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MODULE OVER AN ABELIAN NUMBER FIELD, II

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**ON A QUOTIENT OF THE UNRAMIFIED IWASAWA  
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Let  $p$  be an odd prime number,  $k$  an imaginary abelian field containing a primitive  $p$ -th root of unity, and  $k_\infty/k$  the cyclotomic  $\mathbf{Z}_p$ -extension. Denote by  $L/k_\infty$  the maximal unramified pro- $p$  abelian extension, and by  $L'$  the maximal intermediate field of  $L/k_\infty$  in which all prime divisors of  $k_\infty$  over  $p$  split completely. Let  $N/k_\infty$  (resp.  $N'/k_\infty$ ) be the pro- $p$  abelian extension generated by all  $p$ -power roots of all units (resp.  $p$ -units) of  $k_\infty$ . In the previous paper, we proved that the  $\mathbf{Z}_p$ -torsion subgroup of the odd part of the Galois group  $\text{Gal}(N \cap L/k_\infty)$  is isomorphic, over the group ring  $\mathbf{Z}_p[\text{Gal}(k/\mathbf{Q})]$ , to a certain standard subquotient of the even part of the ideal class group of  $k_\infty$ . In this paper, we prove that the same holds also for the Galois group  $\text{Gal}(N' \cap L'/k_\infty)$ .

**1. Introduction.**

Let  $p$  be a fixed odd prime number,  $k$  an imaginary abelian field containing a primitive  $p$ -th root  $\zeta_p$  of unity, and  $k_\infty/k$  the cyclotomic  $\mathbf{Z}_p$ -extension. Let  $L/k_\infty$  be the maximal unramified pro- $p$  abelian extension, and  $L'$  the maximal intermediate field of  $L/k_\infty$  in which all prime divisors of  $k_\infty$  over  $p$  split completely. We put

$$N = k_\infty(\epsilon^{1/p^n} \mid \epsilon \in E_\infty, n \geq 1), \quad N' = k_\infty(\epsilon^{1/p^n} \mid \epsilon \in E'_\infty, n \geq 1),$$

where  $E_\infty$  (resp.  $E'_\infty$ ) is the group of units (resp.  $p$ -units) of  $k_\infty$ . Put

$$\begin{aligned} \mathcal{X} &= \text{Gal}(L/k_\infty), & \mathcal{Y} &= \text{Gal}(N \cap L/k_\infty), \\ \mathcal{X}' &= \text{Gal}(L'/k_\infty), & \mathcal{Y}' &= \text{Gal}(N' \cap L'/k_\infty), \end{aligned}$$

and let  $\mathcal{X}^-$ ,  $\mathcal{Y}^-$ ,  $\mathcal{X}'^-$ ,  $\mathcal{Y}'^-$  be the odd parts of the respective Galois groups. It is well-known that  $\mathcal{X}^-$  is (finitely generated and) torsion free over  $\mathbf{Z}_p$  (cf. Washington [14, Corollary 13.29]). It is also known (and is shown similarly) that  $\mathcal{X}'^-$  is torsion free over  $\mathbf{Z}_p$ . One naturally asks whether or not the quotients  $\mathcal{Y}^-$  of  $\mathcal{X}^-$  and  $\mathcal{Y}'^-$  of  $\mathcal{X}'^-$  are also torsion free over  $\mathbf{Z}_p$ . This question arised in the previous investigation [5], [6] on a power integral basis problem over cyclotomic  $\mathbf{Z}_p$ -extensions.

Let  $A_\infty$  be the ideal class group of  $k_\infty$ , and  $A_\infty^+$  its even part. It is conjectured by Greenberg [4] that  $A_\infty^+ = \{0\}$ , which is far from being settled

in general. Under this conjecture, it is known that  $\mathcal{Y}^- = \mathcal{X}^-$  and  $\mathcal{Y}'^- = \mathcal{X}'^-$ , and hence  $\mathcal{Y}^-$  and  $\mathcal{Y}'^-$  are torsion free over  $\mathbf{Z}_p$ .

