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Gwenaëlle Genet

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The purpose of this note is to prove in an elementary way and with geometric considerations that the Levi decomposition of a finite group with a split BN-pair of characteristic p (a prime integer) implies the commutator formula.

1. Introduction.

Split BN-pairs are defined by a set of axioms devised to study finite reductive groups (see [**CR**, 69.1], [**R**, 3.1]). Such finite groups G are assumed to contain subgroups B, N and $S \subseteq W := N/T$, with $T := N \cap B$, such that among other things, $T \lhd N$, (W, S) is a Coxeter system, B can be written as a semi-direct $B = UT$ where U is a Sylow p-subgroup of G and $B \cap B^{w_0} = T$ (w_0 the longest element of W). Recalling the geometric representation of W in an euclidean space E, we denote by Φ the associated root system (a subset of the unit sphere), by $\Delta \subseteq \Phi^+$ the fundamental and positive systems of Φ, so that the fundamental reflections $\{s_\delta, \delta \in \Delta\}$ correspond with the elements of S (see [**CR**, 64.28] or [**B**]). For $\delta \in \Delta$, one defines $X_\delta := U \cap U^{w_0 s_\delta}$. An elementary consequence of the axioms of BN-pairs ([**CR**, 69.2]) is:

(R0) *For arbitrary $\gamma \in \Phi$, written as $\gamma = w(\delta)$ with $\delta \in \Delta$, $w \in W$, one may define $X_\gamma :={}^w X_\delta$. This only depends on γ and $X_\gamma \neq \{1\}$.*

A nice extra property, satisfied in all finite reductive groups, is the commutator formula:

(1) $\forall \alpha, \beta \in \Phi,\ \alpha \neq \pm\beta,\ [X_\alpha, X_\beta] \subseteq \left\langle X_{i\alpha+j\beta},\ i > 0,\ j > 0,\ i\alpha + j\beta \in \Phi \right\rangle.$

This is useful to check the crucial property of Levi decompositions, which is that for all subsets $I \subseteq \Delta$, $U \cap U^{w_I}$ is normal in U (w_I the longest element of W_I). In particular,

$$\forall \delta \in \Delta,\ U \cap U^{s_\delta} \lhd U. \tag{2}$$

N. Tinberg ([**T**]) has shown that (2) implies in an elementary way the Levi decomposition.

Here, we show a little more, namely that (1) follows from (2), without using the classification of BN-pairs or the case of rank 2 ([**FS**]). Note that (2) is always satisfied when the root subgroups X_δ have order p. Our arguments are easy considerations in the reflection representation space. We recall two elementary properties of BN-pairs. Denote by l_S the length in W relative to S.

(R1) (See [**CR**, 69.2] or [**R**, 3.3] by iteration.) *There is at least one sequence such that* $\Phi^+ = \{\gamma_1, \ldots, \gamma_N\}$, $N = |\Phi^+|$ *and* $U = X_{\gamma_1} \ldots X_{\gamma_N}$. *If* $\gamma \in \Phi^+$, *then* $X_{-\gamma} \cap U = \{1\} = U \cap U^{w_0}$.

(R2) ([**R**, 2.3]) *If* $w \in W$ *and* $s \in S$ *is such that* $l_S(ws) = l_S(w) + 1$, *then* $U \cap U^{ws} \subseteq U \cap U^s$.

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Notation. If E' is a subset of E, we denote $X(E') := \langle X_\gamma, \gamma \in E' \cap \Phi \rangle$ and if A is a nonempty subset of W, we denote $V_A := \cap_{w \in A} U^w$.

2. First results.

The following proposition shows how to express V_A, $A \subseteq W$, $A \neq \emptyset$, with the root groups X_γ, $\gamma \in \Phi$. For this, we need:

Lemma 1. *Let* $m \geq 1$, *let* $\gamma_1, \ldots, \gamma_m$ *be pairwise distinct in* Φ^+ *and for every* i, $1 \leq i \leq m$, *let* $x_i \in X_{\gamma_i} \backslash \{1\}$. *Let* $w \in W$, *then* $x_1 x_2 \ldots x_m \in U^w$ *if and only if for every* i, $w(\gamma_i) \in \Phi^+$.

