ON A QUOTIENT OF THE UNRAMIFIED IWASAWA MODULE OVER AN ABELIAN NUMBER FIELD, II

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Let $p$ be an odd prime number, $k$ an imaginary abelian field containing a primitive $p$-th root of unity, and $k_\infty/k$ the cyclotomic $\mathbb{Z}_p$-extension. Denote by $L/k_\infty$ the maximal unramified pro-$p$ abelian extension, and by $L'$ the maximal intermediate field of $L/k_\infty$ in which all prime divisors of $k_\infty$ over $p$ split completely. Let $N/k_\infty$ (resp. $N'/k_\infty$) be the pro-$p$ abelian extension generated by all $p$-power roots of all units (resp. $p$-units) of $k_\infty$. In the previous paper, we proved that the $\mathbb{Z}_p$-torsion subgroup of the odd part of the Galois group $\text{Gal}(N \cap L/k_\infty)$ is isomorphic, over the group ring $\mathbb{Z}_p[\text{Gal}(k/\mathbb{Q})]$, to a certain standard subquotient of the even part of the ideal class group of $k_\infty$. In this paper, we prove that the same holds also for the Galois group $\text{Gal}(N' \cap L'/k_\infty)$.

1. Introduction.

Let $p$ be a fixed odd prime number, $k$ an imaginary abelian field containing a primitive $p$-th root of unity, and $k_\infty/k$ the cyclotomic $\mathbb{Z}_p$-extension. Let $L/k_\infty$ be the maximal unramified pro-$p$ abelian extension, and $L'$ the maximal intermediate field of $L/k_\infty$ in which all prime divisors of $k_\infty$ over $p$ split completely. We put

$$N = k_\infty(\epsilon^{1/p^n} \mid \epsilon \in E_\infty, \ n \geq 1), \quad N' = k_\infty(\epsilon^{1/p^n} \mid \epsilon \in E'_\infty, \ n \geq 1),$$

where $E_\infty$ (resp. $E'_\infty$) is the group of units (resp. $p$-units) of $k_\infty$. Put

$$\mathcal{X} = \text{Gal}(L/k_\infty), \quad \mathcal{X}' = \text{Gal}(L'/k_\infty),$$

and let $\mathcal{X}^-, \mathcal{Y}^-, \mathcal{X}'^-, \mathcal{Y}'^-$ be the odd parts of the respective Galois groups. It is well-known that $\mathcal{X}^-$ is (finitely generated and) torsion free over $\mathbb{Z}_p$ (cf. Washington [14, Corollary 13.29]). It is also known (and is shown similarly) that $\mathcal{X}'^-$ is torsion free over $\mathbb{Z}_p$. One naturally asks whether or not the quotients $\mathcal{Y}^-$ of $\mathcal{X}^-$ and $\mathcal{Y}'^-$ of $\mathcal{X}'^-$ are also torsion free over $\mathbb{Z}_p$.

This question arised in the previous investigation [5], [6] on a power integral basis problem over cyclotomic $\mathbb{Z}_p$-extensions.

Let $A_\infty$ be the ideal class group of $k_\infty$, and $A^+_\infty$ its even part. It is conjectured by Greenberg [4] that $A^+_\infty = \{0\}$, which is far from being settled.
in general. Under this conjecture, it is known that $Y^- = X^-$ and $Y'^- = X'^-$, and hence $Y^-$ and $Y'^-$ are torsion free over $\mathbb{Z}_p$.

In the preceding paper [7], we proved that the $\mathbb{Z}_p$-torsion subgroup $\text{Tor} Y^-$ of $Y^-$ is isomorphic, over the group ring $\mathbb{Z}_p[\text{Gal}(k/\mathbb{Q})]$, to a certain standard subquotient of $A^+_\infty$ (under the assumption that $p$ does not divide the degree $[k : \mathbb{Q}]$). Further, we gave some assertions on the vanishing of this subquotient.