In the preceding paper [7], we proved that the  $\mathbf{Z}_p$ -torsion subgroup  $\text{Tor}\mathcal{Y}^-$  of  $\mathcal{Y}^-$  is isomorphic, over the group ring  $\mathbf{Z}_p[\text{Gal}(k/\mathbf{Q})]$ , to a certain standard subquotient of  $A_\infty^+$  (under the assumption that  $p$  does not divide the degree  $[k : \mathbf{Q}]$ ). Further, we gave some assertions on the vanishing of this subquotient.

Let  $\mathcal{O}_\infty$  be the ring of integers of  $k_\infty$ , and  $\mathcal{O}'_\infty = \mathcal{O}_\infty[1/p]$  the ring of  $p$ -integers. The pairs  $(L, N)$  and  $(L', N')$  are objects associated to  $\mathcal{O}_\infty$  and  $\mathcal{O}'_\infty$ , respectively. Since  $k_\infty/k$  is wildly ramified at  $p$ , it is often more natural to use the  $p$ -integers  $\mathcal{O}'_\infty$  than  $\mathcal{O}_\infty$ . Therefore, it is desirable to obtain a corresponding result for the pair  $(\mathcal{X}', \mathcal{Y}')$ . In this paper, we prove that the  $\mathbf{Z}_p$ -torsion subgroup  $\text{Tor}\mathcal{Y}'^-$  of  $\mathcal{Y}'^-$  is also isomorphic to the above mentioned subquotient of  $A_\infty^+$  as a  $\mathbf{Z}_p[\text{Gal}(k/\mathbf{Q})]$ -module. Namely,  $\text{Tor}\mathcal{Y}^-$  and  $\text{Tor}\mathcal{Y}'^-$  are isomorphic to each other over  $\mathbf{Z}_p[\text{Gal}(k/\mathbf{Q})]$ .

## 2. Results.

Let  $k$  be an imaginary abelian field with  $\zeta_p \in k^\times$ , and  $\Delta = \text{Gal}(k/\mathbf{Q})$ ,  $\Gamma = \text{Gal}(k_\infty/k)$ . We assume that

(H)  $p$  does not divide the degree  $[k : \mathbf{Q}]$ .

Then, we have a canonical decomposition

$$\text{Gal}(k_\infty/\mathbf{Q}) = \Delta \times \Gamma.$$

A  $\mathbf{Q}_p$ -valued character of  $\Delta$  defined and irreducible over  $\mathbf{Q}_p$  is simply called a  $\mathbf{Q}_p$ -character. For a  $\mathbf{Q}_p$ -character  $\Phi$  of  $\Delta$  and a  $\mathbf{Z}_p[\Delta]$ -module  $X$ , we denote by  $X^+$ ,  $X^-$  and  $X(\Phi)$  the even part, the odd part and the  $\Phi$ -component  $e_\Phi X$  of  $X$ , respectively. Here,  $e_\Phi$  is the idempotent of  $\mathbf{Q}_p[\Delta]$  defined by

$$e_\Phi = \frac{1}{|\Delta|} \sum_{\sigma \in \Delta} \Phi(\sigma) \sigma^{-1},$$

which is an element of  $\mathbf{Z}_p[\Delta]$  by the assumption (H).

Throughout this paper, we fix an *even*  $\mathbf{Q}_p$ -character  $\Psi$  of  $\Delta$  and its irreducible component  $\psi$  over the algebraic closure  $\overline{\mathbf{Q}_p}$ . Denote by  $\Psi^*$  and  $\psi^*$  the *odd* characters of  $\Delta$  associated to  $\Psi$  and  $\psi$  by

$$\Psi^*(\sigma) = \omega(\sigma)\Psi(\sigma^{-1}), \quad \psi^*(\sigma) = \omega(\sigma)\psi(\sigma^{-1}), \quad (\sigma \in \Delta),$$

respectively, where  $\omega$  is the character of  $\Delta$  representing the Galois action on  $\zeta_p$ . We often regard  $\psi$  and  $\psi^*$  as primitive Dirichlet characters.

