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REMARK ON THE RATE OF DECAY OF SOLUTIONS TO LINEARIZED COMPRESSIBLE NAVIER–STOKES EQUATIONS

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We consider the $L_p - L_q$ estimates of solutions to the Cauchy problem of linearized compressible Navier–Stokes equation. Especially, we investigate the diffusion wave property of the compressible Navier–Stokes flows, which was studied by D. Hoff and K. Zumbrum and Tai-P. Liu and W. Wang.

1. Introduction.

In this paper, we consider the Cauchy problem of the following linearized compressible Navier-Stokes equations:

$$\begin{aligned} &\rho_t + \gamma \operatorname{div} v = 0 && \text{in } (0,\infty)\times\mathbb{R}^n, \\ &v_t - \alpha\Delta v - \beta\nabla \operatorname{div} v + \gamma\nabla\rho = 0 && \text{in } (0,\infty)\times\mathbb{R}^n, \\ &\rho|_{t=0} = \rho_0, \quad v|_{t=0} = v_0 && \text{in } \mathbb{R}^n, \end{aligned} \tag{1.1}$$

where $v = v(t,x) = {}^T(v_1(t,x),\ldots,v_n(t,x))$ a vector valued unknown function, $\rho = \rho(t,x)$ is a scalar valued unknown function; t is time variable; we denote the spatial point of n-dimensional Euclidian Space $\mathbb{R}^n$ by $x = (x_1,\ldots,x_n)$ $(n \geqq 2)$;

$$\rho_t = \frac{\partial\rho}{\partial t}, \qquad v_t = \frac{\partial v}{\partial t}, \qquad \Delta v = \sum_{j=1}^n \frac{\partial^2 v}{\partial x_j^2},$$

$$\operatorname{div} v = \sum_{j=1}^n \frac{\partial v_j}{\partial x_j}, \qquad \nabla\rho = \left(\frac{\partial\rho}{\partial x_1},\ldots,\frac{\partial\rho}{\partial x_n}\right);$$

ρ_0 and v_0 are given initial data; α and γ are positive constants and β a nonnegative constant. Concerning the decay property, asymptotically, the solution decomposed into sum of two parts under the influence of a hyperbolic aspect and a parabolic aspect. One of which dominates in L_p for $2 \leqq p \leqq \infty$, the other $1 \leqq p < 2$. For $p \geqq 2$, the time asymptotic behavior of solutions is similar to the solution of pure diffusion problem. Namely, the decay at the rate of the solution is similar to the solution of a linear, second order, strictly parabolic system with L_1 initial data. Moreover, the decay order of the term that is given by the convolution of Green functions of diffusion equation and wave equation is better than the solution to pure diffusion system. On the other hand, for $p < 2$, the asymptotically dominant term reflects the spreading effect of the solution operator for the standard multi-dimensional wave equation. As a result, the solution may grow without bound in L_p for $p < 2$. This result was investigated by D. Hoff and K. Zumbrun [**2**, **3**] in the case of the Navier-Stokes system describing the compressible fluid flow, and Y. Shibata [**6**] in the case of the linear viscoelastic equation. D. Hoff and K. Zumbrun [**2**, **3**] considered the linear effective artificial viscosity system as the first approximation of the compressible Navier-Stokes equation in several space dimension. The Green function of this system is written exactly by the convolution of the Green function of diffusion equation and wave equation. In view of this, they gave the pointwise estimate and L_p estimate of the Green function in [**2**, **3**], and L_p estimate for the solutions to the nonlinear problem in [**2**]. But, the Green function of the system (1.1) and the linear viscoelastic equation is not written exactly. Tai-P. Liu and W. Wang [**4**] gave the pointwise estimate for the solutions to the system (1.1) and the nonlinear problem in odd multi-dimension case, and Y. Shibata [**6**] gave the L_p estimate for the solution to the linear viscoelastic equations by directly using Fourier transform method. The main difference of the structure to the solutions between (1.1) or effective artificial viscosity system and linear viscoelastic equation is the Riesz kernel $R_j(x) = \mathcal{F}^{-1}\left[\xi_j/|\xi|\right](x)$, where $\mathcal{F}^{-1}$ denotes the Fourier inverse transform. The Green matrix of the system (1.1) and effective artificial viscosity system includes the Riesz kernel. Since the convolution operator $u \to R_j * u$ is not bounded from L_1 to L_1 and from L_∞ to L_∞, if we consider L_1 or L_∞ estimate, then these features will lead to a great deal of cancellation in the convolution operator of the Green function. D. Hoff and K. Zumbrun [**2**] overcame this difficulty by applying the weak version of the Paley-Wiener theorem to the general, symmetrizable, hyperbolic-strictly parabolic systems. In this paper, we shall estimate directly using Fourier transform method in [**6**]. In particular, we shall detect the cancellation in the Green function.

2. Main results.

First of all, we shall introduce the solution operator of (1.1). Applying the Fourier transform with respect to $x = (x_1, \ldots, x_n)$, (1.1) is reduced to the following ordinary differential equation with parameter $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$:

$$(2.1) \qquad \begin{cases} \frac{d\hat{\rho}}{dt}(t,\xi) + i\gamma\xi\cdot\hat{v}(t,\xi) = 0, \\ \frac{d\hat{v}}{dt}(t,\xi) + \alpha|\xi|^2\hat{v}(t,\xi) + \beta\xi\left(\xi\cdot\hat{v}(t,\xi)\right) + i\gamma\xi\hat{\rho}(t,\xi) = 0, \\ \hat{\rho}(0,\xi) = \hat{\rho}_0(\xi),\ \hat{v}(0,\xi) = \widehat{v_0}(\xi), \end{cases}$$

where

$$\hat{u}(t,\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} u(t,x)dx, \quad \widehat{u_j}(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} u_j(x)dx.$$

By (2.1) we have

$$\begin{cases} \frac{d^2\hat{\rho}}{dt^2}(t,\xi) + (\alpha+\beta)|\xi|^2 \frac{d\hat{\rho}}{dt}(t,\xi) + \gamma^2|\xi|^2\hat{\rho}(t,\xi) = 0, \\ \hat{\rho}(0,\xi) = \widehat{\rho_0}(\xi),\ \hat{\rho}_t(0,\xi) = -i\gamma\xi\cdot\widehat{v_0}(\xi). \end{cases} \tag{2.2}$$

The characteristic equation corresponding to the (2.2) is

$$\lambda^2 + (\alpha+\beta)|\xi|^2\lambda + \gamma^2|\xi|^2 = 0. \tag{2.3}$$

The roots $\lambda_\pm(\xi)$ of (2.3) are given by the formula

$$\lambda_\pm(\xi) = -A\left(|\xi|^2 \pm \sqrt{|\xi|^4 - B^2|\xi|^2}\right), \tag{2.4}$$

where $A = (\alpha+\beta)/2, B = 2\gamma/(\alpha+\beta)$. When $|\xi| \neq 0$, B, the solution of (2.2) is given by the formula

(2.5)

$$\hat{\rho}(t,\xi) = \frac{\lambda_+(\xi)e^{\lambda_-(\xi)t} - \lambda_-(\xi)e^{\lambda_+(\xi)t}}{\lambda_+(\xi)-\lambda_-(\xi)}\widehat{\rho_0}(\xi) - i\gamma\frac{e^{\lambda_+(\xi)t} - e^{\lambda_-(\xi)t}}{\lambda_+(\xi)-\lambda_-(\xi)}\xi\cdot\widehat{v_0}(\xi).$$

Since $\lambda_+(\xi) = \lambda_-(\xi)$ when $|\xi| = B$, as the solution of (2.2), when $B/2 < |\xi| < 2B$, we use the following formula

$$\begin{aligned} \hat{\rho}(t,\xi) = {} & \frac{1}{2\pi i}\oint_\Gamma \frac{(z+|\xi|^2)e^{zt}}{z^2+(\alpha+\beta)|\xi|^2 z + \gamma^2|\xi|^2}\,dz\,\widehat{\rho_0}(\xi) \\ & + \frac{i\gamma}{2\pi i}\oint_\Gamma \frac{e^{zt}}{z^2+(\alpha+\beta)|\xi|^2 z + \gamma^2|\xi|^2}\,dz\,\xi\cdot\widehat{v_0}(\xi), \end{aligned} \tag{2.6}$$

where Γ is a closed path containing $\lambda_\pm(\xi)$ and contained in $\{z \in \mathbb{C} \,|\, \mathrm{Re}z \leqq -c_0\}$ and c_0 is a positive number such that

$$\max_{\frac{B}{2}\leqq|\xi|\leqq 2B} \mathrm{Re}\lambda_\pm(\xi) \leqq -2c_0. \tag{2.7}$$

Also, by (2.1) we have

$$\begin{cases} \frac{d\hat{v}}{dt}(t,\xi) + \alpha|\xi|^2\hat{v}(t,\xi) = \hat{f}(t,\xi), \\ \hat{v}(0,\xi) = \widehat{v_0}(\xi), \end{cases} \tag{2.8}$$

where

$$\hat{f}(t,\xi) = \frac{\xi}{i\gamma}\left\{\beta\frac{d\hat{\rho}}{dt}(t,\xi) + \gamma^2\hat{\rho}(t,\xi)\right\}.$$

Therefore, by (2.5) and (2.8), the solution of (2.6) given by the formula:

when $|\xi| \neq 0, B$,

$$
\begin{aligned}
(2.9) \quad \hat{v}(t,\xi) &= e^{-\alpha|\xi|^2 t}\widehat{v_0}(\xi) + \int_0^t e^{-\alpha|\xi|^2(t-s)}\hat{f}(s,\xi)\,ds \\
&= e^{-\alpha|\xi|^2 t}\widehat{v_0}(\xi) - i\gamma\xi\left(\frac{e^{\lambda_+(\xi)t} - e^{\lambda_-(\xi)t}}{\lambda_+(\xi)-\lambda_-(\xi)}\right)\widehat{\rho_0}(\xi) \\
&\quad + \left(\frac{\lambda_+(\xi)e^{\lambda_+(\xi)t} - \lambda_-(\xi)e^{\lambda_-(\xi)t}}{\lambda_+(\xi)-\lambda_-(\xi)} - e^{-\alpha|\xi|^2 t}\right)\frac{\xi\,(\xi\cdot\widehat{v_0}(\xi))}{|\xi|^2},
\end{aligned}
$$

and when $B/2 < |\xi| < 2B$,

$$
\begin{aligned}
\hat{v}(t,\xi) &= e^{-\alpha|\xi|^2 t}\widehat{v_0}(\xi) - \frac{i\gamma\xi}{2\pi i}\oint_\Gamma \frac{e^{zt}}{z^2 + (\alpha+\beta)|\xi|^2 z + \gamma^2|\xi|^2}\,dz\,\widehat{\rho_0}(\xi) \\
&\quad + \left(\frac{1}{2\pi i}\oint_\Gamma \frac{(z+|\xi|^2)e^{zt}}{z^2 + (\alpha+\beta)|\xi|^2 z + \gamma^2|\xi|^2}\,dz\, - e^{-\alpha|\xi|^2 t}\right)\frac{\xi\,(\xi\cdot\hat{v}_0(\xi))}{|\xi|^2}.
\end{aligned}
$$

Let $\varphi_0(\xi)$, $\varphi_M(\xi)$ and $\varphi_\infty(\xi)$ be functions in $C^\infty(\mathbb{R}^n)$ such that

$$
(2.10) \qquad \varphi_0(\xi) = \begin{cases} 1 & |\xi| \leqq B/2, \\ 0 & |\xi| \geqq B/\sqrt{2}, \end{cases} \qquad \varphi_\infty(\xi) = \begin{cases} 1 & |\xi| \geqq 2B, \\ 0 & |\xi| \leqq \sqrt{2}B, \end{cases}
$$
$$
\varphi_M(\xi) = 1 - \varphi_0(\xi) - \varphi_\infty(\xi).
$$

Put

$$
\begin{aligned}
(2.11) \qquad & E_0(t) = (E_{0,\rho}(t), E_{0,v}(t)), \\
& E_\infty(t) = (E_{\infty,\rho}(t), E_{\infty,v}(t)), \\
& E_{0,\rho}(t)(\rho_0, v_0)(x) = \mathcal{F}^{-1}\left[\varphi_0(\xi)\hat{\rho}(t,\xi)\right](x), \\
& E_{0,v}(t)(\rho_0, v_0)(x) = \mathcal{F}^{-1}\left[\varphi_0(\xi)\hat{v}(t,\xi)\right](x), \\
& E_{\infty,\rho}(t)(\rho_0, v_0)(x) = \mathcal{F}^{-1}\left[(\varphi_M(\xi)+\varphi_\infty(\xi))\hat{\rho}(t,\xi)\right](x), \\
& E_{\infty,v}(t)(\rho_0, v_0)(x) = \mathcal{F}^{-1}\left[(\varphi_M(\xi)+\varphi_\infty(\xi))\hat{v}(t,\xi)\right](x).
\end{aligned}
$$

Noting that $\varphi_M(\xi) = 1$ for $B/\sqrt{2} \leqq |\xi| \leqq \sqrt{2}B$ and $\varphi_M(\xi) = 0$ for $|\xi| \geqq 2B$ or $|\xi| \leqq B/2$, by (2.10) and (2.11) we see that $(\rho(t,x), v(t,x)) = E_0(t)(\rho_0, v_0)(x)$ is a solution of (1.1). The main purpose of the paper is to show the following two theorems.

Theorem 2.1 ($L_1 - L_\infty$ and $L_1 - L_1$ estimate of $E_0(t)$)**.**

(1) *For any $t > 0$, we have*

$$\|\partial_t^j \partial_x^\alpha E_{0,\rho}(t)(\rho_0, v_0)\|_{L_\infty(\mathbb{R}^n)} \leqq C_{j,\alpha,n}(1+t)^{-\left(\frac{3n-1}{4}+\frac{j+|\alpha|}{2}\right)} \left[\|\rho_0\|_{L_1(\mathbb{R}^n)} + \|v_0\|_{L_1(\mathbb{R}^n)}\right];$$

$$\|\partial_t^j \partial_x^\alpha E_{0,v}(t)(\rho_0, v_0)\|_{L_\infty(\mathbb{R}^n)} \leqq C_{j,\alpha,n}(1+t)^{-\left(\frac{3n-1}{4}+\frac{j+|\alpha|}{2}\right)} \left[\|\rho_0\|_{L_1(\mathbb{R}^n)} + \|v_0\|_{L_1(\mathbb{R}^n)}\right] + C_{j,\alpha,n}(1+t)^{-\left(\frac{n}{2}+\frac{j+|\alpha|}{2}\right)} \|v_0\|_{L_1(\mathbb{R}^n)}.$$

Here and hereafter, we write

$$\partial_t^j = \frac{\partial^j}{\partial t^j}, \qquad \partial_x^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}},$$
$$\alpha = (\alpha_1, \dots, \alpha_n), \qquad |\alpha| = \alpha_1 + \cdots + \alpha_n,$$

$C_{A,B,\dots}$ *means the constant depending on $A, B, \dots$.*

(2) *For any $t > 0$, we have*

$$\|\partial_t^j \partial_x^\alpha E_0(t)(\rho_0, v_0)\|_{L_1(\mathbb{R}^n)} \leqq C_{j,\alpha,n}(1+t)^{q(n)-\frac{j+|\alpha|}{2}} \left[\|\rho_0\|_{L_1(\mathbb{R}^n)} + \|v_0\|_{L_1(\mathbb{R}^n)}\right],$$

where

$$q(n) = \begin{cases} \dfrac{n-1}{4} & \text{if } n \geqq 3 \text{ and } n \text{ is an odd number,} \\ \dfrac{n}{4} & \text{if } n \geqq 2 \text{ and } n \text{ is an even number.} \end{cases}$$

Remark. The estimate (1) is better than [**3**, Theorem 1.2] when $n = 2$, $j = 0$ and $|\alpha| = 0$.