Let $\mathcal{O}_\infty$ be the ring of integers of $k_\infty$, and $\mathcal{O}'_\infty = \mathcal{O}_\infty[1/p]$ the ring of $p$-integers. The pairs $(L, N)$ and $(L', N')$ are objects associated to $\mathcal{O}_\infty$ and $\mathcal{O}'_\infty$, respectively. Since $k_\infty/k$ is wildly ramified at $p$, it is often more natural to use the $p$-integers $\mathcal{O}'_\infty$ than $\mathcal{O}_\infty$. Therefore, it is desirable to obtain a corresponding result for the pair $(X', Y')$. In this paper, we prove that the $\mathbb{Z}_p$-torsion subgroup $\text{Tor} Y'^-$ of $Y'^-$ is also isomorphic to the above mentioned subquotient of $A^+_\infty$ as a $\mathbb{Z}_p[\text{Gal}(k/\mathbb{Q})]$-module. Namely, $\text{Tor} Y^-$ and $\text{Tor} Y'^-$ are isomorphic to each other over $\mathbb{Z}_p[\text{Gal}(k/\mathbb{Q})]$.

2. Results.

Let $k$ be an imaginary abelian field with $\zeta_p \in k^\times$, and $\Delta = \text{Gal}(k/\mathbb{Q})$, $\Gamma = \text{Gal}(k_\infty/k)$. We assume that

(H) \hspace{1cm} p \text{ does not divide the degree } [k : \mathbb{Q}].

Then, we have a canonical decomposition

$$\text{Gal}(k_\infty/\mathbb{Q}) = \Delta \times \Gamma.$$ 

A $\mathbb{Q}_p$-valued character of $\Delta$ defined and irreducible over $\mathbb{Q}_p$ is simply called a $\mathbb{Q}_p$-character. For a $\mathbb{Q}_p$-character $\Phi$ of $\Delta$ and a $\mathbb{Z}_p[\Delta]$-module $X$, we denote by $X^+$, $X^-$ and $X(\Phi)$ the even part, the odd part and the $\Phi$-component $e_\Phi X$ of $X$, respectively. Here, $e_\Phi$ is the idempotent of $\mathbb{Q}_p[\Delta]$ defined by

$$e_\Phi = \frac{1}{|\Delta|} \sum_{\sigma \in \Delta} \Phi(\sigma)\sigma^{-1},$$

which is an element of $\mathbb{Z}_p[\Delta]$ by the assumption (H).

Throughout this paper, we fix an even $\mathbb{Q}_p$-character $\Psi$ of $\Delta$ and its irreducible component $\psi$ over the algebraic closure $\overline{\mathbb{Q}}_p$. Denote by $\Psi^*$ and $\psi^*$ the odd characters of $\Delta$ associated to $\Psi$ and $\psi$ by

$$\Psi^*(\sigma) = \omega(\sigma)\Psi(\sigma^{-1}), \hspace{1cm} \psi^*(\sigma) = \omega(\sigma)\psi(\sigma^{-1}), \hspace{1cm} (\sigma \in \Delta),$$

respectively, where $\omega$ is the character of $\Delta$ representing the Galois action on $\zeta_p$. We often regard $\psi$ and $\psi^*$ as primitive Dirichlet characters.

Let $k_n$ ($n \geq 0$) be the $n$-th layer of $k_\infty/k$ with $k_0 = k$, and $A_n$ the Sylow $p$-subgroup of the ideal class group of $k_n$. Let

$$A_\infty = \lim_{\longrightarrow} A_n.$$
be the inductive limit with respect to the inclusion maps \( k_n \to k_m \) (\( n < m \)).

Denote by \( \widetilde{A}_0 \) the image of \( A_0 \) in \( A_\infty \). Let \( A_\infty^\Gamma \) be the elements of \( A_\infty \) fixed by the action of \( \Gamma = \text{Gal}(k_\infty/k) \). It is known (cf. [4, Proposition 1]) that \( (A_\infty^\Gamma)^+ \) is a finite abelian group as a consequence of the Leopoldt conjecture for \((k,p)\) proved by Brumer [1]. Hence, so is \( (A_\infty^\Gamma/\widetilde{A}_0)(\Psi) \). On the other hand, Tor\( Y(\Psi^*) \) and Tor\( Y'(\Psi^*) \) are also finite since \( X^- \) is finitely generated over \( \mathbb{Z}_p \) by the theorem of Ferrero and Washington [2].

For the trivial character \( \Psi_0 \), it is known (cf. [14, Proposition 6.16]) that \( A_\infty(\Psi_0) = \{0\} \) and \( X(\Psi_0^*) = \{0\} \). So, in what follows, we assume that \( \Psi \) is nontrivial (and even).