Let  $k_n$  ( $n \geq 0$ ) be the  $n$ -th layer of  $k_\infty/k$  with  $k_0 = k$ , and  $A_n$  the Sylow  $p$ -subgroup of the ideal class group of  $k_n$ . Let

$$A_\infty = \varinjlim A_n$$

be the inductive limit with respect to the inclusion maps  $k_n \rightarrow k_m$  ( $n < m$ ). Denote by  $\tilde{A}_0$  the image of  $A_0$  in  $A_\infty$ . Let  $A_\infty^\Gamma$  be the elements of  $A_\infty$  fixed by the action of  $\Gamma = \text{Gal}(k_\infty/k)$ . It is known (cf. [4, Proposition 1]) that  $(A_\infty^\Gamma)^+$  is a finite abelian group as a consequence of the Leopoldt conjecture for  $(k, p)$  proved by Brumer [1]. Hence, so is  $(A_\infty^\Gamma/\tilde{A}_0)(\Psi)$ . On the other hand,  $\text{Tor}\mathcal{Y}(\Psi^*)$  and  $\text{Tor}\mathcal{Y}'(\Psi^*)$  are also finite since  $\mathcal{X}^-$  is finitely generated over  $\mathbf{Z}_p$  by the theorem of Ferrero and Washington [2]. For the trivial character  $\Psi_0$ , it is known (cf. [14, Proposition 6.16]) that  $A_\infty(\Psi_0) = \{0\}$  and  $\mathcal{X}(\Psi_0^*) = \{0\}$ . So, in what follows, we assume that  $\Psi$  is nontrivial (and even).

In [7], we proved the following:

**Theorem 1.** *The finite abelian groups  $\text{Tor}\mathcal{Y}(\Psi^*)$  and  $(A_\infty^\Gamma/\tilde{A}_0)(\Psi)$  are isomorphic to each other.*

As for the subquotient  $A_\infty^\Gamma/\tilde{A}_0$  of  $A_\infty$ , we proved in [7, Proposition 1] the following:

**Proposition 1.** *When  $\psi(p) \neq 1$ , we have  $(A_\infty^\Gamma/\tilde{A}_0)(\Psi) = \{0\}$ .*

For more on this subquotient, see [7, Proposition 3] and [8].

The main result of this paper is as follows.

**Theorem 2.**  *$\text{Tor}\mathcal{Y}'(\Psi^*)$  is isomorphic to  $(A_\infty^\Gamma/\tilde{A}_0)(\Psi)$  as an abelian group.*

We obtain the following corollary from Theorems 1 and 2.

**Corollary.** *The  $\mathbf{Z}_p[\Delta]$ -modules  $\text{Tor}\mathcal{Y}'^-$  and  $\text{Tor}\mathcal{Y}^-$  are isomorphic to each other.*

We put

$$\mathcal{H} = \text{Gal}(N/k_\infty) \quad \text{and} \quad \mathcal{H}' = \text{Gal}(N'/k_\infty).$$

It is known (cf. [6, Claim (page 97)]) that, by the restriction map,

$$(1) \quad \mathcal{H}'(\Psi^*) = \mathcal{H}(\Psi^*).$$

This is because the Leopoldt conjecture for  $(k_n, p)$  holds for all  $n \geq 0$  by [1]. It is also known (see Section 4.2 (Proof of Lemma 1)) that, by the restriction map,

$$(2) \quad \mathcal{X}(\Psi^*) = \mathcal{X}'(\Psi^*) \quad \text{when } \psi^*(p) \neq 1.$$

Therefore, when  $\psi^*(p) \neq 1$ , we have  $\mathcal{Y}'(\Psi^*) = \mathcal{Y}(\Psi^*)$ . By this and Proposition 1, we see that Theorem 2 follows immediately from Theorem 1 and the following:

**Theorem 3.** *When  $\psi^*(p) = 1$ ,  $\mathcal{Y}'(\Psi^*)$  is torsion free over  $\mathbf{Z}_p$ .*

**Remark 1.** Let  $A'_n$  be the Sylow  $p$ -subgroup of the  $p$ -ideal-class group of  $k_n$  in the sense of Iwasawa [10, Section 4.3], and let  $A'_\infty$  be the inductive limit of  $A'_n$  with respect to the inclusion maps  $k_n \rightarrow k_m$  ( $n < m$ ). Denote by  $\tilde{A}'_0$  the image of  $A'_0$  in  $A'_\infty$ . To talk about the Galois groups  $\mathcal{X}'$ ,  $\mathcal{Y}'$ , it is more natural to use  $A'_\infty$  than  $A_\infty$ . However, it is known (cf. [4, Corollary]) that the natural projections

$$A_\infty^+ \longrightarrow A_\infty'^+ \quad \text{and} \quad \tilde{A}_0^+ \longrightarrow \tilde{A}_0'^+$$

are isomorphisms as a consequence of the Leopoldt conjecture for  $(k_n, p)$  ( $n \geq 0$ ).