Theorem 2.2 ($L_1 - L_1$ and $L_\infty - L_\infty$ estimate of $E_\infty(t)$)**.** *Let $p = 1$ or ∞. For any $t > 0$, we have*

$$\|\partial_t^j \partial_x^\alpha E_{\infty,\rho}(t)(\rho_0, v_0)\|_{L_p(\mathbb{R}^n)} \leqq C_{j,\alpha,n} e^{-ct} \left[C_k t^{-(j-k)} \|\rho_0\|_{W_p^{(2k+|\alpha|-1)^+}(\mathbb{R}^n)} + \|\rho_0\|_{W_p^{|\alpha|}(\mathbb{R}^n)}\right] + C_{j,\alpha,n} e^{-ct} \left[C_k t^{-(j-k)} \|v_0\|_{W_p^{2k+|\alpha|}(\mathbb{R}^n)} + \|v_0\|_{W_p^{|\alpha|}(\mathbb{R}^n)}\right];$$

$$\|\partial_t^j \partial_x^\alpha E_{\infty,v}(t)(\rho_0, v_0)\|_{L_p(\mathbb{R}^n)} \leqq C_{j,\alpha,n} e^{-ct} \left[C_k t^{-(j-k)} \|\rho_0\|_{W_p^{2k+|\alpha|}(\mathbb{R}^n)} + \|\rho_0\|_{W_p^{|\alpha|}(\mathbb{R}^n)}\right] + C_{j,\alpha,n} e^{-ct}(1 + t^{-\frac{1}{2}}) \left[C_k t^{-(j-k)} \|v_0\|_{W_p^{2k+|\alpha|}(\mathbb{R}^n)} + \|v_0\|_{W_p^{|\alpha|}(\mathbb{R}^n)}\right].$$

Here and hereafter, we put $K^+ = \max(K, 0)$ *and*

$$W_p^k(\mathbb{R}^n) = \left\{ u \in L_p(\mathbb{R}^n) \mid \|u\|_{W_p^k(\mathbb{R}^n)} = \sum_{|\alpha| \leqq k} \|\partial_x^\alpha u\|_{L_p(\mathbb{R}^n)} < \infty \right\}.$$

Y. Shibata [**6**] gave the $L_p - L_q$ type estimates for the solution to the linear viscoelastic equation:

$$\begin{cases} v_{tt} - \Delta v - \Delta v_t = 0 & \text{in } [0, \infty) \times \mathbb{R}^n, \\ v(0) = v_0, \ v_t(0) = v_1 & \text{in } \mathbb{R}^n. \end{cases} \tag{2.12}$$

The solution of (2.12) are representated by the Fourier transform as follows: When $|\xi| \neq 0, 2$

$$\hat{v}(t, \xi) = \frac{\lambda_+(\xi) e^{\lambda_-(\xi)t} - \lambda_-(\xi) e^{\lambda_+(\xi)t}}{\lambda_+(\xi) - \lambda_-(\xi)} \widehat{v_0}(\xi) + \frac{e^{\lambda_+(\xi)t} - e^{\lambda_-(\xi)t}}{\lambda_+(\xi) - \lambda_-(\xi)} \widehat{v_1}(\xi),$$

where

$$\lambda_\pm(\xi) = \frac{-|\xi|^2 \pm \sqrt{|\xi|^4 - |\xi|^2}}{2}$$

and when $1 < |\xi| < 4$,

$$\hat{v}(t, \xi) = \frac{1}{2\pi i} \oint_\gamma \frac{(z + |\xi|^2) e^{zt}}{z^2 + |\xi|^2 z + |\xi|^2} \, dz \widehat{v_0}(\xi) + \frac{1}{2\pi i} \oint_\gamma \frac{e^{zt}}{z^2 + |\xi|^2 z + |\xi|^2} \, dz \hat{v}_1(\xi),$$

where γ is a closed path containing $\lambda_\pm(\xi)$ and contained in $\{z \in \mathbb{C} \mid \mathrm{Re} z \leqq -c_0\}$ and c_0 is a positive number such that

$$\max_{1 \leqq |\xi| \leqq 4} \mathrm{Re} \lambda_\pm(\xi) \leqq -2c_0.$$

The difference of the structure to the solutions between (1.1) and (2.12) is the Riesz kernel $R_j(x) = \mathcal{F}^{-1}(\xi_j / |\xi|)(x)$. The Green matrix of the solution of (1.1) includes the Riesz kernel (cf. (2.9)). Since the convolution operator $u \to R_j * u$ is bounded from L_p to L_p for $1 < p < \infty$, the following theorems directly follow from [**6**, Theorems 2.1 and 2.2].

Theorem 2.3 ($L_p - L_q$ estimate of $E_0(t)$)**.**

(1) *Let M be the positive number $\geqq 1$ and let $1 \leqq p \leqq q \leqq \infty$, $(p, q) \neq (\infty, \infty), (1, 1)$. Then, for any $t \in [0, M]$, we have*

$$\|\partial_t^j \partial_x^\alpha E_0(t)(\rho_0, v_0)\|_{L_q(\mathbb{R}^n)} \leqq C_{n,p,q,j,\alpha,M} \left[\|\rho_0\|_{L_p(\mathbb{R}^n)} + \|v_0\|_{L_p(\mathbb{R}^n)} \right].$$

(2) *Let $1 \leqq p \leqq 2 \leqq q \leqq \infty$. For any $t > 0$, we have*

$$\begin{aligned} &\|\partial_t^j \partial_x^\alpha E_0(t)(\rho_0, v_0)\|_{L_q(\mathbb{R}^n)} \\ &\leqq C_{n,p,q,j,\alpha} (1 + t)^{-\left(\frac{n}{2}\left(\frac{1}{p} - \frac{1}{q}\right) + \frac{j + |\alpha|}{2}\right)} \left[\|\rho_0\|_{L_p(\mathbb{R}^n)} + \|v_0\|_{L_p(\mathbb{R}^n)} \right]. \end{aligned}$$

Remark.

(1) The estimate (1) in Theorem 2.1 is better than the estimate (2) in Theorem 2.3 with $(p, q) = (1, \infty)$.

(2) By Theorem 2.1 and Theorem 2.2, the estimate (1) in Theorem 2.3 also holds when $(p, q) = (1, 1)$ or (∞, ∞).

Theorem 2.4 ($L_p - L_p$ estimate of $E_\infty(t)$)**.** *Let* $1 < p < \infty$*. For any* $t > 0$*, we have*

$$\begin{aligned}
&\|\partial_t^j \partial_x^\alpha E_{\infty,\rho}(t)(\rho_0, v_0)\|_{L_p(\mathbb{R}^n)} \\
&\leqq C_{j,\alpha,p,N} e^{-ct} \left[t^{-\frac{N}{2}} \|\rho_0\|_{W_p^{(2j+|\alpha|-N-2)^+}(\mathbb{R}^n)} + \|\rho_0\|_{W_p^{|\alpha|}(\mathbb{R}^n)} \right] \\
&\quad + C_{j,\alpha,p,N} e^{-ct} \left[t^{-\frac{N}{2}} \|v_0\|_{W_p^{(2j+|\alpha|-N-1)^+}(\mathbb{R}^n)} + \|v_0\|_{W_p^{(|\alpha|-1)^+}(\mathbb{R}^n)} \right];
\end{aligned}$$

$$\begin{aligned}
&\|\partial_t^j \partial_x^\alpha E_{\infty,v}(t)(\rho_0, v_0)\|_{L_p(\mathbb{R}^n)} \\
&\leqq C_{j,\alpha,n,N} e^{-ct} \left[t^{-\frac{N}{2}} \|\rho_0\|_{W_p^{(2j+|\alpha|-N-1)^+}(\mathbb{R}^n)} + \|\rho_0\|_{W_p^{(|\alpha|-1)^+}(\mathbb{R}^n)} \right] \\
&\quad + C_{j,\alpha,n,N} e^{-ct} \left[t^{-\frac{N}{2}} \|v_0\|_{W_p^{(2j+|\alpha|-N)^+}(\mathbb{R}^n)} + \|v_0\|_{W_p^{(|\alpha|-2)^+}(\mathbb{R}^n)} \right].
\end{aligned}$$

3. Proof of Theorem 2.1 (1).

To prove Theorem 2.1 (1), we put

$$\begin{aligned}
L_{11}(t,x) &= \mathcal{F}^{-1}\left[\frac{\lambda_+(\xi)e^{\lambda_-(\xi)t} - \lambda_-(\xi)e^{\lambda_+(\xi)t}}{\lambda_+(\xi) - \lambda_-(\xi)} \varphi_0(\xi) \right](x), \\
L_{12}(t,x) &= -i\gamma \mathcal{F}^{-1}\left[{}^t\xi \, \frac{e^{\lambda_+(\xi)t} - e^{\lambda_-(\xi)t}}{\lambda_+(\xi) - \lambda_-(\xi)} \varphi_0(\xi) \right](x), \\
L_{21}(t,x) &= {}^tL_{12}(t,x), \\
L_{22}(t,x) &= K_1(t,x) + K_2(t,x) - K_3(t,x), \\
K_1(t,x) &= \mathcal{F}^{-1}\left[e^{-\alpha|\xi|^2 t} \varphi_0(\xi) \right](x) I, \quad I \text{ is unit matrix}, \\
K_2(t,x) &= \mathcal{F}^{-1}\left[\frac{\lambda_+(\xi)e^{\lambda_+(\xi)t} - \lambda_-(\xi)e^{\lambda_-(\xi)t}}{\lambda_+(\xi) - \lambda_-(\xi)} \frac{\xi_j \xi_k}{|\xi|^2} \varphi_0(\xi) \right](x), \\
K_3(t,x) &= \mathcal{F}^{-1}\left[e^{-\alpha|\xi|^2 t} \frac{\xi_j \xi_k}{|\xi|^2} \varphi_0(\xi) \right](x),
\end{aligned} \tag{3.1}$$

and then, from (2.5) and (2.9) it follows that

$$E_0(t)(\rho_0, v_0) = \begin{pmatrix} L_{11}(t,\cdot) & L_{12}(t,\cdot) \\ L_{21}(t,\cdot) & L_{22}(t,\cdot) \end{pmatrix} * \begin{pmatrix} \rho_0 \\ v_0 \end{pmatrix}, \tag{3.2}$$

where $*$ denotes the spatial convolution. In view of the Young inequality, in order to get Theorem 2.1 (1) it suffices to show that for $t > 0$

$$\|\partial_t^j \partial_x^\alpha L_{11}(t,\cdot)\|_{L_\infty(\mathbb{R}^n)} \leqq C_{j,\alpha,n} t^{-(\frac{3n-1}{4}+\frac{j+|\alpha|}{2})}, \tag{3.3}$$

$$\|\partial_t^j \partial_x^\alpha L_{12}(t,\cdot)\|_{L_\infty(\mathbb{R}^n)} \leqq C_{j,\alpha,n} t^{-(\frac{3n-1}{4}+\frac{j+|\alpha|}{2})}, \tag{3.4}$$

$$\|\partial_t^j \partial_x^\alpha K_1(t,\cdot)\|_{L_\infty(\mathbb{R}^n)} \leqq C_{j,\alpha,n} t^{-(\frac{n}{2}+\frac{|\alpha|}{2}+j)}, \tag{3.5}$$

$$\|\partial_t^j \partial_x^\alpha K_2(t,\cdot)\|_{L_\infty(\mathbb{R}^n)} \leqq C_{j,\alpha,n} t^{-(\frac{3n-1}{4}+\frac{j+|\alpha|}{2})}, \tag{3.6}$$

$$\|\partial_t^j \partial_x^\alpha K_3(t,\cdot)\|_{L_\infty(\mathbb{R}^n)} \leqq C_{j,\alpha,n} t^{-(\frac{n}{2}+\frac{|\alpha|}{2}+j)}. \tag{3.7}$$

It is obvious that

$$\begin{aligned}\left|\partial_t^j \partial_x^\beta \mathcal{F}^{-1}\left[e^{-\alpha|\xi|^2 t}\left(\delta_{ik} - \frac{\xi_i \xi_k}{|\xi|^2}\right)\varphi_0(\xi)\right](x)\right| &\leqq C_{j,\beta,n} \int_{\mathbb{R}^n} e^{-\alpha|\xi|^2 t}|\xi|^{2j+|\beta|}\, d\xi \\ &\leqq C_{j,\beta,n} t^{-(\frac{n}{2}+\frac{|\beta|}{2}+j)},\end{aligned}$$

which show (3.5) and (3.7). In view of (3.1), we put

$$K_{\psi,0}(t,x) = \mathcal{F}^{-1}\left[\frac{e^{\lambda_+(\xi)t} - e^{\lambda_-(\xi)t}}{\lambda_+(\xi) - \lambda_-(\xi)}\psi(\xi)\varphi_0(\xi)\right](x), \tag{3.8}$$

$$K_{\psi,1}(t,x) = \mathcal{F}^{-1}\left[\frac{\lambda_+(\xi)e^{\lambda_+(\xi)t} - \lambda_-(\xi)e^{\lambda_-(\xi)t}}{\lambda_+(\xi) - \lambda_-(\xi)}\psi(\xi)\varphi_0(\xi)\right](x), \tag{3.9}$$

$$K_{\psi,2}(t,x) = \mathcal{F}^{-1}\left[\frac{\lambda_-(\xi)e^{\lambda_+(\xi)t} - \lambda_+(\xi)e^{\lambda_-(\xi)t}}{\lambda_+(\xi) - \lambda_-(\xi)}\psi(\xi)\varphi_0(\xi)\right](x), \tag{3.10}$$

where $\psi = \psi(\omega) \in C^\infty(S^{n-1})$, $S^{n-1} = \{\xi \in \mathbb{R}^n \mid |\xi| = 1\}$ and $\psi(\xi) = \psi(\xi/|\xi|)$. By (2.3) and (2.4), we know that

$$\begin{gathered}\lambda_+(\xi)\lambda_-(\xi) = A^2B^2|\xi|^2, \\ \lambda_+(\xi) + \lambda_-(\xi) = -2A|\xi|^2, \\ \lambda_\pm(\xi)^2 + 2A\lambda_\pm(\xi)|\xi|^2 + \gamma|\xi|^2 = 0,\end{gathered} \tag{3.11}$$

and then

$$\begin{cases} K_{\psi,1}(t,x) = \partial_t K_{\psi,0}(t,x), \\ K_{\psi,2}(t,x) = -\partial_t K_{\psi,0}(t,x) + 2A\Delta K_{\psi,0}(t,x). \end{cases} \tag{3.12}$$

Therefore, in order to show (3.3), (3.4) and (3.6) it suffices to show the following theorem:

Theorem 3.1. *Let $n \geqq 2$. For any $t \geqq 0$, we have*

$$\|\partial_t^j \partial_x^\alpha K_{\psi,0}(t,\cdot)\|_{L_\infty(\mathbb{R}^n)} \leqq C_{j,\alpha,n}(1+t)^{-\left(\frac{3n-3}{4}+\frac{j+|\alpha|}{2}\right)}.$$

To prove this theorem, first of all, we shall estimate $K_{\psi,0}(t,x)$ near the light cone. Namely, we shall show that for $t \geqq \max(1,(R/R_0)^4)$ and $|x| \geqq R_0 t$

$$\left|\partial_t^j \partial_x^\alpha K_{\psi,0}(t,x)\right| \leqq C_{j,\alpha,n}(1+t)^{-\left(\frac{3n-3}{4}+\frac{j+|\alpha|}{2}\right)}, \tag{3.13}$$

where R is the number appearing in Lemma 3.2, below and R_0 is the fixed number such that $R_0 \leqq \gamma/4$. To obtain (3.13), we shall use the following lemma concerning the stationary phase method (cf. Vainberg [**9**, pp. 29-35]):

Lemma 3.2. *Let $g(\omega) \in C^\infty(S^{n-1})$, $S^{n-1} = \{\xi \in \mathbb{R}^n \,|\, |\xi| = 1\}$. Then, there exist a $R > 1$ and a C_g such that*

$$\left|\int_{S^{n-1}} e^{ir(\hat{x}\cdot\omega)} g(\omega)\, dS_\omega\right| \leqq C_g\, r^{-\frac{n-1}{2}}, \quad \hat{x} \in S^{n-1},\ r \geqq R.$$

