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SOME REMARKS ON THE PRESCRIBED MEAN CURVATURE EQUATION ON COMPLETE MANIFOLDS

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Dedicated to the memory of Franca Burrone Rigoli

We obtain conditions on the behavior at infinity of the mean curvature of a graph under volume growth assumptions. An L^q comparison result is also given.

0. Introduction.

Let $(M, \langle , \rangle)$ be a complete (noncompact), m-dimensional, $m \geq 2$, Riemannian manifold and, for a fixed reference point $o \in M$, set $r(x) = \mathrm{dist}_{(M,\langle,\rangle)}(o, x)$. Thus B_R and ∂B_R denote, respectively, the geodesic ball and sphere of radius R centered at o. In what follows we shall always assume M connected.

We associate to a smooth function $u : M \to \mathbb{R}$ its graph $\Gamma_u : M \to M \times \mathbb{R}$, defined by

$$\Gamma_u : x \to (x, u(x)).$$

Indicating with $(,)$ the product metric on $M \times \mathbb{R}$,

$$\Gamma_u : (M, \Gamma_u^*(,)) \to (M \times \mathbb{R}, (,))$$

becomes an isometric embedding. Let $\nabla, \mathrm{div}, | \, |$ denote the gradient, the divergence operator and the norm with respect to $\langle , \rangle$. Then, Γ_u has mean curvature $\frac{1}{m} a(x)$, for some function $a \in \mathcal{C}^\infty(M)$ (and appropriately chosen normal to Γ_u) if and only if

$$\mathrm{div}\left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}\right) = a(x), \quad \text{on } M.$$

Thus, if $a(x)$ is constant, Γ_u has constant mean curvature and if $a(x) \equiv 0$, Γ_u is a minimal graph (and u is a minimal map).

When $M = \mathbb{R}^m$ is the Euclidean space, a result of S. Bernstein [**B1**], [**B2**] for surfaces (later improved in varying ways by E. Heinz [**H**], by S.S. Chern [**S**], and by R. Finn [**F**]) implies that a graph with constant mean curvature defined on all of $M = \mathbb{R}^m$ is necessarily minimal. We point out that at the heart of these arguments there are estimates for the mean curvature in terms of isoperimetric quantities.

This was later generalized to the case of graphs over Riemannian manifolds by I. Salavessa [**S**], who showed that if Γ is a parallel mean curvature graph over $(M, \langle , \rangle)$, then

$$\|H_\Gamma\| \leq \frac{h(M)}{m},$$

where $h(M)$ is the Cheeger constant of M, and $||H_\Gamma||$ the (constant) length of the mean curvature vector of Γ. Now, if M has subexponential volume growth, then $h(M) = 0$, and Γ is a minimal graph.

When M has a faster volume growth there might exist non-minimal constant mean curvature graphs, as the following example shows. Let $\mathbb{H}^m$ be the hyperbolic space with canonical metric $\langle , \rangle$ of constant negative curvature -1, which we realize, in polar coordinates $(r, \theta) \in (0, +\infty) \times S^{m-1}$, as

$$\langle , \rangle = dr^2 + (\sinh r)^2 \ \ d\theta^2,$$

$d\theta^2$ being the standard metric on S^{m-1}. Then, for any $a \in (0, m-1]$ the smooth function

$$u(x) = \int_0^{r(x)} \frac{(\sinh t)^{1-m} \int_0^t a\,(\sinh s)^{m-1}\, ds}{\left\{1 - (\sinh t)^{2(1-m)} \left[\int_0^t a\,(\sinh s)^{m-1}\, ds\right]^2\right\}^{\frac{1}{2}}} dt$$

realizes a graph Γ_u with constant mean curvature $\frac{a}{m}$. In this example, note that

$$h(\mathbb{H}^m) = m - 1 \quad \text{and} \quad u(x) \sim r(x), \text{ as } r(x) \to +\infty.$$

Thus, in order to obtain a result similar to that of Bernstein mentioned above when $(M, \langle , \rangle)$ grows at most exponentially, it seems natural to require some growth conditions on u. Indeed, if we assume that $u(x) = o(r(x))$ as $r(x) \to +\infty$, then the constant mean curvature graph Γ_u is minimal. We refer to [**RSV**] for a precise statement and details.

In this note we prove:

Theorem A. *Let* $\Gamma_u : M \to M \times \mathbb{R}$ *be a graph such that*

$$\sup_M |u| < +\infty \tag{0.1}$$

and with mean curvature $a(x) \in C^\infty(M)$ *of constant sign. Assume that* $(M, \langle , \rangle)$ *satisfies one of the following growth assumptions:*

(i) $\quad \mathrm{vol}\,(\partial B_r) \leq C r^\alpha$, *for some* $\alpha \geq 0$,

(ii) $\quad \mathrm{vol}\,(\partial B_r) \leq C e^{\alpha r}$, *for some* $\alpha > 0$,

(iii) $\quad \mathrm{vol}\,(\partial B_r) \leq C e^{\alpha r^2}$, *for some* $\alpha > 0$, *for some constant* $C > 0$. *Then, corresponding to cases* (i), (ii), (iii) *we respectively have:*

(j) $$\liminf_{r(x)\to+\infty} \frac{|a(x)|}{r(x)^{-2}\,[\log r(x)]^{-1}} = 0,$$

(jj) $$\liminf_{r(x)\to+\infty} \frac{|a(x)|}{r(x)^{-1}\,[\log r(x)]^{-1}} = 0,$$

(jjj) $$\liminf_{r(x)\to+\infty} \frac{|a(x)|}{[\log r(x)]^{-1}} = 0.$$

Note that if (iii) holds then (jjj) implies that a constant mean curvature graph satisfying (0.1) is minimal.

In Section 2 we describe examples which show that Theorem A is fairly sharp.

Many results in the study of assigned mean curvature graphs, rely on comparison-type properties and Theorem A is no exception (see Corollary 1.2). Loosely speaking, given a domain Ω suppose that u and v satisfy

$$(0.2)\qquad \begin{cases} \text{i)}\quad \operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) \geq \operatorname{div}\left(\frac{\nabla v}{\sqrt{1+|\nabla v|^2}}\right), & \text{on } \Omega \\ \text{ii)}\quad u \leq v, & \text{on } \partial\Omega. \end{cases}$$

We look for conditions ensuring the possibility of extending inequality (0.2) ii) to all of Ω.

There is a vast literature on the subject ranging from the geometrical to the analytical approach. We limit ourselves to quote the papers [**CK**], [**Hw**], [**Hw1**], [**M**] which are closely related with the results below. Our contribution is twofold. In Theorem 1.1 we analyze the case where the strict inequality holds in (0.2) i), while in Theorem 1.3 we prove an L^q-comparison result from which the next uniqueness statement follows at once.

Theorem B. *Let* $\Omega \subseteq (M, \langle,\rangle)$ *be a domain with* (*possibly empty*) *boundary* $\partial\Omega$. *Let* $u, v \in \mathcal{C}^2(\Omega) \cap \mathcal{C}^0(\overline{\Omega})$, *be such that*

$$(0.3)\qquad \begin{cases} \operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = \operatorname{div}\left(\frac{\nabla v}{\sqrt{1+|\nabla v|^2}}\right), & \textit{on } \Omega \\ u = v, & \textit{on } \partial\Omega. \end{cases}$$

Assume that, for some $q > 1$,

$$\int_{B_r\cap\Omega} |u - v|^q = O(r^2 \log r), \quad \text{as } r \to +\infty.$$

Then, if $\partial\Omega \neq \varnothing$, $u = v$ *on* Ω, *otherwise* $u = v + A$ *on* M *for some* $A \in \mathbb{R}$.

The sharpness of Theorem 1.3 (and thus of Theorem B) is discussed in Section 2.

