Pacific Journal of Mathematics

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Volume 206 No. 1 September 2002

PACIFIC JOURNAL OF MATHEMATICS
Vol. 206, No. 1, 2002

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Trudinger and Moser, interested in certain nonlinear problems in differential geometry, showed that if $|\nabla u|^q$ is integrable on a bounded domain in R^n with $q \geq n \geq 2$, then u is exponentially integrable there. Symmetrization reduces the problem to a one-dimensional inequality, which Jodeit extended to $q > 1$. Carleson and Chang proved that this inequality has extremals when $q \geq 2$ is an integer. Hence, so does the Moser-Trudinger inequality (with $q = n$).

This paper extends the result of Carleson and Chang to all real numbers $q > 1$. An application and some related results involving noninteger q are also discussed.

Introduction.

Let D be a bounded domain in $\mathbf{R}^n$, $n \geq 2$. Let $W^n(D)$ be the Sobolev space of functions u supported in the closure of D with gradient in $L^n(D)$. Trudinger [**10**] showed that for u in the unit ball of W^n, there are constants α and A (depending only on n) such that

$$(1) \qquad \int_D \exp(\alpha u^{\frac{n}{n-1}})\, dx \leq A|D|.$$

Moser [**9**] found the largest possible value of α by using symmetrization to reduce this to a one-dimensional problem. The integer n can then be replaced by a real number $q \geq 2$. Jodeit [**5**] extended the result to $1 < q < 2$.

Theorem A (Jodeit, Moser)**.** *Let $1 < q < \infty$, $1/p + 1/q = 1$. Let ω be a function in $C^1[0,\infty)$ such that $\omega(0) = 0$ and $\int |w\prime|^q \leq 1$. Then*

$$(2) \qquad A(q) = \sup_w \int_0^\infty \exp(\omega^p(t) - t)\, dt.$$

Carleson and Chang [**2**] proved that this theorem has extremals for integers $q \geq 2$. Through symmetrization, this proves that the Trudinger-Moser theorem for $W^n(D)$ has extremals, at least when D is a ball. Flucher [**3**] extended this to arbitrary smooth bounded domains in $\mathbf{R}^n$. Lin [**6**] did the same for $n \geq 3$. It is natural to ask whether Theorem A has extremals for general q. The main result of this paper is:

Theorem B. *Theorem* A *has extremals for all real numbers* $q > 1$.

The outline of the proof is similar to that in [**2**], especially when $q \geq 2$. In Section 1 we show that if no extremals exist, then $A(q)$ is less than an explicit constant $R(q)$. This requires new methods when $1 < q < 2$. One of the main ideas of [**2**] is linearization, in which the exponent p is replaced by 1, with controllable error for $1 < p \leq 2$. So, it is not surprising that their proof (their inequality (23), for example) breaks down when $1 < q < 2$.

Section 2 provides a specific ω to show $A(q) > R(q)$. This part requires a different construction than in [**2**], but for a different reason. There is less slack: It appears that $A(q) \to R(q)$ as $q \to 1^+$ (but we do not attempt a proof of this).

In a related paper, McLeod and Peletier [**8**] give a somewhat different proof of Theorems A and B for integer $q > 1$. It differs especially in the first part, in showing $A(q) \leq R(q)$. It then refers to the ω in [**2**].

It is not clear whether Theorem B has important applications to functions on $\mathbf{R}^n$ (with $q \neq n$). But there are several related results that show it is reasonable to look at noninteger q. For example, in Section 3, we use Theorem B to generalize the results in [**2**] to $u \in W^q(B^n)$, $1 < q \leq n$, with similar sharpness in α. When $q < n$ this involves a weight.

Also, Theorem A is used by the authors in [**4**] to prove an inequality like Moser's for functions in the Lorentz-Sobolev space $W^{n,q}(D)$. It isn't clear whether Theorem B gives extremals for this problem, due to problems with symmetrization.

Adams [**1**] has extended the Moser-Trudinger theorem to higher-order derivatives, based on a generalization of Theorem A by Garcia. A very interesting question is whether the inequality of Adams, or Garcia, has extremals. Our methods seem promising in showing that Garcia's inequality has them, for some range of q.

Section 1.

This section contains the proof of Proposition 1 below, and follows the strategy in [**2**]. Mainly, the range $1 < q < 2$ requires a new approach. We will generally avoid duplication of [**2**], except that our construction in the next section (unlike the one in [**2**]) is based on this work. So several equations from [**2**] are included here for later reference.

Let $R(q) = 1 + \exp\{\psi(q) + \gamma\}$, where $\psi(q)$ is the psi function $\Gamma'(q)/\Gamma(q)$ and γ is the Euler constant. If $q = n$ is an integer, then $R(q)$ is the Carleson-Chang constant $1 + \exp\{1 + 1/2 + \cdots + 1/(n-1)\}$. $A(q)$ is the constant of Theorem A.

Proposition 1. *If Theorem* A *has no extremal, then* $A(q) \leq R(q)$.

The following notation and results will be used in the proof. Let K_q be the space of continuous piecewise C^1 functions $\omega(t)$ on $[0,\infty)$ satisfying

$$\omega(0)=0, \omega'(t)\geq 0, \quad \text{and} \quad \int_0^\infty |\omega'|^q \leq 1.$$

Roughly, these are the functions of Theorem A.