If we put $|\xi| = r$, we have

$$\lambda_\pm(\xi) = -A\left(r^2 \pm ir\sqrt{B^2 - r^2}\right) = \lambda_\pm(r).$$

Since we may assume that $\varphi_0(\xi) = \varphi_0(|\xi|) = \varphi_0(r)$, by using the polar coordinate we have

$$\partial_t^j \partial_x^\alpha K_{\psi,0}(t,x) = \left(\frac{1}{2\pi}\right)^n \int_0^\infty \frac{\lambda_+(r)e^{\lambda_+(r)t} - \lambda_-(r)e^{\lambda_-(r)t}}{\lambda_+(r) - \lambda_-(r)}\, r^{|\alpha|+n-1}\varphi_0(r)dr \cdot \int_{S^{n-1}} e^{i(\hat{x}\cdot\omega)r|x|}\,(i\omega)^\alpha\, \psi(\omega)\, dS_\omega,$$

where $\hat{x} = x/|x|$. Let $\epsilon > 0$ be a number determined later on. Let us consider the case where $|x|\epsilon \geqq R$, below. Since $r|x| \geqq \epsilon|x| \geqq R$ when $r \geqq \epsilon$, by Lemma 3.2

$$\left|\int_{S^{n-1}} e^{i(\hat{x}\cdot\omega)r|x|}\,(i\omega)^\alpha\, \psi(\omega)\, dS_\omega,\right| \leqq C_\alpha (r|x|)^{-\frac{n-1}{2}}.$$

Noting that $\varphi_0(r) = 0$ when $r \geqq B/\sqrt{2}$ (cf. (2.10)), we have

$$\begin{aligned}&\left|\partial_t^j \partial_x^\alpha K_{\psi,0}(t,x)\right| \\ &\leqq C\left\{\int_0^\epsilon r^{n+j+|\alpha|-2}\, dr + \int_\epsilon^\infty e^{-Ar^2 t} r^{n-2+j+|\alpha|}(r|x|)^{-\frac{n-1}{2}}\, dr\right\}.\end{aligned}$$

If we make the change of variable; $r\sqrt{t} = s$ in the last integration and if we use the assumption: $|x| \geqq R_0 t$, then we have

$$\left|\partial_t^j \partial_x^\alpha K_{\psi,0}(t,x)\right| \leqq C_{j,\alpha,n}\left\{\epsilon^{n+j+|\alpha|-1} + t^{-\left(\frac{3n-3}{4}+\frac{j+|\alpha|}{2}\right)}\right\}.$$

Choose $\epsilon > 0$ in such a way that

$$\epsilon^{n+j+|\alpha|-1} = t^{-\left(\frac{3n-3}{4}+\frac{j+|\alpha|}{2}\right)}.$$

When $|x| \geqq R_0 t$ and $t \geqq \max(1, (R/R_0)^4)$, we see that

$$|x|\epsilon \geqq R_0 t \cdot t^{-\left(\frac{3n-3}{4}+\frac{j+|\alpha|}{2}\right)/(n+j+|\alpha|-1)} = R_0 t^{\frac{\frac{n-1}{4}+\frac{j+|\alpha|}{2}}{n+j+|\alpha|-1}} \geqq R_0 t^{\frac{1}{4}} \geqq R.$$

Therefore, we have (3.13).

Now, we shall show that for $t \geqq 1$ and $|x| \leqq R_0 t$

$$\left|\partial_t^j \partial_x^\alpha K_{\psi,0}(t,x)\right| \leqq C_{j,\alpha,n} t^{-\left(\frac{3n-3}{4}+\frac{j+|\alpha|}{2}\right)}. \tag{3.14}$$

If we put

$$f(\xi) = \sqrt{1-|\xi|^2 B^{-2}} = 1 + |\xi|^2 g(|\xi|^2), \quad g(s) = -\frac{1}{2B^2}\int_0^1 \frac{1}{\sqrt{1-\theta s B^{-2}}}\,d\theta,$$

then by Taylor's formula we have

$$\begin{aligned}\frac{e^{\lambda_+(\xi)t} - e^{-\lambda_-(\xi)t}}{\lambda_+(\xi)-\lambda_-(\xi)} = \sum_{\ell=0}^{N} \frac{1}{\ell!}\left(\partial_t^\ell \frac{\sin\gamma|\xi|t}{|\xi|}\right)\frac{e^{-A|\xi|^2 t}}{\gamma f(|\xi|)}\left(|\xi|^2 g(|\xi|^2)t\right)^\ell \\ + e^{-A|\xi|^2 t} R_N(t,|\xi|),\end{aligned} \tag{3.15}$$

where

$$\begin{aligned}&R_N(t,|\xi|)\\ &= \frac{1}{2i\gamma|\xi|f(|\xi|)N!}\int_0^1 (1-\theta)^N \Big[e^{i\gamma|\xi|t+i\gamma|\xi|^3 g(|\xi|^2)t\theta}\left(i\gamma|\xi|^3 g(|\xi|^2)t\right)^{N+1} \\ &\quad - e^{-i\gamma|\xi|t-i\gamma|\xi|^3 g(|\xi|^2)t\theta}\left(-i\gamma|\xi|^3 g(|\xi|^2)t\right)^{N+1}\Big]\,d\theta.\end{aligned}$$

In fact,

$$e^{\lambda_\pm(\xi)t} = e^{-A|\xi|^2 t} e^{\mp i\gamma|\xi|f(|\xi|)t} = e^{-A|\xi|^2 t} e^{\mp i\gamma|\xi|t + i\gamma|\xi|^3 g(|\xi|^2)t}.$$

Put

$$h(\theta) = e^{\pm i\gamma|\xi|^3 g(|\xi|^2)t\theta}, \quad h^{(k)}(\theta) = \frac{d^k h}{d\theta^k}(\theta).$$

Since

$$h(1) = h(0) + h'(0) + \cdots + \frac{1}{N!}h^{(N)}(0) + \frac{1}{N!}\int_0^1 (1-\theta)^N h^{(N+1)}(\theta)\,d\theta,$$

we have

$$\begin{aligned}e^{\pm i\gamma|\xi|^3 g(|\xi|^2)t} = 1 + \left(\pm i\gamma|\xi|^3 g(|\xi|^2)t\right) + \dots \frac{1}{N!}\left(\pm i\gamma|\xi|^3 g(|\xi|^2)t\right)^N \\ + \frac{1}{N!}\int_0^1 (1-\theta)^N e^{\pm i\gamma|\xi|^3 g(|\xi|^2)t\theta}\left(\pm i\gamma|\xi|^3 g(|\xi|^2)t\right)^{N+1}.\end{aligned}$$

Since

$$\begin{aligned}&\left(i\gamma|\xi|^3 g(|\xi|^2)t\right)^N e^{i\gamma|\xi|t} - \left(i\gamma|\xi|^3 g(|\xi|^2)t\right)^N e^{-i\gamma|\xi|t}\\ &= \left\{\partial_t^N \left(e^{i\gamma|\xi|t} - e^{-i\gamma|\xi|t}\right)\right\} \left(|\xi|^2 g(|\xi|^2)t\right)^N\\ &= 2i\left(\partial_t^N \sin\gamma|\xi|t\right)\left(|\xi|^2 g(|\xi|^2)t\right)^N,\end{aligned}$$

noting that $\lambda_+(\xi) - \lambda_-(\xi) = -2i\gamma|\xi| f(|\xi|)$, we have (3.15).

We shall use the following lemma.

Lemma 3.3 (cf. Mizohata [**5**], Evans [**1**]). *Put*

$$w(t,x) = \mathcal{F}^{-1}\left[\frac{\sin|\xi|t}{|\xi|}\hat{h}(\xi)\right](x).$$

Then, for suitable constants a_α we have

$$w(t,x) = \sum_{0\leqq|\alpha|\leqq\frac{n-3}{2}} a_\alpha t^{|\alpha|+1}\int_{|z|=1} z^\alpha \left(\partial_x^\alpha h\right)(x+tz)\, dS$$

for odd $n \geqq 3$; and

$$w(t,x) = \sum_{0\leqq|\alpha|\leqq\frac{n-2}{2}} a_\alpha t^{|\alpha|+1}\int_{|z|\leqq1} \frac{z^\alpha \left(\partial_x^\alpha h\right)(x+tz)}{\sqrt{1-|z|^2}}\, dz$$

for even $n \geqq 2$.

Regarding (3.15), we put

$$\begin{aligned}G_\ell(t,x) &= \mathcal{F}^{-1}\left[\frac{e^{-A|\xi|^2 t}}{f(|\xi|)}\left(|\xi|^2 g(|\xi|^2)t\right)^\ell \psi(\xi)\varphi_0(\xi)\right](x),\\ \omega_\ell(t,x) &= \frac{1}{\gamma}\mathcal{F}^{-1}\left[\left(\partial_t^\ell \frac{\sin\gamma|\xi|t}{|\xi|}\right)\widehat{G_\ell}(t,\xi)\right](x).\end{aligned}$$

Since

$$\mathcal{F}^{-1}\left[\partial_t^\ell\left(\frac{\sin\gamma|\xi|t}{|\xi|}\right)\hat{h}(\xi)\right](x) = \partial_t^\ell \mathcal{F}^{-1}\left[\frac{\sin\gamma|\xi|t}{|\xi|}\hat{h}(\xi)\right](x),$$

by Lemma 3.3 we have

$$\begin{aligned}\partial_t^j\partial_x^\beta K_{\psi,0}(t,x) &= \sum_{\ell=0}^N \partial_t^j\partial_x^\beta \omega_\ell(t,x)\\ &\quad + \partial_t^j\partial_x^\beta \mathcal{F}^{-1}\left[e^{-A|\xi|^2 t} R_N(t,|\xi|)\right](x),\end{aligned} \tag{3.16}$$

where

(3.17) $\partial_t^j \partial_x^\beta \omega_\ell(t,x)$

$$
= \begin{cases}
\frac{1}{\gamma}\sum_{k=0}^{j}\binom{j}{k}\sum_{|\alpha|\leqq\frac{n-3}{2}} a_\alpha \sum_{m=0}^{\min(\ell+k,|\alpha|+1)}\binom{\ell+k}{m}\partial_t^m(\gamma t)^{|\alpha|+1} \\
\qquad \cdot \sum_{|\delta|=\ell+k-m}\int_{|z|=1} z^{\alpha+\delta}\left(\partial_x^{\alpha+\beta+\delta}\partial_t^{j-k}G_\ell\right)(t,x+\gamma tz)\,dS \\
\qquad\qquad \text{for odd } n\geqq 3; \\
\frac{1}{\gamma}\sum_{k=0}^{j}\binom{j}{k}\sum_{|\alpha|\leqq\frac{n-2}{2}} a_\alpha \sum_{m=0}^{\min(\ell+k,|\alpha|+1)}\binom{\ell+k}{m}\partial_t^m(\gamma t)^{|\alpha|+1} \\
\qquad \cdot \sum_{|\delta|=\ell+k-m}\int_{|z|\leqq 1} \frac{z^{\alpha+\delta}\left(\partial_x^{\alpha+\beta+\delta}\partial_t^{j-k}G_\ell\right)(t,x+\gamma tz)}{\sqrt{1-|z|^2}}\,dz \\
\qquad\qquad \text{for even } n\geqq 2.
\end{cases}
$$

The following proposition and lemma play an essential role to prove Theorem 3.1.

Proposition 3.4 (Shibata-Shimizu [**7**])**.** *Let α be a number $> -n$ and put $\alpha = N+\sigma-n$ where $N \geqq 0$ is an integer and $0<\sigma\leqq 1$. Let $f(\xi)$ be a function in $C^\infty(\mathbb{R}^n-\{0\})$ such that*

$$
\partial_\xi^\gamma f(\xi)\in L_1(\mathbb{R}^n),\quad |\gamma|\leqq N;
$$
$$
\left|\partial_\xi^\gamma f(\xi)\right| \leqq C_\gamma|\xi|^{\alpha-|\gamma|},\quad \xi\neq 0,\quad {}^\forall\gamma.
$$

Then, we have

$$
\left|\mathcal{F}^{-1}[f(\xi)](x)\right| \leqq C_{\alpha,n}\left(\max_{|\gamma|\leqq N+2} C_\gamma\right)|x|^{-(n+|\alpha|)},\quad x\neq 0,
$$

where $C_{\alpha,n}$ is a constant depending essentially only on n and α.

Lemma 3.5. *Let α be a nonnegative number and $\psi(t,\xi)$ be a function such that*

$$
\psi(t,\cdot)\in C^\infty(\mathbb{R}^n-\{0\}),\quad {}^\forall t\geqq 0,
$$
$$
\left|\partial_\xi^\gamma\psi(t,\xi)\right| \leqq C_\gamma|\xi|^{\alpha-|\gamma|},\quad \xi\neq 0,\ {}^\forall\gamma,\ {}^\forall t\geqq 0.
$$

Put

$$
g(t,x)=\mathcal{F}^{-1}\left[e^{-\beta|\xi|^2 t}\psi(t,\xi)\right](x),
$$

where $\beta>0$. Then, we have

$$
|g(t,x)| \leqq C_{\alpha,\beta,n}|x|^{-(\alpha+n)},\quad x\neq 0, \tag{3.18}
$$
$$
|g(t,x)| \leqq C_{\alpha,\beta,n}t^{-\frac{\alpha+n}{2}},\quad t>0.
$$

Moreover,

$$\|g(t,\cdot)\|_{L_1(\mathbb{R}^n)} \leqq C_{\alpha,\beta,n} t^{-\frac{|\alpha|}{2}}, \quad \alpha > 0,\ t > 0,$$
$$\int_{|x|\leqq At} |g(t,x)|dx \leqq C_{\beta,n,A}(1+\log(1+t)), \quad \alpha = 0,\ t > 0.$$

Proof. By the formula of derivative of composed function (cf. Simader [**8**, p. 202]):

$$\partial_\xi^\gamma h(g(\xi)) = \sum_{\nu=1}^{|\gamma|} h^{(\nu)}(g(\xi)) \left[\sum_{\substack{\alpha_1+\cdots+\alpha_\nu=\gamma \\ |\alpha_i|\geqq 1}} \left(\partial_\xi^{\alpha_1} g(\xi)\right) \cdots \left(\partial_\xi^{\alpha_\nu} g(\xi)\right) \right], \tag{3.19}$$

we have

$$\partial_\xi^\gamma e^{-\beta|\xi|^2 t} = \sum_{\nu=1}^{|\gamma|} (\beta t)^\nu e^{-\beta|\xi|^2 t} \left[\sum_{\substack{\alpha_1+\cdots+\alpha_\nu=\gamma \\ |\alpha_i|\geqq 1}} \left(\partial_\xi^{\alpha_1} |\xi|^2\right) \cdots \left(\partial_\xi^{\alpha_\nu} |\xi|^2\right) \right].$$

Since

$$\left|\partial_\xi^{\alpha_i} |\xi|^M\right| \leqq C_{\alpha_i} |\xi|^{M-|\alpha_i|}, \quad \xi \neq 0, \tag{3.20}$$
$$(t|\xi|^2)^M e^{-\beta|\xi|^2 t} \leqq C_{M,\beta} e^{-\frac{\beta}{2}|\xi|^2 t},$$

we have

$$\begin{aligned} \left|\partial_\xi^\gamma e^{-\beta|\xi|^2 t}\right| &\leqq C_\gamma \sum_{\nu=1}^{|\gamma|} (\beta t)^\nu e^{-\beta|\xi|^2 t} \left[\sum_{\substack{\alpha_1+\cdots+\alpha_\nu=\gamma \\ |\alpha_i|\geqq 1}} |\xi|^{2\nu-(|\alpha_1|+\cdots+|\alpha_\nu|)} \right] \\ &\leqq C_\gamma \sum_{\nu=1}^{|\gamma|} (\beta|\xi|^2 t)^\nu e^{-\beta|\xi|^2 t} |\xi|^{-|\gamma|} \\ &\leqq C_{\beta,\gamma} |\xi|^{-|\gamma|} e^{-\frac{\beta}{2}|\xi|^2 t}, \quad \xi \neq 0, \end{aligned} \tag{3.21}$$

and the Leibniz's rule we have

$$\left|\partial_\xi^\gamma \left(e^{-\beta|\xi|^2 t}\psi(t,\xi)\right)\right| \leqq C_{\alpha,\beta,\gamma} e^{-\frac{\beta}{2}|\xi|^2 t} |\xi|^{\alpha-|\gamma|}, \quad \xi \neq 0,\ {}^\forall\gamma. \tag{3.22}$$

Therefore, by Proposition 3.4 we have (3.18).