For further details and related properties, for instance the case $q = 1$, we refer the reader to Section 1.

We would like to stress that in all the results presented in the sequel, we do not impose curvature restrictions on the underlying manifold $(M, \langle , \rangle)$.

We use the variable constant convention and denote with C a positive constant whose actual value may vary from place to place.

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1. Proof of the results.

The proof of Theorem 1.1 below is a variation of some ideas of Grigor'yan originally used, in the linear setting, to guarantee stochastic completeness, [**G**], and, later, [**G1**], to show the non-existence of nontrivial bounded solutions of the Schrödinger equation $\Delta u - b(x)u = 0$, $b(x) \geq 0$, $b(x) \not\equiv 0$, on complete manifolds. We provide details because of some differences and for the sake of completeness. First, we recall the following inequality due to Mikljukov, [**M**], p. 265, Hwang [**Hw**], p. 342, and Collin and Krust [**CK**], p. 452, that we shall use below and in Lemma 1.2. For $u, v \in \mathcal{C}^1(\Omega)$,

$$\begin{aligned}\left\langle \frac{\nabla u}{\sqrt{1+|\nabla u|^2}} - \frac{\nabla v}{\sqrt{1+|\nabla v|^2}}, \nabla u - \nabla v \right\rangle &\geq \sigma(x) \left| \frac{\nabla u}{\sqrt{1+|\nabla u|^2}} - \frac{\nabla v}{\sqrt{1+|\nabla v|^2}} \right|^2 \\ &\geq \left| \frac{\nabla u}{\sqrt{1+|\nabla u|^2}} - \frac{\nabla v}{\sqrt{1+|\nabla v|^2}} \right|^2\end{aligned}$$

where, for the ease of notation, we have set

$$\sigma(x) = \frac{1}{2}\left[\sqrt{1+|\nabla u|^2} + \sqrt{1+|\nabla v|^2}\right](x). \tag{1.1}$$

Theorem 1.1. *Let $\Omega \subseteq (M, \langle , \rangle)$ be an unbounded domain with (possibly empty) boundary $\partial\Omega$. Let $p : [0, +\infty) \to [0, +\infty)$ be a continuous function such that, for some $\overline{R} > 0$ and for each $R \geq \overline{R}$, either one of the following conditions is satisfied:*

$$\frac{e^{D\left\{\int_0^R \sqrt{p(s)}ds\right\}^2}}{\operatorname{vol}(\partial B_R \cap \Omega)} \notin L^1(+\infty), \tag{1.2}$$

for some constant $D > 0$;

$$\frac{\left\{\int_R^{\frac{3}{2}R} \sqrt{p(s)}ds\right\}^2}{R \log \operatorname{vol}(B_{2R} \cap \Omega)} \geq h(R) \notin L^1(+\infty). \tag{1.3}$$

where the function $h : [\overline{R}, +\infty) \to (0, +\infty)$ *is continuous and monotonic non-increasing.*

Let $u, v \in \mathcal{C}^1(\overline{\Omega}) \cap \mathcal{C}^2(\Omega)$ *satisfy*

(1.4)
$$\begin{cases} \text{i)} \quad \operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) - \operatorname{div}\left(\frac{\nabla v}{\sqrt{1+|\nabla v|^2}}\right) = q(x) \geq p(r(x)), \quad \textit{on } \Omega \\ \text{ii)} \quad \sup_{\overline{\Omega}} (u - v) < +\infty. \end{cases}$$

- *If* $\partial\Omega \neq \emptyset$ *and* $u \leq v$, *on* $\partial\Omega$, *then* $u \leq v$ *on* Ω.
- *If* $\partial\Omega = \emptyset$, *i.e.,* $\Omega = M$, *and* $q(x) \equiv 0$, *then either* $u \leq v$ *on* M, *or* $u - v$ *is constant.*
- *If* $\partial\Omega = \emptyset$ *and* $q(x) \not\equiv 0$, *then there are no solutions to* (1.4) i) *satisfying* (1.4) ii).

Proof. Let u, v be solutions of (1.4) i) and ii). We consider first the case where $q(x) \not\equiv 0$ and assume by contradiction that there exists x_0 in the support Supp (q) of q such that

$$u(x_0) > v(x_0). \tag{1.5}$$

We set $\gamma = \sup_{\overline{\Omega}}(u - v)$ so that $0 < \gamma < +\infty$. We introduce a new parameter $t \in [0, +\infty)$ and a new function w on $\overline{\Omega} \times [0, +\infty)$,

$$w(x, t) = e^t[u(x) - v(x)] - \gamma. \tag{1.6}$$

We note that

$$\text{i)} \ \ w(x, 0) \leq 0; \quad \text{ii)} \ \ \frac{\partial w}{\partial t} = e^t(u - v); \quad \text{iii)} \ \ \nabla w = e^t(\nabla u - \nabla v). \tag{1.7}$$

Given $\overline{T} > 0$, let $g : M \times [0, \overline{T}) \to \mathbb{R}$ and $\psi : M \to \mathbb{R}$, be a Lipschitz, respectively, a compactly supported Lipschitz function, to be chosen later. Finally, for ease of notation we write $\varphi(t) = (1 + t^2)^{-\frac{1}{2}}$ for $t \geq 0$. Denoting $w_+ = \max\{0, w\}$, let Z be the Lipschitz vector field defined by

$$Z = \gamma\psi^2 w_+^2 e^{g+t} \left[\varphi\left(|\nabla u|\right) \nabla u - \varphi\left(|\nabla v|\right) \nabla v\right], \quad \text{on } \overline{\Omega}.$$

We compute its divergence and we use (1.4), $u - v \leq \gamma$, $p \geq 0$, (1.7) iii), $\gamma > 0$ to obtain

$$\begin{aligned} \operatorname{div} Z \ \geq \ & \psi^2 w_+^2 e^{g+t} q(x)(u - v) \\ & + 2\gamma\psi w_+^2 e^{g+t} \left\langle \varphi\left(|\nabla u|\right) \nabla u - \varphi\left(|\nabla v|\right) \nabla v, \nabla\psi \right\rangle \\ & + 2\gamma\psi^2 w_+ e^{g+2t} \left\langle \varphi\left(|\nabla u|\right) \nabla u - \varphi\left(|\nabla v|\right) \nabla v, \nabla u - \nabla v \right\rangle \\ & + \gamma\psi^2 w_+^2 e^{g+t} \left\langle \varphi\left(|\nabla u|\right) \nabla u - \varphi\left(|\nabla v|\right) \nabla v, \nabla g \right\rangle. \end{aligned}$$

Using the Schwarz and the Mikljukov-Hwang-Collin-Krust inequalities we obtain

$$\begin{aligned}\operatorname{div} Z \quad \geq \quad & \psi^2 w_+^2 e^{g+t} q(x)(u-v) \\ & -2\gamma\,|\psi|\, w_+^2 e^{g+t}\,|\varphi(|\nabla u|)\nabla u - \varphi(|\nabla v|)\nabla v|\,|\nabla\psi| \\ & -\gamma\psi^2 w_+^2 e^{g+t}\,|\varphi(|\nabla u|)\nabla u - \varphi(|\nabla v|)\nabla v|\,|\nabla g| \\ & +2\gamma\psi^2 w_+ e^{g+2t}\,|\varphi(|\nabla u|)\nabla u - \varphi(|\nabla v|)\nabla v|^2 .\end{aligned}$$

