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Consider the stationary capillary problem of a drop of liquid attached to a fixed surface, so that the drop minimizes an energy functional subject to a volume constraint. There are many such capillary problems in which, due to the symmetry of the fixed surface, one cannot hope for a capillary surface which is a strict local minimum for energy. A weaker concept which is sensible to consider is that of local minimality modulo the isometries of space which map the fixed surface into itself. In other words, it is reasonable to attempt to show that, given a capillary surface, any nearby comparison surface will have energy greater than or equal to the given surface, and if the energy is equal, the comparison surface is simply a translation or rotation of the given surface. Eigenvalue conditions are derived which imply that a capillary surface is a strict local minimum modulo isometries, and are applied to the specific example of a liquid bridge between two parallel planes.

1. Introduction.

In [**18**], sufficient conditions are given for a capillary surface to be a constrained local minimum for the relevant energy functional. More specifically, suppose that Γ is the boundary of a fixed solid region in space, and that we put a drop of liquid in contact with Γ. Let Ω be the region in space occupied by the drop, and let Σ be the free boundary of Ω. In the absence of gravity or any other external potential, the shape of the drop results from minimizing the functional

$$\mathcal{E}(\Omega) = |\Sigma| - c|\Sigma_1| \tag{1.1}$$

where $|\Sigma|$ is the area of the free surface of the drop, $|\Sigma_1|$ is the area of the region on Γ wetted by the drop (i.e., $\partial\Omega \cap \Gamma$), and $c \in [-1, 1]$ is a material constant. The minimization is under the constraint that the volume of the drop is fixed. We will assume that $\overline{\Sigma}$ is compact. The first order necessary conditions for a drop to minimize the energy in (1.1) are that the mean curvature of Σ is a constant H and that the angle between the normals to Σ and to Γ along the curve of contact is constantly $\gamma = \arccos(c)$ (see [**4**]).

Taylor ([**13**]) has shown that a capillary surface in $\mathbb{R}^3$ must be analytic. Boundary regularity is also addressed by Taylor in [**13**]. If Γ is at least Hölder continuously differentiable, Taylor shows that $\partial\Sigma$ consists of a finite number of curves which are also Hölder continuously differentiable. If Γ is C^∞, higher regularity of the contact curve $\partial\Sigma$ follows by the routine argument of straightening Γ with a C^∞ diffeomorphism and using elliptic theory (see, e.g., [**8**]).

In the case that $c \neq -1$, 1, i.e., $\gamma \in (0, \pi)$, sufficient conditions for a drop to be a strict energy minimum subject to the volume constraint are derived in [**18**], and involve the eigenvalues of a certain partial differential operator. However, it is common to consider situations in which one cannot expect a strict local minimum for energy. For example, in considering a liquid bridge between two parallel planes (e.g., [**1**], [**6**], [**14**], [**15**], [**20**]), it's pointless to seek a strict local minimum for energy, since translations of bridges parallel to the planes will leave energy and volume unchanged. A drop hanging from a horizontal plane ([**19**]) is similarly indifferent to translations (although [**19**] includes a gravity term which we will neglect). A comparable situation occurs for a drop inside a sphere ([**11**]) or attached to the outside of a sphere. Rotations of space which map the sphere into itself will again leave energy and volume of a drop unchanged. Other examples are easily constructed.

What we can hope for are conditions under which a given capillary surface is a volume-constrained strict local minimum modulo isometries which map Γ onto itself. In other words, we seek conditions sufficient to show that any "nearby" drop containing the same volume will have energy greater than or equal to Σ, and if it has energy equal to Σ, it is isometric to Σ. For technical reasons, we will interpret "nearby" to mean close in the $C^3(\overline{\Sigma})$ norm. This is in contrast to [**18**], where it was possible to use the $C^1(\Sigma)$ norm.

In [**14**], I derived eigenvalue conditions which I claimed implied local minimality of the energy for bridges modulo translations. However, as Finn pointed out in [**5**], I was confusing minimality with stability. Essentially, a bridge is stable if it is a local minimum for energy to second order. This, however, does not imply that a bridge is a local minimum for energy. Stability is roughly the same as a certain quadratic form being positive definite, whereas a stronger condition is necessary to show local minimality (see [**7**], [**17**], and the introduction to [**18**] for discussion of this important point). Since several papers concerning liquid bridges have been based on the conditions in [**14**] (e.g., [**6**], [**20**]), this means that none of these papers actually dealt with local minimality of energy. Instead, they were concerned with the weaker concept of stability. In Section 5, the conditions in [**14**] will be shown to be equivalent to the conditions derived in Section 4 for local minimality modulo translations in the special case of a liquid bridge between parallel plates. Thus, the papers based on [**14**] are all strengthened, since they now

deal with local minimality modulo translations parallel to the planes. It should be noted that Miersemann ([**10**]) has also addressed the question of whether the conditions in [**14**] imply that a liquid bridge is an energy minimum in some sense.

2. The set-up and an example.

We are considering orientation preserving isometries of $\mathbb{R}^3$ which take Γ into itself but do not fix Σ. We may of course write these isometries as $\vec{x} \to A\vec{x} + \vec{b}$, where A is an orthogonal matrix with determinant $+1$ and $\vec{b}$ is a vector (see [**9**]). We assume that A and $\vec{b}$ depend differentiably on a vector parameter $\vec{z} \in \mathbb{R}^k$. We may also assume that $A(\vec{0}) = I$ and $\vec{b}(\vec{0}) = \vec{0}$. For simplicity we will occasionally write $G_{\vec{z}}(\vec{x})$ for $A(\vec{z})\vec{x} + \vec{b}(\vec{z})$.

The approach to generalizing normal variations that we will follow is the same as in [**18**]. We define a curvilinear coordinate system $X : \overline{\Sigma} \times [-\epsilon, \epsilon] \to \mathbb{R}^3$ so that if $p \in \Sigma$ then $X(p, 0) = p$ and if $p \in \partial\Sigma$ then $X(p, w) \in \Gamma$. For any $\varphi(p) \in C^1(\overline{\Sigma})$, $X(p, \varphi(p))$ describes an embedded surface which is the boundary of a physically realizable perturbation of the drop. This means that we have a natural way of associating energy and surface area to functions in $C^1(\overline{\Sigma})$. (For technical reasons, we will later need to restrict our attention to $C^3(\overline{\Sigma})$.) We will make the simplifying assumption that $X_w(p, 0) \cdot \vec{N} = 1$, where $\vec{N}$ is the unit normal to Σ. We will also assume that Σ and Γ are such that X and X^{-1} may be chosen to be infinitely differentiable (although thrice differentiable will suffice for the purposes of this paper). Finally, we shall restrict our attention to bounded capillary surfaces, so that $\overline{\Sigma}$ will be compact. For further discussion of this approach, see [**18**]. Note that the method being used cannot apply to the mathematically interesting cases $\gamma = 0$ and $\gamma = \pi$, since the regularity of the curvilinear coordinate system would break down in those cases.

We will use the following notation:

- We will refer to the function in $C^1(\overline{\Sigma})$ which is identically zero as o, to avoid confusion with the number 0.
- If $\varphi \in C^1(\overline{\Sigma})$, then Σ_φ is the surface $X(p, \varphi(p))$. (Note that $\Sigma_o = \Sigma$.)
- If $\varphi \in C^1(\overline{\Sigma})$ and $\vec{z} \in \mathbb{R}^k$, it may happen that the surface obtained by applying $G_{\vec{z}}$ to Σ_φ may be written as the graph of a C^1 function in curvilinear coordinates. In that case, we will call that function $Q(p; \varphi, \vec{z})$. Of course, we have $Q(p; \varphi, \vec{0}) = \varphi(p)$.
- There are several different norms used. $\|\vec{z}\|$ is the Euclidean norm in $\mathbb{R}^n$. If Σ may be parameterized by a single domain $D \in \mathbb{R}^2$ with a smooth extension to ∂D, then for $\varphi \in C^k(\overline{\Sigma})$ we shall define $\|\varphi\|_k$ to be the regular $C^k(\overline{D})$ norm of $\varphi(p(u, v))$ considered as a function of

$(u, v) \in \overline{D}$. This has an obvious generalization when Σ is parameterized in several patches.