By the assumptions, we have

$$\begin{aligned} |g(t,x)| &\leqq C \int_{\mathbb{R}^n} e^{-\beta|\xi|^2 t} |\xi|^\alpha \, d\xi = C t^{-\frac{\alpha+n}{2}} \int_{\mathbb{R}^n} e^{-\beta|\eta|^2} |\eta|^\alpha \, d\eta \\ &\leqq C_{\alpha,\beta,n} t^{-\frac{\alpha+n}{2}}. \end{aligned} \tag{3.23}$$

By (3.18) and (3.23), we have

$$\|g(t,\cdot)\|_{L_1(\mathbb{R}^n)} \leqq C_{\alpha,\beta,n} t^{-\frac{\alpha+n}{2}} \int_{|x|\leqq\sqrt{t}} dx + C_{\alpha,\beta,n} \int_{|x|\geqq\sqrt{t}} |x|^{-(\alpha+n)}\,dx$$
$$\leqq C_{\alpha,\beta,n} t^{-\frac{\alpha}{2}}, \quad \alpha > 0,$$

and also

$$\int_{|x|\leqq At} |g(t,x)|\,dx \leqq C_{\beta,n} t^{-\frac{n}{2}} \int_{|x|\leqq At} dx \leqq C_{\beta,n,A}, \quad \alpha = 0,\ t < 1,$$

$$\int_{|x|\leqq At} |g(t,x)|\,dx \leqq C_{\beta,n} t^{-\frac{n}{2}} \int_{|x|\leqq\sqrt{t}} dx + C_{\beta,n} \int_{\sqrt{t}\leqq|x|\leqq At} |x|^{-n} dx$$
$$\leqq C_{\beta,n,A}(1+\log t), \quad \alpha = 0,\ t \geqq 1$$

which completes the Proof of Lemma 3.5.

Concerning the estimate $G_\ell(t,x)$, we have

(3.24)
$$\left|\partial_t^{j-k}\partial_x^{\alpha+\beta+\delta} G_\ell(t, x+\gamma t z)\right| \leqq C_{j,k,\alpha,\beta,\delta,\ell} |x+\gamma t z|^{-(2(j-k)+|\alpha|+|\beta|+|\delta|+n)}.$$

In fact,

$$\partial_t^{j-k}\partial_x^{\alpha+\beta+\delta} G_\ell(t,x)$$
$$= \mathcal{F}^{-1}\Bigg[\sum_{m=0}^{\min(j-k,\ell)} \binom{j-k}{m} \frac{e^{-A|\xi|^2 t}}{f(|\xi|)} (-A|\xi|^2)^{j-k-m} (i\xi)^{\alpha+\beta+\delta}$$
$$\cdot\, \partial_t^m \left(|\xi|^2 g(|\xi|^2) t\right)^\ell \psi(\xi)\varphi_0(\xi)\Bigg](x).$$

By (3.20), (3.21) and (3.22) we have

$$\Bigg|\partial_\xi^\mu \Bigg(\sum_{m=0}^{\min(j-k,\ell)} \binom{j-k}{m} \frac{e^{-\frac{A}{2}|\xi|^2 t}}{f(|\xi|)} (-A|\xi|^2)^{j-k-m} (i\xi)^{\alpha+\beta+\delta}$$
$$\cdot\, \partial_t^m (|\xi|^2 g(|\xi|^2) t)^\ell \psi(\xi)\varphi_0(\xi)\Bigg)\Bigg|$$
$$\leqq C_{j,k,\alpha,\beta,\delta,\ell,\mu} |\xi|^{2(j-k)+|\alpha|+|\beta|+|\delta|-|\mu|}.$$

Therefore, by Lemma 3.5, we have (3.24).

First we consider the case when n is an odd $\geqq 3$. When $|z| = 1$, $|x| \leqq R_0 t$, and $R_0 \leqq \gamma/4$, we have

$$|x+\gamma t z| \geqq \gamma t - |x| \geqq (\gamma - R_0)t \geqq \frac{\gamma}{2} t.$$

Therefore, applying (3.24) to (3.17), we have for $|x| \leqq R_0$ and $t \geqq 1$

(3.25)

$$
\begin{aligned}
&\left|\partial_t^j \partial_x^\beta \omega_\ell(t,x)\right| \\
&\leqq \frac{1}{\gamma}\sum_{k=0}^{j}\binom{j}{k}\sum_{|\alpha|\leqq\frac{n-3}{2}}|a_\alpha|\sum_{m=0}^{\min(\ell+k,|\alpha|+1)}\binom{\ell+k}{m}\frac{(|\alpha|+1)!}{m!}(\gamma t)^{|\alpha|+1}t^{-m} \\
&\qquad\cdot\sum_{|\delta|=\ell+k-m}C_{j,k,\alpha,\beta,\delta,\ell}\int_{|z|=1}|x+\gamma tz|^{-(2(j-k)+|\alpha|+|\beta|+|\delta|+n)}\,dS \\
&\leqq \frac{1}{\gamma}\sum_{k=0}^{j}\binom{j}{k}\sum_{|\alpha|\leqq\frac{n-3}{2}}|a_\alpha|\sum_{m=0}^{\min(\ell+k,|\alpha|+1)}\binom{\ell+k}{m}\frac{(|\alpha|+1)!}{m!}(\gamma t)^{|\alpha|+1}t^{-m} \\
&\qquad\cdot\sum_{|\delta|=\ell+k-m}C_{j,k,\alpha,\beta,\delta,\ell}t^{-(2(j-k)+|\alpha|+|\beta|+|\delta|+n)}\int_{|z|=1}dS \\
&\leqq C_{j,\beta,\ell,n}t^{-(j+|\beta|+\ell+n-1)}.
\end{aligned}
$$

Next, we consider the case when n is even $\geqq 2$. By (3.17) we have

$$
\begin{aligned}
&\left|\partial_t^j \partial_x^\beta \omega_\ell(t,x)\right| \\
&\leqq \frac{1}{\gamma}\sum_{k=0}^{j}\binom{j}{k}\sum_{|\alpha|\leqq\frac{n-2}{2}}|a_\alpha|\sum_{m=0}^{\min(\ell+k,|\alpha|+1)}\binom{\ell+k}{m}\frac{(|\alpha|+1)!}{m!}\gamma^{|\alpha|+1}t^{|\alpha|+1-m} \\
&\quad\cdot\sum_{|\delta|=\ell+k-m}C_{j,k,\alpha,\beta,\delta,\ell}\int_{|z|\leqq1}\frac{\left|\left(\partial_t^{j-k}\partial_x^{\alpha+\beta+\delta}G_\ell\right)(t,x+\gamma tz)\right|}{\sqrt{1-|z|^2}}\,dz.
\end{aligned}
$$

Put

$$\int_{|z|\leqq1}\frac{\left|\left(\partial_t^{j-k}\partial_x^{\alpha+\beta+\delta}G_\ell\right)(t,x+\gamma tz)\right|}{\sqrt{1-|z|^2}}\,dz = I + II,$$

where

$$
\begin{aligned}
I &= \int_{\frac{1}{2}\leqq|z|\leqq1}\frac{\left|\left(\partial_t^{j-k}\partial_x^{\alpha+\beta+\delta}G_\ell\right)(t,x+\gamma tz)\right|}{\sqrt{1-|z|^2}}\,dz, \\
II &= \int_{|z|\leqq\frac{1}{2}}\frac{\left|\left(\partial_t^{j-k}\partial_x^{\alpha+\beta+\delta}G_\ell\right)(t,x+\gamma tz)\right|}{\sqrt{1-|z|^2}}\,dz.
\end{aligned}
$$

When $1/2 \leqq |z| \leqq 1$, $|x| \leqq R_0 t$, $t \geqq 1$ and $R_0 \leqq \gamma/4$, we have

$$|x+\gamma tz| \geqq \frac{\gamma t}{2} - |x| \geqq \left(\frac{\gamma}{2} - R_0\right) t \geqq \frac{\gamma}{4} t.$$

Then, by (3.24) we have

$$\mathrm{I} \leqq C_{j,k,\alpha,\beta,\delta,\ell} t^{-(2(j-k)+|\alpha|+|\beta|+|\delta|+n)} \int_{\frac{1}{2}\leqq|z|\leqq 1} \frac{dz}{\sqrt{1-|z|^2}}.$$

When $|z| \leqq 1/2$, $|x| \leqq R_0 t$, $t \geqq 1$ and $R_0 \leqq \gamma/4$, we have

$$|x+\gamma tz| \leqq \frac{\gamma}{2} t + R_0 t \leqq \frac{3}{4}\gamma t, \quad \sqrt{1-|z|^2} \geqq \frac{\sqrt{3}}{2}.$$

Therefore, putting $p = x + \gamma tz$, by Lemma 3.5 we have

$$\begin{aligned} \mathrm{II} &\leqq \frac{2}{\sqrt{3}} \int_{|p|\leqq\frac{3}{4}\gamma t} \left| \left(\partial_t^{j-k} \partial_x^{\alpha+\beta+\delta} G_\ell\right)(t,p)\right| dp \\ &\leqq C_{j,k,\alpha,\beta,\delta,\ell,n} \\ &\quad \cdot \begin{cases} t^{-n} t^{-\frac{1}{2}(2(j-k)+|\alpha|+|\beta|+|\delta|)} & \text{when } 2(j-k)+|\alpha|+|\beta|+|\delta| \geqq 1, \\ t^{-n}(1+\log t) & \text{when } 2(j-k)+|\alpha|+|\beta|+|\delta| = 0 \\ & \text{and } \ell = 0, \\ t^{-n} & \text{when } 2(j-k)+|\alpha|+|\beta|+|\delta| = 0 \\ & \text{and } \ell \geqq 1. \end{cases} \end{aligned}$$

Combining these estimations, we have for $|x| \leqq R_0 t$ and $t \geqq 1$

(3.26)

$$\begin{aligned} &\left|\partial_t^j \partial_x^\beta w_\ell(t,x)\right| \\ &\leqq \frac{1}{\gamma} \sum_{k=0}^{j} \binom{j}{k} \sum_{1\leqq|\alpha|\leqq\frac{n-2}{2}} |a_\alpha| \sum_{m=0}^{\min(\ell+k,|\alpha|+1)} \binom{\ell+k}{m} \frac{(|\alpha|+1)!}{m!} (\gamma t)^{|\alpha|+1} t^{-m} \\ &\quad \cdot \sum_{|\delta|=\ell+k-m} C_{j,k,\alpha,\beta,\delta,\ell,n} \left\{ t^{-(2(j-k)+|\alpha|+|\beta|+|\delta|+n)} + t^{-n-\frac{1}{2}(2(j-k)+|\beta|+|\delta|)} \right\} \\ &\quad + \sum_{k=0}^{j-1} \binom{j}{k} \sum_{m=0}^{\min(\ell+k,1)} \binom{\ell+k}{m} \frac{|a_\alpha|}{m!} t^{1-m} \\ &\quad \cdot \sum_{|\delta|=\ell+k-m} C_{j,k,\beta,\delta,\ell,n} \left\{ t^{-(2(j-k)+|\beta|+|\delta|+n)} + t^{-n-\frac{1}{2}(2(j-k)+|\beta|+|\delta|)} \right\} \end{aligned}$$

$$+\sum_{m=0}^{\min(j+\ell,1)}\binom{j+\ell}{m}\frac{|a_\alpha|}{m!}t^{1-m}$$
$$\cdot C_{j,\beta,\ell,n}\left\{t^{-(\ell+j-m+|\beta|+n)}+t^{-n-\frac{1}{2}(\ell+j-m+|\beta|)}(1+\log(1+t))\right\}$$
$$\leqq C_{j,\beta,\ell,n}\left\{(1+\log(1+t))t^{-\left(\frac{3n-3}{4}+\frac{j+|\beta|}{2}\right)-\frac{n-1}{4}-\frac{\ell}{2}}+t^{-\left(\frac{3n-3}{4}+\frac{j+|\beta|}{2}\right)-\frac{1}{4}}\right\}.$$

Next, we shall estimate the remainder term. By (3.15) and (3.16), we have

$$\partial_t^j\partial_x^\beta\mathcal{F}^{-1}\left[e^{-A|\xi|^2t}R_N(t,\xi)\right](x)$$
$$=\sum_{k=o}^{j}\binom{j}{k}\mathcal{F}^{-1}\left[e^{-A|\xi|^2t}(-A|\xi|^2)^k(i|\xi|)^\beta\partial_t^{j-k}R_N(t,\xi)\right](x),$$

and

$$\left|\partial_t^{j-k}R_N(t,\xi)\right|$$
$$=\left|\frac{1}{2i\gamma|\xi|f(|\xi|)N!}\int_0^1(1-\theta)^N\sum_{\ell_1=0}^{j-k}\binom{j-k}{\ell_1}\partial_t^{j-k-\ell_1}e^{\pm i\gamma|\xi|^3g(|\xi|^2)t\theta}\right.$$
$$\left.\cdot\sum_{\ell_2=0}^{\ell_1}\binom{\ell_1}{\ell_2}\partial_t^{\ell_1-\ell_2}e^{\pm i\gamma|\xi|t}\partial_t^{\ell_2}(|\xi|^3g(|\xi|^2)t)^{N+1}\,d\theta\right|$$
$$\leqq C_{j,\beta,N}\sum_{\ell_1=0}^{j-k}\sum_{\ell_2=0}^{\ell_1}|\xi|^{3(N+j-k)+2(1-\ell_1)-\ell_2}t^{N+1-\ell_2}.$$

Combining these estimations, we have

(3.27)

$$\left|\partial_t^j\partial_x^\beta\mathcal{F}^{-1}\left[e^{-A|\xi|^2t}R_N(t,\xi)\right](x)\right|$$
$$\leqq C_{j,\beta,N}\sum_{k=0}^{j}\sum_{\ell_1=0}^{j-k}\sum_{\ell_2=0}^{\ell_1}t^{N+1-\ell_2}\int_{\mathbb{R}^n}|\xi|^{3(N+j)+2(1-\ell_1)-\ell_2-k+|\beta|}e^{-A|\xi|^2t}\,d\xi$$
$$\leqq C_{j,\beta,N}\sum_{k=0}^{j}\sum_{\ell_1=0}^{j-k}\sum_{\ell_2=0}^{\ell_1}t^{-\frac{N+j+|\beta|+(j-k-\ell_1)+(j-\ell_1)}{2}}$$
$$\leqq C_{j,\beta,N}t^{-\frac{N+j+|\beta|}{2}}.$$

By (3.25), (3.26), (3.27) and (3.13), we have

$$\left|\partial_t^j \partial_x^\beta K_{\psi,0}(t,x)\right| \leqq C_{j,\beta,n} t^{-\left(\frac{3n-3}{4}+\frac{j+|\beta|}{2}\right)}, \quad t \geqq \max(1,(R/R_0)^4). \tag{3.28}$$

To complete the Proof of Theorem 3.1, we have to estimate the case when $0 \leqq t \leqq \max(1,(R/R_0)^4)$. But, it is obvious that

$$\begin{aligned}
&\left|\partial_t^j \partial_x^\beta K_{\psi,0}(t,x)\right| \\
&\leqq \left(\frac{1}{2\pi}\right)^n \left|\int_{\mathbb{R}^n} (i\xi)^\beta \frac{\lambda_+(\xi)^j e^{\lambda_+(\xi)t} - \lambda_-(\xi)^j e^{\lambda_-(\xi)t}}{\lambda_+(\xi)-\lambda_-(\xi)} \psi(\xi)\varphi_0(\xi)\right| \\
&\leqq C_{j,\beta} \int_0^{\frac{B}{\sqrt{2}}} r^{j+|\beta|+n-2} dr \\
&\leqq C_{j,\beta,n}.
\end{aligned}$$

Therefore, the Proof of Theorem 3.1 is completed.