**Remark 2.** It is conjectured that  $A_\infty^+ = \{0\}$  (cf. [4]). We have many numerical examples of  $(k, p)$  with  $A_\infty^+ = \{0\}$ , but no counter examples (see Kraft and Schoof [11], Kurihara [12], Sumida and the author [9]). However, the conjecture is not yet proved to be true in general.

### 3. Proof of Theorem 3.

We recall a standard notation. Let  $O = O_\psi$  be the subring of  $\overline{\mathbf{Q}}_p$  generated by the values of  $\psi$  over  $\mathbf{Z}_p$ . We identify the subring  $e_{\Psi^*}\mathbf{Z}_p[\Delta]$  of  $\mathbf{Z}_p[\Delta]$  with  $O$  by sending  $e_{\Psi^*}\sigma$  to  $\psi^*(\sigma)$ , ( $\sigma \in \Delta$ ). Then, for a  $\mathbf{Z}_p[\Delta]$ -module  $X$ ,  $X(\Psi^*)$  is regarded as an  $O$ -module. We fix a topological generator  $\gamma$  of  $\Gamma$ . We identify, as usual, the completed group ring  $e_{\Psi^*}\mathbf{Z}_p[\Delta][[\Gamma]]$  with the power series ring  $\Lambda = O[[T]]$  by  $\gamma = 1 + T$  and the above identification. Thus, for a  $\mathbf{Z}_p[\Delta][[\Gamma]]$ -module  $X$  (such as several Galois groups over  $k_\infty$ ), we can regard  $X(\Psi^*)$  as a module over  $O$  or  $\Lambda$ . We denote by  $q$  the element of  $p\mathbf{Z}_p$  such that  $\zeta^\gamma = \zeta^{1+q}$  for all  $\zeta \in \mu_{p^\infty}$ .

Let  $M/k_\infty$  be the maximal pro- $p$  abelian extension unramified outside  $p$ . The fields  $N$ ,  $L$ ,  $N'$  and  $L'$  are intermediate fields of  $M/k_\infty$ . We put

$$\begin{aligned} \mathcal{G} &= \text{Gal}(M/k_\infty), & \mathcal{Z}' &= \text{Gal}(M/N') \\ \mathcal{I} &= \text{Gal}(M/L), & \mathcal{T}' &= \text{Gal}(M/L'). \end{aligned}$$

For a  $\mathbf{Q}_p$ -character  $\Phi$  of  $\Delta$ , denote by  $M(\Phi)$  the intermediate field of  $M/k_\infty$  corresponding to  $\bigoplus'_{\Phi'} \mathcal{G}(\Phi')$  by Galois theory where  $\Phi'$  runs over the  $\mathbf{Q}_p$ -characters of  $\Delta$  with  $\Phi' \neq \Phi$ . Then,  $\text{Gal}(M(\Phi)/k_\infty) = \mathcal{G}(\Phi)$ . We define  $N(\Phi)$ ,  $L(\Phi)$ , etc, in a similar way.

As we have mentioned in Section 2,  $\mathcal{H}'(\Psi^*) = \mathcal{H}(\Psi^*)$ . Therefore, by the assertion [6, Lemma 1] on  $\mathcal{H}(\Psi^*)$ , there exists an injective  $\Lambda$ -homomorphism

$$\iota : \mathcal{H}'(\Psi^*) \hookrightarrow \begin{cases} \Lambda, & \text{when } \psi(p) \neq 1, \\ \Lambda \oplus \Lambda/(T - q), & \text{when } \psi(p) = 1, \end{cases}$$

with a finite cokernel. This is the  $\Delta$ -decomposed version of [10, Theorem 15]. In the next section, we prove the following two lemmas.

