ON THE COMMUTATOR FORMULA OF A SPLIT BN-PAIR

Gwenaëlle Genet
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The purpose of this note is to prove in an elementary way and with geometric considerations that the Levi decomposition of a finite group with a split BN-pair of characteristic \( p \) (a prime integer) implies the commutator formula.

1. Introduction.

Split BN-pairs are defined by a set of axioms devised to study finite reductive groups (see [CR, 69.1], [R, 3.1]). Such finite groups \( G \) are assumed to contain subgroups \( B, N \) and \( S \subseteq W := N/T \), with \( T := N \cap B \), such that among other things, \( T \triangleleft N \), \((W, S)\) is a Coxeter system, \( B \) can be written as a semi-direct \( B = UT \) where \( U \) is a Sylow \( p \)-subgroup of \( G \) and \( B \cap B^{w_0} = T \) (\( w_0 \) the longest element of \( W \)). Recalling the geometric representation of \( W \) in an euclidean space \( E \), we denote by \( \Phi \) the associated root system (a subset of the unit sphere), by \( \Delta \subseteq \Phi^+ \) the fundamental and positive systems of \( \Phi \), so that the fundamental reflections \( \{ s_\delta, \delta \in \Delta \} \) correspond with the elements of \( S \) (see [CR, 64.28] or [B]). For \( \delta \in \Delta \), one defines \( X_\delta := U \cap U^{w_0s_\delta} \). An elementary consequence of the axioms of BN-pairs ([CR, 69.2]) is:

\( (R0) \) For arbitrary \( \gamma \in \Phi \), written as \( \gamma = w(\delta) \) with \( \delta \in \Delta \), \( w \in W \), one may define \( X_\gamma := w^{\ast}X_\delta \). This only depends on \( \gamma \) and \( X_\gamma \neq \{1\} \).

A nice extra property, satisfied in all finite reductive groups, is the commutator formula:

\( \forall \alpha, \beta \in \Phi, \alpha \neq \pm \beta, [X_\alpha, X_\beta] \subseteq \langle X_{i\alpha+j\beta}, i > 0, j > 0, i\alpha + j\beta \in \Phi \rangle \).

This is useful to check the crucial property of Levi decompositions, which is that for all subsets \( I \subseteq \Delta \), \( U \cap U^{w_I} \) is normal in \( U \) (\( w_I \) the longest element of \( W_I \)). In particular,

\( \forall \delta \in \Delta, U \cap U^{s_\delta} \triangleleft U. \)

N. Tinberg ([T]) has shown that (2) implies in an elementary way the Levi decomposition.

Here, we show a little more, namely that (1) follows from (2), without using the classification of BN-pairs or the case of rank 2 ([FS]). Note that (2)
is always satisfied when the root subgroups $X_\delta$ have order $p$. Our arguments are easy considerations in the reflection representation space. We recall two elementary properties of BN-pairs. Denote by $l_S$ the length in $W$ relative to $S$.

(R1) (See [CR, 69.2] or [R, 3.3] by iteration.) There is at least one sequence such that $\Phi^+ = \{\gamma_1, \ldots, \gamma_N\}$, $N = |\Phi^+|$ and $U = X_{\gamma_1} \cdots X_{\gamma_N}$. If $\gamma \in \Phi^+$, then $X_{-\gamma} \cap U = \{1\} = U \cap U^{w_0}$.

(R2) ([R, 2.3]) If $w \in W$ and $s \in S$ is such that $l_S(ws) = l_S(w) + 1$, then $U \cap U^{ws} \subseteq U \cap U^s$.

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Notation. If $E'$ is a subset of $E$, we denote $X(E') := \langle X_{\gamma}, \gamma \in E' \cap \Phi \rangle$ and if $A$ is a nonempty subset of $W$, we denote $V_A := \cap_{w \in A} U^w$.