We note that, by definition of w, $Z=0$ on $\partial\Omega$ for every t in $[0,\overline{T})$ and that $\operatorname{supp} Z \subset\subset \Omega$ if $t>0$. Given $0\le T_1<T_2\le\overline{T}$, we integrate over $\Omega\times[T_1,T_2]$ and use the divergence theorem and (1.7) ii) to deduce

$$\begin{aligned}&\int_\Omega\int_{T_1}^{T_2}\psi^2 w_+^2 e^g q\frac{\partial w}{\partial t} \\ &\le -\int_{T_1}^{T_2}\int_\Omega 2\gamma\psi^2 w_+ e^{g+2t}\,|\varphi(|\nabla u|)\nabla u - \varphi(|\nabla v|)\nabla v|^2 \\ &\quad +\int_{T_1}^{T_2}\int_\Omega 2\gamma\,|\psi|\,w_+^2 e^{g+t}\,|\varphi(|\nabla u|)\nabla u - \varphi(|\nabla v|)\nabla v|\,|\nabla\psi| \\ &\quad +\int_{T_1}^{T_2}\int_\Omega \gamma\psi^2 w_+^2 e^{g+t}\,|\varphi(|\nabla u|)\nabla u - \varphi(|\nabla v|)\nabla v|\,|\nabla g| .\end{aligned}$$

Integrating by parts with respect to t on $[T_1,T_2]$ the LHS, and using the elementary inequality $2ab\le\varepsilon a^2+\varepsilon^{-1}b^2$, $a,b\ge0$, $\varepsilon>0$ with appropriate choices of ε on the last two terms of the RHS of the above inequality, yield

$$\begin{aligned}\int_\Omega \psi^2 w_+^3 e^g q\Big|_{T_1}^{T_2} \le \int_{T_1}^{T_2}\int_\Omega \psi^2 w_+^3\left\{q\frac{\partial g}{\partial t}+\frac{3}{2}\gamma|\nabla g|^2\right\}e^g \\ +2\int_{T_1}^{T_2}\int_\Omega \gamma w_+^3 e^g|\nabla\psi|^2 .\end{aligned} \tag{1.8}$$

Let $R_0=\max\left\{\overline{R},\operatorname{dist}_{(M,<,>)}(o,x_0)\right\}$ and fix $R_2>R_0$. Denoting by $C>0$ a constant to be chosen later, we define

$$f(r)=\begin{cases} C^{-1}\left\{\int_{R_2}^r\sqrt{p(s)}ds\right\} & \text{if } r>R_2 \\ 0 & \text{if } r\le R_2\end{cases} \tag{1.9}$$

and we let

$$g(x,t)=\frac{f(r(x))^2}{t-\overline{T}},\quad \text{for } r\ge0,\quad 0\le t<\overline{T}.$$

If C in (1.9) is chosen so as to have

$$q\frac{\partial g}{\partial t}+\frac{3}{2}\gamma\,|\nabla g|^2\le 0,$$

then, by (1.8),

$$\int_\Omega \psi^2 w_+^3 e^g q \Big|_{T_1}^{T_2} \le 2\int_{T_1}^{T_2}\int_\Omega \gamma w_+^3 e^g |\nabla\psi|^2 . \tag{1.10}$$

We fix $R_1 > R_2$ and we choose ψ radial satisfying

$$|\psi| \le 1 \text{ on } M; \quad \psi = 1 \text{ on } B_{R_2}; \quad \psi = 0 \text{ off } B_{R_1}. \tag{1.11}$$

Since $q(x) \ge 0$, $g \le 0$ and $g(.,T) \equiv 0$ on B_{R_2},

$$\int_\Omega \psi^2 w_+^3 e^g q \Big|_{T_1}^{T_2} \ge E(R_2,T_2) - E(R_1,T_1) \tag{1.12}$$

where we have set

$$E(R,T) = \int_{\Omega\cap B_R} q(x) w_+(x,T)^3 \ge 0, \quad R \ge R_0.$$

With these choices of ψ and g, inserting (1.12) into (1.10) yields

$$E(R_2,T_2) - E(R_1,T_1) \le 2\gamma \int_{T_1}^{T_2}\int_{(B_{R_1}\setminus B_{R_2})\cap\Omega} \left(\psi'(r)\right)^2 w_+^3 e^g. \tag{1.13}$$

We now distinguish two cases.

First case. Condition (1.2) is satisfied. Let $T_2 = \overline{T}$. We use (1.13) together with (1.6) and the co-area formula to get

$$E(R_2,T_2) - E(R_1,T_1) \le C\gamma^4 e^{3T_2} \int_{R_2}^{R_1} \left(\psi'(r)\right)^2 \mathrm{vol}\,(\partial B_r \cap \Omega) e^{\frac{f(r)^2}{T_1 - T_2}} dr,$$

valid for all radial Lipschitz functions satisfying (1.11). Letting ψ be the function defined by $\psi(r) = 1$ if $0 \le r \le R_2$,

$$\psi(r) = \frac{\int_r^{R_1} \mathrm{vol}\,(\partial B_r \cap \Omega)^{-1} e^{-\frac{f(r)^2}{T_1 - T_2}} dr}{\int_{R_2}^{R_1} \mathrm{vol}\,(\partial B_r \cap \Omega)^{-1} e^{-\frac{f(r)^2}{T_1 - T_2}} dr}$$

if $R_2 \le r \le R_1$, and $\psi(r) = 0$ if $r \ge R_1$, we obtain

$$E(R_2,T_2) - E(R_1,T_1) \le C\gamma^4 e^{3T_2} \left\{ \int_{R_2}^{R_1} \frac{e^{-\frac{f(r)^2}{T_1 - T_2}}}{\mathrm{vol}\,(\partial B_r \cap \Omega)} dr \right\}^{-1}. \tag{1.14}$$

Fix D such that (1.3) holds. Then there exist constants $C = C(R_2) > 0$ and $\alpha > 0$ such that

$$\frac{f(r)^2}{\alpha} \ge D \left\{ \int_0^r \sqrt{p(s)} ds \right\}^2 - C.$$

Hence, whenever we choose $0 \leq T_1 < T_2$ so that $T_2 - T_1 \leq \alpha$, it follows from (1.14) that

$$E(R_2, T_2) \leq E(R_1, T_1) + C\gamma^4 e^{3T_2} \left\{ \int_{R_2}^{R_1} \frac{e^{D\left\{\int_0^r \sqrt{p(s)}ds\right\}^2}}{\operatorname{vol}(\partial B_r \cap \Omega)} dr \right\}^{-1}. \tag{1.15}$$

Since $E(R, T)$ is a non-decreasing function of R, $R \geq R_0$, the limit

$$\lim_{R \to +\infty} E(R, T) = E(+\infty, T)$$

exists, and assumption (1.2), together with $T_2 - T_1 \leq \alpha$, (1.15) gives

$$E(R_2, T_2) \leq E(+\infty, T_1). \tag{1.16}$$

Choosing $T_1 = 0$ and $0 < T_2 \leq \alpha$, (1.7) i) and (1.16) yield

$$E(R_2, T_2) \leq E(+\infty, 0) = 0$$

and, since $R_2 > R_0$ is arbitrary, $E(R, T_2) = 0$ for every $R > 0$. Choosing $T_1 = \alpha$ and $\alpha < T_2 \leq 2\alpha$, we obtain

$$E(R_2, T_2) \leq E(+\infty, \alpha) = 0.$$

Proceeding in this way we get $E(R, T) = 0$ for each $T \geq 0$ and $R > 0$. It follows from the definition of E that $w_+(x, T) \equiv 0$ in a neighborhood of x_0 for each $T \geq 0$. Thus at x_0

$$u(x_0) - v(x_0) \leq \gamma e^{-t}, \quad \text{for each } t \geq 0.$$

Using (1.5) we obtain a contradiction.

