, \ldots, \lambda_p\} \subseteq \Lambda$. $\bigwedge^p A_v^{(\Lambda)}$ can be regarded as an $S$-graded $A_v$-module by giving to $e_J$ the degree $\sum_{r=1}^p \lambda_r$. Thus (1) is an $S$-graded free resolution where all the homomorphisms are homogeneous of degree 0.

We shall write $K[S].(\Lambda)$ for the complex obtained by tensoring (1) through with $K[S]$:
\[ \cdots \to \bigwedge^p (K[S])^{(\Lambda)} \xrightarrow{e_p} \bigwedge^{p-1} (K[S])^{(\Lambda)} \to \cdots \to K[S] \xrightarrow{e_0} K \bigotimes_{A_v} K[S] \to 0. \]
The formula for $e_p$ is the same one as $d_p$ but replacing $X_{\lambda_r}$ by $\lambda_r$. Furthermore the homomorphisms $e_p$ are homogeneous of degree 0 under the induced gradings. As a consequence, taking into account the commutative property of the $\text{Tor}$ functor, there exists a homogeneous degree 0 isomorphism of $S$-graded $A_v$-modules between the $i$-th $\text{Tor}$ module $\text{Tor}^A_i(K, K[S])$ and the $i$-th homology module $H_i(K[S].(\Lambda))$.

Finally, for each $\alpha \in S$, we give a $K$-vector space complex isomorphic to that of homogeneous components of degree $\alpha$ in $K[S].(\Lambda)$. Denote by $P(\Lambda)$ the power set of $\Lambda$, $P(\Lambda)$ is an abstract simplicial complex. Set
\[ \Delta_\alpha := \left\{ J \subseteq \Lambda \mid J \text{ is a finite subset of } \Lambda \text{ and } \alpha - \sum_{J \subseteq \Lambda} J \in S \right\}, \]
where $\sum_J = \sum_{\lambda \in J} \lambda$. $\Delta_\alpha$ is a simplicial subcomplex of $P(\Lambda)$. Associate to $\Delta_\alpha$, we consider the complex of vector spaces $C_i.(\Delta_\alpha)$ such that its vector spaces are $C_i(\Delta_\alpha) = \bigoplus_{J \in \Delta_\alpha, \text{card}(J) = i+1} K J$, $i \geq -1$ and its boundaries $\partial : C_i(\Delta_\alpha) \rightarrow C_{i-1}(\Delta_\alpha)$ are given by $\partial(J) = \sum_{\beta \subseteq J} (-1)^{\eta_J(\beta)} J \setminus \{\beta\}$, where $\eta_J(\beta)$ denotes the number of place that $\beta$ has among the elements in $J$. The homology of this complex will be called the augmented homology of $\Delta_\alpha$. This subsection can be summarized in the following:

**Theorem 2.** For each $\alpha \in S$, there exists an explicit isomorphism of $K$-vector spaces between the vector space $(I_i)_\alpha/(M_i I_i)_\alpha$ of $i$-th syzygies of degree $\alpha$ of $K[S]$ as $A_v$-module and the $i$-th augmented homology vector space of the simplicial complex $\Delta_\alpha$, $H_i(\Delta_\alpha)$.

3.1.2. We devote this subsection to show how bases for the homology $H_i(\Delta_\alpha)$ can be explicitly computed from bases of the homology of vector space complexes associated to directed graphs which depend on the set $\Lambda$. 


