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In this note, we compute the Tian's $\alpha_G(M)$ -invariant on $\mathbb{C}\mathbb{P}^2\#2\overline{\mathbb{C}\mathbb{P}^2}$. Our result is an improvement of Abdesselem's result in Abdesselem (1997). As a consequence, we obtain a good estimate of Ricci curvature on $\mathbb{C}\mathbb{P}^2\#2\overline{\mathbb{C}\mathbb{P}^2}$ by studying certain complex Monge–Ampère equation.

1. Introduction.

It is well-known that the $\alpha_G(M)$ -invariant introduced by Tian plays an important role in the study of the existence of Kähler-Einstein metrics on complex manifolds with positive first Chern class ([T1], [T2], [TY]). Based on the estimate of $\alpha_G(M)$ -invariant, Tian in 1990 proved that any complex surface with $c_1(M) > 0$ always admits a Kähler-Einstein metric except in two cases $\mathbb{C}\mathbb{P}^2\#\overline{\mathbb{C}\mathbb{P}^2}$ and $\mathbb{C}\mathbb{P}^2\#2\overline{\mathbb{C}\mathbb{P}^2}$, i.e., the blow-ups of $\mathbb{C}\mathbb{P}^2$ at one point and two points respectively ([T2]). Instead of Kähler-Einstein metric, Koiso constructed a Kähler-Ricci soliton on $\mathbb{C}\mathbb{P}^2\#\overline{\mathbb{C}\mathbb{P}^2}$ ([Ko]). But it is still unknown that there is a Kähler-Ricci soliton on $\mathbb{C}\mathbb{P}^2\#2\overline{\mathbb{C}\mathbb{P}^2}$ or not. Recently, the author studied a sufficient condition for the existence of Kähler-Ricci soliton on a complex manifold with $c_1(M) > 0$ in the sense of Tian's $\alpha_G(M)$ -invariant ([Zh]). In this note, we compute the Tian's $\alpha_G(M)$ -invariant on $\mathbb{C}\mathbb{P}^2\#2\overline{\mathbb{C}\mathbb{P}^2}$ and wish that our estimate was an important step towards finding the Kähler-Ricci soliton on $\mathbb{C}\mathbb{P}^2\#2\overline{\mathbb{C}\mathbb{P}^2}$. Kähler-Ricci soliton can be regarded as a good replacement when a Kähler manifold with $c_1(M) > 0$ doesn't admit a Kähler-Einstein metric ([Ca], [Ha]). The uniqueness problem of such metrics was solved by Tian and the author recently ([TZ1], [TZ2], [TZ3]). Our result is also an improvement of Abdesselem's result ([Ab]). As a consequence, we obtain a good estimate of Ricci curvature on $\mathbb{C}\mathbb{P}^2\#2\overline{\mathbb{C}\mathbb{P}^2}$ by studying certain complex Monge-Ampère equation.

2. Reduction to a local estimate.

Let M be the blow-up of $\mathbb{C}\mathbb{P}^2$ at two points and let π be its natural projection. Without loss of generality, we may assume the two points $p_1 = [0, 0, 1]$ and $p_2 = [0, 1, 0]$. Then $M \setminus (\pi^*p_1 \cup \pi^*p_2)$ is isomorphic to $\mathbb{C}\mathbb{P}^2 \setminus (\{p_1\} \cup \{p_2\})$.

If we choose an inhomogeneous coordinates $(z_1, z_2) = [1, z_1, z_2]$ of $\mathbb{C}P^2$, the Kähler metric

$$\omega_{g_0} = \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial}(\log(1 + |z_1|^2) + \log(1 + |z_2|^2) + \log(1 + |z_1|^2 + |z_2|^2))$$

can be extended to a Kähler metric g on M which belongs to $c_1(M)$. Clearly, if we take the transformation of inhomogeneous coordinates $\rho_1 : (w_2, w_1) = [w_2, w_1, 1] \rightarrow (z_1, z_2) = [1, z_1, z_2]$, i.e., $z_1 = \frac{w_1}{w_2}, z_2 = \frac{1}{w_2}$, then we get a Kähler metric on $\mathbb{C}^2 \setminus \{(0, 0)\}$, given by

$$\omega_{g_1} = \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial}(\log(1 + |w_2|^2) + \log(|w_1|^2 + |w_2|^2) + \log(1 + |w_1|^2 + |w_2|^2)).$$