**Lemma 1.** *There exists a  $\Lambda$ -isomorphism:*

$$\mathcal{I}'(\Psi^*) \cong \begin{cases} \Lambda, & \text{when } \psi(p) \neq 1, \\ \Lambda \oplus \Lambda/(T - q), & \text{when } \psi(p) = 1. \end{cases}$$

**Lemma 2.** *We have  $M(\Psi^*) = N'(\Psi^*)L'(\Psi^*)$ .*

*Proof of Theorem 3.* Assume that  $\psi^*(p) = 1$ . We put

$$\overline{\mathcal{I}'}(\Psi^*) = \mathcal{I}'(\Psi^*)\mathcal{Z}'(\Psi^*)/\mathcal{Z}'(\Psi^*).$$

Then, we have  $\overline{\mathcal{I}'}(\Psi^*) \subseteq \mathcal{H}'(\Psi^*)$ , and

$$\mathcal{Y}'(\Psi^*) \cong \mathcal{H}'(\Psi^*)/\overline{\mathcal{I}'}(\Psi^*).$$

As  $\psi^*(p) = 1$ , we see from Lemmas 1 and 2 that

$$\overline{\mathcal{I}'}(\Psi^*) \cong \mathcal{I}'(\Psi^*) \cong \Lambda.$$

Let  $\iota$  be an embedding of  $\mathcal{H}'(\Psi^*)$  into  $\Lambda$  with a finite cokernel. By the above, the image  $\iota(\overline{\mathcal{I}'}(\Psi^*))$  of  $\overline{\mathcal{I}'}(\Psi^*)$  equals a principal ideal  $(f)$  of  $\Lambda$  for some  $f \in \Lambda$ . Therefore, we obtain an injective  $\Lambda$ -homomorphism

$$\mathcal{Y}'(\Psi^*) \hookrightarrow \Lambda/(f)$$

with a finite cokernel. On the other hand,  $f$  is relatively prime to  $p$  by [2]. Hence,  $\mathcal{Y}'(\Psi^*)$  is torsion free over  $\mathbf{Z}_p$ .  $\square$

#### 4. Proof of lemmas.

**4.1. Preliminaries.** In this subsection, we give and recall some assertions on some groups of local universal norms of  $k_\infty/k$  and the Galois groups  $\mathcal{I} = \text{Gal}(M/L)$ ,  $\mathcal{I}' = \text{Gal}(M/L')$ . For a while, we fix a prime ideal  $\mathfrak{p}$  of  $k$  over  $p$ . We denote the unique prime ideal of  $k_n$  over  $\mathfrak{p}$  simply by  $\mathfrak{p}$ . Let  $k_{n,\mathfrak{p}}$  be the completion of  $k_n$  at  $\mathfrak{p}$ , and  $\mathcal{U}_{n,\mathfrak{p}}$  the group of principal units of  $k_{n,\mathfrak{p}}$ . Let

$$\mathcal{V}_{n,\mathfrak{p}} = \bigcap_{m \geq n} N_{m/n} \mathcal{U}_{m,\mathfrak{p}} \quad \text{and} \quad \mathcal{W}_{n,\mathfrak{p}} = \bigcap_{m \geq n} N_{m/n} ((k_{m,\mathfrak{p}}^\times)^{(p)})$$

be the groups of universal norms. Here,  $N_{m/n}$  denotes the norm map from  $k_m^\times$  to  $k_n^\times$ , and for an abelian group  $X$ ,  $X^{(p)}$  denotes the maximal pro- $p$  quotient. We put

$$\mathcal{U}_n = \prod_{\mathfrak{p}|p} \mathcal{U}_{n,\mathfrak{p}}, \quad \mathcal{V}_n = \prod_{\mathfrak{p}|p} \mathcal{V}_{n,\mathfrak{p}}, \quad \mathcal{W}_n = \prod_{\mathfrak{p}|p} \mathcal{W}_{n,\mathfrak{p}},$$

where  $\mathfrak{p}$  runs over the primes of  $k$  over  $p$ . These are closed subgroups of the maximal pro- $p$  quotient  $\widehat{k_n^\times} = (\prod_{\mathfrak{p}|p} k_{n,\mathfrak{p}}^\times)^{(p)}$ . Denote by  $\varphi_n$  the natural embedding of  $k_n^\times$  into  $\widehat{k_n^\times}$ . Let  $E_n$  (resp.  $E'_n$ ) be the group of units (resp.  $p$ -units) of  $k_n$ , and let  $\mathcal{E}_n$  (resp.  $\mathcal{E}'_n$ ) be the closure of  $\varphi_n(E_n)$  (resp.  $\varphi_n(E'_n)$ )