Let ω_m be a sequence in K_q such that $\int_0^\infty \exp(\omega_m^p(t)-t)\,dt$ converges to $A(q)$ as $m\to\infty$. Assuming Theorem A has no extremal, the following conditions hold:

(a) For each $A>0$, $\int_0^A |\omega_m'(t)|^q\,dt \to 0$ as $m\to\infty$.
(b) For m large enough, there exists a point a_m in $[1,\infty)$ such that $(\omega_m(a_m))^p - a_m = -2\log^+(a_m)$. Moreover, if a_m denotes the first such point, then $a_m\to\infty$ as $m\to\infty$.
(c) $\limsup_{m\to\infty}\int_{a_m}^\infty \exp(\omega_m^p(t)-t)\,dt \leq \exp(\psi(q)+\gamma)$.
(d) $\lim_{m\to\infty}\int_0^{a_m}\exp(\omega_m^p(t)-t)\,dt = 1$.

Proposition 1 follows from (c) and (d). The proofs in [**2**] require only minor modifications except for part (c) for $1<q<2$, which begins with the following lemma.

Lemma 1.1. *For* $1<q<\infty$, $p=q/(q-1)$ *and* $\delta>0$, *let* $K_{\delta,q}$ *be the space of continuous piecewise* C^1 *functions on* $[0,\infty)$ *satisfying* $\phi(0)=0$, $\phi'(t)\geq 0$ *and* $\int(\phi')^q\leq\delta$. *Then for each* $c>0$,

$$\sup_{\phi\in K_{\delta,q}}\int_0^\infty \exp\{c\phi(t)-t\}\,dt < \exp\{(1/p)^{q-1}c^q\delta/q\}R(q). \tag{3}$$

While the proof resembles that in [**2**], we list the modifications required for noninteger q, and also some formulas needed later. Inequality (3) has an extremal ϕ such that,

$$c\phi'(t) = p(1+Be^{t/(q-1)})^{-1}, \tag{4}$$

where $B\geq 0$ is chosen so that

$$c^q\delta = \int_0^\infty (c\phi'(t))^q\,dt. \tag{5}$$

It is also shown $(B+1)/B$ is the numerical value of the supremum of (3). Let

$$\begin{aligned}\beta(t) &= [1+1/(Be^{t/(q-1)})]^{-1}, \quad \text{so that}\\ \phi(t) &= (q/c)[\log(1+1/B)+\log(\beta(t))].\end{aligned} \tag{6}$$

For further reference, define for $B \geq 0$,

$$\varepsilon(q,B) = \int_{B+1}^{\infty} (u-1)^{-1}[1/u - 1/u^q]\,du, \tag{7}$$

$$= \sum_{k=1} \frac{1}{k(B+1)^k} - \frac{1}{(k+q-1)(B+1)^{k+q-1}}.$$

After a change of variables, the right side of (5) is equal to

$$p^q(q-1)\int_B^{\infty} \frac{1}{(u-1)u^q}\,du = p^q(q-1)\sum_{k=1} \frac{1}{(k+q-1)(B+1)^{k+q-1}}, \tag{8}$$

$$= p^q(q-1)[\log(1+1/B) - \varepsilon(q,B)],$$

$$> p^q(q-1)[\log(1+1/B) - (\psi(q)+\gamma)],$$

where $\psi(q)+\gamma = \varepsilon(q,0) > \varepsilon(q,B)$ for all $B > 0$. Solving for $(B+1)/B$ in the above inequality establishes (3).

With the help of (7) and (8), the following extends a lemma of Carleson-Chang to noninteger $q \geq 2$, which is used to prove (c) in this case.

Lemma 1.2.a. *Let $\omega \in K_q$ and $\int_a^{\infty}(\omega')^q = \delta$. For $2 \leq q < \infty$ and $a > 0$, we have*

$$\int_a^{\infty} \exp(\omega^p(t)-t)\,dt \leq \frac{\exp(\omega^p(a)-a)}{1-\delta^{\frac{1}{q-1}}} \exp\left(\frac{C_1^q \beta_q}{p^{q-1}q}\right) R(q), \tag{9}$$

where $\beta_q = \delta/(1-\delta^{1/(q-1)})^{q-1}$ and $C_1 = p\omega^{p-1}(a)$.

Our proof of (c) for $1 < q < 2$ requires a similar lemma:

Lemma 1.2.b. *For $1 < q < 2$ and a large enough with $\omega^p(a) - a = -2\log(a)$, we have*

$$\int_a^{\infty} \exp(\omega^p(t)-t)\,dt \tag{10}$$

$$\leq \frac{\exp(\omega^p(a)-a)}{1-\delta^{\frac{1}{q-1}}} \exp\left(\frac{C_1^q(1+\alpha)^q \beta_q}{p^{q-1}q}\right) R(q) + 2\exp(-a),$$

where $\alpha = C_2(\log(a)/a)^{1/q}$, for some constant C_2 independent of a.