We may obtain information about $Q(p; \varphi, \vec{z})$ from the implicit function theorem. If φ and $\vec{z}$ are such that $Q(p; \varphi, \vec{z})$ exists, then for each $\hat{p} \in \Sigma$ there must be a $p \in \Sigma$ so that

$$X(\hat{p}, Q) = G_{\vec{z}}\left(X(p, \varphi(p))\right). \tag{2.1}$$

This relation defines Q and p implicitly as functions of $\vec{z}$ and $\hat{p}$, so that we may calculate derivatives of Q from (2.1). We will later see (Lemma 3.2) that, under reasonable assumptions, Q is a differentiable function of $\vec{z}$ and p at least for $\vec{z}$ small. A result of this is that, at least for $\vec{z}$ small, $Q(p; o, \vec{z})$ is a k dimensional manifold in $C^1(\Sigma)$ through the origin. We will call this manifold $\mathcal{I}$. An interpretation of $\mathcal{I}$ is that it is the collection of functions in $C^1(\Sigma)$ whose graphs in curvilinear coordinates are simply translations or rotations of the original surface Σ. The tangent space to $\mathcal{I}$ at the origin is spanned by the functions $\mu_j(p) = \frac{\partial}{\partial z_j}\left(Q(p; o, \vec{z})\right)$.

A lower dimensional example may help clarify the approach outlined above. Instead of surfaces in space, we will consider curves in the plane. More specifically, suppose that Σ is the unit circle, and that Γ is the empty set. A natural curvilinear coordinate system in a neighborhood of Σ is given by

$$X(p, w) = (w + 1)p, \tag{2.2}$$

where $p \in \Sigma$ and $w \in \left[-\frac{1}{2}, \frac{1}{2}\right]$. Since we require that $X_w \cdot \vec{N} = 1$, this leads to taking $\vec{N}$ to be the outward normal. If we use the natural parameterization of Σ in (2.2), we may write the coordinate system as

$$X(p(u), w) = \langle (w + 1)\cos u, (w + 1)\sin u \rangle .$$

Now, choose the group $G_{\vec{z}}$ to consist simply of translations by the vector $\vec{z} = \langle z_1, z_2 \rangle$. The function $Q(p; \varphi, \vec{z})$ may be understood as follows. Given a function $\varphi(p)$ defined on Σ, its graph in curvilinear coordinates is the curve $(\varphi(p) + 1)p$, $p \in \Sigma$. Translate this graph by the vector $\vec{z}$. The resulting curve may also be the graph of a function in curvilinear coordinates. If so, we will call the function of which it is a graph $Q(p; \varphi, \vec{z})$, where φ and $\vec{z}$ may be viewed as parameters. More precisely, given $\varphi(p)$, the relation

$$X(\hat{p}, Q) = X(p, \varphi(p)) + \vec{z} \tag{2.3}$$

generically determines Q and p as functions of $\hat{p}$ and $\vec{z}$. Once we have the function $Q(\hat{p})$, we replace the argument by p.

To illustrate, we will determine $Q(p; o, \vec{z})$ for the example of the unit circle. Making the natural identification of p with $\langle \cos u, \sin u \rangle$, (2.3) becomes

$$\begin{aligned} (Q+1)\cos\hat{u} &= \cos u + z_1 \\ (Q+1)\sin\hat{u} &= \sin u + z_2. \end{aligned} \tag{2.4}$$

It is straightforward to eliminate u from (2.4) and obtain that

$$Q = -1 + z_1 \cos\hat{u} + z_2 \sin\hat{u} + \sqrt{1 - (z_1 \sin\hat{u} - z_2 \cos\hat{u})^2}$$

so that, removing the hats and changing back to p,

$$Q(p; o, \vec{z}) = -1 + \vec{z} \cdot p + \sqrt{1 - (\vec{z} \cdot p^{\perp})^2}, \tag{2.5}$$

where $p^{\perp}$ is obtained by rotating p clockwise by $90°$. Note that $Q(p; o, \vec{0})$ is identically zero, as we would expect. The set of functions described in (2.5) is the manifold $\mathcal{I}$ for this example. Tangent vectors to $\mathcal{I}$ at the origin are obtained by differentiating (2.5) with respect to z_i and are $\mu_1(p) = p \cdot \langle 1, 0 \rangle$ and $\mu_2(p) = p \cdot \langle 0, 1 \rangle$.

3. An orthogonality result.

We return now to the general case. It is clear that if $\vec{z}$ is very large, then $Q(p; o, \vec{z})$ might not be defined, since a large motion of Σ may lead to a surface which is not a graph in curvilinear coordinates. Even in the case that $\vec{z}$ is small, $Q(p; \varphi, \vec{z})$ may turn out to be undefined if φ has large derivatives. However, we do have the following result for existence.

Lemma 3.1. *There exist $\delta_1 > 0$ and $\delta_2 > 0$ so that if $\|\vec{z}\| < \delta_1$ and $\|\varphi\|_1 < \delta_2$, then $Q(p; \varphi, \vec{z})$ is defined.*

Proof. This follows from [**18**], Theorem 3.1, which states that a vector perturbation of Σ which is sufficiently small and has sufficiently small derivatives will result in a surface which is a graph in curvilinear coordinates. □

We need to have some idea of how smooth Q will be.

Lemma 3.2. *For any positive integer k, there are $\delta_1 > 0$, $\delta_2 > 0$, so that if $\varphi \in C^k(\overline{\Sigma})$ is a fixed function with $\|\varphi\|_k < \delta_2$, then $Q(p; \varphi, \vec{z})$, considered as a function of p and $\vec{z}$, has continuous derivatives of all orders up to k, at least on $\overline{\Sigma} \times B_{\delta_1}$.*

Proof. Focusing on a particular coordinate patch, we let $Y(u, v, w) = X(p(u, v), w)$. If we are given $\hat{u}$, $\hat{v}$, $\vec{z}$, and φ, then the relation

$$Y(\hat{u}, \hat{v}, Q) = A(\vec{z}) Y(u, v, \varphi(u, v)) + \vec{b}(\vec{z}) \tag{3.1}$$

implicitly determines Q, u, and v as functions of $\hat{u}$, $\hat{v}$, and $\vec{z}$. Let $F : \mathbb{R}^{k+5} \to \mathbb{R}^3$ be defined by

$$F(u, v, Q, \hat{u}, \hat{v}, \vec{z}) = Y(\hat{u}, \hat{v}, Q) - G_{\vec{z}}(Y(u, v, \varphi(p(u, v)))). \tag{3.2}$$

Note that if φ is k times differentiable, so is F. Following the statement of the implicit function theorem in [**2**], we let x correspond to (u, v, Q) and y correspond to $(\hat{u}, \hat{v}, \vec{z})$. Fix a point (u_0, v_0) in the parameter domain, and let $x_0 = (u_0, v_0, \varphi(p(u_0, v_0)))$, $y_0 = (u_0, v_0, \vec{0})$. Then $F(x_0, y_0) = \vec{0}$. To show that $F(x, y) = \vec{0}$ determines x as a C^k function of y in a neighborhood of (x_0, y_0), we need to show that $D_xF(x_0, y_0)$ has a bounded inverse. But D_xF is the linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^3$ whose matrix is

$$D_xF$$
$$= \begin{pmatrix} -A(\vec{z})Y_1(u, v, \varphi(p(u, v))) - A(\vec{z})Y_3(u, v, \varphi(p(u, v)))\frac{\partial}{\partial u}\varphi(p(u, v)) \\ -A(\vec{z})Y_2(u, v, \varphi(p(u, v))) - A(\vec{z})Y_3(u, v, \varphi(p(u, v)))\frac{\partial}{\partial v}\varphi(p(u, v)) \\ Y_3(\hat{u}, \hat{v}, Q) \end{pmatrix}^T.$$

Therefore,

$$D_xF(x_0, y_0) \tag{3.3}$$
$$= \begin{pmatrix} -Y_1(u_0, v_0, \varphi(p(u_0, v_0))) - Y_3(u_0, v_0, \varphi(p(u_0, v_0)))\frac{\partial}{\partial u}(\varphi(p(u_0, v_0))) \\ -Y_2(u_0, v_0, \varphi(p(u_0, v_0))) - Y_3(u_0, v_0, \varphi(p(u_0, v_0)))\frac{\partial}{\partial v}(\varphi(p(u_0, v_0))) \\ Y_3(u_0, v_0, \varphi(p(u_0, v_0))) \end{pmatrix}^T.$$

But

$$\begin{pmatrix} -Y_1(u_0, v_0, \varphi(p(u_0, v_0))) \\ -Y_2(u_0, v_0, \varphi(p(u_0, v_0))) \\ Y_3(u_0, v_0, \varphi(p(u_0, v_0))) \end{pmatrix}^T$$

is nonsingular by the assumptions made on X. Therefore $D_xF(x_0, y_0)$ is nonsingular for $\frac{\partial}{\partial u}(\varphi(p(u_0, v_0)))$ and $\frac{\partial}{\partial v}(\varphi(p(u_0, v_0)))$ sufficiently small, and we may apply Theorem 2.3 of [**2**]. □

We will need to be able to find the functions $\mu_k(p)$ which span the tangent space to $\mathcal{I}$ at o. This may be accomplished by applying the following lemma.

