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SELF-AFFINE TILING VIA SUBSTITUTION DYNAMICAL SYSTEMS AND RAUZY FRACTALS

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In this paper we show that a class of sets known as the Rauzy fractals, which are constructed via substitution dynamical systems, give rise to self-affine multi-tiles and self-affine tilings. This provides an efficient and unconventional way for constructing aperiodic self-affine tilings. Our result also leads to a proof that a Rauzy fractal $\mathfrak{R}$ associated with a primitive and unimodular Pisot substitution has nonempty interior.

1. Introduction.

Let A be an expanding matrix in $M_d(\mathbb{R})$, that is, one with all eigenvalues $|\lambda| > 1$. Let $\mathcal{D}_{ij}$ where $1 \leq i,j \leq J$ be finite (possibly empty) subsets of vectors in $\mathbb{R}^d$. It is known (see [**16**]) that there exist unique nonempty compact sets $X_1, \ldots, X_J$ such that

$$A(X_i) = \bigcup_{j=1}^{J} (X_j + \mathcal{D}_{ij}), \quad i = 1, \ldots, J, \tag{1.1}$$

provided that the *subdivision matrix* $S = [s_{ij}]$ where $s_{ij} := |\mathcal{D}_{ij}|$ is primitive. The J-tuple $(X_1, \ldots, X_J)$ is called the attractor of the graph-directed iterated function system (IFS) (1.1).

An interesting class of graph-directed IFS consists of those whose attractors are tiles of $\mathbb{R}^d$. We say that $(X_1, \ldots, X_J)$ is a *self-affine multi-tile* if each X_i has nonempty interior and the unions on the right side of (1.1) are all measure-wise disjoint. Self-affine multi-tiles arise in many applications, include the construction of orthonormal multi-wavelet bases, self-replicating tilings and quasicrystals. They form an important class of tiles in the theory of tiling. For example, the Penrose aperiodic tiles can be viewed as a self-affine multi-tile. By iterating Equation (1.1) repeatedly we may obtain a *self-affine* tiling, or *self-replicating* tiling. Such inflation technique remains as the most effective way for generating aperiodic and "exotic" tilings. For more on self-affine multi-tiles and tilings, see [**6**], [**8**], [**12**], and the references therein.

Self-affine multi-tiles and self-affine tilings are the easiest to construct when the expanding matrix A is an integer matrix, or is similar to an integer matrix. The studies in some of the recent papers (e.g., [**6**], [**8**]) are mostly confined to such settings. These multi-tiles typically — although not always — yield periodic self-affine tilings. However, the most interesting multi-tiles such as some of the Penrose tiles that yield aperiodic self-affine tilings do not use integer matrices as the expansion matrix. Constructions of self-affine multi-tiles and self-affine tilings systematically using non-integer matrices remain difficult in general.

Our main objective in this paper is to introduce a method for constructing interesting self-affine multi-tiles and self-affine tilings, in which the expansion matrix A is not an integral matrix. The self-affine multi-tiles and tilings are obtained through substitution dynamical systems, via a peculiar class of sets called the *Rauzy fractals*. We first give a brief introduction on substitution dynamical systems and Rauzy fractals below.

A *substitution* on a finite alphabet $\mathcal{A} = \{1, \ldots, n\}$ is a map Π from $\mathcal{A}$ to $\mathcal{A}^* = \bigcup_{i\geq 0} \mathcal{A}^i$, where $\mathcal{A}^i$ denotes the set of words of length i in $\mathcal{A}$, with $\mathcal{A}^0$ containing only the empty word $\emptyset$. This map extends to a map on $\mathcal{A}^*$ into itself by juxtaposition $\Pi(\emptyset) = \emptyset$ and $\Pi(UV) = \Pi(U)\Pi(V)$ for all $U, V \in \mathcal{A}^*$. Let $\mathcal{A}^{\mathbb{N}}$ denote the set of one sided infinite sequences in the alphabet $\mathcal{A}$:

$$\mathcal{A}^{\mathbb{N}} := \Big\{ a_1 a_2 a_3 \ldots \;\Big|\; a_j \in \mathcal{A} \Big\}.$$

Then Π extends to a map on $\mathcal{A}^{\mathbb{N}}$ into itself in the obvious way. An important object associated to a substitution is its *incidence matrix*. For any $U \in \mathcal{A}^*$ let $|U|_j$ denote the number of occurrences of the symbol j in the word U. The incidence matrix $M = [m_{ij}]$ is then defined by $m_{ij} = |\Pi(i)|_j$ for $1 \leq i, j \leq n$. Throughout this paper we shall assume that the substitution Π is *primitive*, namely $M^k > 0$ for some $k \geq 1$.

We call $\mathbf{U} \in \mathcal{A}^{\mathbb{N}}$ a *fixed point* of Π if $\Pi(\mathbf{U}) = \mathbf{U}$. Associated to a fixed point $\mathbf{U}$ of the substitution is a symbolic dynamical system $(\Omega_{\mathbf{U}}, \sigma)$ where σ is the shift map on $\mathcal{A}^{\mathbb{N}}$ given by $\sigma(a_1 a_2 \ldots) = a_2 a_3 \ldots$ and $\Omega_{\mathbf{U}}$ is the closure of $\{\sigma^m(\mathbf{U}) : m \geq 0\}$ in $\mathcal{A}^{\mathbb{N}}$. This symbolic dynamical system has been studied extensively, see Queffélec [**20**]. It is straightforward to show that if the substitution is primitive then the system $(\Omega_{\mathbf{U}}, \sigma)$ is minimal and uniquely ergodic.

An important question is whether and how the symbolic dynamical system $(\Omega_{\mathbf{U}}, \sigma)$ is realizable geometrically. We say that a dynamical system (X, f) is *semiconjugate to* another dynamical system (Y, g) if there exists a continuous surjective map $\phi : X \to Y$ such that $\phi \circ f = g \circ \phi$. By *geometrically realizable* we mean there exists a dynamical system (X, f) defined on a geometrical structure, e.g., a differentiable manifold, for some continuous map f on X, such that $(\Omega_{\mathbf{U}}, \sigma)$ is semiconjugate to (X, f). One of the main motivations of this question is the study of the spectral properties of the symbolic system. One way to do this is to show that symbolic system is mesurably conjugate or semiconjugate to the geometrical system. In particular a way to show that a substitution dynamical systems has discrete spectrum is to find a isomorphism to rotation on a compact group [cf. [**20**, **21**, **31**]]. It has been conjectured that every primitive and unimodular Pisot substitution dynamical system is measurably isomprphic to a toral translation.

Rauzy approached this conjecture via geometrical realization of the symbolic system. In [**23**] he studied the tribonacci substitution, i.e., $1 \to 12$, $2 \to 13$, $3 \to 1$, and he showed that the associated symbolic dynamical system can be realized geometrically as $(\mathbb{T}^2, f)$ where f is an irrational translation on the 2-torus $\mathbb{T}^2$. This result generalizes also to the family of substitutions

$$1 \to 12,\ 2 \to 13,\ \ldots,\ (n-1) \to 1n,\ n \to 1. \tag{1.2}$$

The symbolic dynamical system $(\Omega_{\mathbf{U}}, \sigma)$ associated to the unique fixed point for each n is geometrically realizable as $(\mathbb{T}^{n-1}, f_n)$ for some irrational translation f_n on $\mathbb{T}^{n-1}$.