Second case. Suppose that Condition (1.3) holds. We choose $\tau > 0$, $\eta \in (0, \tau]$ and we set $T_1 = \tau - \eta$, $T_2 = \tau$, $\overline{T} = \tau + \eta$, $R_1 = 2R$, $R_2 = R$, with $R > R_0$. Next, for every R, we choose a radial function $\psi = \psi_R$ satisfying (1.11) and such that

$$\psi \equiv 1 \;\text{ on } B_{3R/2} \quad \text{and} \quad |\psi'| \leq \frac{C}{R},$$

for some constant $C > 0$ independent of R. Then, (1.13), the definitions of w_+ and γ, and the fact that

$$g(r, t) \leq -(2C)^{-1} \frac{\left\{\int_R^{\frac{3}{2}R} \sqrt{p(s)}ds\right\}^2}{\eta}, \quad \text{on } \left[\tfrac{3}{2}R, +\infty\right) \times [\tau - \eta, \tau),$$

give

$$E(R, \tau) \leq E(2R, \tau - \eta) + \frac{c(\tau)}{R^2} \operatorname{vol}(\Omega \cap B_{2R}) e^{-(2C)^{-1} \frac{\left\{\int_R^{\frac{3}{2}R} \sqrt{p(s)}ds\right\}^2}{\eta}}, \tag{1.17}$$

where

$$c(\tau) = c(e^{\tau} - 1)^3 \tau, \qquad c > 0.$$

We choose $\eta = \eta(R, \tau)$ so small that

$$\text{i) } 0 < \eta \le \tau, \quad \text{ii) } \log \operatorname{vol}(\Omega \cap B_{2R}) - (2C)^{-1} \frac{\left\{ \int_R^{\frac{3}{2}R} \sqrt{p(s)} ds \right\}^2}{\eta} \le 0.$$

According to (1.2), this holds provided

$$0 < \eta \le \min \left\{ (2C)^{-1} Rh(R), \tau \right\}.$$

Then, (1.17) gives

$$E(R, \tau) \le E(2R, \tau - \eta) + \frac{c(\tau)}{R^2}. \tag{1.18}$$

Let $R_k = 2^k R$. Since

$$\sum_{k=0}^{+\infty} h(R_k) R_k \ge \int_R^{+\infty} h(r) dr = +\infty,$$

there exists $k_0 = k_0(R, \tau) \in \mathbb{N}$ such that

$$(2C)^{-1} \sum_{k=0}^{k_0} h(R_k) R_k \ge \tau.$$

We set, for $k = 0, \ldots, k_0$,

$$\eta_k = \tau \, \frac{h(R_k) R_k}{\sum_{j=0}^{k_0} h(R_j) R_j}$$

and we define

$$\tau_0 = \tau; \qquad \tau_k = \tau - \sum_{j=0}^{k-1} \eta_j, \; k = 0, \ldots, k_0 + 1.$$

A simple verification shows that

$$\begin{array}{rl} \text{i)} & 0 < \eta_k \le \min \left\{ (2C)^{-1} R_k h(R_k), \tau_k \right\} \\ \text{ii)} & c(\tau_k) \le c(\tau) \\ \text{iii)} & \tau_{k_0+1} = 0. \end{array} \tag{1.19}$$

Because of (1.19) i), ii) we can use inequality (1.18) to obtain

$$E(R_k, \tau_k) \le E(R_{k+1}, \tau_{k+1}) + \frac{c(\tau)}{R_k^2},$$

for each $k = 0, \dots, k_0 + 1$. Recalling (1.19) iii) and (1.7) i), we obtain

$$0 \le E(R,\tau) \le E(R_{k_0+1}, 0) + c(\tau) \sum_{k=0}^{k_0} \frac{1}{R_k^2} \le \frac{4c(\tau)}{3} \frac{1}{R^2},$$

whence, letting R tend to $+\infty$,

$$\int_\Omega q(x) w_+^3(x,\tau) = 0.$$

Since $q > 0$ in a neighborhood of x_0, we deduce

$$w_+(x_0, \tau) = 0$$

that is

$$u(x_0) - v(x_0) \le \gamma e^{-\tau}.$$

Since τ can be arbitrarily large, this yields the contradiction required to complete the proof that $u(x_0) \le v(x_0)$ even in this case.

To conclude, we consider first the case where $\partial\Omega \ne \emptyset$. By what showed above, $u \le v$ on $Supp(q)$, and we need to prove that the inequality holds on the whole of Ω. The authors are indebted to the referee who suggested the following argument. If (1.3) holds, then for every R there exist $R < r_1 < r_2 < 3R/2$, such that $p(r) > 0$ in (r_1, r_2). By what proved above, $u \le v$ on $\left(B_{r_2} \setminus B_{r_1}\right) \cap \Omega$. By the maximum principle, the inequality holds on $B_{r_2} \cap \Omega$, and, letting $R \to +\infty$, it holds on Ω.

Assuming instead the validity of (1.2), we distinguish two cases. If $\sqrt{p(t)}$ is not in $L^1(+\infty)$, then $p(r_i) > 0$ along a sequence $r_i \to +\infty$, and the conclusion follows as above. If $\sqrt{p(t)}$ is in $L^1(+\infty)$, then Condition (1.2) becomes

$$\frac{1}{\mathrm{vol}\,(\partial B_R \cap \Omega)} \notin L^1(+\infty),$$

and the conclusion follows from Theorem 1.6 below (whose proof is independent of Theorem 1.1).

Next, let $\Omega = M$. If $q(x) \not\equiv 0$, the argument above shows that $u \le v$ on M. Since $u + A$ satisfies the same assumptions as u, it follows that $u + A \le v$ for every A. Since this is clearly impossible, this shows that there are no solutions of (1.4) i) satisfying (1.4) ii).

Finally, if $\Omega = \emptyset$ and $q(x) \equiv 0$ on M, the conclusion follows from Theorem 1.6 as above. □

Remark. Conditions (1.2) and (1.3) are independent of each other. To see this consider the following examples.

(1) Let $\mathbb{R}^2 \setminus \{0\} = (0, +\infty) \times S^1$ be endowed with the metric

$$\langle , \rangle_g = dr^2 + g^2(r) d\theta^2,$$

where $d\theta^2$ is the standard metric on S^1 and $g(r) = re^{\sqrt{1+r^2}-1}$. Then, $\langle , \rangle_g$ can be extended to a smooth, complete metric on $\mathbb{R}^2$. Set $(M, \langle , \rangle) = (\mathbb{R}^2, \langle , \rangle_g)$. We note that, for $r > 0$ sufficiently large,

$$\mathrm{vol}\,(\partial B_r) \geq C_1 e^r$$

and

$$\mathrm{vol}\,(B_r) \leq C_2 e^{\alpha r}$$

with $\alpha, C_i > 0$ suitable constants, $i = 1, 2$. We choose

$$p(s) = \begin{cases} \frac{(\log s - 1)^2}{4s(\log s)^3} & \text{for } s > e \\ 0 & \text{for } s \leq e \end{cases}$$

and we observe that

$$\sqrt{p(s)} = \frac{d}{ds}\sqrt{\frac{s}{\log s}}, \qquad s > e.$$

Thus, for each $D > 0$ and for each $R > e$ large enough, it holds

$$\frac{e^{D\left\{\int_0^R \sqrt{p(s)}ds\right\}^2}}{\mathrm{vol}\,(\partial B_R)} \leq C_1^{-1} e^{R\left(\frac{D}{\log R} - 1\right)} \in L^1(+\infty),$$

proving that Condition (1.2) is not satisfied. On the other hand,

$$\frac{\left\{\int_R^{\frac{3}{2}R} \sqrt{p(s)}ds\right\}^2}{R \log \mathrm{vol}\,(B_{2R})} \geq C\frac{1}{R\log R}, \qquad R \text{ large},$$

for some constant $C > 0$, so that, setting h to denote the RHS of this latter inequality, Condition (1.3) is met.