Proof. If for all i, $w(\gamma_i) \in \Phi^+$, then for all i, $X_{w(\gamma_i)} = {}^w X_{\gamma_i} \subseteq U$ because of (R1). We prove the converse by induction on $l_S(w)$. If $w = 1$, this is obvious.

If $w = s_\delta$, $\delta \in \Delta$, one must check δ is none of the γ_i's. Suppose on the contrary that $\delta = \gamma_{i_0}$. Then $s_\delta(\gamma_i) \in \Phi^+$ for any $i \neq i_0$. So $x_1 \ldots x_{i_0 - 1} \in U^{s_\delta}$ and $x_{i_0+1} \ldots x_m \in U^{s_\delta}$ by the 'if'. Then $x_{i_0} \in U^{s_\delta} \cap X_\delta \subseteq U^{s_\delta} \cap U^{w_0 s_\delta} = \{1\}$, a contradiction.

For an arbitrary $w \in W$ with $l_S(w) \geq 1$, we can write $w = w' s_\delta$ for $w' \in W$, $\delta \in \Delta$ and $l_S(w') = l_S(w) - 1$. Then $U \cap U^w \subseteq U \cap U^{s_\delta}$ by (R2). We have just seen that, for all i, $s_\delta(\gamma_i) \in \Phi^+$. Set $\gamma_i' := s_\delta(\gamma_i)$, $x_i' := {x_i}^{\dot{s_\delta}} \in X_{\gamma_i'}$ where $\dot{s_\delta} \in N$ is a representative of s_δ. The result comes by applying the induction hypothesis to $x_1' x_2' \ldots x_m' \in U^{w'}$. □

Proposition 1. *For all nonempty subset* $A \subseteq W$, *we have* $V_A = X(\Psi_A)$ *where* $\Psi_A := \{\gamma \in \Phi \mid \forall w \in A,\ w(\gamma) \in \Phi^+\}$. *Moreover,* $\Psi_A = \{\gamma \in \Phi,\ X_\gamma \subseteq V_A\}$.

Proof. Suppose $1 \in A$, so that $\Psi_A \subseteq \Phi^+$. Take a sequence $\gamma_1, \ldots, \gamma_N \in \Phi^+$ as in (R1), so that $\Psi_A = \{\gamma_{i_1}, \ldots, \gamma_{i_m}\}$ for $1 \leq i_1 < \cdots < i_m \leq N$. Then any $x \in V_A \subseteq U$, can be written as $x = x_1 \ldots x_N$ with $x_i \in X_{\gamma_i}$. The above lemma implies that $x_i = 1$ whenever $\gamma_i \notin \Psi_A$. Then $V_A = X(\Psi_A)$. So it makes clear that $\Psi_A \subseteq \{\gamma \in \Phi,\ X_\gamma \subseteq V_A\}$. But if $\gamma \in \Phi$ is such that

$X_\gamma \subseteq V_A$ then $\gamma \in \Phi^+$ by (R0) and (R1). Lemma 1 tells us $\gamma \in \Psi_A$. The last equality is proved.

Now, suppose $1 \notin A$. Take $a \in A$ and define $A' := Aa^{-1} \subseteq W$. The above applies to A' and gives our claim since $V_A = (V_{A'})^a$ and $\Psi_A = a^{-1}\Psi_{A'}$. □

The following is a slight adaptation to our needs of the standard theorem about separation of convex sets.