4. Proof of Theorem 2.1 (2).

In this section, we shall show Theorem 2.1 (2). In view of (3.1), (3.2) and Young inequality, it suffices to show that

$$\|\partial_t^j \partial_x^\alpha L_{ij}(t,\cdot)\|_{L_1(\mathbb{R}^n)} \leqq C_{j,\alpha,n}(1+t)^{q(n)-\frac{j+|\alpha|}{2}}, \tag{4.1}$$

where $q(n) = (n-1)/4$ for odd $n \geqq 3$ and $= n/4$ for even $n \geqq 2$. Since the kernel of $L_{11}(t,x)$, $L_{12}(t,x) = {}^tL_{21}(t,x)$ are the same as those of (2.12), (4.1) directly follows from the results of [**6**, Theorem 2.1] when $(i,j) = (1,1),(1,2)$ and (2.1). Therefore, our task is to show (4.1) when $(i,j) = (2,2)$. In view of (3.1), we put

$$\begin{aligned}
L_0(t,x) &= K_2(t,x) - K_3(t,x) \\
&= \mathcal{F}^{-1}\left[\left(\frac{\lambda_+(\xi)e^{\lambda_+(\xi)t} - \lambda_-(\xi)e^{\lambda_-(\xi)t}}{\lambda_+(\xi)-\lambda_-(\xi)} - e^{-\alpha|\xi|^2 t}\right)\frac{\xi_j\xi_k}{|\xi|^2}\right](x).
\end{aligned}$$

Then, we have

$$L_{22}(t,x) = K_1(t,x) + L_0(t,x), \quad K_1(t,x) = \mathcal{F}^{-1}\left[e^{-\alpha|\xi|^2 t}\varphi_0(\xi)\right](x)I.$$

Noting that $\varphi_0(\xi) = 0$ when $|\xi| \geqq B/\sqrt{2}$, we have

$$\|\partial_t^j \partial_x^\beta K_1(t,\cdot)\|_{L_1(\mathbb{R}^n)} \leqq C_{j,\beta,n}(1+t)^{-j-\frac{|\beta|}{2}}. \tag{4.2}$$

In fact, putting

$$\chi(x) = \mathcal{F}^{-1}\left[\varphi_0(\xi)\right](x) \in \mathcal{S}(\mathbb{R}^n),$$

we have

$$K_1(t,x) = \frac{1}{(4\pi\alpha t)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{2\alpha t}} \chi(y)\, dy.$$

By Young inequality, we see that

$$\|K_1(t,x)\|_{L_1(\mathbb{R}^n)} \leqq C_n, \quad t>0.$$

When $j+|\beta| \geqq 1$, we have

$$\partial_t^j \partial_x^\beta \mathcal{F}^{-1}\left[e^{-\alpha|\xi|^2 t}\varphi_0(\xi)\right] = \mathcal{F}^{-1}\left[(i\xi)^\beta(-\alpha|\xi|^2)^j e^{-\alpha|\xi|^2 t}\varphi_0(\xi)\right],$$

and

$$\left|\partial_\xi^\mu\left((i\xi)^\beta(-\alpha|\xi|^2)^j\varphi_0(\xi)\right)\right| \leqq C_{j,\beta,n}|\xi|^{2j+|\beta|-|\mu|}, \quad \xi \neq 0.$$

Therefore, by Lemma 3.5 we have

$$\|\partial_t^j \partial_x^\beta K_1(t,x)\|_{L_1(\mathbb{R}^n)} \leqq C_{j,\beta,n} t^{-j-\frac{|\beta|}{2}}, \quad t>0.$$

When $0<t\leqq 1$, since

$$\begin{aligned}\partial_t^j \partial_x^\beta \mathcal{F}^{-1}\left[e^{-\alpha|\xi|^2 t}\varphi_0(\xi)\right](x) &= (\alpha\Delta)^j\partial_x^\beta\left\{\frac{1}{(4\pi\alpha t)^{\frac{n}{2}}}\int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{2\alpha t}}\chi(y)\,dy\right\} \\ &= \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n} e^{-|z|^2}((\alpha\Delta)^j\partial_x^\beta\chi)(x-\sqrt{2\alpha t}z)\,dz,\end{aligned}$$

we have

$$\|\partial_t^j \partial_x^\beta K_1(t,\cdot)\|_{L_1(\mathbb{R}^n)} \leqq C_{j,\beta,n}, \quad 0<t\leqq 1.$$

Combining these estimations, we have (4.2).

Now, we shall show that

$$\|\partial_t^j \partial_x^\beta L_0(t,\cdot)\|_{L_1(\mathbb{R}^n)} \leqq C_{j,\beta,n} t^{q(n)-\frac{j+|\alpha|}{2}}, \quad t \geqq 1. \tag{4.3}$$

By (3.15), we have

$$\begin{aligned}&\frac{\lambda_+(\xi)e^{\lambda_+(\xi)t}-\lambda_-(\xi)e^{\lambda_-(\xi)t}}{\lambda_+(\xi)-\lambda_-(\xi)} \\ &= \partial_t\left(\frac{e^{\lambda_+(\xi)t}-e^{\lambda_-(\xi)t}}{\lambda_+(\xi)-\lambda_-(\xi)}\right) \\ &= \partial_t\left\{\sum_{\ell=0}^N \frac{1}{\ell!}\partial_t^\ell\left(\frac{\sin\gamma|\xi|t}{|\xi|}\right)\frac{e^{-A|\xi|^2 t}}{\gamma f(|\xi|)}(|\xi|^2 g(|\xi|^2)t)^\ell + e^{-A|\xi|^2 t}R_N(t,|\xi|)\right\} \\ &= \sum_{\ell=0}^N \frac{1}{\ell!}\partial_t^{\ell+1}\left(\frac{\sin\gamma|\xi|t}{|\xi|}\right)\frac{e^{-A|\xi|^2 t}}{\gamma f(|\xi|)}(|\xi|^2 g(|\xi|^2)t)^\ell \\ &\quad + \sum_{\ell=0}^N \frac{1}{\ell!}\partial_t^\ell\left(\frac{\sin\gamma|\xi|t}{|\xi|}\right)\partial_t\left(\frac{e^{-A|\xi|^2 t}}{\gamma f(|\xi|)}(|\xi|^2 g(|\xi|^2)t)^\ell\right) \\ &\quad + \partial_t\left(e^{-A|\xi|^2 t}R_N(t,|\xi|)\right).\end{aligned}$$

Since $f(|\xi|) = 1 + |\xi|^2 g(|\xi|^2)$, we have

$$\begin{aligned}&\partial_t\left(\frac{\sin\gamma|\xi|t}{|\xi|}\right)\frac{e^{-A|\xi|^2t}}{\gamma f(|\xi|)}\\&= e^{-A|\xi|^2t}\left\{\partial_t\left(\frac{\sin\gamma|\xi|t}{\gamma|\xi|}\right) - \partial_t\left(\frac{\sin\gamma|\xi|t}{|\xi|}\right)\frac{|\xi|^2 g(|\xi|^2)}{\gamma f(|\xi|)}\right\}.\end{aligned}$$

Combining these two estimations, we have

$$\begin{aligned}L_0(t,x) = L_1(t,x) + L_2(t,x) - M_0^1(t,x) + \sum_{\ell=1}^{N}\frac{1}{\ell!}M_\ell^1(t,x)\\ + \sum_{\ell=0}^{N}\frac{1}{\ell!}M_\ell^2(t,x)\sum_{\ell=1}^{N}\frac{1}{(\ell-1)!}M_\ell^3(t,x) + \mathcal{R}_N(t,x),\end{aligned}$$

where

$$\begin{aligned}
L_1(t,x) &= \mathcal{F}^{-1}\left[\left(\partial_t\left(\frac{\sin\gamma|\xi|t}{|\xi|}\right) - 1\right)e^{-A|\xi|^2t}\frac{\xi_j\xi_k}{|\xi|^2}\varphi_0(\xi)\right](x),\\
L_2(t,x) &= \mathcal{F}^{-1}\left[\left(e^{-A|\xi|^2t} - e^{-\alpha|\xi|^2}t\right)\frac{\xi_j\xi_k}{|\xi|^2}\varphi_0(\xi)\right](x),\\
M_0^1(t,x) &= \mathcal{F}^{-1}\left[\partial_t\left(\frac{\sin\gamma|\xi|t}{|\xi|}\right)\frac{e^{-A|\xi|^2t}g(|\xi|^2)}{\gamma f(|\xi|^2)}\xi_j\xi_k\varphi_0(\xi)\right](x),\\
M_\ell^1(t,x) &= \mathcal{F}^{-1}\left[\partial_t^{\ell+1}\left(\frac{\sin\gamma|\xi|t}{|\xi|}\right)\frac{e^{-A|\xi|^2t}}{\gamma f(|\xi|^2)}\left(|\xi|^2 g(|\xi|^2)t\right)^\ell\frac{\xi_j\xi_k}{|\xi|^2}\varphi_0(\xi)\right](x),\\
M_\ell^2(t,x) &= \mathcal{F}^{-1}\left[\partial_t^{\ell}\left(\frac{\sin\gamma|\xi|t}{|\xi|}\right)\frac{-Ae^{-A|\xi|^2t}}{\gamma f(|\xi|^2)}\left(|\xi|^2 g(|\xi|^2)t\right)^\ell\xi_j\xi_k\varphi_0(\xi)\right](x),\\
M_\ell^3(t,x) &= \mathcal{F}^{-1}\Bigg[\partial_t^{\ell}\left(\frac{\sin\gamma|\xi|t}{|\xi|}\right)\frac{e^{-A|\xi|^2t}g(|\xi|^2)}{\gamma f(|\xi|^2)}\\
&\qquad\cdot\left(|\xi|^2 g(|\xi|^2)t\right)^{\ell-1}\xi_j\xi_k\varphi_0(\xi)\Bigg](x),
\end{aligned}$$

and

$$\mathcal{R}_N(t,x) = \mathcal{F}^{-1}\left[\partial_t\left(e^{-A|\xi|^2t}R_N(t,|\xi|)\frac{\xi_j\xi_k}{|\xi|^2}\varphi_0(\xi)\right)\right](x).$$

First, we shall show that

$$\|L_1(t.\cdot)\|_{L_1(\mathbb{R}^n)} \leqq C_n t^{q(n)}, \quad t \geqq 1. \tag{4.4}$$

Put

$$g_0(t.x) = \mathcal{F}^{-1}\left[e^{-A|\xi|^2t}\frac{\xi_j\xi_k}{|\xi|^2}\varphi_0(\xi)\right](x).$$

Since

$$
\begin{aligned}
&L_1(t,x)+g_0(t,x)\\
&=\partial_t\mathcal{F}^{-1}\left[\left(\frac{\sin\gamma|\xi|t}{\gamma|\xi|}\right)\widehat{g_0}(t,\xi)\right](x)-\mathcal{F}^{-1}\left[\frac{\sin\gamma|\xi|t}{\gamma|\xi|}\widehat{\partial_t g_0}(t,\xi)\right](x),
\end{aligned}
$$

by Lemma 3.3 we have

$$
\begin{aligned}
(4.5)\qquad &L_1(t,x)+g_0(t,x)\\
&=\partial_t\left\{\frac{1}{\gamma}\sum_{|\alpha|\leqq\frac{n-3}{2}}a_\alpha(\gamma t)^{|\alpha|+1}\int_{|z|=1}z^\alpha(\partial_x^\alpha g_0)(t,x+\gamma tz)\,dS\right\}\\
&\quad-\mathcal{F}^{-1}\left[\frac{\sin\gamma|\xi|t}{|\xi|^2}\widehat{\partial_t g_0}(t,\xi)\right](x)\\
&=\sum_{|\alpha|\leqq\frac{n-3}{2}}a_\alpha(|\alpha|+1)(\gamma t)^{|\alpha|+1}\int_{|z|=1}z^\alpha(\partial_x^\alpha g_0)(t,x+\gamma tz)\,dS\\
&\quad+\sum_{|\alpha|\leqq\frac{n-3}{2}}a_\alpha(\gamma t)^{|\alpha|+1}\sum_{|\delta|=1}\int_{|z|=1}z^{\alpha+\delta}(\partial_x^{\alpha+\delta}g_0)(t,x+\gamma tz)\,dS
\end{aligned}
$$

when n is an odd $\geqq 3$. In view of Lemma 3.3, we see that

$$
(4.6)\qquad a_0\int_{|z|=1}dS=1.
$$

In fact, putting

$$
w(t,x)=\mathcal{F}^{-1}\left[\frac{\sin|\xi|t}{|\xi|}\hat{h}(\xi)\right](x),
$$

by Lemma 3.3, we have

$$
\begin{aligned}
h(x)&=w_t(0,x)\\
&=\sum_{|\alpha|\leqq\frac{n-3}{2}}a_\alpha(1+|\alpha|)t^{|\alpha|}\int_{|z|=1}z^\alpha(\partial_x^\alpha h)(x+tz)\,dS\bigg|_{t=0}\\
&\quad+\sum_{|\alpha|\leqq\frac{n-3}{2}}a_\alpha t^{|\alpha|+1}\sum_{|\delta|=1}\int_{|z|=1}z^{\alpha+\delta}(\partial_x^{\alpha+\delta}h)(x+tz)\,dS\bigg|_{t=0}\\
&=\left(a_0\int_{|z|=1}dS\right)h(x),
\end{aligned}
$$

which implies (4.6). Combining (4.5) and (4.6), we have

$$
\begin{aligned}
(4.7)\qquad & L_1(t,x) \\
&= a_0 \int_{|z|=1} \{g_0(t, x+\gamma t z) - g_0(t,x)\}\, dS \\
&\quad + \sum_{1 \leqq |\alpha| \leqq \frac{n-3}{2}} a_\alpha (1+|\alpha|)(\gamma t)^{|\alpha|} \int_{|z|=1} z^\alpha (\partial_x^\alpha g_0)(t, x+\gamma t z)\, dS \\
&\quad + \sum_{|\alpha| \leqq \frac{n-3}{2}} a_\alpha (\gamma t)^{|\alpha|+1} \sum_{|\delta|=1} \int_{|z|=1} z^{\alpha+\delta} (\partial_x^{\alpha+\delta} g_0)(t, x+\gamma t z)\, dS.
\end{aligned}
$$

Similarly, we see that

$$
\begin{aligned}
(4.8)\qquad & L_1(t,x) \\
&= a_0 \int_{|z|\leqq 1} \frac{g_0(t, x+\gamma t z) - g_0(t,x)}{\sqrt{1-|z|^2}}\, dz \\
&\quad + \sum_{1 \leqq |\alpha| \leqq \frac{n-2}{2}} a_\alpha (1+|\alpha|)(\gamma t)^{|\alpha|} \int_{|z|\leqq 1} \frac{z^\alpha (\partial_x^\alpha g_0)(t, x+\gamma t z)}{\sqrt{1-|z|^2}}\, dz \\
&\quad + \sum_{|\alpha| \leqq \frac{n-2}{2}} a_\alpha (\gamma t)^{|\alpha|+1} \sum_{|\delta|=1} \int_{|z|\leqq 1} \frac{z^{\alpha+\delta} (\partial_x^{\alpha+\delta} g_0)(t, x+\gamma t z)}{\sqrt{1-|z|^2}}\, dz
\end{aligned}
$$

when n is an even $\geqq 2$. Concerning the estimate $g_0(t,x)$, we have

$$
(4.9)\qquad \|\partial_t^j \partial_x^\alpha g_0(t,\cdot)\|_{L_1(\mathbb{R}^n)} \leqq C_{j,\alpha,n} t^{-\left(j+\frac{|\alpha|}{2}\right)}, \quad j+|\alpha| \geqq 1.
$$