In [7], we proved the following:

**Theorem 1.** The finite abelian groups Tor\( Y(\Psi^*) \) and \( (A_\infty^\Gamma/\widetilde{A}_0)(\Psi) \) are isomorphic to each other.

As for the subquotient \( A_\infty^\Gamma/\widetilde{A}_0 \) of \( A_\infty \), we proved in [7, Proposition 1] the following:

**Proposition 1.** When \( \psi(p) \neq 1 \), we have \( (A_\infty^\Gamma/\widetilde{A}_0)(\Psi) = \{0\} \).

For more on this subquotient, see [7, Proposition 3] and [8].

The main result of this paper is as follows.

**Theorem 2.** Tor\( Y'(\Psi^*) \) is isomorphic to \( (A_\infty^\Gamma/\widetilde{A}_0)(\Psi) \) as an abelian group.

We obtain the following corollary from Theorems 1 and 2.

**Corollary.** The \( \mathbb{Z}_p[\Delta] \)-modules Tor\( Y'^- \) and Tor\( Y^- \) are isomorphic to each other.

We put

\[ \mathcal{H} = \text{Gal}(N/k_\infty) \quad \text{and} \quad \mathcal{H}' = \text{Gal}(N'/k_\infty). \]

It is known (cf. [6, Claim (page 97)]) that, by the restriction map,

\[ \mathcal{H}'(\Psi^*) = \mathcal{H}(\Psi^*). \tag{1} \]

This is because the Leopoldt conjecture for \((k_n,p)\) holds for all \( n \geq 0 \) by [1]. It is also known (see Section 4.2 (Proof of Lemma 1)) that, by the restriction map,

\[ \mathcal{X}(\Psi^*) = \mathcal{X}'(\Psi^*) \quad \text{when} \quad \psi^*(p) \neq 1. \tag{2} \]

Therefore, when \( \psi^*(p) \neq 1 \), we have \( \mathcal{Y}'(\Psi^*) = \mathcal{Y}(\Psi^*) \). By this and Proposition 1, we see that Theorem 2 follows immediately from Theorem 1 and the following:

**Theorem 3.** When \( \psi^*(p) = 1 \), \( \mathcal{Y}'(\Psi^*) \) is torsion free over \( \mathbb{Z}_p \).
Remark 1. Let $A'_n$ be the Sylow $p$-subgroup of the $p$-ideal-class group of $k_n$ in the sense of Iwasawa [10, Section 4.3], and let $A'_\infty$ be the inductive limit of $A'_n$ with respect to the inclusion maps $k_n \rightarrow k_m$ ($n < m$). Denote by $\tilde{A}'_0$ the image of $A'_0$ in $A'_\infty$. To talk about the Galois groups $X', Y'$, it is more natural to use $A'_\infty$ than $A_\infty$. However, it is known (cf. [4, Corollary]) that the natural projections $A'^+\infty \rightarrow A'^+\infty$ and $\tilde{A}'_0 \rightarrow \tilde{A}'_0$ are isomorphisms as a consequence of the Leopoldt conjecture for $(k_n, p)$ ($n \geq 0$).

Remark 2. It is conjectured that $A'^+\infty = \{0\}$ (cf. [4]). We have many numerical examples of $(k, p)$ with $A'^+\infty = \{0\}$, but no counter examples (see Kraft and Schoof [11], Kurihara [12], Sumida and the author [9]). However, the conjecture is not yet proved to be true in general.

3. Proof of Theorem 3.

We recall a standard notation. Let $O = O_\psi$ be the subring of $\overline{\mathbb{Q}}_p$ generated by the values of $\psi$ over $\mathbb{Z}_p$. We identify the subring $e_\psi^{-1} O_\psi \mathbb{Z}_p[\Delta]$ of $\mathbb{Z}_p[\Delta]$ with $O$ by sending $e_\psi^{-1} \sigma$ to $\psi^{-1}(\sigma)$, $(\sigma \in \Delta)$. Then, for a $\mathbb{Z}_p[\Delta]$-module $X$, $X(\Psi^*)$ is regarded as an $O$-module. We fix a topological generator $\gamma$ of $\Gamma$. We identify, as usual, the completed group ring $e_\psi^{-1} O_\psi \mathbb{Z}_p[\Delta][[\Gamma]]$ with the power series ring $\Lambda = O[[T]]$ by $\gamma = 1 + T$ and the above identification. Thus, for a $\mathbb{Z}_p[\Delta][[\Gamma]]$-module $X$ (such as several Galois groups over $k_\infty$), we can regard $X(\Psi^*)$ as a module over $O$ or $\Lambda$. We denote by $q$ the element of $p\mathbb{Z}_p$ such that $\zeta^\gamma = \zeta^{1+q}$ for all $\zeta \in \mu_p\infty$.