(2) We let $(M, \langle , \rangle) = \left(\mathbb{R}^2, can\right)$ and we define

$$p(s) = \frac{1}{4\,(s+e)^2 \log(s+e)}\,.$$

Then, we have

$$\frac{e^{\left\{\int_0^R \sqrt{p(s)}ds\right\}^2}}{\mathrm{vol}\,(\partial B_R)} \to \frac{e}{2\pi} \qquad \text{as } R \to +\infty$$

proving that Condition (1.2) is met. On the other hand,

$$\frac{\left\{\int_R^{\frac{3}{2}R} \sqrt{p(s)}ds\right\}^2}{R \log \mathrm{vol}\,(B_{2R})} \sim C\frac{1}{R(\log R)^2}, \qquad \text{as } R \to +\infty$$

for some constant $C > 0$.

Proof of Theorem A. Possibly defining $\widetilde{u} = -u$ we may suppose that $a(x) \geq 0$ and, by (0.1), that $\sup_M \widetilde{u} < +\infty$. We first consider Case (i). Assume by contradiction that there exist $C, R > 0$ such that, for $r(x) \geq R$,

$$a(x) \geq \frac{C}{r(x)^2 \log r(x)}\,.$$

We choose $p : [0, +\infty) \to [0, +\infty)$ continuous satisfying $p(r) = Cr^{-2}(\log r)^{-1}$, for $r \geq R$, and $a(x) \geq p(r(x))$, on M. According to (i), (1.2) holds for $D > 0$ sufficiently large. A contradiction is achieved by applying Theorem 1.1.

In the remaining cases, one proceeds in a similar way, using (1.3) instead of (1.2). □

Lemma 1.2. *Let $\Omega \subseteq (M, \langle,\rangle)$ be an unbounded domain and let $u, v \in \mathcal{C}^2(\Omega) \cap \mathcal{C}^0(\overline{\Omega})$ be such that, for some constant B,*

$$\Omega_B = \{x \in \Omega : u(x) > v(x) + B\}$$

is nonempty, with boundary $\partial\Omega_B$ contained in Ω, and $u - v$ is nonconstant on Ω_B. Suppose that

$$\text{(1.20)} \qquad \operatorname{div}\left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}\right) \geq \operatorname{div}\left(\frac{\nabla v}{\sqrt{1 + |\nabla v|^2}}\right), \qquad \text{on } \Omega.$$

Let $\alpha \in \mathcal{C}^1([B, +\infty))$, $\beta \in \mathcal{C}^0([B, +\infty))$ be such that

$$\text{(1.21)} \qquad \alpha(u - v) \geq 0$$

$$\text{(1.22)} \qquad \alpha'(u - v) \geq \beta(u - v) > 0,$$

on $\overline{\Omega}_B$, and let $\lambda \in \mathcal{C}^1(\mathbb{R})$ satisfy

$$\text{(1.23)} \qquad \lambda(t) \geq 0 \text{ on } \mathbb{R}; \quad \lambda(t) = 0 \text{ on } (-\infty, B]; \quad \lambda'(t) \geq 0 \text{ on } (B, +\infty).$$

Then, there exists $R_1 > 0$ such that, for each $r > R \geq R_1$,

$$\text{(1.24)} \qquad \frac{1}{\int_{B_R \cap \Omega_B} \sigma\beta(u - v)\lambda(u - v)\left|\frac{\nabla u}{\sqrt{1+|\nabla u|^2}} - \frac{\nabla v}{\sqrt{1+|\nabla v|^2}}\right|^2} \geq \int_R^r \frac{\inf_{\partial B_s \cap \Omega_B} \sigma}{\int_{\partial B_s \cap \Omega_B} \lambda(u - v)\frac{\alpha(u-v)^2}{\beta(u-v)}} ds.$$

Proof. We keep the notation $\varphi(t) = \left(1 + t^2\right)^{-\frac{1}{2}}$. Let Z be the vector field defined by

$$Z = \lambda(u - v)\alpha(u - v)\left\{\varphi\left(|\nabla u|\right)\nabla u - \varphi\left(|\nabla v|\right)\nabla v\right\}, \quad \text{on } \overline{\Omega}.$$

Note that by our choice of λ, and the assumption that $\partial\Omega_B \subset \Omega$, $\overline{\Omega}_B \subset \Omega$ and $Z \equiv 0$ on $\Omega \setminus \Omega_B$. Computing the divergence of Z and using (1.20), (1.23), (1.21), (1.22) and the Mikljukov-Hwang-Collin-Krust inequality, we obtain

$$\operatorname{div} Z \geq \sigma\lambda(u-v)\beta(u-v)\left|\varphi\left(|\nabla u|\right)\nabla u - \varphi\left(|\nabla v|\right)\nabla v\right|^2 \quad \text{on } \overline{\Omega}_B.$$

Let $s_0 = \inf\{s : B_s \cap \Omega_B \neq \varnothing\}$. Integrating over $\Omega \cap B_s$, and applying the divergence theorem and recalling that $Z \equiv 0$ outside Ω_B yield

$$\int_{B_s\cap\Omega_B} \sigma\lambda(u-v)\beta(u-v)\left|\varphi\left(|\nabla u|\right)\nabla u - \varphi\left(|\nabla v|\right)\nabla v\right|^2 \leq \gamma(s) \tag{1.25}$$

with

$$\gamma(s) = \int_{\partial B_s\cap\Omega_B} \left\langle Z, \frac{\partial}{\partial s}\right\rangle$$

for $s \geq s_0$. Further, Schwarz inequality, the positivity of $\beta(u-v)$ and of σ, and Hölder inequality give

(1.26)

$$\gamma(s) \leq \frac{1}{\{\inf_{\partial B_s\cap\Omega_B}\sigma\}^{\frac{1}{2}}}\left\{\int_{\partial B_s\cap\Omega_B} \lambda(u-v)\frac{\alpha(u-v)^2}{\beta(u-v)}\right\}^{\frac{1}{2}}$$
$$\cdot\left\{\int_{\partial B_s\cap\Omega_B} \sigma\lambda(u-v)\beta(u-v)\left|\varphi\left(|\nabla u|\right)\nabla u - \varphi\left(|\nabla v|\right)\nabla v\right|^2\right\}^{\frac{1}{2}}.$$

Set

$$\zeta(s) = \frac{1}{\{\inf_{\partial B_s\cap\Omega_B}\sigma\}^{\frac{1}{2}}}\left\{\int_{\partial B_s\cap\Omega_B} \lambda(u-v)\frac{\alpha(u-v)^2}{\beta(u-v)}\right\}^{\frac{1}{2}}$$
$$H(s) = \int_{B_s\cap\Omega_B} \sigma\lambda(u-v)\beta(u-v)\left|\varphi\left(|\nabla u|\right)\nabla u - \varphi\left(|\nabla v|\right)\nabla v\right|^2$$

so that, by the co-area formula

$$H'(s) = \int_{\partial B_s\cap\Omega_B} \sigma\lambda(u-v)\beta(u-v)\left|\varphi\left(|\nabla u|\right)\nabla u - \varphi\left(|\nabla v|\right)\nabla v\right|^2.$$

From (1.25) and (1.26) we obtain

$$H(s) \leq \zeta(s)\big(H'(s)\big)^{\frac{1}{2}}. \tag{1.27}$$

Since our assumptions on $u-v$ and on Ω_B imply that $\nabla(u-v) \neq 0$ on Ω_B, we deduce that there exists $R_1 > 0$ such that, for each $s \geq R_1$,

$$H(s) > 0,$$

and therefore, by (1.27), $H'(s)$ and $\zeta(s) > 0$ for $s > R_1$. Rearranging we obtain

$$\frac{H'(s)}{H(s)^2} \geq \frac{1}{\zeta(s)^2},$$

and then, integrating over $[R, r]$ with $R_1 \le R < r$,

$$\frac{1}{H(R)} \ge \frac{1}{H(R)} - \frac{1}{H(r)} \ge \int_R^r \frac{ds}{\zeta(s)^2}.$$

Recalling the definitions of $H(R)$ and $\zeta(s)$ we obtain (1.24). □

Remark. Under the assumption that α and β satisfy (1.21) and (1.22) on M, the proof of Lemma 1.2 goes through with M in place of Ω_B and with $\lambda \equiv 1$. The conclusion (1.24) is modified accordingly.