Key to Rauzy's results is the construction of a set known as a *Rauzy fractal* associated to the fixed point $\mathbf{U}$ of a primitive substitution. In [**23**] the set $\Omega_{\mathbf{U}}$ is mapped to $\mathbb{C}$ via a *valuation*, which is a map $E : \mathcal{A}^* \longrightarrow \mathbb{C}$ having the properties that $E(UV) = E(U) + E(V)$ and $E(\Pi(U)) = \omega E(U)$ for all $U, V \in \mathcal{A}^*$ and some fixed constant ω. It is shown in Holton and Zamboni [**9**] that in a valuation the constant ω must be an eigenvalue of the incidence matrix M, and there exists an ω-eigenvector $\mathbf{v} = [v_1, \ldots, v_n]^T$ such that

$$E(U) = \sum_{i=1}^{n} |U|_i v_i, \text{ for all } U \in \mathcal{A}^*. \tag{1.3}$$

We call the valuation defined above the *valuation corresponding to* $\mathbf{v}$. Suppose that

$$\omega_1, \overline{\omega}_1, \ldots, \omega_p, \overline{\omega}_p, \omega_{p+1}, \ldots, \omega_r \tag{1.4}$$

are all the eigenvalues of M inside the unit disk, where $\omega_1, \overline{\omega}_1, \ldots, \omega_p, \overline{\omega}_p$ are nonreal complex and $\omega_{p+1}, \ldots, \omega_r$ are real. Let $\mathbf{v}_j$ be an ω_j-eigenvector of M and E_j be the valuation corresponding to $\mathbf{v}_j$. For $U \in \mathcal{A}^*$ define $\Delta(U) = [E_1(U), \ldots, E_r(U)]^T$. Note that $\Delta(U)$ is a vector in $\mathbb{C}^p \times \mathbb{R}^{r-p}$, which we identify with $\mathbb{R}^{p+r}$. The valuation done by Rauzy in [**23**] in the tribonacci substitution is shown in Example 2.1.

Definition 1.1. We call the closure of the set $\{\Delta([\mathbf{U}]_m) : \ m \geq 0\}$ in $\mathbb{C}^p \times \mathbb{R}^{r-p}$ (identified with $\mathbb{R}^{p+r}$) a Rauzy fractal, where $[\mathbf{U}]_m$ is the word formed by the first m symbols of the fixed point $\mathbf{U}$.

It is known that a Rauzy fractal is bounded ([**23**, **9**]). The geometric and dynamical properties of Rauzy fractals have been studied in a number of papers, among others see [**1**, **9**, **23**, **26**, **27**, **29**]. For example, Holton and Zamboni [**9**] studied the Hausdorff dimensions of various sets related to Rauzy fractals. They also observed that a Rauzy fractal can be expressed as the attractor of some graph directed iterated function system (IFS). Solomyak [**29**, **31**] showed that certain family of primitive and unimodular Pisot substitutions (defined below) has pure discrete spectrum via geometrical realizations as irrational translations on tori. Arnoux and Ito [**1**] showed that for certain primitive and unimodular Pisot substitutions on a alphabet of k letters can be realized geometrically as an exchange of pieces by translation on the plane $\mathbb{R}^{k-1}$ and as irrational translations on $\mathbb{T}^{k-1}$. The geometry of boundary of the Rauzy fractal for the tribonacci substitution has been studied in [**28**, **17**].

We shall study Rauzy fractals associated to fixed points of primitive and *unimodular Pisot* substitutions. A substitution Π in $n \geq 2$ symbols is unimodular Pisot if its incidence matrix M satisfies (i) $|\det(M)| = 1$ and (ii) the Perron-Frobenius eigenvalue λ of M is a Pisot number, and the characteristic polynomial of M is the minimal polynomial of λ, i.e., it is irreducible over $\mathbb{Q}$. A Pisot number is an algebraic integer greater than 1, such that all its Galois conjugates have modulus less than 1. Most of the studies on Rauzy fractals assume the substitutions are primitive and unimodular Pisot because only such substitutions allow geometric realizations.

In this paper we prove several general properties concerning self-affine multi-tiles in §3. These results are themselves rather fundamental for the study of self-affine multi-tiles and tilings. We then use these results to prove in §4 a main theorem of ours, that the so-called *natural decomposition* of a Rauzy fractal $\mathfrak{R}$ (defined later) gives a self-affine multi-tile. The expansion matrix for the self-affine multi-tile is not an integer matrix; in fact it is not even similar to an integer matrix. As a corollary we prove that every set in the natural decomposition of $\mathfrak{R}$ has nonempty interior and is in fact the closure of its interior. Our method yields a simple new proof of a result, due to Arnoux and Ito [**1**], concerning the disjointness of the natural decomposition of $\mathfrak{R}$.

We should point out that self-similar tilings using non-integral expansions are not new. In an unpublished work Thurston [**34**] outlined a construction of self-similar tilings for a complex Pisot number β. This construction, which was studied further in Kenyon [**11**, **12**], Praggastis [**19**] and Petronio [**18**], employs base-β expansions (called β-expansions) $\sum_{j\in\mathbb{Z}} d_j\,\beta^j$ with d_j belonging to a suitably chosen set $\mathcal{D} \subset \mathbb{Q}(\beta)$ called the "digit set." The expansions are not unique in general. However, it is possible to make the expansions unique by disallowing some subsequences. Once this is done, the remaining β-expansions form a partition of the plane that is a self-similar tiling using the expansion β. Although coming from a completely different angle, the Rauzy fractal induced self-affine tilings turn out to be related to Thurston's framework. This is evident in Rauzy's work on the tribonacci substitution Rauzy fractal. The general Rauzy fractals studied here provide a concrete and very general construction of Thurston-like self-affine tilings. Moreover, these Rauzy fractals allow the expanding matrices to be nonsimilitudes. We also mention that unlike in much previously mentioned work, we prove the tiling property of Rauzy fractals by proving first they have positive Lebesgue measure and then extablishing structural results on the expansions, rather than the other way around.

The fact that $\mathfrak{R}$ has positive Lebesgue measure had been conjectured and is a key part for the geometric realization of substitution dynamical systems. This fact has recently also been proved independently by Canterini and Siegel [**3**], following from a stronger result of theirs concerning the geometric realization of substitutions. Their methods are rather different from ours, and they are motivated by geometric realization of substitutions rather than self-affine tilings. In a later paper [**4**], Canterini and Siegel also gave an alternative proof that every set in the natural decomposition of $\mathfrak{R}$ has nonempty interior using a very different method.

2. Preliminary results.

The objective of this section is to prove that a Rauzy fractal associated to a fixed point of a primitive and unimodular Pisot substitution has positive Lebesgue measure. This fact is needed to prove our main results.

Throughout the rest of the paper, unless otherwise stated, $\mathcal{A}$ denotes the alphabet $\mathcal{A} = \{1, 2, \ldots, n\}$ and Π is a primitive and unimodular Pisot substitution on $\mathcal{A}$, with incidence matrix $M = [m_{ij}]$. We use $\mathbf{U}$ to denote a fixed point of Π and $[\mathbf{U}]_k$ to denote the word formed by the first k symbols of $\mathbf{U}$. The Perron-Frobenius eigenvalue of M is λ, and all the other eigenvalues of M are $\omega_1, \overline{\omega}_1, \ldots, \omega_p, \overline{\omega}_p, \omega_{p+1}, \ldots, \omega_r$, where $\omega_1, \ldots, \omega_p$ are nonreal complex and $\omega_{p+1}, \ldots, \omega_r$ are real. Of course, the unimodular Pisot assumption implies that $0 < |\omega_j| < 1$. The map E_j is a valuation corresponding to some ω_j-eigenvector $\mathbf{v}_j$, as defined in (1.3), and $\Delta = [E_1, \ldots, E_r]^T$.

Observe that each $\mathbf{v}_j$ is unique up to scaling. Therefore the Rauzy fractal associated to the substitution Π on $\mathcal{A}$ is actually *unique* up to an affine transformation. Also, by elementary linear algebra the fact that the characteristic polynomial is irreducible over $\mathbb{Q}$ implies that there exist polynomials $g_i(x) \in \mathbb{Z}[x]$ of degrees less than n for $1 \le i \le n$ such that

$$\mathbf{v}_j = [g_1(\omega_j), \ldots, g_n(\omega_j)]^T c_j, \tag{2.1}$$

where $c_j \neq 0$ and $c_j \in \mathbb{C}$ for $j \le p$ and $c_j \in \mathbb{R}$ for $j > p$.