Let $M/k_\infty$ be the maximal pro-$p$ abelian extension unramified outside $p$. The fields $N, L, N'$ and $L'$ are intermediate fields of $M/k_\infty$. We put

$$\mathcal{G} = \text{Gal}(M/k_\infty), \quad \mathcal{Z}' = \text{Gal}(M/N')$$
$$\mathcal{I} = \text{Gal}(M/L), \quad \mathcal{I}' = \text{Gal}(M/L').$$

For a $\mathbb{Q}_p$-character $\Phi$ of $\Delta$, denote by $M(\Phi)$ the intermediate field of $M/k_\infty$ corresponding to $\bigoplus_{\Phi'} \mathcal{G}(\Phi')$ by Galois theory where $\Phi'$ runs over the $\mathbb{Q}_p$-characters of $\Delta$ with $\Phi' \neq \Phi$. Then, $\text{Gal}(M(\Phi)/k_\infty) = \mathcal{G}(\Phi)$. We define $N(\Phi), L(\Phi)$, etc, in a similar way.

As we have mentioned in Section 2, $\mathcal{H}'(\Psi^*) = \mathcal{H}(\Psi^*)$. Therefore, by the assertion [6, Lemma 1] on $\mathcal{H}(\Psi^*)$, there exists an injective $\Lambda$-homomorphism

$$\iota : \mathcal{H}'(\Psi^*) \hookrightarrow \begin{cases} \Lambda, & \text{when } \psi(p) \neq 1, \\ \Lambda \oplus \Lambda/(T - q), & \text{when } \psi(p) = 1, \end{cases}$$

with a finite cokernel. This is the $\Delta$-decomposed version of [10, Theorem 15]. In the next section, we prove the following two lemmas.
Lemma 1. There exists a $\Lambda$-isomorphism:

$$\mathcal{I}'(\Psi^*) \cong \begin{cases} 
\Lambda, & \text{when } \psi(p) \neq 1, \\
\Lambda \oplus \Lambda/(T - q), & \text{when } \psi(p) = 1.
\end{cases}$$

Lemma 2. We have $M(\Psi^*) = N'(\Psi^*)L'(\Psi^*)$.

Proof of Theorem 3. Assume that $\psi^*(p) = 1$. We put

$$\mathcal{I}'(\Psi^*) = \mathcal{I}'(\Psi^*) \mathcal{Z}'(\Psi^*) / \mathcal{Z}'(\Psi^*).$$

Then, we have $\mathcal{I}'(\Psi^*) \subseteq \mathcal{H}'(\Psi^*)$, and

$$\mathcal{Y}'(\Psi^*) \cong \mathcal{H}'(\Psi^*) / \mathcal{I}'(\Psi^*).$$

As $\psi^*(p) = 1$, we see from Lemmas 1 and 2 that

$$\mathcal{I}'(\Psi^*) \cong \mathcal{I}'(\Psi^*) \cong \Lambda.$$

Let $\iota$ be an embedding of $\mathcal{H}'(\Psi^*)$ into $\Lambda$ with a finite cokernel. By the above, the image $\iota(\mathcal{I}'(\Psi^*))$ of $\mathcal{I}'(\Psi^*)$ equals a principal ideal $(f)$ of $\Lambda$ for some $f \in \Lambda$. Therefore, we obtain an injective $\Lambda$-homomorphism

$$\mathcal{Y}'(\Psi^*) \hookrightarrow \Lambda / (f)$$

with a finite cokernel. On the other hand, $f$ is relatively prime to $p$ by [2]. Hence, $\mathcal{Y}'(\Psi^*)$ is torsion free over $\mathbf{Z}_p$. □

4. Proof of lemmas.

4.1. Preliminaries. In this subsection, we give and recall some assertions on some groups of local universal norms of $k_\infty/k$ and the Galois groups $\mathcal{I} = \text{Gal}(M/L)$, $\mathcal{I}' = \text{Gal}(M/L')$. For a while, we fix a prime ideal $p$ of $k$ over $p$. We denote the unique prime ideal of $k_n$ over $p$ simply by $p$. Let $k_{n,p}$ be the completion of $k_n$ at $p$, and $\mathcal{U}_{n,p}$ the group of principal units of $k_{n,p}$.