Proof of (10). For $a > 1$, set $x = t-a$, $\psi(x) = \omega(t) - \omega(a)$. Then,

$$(\omega(a)+\psi(x))^p = \omega^p(a) + p\omega^{p-1}[1 + f(\psi(x)/\omega(a))]\psi(x) + \psi^p(x), \tag{11}$$

where the function f comes from the binomial expansion of $(1+u)^q$. Note that f is an increasing function and f is $O(x)$ as $x \to 0$.

We have $\omega^p(a) - a = -2\log(a)$ and $\omega^p(a) \leq a(1-\delta)^{\frac{1}{q-1}}$ This shows,

$$\delta \leq 2(q-1)\frac{\log(a)}{a} + C\log^2(a)/a^2, \quad \left(C \leq (2-q)2^{2-q}\right) \tag{12}$$

$$\leq C_1 \log(a)/a.$$

Let E_1 be the set of x for which

$$\psi(x) \geq 4\omega(a)\left(\frac{C_1 \log(a)}{a}\right)^{1/q}.$$

Then on E_1, using Holder's inequality and (12),

$$\begin{aligned} \omega(t) &\leq \psi(x)\left[1 + \frac{1}{4}\left(\frac{a}{C_1 \log(a)}\right)^{1/q}\right], \\ &\leq \delta^{1/q} x^{1/p}\left[1 + \frac{1}{4}(C_1 \log(a))^{1/q}\right], \\ &\leq x^{1/p}\left[\left(\frac{C_1 \log(a)}{a}\right)^{1/q} + \frac{1}{4}\right]. \end{aligned}$$

We now require a to be large enough so that $C_1 \log(a)/a < 1/4^q$. The integral of $\exp\{\omega^p - t\}$ over E_1 is bounded by,

$$\int_a^\infty \exp\left(\frac{t-a}{2} - t\right) dt \leq 2e^{-a}. \tag{13}$$

Let $E_2 = \left\{x : \psi(x) \leq 4\omega(a)\left(\frac{C_1 \log(a)}{a}\right)^{1/q}\right\}$. Replacing $\omega^p(t)$ by the right side of (11), we need to estimate the following integral,

$$\int_{E_2} \exp(\omega^p(a) + p\omega^{p-1}(a)(1 + f(\psi(x)/\omega(a)))\psi(x) + \psi^p(x) - x - a)\, dx.$$

Using $\psi^p(x) \leq \delta^{1/(q-1)} x$, we set

$$y = (1 - \delta^{1/(q-1)})x, \quad c_1 = p\omega^{p-1}(a), \quad \phi(y) = \psi(x),$$

and

$$\alpha = C_2\left(\frac{\log(a)}{a}\right)^{1/q} \geq f\left(4\left(\frac{C_1 \log(a)}{a}\right)^{1/q}\right),$$

for some independent constant C_2. Observe that the previous integral is less than the following

$$\frac{\exp(\omega^p(a) - a)}{1 - \delta^{\frac{1}{q-1}}} \sup \int_0^\infty \exp(c_1(1+\alpha)\phi(y) - y)\, dy, \tag{14}$$

where the supremum is taken over all ϕ satisfying

$$\int_0^\infty (\phi'(y))^q\, dy \leq \beta_q.$$

We have the following inequality from (13) and (14).

$$\int_a^\infty \exp(\omega^p(t) - t)\, dt \leq \frac{\exp(\omega^p(a) - a)}{1 - \delta^{\frac{1}{q-1}}} \sup \int_0^\infty \exp(c\phi(y) - y)\, dy + 2e^{-a},$$

where $c = (1+\alpha)c_1$. We now apply Lemma 1.1 proving (10).

We now return to the proof of (c) for $1 < q < 2$. The conclusion of Lemma 1.2.b implies

$$\int_{a_m}^{\infty} \exp(\omega_m^p(t) - t)\, dt \le \frac{e^K \exp(\psi(q)+\gamma)}{1-\delta_m^{1/(q-1)}} + 2\exp(-a_m), \tag{15}$$

where

$$K = \omega_m^p(a_m) - a_m + \beta_q[p\omega^{p-1}(a_m)(1 + C_2 \log(a_m)/a_m)]^q/(p^{q-1}q).$$

All we need to show is that $\limsup K \le 0$ as $m \to \infty$. The above expression for K reduces to

$$K = \omega_m^p(a_m) - a_m + \frac{\delta_m(1+\alpha)^q(\omega_m(a_m))^p}{(q-1)(1-\delta_m^{1/(q-1)})^{q-1}}.$$

We have $(\omega_m(a_m))^p - a_m = -2\log(a_m)$. Applying a binomial expansion to the denominator with estimate (12) derives

$$K \le -2\log(a_m) + 2\log(a_m) + C(\log(a_m)/a_m)^{1+1/q}.$$

Observing $a_m \to \infty$ as $m \to \infty$ completes the proof of (c).

Section 2.

Here we prove that $A(q) > R(q)$ by studying specific examples ω_q. Combined with Proposition 1 from Section 1, this proves Theorem B. When $q \ge 2$, we can use the Carleson-Chang example, but not their proof which uses induction on $q = n$. When $1 < q < 2$, we will need a new type of example, motivated by Section 1, and some very precise estimates to show $A(q) > R(q)$. In passing, it seems likely that $A(q) - R(q) \to 0$ as $q \to 1$.