Lemma 2.1. *Let E be the valuation corresponding to some ω-eigenvector $\mathbf{v}$. Then $E(UV) = E(U) + E(V)$ and $E(\Pi(U)) = \omega E(U)$ for all $U, V \in \mathcal{A}^*$.*

Proof. This is shown in [**9**], and we include a proof for self-containment. The fact $E(UV) = E(U) + E(V)$ follows from the definition. Also by definition,

$$E(\Pi(i)) = \sum_{j=1}^{n} m_{ij} E(j) = \sum_{j=1}^{n} m_{ij} v_j = \omega v_i = \omega E(i),$$

where $\mathbf{v} = [v_1, \ldots, v_n]^T$. By juxtaposition we see that $E(\Pi(U)) = \omega E(U)$ for all $U \in \mathcal{A}^*$. $\square$

Lemma 2.2. $\Delta([\mathbf{U}]_m) \neq \Delta([\mathbf{U}]_k)$ *for* $m \neq k$.

Proof. We prove the stronger statement that $E_i([\mathbf{U}]_m) \neq E_i([\mathbf{U}]_k)$ for all $m \neq k$. Suppose that this is not true. Then $E_i([\mathbf{U}]_m) = E_i([\mathbf{U}]_k)$ for some $m \neq k$. Without loss of generality let $k > m$. Then by definition and (2.1), $\sum_{j=1}^{n} |V|_j g_j(\omega_i) = 0$, where V is the word formed using the $(m+1)$-th through k-th symbols of $\mathbf{U}$. But $g_j(x) \in \mathbb{Z}[x]$ and λ is a Galois conjugate of ω_i, so we must have $\sum_{j=1}^{n} |V|_j g_j(\lambda) = 0$. However, since λ is the Perron-Frobenius eigenvalue of M and M is primitive, all $g_j(\lambda)$ are positive or negative at the same time. This is impossible. So $E_i([\mathbf{U}]_m) \neq E_i([\mathbf{U}]_k)$ for all $m \neq k$, proving the lemma. $\square$

For each polynomial $f(x)$ we let $\boldsymbol{\omega}_f = [f(\omega_1^{-1}), \ldots, f(\omega_r^{-1})]^T$.

Lemma 2.3. *Let D be a finite subset of $\mathbb{Z}$ and let $D[x]$ denote*

$$D[x] = \Big\{ f(x) = a_0 + \cdots + a_m x^m \,\Big|\, a_j \in D,\, m \geq 0 \Big\}.$$

Then there exists an $\varepsilon_0 > 0$ depending only on D, such that for all $f, g \in D[x]$ either $\boldsymbol{\omega}_f = \boldsymbol{\omega}_g$ or $\|\boldsymbol{\omega}_f - \boldsymbol{\omega}_g\| \geq \varepsilon_0$.

Proof. Note that the unimodular Pisot assumption implies that $1/\lambda$ is an algebraic integer and its Galois conjugates are precisely

$$\omega_1^{-1}, \overline{\omega}_1^{-1}, \ldots, \omega_p^{-1}, \overline{\omega}_p^{-1}, \omega_{p+1}^{-1}, \ldots, \omega_r^{-1}.$$

So for each $f(x)$ in $\mathbb{Z}[x]$ the Galois conjugates of $f(\lambda)$ are $f(\lambda^{-1}), f(\omega_1^{-1}), f(\overline{\omega}_1^{-1}), \ldots, f(\omega_r^{-1})$. Therefore

$$(2.2) \qquad L(f) := f(\lambda^{-1}) + f(\omega_1^{-1}) + f(\overline{\omega}_1^{-1}) + \cdots + f(\omega_r^{-1}) \in \mathbb{Z}.$$

In particular since $f_k(x) = x^k f(x) \in \mathbb{Z}[x]$ we have $L(f_k) := L_k \in \mathbb{Z}$ for all $k \geq 0$.

Using the Vandermonde matrix it is easy to prove that for any k, at least one of $L_k, L_{k+1}, \ldots, L_{k+n-1}$ is not zero, unless $f(\lambda^{-1}) = f(\omega_i^{-1}) = 0$ for all i.

The finiteness of D implies $|f(\lambda^{-1})| \leq c_0$ for some c_0 depending only on D. Now assume that the the lemma is false. Then we can find f, g in $D[x]$ with $\boldsymbol{\omega}_f \neq \boldsymbol{\omega}_g$ such that $\|\boldsymbol{\omega}_f - \boldsymbol{\omega}_g\|$ can be made arbitrarily small. Since

$h(x) = f(x) - g(x)$ satisfies $|h(\lambda^{-1})| \leq 2c_0$, $|\lambda^{-k} h(\lambda^{-1})|$ is sufficiently small for sufficiently large k. We derive a contradiction below.

Fix $k > 0$ such that $2\lambda^{-k} c_0 < \frac{1}{n}$. Then choose $f(x), g(x) \in D[x]$ such that $h(x) = f(x) - g(x)$ has the property that $\boldsymbol{\omega}_h \neq 0$ and $\|\boldsymbol{\omega}_h\|$ is so small that for $h_j(x) := x^j h(x)$ we have

$$\|\boldsymbol{\omega}_{h_{j+k}}\| \leq \max_{1 \leq i \leq r} |\omega_i|^{-j-k} \|\boldsymbol{\omega}_h\| < \frac{1}{n}$$

for $0 \leq j \leq n-1$. Since also

$$|h_{j+k}(\lambda^{-1})| = \lambda^{-j-k} |h(\lambda^{-1})| \leq 2\lambda^{-j-k} c_0 < \frac{1}{n}$$

we have that $|L(h_{j+k})| < 1$ for $0 \leq j \leq n-1$. But all $L(h_{j+k})$ are integers, so all of them must be 0. This contradicts the result that one of them is nonzero, proving the lemma. $\square$

Corollary 2.4. *Let S be a finite subset of $\mathbb{Z}[x]$. If in Lemma 2.3 $D[x]$ is replaced by*

$$S[x] = \Big\{ f(x) = a_0(x) + a_1(x)x + \cdots + a_m(x)x^m \,\Big|\, a_i(x) \in S,\, m \geq 0 \Big\},$$

then the same conclusion still holds.

Proof. Because S is finite, there exists a finite $D \subset \mathbb{Z}$ such that $S[x] \subseteq D[x]$. So the conclusion of Lemma 2.3 still holds. $\square$

We shall need the following theorem of Rauzy [**24**] to prove our results:

Theorem 2.5 (Rauzy)**.** *There exists a finite subset $\mathcal{P}$ of $\mathcal{A}^*$ with $\emptyset \in \mathcal{P}$ and a subset $\mathcal{S}$ of $\mathcal{P}^*$, the set of all finite sequences in $\mathcal{P}$, with the following properties:*

(a) *All sequences $(W_k, W_{k-1}, \ldots, W_0) \in \mathcal{S}$ satisfy $W_k \neq \emptyset$.*
(b) *For any $m \geq 1$ there exists a unique sequence of words $(W_k, W_{k-1} \ldots, W_0) \in \mathcal{S}$ such that*

$$[\mathbf{U}]_m = \Pi^k(W_k)\Pi^{k-1}(W_{k-1}) \ldots \Pi(W_1)W_0. \tag{2.3}$$

(c) *For any sequence of words $(W_k, W_{k-1} \ldots, W_0) \in \mathcal{S}$ there exists an $m \geq 1$ such that $[\mathbf{U}]_m$ is given by* (2.3)*.*

Proof. See the theorem in Rauzy [**24**], First Part, Section 6. $\square$

The above theorem of Rauzy's is proved by representing $[\mathbf{U}]_m$ as a path in certain prefix automaton. The set $\mathcal{P}$ is the set of prefixes for the automaton and $\mathcal{S}$ is the set of all allowable paths generated by the prefix automaton. Using this representation many elegant results can be derived, see [**24**] for more details.