Lemma 3.3. *For each k,*

$$\mu_k(p) = \frac{\partial}{\partial z_k} G_{\vec{z}}(p) \cdot \vec{N}. \tag{3.4}$$

Proof. If we are given $\hat{u}$, $\hat{v}$, and $\vec{z}$, then $Q \in \mathcal{I}$ is determined as a function of these variables by the relation

$$Y(\hat{u}, \hat{v}, Q) = A(\vec{z})p(u, v) + \vec{b}(\vec{z}), \tag{3.5}$$

which comes from substituting o for φ in (3.1). Differentiating (3.5) with respect to z_k, we obtain

$$Y_w(\hat{u}, \hat{v}, Q)\frac{\partial Q}{\partial z_k} = \frac{\partial A}{\partial z_k}p(u, v) + A(\vec{z})\left(p_u\frac{\partial u}{\partial z_k} + p_v\frac{\partial v}{\partial z_k}\right) + \frac{\partial \vec{b}}{\partial z_k}.$$

Setting $\vec{z} = \vec{0}$ and using the fact that for that value of $\vec{z}$ we have that $A = I$, $\vec{b} = \vec{0}$, $\hat{u} = u$, $\hat{v} = v$, and $Q = 0$, there follows that

$$Y_w(u,v,0)\frac{\partial Q}{\partial z_k} = \frac{\partial A}{\partial z_k}p(u,v) + \frac{\partial \vec{b}}{\partial z_k} + \left(p_u\frac{\partial u}{\partial z_k} + p_v\frac{\partial v}{\partial z_k}\right). \tag{3.6}$$

Now, take the dot product of (3.6) with the normal $\vec{N}$ to Σ. Since p_u and p_v are tangent to Σ, these terms disappear in taking the dot product. Moreover, as mentioned above, the curvilinear coordinate system is chosen so that $Y_w(u,v,0)\cdot\vec{N} = 1$. Thus (3.6) becomes

$$\frac{\partial Q}{\partial z_k} = \left(\frac{\partial A}{\partial z_k}p(u,v) + \frac{\partial \vec{b}}{\partial z_k}\right)\cdot\vec{N}$$

from which (3.4) follows. □

Thus far, we have put no particular conditions on the parameterization $G_{\vec{z}}$ of the group of isometries beyond differentiability and the requirement that $G_{\vec{0}}$ be the identity. It will be convenient to have the functions $\mu_j(p)$ form an orthonormal set in $H^1(\Sigma)$. This may be accomplished, without loss of generality, by applying the Gram-Schmidt procedure to the original set $\{\mu_1,\ldots,\mu_k\}$, and then reparameterizing the group $G_{\vec{z}}$ to reflect this. We will therefore assume that the set $\{\mu_1,\ldots,\mu_k\}$ forms an orthonormal basis for the tangent space to $\mathcal{I}$ at the origin.

The purpose of the remainder of this section is to establish that, given a $\varphi \in C^3(\overline{\Sigma})$ with a sufficiently small C^3 norm, there is a vector $\vec{z} \in \mathbb{R}^k$ so that $Q(p;\varphi,\vec{z})$ is orthogonal to $\{\mu_1,\ldots,\mu_k\}$ in H^1. The point of this is that the surface which is the graph of $Q(p;\varphi,\vec{z})$ clearly has the same surface energy and surrounds the same volume as the original Σ_φ, so that if we show that the origin is a strong local minimum in the subspace orthogonal to $\mathcal{I}$, it will follow that the origin is a local minimum modulo the group of isometries over the entire space (see Lemma 4.1). The general approach is as follows. For a given φ we define a map $\mathcal{C}_\varphi : \mathbb{R}^k \to \mathbb{R}^k$ by

$$\mathcal{C}_\varphi(\vec{z}) = \vec{z} - \begin{pmatrix} \langle Q(p;\varphi,\vec{z}),\mu_1\rangle \\ \vdots \\ \langle Q(p;\varphi,\vec{z}),\mu_k\rangle \end{pmatrix}, \tag{3.7}$$

where the inner products are in H^1. If, for given φ, $\vec{z}$ is a fixed point of the map $\mathcal{C}_\varphi$, then clearly $Q(p;\varphi,\vec{z})$ is orthogonal to $\{\mu_1,\ldots,\mu_k\}$. We will show that there is an $\epsilon > 0$ so that, for any φ with $|\varphi|$, $|\nabla\varphi|$ and $|D^2\varphi|$ less than ϵ and with third derivatives bounded, the map $\mathcal{C}_\varphi$ will have a fixed point. This will be done by first showing that for such a φ, $\mathcal{C}_\varphi$ takes a small closed ball of $\mathbb{R}^n$ into itself and then using an obvious compactness argument.

We will first establish a lemma for functions defined on $\mathbb{R}^2$. Given a function $f(x,y)$ and functions $a(x,y)$, $b(x,y)$, $c(x,y)$, we define a function $g(\hat{x},\hat{y})$ to be the solution to

$$\hat{x} = x + a(x,y) \tag{3.8}$$

$$\hat{y} = y + b(x,y) \tag{3.9}$$

$$g = f(x,y) + c(x,y). \tag{3.10}$$

In other words, for given $\hat{x}$, $\hat{y}$, Equations (3.8), (3.9), and (3.10) generically determine x, y, and g as functions of $\hat{x}$ and $\hat{y}$. Less formally, we perturb the graph of a function $f(x,y)$ by a vector perturbation $\langle a(x,y), b(x,y), c(x,y)\rangle$, and consider the resulting surface. Theorem 3.1, [**18**] implies that the resulting surface is a graph, as long as the C^1 norms of a, b, and c are sufficiently small. We let $g(x,y)$ be the function of which the perturbed surface is the graph. We will be interested in determining how far the function $g(x,y)$ is from $f(x,y)$. To note a possible point of confusion, "$g(x,y)$" means that we determine the function $g(\hat{x},\hat{y})$ from (3.8)–(3.10) and then substitute the point (x,y) into that function.

Lemma 3.4. *Suppose that $f(x,y)$ has its C^2 norm bounded by 1, and that g is defined by (3.8)-(3.10). Then, for any $\epsilon > 0$ there is a $\delta > 0$ so that if a, b, c have their C^2 norms bounded by δ then $|f(x,y) - g(x,y)| < \epsilon$ and $\|\nabla f - \nabla g\| < \epsilon$ holds for all (x,y). Moreover, δ depends only on ϵ, not on f.*

Proof. We start by bounding the first derivatives of g. Differentiating (3.8) and (3.9) with respect to $\hat{x}$ we obtain

$$\begin{aligned} 1 =& (1 + a_1(x,y))\frac{\partial x}{\partial \hat{x}} + a_2(x,y)\frac{\partial y}{\partial \hat{x}} \\ 0 =& b_1(x,y)\frac{\partial x}{\partial \hat{x}} + (1 + b_2(x,y))\frac{\partial y}{\partial \hat{x}} \end{aligned}$$

so that

$$\frac{\partial x}{\partial \hat{x}} = \frac{1 + b_2(x,y)}{(1 + a_1(x,y))(1 + b_2(x,y)) - b_1(x,y)a_2(x,y)} \tag{3.11}$$

and

$$\frac{\partial y}{\partial \hat{x}} = \frac{-b_1(x,y)}{(1 + a_1(x,y))(1 + b_2(x,y)) - b_1(x,y)a_2(x,y)}. \tag{3.12}$$

From (3.11) and (3.12) it is clear that we can assume that $\left|\frac{\partial x}{\partial \hat{x}}\right|$ and $\left|\frac{\partial y}{\partial \hat{x}}\right|$ are bounded by 2 by requiring δ to be sufficiently small. (A bound that works is to assume that $\delta < \frac{1}{10}$.)

Next differentiate (3.10) with respect to $\hat{x}$ to obtain that

$$\frac{\partial g}{\partial \hat{x}} = f_1(x,y)\frac{\partial x}{\partial \hat{x}} + f_2(x,y)\frac{\partial y}{\partial \hat{x}} + c_1(x,y)\frac{\partial x}{\partial \hat{x}} + c_2(x,y)\frac{\partial y}{\partial \hat{x}}. \tag{3.13}$$

This shows that $\left|\frac{\partial g}{\partial \hat{x}}\right|$ is bounded by $2+2\delta < 3$. We may bound $\frac{\partial g}{\partial \hat{y}}$ similarly.

