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To Professor Jacques Faraut on his sixtieth birthday

Let \mathcal{D} be a bounded symmetric domain of tube type and \underline{G} its group of holomorphic automorphisms. In this paper, we describe explicitly the Plancherel Theorem of weighted Bergman spaces on \mathcal{D} under the action of certain symmetric subgroups of \underline{G} .

1. Introduction.

Let \underline{G} be a noncompact connected real semi-simple Lie group with finite center and Lie algebra $\underline{\mathfrak{g}}$. Let θ be a Cartan involution of \underline{G} and $\underline{K} = \{g \in \underline{G} \mid \theta(g) = g\}$. We use the same letter θ to denote the differential of θ . Then, we have a direct sum decomposition $\underline{\mathfrak{g}} = \underline{\mathfrak{k}} \oplus \underline{\mathfrak{p}}$ in eigenspaces with respect to θ . We assume that \underline{G} is hermitian, then there exists an element Z_0 in the center $\mathfrak{c}(\underline{\mathfrak{k}})$ of $\underline{\mathfrak{k}}$ such that $\mathfrak{c}(\underline{\mathfrak{k}}) = \mathbb{R}Z_0$.

Let σ be an involutive automorphism of \underline{G} . We may assume that σ commutes with θ and $\underline{\mathfrak{g}} = \underline{\mathfrak{h}} \oplus \underline{\mathfrak{q}}$ is the decomposition of the Lie algebra $\underline{\mathfrak{g}}$ with respect to σ . Since $\sigma^2 = \text{id}$, there are two exclusive possibilities. Either $\sigma(Z_0) = Z_0$ and σ acts holomorphically on the symmetric domain $\mathcal{D} := \underline{G}/\underline{K}$, or $\sigma(Z_0) = -Z_0$ and σ acts anti-holomorphically on \mathcal{D} . In this paper we consider the case where σ is holomorphic. The case where σ is anti-holomorphic is considered by Yu. A. Neretin (cf. [22], [23]). See also [8] and [30].

Let $\mathcal{H}_\ell^2(\mathcal{D})$ be the ordinary Bergman space of \mathcal{D} where \mathcal{D} is of tube type. For $\nu > \ell - 1$, we consider a weighted Bergman space $\mathcal{H}_\nu^2(\mathcal{D})$ of holomorphic functions on \mathcal{D} . The universal covering $\tilde{\underline{G}}$ of \underline{G} can be realized as the set of pairs (g, φ) with $g \in \underline{G}$ and φ a holomorphic function on \mathcal{D} where $e^{\varphi(z)} = \det(Dg(z))$. Here $Dg(z)$ denote the differential of the map $z \mapsto g \cdot z$. The group $\tilde{\underline{G}}$ acts in $\mathcal{H}_\nu^2(\mathcal{D})$ by

$$(U_\nu(\tilde{g})f)(z) = e^{\nu\varphi(z)} f(g \cdot z), \quad \tilde{g}^{-1} = (g, \varphi).$$

The representation U_ν is a unitary and irreducible representation.

Let p be the universal map of $\tilde{\underline{G}}$ in \underline{G} , and let G be a symmetric subgroup of \underline{G} . In this work we study the decomposition of the restriction of

U_ν to the subgroup $\tilde{G} := p^{-1}(G)$ of \tilde{G} . By [15] (see also [16], [17]) the restriction $U_\nu|_{\tilde{G}}$ is decomposed multiplicity-free and discretely into irreducible representations $(\pi_\mu, \mathcal{H}_\mu)$ of \tilde{G} such that $\mathcal{H}_\mu \subset \mathcal{H}_\nu^2(\mathcal{D})$.

Let \mathcal{S} be the Shilov boundary of \mathcal{D} . The action of the group G on \mathcal{S} admits open orbits. We consider one of the orbits which is a causal symmetric space G/H of compact type. Moreover G/H is a symmetric Makarevič space. The geometry and analysis of the domain \mathcal{D} and the Makarevič space G/H can be described using Jordan algebras.

To study the decomposition of $\mathcal{H}_\nu^2(\mathcal{D})$, we consider a G -invariant domain Ξ in the complexification $G_{\mathbb{C}}/H_{\mathbb{C}}$ of G/H introduced by J. Hilgert, B. Ørsted and G. Ólafsson (cf. [14]). A geometric description of the domain Ξ is given by W. Bertram. The domain Ξ can be realized as $\mathcal{D} \setminus \Sigma$ where Σ is an analytic set (cf. [3]).

We consider a covering $\tilde{\Xi}$ of Ξ with infinite order. We show that there is a unitary isomorphism of $\mathcal{H}_\nu^2(\mathcal{D})$ onto a weighted Bergman space $\mathcal{H}_\nu^2(\tilde{\Xi})$. It is a Hilbert space of holomorphic functions on $\tilde{\Xi}$, which satisfy a monodromy condition and are square integrable with respect to a G -invariant measure on $\tilde{\Xi}$.

To describe explicitly the decomposition of $\mathcal{H}_\nu^2(\tilde{\Xi})$ into irreducible subspaces we study the holomorphic discrete series of the universal covering \tilde{G} . Our approach is based on the spherical Laplace transform associated with the ordered symmetric space G^c/H dual of G/H . See [1] for $G/H \simeq U(p, q)$ and [2] for G/H of Cayley type.

This paper is organized as follows: In Section 2, we give a geometric description of the covering $\tilde{\Xi}$ of Ξ using the theory of Jordan algebras. In Section 3, we study the Bergman space $\mathcal{H}_\nu^2(\tilde{\Xi})$ and its reproducing kernel and we establish a unitary isomorphism of $\mathcal{H}_\nu^2(\tilde{\Xi})$ onto $\mathcal{H}_\nu^2(\mathcal{D})$. To describe explicitly the spectrum of $\mathcal{H}_\nu^2(\tilde{\Xi})$ and to express its reproducing kernel as series of spherical functions associated with the ordered symmetric spaces G^c/H , we study in Section 4 the holomorphic discrete series of \tilde{G} . In particular, we obtain a necessary condition for π_μ to appear in the Plancherel formula. In Section 5, we compute explicitly the L^2 -norm of matrix coefficient associated with an H -spherical unitary highest weight representation. Then, we can state an explicit Plancherel Theorem. The case $G = \underline{K}$ is due to W. Schmid (cf. [28]). See also [11], [12], [26] and [29].

2. Geometric realization of the covering $\tilde{\Xi}$.

Let \mathbb{V} be a Euclidean Jordan algebra, and let Ω be the associated symmetric cone. We denote the dimension of \mathbb{V} by n , the rank by r , and the unit element

by e . A Euclidean Jordan algebra is said to be simple if it has no nontrivial ideal (cf. [12], Chapter II).

Let \mathcal{D} be the unit disc of $\mathbb{V}_{\mathbb{C}} := \mathbb{V} + i\mathbb{V}$ with respect to the spectral norm

$$\mathcal{D} := \{ z \in \mathbb{V}_{\mathbb{C}} \mid e - z \square \bar{z} \gg 0 \},$$

where $z \square w := L(zw) + [L(z), L(w)]$. Here $L(z)$ denotes the endomorphism of $\mathbb{V}_{\mathbb{C}}$ defined by $L(z)w = zw$.

Let \underline{G} be the group of holomorphic automorphisms of \mathcal{D} and let \underline{K} be the isotropy subgroup of 0 in \underline{G} . It is a maximal compact subgroup of \underline{G} . The Lie algebra $\underline{\mathfrak{g}}$ of \underline{G} consists of vector fields of the form

$$X(z) = w + Tz - P(z)\bar{w},$$

where $w \in \mathbb{V}_{\mathbb{C}}$, $T \in \underline{\mathfrak{k}} := \text{Lie}(\underline{K})$ and $P(z) := 2L(z)^2 - L(z^2)$. The application P is called a quadratic representation associated with $\mathbb{V}_{\mathbb{C}}$. We identify a vector field X with the triplet (w, T, \bar{w}) .

Let α be an involutive automorphism of the Jordan algebra \mathbb{V} . Denote also α its \mathbb{C} -linear extension to $\mathbb{V}_{\mathbb{C}}$. The Jordan algebra \mathbb{V} and its complexification $\mathbb{V}_{\mathbb{C}}$ decompose into eigenspaces with respect to the involution α

$$\mathbb{V} = \mathbb{V}^+ \oplus \mathbb{V}^-, \quad \mathbb{V}_{\mathbb{C}} = \mathbb{V}_{\mathbb{C}}^+ \oplus \mathbb{V}_{\mathbb{C}}^-.$$

We say that the pair (\mathbb{V}, α) is irreducible if it is not possible to write

$$(\mathbb{V}, \alpha) = (\mathbb{V}_1 \oplus \mathbb{V}_2, \alpha_1 \oplus \alpha_2).$$

We show that if (\mathbb{V}, α) is irreducible then either \mathbb{V} is simple, or $\mathbb{V} = \mathbb{V}_{\circ} \times \mathbb{V}_{\circ}$ where \mathbb{V}_{\circ} is a simple Euclidean Jordan algebra and $\alpha(x, y) = (-y, -x)$. We note that \mathbb{V}^+ is either simple or a direct sum of two simple algebras.

Let $\{c_1, \dots, c_r\}$ be a Jordan frame of \mathbb{V} . It is a complete system of orthogonal primitive idempotent elements. The algebra $R := \bigoplus_{j=1}^r \mathbb{R}c_j$ is a maximal associative subalgebra of \mathbb{V} . Assume α is given such that $\alpha(R) = R$, then $R = R^+ \oplus R^-$ is the decomposition of R into eigenspaces with respect to α . We note $r^+ := \dim R^+$.

Theorem 2.1 (cf. [4]). *Let \mathbb{V} be a Euclidean Jordan algebra and let α be an involutive automorphism of \mathbb{V} .*

- (1) *The rank of the Euclidean Jordan algebra \mathbb{V}^+ is equal to r^+ .*
- (2) *Either $R = R^+$ and $r = r^+$, or $r = 2r^+$ and $\dim R^+ = \dim R^-$.*

Let

$$\underline{G}^{(-\alpha)} := \{g \in \underline{G} \mid (-\alpha) \circ g \circ (-\alpha) = g\},$$

and let G be its connected identity component. In particular if $\alpha = \text{id}_{\mathbb{V}}$ then $G = \underline{K}$.

The Lie algebra \mathfrak{g} of G consists of vector fields X on $\mathbb{V}_{\mathbb{C}}$ such that $(-\alpha) \circ X \circ (-\alpha) = X$. Then \mathfrak{g} is isomorphic to the set of triplets

$$\{(w, T, \bar{w}) \mid w \in \mathbb{V}_{\mathbb{C}}^-, T \in \mathfrak{k} \text{ and } \alpha \circ T \circ \alpha = T\}.$$

We write, for $z \in \mathbb{V}_{\mathbb{C}}$, $j(z) := z^{-1}$ the inverse of z in the Jordan algebra $\mathbb{V}_{\mathbb{C}}$, and τ the conjugation of $\mathbb{V}_{\mathbb{C}}$ with respect to the real form \mathbb{V} . The application $\theta : g \mapsto (-j\tau) \circ g \circ (-j\tau)$ is a Cartan involution of the Lie algebra \mathfrak{g} (cf. [3]). Then

$$\begin{aligned} \mathfrak{k} &:= \mathfrak{g}^{\theta} = \{(0, T, 0) \mid T \in \mathfrak{k} \text{ and } \theta \circ T \circ \theta = T\}, \\ \mathfrak{p} &:= \mathfrak{g}^{-\theta} = \{(w, 0, \bar{w}) \mid w \in \mathbb{V}_{\mathbb{C}}^-\}. \end{aligned}$$

Let H be the stabilizer of the base point ie in G ,

$$H := \{g \in G \mid g \cdot (ie) = ie\}.$$

Proposition 2.1. *The pair (G, H) is a symmetric pair.*

Proof. Let σ be the involution of G defined by

$$\sigma(g) = (-j) \circ g \circ (-j),$$

which commutes with the Cartan involution θ defined before. The differential of σ , also denoted by σ , is given by

$$\sigma(w, T, \bar{w}) = (-\bar{w}, -T', -w),$$

where T' denote the adjoint of T with respect to the scalar product on \mathbb{V} defined by the trace. By definition of H , its Lie algebra \mathfrak{h} is given by

$$\mathfrak{h} = \{(iw, T, iw) \mid w \in \mathbb{V}^-, T \in \text{Der}(\mathbb{V}^+)\},$$

where $\text{Der}(\mathbb{V}^+)$ is the derivation algebra of \mathbb{V}^+ . Then

$$\mathfrak{h} = \mathfrak{g}^{\sigma} := \{X \in \mathfrak{g} \mid \sigma(X) = X\}.$$

□

The pseudo-Riemannian symmetric space G/H is the open orbit $G \cdot ie$ in the Shilov boundary of \mathcal{D} . It is a compactly causal symmetric space. Moreover G/H is a Makarevič symmetric space (cf. [3], [21]). With respect to the involution σ , the Lie algebra \mathfrak{g} decomposes as $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$ where $\mathfrak{q} := \mathfrak{g}^{-\sigma} = \{(w, iL(v), w) \mid w \in \mathbb{V}^-, v \in \mathbb{V}^+\}$.