Lemma 2.6. *There exists a finite subset S of $\mathbb{Z}[x]$ such that*

$$\{\Delta([\mathbf{U}]_m) \mid m \geq 1\} \subseteq \left\{ \sum_{j=0}^{k} \boldsymbol{\omega}_{p_j} B^j \mathbf{v}_0 \;\middle|\; p_j(x) \in S,\ k \geq 0 \right\} \tag{2.4}$$

where B is the matrix $B := \operatorname{diag}(\omega_1, \omega_2, \ldots, \omega_r)$ and $\mathbf{v}_0$ is a fixed vector in $\mathbb{C}^p \times \mathbb{R}^{r-p}$.

Proof. By Theorem 2.5, the word $[\mathbf{U}]_m$ can be expressed as

$$\Pi^k(W_k)\Pi^{k-1}(W_{k-1}) \ldots W_0,$$

where $W_j \in \mathcal{P}$ for some finite $\mathcal{P} \in \mathcal{A}^*$ and $W_k \neq \emptyset$. Observe that for each $W \in \mathcal{P}$, $E_i(W) = q(\omega_i)c_i$ for some $q(x) \in \mathbb{Z}[x]$ and constant c_i. Since $\mathcal{P}$ is finite we may choose a sufficiently large fixed N so that $q(x) = x^N p(x^{-1})$ for some $p(x) \in \mathbb{Z}[x]$. So $E_i(W) = p(\omega_i^{-1})\omega_i^N c_i$. It follows that

$$E_i([\mathbf{U}]_m) = \sum_{j=0}^{k} \omega_i^j E_i(W_j) = \sum_{j=0}^{k} \omega_i^j p(\omega_i^{-1})\omega_i^N c_i.$$

The lemma is proved by setting $\mathbf{v}_0 = [\omega_1^N c_1, \ldots, \omega_r^N c_r]^T$. □

Lemma 2.7. *Let N be a positive integer and define*

$$R_N = \left\{ \Delta(U) \,\middle|\, U = \Pi^k(W_k) \ldots \Pi(W_1)W_0,\ (W_k, \ldots, W_0) \in \mathcal{S} \text{ and } k \leq N \right\}$$

where $\mathcal{S}$ is defined in Theorem 2.5. Then R_N is a subset of $\{\Delta([\mathbf{U}]_m) : m \geq 0\}$, and there exist $C_1, C_2 > 0$ independent of N such that the cardinality of R_N satisfies

$$C_1 \lambda^N \leq |R_N| \leq C_2 \lambda^N.$$

Proof. $R_N \subseteq \{\Delta([\mathbf{U}]_m) : m \geq 0\}$ follows from Theorem 2.5.

For each $U \in \mathcal{A}^*$ the length of $\Pi^l(U)$ is $\sum_{i,j=1}^n m_{ij}^{(l)} |U|_j$, where $m_{ij}^{(l)}$ are the entries of the matrix M^l. So the length is bounded by $C_0 \lambda^l \max(|U|_j)$ for some constant C_0 independent of U and l. The inequality $|R_N| \leq C_2 \lambda^N$ is now obvious, because $\mathcal{P}$ is finite. To prove $|R_N| \geq C_1 \lambda^N$, consider

$$\sum_{i=1}^n |\Pi^k(W_k) \ldots \Pi(W_1)W_0|_i \geq \sum_{i=1}^n |\Pi^k(W_k)|_i \geq C_3 \lambda^k$$

for some $C_3 > 0$. Therefore, for all m such that $[\mathbf{U}]_m$ is not representable as $\Pi^k(W_k) \ldots W_0$ with $k \leq N$ we must have $\sum_{i=1}^n |U_m|_i = m \geq C_3 \lambda^{N+1}$. Hence $R_N \supseteq \{\Delta(U_m) \,|\, m < C_3 \lambda^{N+1}\}$. Since all $\Delta([\mathbf{U}]_m)$ are distinct by Lemma 2.2, we have

$$|R_N| \geq C_3 \lambda^{N+1} = (C_3 \lambda) \lambda^N = C_1 \lambda^N.$$

□

Proposition 2.8. *The Rauzy fractal $\mathfrak{R}$ has positive Lebesgue measure.*

Proof. By Lemma 2.6 and Corollary 2.4 the set $B^{-N}(R_N)$ has the ε_0-separation property for some $\varepsilon_0 > 0$. Let $\delta = \varepsilon_0/2$ and $B_\delta(y)$ be the ball of radius δ centered at y in $\mathbb{C}^p \times \mathbb{R}^{r-p}$ (which we identify with $\mathbb{R}^{n-1}$). Then

$$S_N = \bigcup_{y \in B^{-N}(R_N)} B_\delta(y)$$

is a disjoint union of balls. So $\nu(S_N) = |R_N|\mathrm{vol}(B_\delta(0))$, where ν denotes the Lebesgue measure. Note that for any set K,

$$\nu(B(K)) = |\omega_1|^2 \dots |\omega_p|^2 |\omega_{p+1}| \dots |\omega_r| \nu(K) = \lambda^{-1}\nu(K). \tag{2.5}$$

(Note that the first p coordinates are complex.) Hence

$$\nu(B^N(S_N)) = \lambda^{-N}|R_N|\mathrm{vol}(B_\delta(0)).$$

It follows from Lemma 2.7 that

$$\nu(B^N(S_N)) \geq C_1 \mathrm{vol}(B_\delta(0)) > 0.$$

On the other hand the limit of $B^N(S_N)$, when N goes to infinity, is $\mathfrak{R}$ in the Hausdorff metric. This implies

$$\nu(\mathfrak{R}) \geq \liminf_{N\to\infty} \nu(B^N(S_N)) > 0.$$

□

Example 2.1. We consider the substitution $1 \to 12$, $2 \to 13$, $3 \to 1$. A simple induction argument shows that it satisfies the relation

$$\Pi^{l+3}(1) = \Pi^{l+2}(1)\Pi^{l+1}(1)\Pi^{l}(1) \tag{2.6}$$

for all $l \geq 0$. It is shown (see [**24**]), using this fact, that $[\mathbf{U}]_m = \Pi^k(W_k)\dots \Pi(W_1)W_0$, where W_i are either $\emptyset$ or 1 and k depends on m but having no three consecutive W_i's equal to 1. So the set $\mathcal{S}$ in Theorem 2.5 is the set of infinite one-sided sequences in the symbols $\mathcal{P} = \{\emptyset, 1\}$, such that there are no three consecutive 1's. Therefore $\Delta([\mathbf{U}]_m) = \sum_{i=0}^k \omega^i \Delta(W_i)$ where ω is one of the two conjugate eigenvalues inside the unit circle, with

$$\Delta(W_i) = \begin{cases} 0 & \text{if } W_i = \emptyset, \\ 1 & \text{if } W_i = 1. \end{cases}$$

Hence

$$\mathfrak{R} = \left\{ \sum_{i\geq 0} a_i \omega^i \;\middle|\; a_i \in \{0,1\} \text{ and } a_i a_{i+1} a_{i+2} = 0 \text{ for all } i. \right\}.$$

This set is shown in Figure 1 (the different shades correspond to the decomposition of $\mathfrak{R}$, discussed in the next section).

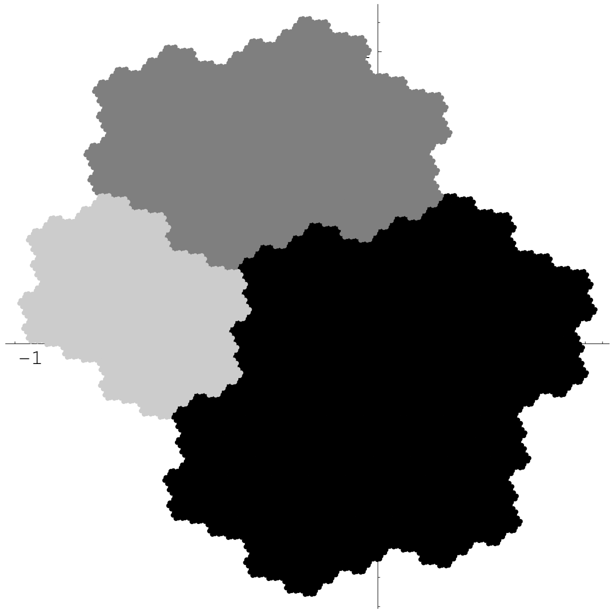

Figure 1. The Rauzy fractal of the substitution $1 \to 12$, $2 \to 13$, $3 \to 1$.