Now,

$$|f(x,y) - g(x,y)| \leq |f(x,y) - g(\hat{x},\hat{y})| + |g(\hat{x},\hat{y}) - g(x,y)|\,. \tag{3.14}$$

The first term on the right side of (3.14) is simply $|c(x,y)|$ and is bounded by δ (to be determined). Using the Mean Value theorem and the fact that the components of ∇g are bounded by 3, we may bound the second term on the right side of (3.14) by a constant times the length of the vector $\langle \hat{x}-x, \hat{y}-y\rangle = \langle a(x,y), b(x,y)\rangle$. Therefore, $|f(x,y) - g(x,y)|$ is bounded by a constant (which may be found explicitly) times δ. Taking δ small enough, we may make $|f(x,y) - g(x,y)|$ less than ϵ.

We use an analogous idea to bound $\|\nabla f - \nabla g\|$. We may bound the second derivatives of g by first bounding the second derivatives of x and y as functions of $\hat{x}$ and $\hat{y}$. It is not necessary to find these second partials explicitly. If we take the partial derivative of (3.11) with respect to $\hat{x}$, we find an expression for $\frac{\partial^2 x}{\partial \hat{x}^2}$ which is a rational function whose numerator is a polynomial in first and second partials of a and b and first partials of x and y. The denominator is $((1+a_1(x,y))(1+b_2(x,y)) - b_1(x,y)a_2(x,y))^2$, which we can bound from zero. Since we already have bounds for the first partials of x and y, this shows that $\frac{\partial^2 x}{\partial \hat{x}^2}$ may be bounded by requiring the first and second derivatives of a, b, and c to be sufficiently small. The other second partials of x and y are similarly bounded. This immediately leads to bounds for the second partials of g. For example, to bound $\frac{\partial^2 g}{\partial \hat{x}^2}$, differentiate (3.13) with respect to $\hat{x}$, and use the bounds for the first and second partials of x and y with respect to $\hat{x}$ and $\hat{y}$ along with the assumed bounds for the second partials of f and of c. Bounds for the other second partials of g may be found similarly.

There holds

$$|f_1(x,y) - g_1(x,y)| \leq |f_1(x,y) - g_1(\hat{x},\hat{y})| + |g_1(\hat{x},\hat{y}) - g_1(x,y)|. \tag{3.15}$$

The second term on the right-hand side of (3.15) may be bounded by a constant times δ in the same manner as was done for (3.14), since we have bounds for the second derivatives of g. To find a bound for the first term, note that

$$g_1(\hat{x},\hat{y}) = (f_1(x,y) + c_1(x,y))\frac{\partial x}{\partial \hat{x}} + (f_2(x,y) + c_2(x,y))\frac{\partial y}{\partial \hat{x}}$$

so that

$$g_1(\hat{x},\hat{y}) - f_1(x,y) = f_1(x,y)\left(\frac{\partial x}{\partial \hat{x}} - 1\right) + c_1(x,y)\frac{\partial x}{\partial \hat{x}} + (f_2(x,y) + c_2(x,y))\frac{\partial y}{\partial \hat{x}}. \tag{3.16}$$

All of $\frac{\partial x}{\partial \hat{x}} - 1$, c_1 and $\frac{\partial y}{\partial \hat{x}}$ may be bounded by explicit constants times δ. From this, (3.16), and (3.15), it follows that $|f_1(x,y) - g_1(x,y)|$ may be bounded by a constant times δ. We may therefore make this difference less than $\frac{\epsilon}{\sqrt{2}}$ by taking δ sufficiently small. Handling $|f_2(x,y) - g_2(x,y)|$ in a similar fashion, we see that we can force $\|\nabla g - \nabla f\|$ to be less than ϵ by requiring δ to be sufficiently small. □

We will now establish the analog of Lemma 3.4 for curvilinear coordinates. Without loss of generality, we will restrict ourselves to a single coordinate patch of Σ, parameterized by u and v. As before, let $Y(u,v,w) = X(p(u,v),w)$ represent the curvilinear coordinate system corresponding to this coordinate patch. In curvilinear coordinates, the analog of the definition of g in (3.8), (3.9) and (3.10) is

$$Y(\hat{u},\hat{v},g) = Y(u,v,f(u,v)) + \langle a(u,v), b(u,v), c(u,v)\rangle\,. \tag{3.17}$$

Lemma 3.5. *Suppose that, given $f(u,v)$, $g(u,v)$ is defined by* (3.17). *Suppose that $Y(u,v,w)$ is invertible, with the Jacobians of Y and of Y^{-1} bounded. Suppose also that the second and third derivatives of the components of Y and of Y^{-1} are also bounded. Then the result analogous to Lemma* 3.4 *holds.*

Proof. Equation (3.17) may be interpreted as

$$\hat{u} = u + \widetilde{a}(u,v) \tag{3.18}$$

$$\hat{v} = v + \widetilde{b}(u,v) \tag{3.19}$$

$$g = f(u,v) + \widetilde{c}(u,v) \tag{3.20}$$

where $\widetilde{a}$, $\widetilde{b}$, and $\widetilde{c}$ are defined by

$$\left\langle \widetilde{a}(u,v), \widetilde{b}(u,v), \widetilde{c}(u,v)\right\rangle = Y^{-1}\big(Y(u,v,f(u,v)) + \langle a(u,v), b(u,v), c(u,v)\rangle\big) - \langle u,v,f(u,v)\rangle\,.$$

We wish to apply Lemma 3.4, so that we need to show that we can force the C^2 norms of $\widetilde{a}$, $\widetilde{b}$, and $\widetilde{c}$ to be small by requiring the C^2 norms of a, b, and c to be small.

Considering a, b, c, f, u and v to be independent variables, define the functions $A(a,b,c,f,u,v)$, $B(a,b,c,f,u,v)$, and $C(a,b,c,f,u,v)$ by

$$\langle A,B,C\rangle = Y^{-1}\left(Y(u,v,f) + \langle a,b,c\rangle\right) - \langle u,v,f\rangle\,.$$

It is clear that the first, second and third partials of A, B, and C are bounded, since they depend only on those of Y and Y^{-1}. Also note that

$$A(0,0,0,f,u,v) = B(0,0,0,f,u,v) = C(0,0,0,f,u,v) = 0 \tag{3.21}$$

holds for all values of f, u, and v. We have that

$$\widetilde{a}(u,v) = A(a(u,v), b(u,v), c(u,v), f(u,v), u, v)$$

with analogous equations for $\widetilde{b}$ and $\widetilde{c}$. It is straightforward to show that

$$|\widetilde{a}(u,v)| \leq K_1 \left\| \langle a(u,v), b(u,v), c(u,v) \rangle \right\|,$$

where K depends on the first partials of A, by applying the Mean Value theorem to (3.21). We may bound $|\widetilde{b}(u,v)|$ and $|\widetilde{c}(u,v)|$ similarly.

We next bound the first partials. There holds

$$\frac{\partial \widetilde{a}}{\partial u} = A_1 a_u + A_2 b_u + A_3 c_u + A_4 f_u + A_5. \tag{3.22}$$

The first three terms on the right side of (3.22) present no problem, since the partials of A are bounded. From (3.21), we see that $A_4(0,0,0,f,u,v) = 0$, so that

$$|A_4(a,b,c,f,u,v)| \leq K_2 \left\| \langle a(u,v), b(u,v), c(u,v) \rangle \right\|$$

using the Mean Value theorem and the bounds for the second derivatives of A. A similar bound holds for A_5. Therefore, from (3.22) we obtain that

$$\left| \frac{\partial \widetilde{a}}{\partial u} \right| \leq K\delta,$$

if the C^2 norms of a, b, and c are all less than some $\delta_1 > 0$. (Actually, all we need at this point is that the C^1 norms of a, b, and c are bounded by δ_1.) We may similarly bound all first partials of $\widetilde{a}$, $\widetilde{b}$, and $\widetilde{c}$.