This will be done adapting the results by Campillo and Gimenez in the case of toric affine varieties [4]. To start with, we describe the type of vector space complexes which we shall use to compute $H_i(\Delta)$, Assume that $\Gamma$ is a subset of $\Lambda$, which is a finite set of generators of a semigroup $T$, and $B$ a subset of $T$. We shall call the directed graph of $T$ associated to the pair $(\Gamma, B)$ to the directed graph $G_{\Gamma B}(T)$ (denoted $G_{\Gamma B}$ if it does not cause confusion) whose vertex set is $\{m \in T| m - \sum L \in B$ for some subset $L \subseteq \Gamma \}$ and such that $(m, m')$ is an edge iff $m' = m + \gamma$ for some $\gamma \in \Gamma$. A $K$-vector space complex $C_i(G_{\Gamma B}(T, m))$ can be associated to the pair $(G_{\Gamma B}, m)$, $m$ being a vertex of $G_{\Gamma B}$, with the following condition holds: Whenever $b \in B$ and $\lambda, \lambda' \in \Gamma$ satisfy $b + \lambda + \lambda' \in B$, then $b + \lambda \in B$ and $b + \lambda' \in B$. In such a case $G_{\Gamma B}$ is called to be a chain graph. Each vector space $C_i(G_{\Gamma B}(T, m))$, $i \geq -1$, is equal to $\bigoplus KL$ where the sum is over all subsets $L$ of $\Gamma$ of cardinality $i + 1$ such that $m - \sum L \in B$. The boundaries are induced by those of the simplicial complex $P(\Lambda)$.

Next, we state the main result of this subsection.

**Theorem 3.** The homology $\tilde{H}_i(\Delta)$ can be explicitly reached from finitely many homologies of $K$-vector space complexes of the type $C_i(G_{\Gamma B}(T, m))$ for suitable $T, \Gamma, B$ and $m$.

To reach a homology from others means to obtain bases of the homology from bases of the others by means of exact sequences. Let’s see how to reach $H_i(\Delta)$. Let $S_\alpha = \{a' \in S| a - a' \in S\}$. $S_\alpha$ is finite since $S$ is C.F. Denote by $S_\alpha$ the subsemigroup of $S$ spanned by $S_\alpha$. It is not difficult to prove that $\Delta_\alpha = \{J \subseteq S_\alpha| a - \sum L \in S_\alpha\}$. Now, pick a partition of $S_\alpha$, $S_\alpha = \Omega_\alpha \cup \Pi_\alpha$, consider the Apery set of $S_\alpha$ relative to $\Pi_\alpha$:

$$A(\alpha) = A = \{a \in S_\alpha| a - e \notin S_\alpha \text{ for all } e \in \Pi_\alpha\}$$

and the related set

$$K_\alpha := \left\{ L \subseteq S_\alpha| L \cap \Pi_\alpha \neq \emptyset \text{ and } a - \sum L \in S_\alpha \right\} \cup \left\{ L \subseteq \Omega_\alpha| a - \sum L \in S_\alpha \setminus A \right\}.$$  

There is no loss of generality in assuming that $\alpha$ is a vertex of $G_{\Omega_\alpha A}(S_\alpha)$ and then, it is clear that the complex associate to $(G_{\Omega_\alpha A}, \alpha)$ makes sense. It will be denoted $C_\alpha(A(\alpha))$ and it is exactly the augmented relative simplicial complex $\tilde{C}_\alpha(\Delta, K_\alpha)$. Therefore, we can state the following long exact sequence, which allows to reach the homology $\tilde{H}_i(\Delta)$ from others.

$$\cdots \rightarrow H_{i+1}(A_\alpha) \rightarrow \tilde{H}_i(K_\alpha) \rightarrow \tilde{H}_i(\Delta) \rightarrow H_i(A_\alpha) \rightarrow \tilde{H}_{i-1}(K_\alpha) \rightarrow \cdots$$
$H_{i+1}(A_\alpha)$ and $H_i(A_\alpha)$ are as we desire. Let us see that $\tilde{H}_i(K_\alpha)$ and $\tilde{H}_{i-1}(K_\alpha)$ so are. Firstly, define the simplicial complex

$$\overline{K}_\alpha := K_\alpha \cup \left\{ L = I \cup J \mid I \subseteq \Omega_\alpha, J \subseteq \Pi_\alpha, \text{card}(J) \geq 2, \alpha - \sum_{I \cup J} \not\in S_\alpha \right\}$$

but $\alpha - \sum_{I} -e \in S_\alpha$ for each $e \in J$

and the subcomplexes of $\overline{K}_\alpha$,

$$K_\alpha(j) := K_\alpha \cup \{ L = I \cup J \in \overline{K}_\alpha \setminus K_\alpha \mid \text{card}(J) \leq j \},$$