Example 2.2. Let the substitution be $1 \to 12$, $2 \to 3$, $3 \to 1$. Let ω be one of the two complex conjugate eigenvalues of the incidence matrix of this substitution. It can be proved, using the techniques described in [**24**], that

$$\mathfrak{R} = \left\{ \sum_{i \geq 0} a_i \omega^i \;\middle|\; a_i \in \{0,1\} \text{ and if } a_i = 1 \text{ for } i \geq 2 \text{ then } a_{i-1} = a_{i-2} = 0 \right\}.$$

This set is shown in Figure 2.

3. Graph-directed IFS and self-affine multi-tiles.

In this section we establish results concerning self-affine multi-tiles. In particular we prove a theorem regarding nonempty interior of the attractor of a graph directed IFS in a more general setting. Let A be an expanding $d \times d$ matrix, i.e., all its eigenvalues are outside the unit disk. Consider the nonempty compact subsets $X_1, \dots, X_J$ of $\mathbb{R}^d$ such that $(X_1, \dots, X_J)$ is the attractor of the graph directed IFS

$$X_i = \bigcup_{j=1}^{J} A^{-1}(X_j + \mathcal{D}_{ij}), \quad i = 1, \dots, J, \tag{3.1}$$

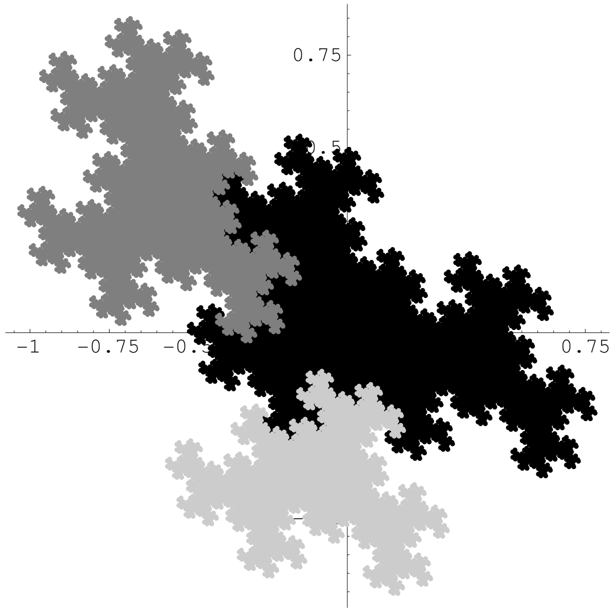

Figure 2. The Rauzy fractal of the substitution $1 \to 12$, $2 \to 3$, $3 \to 1$.

where each $\mathcal{D}_{ij}$ is a finite (possibly empty) subset of $\mathbb{R}^d$. For our study it is more convenient to write (3.1) in the form of

$$A(X_i) = \bigcup_{j=1}^{J} (X_j + \mathcal{D}_{ij}), \;\; i = 1, \ldots, J. \tag{3.2}$$

Iterating (3.2) yields

$$A^m(X_i) = \bigcup_{j=1}^{J} (X_j + \mathcal{D}_{ij}^m), \quad i = 1, \ldots, J, \tag{3.3}$$

where one easily check that

$$\mathcal{D}_{ij}^m = \bigcup_{k_1,\ldots,k_m=1}^{J} (A^{m-1}\mathcal{D}_{ik_m} + A^{m-2}\mathcal{D}_{k_m k_{m-1}} + \cdots + \mathcal{D}_{k_1 j}). \tag{3.4}$$

We say that the graph directed IFS (3.1) is *strongly connected* if its *subdivision matrix* $S = [s_{ij}]$ for $s_{ij} = |\mathcal{D}_{ij}|$ is primitive.

Theorem 3.1. *Let* $(X_1, \ldots, X_J)$ *be the attractor of a strongly connected graph directed* IFS

$$A(X_i) = \bigcup_{j=1}^{J} (X_j + \mathcal{D}_{ij}), \quad i = 1, \ldots, J.$$

Suppose that there exists an $\varepsilon_0 > 0$ *such that the sets* $\mathcal{D}_{ij}^m$ *have* ε_0*-separation for all* i, j, m*, and* X_1 *has positive Lebesgue measure. Then every* X_i *has nonempty interior, and it is the closure of its interior.*

Observe that all X_i have positive Lebesgue measures because of the strong connectivity of the graph directed IFS. Before proving Theorem 3.1 we first establish the following lemma:

Lemma 3.2. *Under the assumption of Theorem* 3.1*, let* δ_m *be a sequence of positive numbers whose limit is* 0*. Then there exist positive constants* r_0 *and* c_0 *such that for each* $m \geq 1$ *there exist finite subsets* $\mathcal{E}_1^m, \dots, \mathcal{E}_J^m$ *of* $\mathbb{R}^d$ *contained in the ball* $B_{r_0}(\mathbf{0})$*, of cardinality at most* c_0 *with* $\|\mathbf{e} - \mathbf{e}'\| \geq \varepsilon_0$ *for all* j, m *and any distinct elements* $\mathbf{e}, \mathbf{e}' \in \mathcal{E}_j^m$*, such that*

$$\nu(B_1(\mathbf{0}) \cap Y_m) \geq (1 - 5^{d+1}\delta_m)\nu(B_1(\mathbf{0})) \tag{3.5}$$

where $Y_m := \bigcup_{j=1}^J (X_j + \mathcal{E}_j^m)$ *and* $B_1(\mathbf{0})$ *is the Euclidean unit ball centered at the origin.*

Proof. Since X_1 has positive Lebesgue measure, it has a Lebesgue point $\mathbf{x}^*$, i.e., there is a sequence $r_m \to 0$ such that

$$\nu(B_{r_m}(\mathbf{x}^*) \cap X_1) \geq (1 - \delta_m)\nu(B_{r_m}(\mathbf{x}^*)).$$

This implies that

$$\nu(A^l(B_{r_m}(\mathbf{x}^*)) \cap X_1) \geq (1 - \delta_m)\nu(A^l(B_{r_m}(\mathbf{x}^*))), \quad \text{for all } l \geq 0. \tag{3.6}$$

We first show that for sufficiently large l, there exists a unit ball $B_1(\mathbf{y}) \subset A^l(B_{r_m}(\mathbf{x}^*))$ with

$$\nu(B_1(\mathbf{y}) \cap A^l(X_1)) \geq (1 - 5^{d+1}\delta_m)\nu(B_1(\mathbf{0})). \tag{3.7}$$

Indeed, since A is expanding $A^l(B_{r_m}(\mathbf{x}^*))$ is an ellipsoid $O_{l,m}$ whose shortest axis goes to infinity as l goes to infinity. Let $O'_{l,m}$ be the homothetically shrunk ellipsoid with shortest axis decreased in length by 2, so that all points in it are at distant at least 1 from the boundary of $O_{l,m}$. By a standard covering lemma (cf. Stein [**33**, p. 9]) applied to $O'_{l,m}$ there is a set $\{B_1(\mathbf{y}')\}$ of disjoint unit balls with centers in $O'_{l,m}$ that cover volume at least $5^{-d}\nu(O'_{l,m})$. Notice that once the shortest axis of the ellipsoid $O_{l,m}$ has length greater that $2d + 1$ we will have $5^{-d}\nu(O'_{l,m}) \geq 5^{-d-1}\nu(O_{l,m})$, since $(\frac{d}{d+1})^d > 1/5$. All these balls lie inside $O_{l,m}$. By (3.6) at most $\delta_m\nu(A^l(B_{r_m}(\mathbf{x}^*)))$ of the volume of $A^l(B_{r_m}(\mathbf{x}^*))$ is uncovered by $A^l(B_{r_m}(\mathbf{x}^*) \cap X_1)$, so at least one of the disjoint balls $\{B_1(\mathbf{y}')\}$ must satisfy (3.7).