Theorem 1.3. *Let Ω be an unbounded domain in M, and let $u, v \in \mathcal{C}^2(\Omega) \cap \mathcal{C}^0(\overline{\Omega})$ be such that*

$$\begin{cases} \operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) \ge \operatorname{div}\left(\frac{\nabla v}{\sqrt{1+|\nabla v|^2}}\right) & \text{on } \Omega \\ u \le v & \text{on } \partial\Omega. \end{cases} \tag{1.28}$$

Assume that, for some $q > 1$,

$$\frac{\inf_{\partial B_s \cap \Omega} \sigma}{\int_{\partial B_s \cap \Omega} |u - v|^q} \notin L^1(+\infty) \tag{1.29}$$

with σ as in (1.1)*. If $\partial\Omega \neq \varnothing$ then $u \le v$ on Ω. If $\partial\Omega = \varnothing$, i.e., $\Omega = M$, and $u - v$ is not constant, then $u \le v$.*

Remark. If Ω is bounded, the result is well-known and it holds true regardless of (1.29).

Proof. We consider first the case where $\partial\Omega \neq \varnothing$. Assume by contradiction that

$$\{x \in \Omega : u(x) > v(x)\} \neq \varnothing.$$

We choose $B > 0$ so small that

$$\Omega_B = \{x \in \Omega : u(x) > v(x) + B\} \neq \varnothing.$$

Since $u \le v$ on $\partial\Omega \neq \varnothing$, $u - v$ is nonconstant on (every connected component of) Ω_B and $\partial\Omega_B \subset \Omega$. We apply Lemma 1.2 with the choices

$$\alpha(s) = s^{q-1}, \quad \beta(s) = (q-1)s^{q-2},\ q > 1, \quad \text{on } [B, +\infty),$$

and with λ satisfying the further requirement $\sup_{\mathbb{R}} \lambda = 1$. According to (1.24), for each $r > R \ge R_1 > 0$, we have

$$\begin{aligned} &\frac{1}{\int_{B_R \cap \Omega_B} \sigma \lambda(u-v)(u-v)^{q-2} \left| \frac{\nabla u}{\sqrt{1+|\nabla u|^2}} - \frac{\nabla v}{\sqrt{1+|\nabla v|^2}} \right|^2} \\ &\ge C \int_R^r \frac{\inf_{\partial B_s \cap \Omega_B} \sigma}{\int_{\partial B_s \cap \Omega_B} (u-v)^q} ds \\ &\ge C \int_R^r \frac{\inf_{\partial B_s \cap \Omega} \sigma}{\int_{\partial B_s \cap \Omega} |u-v|^q} ds, \end{aligned}$$

with $C = (q-1)^2$. Letting $r \to +\infty$ this contradicts (1.29).

Next, we consider the case $\partial\Omega = \varnothing$, that is, $\Omega = M$. Again, we argue by contradiction and assume that $\{x \in M : u(x) > v(x)\} \neq \varnothing$ and that $u - v$ is nonconstant on M. We choose $B > 0$ in such a way that $\Omega_B = \{x : u(x) > v(x) + B\}$ is not empty with (possibly empty) boundary $\partial\Omega_B$. Note that $u - v$ is nonconstant on every connected component of Ω_B. Indeed, this is obvious if $\partial\Omega_B = \varnothing$, for then $\Omega_B = M$, and follows from the definition of Ω_B otherwise. A contradiction is reached as above, applying Lemma 1.2. □

Proof of Theorem B. We apply Theorem 1.3 and use the following chain of implications whose proof is left as an exercise.

(1.30)
$$\begin{cases} \text{Let } F : [0, +\infty) \to [0, +\infty) \text{ be such that } \frac{1}{rF(r)} \notin L^1(+\infty) \text{ and either} \\ \liminf_{r\to+\infty} \frac{\int_{B_r\cap\Omega}|u-v|^q}{r^2} < +\infty, \quad \text{or } \limsup_{r\to+\infty} \frac{\int_{B_r\cap\Omega}|u-v|^q}{r^2F(r)} < +\infty. \end{cases}$$

$$\frac{r}{\int_{B_r\cap\Omega} |u-v|^q} \notin L^1(+\infty). \tag{1.31}$$

$$\frac{1}{\int_{\partial B_r\cap\Omega} |u-v|^q} \notin L^1(+\infty). \tag{1.32}$$

Then (1.30) implies (1.31) which implies (1.32). □

The assumption $q > 1$ in Theorem 1.3 is sharp. This will be discussed in Section 2. However, for $q = 1$, we have the following:

Theorem 1.4. *Let Ω be an unbounded domain in M, and let $u, v \in \mathcal{C}^2(\Omega) \cap \mathcal{C}^0(\overline{\Omega})$ satisfy* (1.28). *Assume that, for some constants $b, C > 0$ and every sufficiently large r we have*

$$\text{i)} \int_{\partial B_r\cap\Omega} |u-v| \leq C\frac{1}{r\,(\log r)^b}; \qquad \text{ii)}\ |u-v| \leq Ce^{r(x)^2}. \tag{1.33}$$

If $\partial\Omega \neq \varnothing$ then $u \leq v$ on Ω. If $\partial\Omega = \varnothing$, that is $\Omega = M$, and $u - v$ is not constant, then $u \leq v$.

Proof. The argument resembles that of Theorem 1.3. We thus consider only the case $\partial\Omega \neq \varnothing$. We apply Lemma 1.2 with

$$\alpha(s) = \{\log[1 + \log(1+s)]\}^b, \quad \beta(s) = \frac{b\,\{\log[1 + \log(1+s)]\}^{b-1}}{[1 + \log(1+s)](1+s)} = \alpha'(s),$$

on $[B,+\infty)$, and with λ satisfying $\sup_{\mathbb{R}}\lambda = 1$ to obtain

(1.34)
$$\left(\int_{B_R\cap\Omega_B}\frac{\sigma\lambda(u-v)\left|\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}-\frac{\nabla v}{\sqrt{1+|\nabla v|^2}}\right|^2}{(1+u-v)\left[1+\log(1+u-v)\right]\left\{\log\left[1+\log(1+u-v)\right]\right\}^{1-b}}\right)^{-1}$$
$$\geq C\int_R^r \frac{\inf_{\partial B_s\cap\Omega_B}\sigma\; ds}{\int_{\partial B_s\cap\Omega_B}(u-v)\left[1+\log(1+u-v)\right]\left\{1+\left[\log(1+\log(1+u-v))\right]^{1+b}\right\}}$$

for $r > R \geq R_0$ sufficiently large. Now, using (1.33) i) and ii), we deduce that
$$\int_{\partial B_s\cap\Omega_B}(u-v)\left[1+\log(1+u-v)\right]\left\{1+\left[\log(1+\log(1+u-v))\right]^{1+b}\right\}$$
$$\leq Cs\log s,$$

for s large. A contradiction is obtained by taking r sufficiently large in (1.34). □

Our next result relates to the conclusion of Theorem 1.3. We stress that we do not assume "a priori" that the graphs Γ_u and Γ_v, associated to u and v, have the same mean curvature.

Theorem 1.5. *Let $u, v \in \mathcal{C}^2(M)$ be such that*

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) \geq \operatorname{div}\left(\frac{\nabla v}{\sqrt{1+|\nabla v|^2}}\right), \quad \text{on } M. \tag{1.35}$$

Assume that

$$\frac{1}{\operatorname{vol}(\partial B_r)} \notin L^1(+\infty). \tag{1.36}$$

If

$$u \leq v, \quad \text{on } M. \tag{1.37}$$

then, $u = v + A$ on M for some constant $A \leq 0$.