Case 1. Suppose $2 \le q < \infty$. Set

$$\omega_q(t) = \begin{cases} [(q-1)^{-1/q}/p]t, & 0 \le t \le q, \\ (t-1)^{1/p}, & q \le t \le N_q, \\ (N_q - 1)^{1/p}, & t \ge N_q, \end{cases}$$

where $N_q = (q-1)\exp(p^q - p) + 1$ is chosen so that $\int_0^\infty |\omega'(t)|^q\, dt = 1$.

One computes the exponential norm as the following:

$$\begin{aligned} I(q) &= \int_0^\infty \exp(\omega_m^p(t) - t)\, dt \\ &= q\int_0^1 e^{v(t)}\, dt + (2-q)/e + (q-1)\exp(p^q - p - 1), \end{aligned}$$

where $v(x) = (q-1)x^p - qx$. We prove the following lemma and thereby establish Theorem B.

Lemma 2.1. *$I(q) > R(q)$ for $q \ge 2$.*

Proof. We observe for $v(x) = (q-1)x^p - qx$ that $v(0) = 0$, $v(1) = -1$, $v'(x) = q(x^{p-1} - 1)$, $v'(0) = -q$, $v'(1) = 0$ and $v'' > 0$. Thus, $v(x) > -qx$ on $(0, 1/q]$ and $v(x) \geq -1$ on $[1/q, 1]$. We estimate

$$\int_0^1 \exp(v) > \int_0^{1/q} \exp(-qx) + \int_{1/q}^1 \exp(-1) = (1 + (q-2)/e)/q,$$

or

$$q \int_0^1 \exp(v) + (2-q)/e > 1.$$

To complete the proof, it is enough to show, for $q \geq 2$,

$$(q-1)\exp(p^q - p - 1) \geq R(q) - 1 = \exp(\psi(q) + \gamma).$$

Both sides are equal to e for $q = 2$. For $q > 2$, the problem reduces to showing

$$\psi'(q) \leq d/dq[p^q - p + \log(q-1)]. \tag{16}$$

We now estimate both sides of (16). Observe,

$$\psi'(q) = \sum_{k=0}^{\infty} \frac{1}{(k+q)^2} \leq \sum_{k=0}^{\infty} \frac{1}{(k+q-\frac{1}{2})(k+q+\frac{1}{2})} = \frac{1}{(q-\frac{1}{2})},$$

$$d/dq(p^q - p) = p^q(\log(p) - p + 1) + (p-1)^2.$$

To prove (16), we must show

$$\frac{1}{(q-\frac{1}{2})} \leq (p-1)^2 + (p-1) + p^q(\log(p) - p + 1),$$

which requires estimates of $\log(p)$ and p^q. Set $x = 1/(2q-1) \leq 1/3$ so that $p = (1+x)/(1-x)$. A Maclaurin series expansion in x shows that

$$\frac{1}{(q-\frac{1}{2})} + \frac{1}{12\left(q-\frac{1}{2}\right)^3} \leq \log(p) \leq \frac{1}{(q-\frac{1}{2})} + \frac{1}{10\left(q-\frac{1}{2}\right)^3}. \tag{17}$$

Therefore, from $d/dq[\log(p^q)] = \log(p) - 1/(q-1) \leq 0$, we have

$$\log(p^q) = \log(4) + \int_2^q \log\left(\frac{t}{(t-1)}\right) - \frac{1}{(t-1)}\, dt,$$

and using (17),

$$\log(p^q) \leq \log(4) + \log((2q-1)/3) + \varepsilon - \log(q-1),$$

where

$$\varepsilon = \int_2^q \frac{1}{(10(t-\frac{1}{2})^3)}\, dt = (1/20)\left[\frac{4}{9} - \frac{1}{\left(q-\frac{1}{2}\right)^2}\right] < 1/45.$$

So $p^q \le 8(q-1/2)\exp(\varepsilon)/[3(q-1)]$. Since $\log(p)-p+1 \le 0$, (16) has been reduced to proving

(18)
$$\frac{1}{(q-\frac{1}{2})} \le \frac{1}{(q-1)} + \left(\frac{1}{(q-1)}\right)^2 + \frac{8(q-\frac{1}{2})}{3(q-1)}\exp(\varepsilon)\left(\log(p) - \frac{1}{(q-1)}\right).$$

Set $\lambda = p-1 = 1/(q-1)$. Since

$$\log(p) - \lambda \ge \frac{1}{(q-\frac{1}{2})} - \frac{1}{(q-1)} = \frac{-1}{[2(q-\frac{1}{2})(q-1)]},$$

the right side of (18) is at least $\lambda + \lambda^2 - (4/3)\lambda^2\exp\{\varepsilon\}$. The left side equals $2\lambda/(\lambda+2)$, so (18) reduces to checking that

$$[(4/3)\exp\{\varepsilon\} - 1](\lambda+2) \le 1. \tag{19}$$

If $q \ge 3$, (19) holds because $\varepsilon \le 1/45$ and $\lambda \le 1/2$.