By (3.3) we can rewrite the inequality (3.7) as

$$\nu\left(B_1(\mathbf{y}) \cap \left(\bigcup_{j=1}^J \left(X_j + \mathcal{D}_{1j}^l\right)\right)\right) \geq (1 - 5^{d+1}\delta_m)\nu(B_1(\mathbf{y})),$$

therefore

$$\nu\left(B_1(\mathbf{y})\cap\left(\bigcup_{j=1}\left(X_j+\mathcal{D}^l_{1j}-\mathbf{y}\right)\right)\right)\geq(1-5^{d+1}\delta_m)\nu(B_1(\mathbf{0})).$$

This shows that if we choose

$$\mathcal{E}^m_j=\left\{\mathbf{d}-\mathbf{y}\,|\,\mathbf{d}\in\mathcal{D}^l_{1j}\text{ with }(X_j+\mathbf{d}-\mathbf{y})\cap B_1(\mathbf{0})\neq\emptyset\right\}$$

then (3.5) holds. Since all $\mathcal{D}^m_{ij}$ have ε_0-separation, we have $\|\mathbf{e}-\mathbf{e}'\|\geq\varepsilon_0$ for all $\mathbf{e},\mathbf{e}'$ in $\mathcal{E}^m_j$. Since all X_j are compact, all $\mathcal{E}^m_j$ are inside the ball $B_{r_0}(\mathbf{0})$ for some fixed $r_0>0$. The ball $B_{r_0}(\mathbf{0})$ can be packed with disjoint balls of radius $\varepsilon_0/2$ centered at the points of $\mathcal{E}^m_j$ for each j. This implies that the cardinality of $\mathcal{E}^m_j$ must be bounded by some constant c_0. □

Proof of Theorem 3.1. We apply the previous lemma and choose a subsequence m_k so that $\{\mathcal{E}^{m_k}_j\}$ converges for all j, and we denote the limit by $\mathcal{E}^\infty_j$. Clearly $\mathcal{E}^\infty_j$ has cardinality at most c_0. So

$$\begin{aligned}\nu\left(B_1(\mathbf{0})\cap\left(\bigcup_{j=1}^J\left(X_j+\mathcal{E}^\infty_j\right)\right)\right)&\geq\liminf_{k\to\infty}\nu\left(B_1(\mathbf{0})\cap\left(\bigcup_{j=1}^J\left(X_j+\mathcal{E}^{m_k}_j\right)\right)\right)\\&\geq\liminf_{k\to\infty}(1-5^{d+1}\delta_{m_k})\nu(B_1(\mathbf{0}))\\&=\nu(B_1(\mathbf{0})).\end{aligned}$$

Since each $X_j+\mathcal{E}^\infty_j$ is a closed set, we must have

$$B_1(\mathbf{0})\cap\left(\bigcup_{j=1}^J\left(X_j+\mathcal{E}^\infty_j\right)\right)=B_1(\mathbf{0}).$$

This means at least one of X_j's must have interior. But if so then the strong connectivity implies that all X_j must have nonempty interior. Let $X'_j=\overline{X^o_j}$. Then $(X'_1,\ldots,X'_J)$ must also satisfy the same graph directed IFS. By the uniqueness (see Mauldin and Williams [**16**]) $X_j=X'_j$ for all j. □

Theorem 3.3. *Let $(X_1,\ldots,X_J)$ be the attractor of a strongly connected graph directed* IFS

$$A(X_i)=\bigcup_{j=1}^J(X_j+\mathcal{D}_{ij}),\quad i=1,\ldots,J \tag{3.8}$$

with a primitive subdivision matrix S. Suppose that some X_i has nonempty interior, and $|\det A|=\rho(S)$ where $\rho(\cdot)$ is the spectral radius. Then $(X_1,\ldots,X_J)$ is a self-affine tile with respect to A and $\mathcal{D}_{ij}$.

Proof. By the primitivity of S we know that all X_i have nonempty interior. Hence $\nu(X_i) > 0$. Let $\mathbf{w} = [\nu(X_1), \ldots, \nu(X_J)]^T$. Taking Lebesgue measure on both sides of (3.8) yields

$$|\det A|\mathbf{w} \leq S\mathbf{w} \tag{3.9}$$

with equality holding if and only if the unions on the right-hand side of (3.8) are all disjoint (measure-wise). But $|\det A|$ is the Perron-Frobenius eigenvector of S and $\mathbf{w} > 0$, so (3.9) must be an equality. Hence all unions on the right-hand side of (3.8) are measure-wise disjoint. Therefore $(X_1, \ldots, X_J)$ is a self-affine multi-tile with respect to A and $\mathcal{D}_{ij}$. □

4. Rauzy fractals.

In this section we introduce a natural decomposition of a Rauzy fractal $\mathfrak{R}$. We show that this natural decomposition leads to a self-affine multi-tile using the results from §3. We derive several corollaries of this result.

For a Rauzy fractal $\mathfrak{R} = \overline{\{\Delta([\mathbf{U}]_m) : m \geq 0\}}$ let

$$\mathfrak{R}_i = \overline{\Big\{\Delta([\mathbf{U}]_m) \;\Big|\; \mathbf{U}(m+1) = i,\ m \geq 0\Big\}},\ 1 \leq i \leq n. \tag{4.1}$$

Clearly $\bigcup_{i=1}^n \mathfrak{R}_i = \mathfrak{R}$. A better way to view $\mathfrak{R}$ and $\mathfrak{R}_i$ is to consider the "tail" of $\mathbf{U}$. For any valuation E corresponding to an eigenvalue $|\omega| < 1$ it is known that E can be extended naturally to a valuation on $\mathcal{A}^{\mathbb{N}}$ (see [**9**]). Since $\Pi(\mathbf{U}) = \mathbf{U}$ we have $E(\mathbf{U}) = \omega E(\mathbf{U})$, implying $E(\mathbf{U}) = 0$. Thus we may view $\mathfrak{R}$ as

$$\mathfrak{R} = -\overline{\big\{\Delta(\sigma^m \mathbf{U}) : m \geq 0\big\}}$$

and $\mathfrak{R}_i$ as

$$\mathfrak{R}_i = -\overline{\Big\{\Delta(\sigma^m \mathbf{U}) \;\Big|\; \mathbf{U}(m+1) = i,\ m \geq 0\Big\}}.$$

In other words, $\mathfrak{R}_i$ is given by the shifts of $\mathbf{U}$ whose beginning elements is the letter i. We shall call $(\mathfrak{R}_1, \ldots, \mathfrak{R}_n)$ the *natural decomposition* of the Rauzy fractal $\mathfrak{R}$.