Differentiating (3.22) with respect to $\hat{u}$, we obtain

$$\frac{\partial^2 \widetilde{a}}{\partial u^2} = (A_{11} a_u + A_{12} b_u + A_{13} c_u + A_{14} f_u + A_{15}) a_u + A_1 a_{uu} + \cdots + A_{55}. \tag{3.23}$$

The only terms of (3.23) which do not contain factors of derivatives of a, b, or c are $A_{44} f_u^2$, $A_{45} f_u$, and A_{55}. These may be bounded by a constant times $\| \langle a(u,v), b(u,v), c(u,v) \rangle \|$ as before, using the fact that the third derivatives of A are bounded. A similar argument may be applied to all the second partials of $\widetilde{a}$, $\widetilde{b}$, and $\widetilde{c}$ to conclude that if the C^2 norms of a, b, and c are all bounded by $\delta_1 > 0$ then there is a K so that the C^2 norms of $\widetilde{a}$, $\widetilde{b}$, and $\widetilde{c}$ are all bounded by $K\delta_1$.

We now apply Lemma 3.4 to Equations (3.18)–(3.20) to conclude that, given $\epsilon > 0$, there is a $\delta > 0$ so that if we bound the C^2 norms of $\widetilde{a}$, $\widetilde{b}$, and $\widetilde{c}$ by δ we bound the C^1 norm of $f(u,v) - g(u,v)$ by ϵ. But we may force the

C^2 norms of $\widetilde{a}$, $\widetilde{b}$, and $\widetilde{c}$ to be bounded by δ by requiring the C^2 norms of a, b, and c to be less than $\frac{\delta}{K}$. □

Lemma 3.6. *Given $\delta > 0$ sufficiently small, there is an $\zeta > 0$ so that if $\|\varphi\|_2 < \zeta$, then $\|Q(p;\varphi,\vec{z}) - Q(p;o,\vec{z})\|_1 \leq \frac{\delta}{2}$ holds for all $\vec{z}$ with $\|\vec{z}\| \leq \delta$. Here $\|\cdot\|_k$, $k = 1, 2$, is the $C^k(\overline{\Sigma})$ norm.*

Proof. First, we will choose $\delta > 0$ to be smaller than δ_1 from Lemma 3.1. We will also need δ small enough so that $\|\vec{z}\| < \delta$ implies that $Q(p;o,\vec{z})$ has its C^2 norm (where p is the variable) bounded by 1. It is possible to choose such a δ since $Q(p;o,\vec{z})$ is continuous on the compact set $\overline{\Sigma} \times \overline{B}_{\delta_1/2}$.

We may now apply Lemma 3.5 to conclude that there is a number $\eta > 0$ so that if we perturb the surface $X(p, Q(p;o,\vec{z}))$ by a vector perturbation whose C^2 norm is less than η then the resulting surface is the curvilinear graph of a function that is within $\delta/2$ of $Q(p;o,\vec{z})$ in the C_1 norm. We therefore need to show that there is $\zeta > 0$ so that if $\|\varphi\|_2 < \zeta$ then the vector perturbation from $X(p, Q(p;o,\vec{z}))$ to $X(p, Q(p;\varphi,\vec{z}))$ has C^2 norm bounded by η, regardless of the choice of $\vec{z}$, as long as $\|\vec{z}\| < \delta$. All we need, however, is to ensure that the perturbation from $X(p,o)$ to $X(p,\varphi(p))$ has its C^2 norm bounded by η, since applying the isometry $G_{\vec{z}}$ to both of these surfaces will not change the perturbation. This is easily done by bounding the derivatives of φ appropriately. □

Theorem 3.1. *For every $\varphi(p) \in C^3(\overline{\Sigma})$ with sufficiently small C^2 norm there is an isometry $G_{\vec{z}}$ so that $Q(p;\varphi,\vec{z})$ is orthogonal to $\{\mu_1,\ldots,\mu_k\}$ in $H^1(\Sigma)$.*

Proof. Rewrite Equation (3.7) as

$$\mathcal{C}_\varphi(\vec{z}) = \vec{z} - \begin{pmatrix} \langle Q(p;o,\vec{z}),\mu_1\rangle \\ \vdots \\ \langle Q(p;o,\vec{z}),\mu_k\rangle \end{pmatrix} - \begin{pmatrix} \langle Q(p;\varphi,\vec{z}) - Q(p;o,\vec{z}),\mu_1\rangle \\ \vdots \\ \langle Q(p;\varphi,\vec{z}) - Q(p;\varphi,\vec{z}),\mu_k\rangle \end{pmatrix}. \tag{3.24}$$

From Lemma 3.2, we know that Q is thrice differentiable as a function of the variables $\vec{z}$ and p. Since Q is twice differentiable as a function of $\vec{z}$ it follows that

$$|Q(p;o,\vec{z}) - (z_1\mu_1 + \cdots + z_k\mu_k)| \leq K\|\vec{z}\|^2$$

for a uniform K holding for all $p \in \Sigma$. Similarly, since third derivatives of Q are bounded, we have

$$\|\nabla_\Sigma Q(p;o,\vec{z}) - (z_1\nabla_\Sigma\mu_1 + \cdots + z_k\nabla_\Sigma\mu_k)\|_\Sigma \leq K'\|\vec{z}\|^2 \tag{3.25}$$

for a uniform constant K', using the fact that converting from the gradient at a point in the parameter domain to the gradient on the surface is done by applying a bounded linear transformation (see [**12**], Volume 4). In (3.25), ∇_Σ is the gradient on Σ, and $\|\cdot\|_\Sigma$ is the metric on Σ.

Therefore, since the μ_j's are orthonormal, the magnitude of the vector

$$\vec{z} - \begin{pmatrix} \langle Q(p; o, \vec{z}), \mu_1 \rangle \\ \vdots \\ \langle Q(p; o, \vec{z}), \mu_k \rangle \end{pmatrix}$$

is bounded by a constant times $\|\vec{z}\|^2$. It follows that for all $\delta > 0$ sufficiently small, $\|\vec{z}\| \leq \delta$ implies that

$$\left\| \vec{z} - \begin{pmatrix} \langle Q(p; o, \vec{z}), \mu_1 \rangle \\ \vdots \\ \langle Q(p; o, \vec{z}), \mu_k \rangle \end{pmatrix} \right\| \leq \frac{\delta}{2}.$$

We now may apply Lemma 3.6 to the last term in (3.24) to conclude that for φ with a sufficiently small C^2 norm, there is a $\delta > 0$ so that $\mathcal{C}_\varphi(\vec{z})$ maps the closed $\mathbb{R}^k$ ball of radius δ centered at the origin continuously into itself. This implies that, for such φ's, $\mathcal{C}_\varphi(\vec{z})$ has a fixed point. But when $\mathcal{C}_\varphi(\vec{z}) = \vec{z}$, clearly $Q(p; \varphi, \vec{z})$ is orthogonal to $\mu_1, \ldots, \mu_k$, as desired. □

4. Eigenvalue conditions.

For simplicity, we will call the subspace of φ's in $H^1(\Sigma)$ which are perpendicular to $\{\mu_1, \ldots \mu_k\}$ $\mathcal{I}^\perp$. Theorem 3.1 tells us that, given a $\varphi \in C^3(\overline{\Sigma})$ which is sufficiently small in $C^2(\overline{\Sigma})$, there is a $\vec{z}$ so that $Q(p; \varphi, \vec{z}) \in \mathcal{I}^\perp$. What this means is that the isometry $G_{\vec{z}}$, when applied to Σ_φ, results in a surface which is the graph in curvilinear coordinates, of a function in $\mathcal{I}^\perp$. It therefore makes sense to focus on $\mathcal{I}^\perp$. For simplicity, we will require closeness in the space $C^3(\overline{\Sigma})$ rather than using the C^2 norm but then requiring the third derivatives to be uniformly bounded.