Let

$$\mathcal{V}_{n,p} = \bigcap_{m \geq n} N_{m/n} \mathcal{U}_{m,p} \quad \text{and} \quad \mathcal{W}_{n,p} = \bigcap_{m \geq n} N_{m/n}((k_{m,p}^\times)^{(p)})$$

be the groups of universal norms. Here, $N_{m/n}$ denotes the norm map from $k_m^\times$ to $k_n^\times$, and for an abelian group $X$, $X^{(p)}$ denotes the maximal pro-$p$ quotient. We put

$$\mathcal{U}_n = \prod_{p | \mathfrak{p}} \mathcal{U}_{n,p}, \quad \mathcal{V}_n = \prod_{p | \mathfrak{p}} \mathcal{V}_{n,p}, \quad \mathcal{W}_n = \prod_{p | \mathfrak{p}} \mathcal{W}_{n,p},$$

where $\mathfrak{p}$ runs over the primes of $k$ over $p$. These are closed subgroups of the maximal pro-$p$ quotient $\widehat{k_n^\times} = (\prod_{p | \mathfrak{p}} k_{m,p}^\times)^{(p)}$. Denote by $\varphi_n$ the natural embedding of $k_n^\times$ into $\widehat{k_n^\times}$. Let $E_n$ (resp. $E'_n$) be the group of units (resp. $p$-units) of $k_n$, and let $\mathcal{E}_n$ (resp. $\mathcal{E}'_n$) be the closure of $\varphi_n(E_n)$ (resp. $\varphi_n(E'_n)$).
in $\hat{k}_n^\times$. Let $\mathcal{U}_\infty$, $\mathcal{E}_\infty$, $\mathcal{W}_\infty$, $\mathcal{E}'_\infty$ be the projective limits of $\mathcal{U}_n$, $\mathcal{E}_n$, $\mathcal{W}_n$, $\mathcal{E}'_n$ with respect to the relative norms, respectively:

$$
\mathcal{U}_\infty = \lim\limits_\leftarrow \mathcal{U}_n \quad (= \lim\limits_\leftarrow \mathcal{V}_n), \quad \mathcal{E}'_\infty = \lim\limits_\leftarrow \mathcal{E}'_n \quad (= \lim\limits_\leftarrow (\mathcal{W}_n \cap \mathcal{E}'_n)), \quad \text{etc}.
$$

These groups are naturally regarded as modules over $\mathbb{Z}_p[\Delta][\Gamma]$.

**Lemma 3.** The projection $P : \mathcal{W}_\infty \to \mathcal{W}_0$ induces an isomorphism

$$
\mathcal{W}_\infty / \mathcal{W}_0 \cong \mathcal{W}_0.
$$

**Proof.** It is clear that the projection $P$ is surjective and that $\mathcal{W}^T_\infty \subseteq \ker P$. So, it suffices to show that ker $P \subseteq \mathcal{W}^T_\infty$. Let $u = (u_n)_{n \geq 0}$ be an element of ker $P$ with $u_n \in \mathcal{W}_n$. As $u_0 = 1$, we see that $u_n$ is contained in $\mathcal{U}_n$. We can write $u_n = w^T_n$ for some $w_n \in \prod_{p \mid p} \hat{k}_n^\times$ by Hilbert Satz 90. Hence, $u_n = \overline{w}^T_n$, $\overline{w}_n$ being the projection of $w_n$ in $\hat{k}_n^\times$. Denote by $x^{(n)}_1$ the element of the product $X = \prod_{\ell} \hat{k}_\ell^\times$ whose $\ell$-th component is $N_n/\ell (w_n)$ (resp. 1) for $\ell \leq n$ (resp. $\ell > n$). Since $X$ is compact, $\{x^{(n)}_1\}$ has an accumulation point $x$ in $X$. We easily see that $x \in \mathcal{W}_\infty$ and $x^T = u$. Therefore, ker $P \subseteq \mathcal{W}^T_\infty$. $\square$