Now suppose that $2 < q \le 3$. Since $\varepsilon \le (q-2)/30$, the mean value theorem shows that $\exp\{\varepsilon\} \le 1+(q-2)/28$. We also have $\lambda+2 = (2q-1)/(q-1) = 3-(q-2)/(q-1)$, so the left side of (19) is at most

$$\begin{aligned}&[1/3 + (q-2)/21][3-(q-2)/(q-1)]\\ &\le 1 + (q-2)(1/7 - 1/[3(q-1)]) \le 1.\end{aligned}$$

This completes the proof of Lemma 2.1.

Case 2. Suppose $1 < q < 2$.

Attempts to use examples like those in Case 1 indicate that the number of pieces required to beat $R(q)$ is unbounded as q approaches 1. We will construct an example that is linear over $[0,a]$ and nonlinear over $[a,\infty)$. We shall show that for large enough a, the exponential norm exceeds $R(q)$. In fact, as $a \to \infty$, the exponential norm of our example converges downward to $R(q)$. This is sufficient to establish the conclusion of Proposition 1 as false thereby proving Theorem B. The idea of how to do this is based upon the method of proof of Proposition 1 for $1 < q < 2$.

To begin, let $a > 1$ and ω be linear on $[0,a]$ satisfying $\omega(0) = 0$ and $\omega^p(a) - a = -2\log(a)$. Define δ by the following:

$$1-\delta = \int_0^a |\omega'(t)|^q\,dt = (w^p(a)/a)^{q-1}. \tag{20}$$

We look to Lemma 1.1 for the definition of our example over $[a,\infty)$. Recall that there is an explicit formula for an extremal for the supremum of (3). We use this formula below. For $t > a$, we define $x = t-a$ and $\omega(t) = \psi(x)+\omega(a)$, where

(21)

$$\psi(x) = (q-1)A_1[\log(1+1/B) + \log(\beta(x))], \quad \text{and} \quad \int_0^\infty (\psi'(x))^q\, dx \le \delta.$$

For ease of notation we have set

$$\beta(x) = [1 + 1/(Be^{\frac{x}{q-1}})]^{-1}.$$

We shall specify the constants A_1 and B later.

The first estimate of the exponential integral is obvious.

$$\int_0^a \exp(w^p(t) - t)\, dt > 1 - e^{-a}. \tag{22}$$

The hard work is estimating the exponential integral over $[a, \infty)$.

The basic idea of [**2**] was to linearize the $(\omega(a) + \psi(x))^p$ and we do the same. However, the obvious inequality

$$(\omega(a) + \psi(x))^p \ge \omega^p(a) + p\omega^{p-1}(a)\psi(x) + \psi^p(x),$$

is too generous for our purposes. Therefore we expand as follows:

$$\omega^p(t) = (\omega(a) + \psi(x))^p = (\mu + (q-1)A_1 \log(\beta(x)))^p,$$

where $\mu = \omega(a) + (q-1)A_1 \log(1+1/B) = \omega(a) + \psi(\infty)$, to obtain

(23)

$$\begin{aligned}(\omega(a) + \psi(x))^p = \mu^p &+ p\mu^{p-1}(q-1)A_1 \log(\beta(x)) \\ &+ (1/2)p(p-1)((q-1)A_1)^2 \mu^{p-2} \log^2(\beta(x)) \\ &+ (1/6)p(p-1)(p-2)((q-1)A_1)^3 (\zeta)^{p-3} \log^3(\beta(x)),\end{aligned}$$

(where $\omega(a) \le \zeta \le \mu$),

$$\ge A_2 + A_3\psi(x) + A_4 \log^2(\beta(x)) + A_5 \log^3(\beta(x)),$$

where

$$\begin{aligned}&A_2 = (\omega(a) + (q-1)A_1 \log(1+1/B))^{p-1}(\omega(a) - qA_1 \log(1+1/B)), \\ &(\text{or} \ \ A_2 = \mu^{p-1}[\omega(a) - (p-1)\psi(\infty)]), \\ &A_3 = p(\omega(a) + (q-1)A_1 \log(1+1/B))^{p-1}, \\ &A_4 = (1/2)p(p-1)((q-1)A_1)^2(\omega(a) + (q-1)A_1 \log(1+1/B))^{p-2}, \\ &A_5 = (1/6)p(p-1)(p-2)((q-1)A_1)^3 W, \\ &\text{where } W = \begin{cases} (\omega(a))^{p-3}, & 2 < p < 3. \\ (\omega(a) + \psi(\infty))^{p-3}, & 3 \le p. \end{cases}\end{aligned} \tag{24}$$

We can now specify A_1 and B. Motivated by Equation (4), with the intention of having $c = A_3 = p/A_1$, we want A_1 and B to satisfy

$$A_3\psi'(x) = p\left(1 + Be^{x/(q-1)}\right)^{-1}, \quad \text{or equivalently,} \tag{25}$$

$$A_1 = (\omega(a) + (q-1)A_1 \log(1 + 1/B))^{1-p} = (\mu)^{1-p}.$$

And also, from (5), we want

$$\frac{\delta}{A_1^q} = (q-1)\int_{B+1}^{\infty} \frac{ds}{(s-1)s^q}. \tag{$*$}$$

It is not yet clear that there exist simultaneous solutions A_1 and B. To see this, let (25) define A_1 as a function of B. Define $L(B)$ as the left side and $R(B)$ as the right side of ($*$). As $B \to \infty$, $A_1 \to \omega(a)^{p-1}$ by (25), and $R(B) \to 0$. So, $L > R$ for large enough B. We compute,

$$dR/dB = -(q-1)/[B(B+1)^q] \quad \text{and} \quad dL/dB = -\delta(q-1)/[B(B+1)].$$

Estimating the integrals as $B \to 0^+$ shows $L < R$ for small enough B.