2. First results.

The following proposition shows how to express $V_A$, $A \subseteq W$, $A \neq \emptyset$, with the root groups $X_{\gamma}$, $\gamma \in \Phi$. For this, we need:

Lemma 1. Let $m \geq 1$, let $\gamma_1, \ldots, \gamma_m$ be pairwise distinct in $\Phi^+$ and for every $i$, $1 \leq i \leq m$, let $x_i \in X_{\gamma_i}\setminus\{1\}$. Let $w \in W$, then $x_1 x_2 \ldots x_m \in U^w$ if and only if for every $i$, $w(\gamma_i) \in \Phi^+$.

Proof. If for all $i$, $w(\gamma_i) \in \Phi^+$, then for all $i$, $X_{w(\gamma_i)} = w X_{\gamma_i} \subseteq U$ because of (R1). We prove the converse by induction on $l_S(w)$. If $w = 1$, this is obvious.

If $w = s_\delta$, $\delta \in \Delta$, one must check $\delta$ is none of the $\gamma_i$’s. Suppose on the contrary that $\delta = \gamma_{i_0}$. Then $s_\delta(\gamma_i) \in \Phi^+$ for any $i \neq i_0$. So $x_1 \ldots x_{i_0-1} \in U^{s_\delta}$ and $x_{i_0} \ldots x_m \in U^{s_\delta}$ by the ‘if’. Then $x_{i_0} \in U^{s_\delta} \cap X_\delta \subseteq U^{s_\delta} \cap U^{s_\delta s_i} = \{1\}$, a contradiction.

For an arbitrary $w \in W$ with $l_S(w) \geq 1$, we can write $w = w' s_\delta$ for $w' \in W$, $\delta \in \Delta$ and $l_S(w') = l_S(w) - 1$. Then $U \cap U^{w} \subseteq U \cap U^{s_\delta}$ by (R2). We have just seen that, for all $i$, $s_\delta(\gamma_i) \in \Phi^+$. Set $\gamma'_i := s_\delta(\gamma_i)$, $x'_i := x_i^{s_\delta} \in X_{\gamma'_i}$ where $s_\delta \in N$ is a representative of $s_\delta$. The result comes by applying the induction hypothesis to $x'_1 x'_2 \ldots x'_m \in U^{w'}$.

Proposition 1. For all nonempty subset $A \subseteq W$, we have $V_A = X(\Psi_A)$ where $\Psi_A := \{\gamma \in \Phi \mid \forall w \in A, w(\gamma) \in \Phi^+\}$. Moreover, $\Psi_A = \{\gamma \in \Phi \mid X_{\gamma} \subseteq V_A\}$.

Proof. Suppose $1 \in A$, so that $\Psi_A \subseteq \Phi^+$. Take a sequence $\gamma_1, \ldots, \gamma_N \in \Phi^+$ as in (R1), so that $\Psi_A = \{\gamma_{i_1}, \ldots, \gamma_{i_m}\}$ for $1 \leq i_1 < \cdots < i_m \leq N$. Then any $x \in V_A \subseteq U$, can be written as $x = x_1 \ldots x_N$ with $x_i \in X_{\gamma_i}$. The above lemma implies that $x_i = 1$ whenever $\gamma_i \notin \Psi_A$. Then $V_A = X(\Psi_A)$. So it makes clear that $\Psi_A \subseteq \{\gamma \in \Phi \mid X_{\gamma} \subseteq V_A\}$. But if $\gamma \in \Phi$ is such that
Proof. One inclusion is clear. For the other, let $x$ find a linear form $f$ be a finite set of $\Psi$. Lemma 2. about separation of convex sets. A last equality is proved. $X_V$ for $w_w$. $C$ open cone generated by the elements of the unit sphere at distance $< r$. $C$ sets us that the set $G \subset \{ f(C\{0\}) \subseteq \mathbb{R}_+^* \text{ and } 0 \not\in f(\Psi) \}$. Then $C = \cap_{f \in F} f^{-1}(\mathbb{R}_+^*) \cup \{0\}$. 