in  $\widehat{k_n^\times}$ . Let  $\mathcal{U}_\infty, \mathcal{E}_\infty, \mathcal{W}_\infty, \mathcal{E}'_\infty$  be the projective limits of  $\mathcal{U}_n, \mathcal{E}_n, \mathcal{W}_n, \mathcal{E}'_n$  with respect to the relative norms, respectively:

$$\mathcal{U}_\infty = \varprojlim \mathcal{U}_n (= \varprojlim \mathcal{V}_n), \quad \mathcal{E}'_\infty = \varprojlim \mathcal{E}'_n (= \varprojlim (\mathcal{W}_n \cap \mathcal{E}'_n)), \quad \text{etc.}$$

These groups are naturally regarded as modules over  $\mathbf{Z}_p[\Delta][[\Gamma]]$ .

**Lemma 3.** *The projection  $P : \mathcal{W}_\infty \rightarrow \mathcal{W}_0$  induces an isomorphism*

$$\mathcal{W}_\infty / \mathcal{W}_\infty^T \cong \mathcal{W}_0.$$

*Proof.* It is clear that the projection  $P$  is surjective and that  $\mathcal{W}_\infty^T \subseteq \ker P$ . So, it suffices to show that  $\ker P \subseteq \mathcal{W}_\infty^T$ . Let  $u = (u_n)_{n \geq 0}$  be an element of  $\ker P$  with  $u_n \in \mathcal{W}_n$ . As  $u_0 = 1$ , we see that  $u_n$  is contained in  $\mathcal{U}_n$ . We can write  $u_n = w_n^T$  for some  $w_n \in \prod_{\mathfrak{p}|p} k_{n,\mathfrak{p}}^\times$  by Hilbert Satz 90. Hence,  $u_n = \overline{w}_n^T$ ,  $\overline{w}_n$  being the projection of  $w_n$  in  $\widehat{k_n^\times}$ . Denote by  $x^{(n)}$  the element of the product  $X = \prod_\ell \widehat{k_\ell^\times}$  whose  $\ell$ -th component is  $N_{n/\ell}(\overline{w}_n)$  (resp. 1) for  $\ell \leq n$  (resp.  $\ell > n$ ). Since  $X$  is compact,  $\{x^{(n)}\}$  has an accumulation point  $x$  in  $X$ . We easily see that  $x \in \mathcal{W}_\infty$  and  $x^T = u$ . Therefore,  $\ker P \subseteq \mathcal{W}_\infty^T$ .  $\square$

By class field theory, it is known (cf. [14, Corollary 13.6]) that the inertia group  $\mathcal{I}$  is canonically isomorphic to  $\mathcal{U}_\infty / \mathcal{E}_\infty$  over  $\mathbf{Z}_p[\Delta][[\Gamma]]$ . As  $\Psi^*$  is odd and  $\Psi^* \neq \omega$ , it follows that  $\mathcal{E}_\infty(\Psi^*) = \{0\}$  by a theorem on units of CM-fields (cf. [14, Theorem 4.12]). Therefore, we obtain a  $\Lambda$ -isomorphism

$$(3) \quad \mathcal{I}(\Psi^*) \cong \mathcal{U}_\infty(\Psi^*).$$

On the  $\Lambda$ -structure of  $\mathcal{U}_\infty$ , it is known (cf. Gillard [3, Proposition 1]) that

$$(4) \quad \mathcal{U}_\infty(\Psi^*) \cong \begin{cases} \Lambda, & \text{when } \psi(p) \neq 1, \\ \Lambda \oplus \Lambda / (T - q), & \text{when } \psi(p) = 1. \end{cases}$$

It is also known (cf. [3, Proposition 2]) that

$$(5) \quad \mathcal{V}_0(\Psi^*) \cong \begin{cases} O, & \text{when } \psi(p) \neq 1 \text{ and } \psi^*(p) \neq 1, \\ O \oplus O/q, & \text{when } \psi(p) = 1, \\ \{0\}, & \text{when } \psi^*(p) = 1. \end{cases}$$

As for the decomposition group  $\mathcal{I}'$ , we need to prove the following:

**Proposition 2.** *The reciprocity law map induces a canonical isomorphism*

$$\mathcal{I}' \cong \mathcal{W}_\infty / \mathcal{E}'_\infty$$

over  $\mathbf{Z}_p[\Delta][[\Gamma]]$ .