Note that (24) and (25) imply $A_1 = p/A_3$. Setting $c = A_3$ in (3) implies that the extremal $\phi(x)$ for Lemma 1.1 is the $\psi(x)$ defined by (21). Thus, ψ satisfies all the formulas in Lemma 1.1. We can now use (23) to proceed with the proof of Theorem B. The terms involving β below were neglected error terms in [**2**], but contribute to an important 'good' integral G defined below. The analysis is very tight.

We now have the following estimate:

(26)

$$\begin{aligned}
&\int_a^{\infty} \exp(\omega^p(t) - t)\, dt \\
&\geq \exp(A_2 - a)\int_0^{\infty} \exp(A_3\psi(x) + A_4\log^2(\beta(x)) + A_5\log^3(\beta(x)) - x)\, dx \\
&= \exp(A_2 - a)\int \exp(\nu(x) + \eta(x))\, dx \\
&= \exp(A_2 - a)\left[\int \exp(\nu(x))\, dx + G\right]
\end{aligned}$$

where $\nu(x) = A_3\psi(x) - x$, $\eta(x) = A_4\log^2(\beta(x)) + A_5\log^3(\beta(x))$ and

$$G = \int_0^{\infty} \exp(\nu(x) + \eta(x)) - \exp(\nu(x))\, dx.$$

We will be done if we show the right-hand side of (26) is larger than $R(q) - 1 + e^{-a} = \exp(\varepsilon(q, 0)) + e^{-a}$. We now expand the right side of (26) into quantities that we must estimate. Since we have chosen ψ to be an extremal, all the estimates of Lemma 1.1 will apply. In particular, we shall need the following identities to establish the estimates that have been organized into Lemmas 2.2 and 2.3.

(27)

(i) $a = \omega^p(a)(1-\delta)^{1-p}$, (by (20));

(ii) $(B+1)/B = \exp(\delta\mu^p/(q-1) + \varepsilon(q,B))$, (by (8) and (25));

(iii) $\delta\mu = \psi(\infty) - \dfrac{(q-1)\varepsilon(q,b)}{\mu^{p-1}}$ (rearranging (ii));

(iv) $\exp(A_2 - a)\displaystyle\int \exp(\nu(x)) = \exp(A_2 - a + \delta\mu^p/(q-1))\exp(\varepsilon(q,B))$;

((iv) is equivalent to (ii)).

Lemma 2.2.

(a) $A_2 - a + \delta\mu^p/(q-1) \geq -\varepsilon^2(q,B)/[\omega(a)\mu^{p-1}]$.

(b) $\varepsilon(q,0) - \varepsilon(q,B) \leq (q-1)B$.

(c) $\varepsilon(q,B) \leq (\pi^2/6 - 1/p)(q-1)$.

Proof of (a). We begin with expanding the left side of (a) and simplifying using the definitions of A_2,μ, $\omega(a)$ and $\psi(\infty)$, see (24).

$$
\begin{aligned}
&A_2 - a + \delta\mu^p/(q-1) \\
&= \mu^{p-1}(\omega(a) - (p-1)\psi(\infty)) - a + \frac{\delta\mu^p}{(q-1)}, \\
&= -a + \mu^{p-1}\left[\omega(a) - (p-1)\psi(\infty) + \frac{\delta\mu}{(q-1)}\right],
\end{aligned}
$$

using $-(p-1) + 1/(q-1) = 0$ and the right side of (27iii) for $\delta\mu$,

$$
\begin{aligned}
&= -a + \mu^{p-1}[\omega(a) - \varepsilon(q,B)\mu^{1-p}], \\
&= -a + \omega^p(a)[1 + \psi(\infty)/\omega(a)]^{p-1} - \varepsilon(q,B),
\end{aligned}
$$

using (27iii) to solve for $\psi(\infty)/\omega(a)$,

$$
= -a - \varepsilon(q,B) + \omega^p(a)\left[\frac{1}{1-\delta} + \frac{(q-1)\varepsilon(q,B)^{p-1}}{(1-\delta)\omega(a)\mu^{p-1}}\right]^{p-1}.
$$

For $p \geq 2$, $(x+y)^{p-1} \geq x^{p-1} + (p-1)x^{p-2}y$ and the above reduces to

(28)

$$
\geq -a - \varepsilon(q,B) + \omega^p(a)(1-\delta)^{1-p} + \varepsilon(q,B)[(1-\delta)(1+\psi(\infty)/\omega(a))]^{1-p}.
$$

Using (27i) and the following version of (27iii),

$$
(1-\delta)(1+\psi(\infty)/\omega(a)) = 1 + \frac{(q-1)\varepsilon(q,B)}{\omega(a)(\omega(a)+\psi(\infty))^{p-1}}.
$$

The right side of (28) is

$$= -\varepsilon(q,B) + \varepsilon(q,B)[1 + (q-1)\varepsilon(q,B)/(\omega(a)\mu^{p-1})]^{1-p},$$
$$\geq -\varepsilon(q,B) + \varepsilon(q,B)[1 - \varepsilon(q,B)/(\omega(a)\mu^{p-1})]$$

which completes the Proof of (a).