Lemma 4.1. *If the origin is a constrained local minimum for energy in the space $C^3(\overline{\Sigma}) \cap \mathcal{I}^\perp$, then Σ is a constrained local minimum (in the $C^3(\overline{\Sigma})$ norm) for energy modulo the group of isometries $G_{\vec{z}}$.*

Proof. Suppose that φ is a small function in $C^3(\overline{\Sigma})$ such that the volume surrounded by Σ_φ equals that surrounded by Σ and that $\mathcal{E}(\Sigma_\varphi) \leq \mathcal{E}(\Sigma)$. Let $\psi \in \mathcal{I}^\perp$ be $Q(p; \varphi, \vec{z})$ for the value of $\vec{z}$ promised by Theorem 3.1. Clearly Σ_φ and Σ_ψ surround the same volume and have the same surface energy. If the origin has been shown to be a constrained local minimum for energy in $\mathcal{I}^\perp$, it follows that $\psi = o$, and Σ_φ is simply an isometry applied to Σ. □

We therefore must seek conditions under which the origin will be a constrained local minimum for energy in $\mathcal{I}^\perp$. The situation is similar to that studied in [**18**], and we will recap some concepts from that paper. In Lemma 2.5 of that paper it is shown that, if Σ_φ surrounds the same volume as Σ,

then

$$\mathcal{E}(\varphi) = \mathcal{E}(o) + \frac{1}{2}\mathcal{M}(\varphi, \varphi) + \eta,$$

where η is a higher order error term and $\mathcal{M}$ is the quadratic form whose associated bilinear form is

$$\mathcal{M}(\varphi, \psi) = \iint_{\Sigma} \nabla\varphi \cdot \nabla\psi - |S|^2 \varphi\psi \, d\Sigma + \oint_{\partial\Sigma} \rho\varphi\psi \, d\sigma.$$

Here $|S|^2$ is the square of the norm of the second fundamental form, which may be written as $2(2H^2 - K)$, where H is mean curvature and K is Gaussian curvature. Also, ρ is given by

$$\rho = \kappa_\Sigma \cot\gamma - \kappa_\Lambda \csc\gamma, \tag{4.1}$$

where κ_Σ is the curvature of the curve $\Sigma \cap \Pi$ and κ_Λ is the curvature of the curve $\Lambda \cap \Pi$, if Π is a plane normal to the contact curve $\partial\Sigma$.

Related to the bilinear form $\mathcal{M}$ is the linear operator $A : H^1(\Sigma) \to H^1(\Sigma)$ which is defined by the relation

$$\langle \varphi, A\psi \rangle = \mathcal{M}(\varphi, \psi). \tag{4.2}$$

The left-hand side of (4.2) is the H^1 inner product. We plan to prove the analog of [**18**], Theorem 2.1, except that instead of working in the entire space $H^1(\Sigma)$, we will be working in the subspace $\mathcal{I}^\perp$. At a crucial point in the proof of [**18**], Theorem 2.1, a small constant is subtracted from a function satisfying $V(\varphi) = V(o)$ to obtain a function φ^* satisfying $\iint_\Sigma \varphi^* \, d\Sigma = 0$. For this to work in the present situation, we must show that subtracting a constant from $\varphi \in \mathcal{I}^\perp$ leads to a function $\varphi^* \in \mathcal{I}^\perp$. In other words, we must show that constant functions are in $\mathcal{I}^\perp$.

Lemma 4.2. *For $i = 1, \ldots, k$, there holds*

$$\iint_{\Sigma} \mu_i \, d\Sigma = 0.$$

As a result, all constants are in $\mathcal{I}^\perp$.

Proof. By assumption, the curvilinear graph of any function in $\mathcal{I}$ is an isometry applied to Σ. In particular, if $\varphi \in \mathcal{I}$, $V(\varphi) = V(o)$. Since the first order term in the expansion of $V(\varphi)$ around o is $\iint_\Sigma \varphi \, d\Sigma$ (see [**18**] (2.9)) the result follows by differentiating $V(Q(p; o, \vec{z}))$ and substituting $\vec{0}$ for $\vec{z}$. □

The analog of [**18**], Theorem 2.1 is the following.

Theorem 4.1. *Suppose that Σ is a capillary surface, with contact angle $\gamma \in (0, \pi)$. Suppose that $G_{\vec{z}}$ is a group of isometries which take the fixed surface Λ into itself, but which do not take Σ into itself. Suppose that $\mathcal{M}(\varphi, \varphi)$ is strongly positive on the subspace of $\mathcal{I}^\perp$ consisting of all $\varphi \in \mathcal{I}^\perp$ for which $\iint_\Sigma \varphi \, d\Sigma = 0$. Then o is a strict local minimum in $C^3(\overline{\Sigma}) \cap \mathcal{I}^\perp$ for $\mathcal{E}$,*

subject to the volume constraint $V(\varphi) = V(o)$, and hence Σ is a constrained local energy minimum in $C^3(\overline{\Sigma})$ modulo the group of isometries $G_{\vec{z}}$.

Proof. The proof that o is a constrained strict local minimum for energy in the space $C^1(\overline{\Sigma}) \cap \mathcal{I}^\perp$ is the same as the proof of Theorem 2.1 of [**18**], which goes through now that we have shown in Lemma 4.2 that constants are in $\mathcal{I}^\perp$. Since o is a local minimum in $C^1(\overline{\Sigma}) \cap \mathcal{I}^\perp$, it is *a fortiori* a local minimum in $C^3(\overline{\Sigma}) \cap \mathcal{I}^\perp$, and we may apply Lemma 4.1. (The reason that we require closeness in C^3 is that the arguments in Section 2 require this, or at least closeness in C^2 with the third derivatives of the difference bounded.) □

Still following [**18**], we seek eigenvalue conditions which will imply the strong positivity for $\mathcal{M}$ that Theorem 4.1 requires. The eigenvalues of the operator A are involved, and also those of a related differential operator. Defining $\mathcal{L} : H^1(\Sigma) \to H^{-1}(\Sigma)$ by

$$\mathcal{L}(\psi) = -\Delta\psi - |S|^2\psi,$$

where Δ is the Laplace-Beltrami operator on Σ, we consider the eigenvalue problem

$$\mathcal{L}(\psi) = \lambda\psi \tag{4.3}$$

on Σ, with boundary condition

$$\mathbf{b}(\psi) = \partial_n\psi + \rho\psi. \tag{4.4}$$

Here $\partial_n\psi$ is the outward normal derivative of ψ (i.e., the directional derivative of ψ in the direction which is tangent to Σ and normal to $\partial\Sigma$). Lemma 2.8 of [**18**] shows that A and the eigenvalue problem (4.3), (4.4) have the same number of negative and zero eigenvalues.

Lemma 4.3. *The functions $\{\mu_1, \ldots, \mu_k\}$ satisfy $\mathcal{L}(\mu_j) = 0$ on Σ, $\mathbf{b}(\mu_j) = 0$ on $\partial\Sigma$ and therefore are in the kernel of A.*

Proof. Since all $\varphi \in \mathcal{I}$ describe surfaces which are isomorphic to Σ, all Σ_φ have the same mean curvature and make the same angle of contact with Γ. Take a particular μ_j, and let $\varphi(t) \subseteq \mathcal{I}$ be a set of functions in $\mathcal{I}$, parameterized by t, such that $\varphi(0) = o$ and $\varphi'(0) = \mu_j$. Differentiating the relations

$$H(\varphi(t)) = H_0$$

on Σ and

$$\gamma(\varphi(t)) = \gamma$$

on $\partial\Sigma$ and substituting in 0 for t gives the desired result. (See the appendix of [**11**] for the derivation of the formulas for variation of mean curvature and of contact angle.) □

Note 4.1. In [**19**], Wente makes an observation similar to Lemma 4.3, attributing it to Concus and Finn.

Since we will want to treat the functions μ_j differently from the other eigenvectors of the eigenvalue problem (4.3), (4.4), we will use the following terminology. Let $\{\lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \dots\}$ be the eigenvalues which do *not* correspond to the eigenvectors μ_j. The list of all eigenvalues of (4.3), (4.4) will therefore be $\{\lambda_0, \lambda_1, \dots\}$ plus a k-fold eigenvalue of 0. Of course it can occur that some λ_i is zero as well. It is interesting to note that if $k > 1$ then $\lambda_0 < 0$. This follows from the fact that the smallest eigenvalue of (4.3), (4.4) cannot be a multiple eigenvalue (see [**3**]).

From Lemma 4.3 it follows that A maps $H^1(\Sigma)$ onto $\mathcal{I}^\perp$. It is natural, therefore to consider the restriction of A onto $\mathcal{I}^\perp$, and seek eigenvalue conditions which will make the map $A : \mathcal{I}^\perp \to \mathcal{I}^\perp$ strongly positive so that Theorem 4.1 may be applied. The situation is analogous to that considered in [**18**], Theorem 2.2.