Proof. One inclusion is clear. For the other, let $x \not\in C$, so $x \neq 0$. We will find a linear form $f$ such that $f(C\{0\}) \subseteq \mathbb{R}_+^*$, $f(x) < 0$ and $0 \not\in f(\Psi)$. One may assume that $x$ has norm 1. If $r > 0$, let $C_r$ (resp. $D_r$) be the open cone generated by the elements of the unit sphere at distance $< r$ from the elements of $C$ (resp. from $x$). For $r$ sufficiently small, we clearly have $C_r \cap D_r = C_r \cap -C_r = \emptyset$. Taking a hyperplane separating the open convex sets $C_r$ and $D_r$, it is clear that such a hyperplane must contain 0. This tells us that the set $G := \{ f \in E^V, f(C\{0\}) \subseteq \mathbb{R}_+^*, f(x) < 0 \}$ is not empty. Now, it is easy to see that $G$ is an open set of the dual $E^V$ and that the set $\{ f \in G, 0 \not\in f(\Psi) \}$ is not empty. Thus our claim.

The consequence of the above on “convex” subsets of $\Phi$ and corresponding subgroups of $G$ is as follows.

**Lemma 2.** Let $C$ be a closed convex cone of $E$ such that $C \cap -C = \{0\}$. Let $\Psi$ be a finite set of $E \setminus \{0\}$ and let $F$ be the subset of the dual $E^V$ of $E$, $F = \{ f \in E^V : f(C\{0\}) \subseteq \mathbb{R}_+^* \text{ and } 0 \not\in f(\Psi) \}$. Then $C = \cap_{f \in F} f^{-1}(\mathbb{R}_+^*) \cup \{0\}$.

**Proof.**

(i) The inclusion $C \cap \Phi \subseteq \Psi_{A(\Phi)}$ is trivial. On the other hand, we have $C \cap \Phi = \cap_{f \in F} f^{-1}(\mathbb{R}_+^*) \cap \Phi$ where the set $F$ consists of the $f \in E^V$ such that $f(C\{0\}) \subseteq \mathbb{R}_+^*$ and $0 \not\in f(\Phi)$ (Lemma 2). But for $f \in F$, $f^{-1}(\mathbb{R}_+^*) \cap \Phi$ is not empty and therefore some unique fundamental system. By transitivity of $W$ on fundamental systems, there is $w_f \in W$ such that $f^{-1}(\mathbb{R}_+^*) \cap \Phi = w_f^{-1}(\Phi^+)$. Obviously, $w_f \in A(\Phi)$. So we get the reverse inclusion we seek.

(ii) By (i) above, we have $X(\Phi) = X(\Psi_{A(\Phi)})$. But Proposition 1 tells us that the latter is indeed $V_{A(\Phi)}$.

**Corollary 1.** Let $C$ and $D$ be two closed convex cones of $E$ such that $C \cap -C = D \cap -D = \{0\}$. Then, $X(C) \cap X(D) = X(C \cap D)$.

**Proof.** With Proposition 2 (ii), we clearly have $X(C) \cap X(D) = V_{A(C)} \cap V_{A(D)} = V_{A(C) \cup A(D)}$, and the latter is $X(\Psi_{A(C) \cup A(D)})$ by Proposition 1. But
\[ \Psi_{A(C) \cup A(D)} = \Psi_{A(C)} \cap \Psi_{A(D)} \] that is again equal to \( C \cap D \cap \Phi \) by Proposition 2 (i). We get our claim. \[ \Box \]

3. The commutator formula.

**Theorem 1.** If \( G \) is a finite group with a split BN-pair satisfying the hypothesis (2), then it satisfies the commutator formula (1).

**Proof.** For all finite subset \( Y \subseteq E \), we denote by \( C(Y) \) the closed convex cone generated by \( Y \). We use the abbreviation \( C(y, z) := C(\{y, z\}) \).