Proof of (b). By (7) we have

$$\begin{aligned}\varepsilon(q,0) - \varepsilon(q,B) &= \int_1^{B+1} \frac{1}{(s-1)}\left[\frac{1}{s} - \frac{1}{s^q}\right] ds \\ &\leq \int_1^{B+1} (q-1)/s\, ds \leq (q-1)B.\end{aligned}$$

Proof of (c). Using the series representation of $\varepsilon(q,0)$, see (7),

$$\begin{aligned}\varepsilon(q,B) \leq \varepsilon(q,0) &= \sum_{k=1}^{\infty}\left(\frac{1}{k} - \frac{1}{k+q-1}\right), \\ &= (q-1)\left(1 + \sum_{k=2}^{\infty}\frac{1}{k(k+q-1)} - \frac{1}{p},\right) \\ &\leq (q-1)\left(\sum_{k=1}^{\infty}\frac{1}{k^2} - \frac{1}{p}\right), \quad \text{which is (c).}\end{aligned}$$

Lemma 2.3. *For large enough a, $G \geq q(q-1)[1-3/a]/[B\mu^p]$.*

Proof. By integration by parts,

$$\int_0^\infty \exp(\nu(x) + \eta(x))\, dx = V(\infty) - \int_0^\infty V(x)\exp(\eta(x))\eta'(x)\, dx,$$

where

$$V(x) = \int_0^x \exp(\nu(t))\, dt,$$

and $\eta(x) = A_4 \log^2(\beta(x)) + A_5 \log^3(\beta(x))$, $\beta(x) = [1 + 1/(Be^{x/q-1})]^{-1}$.

We need to explicitly calculate $V(x)$. To begin, $\nu(t) = A_3\psi(t) - t$, where ψ is an extremal for (3). Therefore, a variational argument shows that ν satisfies,

$$e^{\nu(t)} = A\nu''(t)(\nu'(t)+1)^{q-2}, \quad \text{for some constant } A < 0. \tag{29}$$

Observe $\nu(0) = 0$, $\nu'(\infty) = -1$ and $\nu(\infty) = -\infty$. Multiply (29) by $\nu'(t)$ and integrate to obtain,

$$e^{\nu(t)} = \frac{A(\nu'(t)+1)^q}{q} - \frac{A(\nu'(t)+1)^{q-1}}{q-1} + C. \tag{30}$$

Let $t \to \infty$ to obtain $C = 0$. Equations (29) and (30) imply

$$\nu''(t) = \frac{(\nu'(t)+1)^2}{q} - \frac{(\nu'(t)+1)}{q-1}.$$

Solving this differential equation shows the following version of (4):

$$\nu'(t) + 1 = p(1 + Be^{t/(q-1)})^{-1}. \tag{31}$$

It can be shown, see [**2**], that $V(\infty) = (B+1)/B = J = 1/\beta(0)$. Using (30) and (31) we compute,

$$V(x) = -Ap^{q-1}/(q-1)[(B+1)^{1-q} - (1-\beta(x))^{q-1}].$$

Using $V(\infty) = (B+1)/B$ and the above with $\beta(\infty) = 1$, we have

$$V(x) = (B+1)^q/B[(B+1)^{1-q} - (1-\beta(x))^{q-1}]. \tag{32}$$

Notice $\beta \leq 1$, $\exp(\eta(x)) \geq 1$ and $\eta'(x) \leq 0$, thus the above gives,

$$G \geq 1/B \int_0^\infty [(B+1)^{1-q} - (1-\beta(x))^{q-1}]|\eta'(x)|\, dx,$$

and setting $w = \log(\beta)$,

$$= -1/B \int_{-\log(J)}^0 (2A_4 w - 3A_5 w^2)[(B+1)^{1-q} - (1-e^w)^{q-1}]\, dw.$$

Using a Maclaurin series representation for $(1-x)^{q-1}$ and $(B+1)^{-1} = 1 - 1/J$,

$$G \geq 1/B \int_{-\log(J)}^0 (2A_4 w - 3A_5 w^2)(q-1)[e^w - B/(B+1)]\, dw,$$

integrating by parts and using the definitions of A_4 and A_5 (see (24)) gives,

$$= [q(q-1)/(B\mu^p)][1 - (p-2)(q-1)/\mu^p + O(\log^3(J)/J)].$$

Notice that, $\mu^p \approx \omega^p(a) \approx a$, as a approaches ∞. By (27iii),

$$\mu \geq \omega(a)/(1-\delta) = (a/\omega(a))^{q-1}.$$

So $\mu^p \geq \mu^{p-1}\omega(a) \geq a$. By (20), $\delta \geq 2(q-1)\log(a)/a$. This and (27ii) give B is $O(1/a^2)$. Thus $\log^3(J)/J$ is $o(1/a)$, and so

$$G \geq q(q-1)[1 - 3/a]/(B\mu^p).$$

This completes the proof of Lemma 2.3. We return to the proof of Theorem B.