*Proof.* Let  $M_n$  (resp.  $L'_n$ ) be the maximal abelian extension of  $k_n$  contained in  $M$  (resp.  $L'$ ). It suffices to prove that

$$(6) \quad \text{Gal}(M_n / L'_n) \cong \mathcal{W}_n / (\mathcal{W}_n \cap \mathcal{E}'_n)$$

since  $\mathcal{I}'$  is the projective limit of  $\text{Gal}(M_n/L'_n)$  with respect to the restriction maps. It suffices to show the assertion (6) only when  $n = 0$  by considering  $k_n$  as the base field.

For an integer  $m (\geq 0)$ , we put

$$W^{(m)} = \prod_{\mathfrak{p}|p} N_{m/0} k_{m,\mathfrak{p}}^\times (\supseteq \mathcal{U}_0^{p^m}).$$

For a prime divisor  $\mathfrak{q}$  of  $k$  relatively prime to  $p$ , let  $U_{\mathfrak{q}}$  be the group of local units (resp. the multiplicative group) of the completion  $k_{\mathfrak{q}}$  of  $k$  at  $\mathfrak{q}$  when  $\mathfrak{q}$  is finite (resp. infinite). Let  $J_k$  be the group of idèles of  $k$ . We define its subgroups  $A, B, C$  as follows:

$$A = W^{(m)} \times \prod_{\mathfrak{q} \nmid p} \{1\}, \quad B = \mathcal{U}_0^{p^m} \times \prod_{\mathfrak{q} \nmid p} \{1\}, \quad C = \prod_{\mathfrak{p}|p} \{1\} \times \prod_{\mathfrak{q} \nmid p} U_{\mathfrak{q}},$$

where  $\mathfrak{p}$  (resp.  $\mathfrak{q}$ ) runs over the primes of  $k$  dividing  $p$  (resp. relatively prime to  $p$ ).

Denote by  $H$  the Hilbert  $p$ -class field of  $k$ . Let  $M_{0,m}$  be the maximal intermediate field of  $M_0/H$  whose Galois group over  $H$  is of exponent  $p^m$ . Clearly,  $M_{0,m}$  contains  $k_m$ . Let  $L'_{0,m}$  be the maximal intermediate field of  $M_{0,m}/k_m$  in which all prime divisors of  $k_m$  over  $p$  split completely. We have a natural isomorphism

$$(7) \quad \text{Gal}(M_0/L'_0) \cong \varprojlim \text{Gal}(M_{0,m}/L'_{0,m}),$$

the projective limit being taken with respect to the restriction maps.

It is known that the reciprocity law map induces isomorphisms

$$\text{Gal}(M_{0,m}/k) \cong (J_k/k^\times BC)^{(p)} \quad \text{and} \quad \text{Gal}(L'_{0,m}/k) \cong (J_k/k^\times AC)^{(p)}.$$

For this, see Sumida [13, pp. 692-693]. Therefore, we obtain a canonical isomorphism

$$\text{Gal}(M_{0,m}/L'_{0,m}) \cong (k^\times AC/k^\times BC)^{(p)} \cong (A/(A \cap (k^\times BC)))^{(p)}.$$

We easily see that

$$A \cap (k^\times BC) = (W^{(m)} \cap (E'_0 \mathcal{U}_0^{p^m})) \times \prod_{\mathfrak{q} \nmid p} \{1\}.$$

Here, we are regarding  $E'_0$  as a subgroup of  $\prod_{\mathfrak{p}|p} k_{0,\mathfrak{p}}^\times$  in the natural way. Hence, we have

$$\text{Gal}(M_{0,m}/L'_{0,m}) \cong (W^{(m)}/(W^{(m)} \cap (E'_0 \mathcal{U}_0^{p^m})))^{(p)}.$$