Theorem 4.2. *Suppose that Σ is a capillary surface, with contact angle $\gamma \in (0, \pi)$. Suppose that $G_{\vec{z}}$ is a group of isometries which take the fixed surface Λ into itself, but which do not take Σ into itself. Let $\lambda_0 \leq \lambda_1 \leq \dots$ be the part of the spectrum of the eigenvalue problem* (4.3), (4.4) *which does not include the k-fold zero corresponding to the functions $\{\mu_1, \dots, \mu_k\}$. Then:*

1) *If $0 < \lambda_0$ then o is a constrained local minimum for energy in the set $C^3(\overline{\Sigma}) \cap \mathcal{I}^\perp$, and therefore Σ is a constrained strict local minimum for energy modulo the group $G_{\vec{z}}$ of isometries.* (*Note that this case can only occur in the case that $k = 1$.*)
2) *If $\lambda_1 < 0$ then o is a saddle point for energy in $C^3(\overline{\Sigma}) \cap \mathcal{I}^\perp$, and Σ is not an energy minimum modulo the group of isometries.*
3) *If $\lambda_0 < 0 < \lambda_1$, let $\zeta \in \mathcal{I}^\perp$ satisfy $A\zeta = 1$. Then:*
   a) *If $\iint_\Sigma \zeta < 0$ then Σ is a constrained strict local energy minimum modulo the group of isometries as in Case* 1.
   b) *If $\iint_\Sigma \zeta > 0$ then, as in Case* 2, *Σ is not an energy minimum modulo $G_{\vec{z}}$.*
4) *If $0 = \lambda_0$ and if $\iint_\Sigma \varphi_0 \neq 0$* (*where φ_0 is the eigenvector corresponding to λ_0*), *then Σ is an energy minimum modulo $G_{\vec{z}}$ as in Case* 1.

Proof. The proof is entirely analogous to Theorem 2.2, [**18**], although we consider $A : \mathcal{I}^\perp \to \mathcal{I}^\perp$ rather than from H^1 to H^1 as in [**18**]. The reader is referred to that paper for specifics. In Case 3, the function ζ exists since $1 \in \mathcal{I}^\perp$ by Lemma 4.2, and $A : \mathcal{I}^\perp \to \mathcal{I}^\perp$ is nonsingular. As noted in [**18**], ζ must satisfy $\mathcal{L}(\zeta) = 1$ in Σ, $\mathbf{b}(\zeta) = 0$ on $\partial\Sigma$. □

Note 4.2. The result of Theorem 4.2 seems a bit unsatisfying, since it seems to depend on the choice of the curvilinear coordinate system X. However,

using the arguments in [**18**], Section 3, one can interpret this in a manner which is independent of X. If $\vec{\eta}(p)$ is a vector field defined on $\overline{\Sigma}$ with first, second, and third derivatives all uniformly small, then the result from [**18**] implies that the perturbed surface given by $p + \vec{\eta}(p)$, $p \in \Sigma$ will be the graph, in curvilinear coordinates, of a function whose first, second and third derivatives are uniformly small. We can therefore consider comparison surfaces which are small vector perturbations of Σ, without concerning ourselves with the specific choice of X.

5. An application.

In this section, we will examine a specific problem: A liquid bridge between two parallel planes. There are two purposes in this. The first is to give an example of computing the eigenvalue problem from Section 3, and the second is to show that the eigenvalue conditions from [**14**] in fact imply local minimality modulo translations, as discussed in the introduction. Many of the calculations in this section were performed with the aid of Maple (a computer algebra system). Miersemann ([**10**]) also looked at the question of whether the conditions from [**14**] imply local minimality. His approach was somewhat different, in that he dealt with rotationally symmetric comparison surfaces.

As pointed out in [**14**], a bridge between parallel planes and making constant contact angles γ_1 and γ_2 with the planes must be rotationally symmetric. Assume that the material planes are $z = 0$ and $z = h$, and that the bridge is parameterized as $\langle f(u)\cos v, f(u)\sin v, u\rangle$, $0 \le u \le h$, $0 \le v \le 2\pi$. First, we will convert (4.3), (4.4) to these coordinates. Either by using known formulas for curvature for rotationally symmetric surfaces or by using general formulas such as [**9**], (14.26) and (14.27), one obtains that the mean curvature H is

$$H = \frac{-ff'' + (f')^2 + 1}{2f((f')^2+1)^{3/2}} \tag{5.1}$$

and the Gaussian curvature K is

$$K = -\frac{f''}{f((f')^2+1)^2}.$$

From this we obtain

$$|S|^2 = \left(\frac{1}{f^2((f')^2+1)} + \frac{(f'')^2}{((f')^2+1)^3}\right).$$

We also need the Laplace-Beltrami operator expressed in these coordinates. Computing the metric tensor as in [**16**], we obtain that for a function

$\psi(u,v)$ defined on Σ,

$$\Delta\psi = \frac{1}{(f')^2+1}\psi_{uu} + \frac{1}{f^2}\psi_{vv} + \left(\frac{(f')^3+f'-ff'f''}{f((f')^2+1)^2}\right)\psi_u.$$

Next we must compute the normal derivative $\partial_n\psi$. The gradient of ψ is

$$\nabla\psi = \frac{1}{(f')^2+1}\psi_u\frac{\partial}{\partial u} + \frac{1}{f^2}\psi_v\frac{\partial}{\partial v}$$

and the unit normal is

$$\pm\frac{1}{\sqrt{(f')^2+1}}\frac{\partial}{\partial u},$$

with the sign depending on which plane is being contacted. Using the metric tensor to take the inner product, we obtain that

$$\partial_n\psi = \pm\frac{1}{\sqrt{(f')^2+1}}\psi_u$$

for the outward normal derivative of ψ across the curves of contact of Σ with $z=0$ and $z=h$. Finally, we need to determine ρ in (4.4). In (4.1), κ_Σ will be $\frac{f''}{((f')^2+1)^{3/2}}$, κ_Λ will be zero, and $\cot\gamma$ will be $\pm f'$, where the sign depends on the plane being contacted.

Putting this all together, we obtain that (4.3), (4.4) becomes

$$\begin{aligned}(5.2)\quad -\frac{1}{(f')^2+1}\psi_{uu} - \frac{1}{f^2}\psi_{vv} - \left(\frac{(f')^3+f'-ff'f''}{f((f')^2+1)^2}\right)\psi_u \\ - \left(\frac{1}{f^2((f')^2+1)} + \frac{(f')^2}{((f'')^2+1)^3}\right)\psi = \lambda\psi\end{aligned}$$

on the rectangle $0\le u\le h$, $0\le v\le 2\pi$, with boundary conditions

$$(5.3)\qquad \frac{1}{\sqrt{(f')^2+1}}\psi_u + \frac{f'f''}{((f')^2+1)^{3/2}}\psi = 0.$$

The situation is very similar to that considered in [**19**], Section 4, although we can't simply quote results from that paper. However, the approach we will use is certainly inspired by the approach in [**19**].

Because of the simple geometry, all eigenvalues of (5.2), (5.3) may be obtained by separation of variables (see [**3**]). Thus we write ψ in (5.2), (5.3) as $\psi(u,v) = U(u)V(v)$. Noting that f depends only on u, we obtain the equation

$$\begin{aligned}(5.4)\quad -\frac{f^2}{(f')^2+1}\frac{U''}{U} - f^2\left(\frac{(f')^3+f'-ff'f''}{f((f')^2+1)^2}\right)\frac{U'}{U} \\ - f^2\left(\frac{1}{f^2((f')^2+1)} + \frac{(f'')^2}{((f')^2+1)^3}\right) - \lambda f^2 = \frac{V''}{V} = K\end{aligned}$$

where K is the separation constant. Since $V(v)$ must satisfy periodic boundary conditions, it will be a linear combination of $\sin(mv)$ and $\cos(mv)$, so that K will be $-m^2$ for $m = 0, 1, \ldots$. The boundary conditions for U are the obvious ones from (5.3).

If the work in [**14**] had not already been done, we would analyze (5.4) at this point. Instead, we shall show that the above equation is related to the eigenvalue problem encountered in [**14**]. One difference between our approach and the approach in [**14**] is that in [**14**], the perturbations are radial as opposed to the curvilinear coordinate approach that we are using. We are assuming that the coordinate system is normalized so that $\vec{x}(p,0)\cdot\vec{N} = 1$ on Σ. The coordinate system corresponding to the radial perturbations considered in [**14**] will not satisfy this, and we will introduce a factor of $\frac{1}{\sqrt{(f')^2+1}}$ to convert from the coordinate system of [**14**] to one of the type we consider now. All of which is to say that it is natural to replace U in (5.4) by $\frac{1}{\sqrt{(f')^2+1}}\varphi$. The computation is tedious and was performed using Maple. At one point, an expression for f''' in terms of lower order derivatives is needed. This is obtained by using the fact that mean curvature is constant on the surface, and differentiating (5.1) with respect to u. The result of this computation is the equation

$$-\frac{1}{((f')^2+1)^{3/2}}\varphi'' - \frac{(f')^3 + f' - 3ff'f''}{f((f')^2+1)^{5/2}}\varphi' + \frac{m^2-1}{f^2\sqrt{(f')^2+1}}\varphi = \frac{\lambda}{\sqrt{(f')^2+1}}\varphi \tag{5.5}$$

with boundary conditions

$$\varphi'(0) = \varphi'(h) = 0. \tag{5.6}$$

Equation (5.5) may be rewritten as the Sturm-Liouville problem

$$L_m(\varphi) \equiv -\left(\frac{f\varphi'}{((f')^2+1)^{3/2}}\right)' + \frac{m^2-1}{f\sqrt{(f')^2+1}}\varphi = \lambda\frac{f}{\sqrt{(f')^2+1}}\varphi. \tag{5.7}$$

Even for the case $m = 0$, this is not quite the same Sturm-Liouville problem as considered in [**14**]. In that paper, the problem considered was

$$L_0(\varphi) = \beta\varphi. \tag{5.8}$$

However, the problems will be shown to be related.