Similarly, after the transformation of inhomogeneous coordinates $\rho_2 : (w_2, w_1) = [w_2, 1, w_1] \rightarrow (z_1, z_2) = [1, z_1, z_2]$, i.e., $z_1 = \frac{1}{w_2}, z_2 = \frac{w_1}{w_2}$, then we also get a Kähler metric on $\mathbb{C}^2 \setminus \{0, 0\}$, given by

$$\omega_{g_2} = \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial}(\log(1 + |w_2|^2) + \log(|w_1|^2 + |w_2|^2) + \log(1 + |w_1|^2 + |w_2|^2)).$$

Let $\gamma_{j,\theta}(j = 0, 1, 2)$ and σ_0 be automorphisms of $\mathbb{C}P^2$ given by,

$$\begin{aligned} \gamma_{j,\theta} &: [z_0, z_j, z_2] \rightarrow [z_0, e^{i\theta} z_j, z_2], \\ \sigma_0 &: [z_2, 1, z_1] \rightarrow [z_2, z_1, 1]. \end{aligned}$$

Then $\gamma_{j,\theta}$ and σ_0 generalize a maximal compact subgroups G of automorphisms group of M . Let

$$P_G(M, g) = \left\{ \phi \in C^\infty(M) \mid \omega_g + \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial}\phi > 0, \sup_M \phi = 0, \right. \\ \left. \text{and } \phi \text{ is } G\text{-invariant} \right\}.$$

In [T1], Tian introduced a holomorphic invariant

$$\alpha_G(M) = \sup \left\{ \alpha \mid \int_M e^{-\alpha\phi} dv_g \leq C(\alpha), \forall \phi \in P_G(M, g) \right\},$$

which is independent of the choice of Kähler form ω_g . In this note, we shall estimate the number of $\alpha_G(M)$.

Let $x_i (i = 1, 2) = |z_i|^2$ (resp. $y_i = |w_i|^2$). Then any G -invariant function is of form $\phi(x_1, x_2)$ and the integral can be divided into three parts,

$$\begin{aligned} \int_M e^{-\alpha\phi} dv_g &= \int_{0 \leq x_1 \leq 1, 0 \leq x_2 \leq 1} e^{-\alpha\phi} dv_{g_0} \\ &\quad + \int_{0 < y_1 \leq 1, 0 < y_2 \leq 1} e^{-\alpha\phi} dv_{g_1} + \int_{0 < y_1 \leq 1, 0 < y_2 \leq 1} e^{-\alpha\phi} dv_{g_2}. \end{aligned}$$

So it suffices to estimate each of these three parts of the integral. Note that the computation of part three of the integral is similar to part two.

Let $K_0(x_1, x_2) = \log(1+x_1+x_2) + \log(1+x_1) + \log(1+x_2)$ and $K(x_1, x_2) = K_0(x_1, x_2) + \phi(x_1, x_2)$. Then functions $x_i \frac{\partial K(x_1, x_2)}{\partial x_i}$ ($i = 1, 2$) are both strictly increasing for variable $x_i \in [0, +\infty)$. Clearly,

$$x_1 \frac{\partial K_0(x_1, x_2)}{\partial x_1} = \frac{x_1}{1+x_1+x_2} + \frac{x_1}{1+x_1},$$

$$x_2 \frac{\partial K_0(x_1, x_2)}{\partial x_2} = \frac{x_2}{1+x_1+x_2} + \frac{x_2}{1+x_2}.$$

Since

$$x_1 \frac{\partial \phi(x_1, x_2)}{\partial x_1} \Big|_{x_1=+\infty} = 0,$$

$$x_2 \frac{\partial \phi(x_1, x_2)}{\partial x_2} \Big|_{x_2=+\infty} = 0,$$

by using the monotonicity, we get

$$(2.1) \quad 0 \leq x_1 \frac{\partial K(x_1, x_2)}{\partial x_1} \leq 2,$$

$$0 \leq x_2 \frac{\partial K(x_1, x_2)}{\partial x_2} \leq 2.$$

Furthermore, we have:

Lemma 2.1.

$$\frac{\partial K}{\partial x_1} \leq \frac{3}{2x_1}, \quad x_1 \leq x_2;$$

$$\frac{\partial K}{\partial x_2} \leq \frac{3}{2x_2}, \quad x_2 \leq x_1.$$

Proof. Since ϕ is G -invariant, by the transformation, $w_1 = \frac{z_1}{z_2}, w_2 = \frac{1}{z_2}$, i.e.,

$$y_1 = \frac{x_1}{x_2}, y_2 = \frac{1}{x_2},$$

we have $\phi(y_1, y_2) = \phi\left(\frac{1}{y_1}, \frac{y_2}{y_1}\right)$ (for simplicity, we still use $\phi(y_1, y_2)$ to mean $\phi(x_1(y_1, y_2), x_2(y_1, y_2))$ here; similarly, $K(y_1, y_2)$ and $K_0(y_1, y_2)$ will denote $K(x_1(y_1, y_2), x_2(y_1, y_2))$ and $K_0(x_1(y_1, y_2), x_2(y_1, y_2))$, respectively). It follows

$$(2.2) \quad 2\partial_1\phi(1, y_2) + y_2\partial_2\phi(1, y_2) = 0,$$

and

$$2\partial_1K(1, y_2) + y_2\partial_2K(1, y_2) = 2\partial_1K_0(1, y_2) + y_2\partial_2K_0(1, y_2) = 3.$$

On the other hand, by using the convexity of K , one can check the function with variable u ,

$$u \frac{d}{du} K(u^2y_1, uy_2) = 2u^2y_2\partial_1K(u^2y_1, uy_2) + uy_2\partial_2K(u^2y_1, uy_2)$$

is strictly increasing (cf. [Re]). Hence we obtain that for any $y_1 \leq 1$,

$$2y_1 \partial_1 K(y_1, y_2) + y_2 \partial_2 K(y_1, y_2) \leq 2\partial_1 K(1, y_2) + y_2 \partial_2 K(1, y_2) = 3.$$

In particular, for any $0 < y_1 \leq 1$, we have

$$(2.3) \quad \partial_1 K(y_1, y_2) \leq \frac{3}{2y_1}.$$

Since

$$\frac{\partial K(x_1, x_2)}{\partial x_1} = \frac{1}{x_2} \frac{\partial K(y_1, y_2)}{\partial y_1},$$

by (2.3), we get

$$\frac{\partial K(x_1, x_2)}{\partial x_1} \leq \frac{3}{2x_1}, \quad x_1 \leq x_2.$$

On the other hand, by using the symmetry of $K(x_1, x_2)$ for variables x_1 and x_2 , we have

$$\frac{\partial K(x_1, x_2)}{\partial x_2} = \frac{\partial K(x_2, x_1)}{\partial x_2} = \frac{1}{x_1} \frac{\partial K(y_1, y_2)}{\partial y_1}.$$

Again by (2.3), we get

$$\frac{\partial K}{\partial x_2} \leq \frac{3}{2x_2}, \quad x_2 \leq x_1.$$

□

Lemma 2.2. *Let $C_1 = \{[z_0, 1, 0]\}, C_2 = \{[0, 1, z_2]\}, C_3 = \{[z_0, 0, 1]\}$ be three lines of \mathbb{CP}^2 . Then $\phi \in P_G(M, g)$ are uniformly locally bounded away from the set of five curves $\cup_{i=1}^3 \pi^* C_i \cup_{i=0}^2 \pi^* p_i$.*

Proof. Since ϕ are almost subharmonic functions, by the normalization condition $\sup_M \phi = 0$, one sees that there is a subset $K \subset [0, 2] \times [0, 2]$ with Lebesgue measure bigger than 1 such that ϕ are uniform bounded on K . Then by (2.1), it is easy to see that $\phi(x_1, x_2)$ are uniform locally bounded on $[0, 2] \times [0, 2] \setminus ((x_1, 0) \cup (0, x_2))$. On the other hand, similar to (2.1), we have

$$0 \leq y_1 \frac{\partial K(y_1, y_2)}{\partial y_1} \leq 2,$$

$$0 \leq y_2 \frac{\partial K(y_1, y_2)}{\partial y_2} \leq 3.$$

Hence we can also prove that $\phi(y_1, y_2)$ are uniform locally bounded on $[0, 2] \times [0, 2] \setminus ((y_1, 0) \cup (0, y_2))$. This completes the proof of lemma. □