$1 \leq j \leq \text{card}(\Pi_\alpha)$. $\overline{K}_\alpha$ is acyclic and so $\tilde{H}_{i+1}(\overline{K}_\alpha, K_\alpha) \cong \tilde{H}_i(K_\alpha)$.

Also $\tilde{H}_i(\overline{K}_\alpha, K_\alpha) \cong \tilde{H}_i(K_\alpha(\text{card}(\Pi_\alpha)), K_\alpha(1))$. This last homology can be reached from $\tilde{H}_i(K_\alpha(j), K_\alpha(j - 1)), 2 \leq j \leq \text{card}(\Pi_\alpha)$, since the following exact sequence of vector space complexes

$$0 \to C.(K_\alpha(j), K_\alpha(i)) \to C.(K_\alpha(k), K_\alpha(i)) \to C.(K_\alpha(k), K_\alpha(j)) \to 0$$

holds for sequences $(i, j, k)$ equal to $(1, 2, 3), (1, 3, 4), \ldots, (1, \text{card}(\Pi_\alpha) - 1, \text{card}(\Pi_\alpha))$. As a consequence, we only need to show that the homology $\tilde{H}_i(K_\alpha(j), K_\alpha(j - 1))$ can be computed from finitely many homologies of complexes associated to chain graphs. Indeed, a subset $J \subseteq \Pi_\alpha$ with $\text{card}(J) \geq 2$ is said to be associated to $d \in S_\alpha$, if $d - \sum_{J \not\in S_\alpha} -e \in S_\alpha$ but $d - e \in S_\alpha$ for each $e \in J$. If we denote by $D^J_\alpha$ the set of elements $d$ in $S_\alpha$ such that $J$ is associated to $d$, then

$$\tilde{H}_i(K_\alpha(j), K_\alpha(j - 1)) \cong \bigoplus_{J \subseteq \Pi_\alpha, \text{card}(J) = j} \tilde{H}_{i-j}\left( G_{\Omega_\alpha D^J_\alpha}(S_\alpha, \alpha) \right).$$

A further study leads us to obtain finite subsets of $S_\alpha$, such that $\tilde{H}_i(\Delta_\alpha)$ vanishes when $\alpha$ does not belong to them. In fact, for $-1 \leq l \leq \text{card}(\Omega_\alpha)$ write

$$M_\alpha(l) := K_\alpha \cup \{ L = I \cup J \in \overline{K}_\alpha \setminus K_\alpha \mid \text{card}(I) \leq l \}.$$ 

As above,

$$(3) \quad \tilde{H}_i(\overline{K}_\alpha, K_\alpha) \cong \tilde{H}_i(M_\alpha(\text{card}(\Omega_\alpha)), M_\alpha(-1)).$$

This last homology can be reached from $\tilde{H}_i(M_\alpha(l), M_\alpha(l - 1))$ and

$$\tilde{H}_i(M_\alpha(l), M_\alpha(l - 1)) \cong \bigoplus \tilde{H}_{i-l}(\Theta_{\alpha - \sum I}),$$

where the sum is over all subsets $I \subseteq \Omega_\alpha$ such that $\text{card}(I) = l$ and $\alpha - \sum_{I} \in S_\alpha$, and where $\Theta_d = \{ J \subseteq \Pi_\alpha \mid d - \sum_{J} \in S_\alpha \}$. Consequently, (2) and (3)
prove that if we consider
\[
C_i(\alpha) := \left\{ m \in S_\alpha \mid m = a + \sum_{l} a \in A(\alpha), I \subseteq \Omega_\alpha \text{ and card}(I) = i + 1 \right\}
\]
\[
\cup \left\{ m \in S_\alpha \mid \exists I \subseteq \Omega_\alpha, \text{card}(I) = l \leq i \text{ with } \bar{H}_{l-i}(\Omega_{m-\sum I}) \neq 0 \right\},
\]
then \(\bar{H}_i(\Delta_\alpha) = 0\) if \(\alpha \notin C_i(\alpha)\). The simplicity of the set \(\Theta_d\) has an important consequence:

**Proposition 3** (See [4, Pr. 6.2]). The set \(C_i(\alpha)\) is finite when we choose a suitable partition of the set \(S_\alpha\).

A crucial fact in the above proposition is that \(S_\alpha\) is finitely generated. A suitable partition of \(S_\alpha\) would be a convex partition, that is, a partition \(S_\alpha = \Omega_\alpha \cup \Pi_\alpha\) where the cone generated by \(S_\alpha\) (in \(V_{S_\alpha} := G(S_\alpha) \otimes_{\mathbb{Z}} \mathbb{Q}\)) is equal to the cone generated by \(\Omega_\alpha\) (in \(V_{S_\alpha}\)) and \(\text{card}(\Omega_\alpha)\) equals to the number of extremal rays of the cone spanned by \(S_\alpha\).

**3.2. The defining ideal of the semigroup.** The \(K\)-algebra \(K[S]\) is isomorphic to \(A_v/I_0\). The ideal \(I_0\), usually called the defining ideal of \(S\), is spanned by a set of binomials which are difference of two monomials of the same degree. This set need not be finite. In the first part of this subsection, we shall use [2] to give a method to compute a minimal homogeneous generating set of \(I_0\), \(B\), formed by binomials of the type described above. This method uses the structure of graph of the simplicial complex \(\Delta_\alpha\). On the other hand, denote by \(L_v = K[\{X^\pm_\lambda\}_{\lambda \in \Lambda}]\) the Laurent polynomial ring associate to the set \(\Lambda\) and write \(X^a = \prod_{\lambda \in \Lambda'} X^a_\lambda \in L_v\) whenever \(a = \sum_{\lambda \in \Lambda'} a_\lambda e_\lambda \in \mathbb{Z}[\Lambda], \Lambda'\) being a finite subset of \(\Lambda\). Obviously, \(A_v \subseteq L_v = K[\mathbb{Z}[\Lambda]]\). Recalling the notation at the beginning of Section 3, we observe that

\[
(I_0) = \langle X^a - X^b \mid a - b \in L \rangle \subset A_v.
\]

Following the ideas of [1], this fact will serve us, in the second part of this subsection, to obtain minimal free resolutions of \(K[S]\) as \(A_v\)-module from suitable cell complexes on \(M_L\).

**3.2.1. Minimal generating sets of the defining ideal.** A minimal homogeneous generating set of \(I_0\), \(B\), can be expressed \(B = \cup_{\alpha \in S} B_\alpha\), where \(B_\alpha\) is the set of elements in \(B\) of degree \(\alpha\). As a consequence of 3.1.1, we have that \(B_\alpha\) is a finite set and card \(B_\alpha = \dim_K \bar{H}_0(\Delta_\alpha)\). Moreover, \(\Delta_\alpha\) is a graph which has \(\dim_K \bar{H}_0(\Delta_\alpha) + 1\) connected components. If \(a = \sum_{\lambda \in \Lambda'} a_\lambda e_\lambda \in \mathbb{N}[\Lambda] (a_\lambda \neq 0)\), then \(X^a \in A_v\), the support of \(X^a\), \(\text{Supp}(X^a)\), is the set \(\Lambda'\) and the degree of \(X^a\), \(\text{deg}(X^a)\), is \(\sum_{\lambda \in \Lambda'} a_\lambda \lambda \in S\).