Proof. Assume, by contradiction, that $u - v$ is nonconstant on Ω. The required contradiction follows applying the remark following the proof of Lemma 1.2 with $\alpha(t) = \beta(t) = e^t$, and using (1.36) and (1.37). □

Because of Condition (1.29), Theorem 1.3 can be naturally regarded as an L^q-comparison result. In the literature we can find L^∞-versions of this result, where (1.29) is replaced by a condition involving $\|u-v\|_\infty$. In a sense, Theorem 1.5 goes in this direction. To the best of our knowledge the most general result of this kind is Theorem 3 in [**Hw1**], which improves on

Collin and Krust, [**CK**], Hwang, [**Hw**], Mikljukov, [**M**], and Langevin and Rosenberg, [**LR**]. All these results pertain to unbounded domains of $\mathbb{R}^2$. As a matter of fact they can be extended to domains in a complete manifold $(M, \langle , \rangle)$. We briefly describe the argument, which is based on Lemma 1.2 and on a simple companion estimate. In (1.24) of Lemma 1.2, we obtain an upper bound for the quantity

$$\int_{B_R \cap \Omega_B} \sigma\lambda(u-v)\beta(u-v) \left| \frac{\nabla u}{\sqrt{1+|\nabla u|^2}} - \frac{\nabla v}{\sqrt{1+|\nabla v|^2}} \right|^2 .$$

Reasoning in a similar way we can deduce a lower bound of the form

$$\tag{1.38} \begin{aligned} &\int_{(B_r \setminus B_R) \cap \Omega_B} \sigma\lambda(u-v)\beta(u-v) \left| \frac{\nabla u}{\sqrt{1+|\nabla u|^2}} - \frac{\nabla v}{\sqrt{1+|\nabla v|^2}} \right|^2 \\ &\geq C \int_R^r \frac{\inf_{\partial B_s \cap \Omega} \sigma}{\int_{\partial B_s \cap \Omega} \beta(u-v)\lambda(u-v)} \end{aligned}$$

valid for $r > R \geq R_2 > 0$ sufficiently large and appropriate λ and β.

Combining (1.24) and (1.38) with a suitable choice of α, β, λ, and proceeding as in the proof of Theorem 3 of [**Hw1**], we obtain the following:

Theorem 1.6. *Let $\Omega \subseteq (M, \langle , \rangle)$ be an unbounded domain, and let $u, v \in \mathcal{C}^2(\Omega) \cap \mathcal{C}^0(\overline{\Omega})$ satisfy* (1.28). *Assume that*

$$\frac{\inf_{\partial B_r \cap \Omega} \sigma}{\mathrm{vol}\,(\partial B_r \cap \Omega)} \notin L^1(+\infty)$$

and that

$$\tag{1.39} \liminf_{r \to +\infty} \frac{\sup_{\partial B_r \cap \Omega} (u-v)}{\int_R^r \frac{\inf_{\partial B_r \cap \Omega} \sigma}{\mathrm{vol}\,(\partial B_r \cap \Omega)} ds} = 0,$$

for some $R > 0$ sufficiently large. If $\partial\Omega \neq \varnothing$, then, $u \leq v$ on Ω. If $\Omega = M$, and $u - v$ is not constant, then $u \leq v$.

Remark. Replacing (1.28) with (0.3) and (1.39) with

$$\liminf_{r \to +\infty} \frac{\sup_{\partial B_r \cap \Omega} |u-v|}{\int_R^r \frac{\inf_{\partial B_r \cap \Omega} \sigma}{\mathrm{vol}\,(\partial B_r \cap \Omega)} ds} = 0,$$

Theorem 1.6 yields a uniqueness result which extends to manifolds Theorem 3 of [**Hw1**] and Théorème 2 of [**CK**].

2. Some examples.

In this section we describe examples showing that Theorems A and 1.3 are fairly sharp.

Towards this aim, let $g(r) \in \mathcal{C}^{\infty}([0,+\infty))$ be such that

(2.1)
i) $g(r) > 0$ for $r > 0$; ii) $g'(0) = 1$; iii) $g^{(2k)}(0) = 0$, for each $k = 0, 1, \dots$

On $\mathbb{R}^m \setminus \{0\} = (0,+\infty) \times S^{m-1}$, $m \geq 2$, we define the metric

$$\langle , \rangle = dr^2 + g(r)^2 d\theta^2$$

with $d\theta^2$ the standard metric on S^{m-1}. Because of (2.1) ii), iii) we can extend $\langle , \rangle$ to a smooth, obviously complete, metric on $\mathbb{R}^m$. Given a function $a(r) \in \mathcal{C}^{\infty}([0,+\infty))$, even at 0, that is, $a^{(2k+1)}(0) = 0$, $k = 0, 1, 2, \dots$, we let

$$h(t) = g(t)^{1-m} \int_0^t g(y)^{m-1} a(y)\, dy. \tag{2.2}$$

If

$$h(t) < 1, \quad \text{on } [0,+\infty), \tag{2.3}$$

we may define

$$\beta(r) = \beta_0 + \int_0^r \frac{h(t)}{\sqrt{1-h(t)^2}}\, dt, \quad \beta_0 \in \mathbb{R},$$

and the function

$$u(x) = \beta(r(x)) \tag{2.4}$$

is smooth on $\mathbb{R}^m$ and defines a graph Γ_u with mean curvature $\frac{1}{m} a(r(x))$.

For the sake of simplicity, in what follows we consider the case $m = 2$.

The first examples refer to Theorem A. We begin by showing that the boundedness of u is necessary to conclude.

(1) Let $g(r) = r$ on $[0,+\infty)$. Since $m = 2$,

$$\operatorname{vol}(\partial B_r) = 2\pi r$$

and (i) holds with $\alpha = 1$, $C = 1$. Let

$$a(r) = \frac{1}{(e+r^2)\log(e+r^2)} \sim \frac{1}{2r^2 \log r}, \quad \text{as } r \to +\infty,$$

so that (j) is not satisfied. With the previous notations

$$h(t) = \frac{1}{2t} \log\log(e+t^2),$$

and it is a calculus exercise to verify that (2.3) is satisfied. An easy computation shows that $u(x) \sim [\log r(x)][\log\log r(x)]$, as $r(x) \to +\infty$, and (0.1) is not satisfied.

(2) We choose $g(r) = re^{\sqrt{1+r^2}-1}$. It is readily seen that requirements (2.1) are satisfied.

$$\mathrm{vol}\,(\partial B_r) \sim 2\pi e^{-1}\, re^{r}, \quad \text{as } r \to +\infty$$

and it satisfies (ii) for any $\alpha > 1$, $C > 0$ sufficiently large. Let

$$a(r) = \frac{1}{\sqrt{1+r^2}} \sim \frac{1}{r}, \quad \text{as } r \to +\infty,$$

so that (jj) is not satisfied. Next,

$$h(t) = \frac{e^{\sqrt{1+t^2}-1} - 1}{te^{\sqrt{1+t^2}-1}},$$

and one checks that (2.3) is satisfied and $h(t) \to 0$, as $t \to +\infty$. Here $u(x) \sim \log r(x)$, as $r(x) \to +\infty$, and (0.1) is not satisfied.

(3) Let $g(r) = re^{r^2}$. Then Conditions (2.1) are met, and since

$$\mathrm{vol}\,(\partial B_r) = 2\pi r e^{r^2},$$

(iii) is satisfied for $\alpha > 1$, $C = 2\pi$. Let $a(r) \equiv 1$. so that (jjj) is not satisfied and

$$h(t) = \frac{1}{2}\,\frac{e^{t^2} - 1}{te^{t^2}}\,.$$

Again, (2.3) is satisfied and since $u(x) \sim \log r(x)$, as $r(x) \to +\infty$, (0.1) is not met.