Proposition 2.1. *For any $\alpha < \frac{4}{7}$, there is a uniform C such that*

$$\int_{0 < x_1 \leq 1, 0 < x_2 \leq 1} e^{-\alpha \phi} dv_{g_0} \leq C.$$

Proof. Let $(x_1, x_2) \in S = \{0 \leq x_1 \leq 1, 0 \leq x_2 \leq 1, x_1 \leq x_2\}$. Then by Lemma 2.1, we have

$$\begin{aligned} -K(x_1, x_2) &= \int_{x_2}^1 \partial_2 K(x_1, y) dy + \int_{x_1}^1 \partial_1 K(x_1, 1) dx - K(1, 1) \\ &\leq -\frac{3}{2} \ln x_1 - 2 \ln x_2 - K(1, 1). \end{aligned}$$

Similarly, if $(x_1, x_2) \in S' = \{0 \leq x_1 \leq 1, 0 \leq x_2 \leq 1, x_2 \leq x_1\}$, we have

$$-K(x_1, x_2) \leq -\frac{3}{2} \ln x_2 - 2 \ln x_1 - K(1, 1).$$

Since $dv_{g_0} \leq C_1 dx_1 \wedge dx_2 \wedge d\Theta$ (where $d\Theta = d\theta_1 \wedge d\theta_2$, $0 \leq \theta_1 \leq 2\pi, 0 \leq \theta_2 \leq 2\pi$), we have

$$\begin{aligned} &\int_{0 \leq x_1 \leq 1, 0 \leq x_2 \leq 1} e^{-\alpha\phi} dv_{g_0} \\ &\leq C_2 \left(\int_S + \int_{S'} \right) e^{-\alpha K(x_1, x_2)} dx_1 dx_2 \\ &\leq 2C_2 \int_0^1 \int_0^{x_2} x_1^{-\frac{3\alpha}{2}} x_2^{-2\alpha} dx_1 dx_2 \\ &= \frac{4}{2-3\alpha} C_2 \int_0^1 x_2^{1-\frac{3\alpha}{2}-2\alpha} dx_2. \end{aligned}$$

Clearly, if $\alpha < \frac{4}{7}$, we get

$$\int_{0 \leq x_1 \leq 1, 0 \leq x_2 \leq 1} e^{-\alpha\phi} dv_{g_0} \leq C.$$

□

3. Blow-up transformation.

Lemma 3.1. *Let $k < 1$ be a positive number and $\Delta_k = \{0 < y_1 \leq k, 0 < y_2 \leq 1, \text{ and } y_1 \leq ky_2\}$. Then for any $(y_1, y_2) \in \Delta_k$, it holds*

$$y_2 \partial_2 K(y_1, y_2) \leq \frac{3}{2} + \frac{2+3k}{4+2k} + \frac{k}{2+2k}.$$

Proof. Make transformation $y'_1 = y_2 \leq 1, y'_2 = \frac{y_1}{y_2} \leq k$. Then $y_1 = y'_1 y'_2, y_2 = y'_1$. Moreover, one can check

$$\phi(y'_1, y'_2) = \phi\left(\frac{1}{y'_1}, \frac{y'_2}{y'_1}\right).$$

Hence

$$(3.1) \quad 2\partial_1 \phi(1, y'_2) + y'_2 \partial_2 \phi(1, y'_2) = 0.$$

Since

$$\begin{aligned}\tilde{K}_0(y'_1, y'_2) &= \log y'_1 + \log(1 + y'_1) + \log(1 + y'_2) + \log(1 + y'_1 y'_2 + y'_1) \\ &= \log y'_1 + K_0(y'_1, y'_2),\end{aligned}$$

then

$$\begin{aligned}(3.2) \quad y_2 \partial_2 K(y_1, y_2) &= y'_1 \frac{\partial y'_1}{\partial y_2} \partial_1 \tilde{K}(y'_1, y'_2) + y_2 \frac{\partial y'_2}{\partial y_2} \partial_2 \tilde{K}(y'_1, y'_2) \\ &= y'_1 \partial_1 K(y'_1, y'_2) - y'_2 \partial_2 K(y'_1, y'_2) + 1 \\ &\leq y'_1 \partial_1 K(y'_1, y'_2) + 1 \\ &\leq \partial_1 K(1, y'_2) + 1 \\ &\leq \partial_1 K(1, y'_2) + \frac{y'_2}{2} \partial_2 K(1, y'_2) + 1.\end{aligned}$$