It is clear that \(I_0\) is an ideal generated by the set of binomials \(B = \{X^a - X^b \mid \text{deg}(X^a) = \text{deg}(X^b)\}\). Let \(C\) be a subset of \(B\) whose binomials
have a fixed degree $\alpha$. We shall call graph associated to $C$ to a graph whose vertex set is the set of connected components of $\Delta_\alpha$ which contain the support of a monomial belonging to a binomial in $C$. Two connected components, those associated to the monomials $X^a$ and $X^b$, are adjacent by an edge whenever $X^a - X^b \in C$. $C$ will be a generating tree for $\Delta_\alpha$ if the graph associated to $C$ is, in fact, a tree.

**Theorem 4.** A subset $B = \cup_{\alpha \in S} B_\alpha \subseteq B$ is a minimal homogeneous generating set of $I_0$ if, and only if, $B_\alpha$ is a generating tree for $\Delta_\alpha$ whenever $\dim_K \tilde{H}_0(\Delta_\alpha) \neq 0$ and $B_\alpha = \emptyset$, otherwise.

This theorem is analogous to the stated in [2] for finitely generated semigroups and the proof runs similarly. It is based on the fact that two monomials $M$ and $M'$ of degree $a \in S$ satisfy $M - M' \in (M, I_0)_\alpha$ if, and only if, $\text{Supp}(M)$ and $\text{Supp}(M')$ are in the same connected component of $\Delta_\alpha$. Furthermore, it is possible to decide whether $\dim_K \tilde{H}_0(\Delta_\alpha) \neq 0$ by a close method to that given in [2, Th. 3.11].

### 3.2.2. Cellular resolutions of $K[S]$. For a start, we establish a relation between the module $M_L = K[\mathbb{N}^{(\lambda)} + L]$ and the semigroup algebra of $v$, $K[S]$. Denote by $A_v[L]$ the group algebra of $L$ over $A_v$. $A_v[L]$ is the subalgebra of $K[\{X_\lambda\}_{\lambda \in \Lambda}, \{Z_\lambda^\pm\}_{\lambda \in \Lambda}]$ generated by the monomials $X^a Z^l$ where $a \in \mathbb{N}^{(\lambda)}$ and $l \in L$. Thus, we can give a $Z^{(\lambda)}$-grading on $A_v[L]$ by writing $\deg(X^a Z^l) = a + l$. On the other hand, the morphism $h : A_v[L] \rightarrow M_L$, $X^a Z^l \rightarrow X^{a+l}$ gives to $M_L$ an structure of $Z^{(\lambda)}$-graded $A_v[L]$-module. Moreover, if $J = \text{Ker}(h)$, then the following equality chain holds,

$$M_L \otimes_{A_v[L]} A_v = A_v[L]/J \otimes_{A_v[L]} A_v = A_v/I_0 = K[S].$$

Next, we shall consider two equivalent categories $\mathcal{A}$ and $\mathcal{B}$. $\mathcal{A}$ contains $M_L$, and $K[S]$, viewed as $A_v$-module, is in $\mathcal{B}$. This shall give the desired relation between $M_L$ and $K[S]$. $\mathcal{A}$ will be the category of $Z^{(\lambda)}$-graded $A_v[L]$-modules, where the morphisms are $Z^{(\lambda)}$-graded $A_v[L]$-module homomorphisms of degree 0, and $\mathcal{B}$ the category of $G(S)$-graded $A_v$-modules, where the morphisms are, also, of degree 0. Note that $K[S]$ is $S$-graded and therefore $G(S)$-graded. The functor $\pi : \mathcal{A} \rightarrow \mathcal{B}$ which gives the equivalence is $\pi(M) = M \otimes_{A_v[L]} A_v$. Notice that if $M \in \mathcal{A}$, $M = \bigoplus_{a \in \mathbb{Z}^{(\lambda)}} M_a$, then $\pi$ identifies as $\pi(M)_\alpha$, $\alpha \in G(S)$, all the vector spaces $M_a$ such that $\psi(a) = \alpha$, where $\psi$ is the mapping given at the beginning of Section 3. A complete proof of this equivalence is similar to that of the case of finitely generated semigroups [1, Th. 3.2] and we omit it.