The Lie algebra \mathfrak{g} is semisimple and hermitien. By a theorem of Vinberg and Kostant, there is a regular \underline{G} -invariant cone (i.e., convex, closed, proper, and with nonempty interior) in \mathfrak{g} . Let C_{\max} be a maximal regular \underline{G} -invariant cone in \mathfrak{g} containing $(0, iI, 0)$. By [25],

$$\Gamma(C_{\max}) := \underline{G} \exp(iC_{\max}) = \{g \in \underline{G}_{\mathbb{C}} \mid g \cdot \bar{\mathcal{D}} \subset \mathcal{D}\}.$$

In the complexified space $G_{\mathbb{C}}/H_{\mathbb{C}}$ of G/H we consider the complex domain

$$\Xi := \Gamma(C^0) \cdot ie$$

where C^0 is the interior of $C := i(C_{\max} \cap \mathfrak{g})$. (This domain is introduced by J. Hilgert, B. Ørsted, and G. Ólafsson in [14].)

The domain Ξ can be realized as $\mathcal{D} \setminus \Sigma$ where Σ is the analytic set given by

$$(1) \quad \Sigma = \{z \in \mathcal{D} \mid \det(P(z + \alpha z)) = 0\},$$

where the notation “det” denotes the determinant with respect to \mathbb{V} (cf. [3]).

The domain Ξ can also be realized as a subset of the imaginary tangent bundle of G/H

$$\Xi \simeq G \times C^{\mathfrak{q}} / \sim,$$

where $C^{\mathfrak{q}} := C^0 \cap i\mathfrak{q}$, and $G \times C^{\mathfrak{q}} / \sim$ is the quotient of $G \times C^{\mathfrak{q}}$ by the equivalence relation: $(g_1, X_1) \sim (g_2, X_2)$ if and only if there exists $h \in H$ such that

$$(2) \quad g_2 = g_1 h \quad \text{and} \quad X_2 = \text{Ad}(h^{-1})X_1,$$

(cf. [14]).

The open set Ξ is connected since it is homeomorphic to $\mathcal{D} \setminus \Sigma$, observing that \mathcal{D} is connected and $\text{codim}_{\mathbb{R}}(\Sigma) = 2$.

Let

$$\tilde{\Xi} := \left\{ (z, \zeta) \in \Xi \times \mathbb{C} \mid e^{\frac{2n}{r^+}\zeta} = \det(P(z + \alpha z)) \right\}.$$

Note that $\frac{2n}{r^+}$ is an integer.

Theorem 2.2. *The set $\tilde{\Xi}$ is a connected covering of infinite order of the domain Ξ .*

Proof. Let p be the map defined by

$$\begin{aligned} p : \tilde{\Xi} &\longrightarrow \Xi, \\ (z, \zeta) &\longmapsto z. \end{aligned}$$

Then p is surjective. In fact for $z \in \Xi$, we have $\det(P(z + \alpha z)) \neq 0$, then there exists $\zeta \in \mathbb{C}$ such that $e^{\frac{2n}{r^+}\zeta} = \det(P(z + \alpha z))$. Let $z_0 \in \Xi$, we can find an open neighbourhood U of z_0 such that $p^{-1}(U)$ is homeomorphic to $U \times \mathbb{Z}$. In fact, since p is surjective, there exists $(z_0, \zeta_0) \in \tilde{\Xi}$ such that $p(z_0, \zeta_0) = z_0$. We consider a determination of $\log \left(\det(P(z + \alpha z)) \right)$ in the neighbourhood U of z_0 , we can define a homeomorphism of $U \times \mathbb{Z}$ in $p^{-1}(U)$ as

$$(z, m) \longmapsto \left(z, \log(\det(P(z + \alpha z))) + 2\pi i m \right).$$

Hence $\tilde{\Xi}$ is a covering of infinite order of Ξ .

Let $\{e_1, e_2, \dots, e_{r^+}\}$ be a Jordan frame of \mathbb{V}^+ . An element z of the form $z = \sum_{j=1}^{r^+} z_j e_j$ belongs to Ξ if and only if $0 < |z_j| < 1$. Let

$$z(t) = \sum_{j=1}^{r^+-1} e_j + e^{2\pi i t} e_{r^+} \in \bar{\Xi},$$

and

$$z_0 = \sum_{j=1}^{r^+} z_j e_j \in \Xi.$$

The curve $\varphi(t) := z(t)z_0$ belongs to Ξ and satisfies $\varphi(0) = \varphi(1) = z_0$. Let $\tilde{\varphi}$ be the lifting of φ to $\tilde{\Xi}$,

$$\begin{aligned} \tilde{\varphi} : [0, 1] &\longrightarrow \tilde{\Xi}, \\ t &\longmapsto (\varphi(t), \zeta(t)). \end{aligned}$$

Using the fact that $e_j = c_j$ if $r = r^+$ and $e_j = c_j + c_{j+r^+}$ if $r = 2r^+$, for all $1 \leq j \leq r^+$, we deduce that

$$\begin{aligned} e^{\frac{2n}{r^+}\zeta(t)} &= \det(P(z(t) + \alpha z(t))) \\ &= C(z_1, \dots, z_{r^+}) (e^{2\pi i t})^{\frac{2n}{r^+}}, \end{aligned}$$

where $C(z_1, \dots, z_{r^+})$ is a nonzero constant depending on z_1, \dots, z_{r^+} . There exists $\tilde{h} \in \mathbb{C}^*$ such that $C(z_1, \dots, z_{r^+}) = e^{\tilde{h}}$ and $\frac{2n}{r^+}(\zeta(t) - 2\pi i t) = \tilde{h} + 2\pi i \kappa(t)$. Here $\kappa(t)$ is an integer valued continuous function on $[0, 1]$, therefore constant. Thus $\zeta(1) - \zeta(0) = 2\pi i$ and if $\tilde{\varphi}(0) = (z_0, \zeta_0)$, then $\tilde{\varphi}(1) = (z_0, \zeta_0 + 2\pi i)$. Thus if z_0 is an element of $\Xi \cap \bigoplus_{j=1}^{r^+} \mathbb{C}e_j$ and if (z_0, ζ_0^1) and

(z_0, ζ_0^2) are two points of $\tilde{\Xi}$, there exists a curve $\tilde{\varphi}$ such that $\tilde{\varphi}(0) = (z_0, \zeta_0^1)$ and $\tilde{\varphi}(1) = (z_0, \zeta_0^2)$.

Let (z_1, ζ_1) and (z_2, ζ_2) be two points of $\tilde{\Xi}$. Since Ξ is connected, there exists a curve φ_1 (resp. φ_2) such that

$$\varphi_1(0) = z_0, \quad \varphi_1(1) = z_1 \quad (\text{resp.} \quad \varphi_2(0) = z_0, \quad \varphi_2(1) = z_2).$$

Let $\tilde{\varphi}_1$ (resp. $\tilde{\varphi}_2$) be the lifting of φ_1 (resp. φ_2) to $\tilde{\Xi}$ such that

$$\begin{aligned} \tilde{\varphi}_1(0) &= (z_0, \zeta_0^1) & \tilde{\varphi}_1(1) &= (z_1, \zeta_1) \\ \tilde{\varphi}_2(0) &= (z_0, \zeta_0^2) & \tilde{\varphi}_2(1) &= (z_2, \zeta_2). \end{aligned}$$

Using the fact that $z_0 \in \Xi \cap \bigoplus_{j=1}^{r^+} \mathbb{C}e_j$, we deduce that $\tilde{\Xi}$ is connected. \square

Let

$$\mathbb{T}_\alpha(z) = \det(P(z + \alpha z)).$$

With respect to Lebesgue measure, the restriction to Ξ of the $G_{\mathbb{C}}$ -invariant measure of $G_{\mathbb{C}}/H_{\mathbb{C}}$ is given by

$$d\xi = \frac{d\lambda(z)}{|\mathbb{T}_\alpha(z)|}.$$

Let

$$(3) \quad \mathbb{T}_\alpha(\tilde{z})^{\frac{r}{4n}\nu} := e^{\frac{r}{2r^+}\nu\zeta}, \quad \tilde{z} = (z, \zeta) \in \tilde{\Xi}.$$

The realization of the domain Ξ inside the imaginary tangent bundle of G/H will be used below which permits to show an integral formula.

Let \mathfrak{a} be a Cartan subalgebra in $\mathfrak{k} \cap \mathfrak{q}$. Let Δ be the root system $\Delta(\mathfrak{g}_{\mathbb{C}}, i\mathfrak{a})$, and \mathfrak{a}^+ a positive Weyl chamber. Let Δ^+ be the positive root system with respect to $i\mathfrak{a}^+$.

Theorem 2.3. *For an integrable function f on Ξ ,*

$$\int_{\Xi} f(\xi) d\xi = c_0 \int_G \int_{C^+} f(g \exp(X) \cdot ie) \prod_{\beta \in \Delta^+} (\operatorname{sh} \langle \beta, 2X \rangle)^{m_\beta} dg dX,$$

where $C^+ := C^0 \cap i\mathfrak{a}^+$ and $m_\beta = \dim(\mathfrak{g}_\beta)$.

Proof. Let $Z := Z_H(i\mathfrak{a})$ be the centralizer subgroup of $i\mathfrak{a}$ in H . The map

$$\begin{aligned} \varphi : G/Z \times C^+ &\longrightarrow \Xi, \\ (g \cdot Z, X) &\longmapsto g \exp(X) \cdot ie \end{aligned}$$

is a diffeomorphism onto its open image. In fact, let $g_1 \exp(X_1) \cdot ie$ and $g_2 \exp(X_2) \cdot ie$ be two elements of Ξ such that $g_1 \exp(X_1) \cdot ie = g_2 \exp(X_2) \cdot ie$. Since the group G acts on $G/Z \times C^+$ (resp. Ξ) by

$$g_0 \cdot (g \cdot Z, X) = (g_0 g \cdot Z, X), \quad \left(\text{resp. } g_0 \cdot (g \exp(X) \cdot ie) = g_0 g \exp(X) \cdot ie \right),$$

we may assume that $g_1 = 1$. But since X_1 and X_2 are regular and in the same positive Weyl chamber, by the equivalence relation (2), we deduce that $X_1 = X_2$. Thus establishing injectivity.

We will compute the differential of φ . For this we consider the commutative diagram defined by

$$\begin{array}{ccc} G \times C^+ & \xrightarrow{\Phi} & G^{\mathbb{C}} \\ \phi \downarrow & & \downarrow \Psi \\ G/Z \times C^+ & \xrightarrow{\psi} & G^{\mathbb{C}}/H^{\mathbb{C}} \end{array}$$

$$\begin{array}{ccc} (g, X) & \xrightarrow{\Phi} & g \exp(X) \\ \phi \downarrow & & \downarrow \Psi \\ (g \cdot Z, X) & \xrightarrow{\psi} & g \exp(X) \cdot H_{\mathbb{C}}. \end{array}$$

Then $d(\psi \circ \phi)(1, X) = d(\Psi \circ \Phi)(1, X)$.

Let λ_g (resp. Λ_g) be the left translation by g in G (resp. $G_{\mathbb{C}}/H_{\mathbb{C}}$), then for $(Y, U) \in \mathfrak{g} \times i\mathfrak{a}$,

$$\begin{aligned} & d(\Psi \circ \Phi)(1, X)(Y, U) \\ &= \frac{d}{dt} \Big|_{t=0} \Psi \left(\exp(tY) \exp(X + tU) \right) \\ &= d\Psi(\exp(X)) \cdot d\lambda_{\exp(X)}(1) \cdot \left(e^{-\text{ad}(X)} \cdot Y + \frac{1 - e^{-\text{ad}(X)}}{\text{ad}(X)} \cdot U \right) \\ &= d\Lambda_{\exp(X)} \Psi(1) \cdot d\Psi(1) \cdot \left(e^{-\text{ad}(X)} \cdot Y + U \right), \end{aligned}$$

where $d\Psi(1)$ sends $\mathfrak{g}_{\mathbb{C}}$ onto $\mathfrak{q}_{\mathbb{C}}$. Note $P_{\mathfrak{q}_{\mathbb{C}}}$ the projection of $\mathfrak{g}_{\mathbb{C}}$ onto $\mathfrak{q}_{\mathbb{C}}$ along $\mathfrak{h}_{\mathbb{C}}$, then

$$\begin{aligned} & P_{\mathfrak{q}_{\mathbb{C}}} \left(e^{-\text{ad}(X)} Y + U \right) \\ &= \frac{e^{-\text{ad}(X)} Y + U - e^{\text{ad}(X)} \sigma(Y) + U}{2} \\ &= \frac{e^{-\text{ad}(X)} Y - e^{\text{ad}(X)} \sigma(Y)}{2} + U \\ &= \text{sh}(-\text{ad}(X)) \left(\frac{Y + \sigma(Y)}{2} \right) + \text{ch}(\text{ad}(X)) \left(\frac{Y - \sigma(Y)}{2} \right) + U. \end{aligned}$$

Using the fact that $\sigma(\mathfrak{g}_{\beta}) = \mathfrak{g}_{-\beta}$, the Lie algebra \mathfrak{g} can be written as

$$\mathfrak{g} = \mathfrak{z} \oplus \sum_{\beta \in \Delta^+} (1 + \sigma) \mathfrak{g}_{\beta} \oplus \mathfrak{a} \oplus \sum_{\beta \in \Delta^+} (1 - \sigma) \mathfrak{g}_{\beta},$$

where $\mathfrak{z} := \text{Lie}(Z)$. Then for all $Y \in \mathfrak{g}$ and $Y_\beta \in \mathfrak{g}_\beta$,

$$P_{\mathfrak{q}_\mathbb{C}}\left(e^{-\text{ad}(X)}Y + U\right) = \text{sh}(-\beta(X)) (Y_\beta + \sigma(Y_\beta)) + \text{ch}(\beta(X)) (Y_\beta - \sigma(Y_\beta)) + U.$$

Let ω be the volume form on Ξ which defines an invariant Haar measure on Ξ . Again, the volume form $\varphi^*\omega$ on $G/Z \times C^+$ is given by

$$\varphi^*\omega = c_0 \prod_{\beta \in \Delta^+} (\text{sh } 2\beta(X))^{m_\beta} \omega_1 \otimes \omega_2,$$

where ω_1 is a volume form on G/Z which defines an invariant measure, and ω_2 is a volume form on $i\mathfrak{a}$ which defines a Lebesgue measure. Using the fact that Z is compact, the integral formula holds. \square

3. Weighted Bergman spaces and reproducing kernels.

In this section we introduce the Bergman space $\mathcal{H}_\nu^2(\tilde{\Xi})$ associated with the covering $\tilde{\Xi}$. We establish a unitary isomorphism of $\mathcal{H}_\nu^2(\mathcal{D})$ on $\mathcal{H}_\nu^2(\tilde{\Xi})$. Then we compute the explicit expression of the reproducing kernel of $\mathcal{H}_\nu^2(\tilde{\Xi})$.