By class field theory, it is known (cf. [14, Corollary 13.6]) that the inertia group $I$ is canonically isomorphic to $\mathcal{U}_\infty / \mathcal{E}_\infty$ over $\mathbb{Z}_p[\Delta][\Gamma]$. As $\Psi^*$ is odd and $\Psi^* \neq \omega$, it follows that $\mathcal{E}_\infty (\Psi^*) = \{0\}$ by a theorem on units of CM-fields (cf. [14, Theorem 4.12]). Therefore, we obtain a $\Lambda$-isomorphism

$$
I(\Psi^*) \cong \mathcal{U}_\infty (\Psi^*).
$$

(3)

On the $\Lambda$-structure of $\mathcal{U}_\infty$, it is known (cf. Gillard [3, Proposition 1]) that

$$
\mathcal{U}_\infty (\Psi^*) \cong \begin{cases} 
\Lambda, & \text{when } \psi(p) \neq 1, \\
\Lambda \oplus \Lambda / (T - q), & \text{when } \psi(p) = 1.
\end{cases}
$$

(4)

It is also known (cf. [3, Proposition 2]) that

$$
\mathcal{V}_0 (\Psi^*) \cong \begin{cases} 
O, & \text{when } \psi(p) \neq 1 \text{ and } \psi^* (p) \neq 1, \\
O \oplus O / q, & \text{when } \psi(p) = 1, \\
\{0\}, & \text{when } \psi^* (p) = 1.
\end{cases}
$$

(5)

As for the decomposition group $I'$, we need to prove the following:

**Proposition 2.** The reciprocity law map induces a canonical isomorphism

$$
I' \cong \mathcal{W}_\infty / \mathcal{E}'_\infty
$$

over $\mathbb{Z}_p[\Delta][[\Gamma]]$.

**Proof.** Let $M_n$ (resp. $L'_n$) be the maximal abelian extension of $k_n$ contained in $M$ (resp. $L'$). It suffices to prove that

$$
\text{Gal}(M_n / L'_n) \cong \mathcal{W}_n / (\mathcal{W}_n \cap \mathcal{E}'_n)
$$

(6)
since $\mathcal{I}'$ is the projective limit of $\text{Gal}(M_n/L'_n)$ with respect to the restriction maps. It suffices to show the assertion (6) only when $n = 0$ by considering $k_n$ as the base field.

For an integer $m \geq 0$, we put

$$W^{(m)} = \prod_{p | p} N_{m/0} k^\times_{m,p} \supseteq U_0^{m}.$$  

For a prime divisor $q$ of $k$ relatively prime to $p$, let $U_q$ be the group of local units (resp. the multiplicative group) of the completion $k_q$ of $k$ at $q$ when $q$ is finite (resp. infinite). Let $J_k$ be the group of idèles of $k$. We define its subgroups $A, B, C$ as follows:

$$A = W^{(m)} \times \prod_{q \not| p} \{1\}, \quad B = U_0^{m} \times \prod_{q \not| p} \{1\}, \quad C = \prod_{p | p} \{1\} \times \prod_{q | p} U_q,$$

where $p$ (resp. $q$) runs over the primes of $k$ dividing $p$ (resp. relatively prime to $p$).

Denote by $H$ the Hilbert $p$-class field of $k$. Let $M_{0,m}$ be the maximal intermediate field of $M_0/H$ whose Galois group over $H$ is of exponent $p^m$. Clearly, $M_{0,m}$ contains $k_m$. Let $L'_{0,m}$ be the maximal intermediate field of $M_{0,m}/k_m$ in which all prime divisors of $k_m$ over $p$ split completely. We have a natural isomorphism

$$(7) \quad \text{Gal}(M_0/L'_0) \cong \lim_{\leftarrow} \text{Gal}(M_{0,m}/L'_{0,m}),$$

the projective limit being taken with respect to the restriction maps.