Now, taking into account that the degrees of $M_L$ are in $\mathbb{N}^{(\lambda)} + L$, we can state:
Theorem 5. Let \( \pi : \mathcal{A} \to \mathcal{B} \) be the equivalence of categories above given. Then \( \pi \) transforms \( \mathbf{Z}^{(\Lambda)} \)-graded (minimal) free resolutions of \( M_L \) as \( A_v[L] \)-module into \( S \)-graded (minimal) free resolutions of \( K[S] \) as \( A_v \)-module, and conversely.

Finally, we shall see how to get free resolutions of \( M_L \) from regular cell complexes and, consequently, how to get free resolutions of \( K[S] \). First at all, denote by \( \leq \) the ordering in \( \mathbf{Z}^{(\Lambda)} \) defined so: \( a \leq b \) if, and only if, \( b-a \in \mathbf{N}^{(\Lambda)} \). Also, set \( \min(M_L) := \{ X^a \in M_L \mid X^a/X^b \notin M_L \text{ for all } \lambda \in \Lambda \} \).

Proposition 4. The \( \mathbf{Z}^{(\Lambda)} \)-graded \( A_v \)-module \( M_L \) satisfies the following properties:

i) The set of monomials in \( M_L \) of degree \( \leq a \) is finite for each \( a \in \mathbf{Z}^{(\Lambda)} \).

ii) \( M_L \) is generated as \( A_v \)-module by the set \( \min(M_L) \).

Proof.

i) Write \( a = \sum_{\lambda \in \Lambda} \alpha_\lambda e_\lambda \) and set \( a^+ = \sum_{\lambda \in \Lambda, \lambda < 0} a_\lambda e_\lambda \) and \( a^- = \sum_{\lambda \in \Lambda, \lambda > 0} a_\lambda e_\lambda \). If \( d \) is the degree of a monomial in \( M_L \), then \( d = l + b^+ \), where \( l \in L \) and \( b^+ \in \mathbf{N}^{(\Lambda)} \). It is clear that, as above, \( l = l^+ + l^- \). If \( d \) is finite, then \( l^+ + b^+ + l^- \leq a^+ + a^- \). As a consequence, the set \( \{ l^+ \mid d \leq a \} \) is finite and so is the set \( \{ \psi(l^+) \mid d \leq a \} \) \( \subseteq S \). Finally, \( \{ l^+ \mid d \leq a \} \) is also a finite set, since \( S \) is a C.F. semigroup.

ii) This is a straightforward consequence of the fact that, there is no infinite decreasing sequence under divisibility of monomials in \( M_L \), which follows from i).

Put \( \min(M_L) = \{ X^a \mid a \in I \subset \mathbf{Z}^{(\Lambda)} \} \). \( I \) is, generally, an infinite set. Consider a regular cell complex \( X \) such that \( I \) is its set of vertices and \( \epsilon \) an incidence function on pairs of faces. A typical example of a regular cell complex is the set of faces of a convex polytope.

Associated to \( X \), a cellular complex of \( A_v \)-modules \( M(X) \) can be defined in the following way: The modules are \( M_i(X) = \bigoplus_{j \in X, \dim j = i} A_v J, \) \( i \geq 0 \), (we have identified the face \( J \) in \( X \) with its set of vertices) and the boundaries are given by

\[
\partial J = \sum_{j' \in X, j' \neq \emptyset} \epsilon(J, J')(m_J/m_{J'})J',
\]