For a real ν , let $\mathcal{O}_\nu(\tilde{\Xi})$ be the space of holomorphic functions F on $\tilde{\Xi}$ which satisfy

$$F(z, \zeta + 2\pi i) = e^{2\pi i \frac{r}{2r+1} \nu} F(z, \zeta).$$

This condition will be called a monodromy condition. Remark that the function $\top_\alpha \frac{r}{4n} \nu$ belongs to $\mathcal{O}_\nu(\tilde{\Xi})$.

For $\nu > \frac{2n}{r} - 1$, let $\mathcal{H}_\nu^2(\tilde{\Xi})$ be the Hilbert space of functions $F \in \mathcal{O}_\nu(\tilde{\Xi})$ such that

$$\|F\|_\nu^2 := \int_{\tilde{\Xi}} |F(\tilde{\xi})|^2 p_\nu(\xi) d\xi < \infty,$$

where

$$p_\nu(\xi) = \det(B(\xi, \bar{\xi}))^{\frac{r}{2n}\nu-1} |\top_\alpha(\xi)|^{-\frac{r}{2n}\nu+1},$$

and $B(z, w)$ is the Bergman operator defined by $B(z, w) := \text{id}_{\mathbb{V}_\mathbb{C}} - 2z \square w + P(z)P(w)$.

Proposition 3.1. *Let z and w be two invertible elements of $\mathbb{V}_\mathbb{C}$. Thus*

$$\det(B(z, w)) = \Delta(z)^{\frac{2n}{r}} \Delta(z^{-1} - w)^{\frac{2n}{r}},$$

where Δ is the determinant polynomial associated with \mathbb{V} .

Proof. By definition $\det(B(z, w)) = \det(\text{id}_{\mathbb{V}_\mathbb{C}} - 2z \square w + P(z)P(w))$. According to [18] Proposition 4.13,

$$z \square w = P(w^{-1}, z)P(w),$$

where $P(z, w) = \frac{1}{2} \left(P(z+w) - P(z) - P(w) \right)$. Then

$$\text{id}_{\mathbb{V}_\mathbb{C}} - 2z \square w + P(z)P(w) = \text{id}_{\mathbb{V}_\mathbb{C}} - 2P(w^{-1}, z)P(w) + P(z)P(w).$$

Moreover

$$\begin{aligned} P(w^{-1} - z) &= P(w)^{-1} + 2P(w^{-1}, -z) + P(z) \\ &= P(w^{-1}) - 2P(w^{-1}, z) + P(z). \end{aligned}$$

Hence we deduce the following equalities

$$-2P(w^{-1}, z)P(w) = P(w^{-1} - z)P(w) - P(z)P(w) - \text{id}_{V_{\mathbb{C}}},$$

and

$$\text{id}_{V_{\mathbb{C}}} - 2z \square w + P(z)P(w) = P(w^{-1} - z)P(w).$$

Then, we have

$$\det(\text{id}_{V_{\mathbb{C}}} - 2z \square w + P(z)P(w)) = \det(P(w^{-1} - z)) \det(P(w)),$$

where $\det(P(w)) = \Delta(w)^{\frac{2n}{r}}$. Finally

$$\det(B(z, w)) = \Delta(w)^{\frac{2n}{r}} \Delta(w^{-1} - z)^{\frac{2n}{r}} = \Delta(z)^{\frac{2n}{r}} \Delta(z^{-1} - w)^{\frac{2n}{r}}.$$

□

The universal covering \tilde{G} of G can be realized as the set of pairs (g, φ) with $g \in G$ and φ a holomorphic function on \mathcal{D} defined by

$$e^{\varphi(z)} = \det(Dg(z)),$$

where $Dg(z)$ is the differential of the map $z \mapsto g \cdot z$. The product on \tilde{G} is given by

$$(g_1, \varphi_1) \cdot (g_2, \varphi_2) = (g_1 g_2, \varphi_3),$$

where $\varphi_3(z) = \varphi_1(g_2 \cdot z) + \varphi_2(z)$. For $\tilde{g} = (g, \varphi) \in \tilde{G}$, and $\kappa \in \mathbb{R}$, we will write

$$\det(Dg(z))^{\kappa} := e^{\kappa\varphi(z)}.$$

Let $\widetilde{\Gamma(C)} := \tilde{G} \widetilde{\exp(C)}$ be the semigroup associated with the covering \tilde{G} where $\widetilde{\exp} : \mathfrak{g} \rightarrow \tilde{G}$. We denote by $\widetilde{\Gamma(C^0)}$ the interior of $\widetilde{\Gamma(C)}$. The linear action of \tilde{G} on the space $\mathcal{H}_{\nu}^2(\tilde{\Xi})$ is given by

$$(\pi_0(\tilde{g})F)(\tilde{\xi}) = F(g \cdot \tilde{\xi}), \quad \tilde{g}^{-1} = (g, \varphi),$$

where $g \cdot \tilde{\xi} = (g \cdot \xi, \zeta')$ and $e^{\frac{2n}{r^+} \zeta'} = \det(P(g \cdot \xi + \alpha(g \cdot \xi)))$ (cf. [19] Lemma 5.1).

The representation π_0 extends to a continuous representation of $\widetilde{\Gamma(-C)}$ and a holomorphic one of $\widetilde{\Gamma(-C^0)}$ (cf. [25]).

We recall that the Bergman space $\mathcal{H}_{\nu}^2(\mathcal{D})$ is the Hilbert space of holomorphic functions f on \mathcal{D} such that

$$\|f\|^2 = \int_{\mathcal{D}} |f(z)|^2 \det(B(z, \bar{z}))^{\frac{r}{2n} \nu - 1} d\lambda(z) < \infty,$$

where λ denote the Lebesgue measure (cf. [12]). The action of \widetilde{G} on $\mathcal{H}_\nu^2(\mathcal{D})$ is given by

$$(\pi_\nu(\widetilde{g})f)(z) = e^{\frac{r}{2n}\nu\varphi(z)}f(g \cdot z), \quad \widetilde{g}^{-1} = (g, \varphi).$$

The unitary representation π_ν extends to a continuous representation of $\widetilde{\Gamma(-C)}$ and a holomorphic one of $\widetilde{\Gamma(-C^0)}$.

Let \mathcal{A}_ν be the operator given by

$$\begin{aligned} \mathcal{A}_\nu : \mathcal{H}_\nu^2(\mathcal{D}) &\longrightarrow \mathcal{O}_\nu(\widetilde{\Xi}), \\ f &\longmapsto \mathcal{A}_\nu(f) = \top_\alpha \frac{r}{4n}\nu f. \end{aligned}$$

Since $\top_\alpha \frac{r}{4n}\nu \in \mathcal{O}_\nu(\widetilde{\Xi})$, the operator \mathcal{A}_ν is well defined.

Theorem 3.1. *The operator \mathcal{A}_ν is a unitary isomorphism of $\mathcal{H}_\nu^2(\mathcal{D})$ onto $\mathcal{H}_\nu^2(\widetilde{\Xi})$ intertwining the representations π_ν and π_0 .*

Proof. Since Σ is an analytic set of measure zero and $d\xi = \frac{d\lambda(z)}{|\top_\alpha(z)|}$,

$$\begin{aligned} \|\mathcal{A}_\nu(f)\|_\nu^2 &= \int_{\widetilde{\Xi}} |\mathcal{A}_\nu(f)(\widetilde{\xi})|^2 p_\nu(\xi) d\xi \\ &= \int_{\widetilde{\Xi}} |f(\xi)|^2 |\top_\alpha(\widetilde{\xi})|^{\frac{r}{2n}\nu} \det(B(\xi, \bar{\xi}))^{\frac{r}{2n}\nu-1} \frac{d\xi}{|\top_\alpha(\widetilde{\xi})|^{\frac{r}{2n}\nu-1}} \\ &= \int_{\mathcal{D}} |f(z)|^2 \det(B(z, \bar{z}))^{\frac{r}{2n}\nu-1} d\lambda(z) = \|f\|^2. \end{aligned}$$

If f belongs to $\mathcal{H}_\nu^2(\mathcal{D})$ then $\mathcal{A}_\nu(f) \in \mathcal{O}_\nu(\widetilde{\Xi})$ and

$$\int_{\widetilde{\Xi}} |\mathcal{A}_\nu f(\widetilde{\xi})|^2 p_\nu(\xi) d\xi < \infty.$$

Hence the image of \mathcal{A}_ν is contained in $\mathcal{H}_\nu^2(\widetilde{\Xi})$ and \mathcal{A}_ν is isometric.

Moreover \mathcal{A}_ν is surjective. In fact, let $F \in \mathcal{H}_\nu^2(\widetilde{\Xi})$, then in particular $F \in \mathcal{O}_\nu(\widetilde{\Xi})$. Since $\det(P(\widetilde{z} + \alpha\widetilde{z}))^{-\frac{r}{4n}\nu} \in \mathcal{O}_{-\nu}(\widetilde{\Xi})$, the function

$$f(z) := \top_\alpha(\widetilde{z})^{-\frac{r}{4n}\nu} F(\widetilde{z}), \quad \widetilde{z} = (z, \zeta) \in \widetilde{\Xi},$$

is holomorphic on $\Xi = \mathcal{D} \setminus \Sigma$. Moreover, the function f belongs to $\mathcal{H}_\nu^2(\mathcal{D})$.

In fact

$$\begin{aligned} \|f\|^2 &= \int_{\mathcal{D}} |F(\widetilde{z})|^2 |\top_\alpha(z)|^{-\frac{r}{2n}\nu} \det(B(z, \bar{z}))^{\frac{r}{2n}\nu-1} d\lambda(z) \\ &= \int_{\widetilde{\Xi}} |F(\widetilde{\xi})|^2 |\top_\alpha(\xi)|^{-\frac{r}{2n}\nu+1} \det(B(\xi, \bar{\xi}))^{\frac{r}{2n}\nu-1} d\xi = \|F\|_\nu^2 < \infty. \end{aligned}$$

Then f is a holomorphic function on $\mathcal{D} \setminus \Sigma$ and belongs to $L_\nu^2(\mathcal{D})$. Hence f extends to a holomorphic function on \mathcal{D} . This is the content of the following lemma.