A key ingredient for proving our main theorem is to express the natural decomposition $(\mathfrak{R}_1, \ldots, \mathfrak{R}_n)$ of a Rauzy fractal $\mathfrak{R}$ in terms of the attractor of a graph directed IFS. This fact is proved in [**9**]. Let $F = \{(j,k) : j \in \mathcal{A}, 1 \leq k \leq |\Pi(j)|\}$ where $|U|$ denote the length of U for $U \in \mathcal{A}^*$, and let $F_i = \{(j,k) \in F : \Pi(j)(k) = i\}$, where $\Pi(j)(k)$ denotes the k-th symbol of $\Pi(j)$. Then $(\mathfrak{R}_1, \ldots, \mathfrak{R}_n)$ satisfies

$$\mathfrak{R}_i = \bigcup_{(j,k)\in F_i} \Big(B\mathfrak{R}_j + \Delta([\Pi(j)]_{k-1})\Big),\ i = 1, \ldots, n, \tag{4.2}$$

where $B = \operatorname{diag}(\omega_1, \omega_2, \ldots, \omega_r)$ with ω_j being defined in (1.4). Recall also that for any word U in $\mathcal{A}^*$ the notation $[U]_m$ denotes the word formed by the first m letters of U, with $[U]_0 := \emptyset$. It is shown in addition in [**9**]

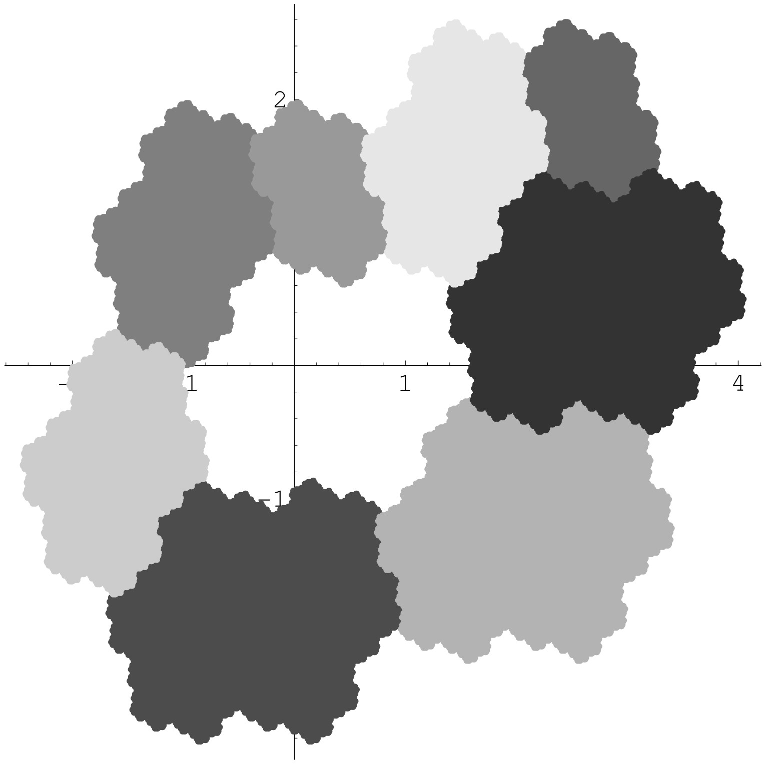

Figure 3. Tiling for the substitution $1 \to 12$, $2 \to 13$, $3 \to 1$.

that the graph directed IFS (4.2) is strongly connected in the sense that its subdivision matrix (defined below) is primitive, see [**9**] for details.

Our main theorem shows that the natural decomposition $(\mathfrak{R}_1, \ldots, \mathfrak{R}_n)$ forms a self-affine multi-tile.

Theorem 4.1. *Let $\mathfrak{R}$ be a Rauzy fractal associated to a fixed point of a primitive and unimodular Pisot substitution in $n \geq 2$ symbols. Then the natural decomposition $(\mathfrak{R}_1, \mathfrak{R}_2, \ldots, \mathfrak{R}_n)$ of $\mathfrak{R}$ is a self-affine multi-tile. Furthermore $\mathfrak{R}_i = \overline{\mathfrak{R}_i^o}$ for all $1 \leq i \leq n$.*

Proof. We already know that $(\mathfrak{R}_1, \ldots, \mathfrak{R}_n)$ is the attractor of the strongly connected graph directed IFS (4.2), which is rewritten as

$$B^{-1}\big(\mathfrak{R}_i\big) = \bigcup_{(j,k)\in F_i} \Big(\mathfrak{R}_j + B^{-1}\big(\Delta([\Pi(j)]_{k-1})\big)\Big), \ i = 1, \ldots, n. \tag{4.3}$$

Note that by (2.1) for each $j \in \mathcal{A}$ there exists a $g_j(x) \in \mathbb{Z}[x]$ and fixed constants $c_1, \ldots, c_r$ such that

$$B^{-1}\Delta(j) = [\omega_1^{-1} g_j(\omega_1) c_1, \ldots, \omega_r^{-1} g_j(\omega_r) c_r]^T.$$

As in the proof of Lemma 2.6, we may choose a sufficiently large but fixed N such that

$$B^{-1}\Delta(j) = [\omega_1^N f_j(\omega_1)c_1, \dots, \omega_r^N f_j(\omega_r)c_r]^T = D_0 \boldsymbol{\omega}_{f_j},$$

for some $f_j \in \mathbb{Z}[x]$, where $D_0 = \operatorname{diag}(\omega_1^N c_1, \dots, \omega_r^N c_r)$. This means that if we write (4.3) as

$$B^{-1}(\mathfrak{R}_i) = \bigcup_{j=1}^{n} \Big(\mathfrak{R}_j + \mathcal{D}_{ij}\Big),\ i = 1, \dots, n,$$

then each $\mathcal{D}_{ij}$ is either empty, or all its elements have the form $D_0\boldsymbol{\omega}_f$ for some finite collection of polynomials $f(x) \in \mathbb{Z}[x]$. Hence by Corollary 2.4 and (3.4) there exists an $\varepsilon_0 > 0$ such that all $\mathcal{D}_{ij}^m$ are ε_0-separated. Since the graph directed IFS (4.2) is strongly connected and $\nu(\mathfrak{R}_i) > 0$ for some i, it now follows from Theorem 3.1 that $\overline{\mathfrak{R}_i^o} \neq \emptyset$ and $\overline{\mathfrak{R}_i^o} = \mathfrak{R}_i$ for all i. Now observe that the subdivision matrix of the graph directed IFS (4.3) is precisely M^T, where M is the incidence matrix of the substitution. Taking the Lebesgue measure on both side of (4.3), using (2.5), we obtain

$$\lambda \mathbf{w} \leq M^T \mathbf{w}, \tag{4.4}$$

where $\mathbf{w} = [\nu(\mathfrak{R}_1), \dots, \nu(\mathfrak{R}_n)]^T$, with equality holding if and only if the unions on the right-hand side of (4.3) are all disjoint (measure-wise). But λ is the Perron-Frobenius eigenvector of M^T and $\mathbf{w} > 0$, so (4.4) must be an equality. Hence all unions on the right-hand side of (4.3) are measure-wise disjoint. Therefore $(\mathfrak{R}_1, \dots, \mathfrak{R}_n)$ is a self-affine multi-tile with respect to B and

$$\mathcal{D}_{ij} := \Big\{B^{-1}\big(\Delta([\Pi(j)]_{k-1})\big) :\ (j,k) \in F_i\Big\}.$$

□

Corollary 4.2. *Let $\mathfrak{R}$ be a Rauzy fractal associated to a fixed point of a primitive and unimodular Pisot substitution in $n \geq 2$ symbols. Then $\mathfrak{R}$ has nonempty interior and $\mathfrak{R} = \overline{\mathfrak{R}^o}$.*

The tilings associated to Examples 2.1 and 2.2 are shown in Figures 3 and 4.

Corollary 4.3. *Let $\mathfrak{R}$ be a Rauzy fractal associated to a fixed pint of a primitive and unimodular Pisot substitution in $n \geq 2$ symbols. Then there exist infinite discrete sets $\mathcal{J}_1, \dots, \mathcal{J}_n \subset \mathbb{C}^p \times \mathbb{R}^{r-p}$ such that*

$$\bigcup_{i=1}^{n} \bigcup_{\mathbf{j} \in \mathcal{J}_i} (\mathfrak{R}_i + \mathbf{j}) = \mathbb{C}^p \times \mathbb{R}^{r-p} \tag{4.5}$$

is a tiling of $\mathbb{C}^p \times \mathbb{R}^{r-p}$.