The next example shows that the volume growth conditions in the statement of the theorem are almost optimal. We consider only the growth Condition (iii), leaving the remaining cases to the interested reader.

(4) Let $g(r) = \frac{1}{2}\frac{d}{dr}e^{r^2\left\{1+\left[\log(1+r^2)\right]^2\right\}}$. Clearly, (2.1) holds. Since

$$\mathrm{vol}\,(\partial B_r) \sim Cre^{4r^2(\log r)^2}\left(\log r\right)^2, \quad \text{as } r \to +\infty,$$

for some constant $C > 0$, (iii) is not satisfied for any choice of $\alpha > 0$. In this case, we choose $a(r) \equiv 1$ so that (jjj) fails. Furthermore,

$$h(t) = \frac{1}{2}g(t)^{-1}\left[e^{t^2\left\{1+\left[\log(1+t^2)\right]^2\right\}} - 1\right] \sim \frac{C}{t\left(\log t\right)^2}, \quad \text{as } t \to +\infty.$$

Thus, $h(t) \to 0$ as $t \to +\infty$ and $h(t) \in L^1(+\infty)$. Again a calculus exercise shows the validity of (2.3). Furthermore, $u(x) = \beta(r(x))$ is nonnegative and bounded above, so that (0.1) is satisfied.

The next three examples show that the decay rate conditions for a in Theorem A are almost optimal.

(5) We fix any $\mu > 0$ and we set $g(r) = r\left[\log\left(e + r^2\right)\right]^{1+\frac{\mu}{2}}$. Clearly, (2.1) is satisfied. Also,

$$\mathrm{vol}\left(\partial B_r\right) \sim \pi 2^{2+\frac{\mu}{2}}\ r\left(\log r\right)^{1+\frac{\mu}{2}}, \quad \text{as } r \to +\infty$$

so that Condition (i) holds true. We let

$$a(r) = \frac{C_0}{(e + r^2)\left[\log(e + r^2)\right]^{2+\mu}},$$

where $C_0 > 0$ is a suitable constant. A computation shows that

$$h(t) = \frac{C_0}{\mu}\ \frac{1 - \left[\log(e + t^2)\right]^{-\frac{\mu}{2}}}{t\left[\log(e + t^2)\right]^{1+\frac{\mu}{2}}},$$

and we can choose $C_0 > 0$ so small that (2.3) is met. Moreover, since $h(t) \to 0$, as $t \to +\infty$ and $h(t) \in L^1(+\infty)$ we have

$$\beta_0 \le u \le \sup_{\mathbb{R}^2} u < +\infty, \quad \beta_0 \in \mathbb{R}.$$

Condition (0.1) is satisfied.

(6) Let $g(r) = re^{\sqrt{1+r^2}-1}\log(e + r^2)[\log(\log(e^2 + r^2))]^2$, so that both (2.1) and the volume growth Condition (ii) are satisfied. Let also

$$a(r) = \frac{1}{\sqrt{1 + r^2}\log(e + r^2)\left[\log\left(\log(e^2 + r^2)\right)\right]^2}$$

and note that

$$\lim_{r(x)\to+\infty} \frac{a(r(x))}{\left\{r(x)\log r(x)\left[\log\left(\log r(x)\right)\right]^2\right\}^{-1}} > 0.$$

A computation shows that

$$h(t) = \frac{1 - e^{-\sqrt{1+t^2}+1}}{t\log(e + t^2)\left[\log\left(\log(e^2 + t^2)\right)\right]^2}.$$

Thus, (2.3) is satisfied. Furthermore, $h(t) \to 0$ as $t \to +\infty$ and $h(t) \in L^1(+\infty)$, so that,

$$\beta_0 \le u \le \sup_{\mathbb{R}^2} u < +\infty, \quad \beta_0 \in \mathbb{R},$$

and Condition (0.1) holds.

(7) Let $g(r) = re^{r^2}\log(e + r^2)\left[\log(\log(e^2 + r^2))\right]^2$, so that (2.1) and the volume growth Condition (iii) are satisfied, and define

$$a(r) = \frac{1}{\log(e + r^2)\left[\log\left(\log(e^2 + r^2)\right)\right]^2}$$

so that

$$\lim_{r(x)\to+\infty} \frac{a(r(x))}{\left\{\log r(x)\left[\log\left(\log r(x)\right)\right]^2\right\}^{-1}} > 0.$$

A computation shows that

$$h(t) = \frac{1-e^{-t^2}}{t\log(e+t^2)\left[\log\left(\log(e^2+t^2)\right)\right]^2},$$

so that (2.3) holds true. Moreover $h(t) \to 0$ as $t \to +\infty$ and $h(t) \in L^1(+\infty)$ and therefore u is bounded.

We now consider Theorem 1.3.

(1) Let $(M, \langle,\rangle) = (\mathbb{R}^2, can)$, $\Omega = \mathbb{R}^2 \backslash B_1(0)$, $v(x) \equiv 0$, $u(x) = \operatorname{arccosh} r(x)$. Then, Γ_u is minimal on $\overline{\Omega}$, $u(x) \equiv v(x) \equiv 0$ on $\partial\Omega$. Since $\operatorname{arccosh} r \sim \log r$ as $r \to +\infty$, for any $q > 1$ there exists $r \gg 1$ and $C > 0$ such that

$$\frac{1}{\int_{\partial B_r \cap \Omega} |u-v|^q} \le \frac{C}{r(\log r)^q} \in L^1(+\infty).$$

Thus, (1.29) is not satisfied. Here, $\partial\Omega \neq \varnothing$ but $u \le v$ on Ω is false.
(2) Let $g(r) \in \mathcal{C}^\infty([0,+\infty))$, $g(r) > 0$ for $r > 0$, $g'(r) \ge 0$ and

$$g(r) = \left\{ \begin{array}{ll} r & \text{on } [0,\frac{1}{2}] \\ r^{\eta-1} & \text{on } [1,+\infty) \end{array} \right. , \ 1 \le \eta < 2,$$

so that

$$\operatorname{vol}(\partial B_r) = 2\pi r^{\eta-1}.$$

Let also $a(r) \in \mathcal{C}^\infty([0,+\infty))$, be such that $a(r) \ge 0$ and

$$a(r) = C\left\{ \begin{array}{ll} 1 & \text{on } [0,1] \\ 0 & \text{on } (2,+\infty). \end{array} \right.$$

By taking C sufficiently small we may arrange that, having defined $h(t)$ as in (2.2), (2.3) holds. Then the function $u(x) = \beta(r(x))$ defined by (2.4) is smooth on $\mathbb{R}^2$, and defines a graph with nonnegative mean curvature (equal to $a(x)/m$). It is easily verified that

$$u(x) \sim Cr(x)^{2-\eta},$$

for some $C > 0$. Thus, if $v \equiv 0$, for each $q > 1$ we have

$$\frac{1}{\int_{\partial B_r} (u-v)^q} \le \frac{C}{r^{q(2-\eta)+\eta-1}} \in L^1(+\infty).$$

Here, $\partial\Omega = \varnothing$ since $\Omega = \mathbb{R}^2$. We also note, in this last example, that

$$\frac{1}{\int_{\partial B_r} (u-v)} \notin L^1(+\infty)$$

so that, Condition (1.29) in Theorem 1.3 cannot be improved to $q \geq 1$.

We would like to conclude with the following observation. In a recent paper, Alencar and do Carmo, [**AdC**], prove that if $f : M \to (N, h)$ is an isometrically immersed complete, noncompact hypersurface with constant mean curvature and of at most polynomial growth, then f is minimal provided $\mathrm{Ricc}_{(N,h)} \geq 0$ and a certain operator has finite index (graphs satisfy this property).

It seems an interesting problem to extend Theorem A in this direction.

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