In fact, we have

$$
\partial_t^\ell \partial_x^\alpha g_0(t,x) = \mathcal{F}^{-1}\left[e^{-A|\xi|^2 t} (-A|\xi|^2)^\ell (i\xi)^\alpha \frac{\xi_j \xi_k}{|\xi|^2} \varphi_0(\xi) \right](x)
$$

and

$$
\left| \partial_\xi^\mu \left((-A|\xi|^2)^\ell (i\xi)^\alpha \frac{\xi_j \xi_k}{|\xi|^2} \varphi_0(\xi) \right) \right| \leqq C_{j,k,\alpha,\mu} |\xi|^{2\ell+|\alpha|-|\mu|}, \quad \xi \neq 0.
$$

Therefore, (4.9) follows from Lemma 3.5. Since

$$
\begin{aligned}
g_0(t, x+\gamma t z) - g_0(t,x) &= \int_0^1 \frac{d}{d\theta} \{g_0(t, x+\gamma t z\theta)\}\, d\theta \\
&= \int_0^1 (\nabla_x g_0)(t, x+\gamma t z\theta)\, d\theta \cdot \gamma t z
\end{aligned}
$$

by (4.9) we have

$$\left\| a_0 \int_{|z|=1} \{g_0(t, \cdot + \gamma t z) - g_0(t, \cdot)\}\, dS \right\|_{L_1(\mathbb{R}^n)} \tag{4.10}$$
$$\leqq a_0 \gamma t \int_{\mathbb{R}^n} \int_{|z|=1} \int_0^1 |z|\, |(\nabla_x g_0)(t, x + \gamma t z \theta)|\, d\theta dS dx$$
$$\leqq C_n t^{\frac{1}{2}}$$

when n is an odd $\geqq 3$; and

$$\left\| a_0 \int_{|z|\leqq 1} \frac{g_0(t, \cdot + \gamma t z) - g_0(t, \cdot)}{\sqrt{1 - |z|^2}}\, dz \right\|_{L_1(\mathbb{R}^n)} \tag{4.11}$$
$$\leqq a_0 \gamma t \int_{\mathbb{R}^n} \int_{|z|\leqq 1} \int_0^1 \frac{|z|\, |(\nabla_x g_0)(t, x + \gamma t z \theta)|}{\sqrt{1 - |z|^2}}\, d\theta dz dx$$
$$\leqq C_n t^{\frac{1}{2}}$$

when n is an even $\geqq 2$. Therefore, by (4.7), (4.9) and (4.10) we have

$$\|L_1(t, \cdot)\|_{L_1(\mathbb{R}^n)} \leqq C_n \left\{ t^{\frac{1}{2}} + \sum_{1 \leqq |\alpha| \leqq \frac{n-3}{2}} t^{\frac{|\alpha|}{2}} + \sum_{|\alpha| \leqq \frac{n-3}{2}} t^{\frac{|\alpha|+1}{2}} \right\}$$
$$\leqq C_n t^{\frac{n-1}{4}}$$

when n is an odd $\geqq 3$; and by (4.7), (4.8) and (4.11) we have

$$\|L_1(t, \cdot)\|_{L_1(\mathbb{R}^n)} \leqq C_n \left\{ t^{\frac{1}{2}} + \sum_{1 \leqq |\alpha| \leqq \frac{n-2}{2}} t^{\frac{|\alpha|}{2}} + \sum_{|\alpha| \leqq \frac{n-2}{2}} t^{\frac{|\alpha|+1}{2}} \right\}$$
$$\leqq C_n t^{\frac{n}{4}}$$

when n is an even $\geqq 2$, which implies (4.4).

Next, we shall show that

$$\|\partial_t^j \partial_x^\beta L_1(t, \cdot)\|_{L_1(\mathbb{R}^n)} \leqq C_{j,\beta,n} t^{q(n) - \frac{j+|\beta|}{4}}, \quad t \geqq 1,\ j + |\beta| \geqq 1. \tag{4.12}$$

By (4.7) we have

$$\begin{aligned}
(4.13)\qquad &\partial_t^j \partial_x^\beta L_1(t,x) \\
&= a_0 \int_{|z|=1} \partial_t^j \partial_x^\beta \{g_0(t,x+\gamma tz) - g_0(t,x)\}\, dS \\
&\quad + \sum_{k=0}^{j} \binom{j}{k} \sum_{1 \leqq |\alpha| \leqq \frac{n-3}{2}} a_\alpha (1+|\alpha|) \sum_{m=0}^{\min(k+1,|\alpha|)} \frac{|\alpha|!}{m!} (\gamma t)^{|\alpha|} t^{-m} \\
&\quad \cdot \sum_{|\delta|=k-m} \int_{|z|=1} z^{\alpha+\delta} (\partial_x^{\alpha+\beta+\delta} \partial_t^{j-k} g_0)(t,x+\gamma tz)\, dS \\
&\quad + \sum_{k=0}^{j} \binom{j}{k} \sum_{0 \leqq |\alpha| \leqq \frac{n-3}{2}} a_\alpha \sum_{m=0}^{\min(k,|\alpha|+1)} \frac{(|\alpha|+1)!}{m!} (\gamma t)^{|\alpha|} t^{1-m} \\
&\quad \cdot \sum_{|\delta|=k-m+1} \int_{|z|=1} z^{\alpha+\delta} (\partial_x^{\alpha+\beta+\delta} \partial_t^{j-k} g_0)(t,x+\gamma tz)\, dS
\end{aligned}$$

when n is an odd $\geqq 3$; and by (4.8) we have

$$\begin{aligned}
(4.14)\qquad &\partial_t^j \partial_x^\beta L_1(t,x) \\
&= a_0 \int_{|z|\leqq 1} \frac{\partial_t^j \partial_x^\beta \{g_0(t,x+\gamma tz) - g_0(t,x)\}}{\sqrt{1-|z|^2}}\, dz \\
&\quad + \sum_{k=0}^{j} \binom{j}{k} \sum_{1 \leqq |\alpha| \leqq \frac{n-2}{2}} a_\alpha (1+|\alpha|) \sum_{m=0}^{\min(k+1,|\alpha|)} \frac{|\alpha|!}{m!} (\gamma t)^{|\alpha|} t^{-m} \\
&\quad \cdot \sum_{|\delta|=k-m} \int_{|z|\leqq 1} \frac{z^{\alpha+\delta} (\partial_x^{\alpha+\beta+\delta} \partial_t^{j-k} g_0)(t,x+\gamma tz)}{\sqrt{1-|z|^2}}\, dz \\
&\quad + \sum_{k=0}^{j} \binom{j}{k} \sum_{0 \leqq |\alpha| \leqq \frac{n-2}{2}} a_\alpha \sum_{m=0}^{\min(k,|\alpha|+1)} \frac{(|\alpha|+1)!}{m!} (\gamma t)^{|\alpha|} t^{1-m} \\
&\quad \cdot \sum_{|\delta|=k-m+1} \int_{|z|\leqq 1} \frac{z^{\alpha+\delta} (\partial_x^{\alpha+\beta+\delta} \partial_t^{j-k} g_0)(t,x+\gamma tz)}{\sqrt{1-|z|^2}}\, dz
\end{aligned}$$

when n is an even $n \geqq 2$. By (4.9) we have

$$\begin{aligned}(4.15)\quad &\left\| a_0 \int_{|z|=1} \partial_t^j \partial_x^\beta \{g_0(t, \cdot + \gamma t z) - g_0(t,\cdot)\}\, dS \right\|_{L_1(\mathbb{R}^n)} \\ &\leqq a_0 \int_{|z|=1} \left\{ \|\partial_t^j \partial_x^\beta g_0(t, \cdot + \gamma t z)\|_{L_1(\mathbb{R}^n)} + \|\partial_t^j \partial_x^\beta g_0(t,\cdot)\|_{L_1(\mathbb{R}^n)} \right\} dS \\ &\leqq C_{j,\beta,n} t^{-\left(j + \frac{|\beta|}{2}\right)};\end{aligned}$$

and

$$\begin{aligned}(4.16)\quad &\left\| a_0 \int_{|z|\leqq 1} \frac{\partial_t^j \partial_x^\beta \{g_0(t, \cdot + \gamma t z) - g_0(t,\cdot)\}}{\sqrt{1-|z|^2}}\, dz \right\|_{L_1(\mathbb{R}^n)} \\ &\leqq a_0 \int_{|z|=1} \frac{\|\partial_t^j \partial_x^\beta g_0(t, \cdot + \gamma t z)\|_{L_1(\mathbb{R}^n)} + \|\partial_t^j \partial_x^\beta g_0(t,\cdot)\|_{L_1(\mathbb{R}^n)}}{\sqrt{1-|z|^2}}\, dz \\ &\leqq C_{j,\beta,n} t^{-\left(j + \frac{|\beta|}{2}\right)}.\end{aligned}$$

Putting

$$p(n) = \begin{cases} \frac{n-3}{2}, & \text{when n is an odd } \geqq 3, \\ \frac{n-2}{2}, & \text{when n is an even } \geqq 2, \end{cases}$$

by (4.9), (4.13), (4.14), (4.15) and (4.16), we have

$$\begin{aligned}&\|\partial_t^j \partial_x^\beta L_1(t,\cdot)\|_{L_1(\mathbb{R}^n)} \\ &\leqq C_{j,\beta,n} \left\{ t^{-\left(j+\frac{|\beta|}{2}\right)} + \sum_{k=0}^{j} \sum_{1\leqq|\alpha|\leqq p(n)} \sum_{m=0}^{\min(k+1,|\alpha|)} t^{-\left(j + \frac{|\beta|-|\alpha|+m-k}{2}\right)} \right. \\ &\quad \left. + \sum_{k=0}^{j} \sum_{0\leqq|\alpha|\leqq p(n)} \sum_{m=0}^{\min(k,|\alpha|+1)} t^{-\left(j + \frac{|\beta|-|\alpha|+m-k-1}{2}\right)} \right\} \\ &\leqq C_{j,\beta,n} t^{q(n) - \frac{j+|\beta|}{2}},\end{aligned}$$

which implies (4.12).

Now we shall estimate

$$\partial_t^\ell \partial_x^\beta L_2(t,x) = (\alpha - A)^j \mathcal{F}^{-1}\left[\left(e^{-A|\xi|^2 t} - e^{-\alpha|\xi|^2 t}\right) |\xi|^{2\ell} (i\xi)^\beta \frac{\xi_j \xi_k}{|\xi|^2} \varphi_0(\xi)\right](x).$$

Since

$$e^{-A|\xi|^2 t} - e^{-\alpha|\xi|^2 t} = (\alpha - A)|\xi|^2 t \int_0^1 e^{-\theta A|\xi|^2 t - (1-\theta)\alpha|\xi|^2 t}\, d\theta,$$

we have

$$\left|\partial_\xi^\eta\left\{\left(e^{-A|\xi|^2t}-e^{-\alpha|\xi|^2t}\right)|\xi|^{2\ell}(i\xi)^\beta\frac{\xi_j\xi_k}{|\xi|^2}\varphi_0(\xi)\right\}\right|$$
$$\leqq C_{\ell,\beta,\eta}|\xi|^{2\ell+|\beta|+2-|\eta|}t,\quad \xi\neq 0.$$

Therefore, by Lemma 3.5 we have

$$\|\partial_t^\ell\partial_\xi^\beta L_2(t,\cdot)\|_{L_1(\mathbb{R}^n)}\leqq C_{\ell,\beta,n}t^{-\left(j+\frac{|\beta|}{2}\right)},\ t>0. \tag{4.17}$$

Next, we shall show that for $t\geqq 1$

$$\begin{cases}\|\partial_t^j\partial_x^\beta M_0^1(t,\cdot)\|_{L_1(\mathbb{R}^n)}\leqq C_{j,\beta,n}t^{q(n)-\frac{j+|\beta|+1}{2}},\\ \|\partial_t^j\partial_x^\beta M_\ell^1(t,\cdot)\|_{L_1(\mathbb{R}^n)}\leqq C_{j,\beta,\ell,n}t^{q(n)-\frac{j+|\beta|+\ell-1}{2}},\quad \ell\geqq 1,\\ \|\partial_t^j\partial_x^\beta M_\ell^2(t,\cdot)\|_{L_1(\mathbb{R}^n)}\leqq C_{j,\beta,\ell,n}t^{q(n)-\frac{j+|\beta|+\ell}{2}},\quad \ell\geqq 0,\\ \|\partial_t^j\partial_x^\beta M_\ell^3(t,\cdot)\|_{L_1(\mathbb{R}^n)}\leqq C_{j,\beta,\ell,n}t^{q(n)-\frac{j+|\beta|+\ell}{2}},\quad \ell\geqq 1.\end{cases} \tag{4.18}$$

To do this, we put

$$\psi_0^1(t,x)=e^{-\frac{A}{2}|\xi|^2t}\frac{g(|\xi|^2)}{\gamma f(|\xi|)}\xi_j\xi_k\varphi_0(\xi),$$
$$\psi_\ell^1(t,x)=e^{-\frac{A}{2}|\xi|^2t}\frac{1}{\gamma f(|\xi|)}(|\xi|^2g(|\xi|^2)t)^\ell\xi_j\xi_k\varphi_0(\xi),$$
$$\psi_\ell^2(t,x)=e^{-\frac{A}{2}|\xi|^2t}\frac{-A}{\gamma f(|\xi|)}(|\xi|^2g(|\xi|^2)t)^\ell\xi_j\xi_k\varphi_0(\xi),$$
$$\psi_\ell^3(t,x)=e^{-\frac{A}{2}|\xi|^2t}\frac{g(|\xi|^2)}{\gamma f(|\xi|)}(|\xi|^2g(|\xi|^2)t)^{\ell-1}\xi_j\xi_k\varphi_0(\xi).$$

Then, we have

$$\begin{cases}M_0^1(t,x)=\mathcal{F}^{-1}\left[\partial_t\left(\frac{\sin\gamma|\xi|t}{|\xi|}\right)e^{-\frac{A}{2}|\xi|^2t}\psi_0^1(t,\xi)\right](x),\\ M_\ell^1(t,x)=-\mathcal{F}^{-1}\left[\partial_t^{\ell-1}\left(\frac{\sin\gamma|\xi|t}{|\xi|}\right)e^{-\frac{A}{2}|\xi|^2t}\psi_\ell^1(t,\xi)\right](x),\\ M_\ell^2(t,x)=\mathcal{F}^{-1}\left[\partial_t^\ell\left(\frac{\sin\gamma|\xi|t}{|\xi|}\right)e^{-\frac{A}{2}|\xi|^2t}\psi_\ell^2(t,\xi)\right](x),\\ M_\ell^3(t,x)=\mathcal{F}^{-1}\left[\partial_t^\ell\left(\frac{\sin\gamma|\xi|t}{|\xi|}\right)e^{-\frac{A}{2}|\xi|^2t}\psi_\ell^3(t,\xi)\right](x).\end{cases} \tag{4.19}$$

Concerning the estimate $\psi_\ell^k(t,\xi)$, we have

$$\left|\partial_\xi^\mu\psi_\ell^k(t,\xi)\right|\leqq C_{\mu,\ell}|\xi|^{2-|\mu|},\quad {}^\forall\mu.$$

Therefore, if we put

$$g_\ell^k(t,x)=\mathcal{F}^{-1}\left[\psi_\ell^k(t,\xi)\right](x),$$

then, by Lemma 3.5 we have

$$\|\partial_t^j\partial_x^\beta g_\ell^k(t,\cdot)\|_{L_1(\mathbb{R}^n)}\leqq C_{j,\beta,k,\ell,n}t^{-\left(1+j+\frac{|\beta|}{2}\right)}. \tag{4.20}$$

In view of (4.19) and (4.20), we consider the function:

$$N_\ell(t,x) = \mathcal{F}^{-1}\left[\partial_t^\ell\left(\frac{\sin\gamma|\xi|t}{|\xi|}\right)\hat{G}(t,\xi)\right](x),$$

where $G(t,x)$ satisfies the following conditions:

$$\|\partial_t^j\partial_x^\beta G(t,\cdot)\|_{L_1(\mathbb{R}^n)} \leqq C_{j,\beta,n}t^{-\left(1+j+\frac{|\beta|}{2}\right)}. \tag{4.21}$$