On the other hand,

$$\begin{aligned}(3.3) \quad 2\partial_1 K_0(1, y'_2) + y'_2 \partial_2 K_0(1, y'_2) \\ = 1 + \frac{2 + 3y'_2}{2 + y'_2} + \frac{y'_2}{1 + y'_2}.\end{aligned}$$

Hence combining (3.1), (3.2), (3.3), the lemma is proved. \square

Lemma 3.2. *Let $0 < \delta < \frac{3}{2}$. Then for any $(y_1, y_2) \in \Delta_k$, we have*

$$-K(y_1, y_2) \leq \begin{cases} -\frac{3}{2} \log y_1 - (\frac{3}{2} - \delta) \log y_2 - K(k, 1), & \text{or} \\ -\frac{1}{2}(\frac{3}{2} + \delta) \log y_1 - c_k \log y_2 - K(k, 1), \end{cases}$$

where $c_k = \frac{3}{2} + \frac{2+3k}{4+2k} + \frac{k}{2+2k}$.

Proof. First we assume that $\partial_2 K(k, 1) \geq \frac{3}{2} - \delta$. Then by the fact

$$2k \partial_1 K(k, 1) + \partial_2 K(k, 1) \leq 3,$$

we have

$$k \partial_1 K(k, 1) \leq \frac{1}{2} \left(\frac{3}{2} + \delta \right).$$

By using the monotonicity, we get

$$(3.4) \quad x \partial_1 K(x, 1) \leq \frac{1}{2} \left(\frac{3}{2} + \delta \right), \quad \forall 0 < x \leq k.$$

On the other hand, by Lemma 3.1, for any $0 < y \leq 1$, we have

$$(3.5) \quad \partial_2 K(y_1, y) \leq \frac{1}{y} \left(\frac{3}{2} + \frac{2+3k}{4+k} + \frac{k}{2+2k} \right).$$

Combining (3.4) and (3.5), we get

$$\begin{aligned} & -K(y_1, y_2) \\ &= \int_{y_2}^1 \partial_2 K(y_1, y) dy + \int_{y_1}^k \partial_1 K(x, 1) dx - K(k, 1) \\ &\leq -\frac{1}{2} \left(\frac{3}{2} + \delta \right) \log y_1 - \left(\frac{3}{2} + \frac{2+3k}{4+2k} + \frac{k}{2+k} \right) \log y_2 - K(k, 1). \end{aligned}$$

In the other case of $\partial_2 K(k, 1) < \frac{3}{2} - \delta$, by the monotonicity, we have

$$\partial_2 K(k, y) < \left(\frac{3}{2} - \delta \right) \frac{1}{y}, \quad \forall 0 < y \leq 1.$$

Combining $\partial_1 K(y_1, y_2) \leq \frac{3}{2y_1}$, we get

$$\begin{aligned} & -K(y_1, y_2) \\ &= \int_{y_2}^1 \partial_2 K(y_1, y) dy + \int_{y_1}^k \partial_1 K(x, 1) dx - K(k, 1) \\ &\leq -\frac{3}{2} \log y_1 - \left(\frac{3}{2} - \delta \right) \log y_2 - K(k, 1). \end{aligned}$$