Lemma 3.2 (cf. [6], [27]). *Let U be a domain in \mathbb{C}^n and let A be an analytic set such that $\text{codim}_{\mathbb{R}}(A) \geq 1$. If $f \in \mathcal{O}(U \setminus A)$ and if $f \in L^2(U)$, then f extends to a holomorphic function on U .*

It remains to show that \mathcal{A}_ν intertwining the representations π_0 and π_ν . In fact it follows from [3] 1.3 (9) that $\top_\alpha(g \cdot \tilde{\xi})^{\frac{r}{4n}\nu} = e^{\frac{r}{2n}\nu\varphi(\xi)} \top_\alpha(\tilde{\xi})^{\frac{r}{4n}\nu}$. Hence

$$\begin{aligned} \mathcal{A}_\nu(\pi_\nu(\tilde{g})f)(\tilde{\xi}) &= \top_\alpha(\tilde{\xi})^{\frac{r}{4n}\nu} e^{\frac{r}{2n}\nu\varphi(\xi)} f(g \cdot \xi) \quad \left(\tilde{g}^{-1} = (g, \varphi)\right) \\ &= e^{-\frac{r}{2n}\nu\varphi(\xi)} \top_\alpha(g \cdot \tilde{\xi})^{\frac{r}{4n}\nu} e^{\frac{r}{2n}\nu\varphi(\xi)} f(g \cdot \xi) \\ &= (\pi_0(\tilde{g})\mathcal{A}_\nu f)(\tilde{\xi}). \end{aligned}$$

□

Proposition 3.3. *The reproducing kernel of the Bergman space $\mathcal{H}_\nu^2(\tilde{\Xi})$ is equal to*

$$K_\nu(\tilde{\xi}_1, \tilde{\xi}_2) = c_\nu \top_\alpha(\tilde{\xi}_1)^{\frac{r}{4n}\nu} \det(B(\xi_1, \bar{\xi}_2))^{-\frac{r}{2n}\nu} \overline{\top_\alpha(\tilde{\xi}_2)^{\frac{r}{4n}\nu}},$$

where c_ν is the positive constant

$$(4) \quad c_\nu = \frac{1}{\pi^n} \prod_{j=1}^r \frac{\Gamma\left(\nu - (j-1)\frac{n-r}{r(r-1)}\right)}{\Gamma\left(\nu - \frac{n}{r} - (j-1)\frac{n-r}{r(r-1)}\right)}.$$

The definition of $\overline{\top_\alpha(\tilde{\xi}_2)^{\frac{r}{4n}\nu}}$ is similar to that given on (3).

Proof. The reproducing kernel of $\mathcal{H}_\nu^2(\mathcal{D})$ is given by

$$K_\nu^{\mathcal{D}}(z, z') = c_\nu \det(B(z, \bar{z}'))^{-\frac{r}{2n}\nu}.$$

From the definition of \mathcal{A}_ν , the reproducing kernel of $\mathcal{H}_\nu^2(\tilde{\Xi})$ is equal to

$$\begin{aligned} K_\nu(\tilde{\xi}, \tilde{\xi}') &= \top_\alpha(\tilde{\xi})^{\frac{r}{4n}\nu} K_\nu^{\mathcal{D}}(\xi, \xi') \overline{\top_\alpha(\tilde{\xi}')^{\frac{r}{4n}\nu}} \\ &= c_\nu \top_\alpha(\tilde{\xi})^{\frac{r}{4n}\nu} \det(B(\xi, \bar{\xi}'))^{-\frac{r}{2n}\nu} \overline{\top_\alpha(\tilde{\xi}')^{\frac{r}{4n}\nu}}. \end{aligned}$$

□

4. Holomorphic discrete series of \tilde{G} .

Recall that $\{e_1, \dots, e_{r+}\}$ is the Jordan frame of R^+ and $\mathfrak{k} \cap \mathfrak{q} = \{(0, iL(v), 0) \mid v \in \mathbb{V}^+\}$.

Let \mathfrak{a} be the Cartan subalgebra in $\mathfrak{k} \cap \mathfrak{q}$ defined by

$$\mathfrak{a} = \left\{ \left(0, i \sum_{j=1}^{r+} t_j L(e_j), 0 \right) \mid t_j \in \mathbb{R} \right\}.$$

We denote by Δ the root system $\Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{ia})$, Δ^+ the positive system with respect to the positive Weyl chamber $(\mathfrak{ia})^+$ defined by

$$(5) \quad (\mathfrak{ia})^+ = \left\{ \left(0, \sum_{j=1}^{r^+} t_j L(e_j), 0 \right) \mid 0 < t_1 < \cdots < t_{r^+} \right\}.$$

Let $X_0 := (0, I, 0) \in \mathfrak{g}_{\mathbb{C}}$. The eigenvalues of $\text{ad}(X_0)$ are 1, 0, and -1 . Let $\Delta_0 := \{\alpha \in \Delta^+ \mid \alpha(X_0) = 0\}$ and $\Delta_1 := \{\alpha \in \Delta^+ \mid \alpha(X_0) = 1\}$. Then $\Delta^+ = \Delta_0 \cup \Delta_1$. The roots belonging to Δ_0 are called compact and the roots belonging to Δ_1 noncompact. Let $\rho := \frac{1}{2} \sum_{\alpha \in \Delta^+} m_{\alpha} \alpha$ be one half of the positive roots weighted by the dimension m_{α} of the root spaces. For the description and computation of ρ we refer to [4]. See also [8].

Let π be a unitary representation of the Lie group G on a Hilbert space \mathcal{H} , and let C be an invariant and regular cone in \mathfrak{ig} . The representation π is called C -positive if for all $X \in C$ and for all C^{∞} vector v ,

$$\frac{d}{dt} \Big|_{t=0} \langle \pi(\exp(tX))v \mid v \rangle \leq 0.$$

Let \mathcal{R} be the set of the weights $\mu = (\mu_1, \mu_2, \dots, \mu_{r^+}) \in \mathbb{R}^{r^+}$ such that

$$\mu_i - \mu_{i+1} \in \mathbb{N}, \quad 1 \leq i \leq r^+ - 1.$$

(If \mathbb{V}^+ is a direct sum of two simple algebras with ranks p and q such that $p + q = r^+ (= r)$, then $i \neq p$.)

For $\mu \in \mathcal{R}$ and β a noncompact positive root. By [14], the ‘‘Harish-Chandra’’ condition $\langle \rho - \mu, \beta \rangle \leq 0$ can be written as

$$\begin{aligned} (\star) \quad \mathbb{V}^+ \text{ is simple} \quad & \mu_{r^+} > \frac{n}{2r} - \frac{d+1}{8} \quad \text{if} \quad r = r^+, \quad \left(d := \frac{2(n-r)}{r(r-1)} \right) \\ & \mu_{r^+} > \frac{n}{r} - \frac{1}{2} \quad \text{if} \quad r = 2r^+. \\ (\star\star) \quad \mathbb{V}^+ \text{ is not simple} \quad & \mu_1 + \mu_{r^+} > -2dp \quad \text{where} \quad r = r^+. \end{aligned}$$

If $\mu \in \mathcal{R}$ and satisfies the ‘‘Harish-Chandra’’ condition, then we can associate to μ a unitary and C -positive representation $(\pi_{\mu}, \mathcal{W}_{\mu})$ of \tilde{G} with highest weight μ . This representation extends to a continuous representation of $\widetilde{\Gamma(C)}$ which is holomorphic on $\widetilde{\Gamma(C^0)}$.

Let $A := \exp \mathfrak{a}$, and $\mathfrak{g}_+^{\mathbb{C}} := \sum_{\beta \in \Delta^+} \mathfrak{g}_{\beta}^{\mathbb{C}}$.

Definition 4.1. A holomorphic function Φ in $\tilde{\Xi}$ will be called a conical function if there exists a continuous character χ_{μ} of A such that

$$\begin{aligned} \mathcal{I}(a)\Phi &= \chi_{\mu}(a)\Phi, \quad (a \in A), \\ d\mathcal{I}(X)\Phi &= 0, \quad (X \in \mathfrak{g}_+^{\mathbb{C}}), \end{aligned}$$

where $(\mathcal{I}(g)F)(\tilde{\xi}) = F(g^{-1} \cdot \tilde{\xi})$.

For all $\mathbf{s} = (s_1, s_2, \dots, s_r) \in \mathbb{C}^r$ and z in $\mathbb{V}_{\mathbb{C}}$, we write

$$\Delta_{\mathbf{s}}(z) := \Delta_1(z)^{s_1-s_2} \Delta_2(z)^{s_2-s_3} \dots \Delta_r(z)^{s_r},$$

where Δ_j is the principal minor of order j (cf. [12]).

For $\mu = (\mu_1, \mu_2, \dots, \mu_{r^+})$, let,

$$\Phi_{\mu}(\tilde{z}) := \Delta_{\mu} \left(\frac{\tilde{z} + \alpha(\tilde{z})}{2} \right).$$

The function Φ_{μ} satisfies the monodromy condition.

Proposition 4.2. *The function Φ_{μ} is conical, and any conical function is proportional to Φ_{μ} .*

The proof is similar to that given for Proposition XI.2.1 in [12].

Note that $\mathcal{W}_{\mu}^{\infty}$ (resp. $\mathcal{W}_{\mu}^{-\infty}$) is the vector space of \mathcal{C}^{∞} (resp. distribution) vectors of \mathcal{W}_{μ} , and $(\mathcal{W}_{\mu}^{-\infty})^H$ the vector space of H -invariant distribution vectors of \mathcal{W}_{μ} . Let \mathcal{R}_H be the subset of highest weight $\mu \in \mathcal{R}$ such that $(\mathcal{W}_{\mu}^{-\infty})^H \neq \{0\}$.

For $\mu \in \mathcal{R}_H$, we denote ψ_{μ} an H -invariant distribution vector. For all element $w \in \mathcal{W}_{\mu}$, the holomorphic mapping $\mathcal{F} : \mathcal{W}_{\mu} \longrightarrow \mathcal{O}(\tilde{\Xi})$, $w \mapsto \mathcal{F}(w)(\tilde{\xi}) := \langle \pi_{\mu}(\tilde{\gamma}_1^{-1}) w | \psi_{\mu} \rangle$ where $\tilde{\xi} = \tilde{\gamma} \cdot H$, is a continuous embedding. Then the representation π_{μ} is realized on a Hilbert space \mathcal{H}_{μ} of holomorphic functions on $\tilde{\Xi}$. In the case where $w = v_{\mu}$, a normalized highest weight vector, we denote $\mathcal{F}_{\mu}(\tilde{\xi}) := \langle \pi_{\mu}(\tilde{\gamma}^{-1}) v_{\mu} | \psi_{\mu} \rangle$. The function \mathcal{F}_{μ} is a conical function.

Let $\mathcal{R}_{\nu, H}$ be the highest weight subset of \mathcal{R}_H such that $\mathcal{F}_{\mu} \in \mathcal{O}_{\nu}(\tilde{\Xi})$.

Proposition 4.3. *The function \mathcal{F}_{μ} satisfies the monodromy condition, i.e., $\mathcal{F}_{\mu} \in \mathcal{O}_{\nu}(\tilde{\Xi})$, if and only if*

$$\mu_i \in \mathbb{Z} + \frac{\nu}{2}, \quad (1 \leq i \leq r^+) \quad \text{if } r = r^+$$

and

$$\mu_i \in \mathbb{Z} + \nu, \quad (1 \leq i \leq r^+) \quad \text{if } r = 2r^+.$$

Proof. Since \mathcal{F}_{μ} is conical, then it is proportional to Φ_{μ} . If \mathbb{V}^+ is simple, then \top_{α} is proportional to the Jordan determinant Δ of \mathbb{V}^+ . In fact, \top_{α} is homogeneous of degree $2n$ and Δ is homogeneous of degree r^+ , then $\top_{\alpha}(\tilde{z}) = \Delta(\tilde{z} + \alpha(\tilde{z}))^{\frac{2n}{r^+}}$ and $\mathcal{F}_{\mu} \in \mathcal{O}_{\nu}(\tilde{\Xi})$ if and only if $\mu_{r^+} \in \mathbb{Z} + \frac{r}{2r^+}\nu$.

Using the fact that $\mu_i - \mu_{i+1} \in \mathbb{N}$, the result holds for \mathbb{V}^+ simple.

If \mathbb{V}^+ is a direct sum of two simple Jordan algebras \mathbb{V}_1^+ of rank p and \mathbb{V}_2^+ of rank q such that $p+q = r^+(=r)$, then there exist $z_1 \in \mathbb{V}_{1, \mathbb{C}}^+$ and $z_2 \in \mathbb{V}_{2, \mathbb{C}}^+$ such that $z + \alpha(z) = z_1 + z_2$ and $\top_{\alpha}(\tilde{z}) = \Delta^{(1)}(\tilde{z}_1)^{\frac{2n}{r}} \Delta^{(2)}(\tilde{z}_2)^{\frac{2n}{r}}$ where $\Delta^{(1)}$ (resp. $\Delta^{(2)}$) is the Jordan determinant of \mathbb{V}_1^+ (resp. \mathbb{V}_2^+). Hence $\mathcal{F}_{\mu} \in \mathcal{O}_{\nu}(\tilde{\Xi})$

if and only if $\mu_p \in \mathbb{Z} + \frac{\nu}{2}$ and $\mu_r \in \mathbb{Z} + \frac{\nu}{2}$. The assertion follows from the fact that $\mu_i - \mu_{i+1} \in \mathbb{N}$ for all $i \neq p$. \square

Remark. In [1] we consider the case $G/H \simeq U(p, q)$ and we establish another isomorphism between $\mathcal{H}_\nu^2(\mathcal{D})$ and $\mathcal{H}_\nu^2(\tilde{\Xi})$. The correspondence between the present isomorphism \mathcal{A}_ν and the one used in [1] is given by $f \mapsto \det(A)^{\frac{\nu}{2}} \det(D)^{-\frac{\nu}{2}} f(z)$ for all $z = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathcal{D}$. This correspondence explains the shift between the highest weight μ shown in [1] and the present form of μ .

Since ν is very large ($\nu > \frac{2n}{r} - 1$), the representation π_μ satisfies the Harish-Chandra condition for all $\mu \in \mathcal{R}_{\nu, H}$.