In order to prove (4.18), it suffices to show that

$$\|\partial_t^j\partial_x^\beta N_\ell(t,\cdot)\|_{L_1(\mathbb{R}^n)} \leqq C_{j,\beta,\ell,n}t^{q(n)-\frac{j+|\beta|+\ell}{2}}. \tag{4.22}$$

By (3.17), we have

$$\begin{aligned}
&\partial_t^j\partial_x^\beta N_\ell(t,x)\\
&=\sum_{k=0}^{j}\binom{j}{k}\sum_{|\alpha|\leqq\frac{n-3}{2}}a_\alpha\sum_{m=0}^{\min(\ell+k,|\alpha|+1)}\binom{\ell+k}{m}\partial_t^m(\gamma t)^{|\alpha|+1}\\
&\quad\cdot\sum_{|\delta|=\ell+k-m}\int_{|z|=1}z^{\alpha+\delta}\left(\partial_x^{\alpha+\beta+\delta}\partial_t^{j-k}G\right)(t,x+\gamma tz)\,dS
\end{aligned}$$

when n is an odd $\geqq 3$; and

$$\begin{aligned}
&\partial_t^j\partial_x^\beta N_\ell(t,x)\\
&=\sum_{k=0}^{j}\binom{j}{k}\sum_{|\alpha|\leqq\frac{n-2}{2}}a_\alpha\sum_{m=0}^{\min(\ell+k,|\alpha|+1)}\binom{\ell+k}{m}\partial_t^m(\gamma t)^{|\alpha|+1}\\
&\quad\cdot\sum_{|\delta|=\ell+k-m}\int_{|z|\leqq1}\frac{z^{\alpha+\delta}\left(\partial_x^{\alpha+\beta+\delta}\partial_t^{j-k}G\right)(t,x+\gamma tz)}{\sqrt{1-|z|^2}}\,dz
\end{aligned}$$

when n is an even $\geqq 2$. Therefore, by (4.21) we have

$$\begin{aligned}
&\|\partial_t^j\partial_x^\beta N_\ell(t,\cdot)\|_{L_1(\mathbb{R}^n)}\\
&\leqq C_{j,\beta,\ell,n}\sum_{k=0}^{j}\sum_{|\alpha|\leqq p(n)}\sum_{m=0}^{\min(\ell+k,|\alpha|+1)}t^{\frac{|\alpha|-|\beta|-\ell-j}{2}-\frac{j+m-k}{2}}\\
&\leqq C_{j,\beta,\ell,n}t^{q(n)-\frac{j+|\beta|+\ell}{2}},
\end{aligned}$$

which implies (4.22).

In order to estimate the remainder term $\mathcal{R}_N(t,x)$, we consider the function:

$$R_{N,\psi}^{\pm}(t,x) = \mathcal{F}^{-1}\left[r_{N,\psi}^{\pm}(t,\xi)\right](x),$$

where

$$\begin{aligned} &r^{\pm}_{N,\psi}(t,\xi) \\ &= \frac{e^{-A|\xi|^2 t}}{2i\gamma|\xi|f(|\xi|)} \int_0^1 (1-\theta)^N e^{\pm i\gamma|\xi|(1+\theta|\xi|^2 g(|\xi|^2))t}\, d\theta \\ &\quad \cdot (\pm i\gamma|\xi|^3 g(|\xi|^2))^{N+1} \psi(\xi)\varphi_0(\xi), \end{aligned}$$

and $\psi \in C^\infty(\mathbb{R}^n - \{0\})$ satisfies the condition:

$$\left|\partial_\xi^\gamma \psi(\xi)\right| \leqq C_\gamma |\xi|^{-|\gamma|}, \quad {}^\forall \xi \neq 0.$$

If $\psi(\xi) = \frac{\xi_j \xi_k}{|\xi|^2}$, then we have

$$\mathcal{R}_N(t,x) = \partial_t \left\{ \left(R^+_{N,\psi}(t,x) - R^-_{N,\psi}(t,x) \right) t^{N+1} \right\}. \tag{4.23}$$

First, we observe that

$$\begin{aligned} &\left|\partial_\xi^\delta \left\{ (-A|\xi|^2 \pm i\gamma|\xi|(1+\theta|\xi|^2) g(|\xi|^2))^j (i\xi)^\beta r^{\pm}_{N,\psi}(t,\xi) \right\}\right| \\ &\leqq C_{j,\beta,N,\delta} |\xi|^{3N+2+j+|\beta|-2|\delta|} e^{-\frac{A}{4}|\xi|^2 t}, \quad 0 \leqq \theta \leqq 1,\ \xi \neq 0. \end{aligned} \tag{4.24}$$

In fact, by the formula of derivative of composed function (cf. (3.19)), we have

(4.25)

$$\begin{aligned} &\partial_\xi^\delta e^{\pm i\gamma|\xi|(1+\theta|\xi|^2 g(|\xi|^2)t)} \\ &= \sum_{\ell=1}^{|\delta|} (\pm i\gamma t)^\ell e^{\pm i\gamma|\xi|(1+\theta|\xi|^2 g(|\xi|^2)t)} \\ &\quad \cdot \sum_{\substack{|\alpha_1|+\cdots+|\alpha_\ell|=|\delta| \\ |\alpha_i| \geqq 1}} \partial_\xi^{\alpha_1} \left\{ |\xi|(1+\theta|\xi|^2 g(|\xi|^2)) \right\} \ldots \partial_\xi^{\alpha_\ell} \left\{ |\xi|(1+\theta|\xi|^2 g(|\xi|^2)) \right\}. \end{aligned}$$

Since

$$\left|\partial_\xi^{\alpha_\nu} \left\{ |\xi|(1+\theta|\xi|^2 g(|\xi|^2)) \right\}\right| \leqq C_{\alpha,\nu} |\xi|^{1-|\alpha_\nu|}, \quad \xi \in \operatorname{supp} \varphi_0,$$

by (4.25) we have

$$\left|\partial_\xi^\delta e^{\pm i\gamma|\xi|(1+\theta|\xi|^2 g(|\xi|^2)t)}\right| \leqq C_\delta \sum_{\ell=1}^{|\delta|} (t|\xi|)^\ell |\xi|^{-|\delta|}, \quad \xi \in \operatorname{supp} \varphi_0.$$

Therefore, we have for $\xi \in \operatorname{supp}\varphi_0$ and $\xi \neq 0$

$$\begin{aligned}
&\left|\partial_\xi^\delta \left\{(-A|\xi|^2 \pm i\gamma|\xi|(1+\theta|\xi|^2)g(|\xi|^2))^j (i\xi)^\beta r_{N,\psi}^{\pm}(t,\xi)\right\}\right| \\
&\leqq C_\delta \sum_{0 \leqq \nu \leqq \delta} |\xi|^{3N+2+j+|\beta|-|\nu|} e^{-\frac{A}{2}|\xi|^2 t} \sum_{\ell=1}^{|\delta-\nu|} (t|\xi|)^\ell |\xi|^{-|\delta-\nu|} \\
&\leqq C_\delta \sum_{0 \leqq \nu \leqq \delta} |\xi|^{3N+2+j+|\beta|-|\delta|} e^{-\frac{A}{2}|\xi|^2 t} \sum_{\ell=1}^{|\delta-\nu|} (t|\xi|)^\ell |\xi|^{-\ell}.
\end{aligned}$$

Since $|\xi|^{-\ell} \leqq C_\delta |\xi|^{-|\delta|}$ $(0 \leqq \ell \leqq |\delta|)$ when $\xi \in \operatorname{supp}\varphi_0$ and $\xi \neq 0$, by the above inequality we have (4.24). Since

$$e^{ix\cdot\xi} = \sum_{j=1}^{n} \frac{x_j}{i|x|^2} \partial_{\xi_j} e^{ix\cdot\xi}, \tag{4.26}$$

in view of (4.24) by $n+1$-times integration by parts, we have

$$\begin{aligned}
&\partial_t^j \partial_x^\beta R_{N,\psi}^{\pm}(t,x) \\
&= \sum_{|\delta|=n+1} \left(\frac{ix}{|x|^2}\right)^\delta \left(\frac{1}{2\pi}\right)^n \int_0^1 \int_{\mathbb{R}^n} e^{ix\cdot\xi} \partial_\xi^\delta \\
&\qquad \cdot \left\{\left(-A|\xi|^2 \pm i\gamma|\xi|(1+\theta|\xi|^2 g(|\xi|^2))\right)^j (i\xi)^\beta r_{N,\psi}^{\pm}(t,\xi)\right\} d\xi d\theta
\end{aligned}$$

when $N > n/3$. Therefore, by (4.24) we have

$$\begin{aligned}
\left|\partial_t^j \partial_x^\beta R_{N,\psi}^{\pm}(t,x)\right| &\leqq C_{j,\beta,N,n} |x|^{-(1+n)} \int_{\mathbb{R}^n} |\xi|^{3N+2+j+|\beta|-2(n+1)} e^{-\frac{A}{4}|\xi|^2 t} d\xi \\
&\leqq C_{j,\beta,N,n} |x|^{-(n+1)} t^{-\frac{3N+j+|\beta|-n}{2}}.
\end{aligned}$$

On the other hand, by (4.24) with $\delta = 0$ we have

$$\begin{aligned}
\left|\partial_t^j \partial_x^\beta R_{N,\psi}^{\pm}(t,x)\right| &\leqq C_{j,\beta,N} \int_{\mathbb{R}^n} |\xi|^{3N+2+j+|\alpha|} e^{-\frac{A}{4}|\xi|^2 t} d\xi \\
&\leqq C_{j,\beta,N} t^{-\frac{3N+2+j+|\beta|+n}{2}}.
\end{aligned}$$

Combining these two estimations, we have

$$\begin{aligned}
&\|\partial_t^j \partial_x^\beta R_{N,\psi}^{\pm}(t,\cdot)\|_{L_1(\mathbb{R}^n)} \\
&\leqq C_{j,\beta,N,n} \left\{\int_{|x|\leqq\sqrt{t}} t^{-\frac{3N+2+j+|\beta|+n}{2}} dx + \int_{|x|\geqq\sqrt{t}} t^{-\frac{3N+j+|\alpha|-n}{2}} |x|^{-n+1} dx\right\} \\
&\leqq C_{j,\beta,N,n} t^{-\frac{3N+|\alpha|+j+1-n}{2}}, \quad t \geqq 1.
\end{aligned} \tag{4.27}$$

By (4.23) and (4.27), we have

$$\|\partial_t^j \partial_x^\beta \mathcal{R}_N(t,\cdot)\|_{L_1(\mathbb{R}^n)} \leqq C_{j,\beta,N} t^{-\frac{N-n+j+|\beta-1}{2}}, \quad t \geqq 1,\ N > \frac{n}{3}. \tag{4.28}$$

Combining (4.4), (4.12), (4.17), (4.18) and (4.28), we have (4.3). To complete the Proof of Theorem 2.1 (2), we have to show that

$$\|\partial_t^j \partial_x^\alpha L_0(t,\cdot)\|_{L_1(\mathbb{R}^n)} \leqq C_{j,\alpha,n}, \quad 0 \leqq t \leqq 1. \tag{4.29}$$

Regarding the relations:

$$\lambda_\pm(\xi) = -A|\xi|^2 \mp i\gamma|\xi| f(|\xi|),$$

we put

$$L_0(t,x) = \sum_{j=1}^{3} \mathcal{F}^{-1}\left[\psi_j(t,\xi)\right](x),$$

where

$$\psi_1(t,\xi) = \frac{A|\xi| e^{-A|\xi|^2 t}}{2i\gamma f(|\xi|)} \left(e^{-i\gamma|\xi| f(|\xi|)t} - e^{i\gamma|\xi| f(|\xi|)t}\right) \frac{\xi_j \xi_k}{|\xi|^2} \varphi_0(\xi),$$

$$\psi_2(t,\xi) = e^{-A|\xi|^2 t} \left(\frac{e^{-i\gamma|\xi| f(|\xi|)t} - e^{i\gamma|\xi| f(|\xi|)t}}{2} - 1\right) \frac{\xi_j \xi_k}{|\xi|^2} \varphi_0(\xi),$$

$$\psi_3(t,\xi) = \left(e^{-A|\xi|^2 t} - e^{-\alpha|\xi|^2 t}\right) \frac{\xi_j \xi_k}{|\xi|^2} \varphi_0(\xi).$$

First, we shall estimate $\mathcal{F}^{-1}\left[\psi_1(t,\xi)\right](x)$. By the formula of derivative of composed function (cf. (3.19)), we have

(4.30)

$$\partial_\xi^\delta e^{\pm\gamma|\xi| f(|\xi|)t} = \sum_{\ell=1}^{|\delta|} (\pm i\gamma t)^\ell e^{\pm i\gamma|\xi| f(|\xi|)t} \sum_{\substack{|\alpha_1|+\cdots+|\alpha_\ell|=|\delta| \\ |\alpha_i| \geqq 1}} \partial_\xi^{\alpha_1}\{|\xi| f(|\xi|)\} \ldots \partial_\xi^{\alpha_\ell}\{|\xi| f(|\xi|)\}.$$

Since

$$\left|\partial_\xi^{\alpha_\nu} |\xi| f(|\xi|)\right| \leqq C_{\alpha_\nu} |\xi|^{1-|\alpha_\nu|}, \quad |\xi| \leqq \frac{B}{\sqrt{2}},\ \xi \neq 0,$$

by (4.30) we have

$$\left|\partial_\xi^\delta e^{\pm i\gamma|\xi| f(|\xi|)t}\right| \leqq C_\delta \sum_{\ell=1}^{|\delta|} |\xi|^{\ell-|\delta|} \leqq C_\delta |\xi|^{-|\delta|}, \quad |\xi| \leqq \frac{B}{\sqrt{2}},\ \xi \neq 0. \tag{4.31}$$

By (4.31) and Leibniz' rule, we have

$$\left|\partial_\xi^\delta \left\{\partial_t^j \psi_1(t,\xi)(i\xi)^\alpha\right\}\right| \leqq C_{j,\delta,\alpha} |\xi|^{1-|\delta|}, \quad |\xi| \leqq \frac{B}{\sqrt{2}},\ \xi \neq 0.$$

Since $\operatorname{supp}\psi_1(t,\cdot) \subset \{\xi \in \mathbb{R}^n \mid |\xi| \leqq B/\sqrt{2}\}$, by Proposition 3.4 with $(\alpha, N, \sigma) = (1, n, 1)$ we have

$$\left|\partial_t^j \partial_x^\alpha \mathcal{F}^{-1}\left[\psi_1(t,\xi)\right](x)\right| \leqq C_{j,\alpha,n}|x|^{-n-1}, \quad {}^\forall x \neq 0.$$

On the other hand, we have

$$\left|\partial_t^j \partial_x^\alpha \mathcal{F}^{-1}\left[\psi_1(t,\xi)\right](x)\right| \leqq C_{j,\alpha,n} \int_{|\xi| \leqq \frac{B}{\sqrt{2}}} |\xi|\, d\xi \leqq C_{j,\alpha,n}.$$

Therefore, combining these two estimations, we have

(4.32)

$$\begin{aligned} \|\partial_t^j \partial_x^\alpha \mathcal{F}^{-1}\left[\psi_1(t,\xi)\right](\cdot)\|_{L_1(\mathbb{R}^n)} &\leqq C_{j,\alpha,n}\left\{\int_{|x|\leqq 1} dx + \int_{|x| \geqq 1} |x|^{-(n+1)}\, dx\right\} \\ &\leqq C_{j,\alpha,n}. \end{aligned}$$

Next, we shall estimate $\mathcal{F}^{-1}\left[\psi_2(t,\xi)\right](x)$. By Taylor's formula, we have

$$\psi_2(t,\xi) = -i\gamma f(|\xi|)te^{-A|\xi|^2 t}\int_0^1 \sin(\theta\gamma|\xi|f(|\xi|)t)\, d\theta \frac{\xi_j\xi_k}{|\xi|}\varphi_0(\xi).$$

Therefore, we have

$$\left|\partial_\xi^\delta\left\{(i\xi)^\alpha \partial_t^j \psi_2(t,\xi)\right\}\right| \leqq C_{j,\alpha,\delta}|\xi|^{1-|\delta|}, \quad \xi \neq 0.$$

Employing the same argument as in $\mathcal{F}^{-1}\left[\psi_1(t,\xi)\right](x)$, we have

$$\|\partial_t^j \partial_x^\alpha \mathcal{F}^{-1}\left[\psi_2(t,\xi)\right](\cdot)\|_{L_1(\mathbb{R}^n)} \leqq C_{j,\alpha,n}. \tag{4.33}$$

Finally, we shall estimate $\mathcal{F}^{-1}\left[\psi_3(t,\xi)\right](x)$. Since

$$\psi_3(t,x) = (A-\alpha)t\int_0^1 e^{-((1-\theta)A+\theta\alpha)|\xi|^2 t}\, d\theta \xi_j\xi_k\varphi_0(\xi),$$

we have

$$\left|\partial_\xi^\delta\left\{(i\xi)^\alpha \partial_t^j \psi_3(t,\xi)\right\}\right| \leqq C_{j,\alpha,\delta}|\xi|^{2-|\delta|}.$$

Therefore, employing the same argument as in $\mathcal{F}^{-1}\left[\psi_1(t,\xi)\right](x)$, we have

$$\|\partial_t^j \partial_x^\alpha \mathcal{F}^{-1}\left[\psi_3(t,\xi)\right](\cdot)\|_{L_1(\mathbb{R}^n)} \leqq C_{j,\alpha,n}. \tag{4.34}$$

Combining (4.32), (4.33) and (4.34), we have (4.29), which completes the proof.