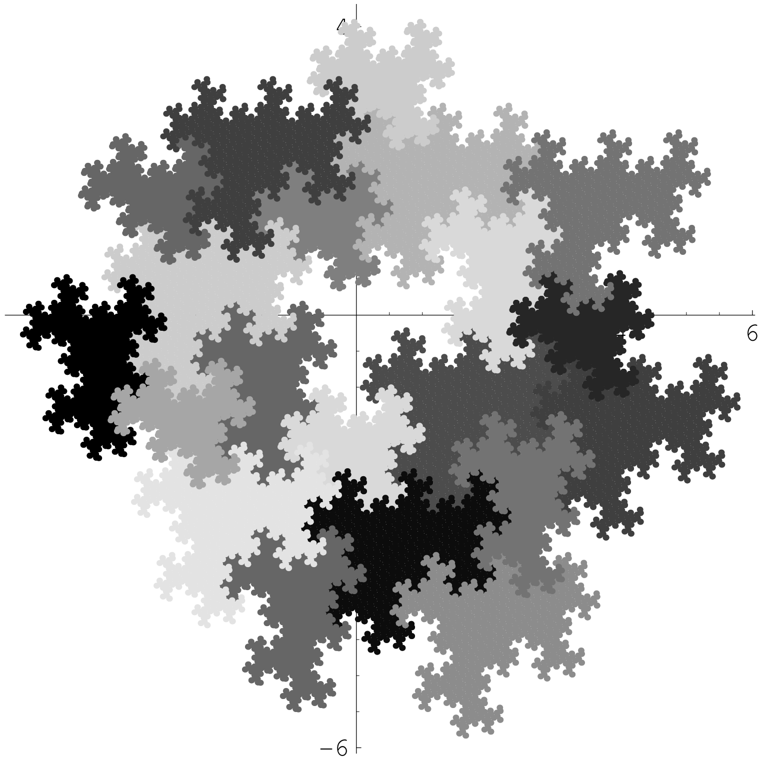

Figure 4. Tiling for the substitution $1 \to 12$, $2 \to 3$, $3 \to 1$.

Proof. The corollary follows from the standard technique of repeated iteration of (4.3). The existence of the infinite discrete sets $\mathcal{J}_1, \ldots, \mathcal{J}_n$ of $\mathbb{C}^p \times \mathbb{R}^{r-p}$ such that

$$\bigcup_{i=1}^{n} \bigcup_{\mathbf{j} \in \mathcal{J}_i} (\mathfrak{R}_i + \mathcal{J}_i) = \mathbb{C}^p \times \mathbb{R}^{r-p}$$

is a tiling of $\mathbb{C}^p \times \mathbb{R}^{r-p}$ is proved in Flaherty and Wang [**6**]. $\square$

An interesting unsolved problem concerns the measure disjointness of the natural decomposition of a Rauzy fractal $\mathfrak{R}$. Here we state it explicitly as a conjecture:

Conjecture. *Let $\mathfrak{R}$ be a Rauzy fractal associated to a fixed pint of a primitive and unimodular Pisot substitution in $n \geq 2$ symbols. Let $(\mathfrak{R}_1, \ldots, \mathfrak{R}_n)$ be the natural decomposition of $\mathfrak{R}$. Then all $\mathfrak{R}_i$ are measure-wise disjoint.*

The above conjecture has been studied in Arnoux and Ito [**1**]. It is shown there that the disjointness of the natural decomposition is equivalent to a combinatorial condition called the *positive strong coincidence* condition. A substitution Π is said to satisfy the positive strong coincidence condition if for any two letters j_1 and j_2 in $\mathcal{A}$ there exist a letter i and integers $m, k_1, k_2 \geq 1$ such that

$$\Pi^m(j_1)(k_1) = \Pi^m(j_2)(k_2) = i \text{ and } \Delta([\Pi^m(j_1)]_{k_1-1}) = \Delta([\Pi^m(j_2)]_{k_2-1}).$$

We state their theorem below, and show that the connection with self-affine multi-tiles also yield a simple proof of the result.

Theorem 4.4 (Arnoux and Ito)**.** *Let $\mathfrak{R}$ be a Rauzy fractal associated to a fixed point $\mathbf{U}$ of a primitive and unimodular Pisot substitution Π in $n \geq 2$ symbols. Let $(\mathfrak{R}_1, \ldots, \mathfrak{R}_n)$ be the natural decomposition of $\mathfrak{R}$. Suppose that Π satisfies the positive strong coincidence condition. Then all $\mathfrak{R}_i$ are measure-wise disjoint.*

Proof. We prove the measure-wise disjointness of $\{\mathfrak{R}_i\}$ using the special property that a Rauzy fractal for the substitution Π is also a Rauzy fractal for the substitution Π^m, $m \geq 1$. This follows easily from the fact that if $\Pi(\mathbf{U}) = \mathbf{U}$ then $\Pi^m(\mathbf{U}) = \mathbf{U}$, and the fact that a valuation on $\Omega_{\mathbf{U}}$ for Π with respect to an ω-eigenvector $\mathbf{v}$ is also a valuation on $\Omega_{\mathbf{U}}$ for Π^m with respect to the ω^m-eigenvector $\mathbf{v}$.

Notice that the natural decomposition $(\mathfrak{R}_1, \ldots, \mathfrak{R}_n)$ of $\mathfrak{R}$ as a Rauzy fractal for Π is also the natural decomposition of $\mathfrak{R}$ as a Rauzy fractal for Π^m, since it depends only on the valuation vector Δ, which is the same for both cases. Now, Π^m has incidence matrix M^m where M is the incidence matrix of Π. Clearly, Π^m is also primitive and unimodular Pisot, with the Pisot eigenvalue λ^m and its conjugates

$$\omega_1^m, \overline{\omega}_1^m, \ldots, \omega_p^m, \overline{\omega}_p^m, \omega_{p+1}^m, \ldots, \omega_r^m$$

where ω_j are as in (1.4). So $(\mathfrak{R}_1, \ldots, \mathfrak{R}_n)$ satisfies

$$\mathfrak{R}_i = \bigcup_{(j,k) \in F_i^m} \Big(B^m \mathfrak{R}_j + \Delta([\Pi^m(j)]_{k-1}) \Big), \ i = 1, \ldots, n, \tag{4.6}$$

where $B = \mathrm{diag}\ (\omega_1, \omega_2, \ldots, \omega_r)$ and $F_i^m = \{(j,k) :\ \Pi^m(j)(k) = i\}$, with the unions on the right side of (4.6) measure-wise disjoint. Since Π satisfies the positive strong coincidence condition, for any two letters j_1 and j_2 in $\mathcal{A}$ there exist a letter i and integers $m, k_1, k_2 \geq 1$ such that

$$\Pi^m(j_1)(k_1) = \Pi^m(j_2)(k_2) = i \text{ and } \Delta([\Pi^m(j_1)]_{k_1-1}) = \Delta([\Pi^m(j_2)]_{k_2-1}).$$

Therefore the unions on the right side of (4.6) contain the terms $B^m \mathfrak{R}_{j_1} + \mathbf{d}$ and $B^m \mathfrak{R}_{j_2} + \mathbf{d}$, where $\mathbf{d} = \Delta([\Pi^m(j_1)]_{k_1-1}) = \Delta([\Pi^m(j_2)]_{k_2-1})$. Hence $B^m \mathfrak{R}_{j_1}$ and $B^m \mathfrak{R}_{j_2}$ are measure-wise disjoint, and so $\mathfrak{R}_{j_1}$ and $\mathfrak{R}_{j_2}$ are measure-wise disjoint. This proves the theorem. $\square$

The positive strong coincidence condition is very difficult to verify in general. We verify it for a special case, in which the substitution Π has a unique periodic point $\mathbf{U}$.

Corollary 4.5. *Let $\mathfrak{R}$ be a Rauzy fractal associated to a fixed pint $\mathbf{U}$ of a primitive and unimodular Pisot substitution Π in $n \geq 2$ symbols. Suppose that $\mathbf{U}$ is the unique fixed point of Π^m for all $m \geq 1$. Then the natural decomposition $(\mathfrak{R}_1, \dots, \mathfrak{R}_n)$ of $\mathfrak{R}$ is measure-wise disjoint.*

Proof. Without loss of generality we assume that $\mathbf{U}(1) = 1$. This means that there exists an $m \geq 1$ such that all $\Pi^m(j)$ begins with the letter 1. Therefore Π satisfies the positive strong coincidence condition by taking $i = 1$ and $k_1 = k_2 = 1$. □

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