Let

$$C_\mu(\nu) = \int_{\Xi} |\langle \pi_\mu(\tilde{\gamma}^{-1}) v_\mu | \psi_\mu \rangle|^2 p_\nu(\xi) d\xi.$$

Proposition 4.4. *For $\mu \in \mathcal{R}_{\nu, H}$, the Hilbert space \mathcal{H}_μ belongs to $\mathcal{H}_\nu^2(\tilde{\Xi})$ if and only if $C_\mu(\nu)$ is finite.*

In this case we denote $\mu \in \mathcal{R}'_{\nu, H}$.

Proof. This is proved in [1] Proposition 4.2. \square

Hence the Plancherel Theorem can be written as:

Theorem 4.1 (cf. [13]). *The Bergman space $\mathcal{H}_\nu^2(\tilde{\Xi})$ is decomposed multiplicity-free and discretely into irreducible Hilbert subspaces,*

$$\mathcal{H}_\nu^2(\tilde{\Xi}) = \bigoplus_{\mu \in \mathcal{R}'_{\nu, H}} \mathcal{H}_\mu.$$

Moreover, the reproducing kernel can be written as

$$K_\nu(\tilde{\xi}_1, \tilde{\xi}_2) = \sum_{\mu \in \mathcal{R}'_{\nu, H}} \frac{1}{C_\mu(\nu)} \langle \pi_\mu(\tilde{\gamma}_2^\# \tilde{\gamma}_1)^{-1} \psi_\mu | \psi_\mu \rangle.$$

The series converges uniformly on compact subsets of $\tilde{\Xi} \times \tilde{\Xi}$.

5. Computation of the constant $C_\mu(\nu)$.

Let \mathcal{M} be a differentiable manifold. A causal structure on \mathcal{M} is a field of cones $\mathcal{M} \ni x \mapsto C_x \subset T_x \mathcal{M}$. The cones C_x are assumed to be closed, convex, proper, and with nonempty interior. Furthermore the cones C_x depend smoothly on x . A piecewise \mathcal{C}^1 curve $\gamma : [0, 1] \rightarrow \mathcal{M}$ is said to be causal if for all t , the derivative $\dot{\gamma}(t)$ belongs to the cone $C_{\gamma(t)}$. The causal structure is said to be global if there exists no nontrivial closed causal curve.

In that case one defines a partial ordering on \mathcal{M} in the following way: One writes $x \leq y$ if there exists a causal curve from x to y .

Let \mathfrak{g} be the Lie algebra defined in Section 2. Let σ be an involutive automorphism of \mathfrak{g} that commutes with the Cartan involution θ where $\mathfrak{g} = \mathfrak{k} \oplus^\theta \mathfrak{p}$, and $\mathfrak{g} = \mathfrak{h} \oplus^\sigma \mathfrak{q}$.

Let $G^c := (\underline{G}^{(\alpha)})_0$ where $\underline{G}^{(\alpha)} = \{g \in \underline{G} \mid \alpha \circ g \circ \alpha = g\}$ and the subscript 0 means the identity component. The group G^c is the group of holomorphic automorphisms of the tube domain T_{Ω^+} associated with the involution α defined by

$$T_{\Omega^+} := \mathbb{V}^- + \Omega^+ = \{x + y \mid x \in \mathbb{V}^-, \quad y \in \Omega^+\}$$

where $\Omega^+ := \mathbb{V}^+ \cap \Omega$ and Ω is the symmetric cone associated with \mathbb{V} . (If \mathbb{V} and \mathbb{V}^+ are simple, the cone Ω^+ coincides with the open cone associated with the Jordan algebra \mathbb{V}^+ .) The group G^c is the c -dual group of G . We consider on $\mathcal{M} := G^c/H$ the causal structure defined by the field of cones

$$C_x = -\bar{\Omega}.$$

The noncompactly causal symmetric space \mathcal{M} is an ordered symmetric space. By [3], the intersection $\mathcal{M} \cap \mathbb{V}$ is a union of connected components of the set $\{x \in \mathbb{V} \mid \det B_\alpha(x, x) \neq 0\}$ where $B_\alpha(x, y) := B(x, \alpha y)$ and $B(x, y)$ is the Bergman operator. In particular

$$(\mathcal{M} \cap \mathbb{V})_0 = \{x \in \mathbb{V} \mid \det B_\alpha(x, x) \neq 0\}_0.$$

Let $\mathfrak{g}^c = \text{Lie}(G^c)$ (the c -dual algebra of \mathfrak{g}). We denote also by σ the \mathbb{C} -linear extension of σ to the complexified algebra $\mathfrak{g}_{\mathbb{C}}$ of \mathfrak{g} . The involution $\theta^c := \theta|_{\mathfrak{g}^c}$ is a Cartan involution of \mathfrak{g}^c . Let \mathfrak{a} be a maximal abelian subspace in $\mathfrak{p}^c \cap i\mathfrak{q}$ where $\mathfrak{p}^c := (\mathfrak{g}^c)^{\theta^c}$ (note that $i\mathfrak{a}$ is a maximal abelian subspace in $\mathfrak{k} \cap \mathfrak{q}$), Δ the root system for the pair $(\mathfrak{g}^c, \mathfrak{a})$, and let Δ^+ be the positive root system with respect to the positive Weyl chamber \mathfrak{a}^+ (see (5)).

Let

$$\mathfrak{n} := \bigoplus_{\beta \in \Delta^+} \mathfrak{g}_\beta^c, \quad \bar{\mathfrak{n}} := \bigoplus_{\beta \in -\Delta^+} \mathfrak{g}_\beta^c,$$

$$N := \exp \mathfrak{n}, \quad \bar{N} := \exp \bar{\mathfrak{n}}, \quad A := \exp \mathfrak{a}.$$

Let $x_0 := eH$ be the base point of G^c/H . The map

$$N \times A \longrightarrow \mathcal{M}, \quad (n, a) \mapsto na \cdot x_0,$$

is a diffeomorphism of $N \times A$ onto its open image $NA \cdot x_0$. For all $x = n \exp(X) \cdot x_0$ ($X \in \mathfrak{a}$), we write $X = A(x)$. We denote $a_H(x) := \exp A(x)$.

Let \mathcal{M}^+ be the subset of \mathcal{M} defined by

$$\mathcal{M}^+ := \{x \in \mathcal{M} \mid x \geq x_0\},$$

called the future of x_0 . By [9], $\mathcal{M}^+ \subset NA \cdot x_0$.

The spherical Laplace transform of an H -invariant function f is defined by

$$\widehat{f}(\lambda) = \int_{\mathcal{M}^+} f(x) a_H(x)^{-\lambda} dx, \quad \lambda \in \mathfrak{a}_{\mathbb{C}}^*.$$

Using the following integral formula,

$$(6) \quad \int_{\mathcal{M}^+} f(x) dx = \int_{-\mathfrak{a}^+} \int_H f(h \exp(X) \cdot x_0) dh \prod_{\beta \in -\Delta^+} (\operatorname{sh} \langle \beta, X \rangle)^{m_\beta} dX,$$

the spherical Laplace transform can be written as

$$\widehat{f}(\lambda) = c \int_{-\mathfrak{a}^+} f(\exp(X) \cdot x_0) \varphi_\lambda(\exp(X)) \prod_{\beta \in -\Delta^+} (\operatorname{sh} \langle \beta, X \rangle)^{m_\beta} dX,$$

where φ_λ is the spherical function of the ordered symmetric space \mathcal{M} , defined in the interior S^0 of $S := \{g \in G^c \mid g \cdot x_0 \geq x_0\} \subset NAH$,

$$\varphi_\lambda(g) = \int_H a_H(hg)^{-\lambda} dh$$

(cf. [10]). The c -function of the symmetric space \mathcal{M} , which we denote by $c_{\mathcal{M}}$, is defined by the integral

$$c_{\mathcal{M}}(\lambda) = \int_{\overline{N} \cap HAN} a_H(\overline{n})^{-(\lambda+\rho)} d\overline{n}.$$

Remark. From Theorem 2.3 and the integral formula (6), we obtain

$$\int_{\Xi} f(\xi) d\xi = c_0 \int_{G/H} \int_{\mathcal{M}^+} f(g \cdot ix^{\frac{1}{2}}) dg dx.$$

This integral formula is a generalization of that given in Proposition X.3.4 of [12] where $G = \underline{K}$.

Let \mathcal{W}_μ^ω (resp. $\mathcal{W}_\mu^{-\omega}$) be the space of analytic (resp. hyperfunction) vectors of \mathcal{W}_μ . By [5] Theorem 1.1, $(\mathcal{W}_\mu^{-\omega})^H = (\mathcal{W}_\mu^{-\infty})^H$ where $(\mathcal{W}_\mu^{-\omega})^H$ is the subspace of H -invariant hyperfunction vectors of $\mathcal{W}_\mu^{-\omega}$. Moreover, if the representation $(\pi_\mu, \mathcal{W}_\mu)$ satisfies the Harish-Chandra condition, the linear form

$$L_\mu(f) = \int_H \langle \pi_\mu(h) f | v_\mu \rangle dh, \quad f \in \mathcal{W}_\mu^\omega$$

defines an H -invariant hyperfunction vector (cf. [20]). Using the fact that $\dim (\mathcal{W}_\mu^{-\infty})^H \leq 1$, and we deduce that if ψ_μ is an H -invariant distribution vector, there exist a constant c_0 such that

$$(7) \quad \langle f | \psi_\mu \rangle = c_0 L_\mu(f), \quad f \in \mathcal{W}_\mu^\omega.$$

In particular if $f = v_\mu$ then

$$\langle v_\mu | \psi_\mu \rangle = c_0 \int_H \langle \pi_\mu(h) v_\mu | v_\mu \rangle dh.$$

Using the integral formula for all functions $f \in L^1(H)$,

$$\int_H f(h)dh = \int_{\overline{N} \cap HAN} f(h(\overline{n}))a_H(\overline{n})^{-2\rho} d\overline{n},$$

(cf. [24]), we deduce that

$$\begin{aligned} L_\mu(v_\mu) &= \int_{\overline{N} \cap HAN} \langle \pi_\mu(h(\overline{n}))v_\mu | v_\mu \rangle a_H(\overline{n})^{-2\rho} d\overline{n} \\ &= \int_{\overline{N} \cap HAN} \langle \pi_\mu(a_H(\overline{n})^{-1}\overline{n})v_\mu | v_\mu \rangle a_H(\overline{n})^{-2\rho} d\overline{n} \quad (\pi_\mu(n(\overline{n}))v_\mu = v_\mu) \\ &= \int_{\overline{N} \cap HAN} \langle \pi_\mu(\overline{n})v_\mu | v_\mu \rangle a_H(\overline{n})^{-2\rho-\mu} d\overline{n} \\ &= \int_{\overline{N} \cap HAN} \langle v_\mu | \pi_\mu(\overline{n})^* v_\mu \rangle a_H(\overline{n})^{-2\rho-\mu} d\overline{n} \\ &= \int_{\overline{N} \cap HAN} a_H(\overline{n})^{-(2\rho+\mu)} d\overline{n} \\ &= c_{\mathcal{M}}(\mu + \rho). \end{aligned}$$

That implies,

$$(8) \quad c_0 = \frac{\langle v_\mu | \psi_\mu \rangle}{c_{\mathcal{M}}(\mu + \rho)}.$$

Lemma 5.1. *For all $\gamma \in \Gamma(C)$,*

$$\langle \pi_\mu(\gamma^{-1})\psi_\mu | \psi_\mu \rangle = \frac{|\langle \psi_\mu | v_\mu \rangle|^2}{c_{\mathcal{M}}(\mu + \rho)} \varphi_{-\mu}(\gamma^{-1}).$$

Proof. By [10], for all $\gamma \in \Gamma(C^0) \cap G^c$ there exist $\overline{n} \in \overline{N}$, $a_H(\gamma) \in A$, and $h \in H$, such that $\gamma = \overline{n}a_H(\gamma)h$. Hence for $\gamma \in \Gamma(C^0) \cap G^c$,

$$\langle \pi_\mu(\gamma^{-1})\psi_\mu | v_\mu \rangle = a_H(\gamma^{-1})^\mu \langle \psi_\mu | v_\mu \rangle.$$

The equalities (7) and (8) yield

$$\begin{aligned} \langle \pi_\mu(\gamma^{-1})\psi_\mu | \psi_\mu \rangle &= \frac{\langle v_\mu | \psi_\mu \rangle}{c_{\mathcal{M}}(\mu + \rho)} \int_H \langle \pi_\mu(h\gamma^{-1})\psi_\mu | v_\mu \rangle dh \\ &= \frac{\langle v_\mu | \psi_\mu \rangle}{c_{\mathcal{M}}(\mu + \rho)} \int_H a_H(h\gamma^{-1})^\mu \langle \psi_\mu | v_\mu \rangle dh \\ &= \frac{|\langle \psi_\mu | v_\mu \rangle|^2}{c_{\mathcal{M}}(\mu + \rho)} \varphi_{-\mu}(\gamma^{-1}). \end{aligned}$$

Now the assertion follows from the fact that the function $\gamma \mapsto \langle \pi_\mu(\gamma^{-1})\psi_\mu | \psi_\mu \rangle$ is holomorphic on $\Gamma(C^0)$ and coincides with $\frac{|\langle \psi_\mu | v_\mu \rangle|^2}{c_{\mathcal{M}}(\mu + \rho)} \varphi_{-\mu}(\gamma^{-1})$ in $\Gamma(C^0) \cap G^c$. \square

Let $(\pi_\mu, \mathcal{W}_\mu)$ be an H -spherical unitary highest weight representation of \tilde{G} such that $(\pi_\mu, \mathcal{W}_\mu)$ belongs to the relative discrete series, and

$$\int_{G/H} |\langle \pi_\mu(g)v_\mu | \psi_\mu \rangle|^2 d\dot{g} = \frac{1}{\delta_\mu}$$

where δ_μ is the relative formal dimension calculated in [20], $\delta_\mu = d_\mu \cdot c_{\mathcal{M}}(\mu + \rho)$, with $d_\mu = \prod_{\beta \in \Delta^+} \frac{\langle \mu + \rho, \beta \rangle}{\langle \rho, \beta \rangle}$ the formal dimension of the representation π_μ .