Using (22), (26), and (27iv),

$$\int_0^\infty \exp(\omega^p(x) - x)\, dx$$
$$> 1 - e^{-a} + \exp(A_2 - a + \delta\mu^p/(q-1))e^{\varepsilon(q,B)} + \exp(A_2 - a)G.$$

We shall show the right side is greater than or equal to $1 + e^{\varepsilon(q,0)}$. Using algebra and (27ii) this goal becomes,

$$\text{(33)}\quad \exp(A_2 - a + \delta\mu^p/(q-1))[1 + BG/(B+1)] \geq e^{-\varepsilon(q,B)}[e^{\varepsilon(q,0)} + e^{-a}].$$

By Lemma 2.2(a) and Lemma 2.3, the left side of the above is greater than or equal to

$$\exp(-\varepsilon^2(q,B)/(\omega(a)\mu^{p-1}))[1 + q(q-1)(1-3/a)/[\mu^p(1+B)]],$$

and,

$$\varepsilon^2(q,B) \leq (\pi^2/6 - 1/p)^2(q-1)^2, \quad \text{(by Lemma 2.2(c))}.$$

Since $\pi^2/6 - 1/p < p$ for $p > 2$ and $p(q-1) = q$, for large enough a the above is

$$\leq (q - \log(a)/a)(q-1).$$

We also claim $1/a - 1/\mu^p$ is $O(1/a^2)$. To see this, from (27i) and (27iii),

$$\mu^p \leq (a/\omega(a))^p(1 + (q-1)\varepsilon(q,B)/a)^p.$$

A binomial expansion shows $(1 + (q-1)\varepsilon(q,B)/a)^p$ is $1 + O(1/a)$. So,

$$\begin{aligned}\limsup_{a\to\infty}(\mu^p - a) &\leq \limsup_{a\to\infty}[(a/\omega(a))^p - a] + O(1)\\ &= O(1).\end{aligned}$$

Therefore, $1/a - 1/\mu^p \approx (\mu^p - a)/a^2$ is $O(1/a^2)$.

Recall that B is $O(1/a^2)$, so the factor $B+1$ is negligible and the left side of (33) is at least

$$\begin{aligned}&[1 - (q - \log(a)/a)(q-1)/a][1 + q(q-1)(1-3/a)/a] + O(1/a^2),\\ &\geq 1 + (q-1)\log(a)/a^2 + O(1/a^2),\\ &\geq 1 + (q-1)[B + (\log(a)-1)/a^2] + O(1/a^2).\end{aligned}$$

Lemma 2.2(b) implies the right side of (33) is at most $1+(q-1)B+O(1/a^4)$, completing the proof.

Section 3. An application of Theorem B.

For real valued functions f on R^n, let f^* be the nonincreasing rearrangement of f defined as $f^*(t) = \inf\{s : m\{|f| > s\} \leq t\}$. We define $f^\#(x)$ to be the spherically symmetric nondecreasing rearrangement of f defined as $f^\#(x) = f^*(\sigma_{n-1}|x|^n/n)$ where σ_{n-1} is the $n-1$ measure of the unit sphere.

We have the following theorem which includes the case $q = n$ which is the application of Carleson and Chang, [**2**].

Theorem C. *Let $1 < q \leq n$. For functions u supported in B^n such that $\|\nabla u\|_q \leq 1$,*

$$\int_{B^n} \exp(\alpha u^{\#p}(x)) m(|x|)\, dx \leq A(q)|B^n|,$$

where

$$m(r) = \frac{\exp\{-(r^{-kn} - 1)\}}{r^{-n(k+1)}}$$

and

$$k = \frac{(n-q)}{n(q-1)}, \alpha = n(\sigma_{n-1})^{1/(q-1)}.$$

If $q = n$, set $m(r) = 1$.

This is sharp in the sense that it does not hold for any larger α. There is an extremal for each $1 < q \leq n$. Also, $m(r)$ is continuous as a function of q.

By standard symmetrization, we can assume $u = u^{\#}$. Set $|x| = e^{-t/n}$, $v(t) = \alpha^{1/p} u^{\#}(x)$ and note $|v'(t)| = (\alpha^{1/p}|x|/n)|\nabla u^{\#}(x)|$, $dx = -|B^n|e^{-t}dt$. So,

$$\int_0^\infty |v'(t)|^q e^{t(q-n)/n}\, dt \leq 1.$$

For $1 < q < n$, set $t = \ln(ks+1)/k$, so $s = (e^{kt-1})/k$. Set $\omega(s) = v(t)$. Then,

$$\int_0^\infty |\omega'(s)|^q\, ds \leq 1.$$

By Theorem A, $\int \exp(\omega^p(s) - s)\, ds \leq A(q)$, and this has an extremal by Theorem B. Thus,

$$\int_0^\infty \exp(v^p(t) - s(t))\, ds(t) \leq A(q),$$

and this has an extremal. This is the conclusion of Theorem C.

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Received July 19, 1999.

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