5. Proof of Theorem 2.2.

In this section, we shall prove Theorem 2.2. First, we consider the part where $|\xi| \geqq \sqrt{2}B$. Since $\lambda_\pm(\xi) = -A\left(|\xi|^2 \pm \sqrt{|\xi|^4 - B^2|\xi|^2}\right)$ when $|\xi| \geqq \sqrt{2}B$, we write:

$$\lambda_+(\xi) = -A|\xi|^2 + 1 + \mu(\xi), \quad \lambda_-(\xi) = -1 - \mu(\xi), \tag{5.1}$$

where

$$\mu(\xi) = \frac{AB^4}{4|\xi|^2} g\left(\frac{B^2}{|\xi|^2}\right), \quad g(s) = \int_0^1 (1-\theta s)^{-\frac{3}{2}}(1-\theta)\, d\theta.$$

Note that $g(B^2/|\xi|^2) \in C^\infty$ when $|\xi| \geqq 2$. In view of (2.5) and (2.9), we put

$$\begin{aligned}
L_\pm(t)u(x) &= \mathcal{F}^{-1}\left[\frac{\lambda_\mp(\xi)e^{\lambda_\pm(\xi)t}}{\lambda_+(\xi)-\lambda_-(\xi)}\varphi_\infty(\xi)\hat{u}(\xi)\right](x),\\
M_{\pm,\beta}(t)u(x) &= \mathcal{F}^{-1}\left[\frac{\xi^\beta e^{\lambda_\pm(\xi)t}}{\lambda_+(\xi)-\lambda_-(\xi)}\varphi_\infty(\xi)\hat{u}(\xi)\right](x), \quad |\beta| = 1,\\
K_{\pm,\infty}(t)v_0(x) &= \mathcal{F}^{-1}\left[\frac{\lambda_\pm(\xi)e^{\lambda_\pm(\xi)t}}{\lambda_+(\xi)-\lambda_-(\xi)}\frac{\xi_j\xi_k}{|\xi|^2}\varphi_\infty(\xi)\widehat{v_0}(\xi)\right](x),\\
K_{1,\infty}(t)v_0(x) &= \mathcal{F}^{-1}\left[e^{-\alpha|\xi|^2t}\left(\delta_{jk} - \frac{\xi_j\xi_k}{|\xi|^2}\right)\varphi_\infty(\xi)\widehat{v_0}(\xi)\right](x).
\end{aligned}$$

By [**6**, Theorem 4.2.1], we have for $p = 1$ or ∞

$$\begin{aligned}
&\|\partial_t^j\partial_x^\alpha L_+(t)u\|_{L_p(\mathbb{R}^n)} \leqq C_{j,k,\alpha}t^{-(j-k)}e^{-ct}\|u\|_{W_p^{2k+(|\alpha|-1)^+}(\mathbb{R}^n)},\\
&\|\partial_t^j\partial_x^\alpha M_{+,\beta}(t)u\|_{L_p(\mathbb{R}^n)} \leqq C_{j,k,\alpha}t^{-(j-k)}e^{-ct}\|u\|_{W_p^{2k+|\alpha|}(\mathbb{R}^n)}, \quad |\beta| = 1,\\
&\|\partial_t^j\partial_x^\alpha\left(L_-(t)u - e^{-t}u\right)\|_{L_p(\mathbb{R}^n)} \leqq C_{j,\alpha}e^{-ct}\|u\|_{W_p^{(|\alpha|-1)^+}(\mathbb{R}^n)},\\
&\|\partial_t^j\partial_x^\alpha M_{-,\beta}(t)u\|_{L_p(\mathbb{R}^n)} \leqq C_{j,\alpha}e^{-ct}\|u\|_{W_p^{|\alpha|}(\mathbb{R}^n)}, \quad |\beta| = 1.
\end{aligned} \tag{5.2}$$

Now we shall show that for $p = 1$ or ∞

(5.3)

$$\begin{aligned}
\|\partial_t^j\partial_x^\alpha k_{+,\infty}(t)v_0\|_{L_p(\mathbb{R}^n)} &\leqq C_{j,\alpha,n}\left(1+t^{-\frac{1}{2}}\right)t^{-(j-k)}e^{-ct}\|v_0\|_{W_p^{2k+|\alpha|}(\mathbb{R}^n)},\\
\|\partial_t^j\partial_x^\alpha k_{1,\infty}(t)v_0\|_{L_p(\mathbb{R}^n)} &\leqq C_{j,\alpha,n}\left(1+t^{-\frac{1}{2}}\right)t^{-(j-k)}e^{-ct}\|v_0\|_{W_p^{2k+|\alpha|}(\mathbb{R}^n)}.
\end{aligned}$$

Put

$$K_{+,j,k}(t,x) = \mathcal{F}^{-1}\left[\frac{\lambda_+(\xi)^j e^{\lambda_+(\xi)t}(1+|\xi|^2)^{j-k}}{(1+|\xi|^2)^j}\frac{\xi_m\xi_\ell}{|\xi|^2}\varphi_\infty(\xi)\right](x),$$

$$K_{1,j,k}(t,x) = \mathcal{F}^{-1}\Bigg[\frac{(-\alpha|\xi|^2)^j e^{-\alpha|\xi|^2 t}(1+|\xi|^2)^{j-k}}{(1+|\xi|^2)^j} \cdot \left(\delta_{m\ell} - \frac{\xi_m\xi_\ell}{|\xi|^2}\right)\varphi_\infty(\xi)\Bigg](x),$$

for $j \leqq k \leqq 0$, and then

$$\partial_t^j\partial_x^\alpha K_{+,\infty}(t)v_0 = K_{+,j,k}(t,\cdot) * \partial_x^\alpha(1-\Delta)^k v_0; \tag{5.4}$$

$$\partial_t^j\partial_x^\alpha K_{1,\infty}(t)v_0 = K_{1,j,k}(t,\cdot) * \partial_x^\alpha(1-\Delta)^k v_0. \tag{5.5}$$

By (5.1) we have for $|\xi| \geqq \sqrt{2}B$

$$\begin{aligned}&\left|\partial_\xi^\nu\left\{\frac{\lambda_+(\xi)^j e^{\lambda_+(\xi)t}(1+|\xi|^2)^{j-k}}{(1+|\xi|^2)^j}\frac{\xi_m\xi_\ell}{|\xi|^2}\varphi_\infty(\xi)\right\}\right| \\ &\leqq C_{j,k,\nu}t^{-(j-k)}e^{-c_1 t}|\xi|^{-|\nu|}e^{-c_2|\xi|^2 t}, \quad {}^\forall\nu,\end{aligned} \tag{5.6}$$

and also we have

$$\begin{aligned}&\left|\partial_\xi^\nu\left\{\frac{(-\alpha|\xi|^2)^j e^{-\alpha|\xi|^2 t}(1+|\xi|^2)^{j-k}}{(1+|\xi|^2)^j}\frac{\xi_m\xi_\ell}{|\xi|^2}\varphi_\infty(\xi)\right\}\right| \\ &\leqq C_{j,k,\nu}t^{-(j-k)}e^{-c_1 t}|\xi|^{-|\nu|}e^{-c_2|\xi|^2 t}, \quad {}^\forall\nu.\end{aligned} \tag{5.7}$$

Therefore, using (4.26) and the integration by parts $n+1$ times, by (5.6) we have

$$\begin{aligned}|L_{+,j,k}(t,x)| &\leqq C_{j,k,n}\frac{t^{-(j-k)}e^{-c_1 t}}{|x|^{n+1}}\int_{|\xi|\geqq\sqrt{2}B}|\xi|^{-n-1}e^{-c_2|\xi|^2 t}\,d\xi \\ &\leqq C_{j,k,n}\frac{t^{-(j-k)}e^{-c_1 t}}{|x|^{n+1}},\end{aligned} \tag{5.8}$$

and by (5.7) we have

$$|L_{1,j,k}(t,x)| \leqq C_{j,k,n}\frac{t^{-(j-k)}e^{-c_1 t}}{|x|^{n+1}}. \tag{5.9}$$

On the other hand, by (5.6) we have

$$\begin{aligned}|L_{+,j,k}(t,x)| &\leqq C_{j,k,n}t^{-(j-k)}e^{-c_1 t}\int_{|\xi|\geqq\sqrt{2}B}e^{-c_2|\xi|^2 t}\,d\xi \\ &\leqq C_{j,k,n}t^{-(j-k)-\frac{n}{2}}e^{-c_1 t},\end{aligned} \tag{5.10}$$

and by (5.7) we have

$$|L_{1,j,k}(t,x)| \leqq C_{j,k,n} t^{-(j-k)-\frac{n}{2}} e^{-c_1 t}. \tag{5.11}$$

Therefore, by (5.8) and (5.10) we have

$$\begin{aligned} &\|L_{+,j,k}(t,\cdot)\|_{L_1(\mathbb{R}^n)} \\ &\leqq C_{j,k,n} t^{-(j-k)} e^{-c_1 t} \left\{ t^{-\frac{n}{2}} \int_{|x|\leqq\sqrt{t}} dx + \int_{|x|\geqq\sqrt{t}} \frac{1}{|x|^{n+1}}\, dx \right\} \\ &\leqq C_{j,k,n} \left(1 + t^{-\frac{1}{2}}\right) t^{-(j-k)} e^{-c_1 t}, \end{aligned} \tag{5.12}$$

and by (5.9) and (5.11) we have

$$\|L_{1,j,k}(t,\cdot)\|_{L_1(\mathbb{R}^n)} \leqq C_{j,k,n} \left(1 + t^{-\frac{1}{2}}\right) t^{-(j-k)} e^{-c_1 t}. \tag{5.13}$$

By (5.12), (5.13) and the Young inequality, we have (5.3).

Next, we shall show that for $p = 1$ or ∞

$$\|\partial_t^j \partial_x^\alpha K_{-,\infty}(t) v_0\|_{L_p(\mathbb{R}^n)} \leqq C_{j,k,\alpha,n} t^{-(j-k)} e^{-c_1 t} \|v_0\|_{W_p^{2k+|\alpha|}(\mathbb{R}^n)}. \tag{5.14}$$

Put

$$\ell_-(t,x) = \mathcal{F}^{-1} \left[\frac{\lambda_-(\xi) e^{\lambda_-(\xi)t}}{\lambda_+(\xi) - \lambda_-(\xi)} \frac{\xi_j \xi_k}{|\xi|^2} \varphi_\infty(\xi) \right](x),$$

and then

$$K_{-,\infty}(t) v_0(x) = \ell_-(t,\cdot) * v_0. \tag{5.15}$$

Now, we shall prove that

$$\|\partial_t^j \ell_-(t,\cdot)\|_{L_1(\mathbb{R}^n)} \leqq C_j e^{-ct}. \tag{5.16}$$

By (5.1) we have for $|\xi| \geqq \sqrt{2}B$

$$\left| \partial_\xi^\beta \left\{ \frac{\lambda_-(\xi)^{j+1} e^{\lambda_-(\xi)t}}{\lambda_+(\xi) - \lambda_-(\xi)} \frac{\xi_m \xi_k}{|\xi|^2} \varphi_\infty(\xi) \right\} \right| \leqq C_{j,\beta} (1+t)^{|\beta|} e^{-ct} |\xi|^{-2-|\beta|}. \tag{5.17}$$

Therefore, using (4.26) and the integration by parts $n-1$ times, by (5.17) we have

$$\left|\partial_t^j \ell_-(t,x)\right| \leqq C_{j,n} e^{-ct} \begin{cases} |x|^{-(n-1)}, & 0 < |x| \leqq 1; \\ |x|^{-(n+1)}, & |x| \geqq 1, \end{cases}$$

which implies (5.16). By (5.16) and the Young inequality, we have (5.14).

In order to complete the Proof of Theorem 2.2, we have to estimate the part where $B/2 \leqq |\xi| \leqq 2B$ (cf. (2.10)). In view of (2.6) and (2.9), below, if we put

(5.18)

$$N_{0,\psi}(t,x) = \frac{1}{2\pi i}\mathcal{F}^{-1}\left[\oint_\Gamma \frac{(z+|\xi|^2)e^{zt}}{z^2+(\alpha+\beta)|\xi|^2z+\gamma^2|\xi|^2}\,dz\psi(\xi)\varphi_M(\xi)\right](x);$$

$$N_{1,\psi}(t,x) = \frac{1}{2\pi i}\mathcal{F}^{-1}\left[i\gamma\xi\oint_\Gamma \frac{e^{zt}}{z^2+(\alpha+\beta)|\xi|^2z+\gamma^2|\xi|^2}\,dz\psi(\xi)\varphi_M(\xi)\right](x);$$

$$N_{2,\psi}(t,x) = \mathcal{F}^{-1}\left[e^{-\alpha|\xi|^2t}\psi(\xi)\varphi_M(\xi)\right](x),$$

where $\psi \in C^\infty(S^{n-1})$ and $\psi = \psi(\xi/|\xi|)$, then we have

$$\mathcal{F}^{-1}\left[\varphi_M(\xi)\hat{\rho}(t,\xi)\right](x) = N_{0,\psi}(t,\cdot)*\rho_0 + N_{1,\psi}(t,\cdot)*v_0;$$

$$\mathcal{F}^{-1}\left[\varphi_M(\xi)\hat{v}(t,\xi)\right](x) = N_{1,\psi}(t,\cdot)*\rho_0 + N_{0,\psi}(t,\cdot)*v_0 + N_{2,\psi}(t,\cdot)*v_0.$$

If we use (4.26) and (2.7), then we see easily that

$$\left|\partial_t^j\partial_x^\alpha N_{\ell,\psi}(t,x)\right| \leqq C_{j,\alpha,N}e^{-ct}|x|^{-N}, \quad {}^\forall N \geqq 0, \text{ integer}.$$

Therefore, applying the Young inequality to (5.18) we have

(5.19)

$$\|\mathcal{F}^{-1}\left[\varphi_M(\xi)(\hat{\rho},\hat{v})(t,\xi)\right]\|_{L_p(\mathbb{R}^n)} \leqq C_{j,\alpha,p}e^{-ct}\|(\rho_0,v_0)\|_{L_p(\mathbb{R}^n)}, \quad 1 \leqq p \leqq \infty.$$

Combining (5.2), (5.3), (5.14) and (5.19), we have Theorem 2.2.

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Department of Mathematics
Kyushu Institute of Technology
1-1 Sensui-cho, Tobata-ku
Kitakyusyu 804-8550
Japan
E-mail address: kobayasi@math.tsukuba.ac.jp

Department of Mathematics
School of Science and Engineering
Waseda University
3-4-1 Ohkubo Shinjuku-ku
Tokyo 169-8555
Japan
E-mail address: yshibata@waseda.jp