Lemma 2. *Let C be a closed convex cone of E such that $C \cap -C = \{0\}$. Let Ψ be a finite set of $E \setminus \{0\}$ and let $\mathcal{F}$ be the subset of the dual $E^\vee$ of E, $\mathcal{F} = \{f \in E^\vee : f(C\setminus\{0\}) \subseteq \mathbb{R}_+^*$ and $0 \notin f(\Psi)\}$. Then $C = \cap_{f\in\mathcal{F}} f^{-1}(\mathbb{R}_+^*) \cup \{0\}$.*

Proof. One inclusion is clear. For the other, let $x \notin C$, so $x \neq 0$. We will find a linear form f such that $f(C\setminus\{0\}) \subseteq \mathbb{R}_+^*$, $f(x) < 0$ and $0 \notin f(\Psi)$. One may assume that x has norm 1. If $r > 0$, let C_r (resp. D_r) be the open cone generated by the elements of the unit sphere at distance $< r$ from the elements of C (resp. from x). For r sufficiently small, we clearly have $C_r \cap D_r = C_r \cap -C_r = \emptyset$. Taking a hyperplane separating the open convex sets C_r and D_r, it is clear that such a hyperplane must contain 0. This tells us that the set $\mathcal{G} := \{f \in E^\vee,\ f(C\setminus\{0\}) \subseteq \mathbb{R}_+^*,\ f(x) < 0\}$ is not empty. Now, it is easy to see that $\mathcal{G}$ is an open set of the dual $E^\vee$ and that the set $\{f \in \mathcal{G},\ 0 \in f(\Psi)\} \neq \mathcal{G}$ since Ψ is finite. So the outcome is still nonempty. Thus our claim. □

The consequence of the above on "convex" subsets of Φ and corresponding subgroups of G is as follows.

Proposition 2. *Let C be a closed convex cone of E such that $C \cap -C = \{0\}$. Denote $A(C) := \{w \in W \mid C \cap \Phi \subseteq w^{-1}(\Phi^+)\}$. Then*

(i) *$C \cap \Phi = \Psi_{A(C)}$ with the notation of Proposition 1.*

(ii) *$X(C) = V_{A(C)}$.*

Proof. (i) The inclusion $C \cap \Phi \subseteq \Psi_{A(C)} = \cap_{w\in A(C)} w^{-1}(\Phi^+)$ is trivial. On the other hand, we have $C \cap \Phi = \cap_{f\in\mathcal{F}} f^{-1}(\mathbb{R}_+^*) \cap \Phi$ where the set $\mathcal{F}$ consists of the $f \in E^\vee$ such that $f(C\setminus\{0\}) \subseteq \mathbb{R}_+^*$ and $0 \notin f(\Phi)$ (Lemma 2). But for $f \in \mathcal{F}$, $f^{-1}(\mathbb{R}_+^*) \cap \Phi$ defines an order relation on Φ ([**B**]) and therefore some unique fundamental system. By transitivity of W on fundamental systems, there is $w_f \in W$ such that $f^{-1}(\mathbb{R}_+^*) \cap \Phi = {w_f}^{-1}(\Phi^+)$. Obviously, $w_f \in A(C)$. So we get the reverse inclusion we seek.

(ii) By (i) above, we have $X(C) = X(\Psi_{A(C)})$. But Proposition 1 tells us that the latter is indeed $V_{A(C)}$. □

Corollary 1. *Let C and D be two closed convex cones of E such that $C \cap -C = D \cap -D = \{0\}$. Then, $X(C) \cap X(D) = X(C \cap D)$.*

Proof. With Proposition 2 (ii), we clearly have $X(C) \cap X(D) = V_{A(C)} \cap V_{A(D)} = V_{A(C)\cup A(D)}$, and the latter is $X(\Psi_{A(C)\cup A(D)})$ by Proposition 1. But

$\Psi_{A(C)\cup A(D)} = \Psi_{A(C)} \cap \Psi_{A(D)}$ that is again equal to $C \cap D \cap \Phi$ by Proposition 2 (i). We get our claim. □

3. The commutator formula.

Theorem 1. *If G is a finite group with a split BN-pair satisfying the hypothesis* (2)*, then it satisfies the commutator formula* (1)*.*

Proof. For all finite subset $Y \subseteq E$, we denote by $C(Y)$ the closed convex cone generated by Y. We use the abbreviation $C(y, z) := C(\{y, z\})$.