Theorem 5.1. *Let P_ν be the G -invariant function such that $P_\nu(\exp(2X) \cdot ie) := p_\nu(\exp(X) \cdot ie)$. The weight $\mu \in \mathcal{R}'_{\nu, H}$ if and only if the spherical Laplace transform $\widehat{P}_\nu(-\mu)$ is finite. Moreover*

$$C_\mu(\nu) = \frac{1}{\delta_\mu} \widehat{P}_\nu(-\mu),$$

where δ_μ is the relative formal dimension.

Proof. We assume that the H -invariant distribution vector ψ_μ is normalized by $\langle \psi_\mu | v_\mu \rangle = 1$. By the integral formula of Theorem 2.3 and the fact that p_ν is G -invariant, we deduce that

$$\begin{aligned} C_\mu(\nu) &= \int_{\Xi} |\langle \pi_\mu(\gamma^{-1})v_\mu | \psi_\mu \rangle|^2 p_\nu(\xi) d\xi \\ &= \int_G \int_{C^+} |\langle \pi_\mu(\gamma^{-1})v_\mu | \psi_\mu \rangle|^2 p_\nu(\exp(X) \cdot ie) \prod_{\beta \in \Delta^+} \text{sh}(2\beta(X))^{m_\beta} dg dX \\ &= \frac{1}{d_\mu} \int_{C^+} \|\pi_\mu(\exp(X))\psi_\mu\|^2 p_\nu(\exp(X) \cdot ie) \prod_{\beta \in \Delta^+} \text{sh}(2\beta(X))^{m_\beta} dX. \end{aligned}$$

According to the last lemma, this yields

$$\begin{aligned} &\int_{\Xi} |\langle \pi_\mu(\gamma)\psi_\mu | v_\mu \rangle|^2 p_\nu(\xi) d\xi \\ &= \frac{1}{\delta_\mu} \int_{C^+} p_\nu(\exp(X) \cdot ie) \varphi_{-\mu}(\exp(2X)) \prod_{\beta \in \Delta^+} \text{sh}(2\beta(X))^{m_\beta} dX \\ &= \frac{1}{\delta_\mu} \widehat{P}_\nu(-\mu). \end{aligned}$$

□

For any $x \in \mathbb{V}$ we have $x = x^+ + x^-$, where $x^+ := \frac{x + \alpha(x)}{2} \in \mathbb{V}^+$ and $x^- := \frac{x - \alpha(x)}{2} \in \mathbb{V}^-$. We denote $x := (x^+, x^-)$.

Let \mathcal{J} be the bounded set in \mathbb{V} defined by

$$\mathcal{J} = \left\{ x = (x^+, x^-) \in \mathbb{V} \mid x^+ \in \Omega, x^+ + x^- \in e - \bar{\Omega} \right\} \subset (\mathbb{V}^+ \cap \Omega) \times ((e + \mathbb{V}^-) \cap \Omega).$$

Proposition 5.2. *The set \mathcal{J} coincides with*

$$\left\{ x \in \mathcal{M} \mid x \geq x_0 \right\},$$

the future of x_0 .

Proof. Let $\varphi : [0, 1] \rightarrow \mathbb{V}$ be the curve defined by

$$\varphi(t) = tx + (1-t)e,$$

where $\varphi(0) = e$ and $\varphi(1) = x$. Since Ω is convex,

$$\varphi(t) + \alpha(\varphi(t)) = t(x + \alpha(x)) + (1-t)2e \in \Omega.$$

Thus $\varphi(t) \in \mathcal{M}$ for all $t \in [0, 1]$. Moreover $\dot{\varphi}(t) = x - e \in -\bar{\Omega} \simeq C^c$, where

$$C^c = \left\{ (v, 0, -\alpha(v)) \mid v \in \bar{\Omega} \right\} \subset \mathfrak{g}^c,$$

the regular cone in \mathfrak{g}^c such that $C^c \cap \mathfrak{p}^c \neq \emptyset$, where $\mathfrak{p}^c = \{(v, L(w), -v) \mid v \in \mathbb{V}^+, w \in \mathbb{V}^-\}$ (cf. [3], p. 26). Then, φ is a nontrivial causal curve in \mathcal{M} from x to x_0 and x belongs to the future of x_0 .

Conversely, let $\varphi : [a, b] \rightarrow \mathcal{M}$ be a causal curve. Assume that there exists $t > a$ such that $\varphi(t) \notin \mathcal{M} \cap \mathbb{V}$ and

$$\kappa = \inf \left\{ t \in [a, b] \mid \varphi(t) \notin \mathcal{M} \cap \mathbb{V} \right\}.$$

Since $\mathcal{M} \cap \mathbb{V}$ is open in \mathcal{M} , then $\varphi(t) \in \mathcal{M} \cap \mathbb{V}$ if $t < \kappa$ and $\varphi(t) \notin \mathcal{M} \cap \mathbb{V}$ if not. Hence

$$\lim_{\substack{t \rightarrow \kappa \\ t < \kappa}} \|\varphi(t)\| = \infty.$$

Moreover the curve $\varphi : [a, \kappa[\rightarrow \mathbb{V}$ is causal with respect to the causal structure defined by the cone $-\bar{\Omega}$. Then, $\varphi(t) \in e - \bar{\Omega}$ and for all $t \in [a, \kappa[$, $\varphi(t)$ belongs to the connected component of x_0 in $\mathcal{M} \cap \mathbb{V}$ given by

$$\left\{ x \in \mathbb{V} \mid x + \alpha(x) \in \Omega \right\}.$$

Hence for all $t \in [a, \kappa[$, $\varphi(t) \in \mathcal{J}$ such that \mathcal{J} is a bounded set. This leads to contradiction. \square

Lemma 5.3. *Let $\mathfrak{s} \in \mathbb{C}^{r^+}$ and $x = n \exp A(x) \cdot x_0 \in NA \cdot x_0$. To identify $\mathfrak{a}_{\mathbb{C}}$ with \mathbb{C}^{r^+} we have*

$$e^{\langle \mathfrak{s} | A(x) \rangle} = \Delta_{\mathfrak{s}} \left(\frac{x + \alpha(x)}{2} \right).$$

Proof. The function $x \mapsto \Delta_{\mathbf{s}}\left(\frac{x + \alpha(x)}{2}\right)$ is N -invariant. Then, for all $x = n \exp A(x) \cdot x_0$,

$$\Delta_{\mathbf{s}}\left(\frac{x + \alpha(x)}{2}\right) = \Delta_{\mathbf{s}}(\exp A(x) \cdot x_0).$$

Since $A(x) \in \mathfrak{a}_{\mathbb{C}}$, there exists $(t_1, t_2, \dots, t_{r^+}) \in \mathbb{C}^{r^+}$ such that $A(x) = \sum_{j=1}^{r^+} t_j L(e_j)$, and $\exp A(x) \cdot x_0 = \sum_{j=1}^{r^+} e^{t_j} e_j$. Here $\{e_j\}_{1 \leq j \leq r^+}$ is the Jordan frame of R^+ . Thus for all $\mathbf{s} = (s_1, \dots, s_{r^+})$, we deduce that

$$\Delta_{\mathbf{s}}\left(\frac{x + \alpha(x)}{2}\right) = \Delta_{\mathbf{s}}\left(\sum_{j=1}^{r^+} e^{t_j} e_j\right) = \prod_{j=1}^{r^+} e^{t_j s_j} = e^{\langle \mathbf{s} | A(x) \rangle}.$$

□

To give an explicit formula of the spherical Laplace transform $\widehat{P}_{\nu}(-\mu)$ and an explicit description of the spectrum $\mathcal{R}'_{\nu, H}$, we recall some notations.

The Gindikin gamma function of the symmetric cone Ω is defined by the following integral, generalizing the classical Siegel integral

$$\begin{aligned} \Gamma_{\Omega}(\mathbf{s}) &= \int_{\Omega} e^{-\mathrm{tr}(x)} \Delta_{\mathbf{s} - \frac{n}{r}}(x) dx \\ &= \prod_{j=1}^r \Gamma\left(s_j - \frac{d}{2}(j-1)\right), \quad d := \frac{2(n-r)}{r(r-1)} \end{aligned}$$

for $\mathbf{s} = (s_1, \dots, s_r)$ where $\Re s_j > (j-1)\frac{d}{2}$ for $j = 1, \dots, r$.

The Gindikin beta function of the symmetric cone Ω is defined by the following integral

$$\begin{aligned} \mathcal{B}_{\Omega}(\mathbf{s}, \mathbf{m}) &= \int_{\Omega \cap (e - \Omega)} \Delta_{\mathbf{s} - \frac{n}{r}}(x) \Delta_{\mathbf{m} - \frac{n}{r}}(e - x) dx \\ &= \frac{\Gamma_{\Omega}(\mathbf{s}) \Gamma_{\Omega}(\mathbf{m})}{\Gamma_{\Omega}(\mathbf{s} + \mathbf{m})}. \end{aligned}$$

Case (I). \mathbb{V}^+ is a simple Jordan algebra.

Let $n^{\pm} := \dim(\mathbb{V}^{\pm})$, and Γ_{Ω^+} (resp. \mathcal{B}_{Ω^+}) be the Gindikin gamma (resp. beta) function of the symmetric cone $\Omega^+ := \Omega \cap \mathbb{V}^+$ of \mathbb{V}^+ .

Lemma 5.4. For $\lambda \in \mathbb{C}$ such that $\Re(\lambda) > \frac{n}{r} - 1$,

$$\int_{(-e + \Omega) \cap \mathbb{V}^-} \Delta(e + v)^{\frac{r}{r^+} \lambda - \frac{n}{r^+}} dv = \frac{\Gamma_{\Omega}(\lambda)}{\Gamma_{\Omega^+}\left(\frac{r}{r^+} \lambda\right)}.$$

Proof. With respect to the decomposition of the Jordan algebra \mathbb{V} as $\mathbb{V} = \mathbb{V}^+ \oplus \mathbb{V}^-$, the Gindikin gamma function of the symmetric cone Ω can be written as

$$\begin{aligned} \Gamma_{\Omega}(\lambda) &= \int_{\Omega} e^{-tr(x)} \Delta(x)^{\lambda - \frac{n}{r}} dx \\ &= \int_{\Omega^+} e^{-tr(x^+)} \left[\int_{\{x^- \mid x^+ + x^- \in \Omega\}} \Delta(x^+ + x^-)^{\lambda - \frac{n}{r}} dx^- \right] dx^+. \end{aligned}$$

Moreover $x^+ + x^- = P((x^+)^{\frac{1}{2}})(e + v)$, where P is the quadratic representation and $v = P((x^+)^{-\frac{1}{2}})x^-$. Hence

$$\begin{aligned} \Gamma_{\Omega}(\lambda) &= \int_{\Omega^+} e^{-tr(x^+)} \Delta(x^+)^{\frac{r}{r^+} \lambda - \frac{n}{r^+} + \frac{n^-}{r^+}} dx^+ \int_{(-e+\Omega) \cap \mathbb{V}^-} \Delta(e + v)^{\frac{r}{r^+} \lambda - \frac{n}{r^+}} dv \\ &= \Gamma_{\Omega^+} \left(\frac{r}{r^+} \lambda \right) \int_{(-e+\Omega) \cap \mathbb{V}^-} \Delta(e + v)^{\frac{r}{r^+} \lambda - \frac{n}{r^+}} dv. \end{aligned}$$

This proves that

$$\int_{(-e+\Omega) \cap \mathbb{V}^-} \Delta(e + v)^{\frac{r}{r^+} \lambda - \frac{n}{r^+}} dv = \frac{\Gamma_{\Omega}(\lambda)}{\Gamma_{\Omega^+} \left(\frac{r}{r^+} \lambda \right)}.$$

□

Proposition 5.5. For $\mu = (m_1 + (r/2r^+)\nu, \dots, m_{r^+} + (r/2r^+)\nu)$, the spherical Laplace transform $\widehat{P}_{\nu}(-\mu)$ is finite if and only if

$$m_1 \geq m_2 \geq \dots \geq m_{r^+} \geq 0.$$

Moreover

$$\widehat{P}_{\nu}(-\mu) = c_0 \Gamma_{\Omega} \left(\nu - \frac{n}{r} \right) \frac{\Gamma_{\Omega^+} \left(\mu - \nu \frac{r}{2r^+} + \frac{n^+}{r^+} \right)}{\Gamma_{\Omega^+} \left(\mu + \nu \frac{r}{2r^+} - \frac{n^-}{r^+} \right)},$$

where c_0 is a positive constant.