where \( m_J \) is the least common multiple of the set \( \{ X^a \mid a \in J \} \). \( M(X) \) is \( \mathbf{Z}^{(\Lambda)} \)-graded, the degree of a face \( J \) being the exponent vector of \( m_J \). When \( M(X) \) is a free resolution of \( M_L \), it is called to be a cellular resolution of \( M_L \). Set \( \Delta = \{ J \in P(I) \mid J \text{ is a finite set} \} \) and associate to \( \Delta \) an incidence function as in the definition of \( \Delta_\alpha \) (see 3.1.1). \( \Delta \) is a cell complex and its associated cellular complex \( M(\Delta) \) is a cellular resolution of \( M_L \) called the Taylor resolution of \( M_L \). This is an easy consequence of the fact that
the subcomplex $\Delta \leq a$ of $\Delta$ on the vertices of degree $\leq a$ is acyclic for all $a \in \mathbb{N}^{(\Lambda)}$.

We desire to apply Theorem 5 to get free resolutions of $K[S]$. In order to do it, we observe that the mapping $\bigoplus_{J \in \mathcal{R}} A_v[L]J \rightarrow M_i(X)$, $Z^lJ \rightarrow J + l$ is an isomorphism of $\mathbb{Z}^{(\Lambda)}$-graded $A_v$-modules if $X$ satisfies that

$$J + l \in X \text{ whenever } J \in X \text{ and } l \in L,$$

$\mathcal{R}$ being a set of representatives of the set of $i$-dimensional orbits defined by the action of $L$ over $X$. Thus, we shall call to $X$ equivariant if it satisfies (5) and $\epsilon(J, J') = \epsilon(J + l, J' + l)$ for all $l \in L$. If $X$ is equivariant, it is straightforward that $M.(X)$ is a $\mathbb{Z}^{(\Lambda)}$-graded complex of $A_v[L]$-modules and that $M.(X)$ is exact over $A_v$ if, and only if, it is exact over $A_v[L]$. In this case, $M.(X)$ is called an equivariant cellular resolution of $M_L$. Applying Theorem 5, we have proved the following:

**Theorem 6.** Let $S$ be the value semigroup of a valuation. If $M.(X)$ is a (minimal) equivariant cellular resolution of $M_L$, then $\pi(M.(X))$ is a (minimal) free resolution of $K[S]$ as $A_v$-module.

$\triangle$ is an equivariant cell complex. Its simplicity allows us to give an explicit resolution of $K[S]$ as $A_v$-module. For each $\alpha \in S$, denote by mon$(A_v)_{\alpha}$ the set of monomials in $(A_v)_{\alpha}$ and by $E_i(\alpha)$ the set of cardinality $i$ subsets of mon$(A_v)_{\alpha}$ whose greatest common divisor is 1. Now, if $F_i(a)$ denotes the set of cardinality $i$ subsets of min$(M_L)$ whose least common multiple is $a \in \mathbb{Z}^{(\Lambda)}$, it is clear, from the definition of $M.(\triangle)$, that $M_i(\triangle) = \bigoplus_{a \in \mathbb{N}^{(\Lambda)} + L} A_vF_i(a)$.

Regarding $M_i(\triangle)$ as $A_v[L]$-module and by Theorem 5, it is clear that $\pi$ takes $F_i(a)$ bijectively to $E_i(\tilde{\psi}(a))$, $\pi(J) = \{X^a/X^c \mid X^c \in J\}$. As a consequence $\pi(M.(\triangle))$ can be expressed so: The $A_v$-modules are $\bigoplus_{\alpha \in S} A_vE_i(\alpha)$ and the boundaries are given by

$$\partial(I) = \sum_{X^c \in I} (-1)^{\eta(I)} \gcd(I \setminus \{X^c\})[I \setminus \{X^c\}],$$

where $I \in E_i(\alpha)$, $\eta_I$ is defined as in 3.1.1 and $[I \setminus \{X^c\}]$ means to remove the common factor $\gcd(I \setminus \{X^c\})$ from $I \setminus \{X^c\}$.

**References**


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