Suppose G satisfies (2). Denote $U_\delta = U \cap U^{s_\delta}$ when $\delta \in \Delta$. Let $\alpha \in \Delta$ and $\beta \in \Phi^+$, $\alpha \neq \beta$. We have $U_\alpha = V_{\{1,s_\alpha\}} = X(\Psi_{\{1,s_\alpha\}})$ and $X_\beta \subseteq U_\alpha$ by Proposition 1 because $\Psi_{\{1,s_\alpha\}} = \Phi^+ \backslash \{\alpha\}$. But $\Phi^+ \backslash \{\alpha\} = \Phi \cap C(\Phi^+ \backslash \{\alpha\})$ because any positive linear combination of positive roots that is again a root is a positive one and α is a fundamental root, so is a minimal positive linear combination of positive roots. Therefore, $U_\alpha = X(C(\Phi^+ \backslash \{\alpha\}))$. Besides, since $X_\alpha \subseteq U$, $[X_\alpha, X_\beta] \subseteq [X_\alpha, U_\alpha] \subseteq U_\alpha$ by hypothesis (2). On the other hand, both X_α and X_β are subgroups of $X(C(\alpha, \beta))$, so $[X_\alpha, X_\beta] \subseteq X(C(\alpha, \beta))$. The closed convex cones $C(\Phi^+ \backslash \{\alpha\})$ and $C(\alpha, \beta)$ satisfy $C \cap -C = \{0\}$ by property of positive roots ([**B**]) then Corollary 1 implies $[X_\alpha, X_\beta] \subseteq U_\alpha \cap X(C(\alpha, \beta)) = X(C(\Phi^+ \setminus \{\alpha\}) \cap C(\alpha, \beta)) = X(C(\alpha, \beta) \setminus \{\alpha\})$. So we get, for all $\alpha \in \Delta$, $\beta \in \Phi^+ \backslash \{\alpha\}$,

$$[X_\alpha, X_\beta] \subseteq X(C(\alpha, \beta) \backslash \{\alpha\}). \tag{3}$$

If $\alpha \in \Delta$, $\beta \in \Phi^- \backslash \{-\alpha\}$, we also have (3) by applying the above to $w_0 s_\alpha(\alpha) \in \Delta$, $w_0 s_\alpha(\beta) \in \Phi^+$ and conjugating the corresponding subgroups of G by $s_\alpha w_0 \in W$. So we have (3) for any $\alpha \in \Delta$, $\beta \in \Phi \backslash \{\pm\alpha\}$. If α, β are any non-proportional arbitrary roots, there is $w \in W$ such that $w(\alpha) \in \Delta$, then again we have (3). Now, exchanging the rôles of α and β, (3) becomes $[X_\alpha, X_\beta] \subseteq X(C(\alpha, \beta) \backslash \{\beta\})$. It is clear that each set $\Phi \cap \big(C(\alpha, \beta) \backslash \{\alpha\}\big)$, $\Phi \cap \big(C(\alpha, \beta) \backslash \{\beta\}\big)$, and $\Phi \cap \big(C(\alpha, \beta) \backslash \{\alpha, \ \beta\}\big)$ is of the type $\Phi \cap C$ where C is a closed convex cone such that $C \cap -C = \{0\}$ (draw a picture in the plane generated by α and β). Then Corollary 1 gives $X(C(\alpha, \beta) \backslash \{\alpha\}) \cap X(C(\alpha, \beta) \backslash \{\beta\}) = X(C(\alpha, \beta) \backslash \{\alpha, \beta\})$. We get our claim. □

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UFR de Mathématiques - Case 7012
Université Denis Diderot - Paris 7
2, Place Jussieu
75251 Paris Cedex 05
France
E-mail address: genet@math.jussieu.fr