The lemma is proved. □

Lemma 3.3. *Let $k > 1$ and $\overline{\Delta}_k = \{0 < y_1 \leq \frac{1}{2}, 0 < y_2 \leq \frac{1}{2k}, \text{ and } y_1 \leq ky_2\}$. Then for any $(y_1, y_2) \in \overline{\Delta}_k$, we have*

$$\begin{aligned} & -K(y_1, y_2) \\ &< \begin{cases} -\frac{3}{2} \log y_1 - b_1 \log y_2 - K(\frac{1}{2}, \frac{1}{2k}), & \text{if } \frac{1}{2k} \partial_2 K(\frac{1}{2}, \frac{1}{2k}) < b_1 \\ -\frac{1}{2}(3 - b_j) \log y_1 - b_{j+1} \log y_2 \\ \quad -K(\frac{1}{2}, \frac{1}{2k}), & \text{if } b_j \leq \frac{1}{2k} \partial_2 K(\frac{1}{2}, \frac{1}{2k}) < b_{j+1}, \end{cases} \end{aligned}$$

where $b_1 = \frac{2}{3} - \delta$, and $b_{j+1} = 3 - (\frac{1}{2})^j (3 - b_1)$, $j = 1, 2, \dots$.

Proof. The proof is similar to that of Lemma 3.2. We omit it. □

Lemma 3.4. *There are a positive number $\alpha > \frac{1}{2}$ and a uniform constant C such that*

$$\int_{\Delta_{\frac{1}{4}}} e^{-\alpha\phi} dv_{g_1} \leq C,$$

where $\Delta_{\frac{1}{4}} = \{0 < y_1 \leq \frac{1}{4}, 0 < y_2 \leq 1, \text{ and } y_1 \leq \frac{1}{4}y_2\}$.

Proof. Let $c_0 = 2 + \frac{19}{90} < \frac{9}{4}$. Then it is clear that there are two positive numbers $\alpha_0 > \frac{1}{2}$ and δ_0 such that $c_0 - \frac{1}{4} < \frac{1}{\alpha_0} - \frac{1}{2}\delta_0$. We first suppose that for all $(y_1, y_2) \in (0, \frac{1}{4}] \times (0, 1]$,

$$-K(y_1, y_2) \leq \frac{3}{2} \log y_1 - \left(\frac{3}{2} - \delta_0 \right) \log y_2 - K\left(\frac{1}{4}, 1\right).$$

Since

$$dv_{g_1} \leq C_1 e^{-K_0} dy_1 \wedge dy_2 \wedge d\Theta \leq C'_1 (y_1 + y_2)^{-1} dy_1 \wedge dy_2 \wedge d\Theta,$$

we have

$$\begin{aligned} & \int_{0 < y_1 \leq \frac{1}{4}, 0 < y_2 \leq 1} e^{-\alpha\phi} dv_{g_1} \\ & \int_{0 < y_1 \leq \frac{1}{4}, 0 < y_2 \leq 1} e^{-\alpha K} e^{-(1-\alpha)K_0} dy_1 \wedge dy_2 \wedge d\Theta \\ & \leq C_2 \int_0^{\frac{1}{4}} \int_0^1 (y_1 + y_2)^{-(1-\alpha)} y_1^{-\frac{3\alpha}{2}} y_2^{-(\frac{3}{2}-\delta_0)} dy_1 dy_2 \\ & \leq C_3 \int_0^{\frac{1}{4}} \int_0^1 y_1^{-\frac{3\alpha}{2}-\frac{1-\alpha}{s}} y_2^{-(\frac{3}{2}-\delta_0)-\frac{1-\alpha}{t}} dy_1 dy_2, \end{aligned}$$

where s and t are two positive numbers satisfying $\frac{1}{s} + \frac{1}{t} = 1$. By choosing $t < 2$ sufficiently closely to 2, we see that there are positive numbers s, t and $\alpha > \frac{1}{2}$ such that

$$\frac{3\alpha}{2} + \frac{1-\alpha}{s} < 1, \text{ and } \alpha \left(\frac{3}{2} - \delta_0 \right) + \frac{1-\alpha}{t} < 1.$$

Hence we obtain a uniform constant such that

$$(3.6) \quad \int_{0 < y_1 \leq k, 0 < y_2 \leq 1} e^{-\alpha\phi} dv_{g_1} \leq C.$$

By (3.6) and Lemma 3.2, we may assume that for any $(y_1, y_2) \in \Delta_{\frac{1}{4}}$,

$$-K(y_1, y_2) < -\left(\frac{3}{2} + \delta_0 \right) \log y_1 - c_0 \log y_2 - K \left(\frac{1}{4}, 1 \right).$$