Proof. By Proposition 3.1 and for all $x = n \exp A(x) \cdot x_0$, the function p_{ν} is given by

$$p_{\nu}(x) = \det \left(P(x + \alpha(x)) \right)^{-\frac{r}{2n} \nu + 1} \Delta(e - x^2)^{\nu - \frac{2n}{r}}.$$

Since $A(x) \in \mathfrak{a}_{\mathbb{C}}$, there exists $(t_1, \dots, t_{r^+}) \in \mathbb{C}^{r^+}$ such that $A(x) = \sum_{j=1}^{r^+} t_j L(e_j)$, and

$$\exp \left(\frac{A(x)}{2} \right) \cdot x_0 = \sum_{j=1}^{r^+} e^{\frac{t_j}{2}} e_j.$$

Let $x = \sum_{j=1}^{r^+} e^{t_j} e_j$ and $u = \sum_{j=1}^{r^+} e^{\frac{t_j}{2}} e_j$. Using the fact that p_ν is G -invariant, we deduce that

$$p_\nu(u) = \Delta(x^+)^{-\nu \frac{r}{2r^+} + \frac{n}{r^+}} \Delta(e-x)^{\nu - \frac{2n}{r}} = P_\nu(x).$$

For $x \in \mathbb{V}$, let $d\delta(x)$ be the G -invariant measure on \mathcal{M} , where its restriction to $\mathcal{M} \cap \mathbb{V}$ is equal to $d\delta(x) = \Delta(x^+)^{-\frac{n}{r^+}} dx^+ dx^-$. Using Lemma 5.3, the spherical Laplace transformation of P_ν can be written as

$$\begin{aligned} \widehat{P}_\nu(-\mu) &= \int_{x \geq x_0} e^{\langle \mu, A(x) \rangle} P_\nu(x) d\delta(x) \\ &= \int_{\Omega^+ \cap (e-\Omega^+)} \int_{\{x^- | e-x^+-x^- \in \Omega\}} \Delta_\mu(x^+) \Delta(x^+)^{-\nu \frac{r}{2r^+} + \frac{n}{r^+}} \\ &\quad \cdot \Delta(e-x)^{\nu - \frac{2n}{r}} \Delta(x^+)^{-\frac{n}{r^+}} dx^+ dx^- \\ &= \int_{\Omega^+ \cap (e-\Omega^+)} \Delta_{\mu - \nu \frac{r}{2r^+}}(x^+) \\ &\quad \cdot \left[\int_{\{x^- | e-x^+-x^- \in \Omega\}} \Delta(e-x^+-x^-)^{\nu - \frac{2n}{r}} dx^- \right] dx^+. \end{aligned}$$

Using the last lemma, and the fact that $e-x^+-x^- = P((e-x^+)^{\frac{1}{2}})(e-v)$ where $v = P((e-x^+)^{-\frac{1}{2}})x^-$, we obtain

$$\begin{aligned} \widehat{P}_\nu(-\mu) &= \int_{\Omega^+ \cap (e-\Omega^+)} \Delta_{\mu - \nu \frac{r}{2r^+}}(x^+) \Delta(e-x^+)^{\nu \frac{r}{r^+} - \frac{2n}{r^+} + \frac{n^-}{r^+}} dx^+ \\ &\quad \cdot \int_{(e-\Omega) \cap \mathbb{V}^-} \Delta(e-v)^{\nu \frac{r}{r^+} - \frac{2n}{r^+}} dv \\ &= \frac{\Gamma_\Omega\left(\nu - \frac{n}{r}\right)}{\Gamma_{\Omega^+}\left(\nu \frac{r}{r^+} - \frac{n}{r^+}\right)} \mathcal{B}_{\Omega^+}\left(\mu - \nu \frac{r}{2r^+} + \frac{n^+}{r^+}, \nu \frac{r}{r^+} - \frac{n}{r^+}\right), \end{aligned}$$

such that $\mu_j - \nu \frac{r}{2r^+} + \frac{n^+}{r^+} > (j-1) \frac{n^+ - r^+}{r^+(r^+ - 1)}$ for $1 \leq j \leq r^+$. By the fact that $\mu_j = m_j + \nu \frac{r}{2r^+}$, where $m_j \in \mathbb{Z}$ and $\mu_j - \mu_{j+1} \in \mathbb{N}$, this condition will be written as

$$m_1 \geq m_2 \geq \dots \geq m_{r^+} \geq 0.$$

□

It follows from Proposition 5.5 that the spectrum of the Bergman space $\mathcal{H}_\nu^2(\tilde{\Xi})$ is given by

$$\mathcal{R}'_{\nu,H} = \left\{ \mu = \left(m_1 + (r/2r^+)\nu, m_2 + (r/2r^+)\nu, \dots, m_{r^+} + (r/2r^+)\nu \right) \middle| \right. \\ \left. m_1 \geq m_2 \geq \dots \geq m_{r^+} \geq 0 \right\}.$$

Recall that the constant $c_\nu = \frac{1}{\pi^n} \frac{\Gamma_\Omega(\nu)}{\Gamma_\Omega\left(\nu - \frac{n}{r}\right)}$ (see (4)), the reproducing kernel $K_\nu(\tilde{\xi}_1, \tilde{\xi}_2)$ can be written as

$$\begin{aligned} & \top_\alpha(\tilde{\xi}_1)^{\frac{r}{4n}\nu} \det(B(\xi_1, \bar{\xi}_2))^{-\frac{r}{2n}\nu} \overline{\top_\alpha(\tilde{\xi}_2)^{\frac{r}{4n}\nu}} \\ &= \sum_{\mathbf{m} \geq 0} \frac{c_0 d_\mu}{\Gamma_\Omega(\nu)} \frac{\Gamma_{\Omega^+}\left(\mathbf{m} + \nu \frac{r}{r^+} - \frac{n^-}{r^+}\right)}{\Gamma_{\Omega^+}\left(\mathbf{m} + \frac{n^+}{r^+}\right)} \varphi_{-\mu}\left((\tilde{\gamma}_2^\# \tilde{\gamma}_1)^{-1}\right), \end{aligned}$$

for all $\tilde{\xi}_1 = \tilde{\gamma}_1 \cdot H$ and all $\tilde{\xi}_2 = \tilde{\gamma}_2 \cdot H$ in $\tilde{\Xi}$. The summation index runs over the integers of type $\mathbf{m} := (m_1, m_2, \dots, m_{r^+})$ such that $m_1 \geq \dots \geq m_{r^+} \geq 0$. We use $\mathbf{m} \geq 0$ to denote the summation index. The function

$$\nu \mapsto \top_\alpha(\tilde{\xi}_1)^{\frac{r}{4n}\nu} \det(B(\xi_1, \bar{\xi}_2))^{-\frac{r}{2n}\nu} \overline{\top_\alpha(\tilde{\xi}_2)^{\frac{r}{4n}\nu}},$$

is holomorphic on \mathbb{C} and coincides with

$$\sum_{\mathbf{m} \geq 0} \frac{c_0 d_\mu}{\Gamma_\Omega(\nu)} \frac{\Gamma_{\Omega^+}\left(\mathbf{m} + \nu \frac{r}{r^+} - \frac{n^-}{r^+}\right)}{\Gamma_{\Omega^+}\left(\mathbf{m} + \frac{n^+}{r^+}\right)} \varphi_{-\mu}\left((\tilde{\gamma}_2^\# \tilde{\gamma}_1)^{-1}\right),$$

for $\nu \in \left] \frac{2n}{r} - 1, +\infty \right[$, therefore these two functions coincide everywhere. The following theorem holds.

Theorem 5.2. *Assume α is given such that \mathbb{V}^+ is a simple algebra. Then for all $\nu \in \mathbb{C}$ such that $\Re(\nu) > \frac{n}{r} - 1$,*

$$\begin{aligned} & \top_\alpha(\tilde{\xi}_1)^{\frac{r}{4n}\nu} \det(B(\xi_1, \bar{\xi}_2))^{-\frac{r}{2n}\nu} \overline{\top_\alpha(\tilde{\xi}_2)^{\frac{r}{4n}\nu}} \\ &= \sum_{\mathbf{m} \geq 0} \frac{c_0 d_\mu}{\Gamma_\Omega(\nu)} \frac{\Gamma_{\Omega^+}\left(\mathbf{m} + \nu \frac{r}{r^+} - \frac{n^-}{r^+}\right)}{\Gamma_{\Omega^+}\left(\mathbf{m} + \frac{n^+}{r^+}\right)} \varphi_{-\mu}\left((\tilde{\gamma}_2^\# \tilde{\gamma}_1)^{-1}\right). \end{aligned}$$

The series converges uniformly on compact subsets of $\tilde{\Xi} \times \tilde{\Xi}$.

An application (Makarevič symmetric spaces of Cayley type). Let \mathbb{V} be a simple Euclidean Jordan algebra of dimension n and rank r . The bounded symmetric domain associated with $\mathbb{V}_{\mathbb{C}} \times \mathbb{V}_{\mathbb{C}}$ is the bidisc $\mathcal{D} \times \mathcal{D}$ where \mathcal{D} is the unit disc of $\mathbb{V}_{\mathbb{C}}$. Let α be the involution on $\mathcal{D} \times \mathcal{D}$ defined by $\alpha(z, w) = (-w, -z)$. In this case, the domain Ξ is realized as $\mathcal{D} \times \mathcal{D} \setminus \Sigma$ where

$$\Sigma = \{(z, w) \in \mathcal{D} \times \mathcal{D} \mid \Delta(z - w) = 0\},$$

and Δ is the determinant polynomial associated with \mathbb{V} (see (1)).

Let

$$\left\{ z, w; z', w' \right\} := \frac{\Delta(z' - z)\Delta(w - w')}{\Delta(z - w)\Delta(z' - w')}$$

be the cross-ratio of four points z, w, z' , and w' of $\mathbb{V}_{\mathbb{C}}$. This definition generalizes the classical cross-ratio-matrix and satisfies the G -invariance property $\left\{ g \cdot z, g \cdot w; g \cdot z', g \cdot w' \right\} = \left\{ z, w; z', w' \right\}$. The reproducing kernel of the Bergman space $\mathcal{H}_{\nu}^2(\tilde{\Xi})$ is equal to

$$K_{\nu}(\tilde{z}_1, \tilde{w}_1; \tilde{z}_2, \tilde{w}_2) = c_{\nu}^2 \left\{ \tilde{z}_1, \tilde{w}_1; \tilde{z}_2^{-1}, \tilde{w}_2^{-1} \right\}^{\nu},$$

for $(\tilde{z}_1, \tilde{w}_1)$ and $(\tilde{z}_2, \tilde{w}_2)$ in $\tilde{\Xi}$.

Using Theorem 5.2, we deduce a formula of a complex power of a cross-ratio of four points,

$$\left\{ \tilde{z}_1, \tilde{w}_1; \tilde{z}_2^{-1}, \tilde{w}_2^{-1} \right\}^{\nu} = \sum_{\mathbf{m} \geq 0} \frac{c_0 d_{\mu}}{\Gamma_{\Omega}(\nu)^2} \frac{\Gamma_{\Omega}\left(\mathbf{m} + 2\nu - \frac{n}{r}\right)}{\Gamma_{\Omega}\left(\mathbf{m} + \frac{n}{r}\right)} \varphi_{-\mu}\left((\tilde{\gamma}_2^{\sharp} \tilde{\gamma}_1)^{-1}\right),$$

for all $(\tilde{z}_1, \tilde{w}_1) = \tilde{\gamma}_1 \cdot H$ and all $(\tilde{z}_2, \tilde{w}_2) = \tilde{\gamma}_2 \cdot H$. We remark that this formula is a generalization of that given in Theorem 3.1 of [7].

Let \mathcal{W} be the Wallach set

$$\mathcal{W} = \left\{ 0, \frac{n-r}{r(r-1)}, \dots, \frac{n}{r} - 1 \right\} \cup \left] \frac{n}{r} - 1, +\infty \right[,$$

(cf. [12] p. 268), and let $\mathcal{K}_{\nu}(\tilde{z}_1, \tilde{w}_1; \tilde{z}_2, \tilde{w}_2) := \left\{ \tilde{z}_1, \tilde{w}_1; \tilde{z}_2^{-1}, \tilde{w}_2^{-1} \right\}^{\nu}$. The kernel \mathcal{K}_{ν} is of positive type on $\tilde{\Xi} \times \tilde{\Xi}$ if and only if ν belongs to \mathcal{W} . Then \mathcal{K}_{ν} is a reproducing kernel of some Hilbert space $\mathcal{H}_{\nu}(\tilde{\Xi})$. In particular if $\nu > \frac{2n}{r} - 1$, $\mathcal{H}_{\nu}(\tilde{\Xi}) = \mathcal{H}_{\nu}^2(\tilde{\Xi})$.