Lemma 5.1. *Denote by λ_{jm} the jth eigenvalue of the Sturm-Liouville problem*

$$L_m(\varphi) = \lambda\frac{f}{\sqrt{(f')^2+1}}\varphi,$$

with boundary conditions $\varphi'(0) = \varphi'(h) = 0$. *Then*

$$\lambda_{0m} < \lambda_{1m} < \lambda_{2m} < \dots, \tag{5.9}$$

$$\lambda_{00} < \lambda_{01} < \lambda_{02} < \dots, \tag{5.10}$$

and $\lambda_{01} = 0$.

Proof. Equation (5.9) follows from standard Sturm-Liouville theory. To prove the rest, we use the result that λ_{0j} will be the infimum of the quadratic form

$$\int_0^h \frac{f}{((f')^2+1)^{3/2}}(\varphi')^2 + \frac{m^2-1}{f\sqrt{(f')^2+1}}\varphi^2\,du \tag{5.11}$$

subject to the condition

$$\int_0^h \frac{f}{\sqrt{(f')^2+1}}\varphi^2\,du = 1 \tag{5.12}$$

(see [**3**]). Increasing m will increase the infimum, implying (5.10). Finally, $\lambda_{01} = 0$ since the infimum of (5.11) subject to (5.12) is clearly 0, attained by φ being the correct constant. □

The relation between the eigenvalues of Sturm-Liouville problem involving (5.5), setting m to be 0, and those involving (5.8) may be found using the following lemma.

Lemma 5.2. *Consider two Sturm-Liouville problems on* $[0, h]$ *given by*

$$-(p\varphi')' + q\varphi = \lambda\rho_1\varphi,$$

and

$$-(p\varphi')' + q\varphi = \beta\rho_2\varphi,$$

with the same boundary conditions, either $\varphi' + \sigma\varphi = 0$ *at the endpoints, or* $\varphi = 0$ *at the endpoints. Suppose that* $p > 0$, *and* $\rho_i > 0$. *Then the two problems have the same number of negative and zero eigenvalues.*

Proof. This follows from the maximum-minimum property of eigenvalues ([**3**]). To illustrate, consider the case $\sigma = 0$. Define the bilinear form $\mathcal{D}(\varphi, \psi)$ by

$$\mathcal{D}(\varphi, \psi) = \int_0^h p\varphi'\psi' + q\varphi\psi\,du$$

and the bilinear form $H_i(\varphi, \psi)$ by

$$H_i(\varphi, \psi) = \int_0^h \rho_i\varphi\psi\,du.$$

Then, according to [**3**], the nth eigenvalue of each of the above Sturm-Liouville problems may be characterized as follows. Given $n-1$ piecewise continuous functions $\{v_1, \ldots, v_{n-1}\}$, let $d_i\{v_1, \ldots, v_{n-1}\}$ be the greatest lower bound of the set of values assumed by the quadratic form $\mathcal{D}(\varphi, \varphi)$, where φ is any function continuous on $[0, h]$ satisfying $H_i(\varphi, \varphi) = 1$ and $H_i(\varphi, v_k) = 0$ for $k = 1, \ldots, n-1$. Then the nth eigenvalue is the maximum of all values d_i as the functions v_k range over all admissible values.

Thus, the nth eigenvalue is positive if and only if $d_i > 0$ for some choice of the set $\{v_1, \ldots, v_{n-1}\}$. In turn, this occurs if and only if $D(\varphi, \varphi) > 0$ for all nontrivial φ with $H_i(\varphi, v_k) = 0$ for $k = 1, \ldots, n-1$. Now, suppose that $\lambda_n > 0$. Take the set $\{v_1, \ldots, v_{n-1}\}$ satisfying the above property with $i = 1$. Let $w_k = \frac{\rho_1}{\rho_2} v_k$. Since $H_1(\varphi, v_k) = H_2(\varphi, w_k)$, we have that d_2 corresponding to the set $\{w_1, \ldots, w_k\}$ is positive. Thus $\beta_n > 0$. This clearly works both ways, so that $\lambda_n > 0$ if and only if $\beta_n > 0$. Moreover it is obvious that zero is an eigenvalue of the same order for both problems, since in that case the equations are the same. The case of $\sigma \neq 0$ and $\varphi = 0$ at the endpoints can be treated similarly (although not needed in this paper). $\square$

Theorem 5.1. *If the conditions of* [**14**] *Theorem* 4.1 *are met, then the liquid bridge is a strict local minimum, in the space* $C^3(\overline{\Sigma})$*, for energy modulo translations.*

Proof. What we must do is show that the eigenvalue conditions used in [**14**], Theorem 3.1 to imply positive definiteness of the second derivative are actually enough to imply that the bridge is a strict local minimum modulo translations. In our notation, these are that the eigenvalues β_0, β_1, $\ldots$ of the problem (5.8), with boundary conditions $\varphi'(0) = \varphi'(h) = 0$. From Lemma 5.1 it is clear that the only negative eigenvalues of (5.2), (5.3) must have $m = 0$. From Lemma 5.2, the Sturm-Liouville problem (5.5), (5.6) and the Sturm-Liouville problem considered in [**14**] have the same number of negative and non-positive eigenvalues. Therefore, the eigenvalue problem (5.2), (5.3) and the Sturm-Liouville problem of [**14**] have the same number of negative eigenvalues. The fact that $\lambda_{01} = 0$ means that zero is an eigenvalue of multiplicity at least two, corresponding to the group of translations parallel to the planes. However, we may neglect the two dimensional subspace of the kernel corresponding to translations as in Theorem 4.2.

Thus, neglecting translations, the eigenvalue problem (5.2), (5.3) and the Sturm-Liouville problem of [**14**] have both the same number of negative eigenvalues and the same multiplicity at zero, so that the conditions on the number of negative eigenvalues in Theorem 4.2 and the conditions on the number of negative eigenvalues in [**14**] are the same. The last thing to check is that Condition 3 of Theorem 4.2 corresponds to the condition in [**14**], Theorem 3.1 when there is precisely one negative eigenvalue. This is that if the function $\varphi(u)$ which solves $L_0(\varphi) = f$, $\varphi'(0) = \varphi'(h) = 0$

satisfies $\int_0^h \varphi f\,dx < 0$, then the quadratic form is positive definite, and if $\int_0^h \varphi f\,dx > 0$ then the quadratic form is indefinite.

Given such a $\varphi(u)$, let $\zeta = \frac{1}{\sqrt{1+(f')^2}}\varphi$. By the same tedious computations which derived (5.5), $f\mathcal{L}(\zeta) = L_0(\varphi)$, so that $\mathcal{L}(\zeta) = 1$. Similarly, $\widetilde{b}(\zeta) = 0$ on $\partial\Sigma$, so that this is the function ζ sought in Condition 3 of Theorem 4.2. To apply Theorem 4.2, we must also ensure that $\zeta \in \mathcal{I}^\perp$. Using Lemma 3.3 we find that the tangent space to $\mathcal{I}$ is spanned by $\{\mu_1, \mu_2\}$ where

$$\mu_1(u,v) = \frac{\cos v}{\sqrt{1+(f')^2}}$$

and

$$\mu_2(u,v) = \frac{\sin v}{\sqrt{1+(f')^2}}.$$

From this one can verify that any function of u alone will be orthogonal to both μ_1 and μ_2 using the H^1 inner product. Thus $\zeta \in \mathcal{I}^\perp$. Finally, it is simple to check that

$$\iint_\Sigma \zeta\,d\Sigma = 2\pi \int_0^h \varphi f\,du$$

so that the condition on the sign of $\int_0^h \varphi f\,du$ in [**14**] is the same as the condition on the sign of $\iint_\Sigma \zeta$ in Theorem 4.2.

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