Suppose \( G \) satisfies (2). Denote \( U_\delta = U \cap U^\delta \) when \( \delta \in \Delta \). Let \( \alpha \in \Delta \) and \( \beta \in \Phi^+ \), \( \alpha \neq \beta \). We have \( U_\alpha = V_{\{1,s_\alpha\}} = X(\Psi_{\{1,s_\alpha\}}) \) and \( X_\beta \subseteq U_\alpha \) by Proposition 1 because \( \Psi_{\{1,s_\alpha\}} = \Phi^+ \setminus \{\alpha\} \). But \( \Phi^+ \setminus \{\alpha\} = \Phi \cap C(\Phi^+ \setminus \{\alpha\}) \) because any positive linear combination of positive roots that is again a root is a positive one and \( \alpha \) is a fundamental root, so is a minimal positive linear combination of positive roots. Therefore, \( U_\alpha = X(C(\Phi^+ \setminus \{\alpha\})) \). Besides, since \( X_\alpha \subseteq U \), \([X_\alpha, X_\beta] \subseteq [X_\alpha, U_\alpha] \subseteq U_\alpha \) by hypothesis (2). On the other hand, both \( X_\alpha \) and \( X_\beta \) are subgroups of \( X(C(\alpha, \beta)) \), so \([X_\alpha, X_\beta] \subseteq X(C(\alpha, \beta)) \). The closed convex cones \( C(\Phi^+ \setminus \{\alpha\}) \) and \( C(\alpha, \beta) \) satisfy \( C \cap -C = \{0\} \) by property of positive roots ([B]) then Corollary 1 implies \([X_\alpha, X_\beta] \subseteq U_\alpha \cap X(C(\alpha, \beta)) = X(C(\Phi^+ \setminus \{\alpha\}) \cap C(\alpha, \beta)) = X(C(\alpha, \beta) \setminus \{\alpha\}) \). So we get, for all \( \alpha \in \Delta \), \( \beta \in \Phi^+ \setminus \{\alpha\} \),

\[ [X_\alpha, X_\beta] \subseteq X(C(\alpha, \beta) \setminus \{\alpha\}). \]

If \( \alpha \in \Delta \), \( \beta \in \Phi^- \setminus \{-\alpha\} \), we also have (3) by applying the above to \( w_0s_\alpha(\alpha) \in \Delta \), \( w_0s_\alpha(\beta) \in \Phi^+ \) and conjugating the corresponding subgroups of \( G \) by \( s_\alpha w_0 \in W \). So we have (3) for any \( \alpha \in \Delta \), \( \beta \in \Phi \setminus \{\pm \alpha\} \).

If \( \alpha, \beta \) are any non-proportional arbitrary roots, there is \( w \in W \) such that \( w(\alpha) \in \Delta \), then again we have (3). Now, exchanging the rôles of \( \alpha \) and \( \beta \), (3) becomes \([X_\alpha, X_\beta] \subseteq X(C(\alpha, \beta) \setminus \{\beta\}) \). It is clear that each set \( \Phi \cap (C(\alpha, \beta) \setminus \{\alpha\}) \), \( \Phi \cap (C(\alpha, \beta) \setminus \{\beta\}) \), and \( \Phi \cap (C(\alpha, \beta) \setminus \{\alpha, \beta\}) \) is of the type \( \Phi \cap C \) where \( C \) is a closed convex cone such that \( C \cap -C = \{0\} \) (draw a picture in the plane generated by \( \alpha \) and \( \beta \)). Then Corollary 1 gives \( X(C(\alpha, \beta) \setminus \{\alpha\}) \cap X(C(\alpha, \beta) \setminus \{\beta\}) = X(C(\alpha, \beta) \setminus \{\alpha, \beta\}) \). We get our claim. \[ \Box \]

**References**


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UFR de Mathématiques - Case 7012
Université Denis Diderot - Paris 7
2, Place Jussieu
75251 Paris Cedex 05
France
E-mail address: genet@math.jussieu.fr