Let $p = 1 - \frac{\alpha_0}{2} (\frac{3}{2} + \delta_0) > 0$. Then

$$\begin{aligned} (3.7) \quad & \int_{\Delta_{\frac{1}{4}}} e^{-\alpha_0\phi} dv_{g_1} \\ & \leq C_4 \int_0^1 dy_2 \int_0^{\frac{1}{4}y_2} (y_1 + y_2)^{-(1-\alpha_0)} y_1^{-\frac{\alpha_0}{2}(\frac{3}{2}+\delta_0)} y_2^{-c_0\alpha_0} dy_1 \\ & = \frac{C_4}{p} \int_0^1 dy_2 \int_0^{\frac{1}{4^p}y_2^p} (y_1^p + y_2)^{-(1-\alpha_0)} y_2^{-c_0\alpha_0} dy_1 \\ & \leq \frac{C_5}{p} \int_0^1 dy_2 \int_0^{\frac{1}{4^p}y_2^p} y_1^{-\frac{(1-\alpha_0)}{ps}} y_2^{-\frac{1-\alpha_0}{t}} y_2^{-c_0\alpha_0} dy_1 \\ & \leq C_6 \int_0^1 y_2^{p-\alpha_0c_0-(1-\alpha_0)} dy_2, \end{aligned}$$

where s and t are two positive numbers satisfying $\frac{1}{s} + \frac{1}{t} = 1$. By the choice of numbers δ_0 and α_0 , it is clear $p - \alpha_0 c_0 - (1 - \alpha_0) > -1$. Hence

$$(3.8) \quad \int_{\Delta_{\frac{1}{4}}} e^{-\alpha_0 \phi} dv_{g_1} \leq C.$$

By combining (3.6) and (3.8), the lemma is proved. □

Lemma 3.5. *For any positive number ϵ , there is a uniform constant C depending only on ϵ such that*

$$\int_{\overline{\Delta}_5} e^{-(\frac{1}{2}-\epsilon)\phi} dv_{g_1} \leq C,$$

where $\overline{\Delta}_5 = \{0 < y_1 \leq \frac{1}{2}, 0 < y_2 \leq \frac{1}{10}, \text{ and } y_1 \leq 5y_2\}$.

Proof. From the proof of Lemma 3.4, we may assume that

$$b_j \leq \frac{1}{10} \partial_2 K \left(\frac{1}{2}, \frac{1}{10} \right) < b_{j+1},$$

for some integer j , and

$$-K(y_1, y_2) < -\frac{1}{2}(3 - b_j) \log y_1 - b_{j+1} \log y_2 - K \left(\frac{1}{2}, \frac{1}{10} \right),$$

where $b_{j+1} = 3 - (\frac{1}{2})^j (3 - b_1)$, and $b_1 = \frac{3}{2} - \delta$.

Let $\alpha_0 = \frac{1}{2} - \epsilon$ and $p = 1 - \frac{\alpha_0}{2}(3 - b_j) > 0$. Then one can check $p - \alpha_0 b_{j+1} - (1 - \alpha_0) \geq -1 + \epsilon'$, for some positive number ϵ' depending only on ϵ . Hence similar to (3.7), we get,

$$\int_{\overline{\Delta}_5} e^{-\alpha_0 \phi} dv_{g_1} \leq C \int_0^{\frac{1}{10}} y_2^{p - \alpha_0 b_{j+1} - (1 - \alpha_0)} dy_2 \leq C'.$$

The lemma is proved. □

Lemma 3.6. *There is a positive number $\alpha > \frac{1}{2}$ and a uniform constant C such that*

$$\int_{\Delta'_5} e^{-\alpha \phi} dv_{g_1} \leq C,$$

where $\Delta'_5 = \{0 < y_1 \leq 1, 0 < y_2 \leq \frac{1}{5}, \text{ and } y_1 \geq 5y_2\}$.