It is known that the reciprocity law map induces isomorphisms

$$\text{Gal}(M_{0,m}/k) \cong \left(J_k/k^\times BC\right)(p) \quad \text{and} \quad \text{Gal}(L'_{0,m}/k) \cong \left(J_k/k^\times AC\right)(p).$$

For this, see Sumida [13, pp. 692-693]. Therefore, we obtain a canonical isomorphism

$$\text{Gal}(M_{0,m}/L'_{0,m}) \cong (k^\times AC/k^\times BC)(p) \cong \left(A/(A \cap (k^\times BC))\right)(p).$$

We easily see that

$$A \cap (k^\times BC) = (W^{(m)} \cap (E'_0 U_0^{m})) \times \prod_{q | p} \{1\}.$$  

Here, we are regarding $E'_0$ as a subgroup of $\prod_{p | p} k^\times_{0,p}$ in the natural way. Hence, we have

$$\text{Gal}(M_{0,m}/L'_{0,m}) \cong (W^{(m)}/(W^{(m)} \cap (E'_0 U_0^{m})))^{(p)}.$$  

From this and (7), we obtain

$$\text{Gal}(M_0/L'_0) \cong \mathcal{W}_0/(\mathcal{W}_0 \cap \mathcal{E}'_0)$$

by an elementary but tedious argument on the topology of $\hat{k}_0^\times = (\prod_{p | p} k^\times_{0,p})^{(p)}$, which we leave to the reader. \qed
4.2. Proof of Lemmas 1 and 2.

Proof of Lemma 1 (and the formula (2)). Let $B_n$ be the subgroup of $A_n$ consisting of classes which contain a product of prime ideals of $k_n$ over $p$, and let $B_\infty$ be the projective limit of $B_n$ with respect to the relative norms.

From class field theory, we see that $\mathcal{I}'/\mathcal{I}$ is canonically isomorphic to $B_\infty$. Let $D (\subset \Delta)$ be the decomposition group of $p$ at $k$. Then, we have a natural surjection

$$
\mathbb{Z}_p[\Delta/D] \twoheadrightarrow B_\infty \cong \mathcal{I}'/\mathcal{I}
$$

over $\mathbb{Z}_p[\Delta]$. We see that $\mathbb{Z}_p[\Delta/D](\Psi^*) = \{0\}$ or $O$ according as $\psi^*(p) \neq 1$ or $\psi^*(p) = 1$. Let $\psi^*(p) \neq 1$. Then, from the above surjection, we see that $\mathcal{I}'(\Psi^*) = I(\Psi^*)$ (from which (2) follows). Hence, the assertion of Lemma 1 follows from (3) and (4) in this case.

Let $\psi^*(p) = 1$. We have the following exact sequence of $\mathbb{Z}_p[\Delta]$-modules.

$$
\{0\} \rightarrow \mathcal{U}_0 \rightarrow \left( \prod_{p | \mathcal{P}} \mathbb{Z}_p^\times \right)^{(p)} \rightarrow \mathbb{Z}_p[\Delta/D] \rightarrow \{0\}.
$$

As $\psi^*(p) = 1$, we see from (5) that

$$(\mathcal{W}_0 \cap \mathcal{U}_0)(\Psi^*) = \mathcal{V}_0(\Psi^*) = \{0\}.$$

Therefore, by the above exact sequence, we see that the $O$-module $\mathcal{W}_0(\Psi^*)$ is free of rank one (or $\mathcal{W}_0(\Psi^*) = \{0\}$). Hence, $\mathcal{W}_\infty(\Psi^*)$ is cyclic over $\Lambda$ by Lemma 3 and Nakayama’s lemma (cf. [14, Lemma 13.16]). By this and Proposition 2, $\mathcal{I}'(\Psi^*)$ is cyclic over $\Lambda$. Then, we obtain $\mathcal{I}'(\Psi^*) \cong \Lambda$ since $\mathcal{I} \subseteq \mathcal{I}'$ and $\mathcal{I}(\Psi^*) \cong \Lambda$ by (3) and (4). □

Proof of Lemma 2. It is known (cf. [6, Proposition 3]) that

$$M(\Psi^*) = N(\Psi^*)L(\Psi^*).$$

Let $\psi^*(p) \neq 1$. Then, $N'(\Psi^*) = N(\Psi^*)$ and $L'(\Psi^*) = L(\Psi^*)$ by (1) and (2). Hence, the assertion follows from the above in this case. Let $\psi^*(p) = 1$. Then, by Lemma 1, $\mathcal{I}'(\Psi^*) \cong \Lambda$. On the other hand, $Z'(\Psi^*)$ is finitely generated and torsion over $\Lambda$ by [10, Theorems 5, 14]. Therefore, we obtain $Z'(\Psi^*) \cap \mathcal{I}'(\Psi^*) = \{0\}$, and hence $M(\Psi^*) = N'(\Psi^*)L'(\Psi^*)$. □

References


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