We note that $\frac{n}{r} \in \frac{1}{2}\mathbb{N}$. In [12], J. Faraut and A. Korányi showed that the analytic continuation Bergman space $\mathcal{H}_{\frac{n}{r}}(\mathcal{D} \times \mathcal{D})$ ($\simeq \mathcal{H}_{\frac{n}{r}}(\tilde{\Xi})$ by Theorem 3.1) coincides with the Hardy space $H^2(\mathcal{D} \times \mathcal{D})$. It is a Hilbert space of holomorphic functions f on $\mathcal{D} \times \mathcal{D}$ such that

$$\sup_{0 < r_1, r_2 < 1} \int_{\mathcal{S} \times \mathcal{S}} |f(r_1 u_1, r_2 u_2)|^2 d\sigma(u_1) d\sigma(u_2) < \infty,$$

where \mathcal{S} is the Shilov boundary of \mathcal{D} .

Using the same notations as in Section 3, the covering \tilde{G} of order 2 of G acts on $H^2(\mathcal{D} \times \mathcal{D})$ by the representation $\tilde{\pi}_{\frac{n}{r}} := \pi_{\frac{n}{r}} \otimes \pi_{\frac{n}{r}}$: for $f \in H^2(\mathcal{D} \times \mathcal{D})$ and $\tilde{g}^{-1} = (g, \varphi) \in \tilde{G}$,

$$(\tilde{\pi}_{\frac{n}{r}}(\tilde{g})f)(z, w) = \left(\frac{\Delta(g \cdot z - g \cdot w)}{\Delta(z - w)} \right)^{\frac{n}{r}} f(g \cdot z, g \cdot w).$$

Using [4] Theorem 3.2.1, we can deduce that the representation $\tilde{\pi}_{\frac{n}{r}}$ decomposes into a discrete sum of irreducible unitary representations π_{μ} of \tilde{G} with highest weights

$$\mu = \left(m_1 + \frac{n}{r}, m_2 + \frac{n}{r}, \dots, m_r + \frac{n}{r} \right)$$

such that $m_1 \geq m_2 \geq \dots \geq m_r \geq 0$. In particular, the decomposition of the Hardy space $H^2(\mathcal{D} \times \mathcal{D})$ holds. Hence, we find the decomposition of $H^2(\mathcal{D} \times \mathcal{D})$ shown in [7] when $\frac{n}{r}$ belongs to \mathbb{N} .

Case (II): \mathbb{V}^+ is a direct sum of two simple algebras.

Let \mathbb{V}^+ be a direct sum of two simple algebras \mathbb{V}_1^+ of rank p and \mathbb{V}_2^+ of rank q such that $p + q = r^+$ ($= r$). Let $n_1 = p + \frac{1}{2}dp(p-1)$ denote the dimension of \mathbb{V}_1^+ , and $n_2 = q + \frac{1}{2}dq(q-1)$ the dimension of \mathbb{V}_2^+ . Then the involution α is given by $\alpha = P(w)$, where $w = c_1 + c_2 + \dots + c_p - c_{p+1} - c_{p+2} - \dots - c_r$. The cone Ω^+ is a direct sum of Ω_1^+ and Ω_2^+ , where $\Omega_1^+ := \Omega^+ \cap \mathbb{V}_1^+$ and $\Omega_2^+ := \Omega^+ \cap \mathbb{V}_2^+$.

For $\mathbf{s} \in \mathbb{C}^p$ (resp. \mathbb{C}^q), we write $\Delta_{\mathbf{s}}^{(1)}$ (resp. $\Delta_{\mathbf{s}}^{(2)}$) for the generalized power function associated with Ω_1^+ (resp. Ω_2^+).

Using the same techniques as in Lemma 5.4, we show that for all $\lambda \in \mathbb{C}$ such that $\Re(\lambda) > \frac{n}{r} - 1$,

$$(9) \quad \int_{(-e+\Omega) \cap \mathbb{V}^-} \Delta(e+v)^{\lambda - \frac{n}{r}} dv = \frac{\Gamma_{\Omega}(\lambda)}{\Gamma_{\Omega_1^+} \left(\lambda - \frac{n^+}{r} + \frac{n_1}{p} \right) \Gamma_{\Omega_2^+} \left(\lambda - \frac{n^+}{r} + \frac{n_2}{q} \right)}.$$

Here $\Gamma_{\Omega_1^+}$ (resp. $\Gamma_{\Omega_2^+}$) denotes the Gindikin gamma function associated with the symmetric cone Ω_1^+ (resp. Ω_2^+).

Proposition 5.6. *For all $\mu = \left(m_1 + (\nu/2), \dots, m_p + (\nu/2); m_{p+1} + (\nu/2), \dots, m_r + (\nu/2) \right) := (\underline{\mu}_1; \underline{\mu}_2)$, the spherical Laplace transform $\hat{P}_{\nu}(-\mu)$ is finite if and only if*

$$m_1 \geq \dots \geq m_p \geq 0, \quad m_{p+1} \geq \dots \geq m_r \geq 0.$$

Moreover

$$\widehat{P}_\nu(-\mu) = c_0 \Gamma_\Omega \left(\nu - \frac{n}{r} \right) \cdot \frac{\Gamma_{\Omega_1^+} \left(\mu_1 - \frac{\nu}{2} + \frac{n_1}{p} \right) \Gamma_{\Omega_2^+} \left(\mu_2 - \frac{\nu}{2} + \frac{n_2}{q} \right)}{\Gamma_{\Omega_1^+} \left(\mu_1 + \frac{\nu}{2} - \frac{n^+}{r} - \frac{n}{r} + \frac{2n_1}{p} \right) \Gamma_{\Omega_2^+} \left(\mu_2 + \frac{\nu}{2} - \frac{n^+}{r} - \frac{n}{r} + \frac{2n_2}{q} \right)},$$

where c_0 is a positive constant.

Proof. As proved in Proposition 5.5, we have

$$\begin{aligned} \widehat{P}_\nu(-\mu) &= \int_{\Omega_1^+ \cap (e - \Omega_1^+)} \Delta_{\mu_1 - \frac{\nu}{2}}^{(1)}(x_1^+) \Delta^{(1)}(e - x_1^+)^{\nu + \frac{n^-}{r} - \frac{2n}{r}} dx_1^+ \\ &\quad \int_{\Omega_2^+ \cap (e - \Omega_2^+)} \Delta_{\mu_2 - \frac{\nu}{2}}^{(2)}(x_2^+) \Delta^{(2)}(e - x_2^+)^{\nu + \frac{n^-}{r} - \frac{2n}{r}} dx_2^+ \\ &\quad \int_{(e - \Omega) \cap \mathbb{V}^-} \Delta(e - v)^{\nu - \frac{2n}{r}} dv \\ &= \mathcal{B}_{\Omega_1^+} \left(\mu_1 - \frac{\nu}{2} + \frac{n_1}{p}, \nu - \frac{n^+}{r} - \frac{n}{r} + \frac{n_1}{p} \right) \\ &\quad \mathcal{B}_{\Omega_2^+} \left(\mu_2 - \frac{\nu}{2} + \frac{n_2}{q}, \nu - \frac{n^+}{r} - \frac{n}{r} + \frac{n_2}{q} \right) \\ &\quad \int_{(e - \Omega) \cap \mathbb{V}^-} \Delta(e - v)^{\nu - \frac{2n}{r}} dv. \end{aligned}$$

Then the assertion follows from formula (9). \square

Let $\mathbf{m} := (\mathbf{m}_1, \mathbf{m}_2) = (m_1, \dots, m_p; m_{p+1}, \dots, m_r)$. We use $\mathbf{m} \geq 0$ to denote $m_1 \geq \dots \geq m_p \geq 0$; $m_{p+1} \geq \dots \geq m_r \geq 0$. We apply the similar arguments as in Theorem 5.2, the following theorem holds.

Theorem 5.3. *Assume α is given such that \mathbb{V}^+ is a direct sum of two simple algebras. Then for all $\nu \in \mathbb{C}$ such that $\Re(\nu) > \frac{n}{r} - 1$,*

$$\begin{aligned} &\Upsilon_\alpha(\tilde{\xi}_1)^{\frac{r}{4n}\nu} \det(B(\xi_1, \bar{\xi}_2))^{-\frac{r}{2n}\nu} \overline{\Upsilon_\alpha(\tilde{\xi}_2)^{\frac{r}{4n}\nu}} \\ &= \sum_{\mathbf{m} \geq 0} \frac{c_0 d_\mu}{\Gamma_\Omega(\nu)} \\ &\quad \cdot \frac{\Gamma_{\Omega_1^+} \left(\mathbf{m}_1 + \nu - \frac{n^+}{r} - \frac{n}{r} + \frac{2n_1}{p} \right) \Gamma_{\Omega_2^+} \left(\mathbf{m}_2 + \nu - \frac{n^+}{r} - \frac{n}{r} + \frac{2n_2}{q} \right)}{\Gamma_{\Omega_1^+} \left(\mathbf{m}_1 + \frac{n_1}{p} \right) \Gamma_{\Omega_2^+} \left(\mathbf{m}_2 + \frac{n_2}{q} \right)} \varphi_{-\mathbf{m} - \frac{\nu}{2}} \\ &\quad \left((\tilde{\gamma}_2^\# \tilde{\gamma}_1)^{-1} \right). \end{aligned}$$

The series converges uniformly on compact subsets of $\tilde{\Xi} \times \tilde{\Xi}$.

Remark. If $p = 0$ or $q = 0$, then $\alpha = \text{id}_V$ (the compact case). The last series can be written as

$$\Delta(e - z)^{-\nu} = c_0 \sum_{\mathbf{m} \geq 0} d_{\mathbf{m}} \frac{(\nu)_{\mathbf{m}}}{\left(\frac{n}{r}\right)_{\mathbf{m}}} \varphi_{\mathbf{m}}(z),$$

with $\mathbf{m} = (m_1, m_2, \dots, m_r)$. Here $(\mathbf{s})_{\mathbf{m}}$ denote $(\mathbf{s})_{\mathbf{m}} = \frac{\Gamma_{\Omega}(\mathbf{s} + \mathbf{m})}{\Gamma_{\Omega}(\mathbf{s})}$, for all $\mathbf{s} \in \mathbb{C}^r$ and $\mathbf{m} \in \mathbb{N}^r$. Hence, we find the generalized binomial formula shown in [12]. See also [2].

We present here the table of Makarevič symmetric space G/H , and its dual symmetric space G^c/H .

V	G/H	G^c/H
1) Herm(n, \mathbb{C})	$U(p, q) \quad (p+q=n)$ $SO^*(2n)/SO(n, \mathbb{C})$ $Sp(4n, \mathbb{R})/Sp(2n, \mathbb{C})$	$GL(n, \mathbb{C})/U(p, q)$ $SO(n, n)/O(n, \mathbb{C})$ $Sp(n, n)/Sp(2n, \mathbb{C})$
2) Sym(n, \mathbb{R})	$U(p, q)/O(p, q)$ $Sp(2n, \mathbb{R})$	$GL(n, \mathbb{R})/O(p, q)$ $Sp(2n, \mathbb{C})/Sp(2n, \mathbb{R})$
3) Herm(n, \mathbb{H})	$U(2p, 2q)/Sp(p, q)$ $SO^*(2n)$	$U^*(2n)/Sp(p, q)$ $SO(2n, \mathbb{C})/O^*(2n)$
4) $\mathbb{R} \times \mathbb{R}^{n-1}$	$SO(p) \times SO(2, q)/$ $SO(p-1) \times SO(1, q)$	$SO(1, p-1) \times SO(1, q+1)/$ $SO(p-1) \times SO(1, q)$
5) Herm($3, \mathbb{O}$)	$E_6 \times U(1)/F_4$ $E_{6(-14)} \times U(1)/F_{4(-20)}$ $SU(6, 2)/Sp(3, 1)$	$E_6 \times \mathbb{R}^+/F_4$ $E_{6(-14)} \times \mathbb{R}^+/F_{4(-20)}$ $SU^*(8)/Sp(3, 1)$
$V \times V$, where V is either of the type 1), 2), 3), 4), or 5).	<i>Cayley type</i>	
	$SU(n, n)/SL(n, \mathbb{C}) \times \mathbb{R}_+^*$ $Sp(2n, \mathbb{R})/GL(n, \mathbb{R})$ $SO^*(4n)/SL(n, \mathbb{H}) \times \mathbb{R}_+^*$ $SO(2, n)/SO(1, n-1) \times \mathbb{R}_+^*$ $E_{7(-25)}/E_{6(-26)} \times \mathbb{R}_+^*$	$SU(n, n)/SL(n, \mathbb{C}) \times \mathbb{R}_+^*$ $Sp(2n, \mathbb{R})/GL(n, \mathbb{R})$ $SO^*(4n)/SL(n, \mathbb{H}) \times \mathbb{R}_+^*$ $SO(2, n)/SO(1, n-1) \times \mathbb{R}_+^*$ $E_{7(-25)}/E_{6(-26)} \times \mathbb{R}_+^*$

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