Proof. As in the proof of Lemma 3.1, we make a transformation, $y'_1 = y_1, y'_2 = \frac{y_2}{y_1} \leq \frac{1}{5}$. Then $y_1 = y'_1, y_2 = y'_1 y'_2$. Moreover, one can check

$$\phi(y'_1, y'_2) = \phi \left(\frac{1}{y'_1}, \frac{y'_2}{y'_1} \right).$$

Hence

$$2\partial_1 \phi(1, y'_2) + y'_2 \partial_2 \phi(1, y'_2) = 0.$$

Since

$$K_0(y'_1, y'_2) = \log(1 + y'_2) + \log(1 + y'_1 y'_2) + \log(1 + y'_1 y'_2 + y'_1),$$

then, for any $y'_2 \leq \frac{1}{5}$,

$$2\partial_1 K_0(1, y'_2) + y'_2 \partial_2 K_0(1, y'_2) = 1 + \frac{2y'_2}{2 + y'_2} + \frac{4y'_2}{1 + y'_2} < 2.$$

It follows that for any $y'_1 \leq 1$, and $y'_2 \leq \frac{1}{5}$,

$$2y'_1 \partial_1 K(y'_1, y'_2) + y'_2 \partial_2 K(y'_1, y'_2) < 2.$$

In particular, there is a positive number δ such that

$$\partial_1 K(y'_1, y'_2) < \frac{1 - \delta}{y'_1} \text{ and } \partial_2 K(y'_1, y'_2) < \frac{2(1 - \delta)}{y'_2}.$$

Hence one can choose a positive number $\alpha > \frac{1}{2}$ such that

$$\int_{\Delta'_5} e^{-\alpha\phi} dv_{g_1} \leq C_1 \int_0^{\frac{1}{5}} dy'_2 \int_0^1 e^{-\alpha K(y'_1, y'_2)} dy'_1 \leq C.$$

□

Combining Lemma 3.4, Lemma 3.5 and Lemma 3.6, we obtain:

Proposition 3.1. *For any positive number ϵ , there is a uniform constant C depending only on ϵ such that*

$$\int_{0 < y_1 \leq 1, 0 < y_2 \leq 1} e^{-(\frac{1}{2} - \epsilon)\phi} dv_{g_1} \leq C.$$

Proof. By Lemma 3.4, Lemma 3.5 and Lemma 3.6, it suffices to prove that for any $(y_1, y_2) \in (0, 1] \times (0, 1] \setminus (\cup \Delta_{\frac{1}{4}} \cup \overline{\Delta}_5 \cup \Delta'_5)$, $\phi(y_1, y_2)$ is uniformly bounded. This follows from Lemma 2.2 immediately. □

Combining Propositions 2.1 and 3.1, we prove:

Theorem 3.1. *Let M be the blow-up of CP^2 at two points. Then $\alpha_G(M) \geq \frac{1}{2}$.*

Remark 3.1. In [Ab], Abdesselem proved $\alpha(M) \geq \frac{1}{4}$ on $CP^2 \# 2\overline{CP^2}$. Theorem 3.1 is an improvement of Abdesselem's result. we guess that $\alpha_G(M) = \frac{1}{2}$ on $CP^2 \# 2\overline{CP^2}$.

4. Estimate of Ricci curvature.

In this section, we prove:

Theorem 4.1. *Let $M = \mathbb{C}P^2 \# 2\overline{\mathbb{C}P^2}$. Then there exists a Kähler metric with its Kähler form $\omega \in c_1(M)$ such that Ricci curvature of ω is not less than $\frac{3}{4}$.*

Proof. Choose a G-invariant Kähler form $\omega_g \in c_1(M)$ of M . Then there is a smooth function h such that

$$\begin{cases} \text{Ric}(\omega_g) - \omega_g = \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial}h, \\ \int_M e^h \omega_g^n = \int_M \omega_g^n. \end{cases}$$

We consider the following complex Monge-Ampère equations with one parameter $t \in [0, 1]$,

$$\begin{cases} \det(g_{i\bar{j}} + \phi_{i\bar{j}}) = \det(g_{i\bar{j}}) e^{h-t\phi}, \\ \det(g_{i\bar{j}} + \phi_{i\bar{j}}) > 0. \end{cases}$$

Then by a result in [T1] together with Theorem 3.1, we conclude that for any $t < \frac{3}{4}$, there is a smooth function ϕ solves the above equation on t . It follows

$$\text{Ric}(\omega_\phi) = t\omega_\phi + (1-t)\omega_g > t\omega_\phi.$$

The theorem is proved. \square

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