From this and (7), we obtain

$$\text{Gal}(M_0/L'_0) \cong \mathcal{W}_0/(\mathcal{W}_0 \cap \mathcal{E}'_0)$$

by an elementary but tedious argument on the topology of  $\widehat{k_0^\times} = (\prod_{\mathfrak{p}|p} k_{0,\mathfrak{p}}^\times)^{(p)}$ , which we leave to the reader. □

#### 4.2. Proof of Lemmas 1 and 2.

*Proof of Lemma 1 (and the formula (2)).* Let  $B_n$  be the subgroup of  $A_n$  consisting of classes which contain a product of prime ideals of  $k_n$  over  $p$ , and let  $B_\infty$  be the projective limit of  $B_n$  with respect to the relative norms.

From class field theory, we see that  $\mathcal{I}'/\mathcal{I}$  is canonically isomorphic to  $B_\infty$ . Let  $D (\subseteq \Delta)$  be the decomposition group of  $p$  at  $k$ . Then, we have a natural surjection

$$\mathbf{Z}_p[\Delta/D] \longrightarrow B_\infty \cong \mathcal{I}'/\mathcal{I}$$

over  $\mathbf{Z}_p[\Delta]$ . We see that  $\mathbf{Z}_p[\Delta/D](\Psi^*) = \{0\}$  or  $O$  according as  $\psi^*(p) \neq 1$  or  $\psi^*(p) = 1$ . Let  $\psi^*(p) \neq 1$ . Then, from the above surjection, we see that  $\mathcal{I}'(\Psi^*) = \mathcal{I}(\Psi^*)$  (from which (2) follows). Hence, the assertion of Lemma 1 follows from (3) and (4) in this case.

Let  $\psi^*(p) = 1$ . We have the following exact sequence of  $\mathbf{Z}_p[\Delta]$ -modules.

$$\{0\} \longrightarrow \mathcal{U}_0 \longrightarrow \left( \prod_{\mathfrak{p}|p} k_{0,\mathfrak{p}}^\times \right)^{(p)} \longrightarrow \mathbf{Z}_p[\Delta/D] \longrightarrow \{0\}.$$

As  $\psi^*(p) = 1$ , we see from (5) that

$$(\mathcal{W}_0 \cap \mathcal{U}_0)(\Psi^*) = \mathcal{V}_0(\Psi^*) = \{0\}.$$

Therefore, by the above exact sequence, we see that the  $O$ -module  $\mathcal{W}_0(\Psi^*)$  is free of rank one (or  $\mathcal{W}_0(\Psi^*) = \{0\}$ ). Hence,  $\mathcal{W}_\infty(\Psi^*)$  is cyclic over  $\Lambda$  by Lemma 3 and Nakayama's lemma (cf. [14, Lemma 13.16]). By this and Proposition 2,  $\mathcal{I}'(\Psi^*)$  is cyclic over  $\Lambda$ . Then, we obtain  $\mathcal{I}'(\Psi^*) \cong \Lambda$  since  $\mathcal{I} \subseteq \mathcal{I}'$  and  $\mathcal{I}(\Psi^*) \cong \Lambda$  by (3) and (4).  $\square$

*Proof of Lemma 2.* It is known (cf. [6, Proposition 3]) that

$$M(\Psi^*) = N(\Psi^*)L(\Psi^*).$$

Let  $\psi^*(p) \neq 1$ . Then,  $N'(\Psi^*) = N(\Psi^*)$  and  $L'(\Psi^*) = L(\Psi^*)$  by (1) and (2). Hence, the assertion follows from the above in this case. Let  $\psi^*(p) = 1$ . Then, by Lemma 1,  $\mathcal{I}'(\Psi^*) \cong \Lambda$ . On the other hand,  $\mathcal{Z}'(\Psi^*)$  is finitely generated and torsion over  $\Lambda$  by [10, Theorems 5, 14]. Therefore, we obtain  $\mathcal{Z}'(\Psi^*) \cap \mathcal{I}'(\Psi^*) = \{0\}$ , and hence  $M(\Psi^*) = N'(\Psi^*)L'(\Psi^*)$ .  $\square$

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